STABLE LAWS FOR RANDOM DYNAMICAL SYSTEMS

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Abstract. In this paper we consider random dynamical systems formed by concatenating maps acting on the unit interval $[0,1]$ in an iid fashion. Considered as a stationary Markov process, the random dynamical system possesses a unique stationary measure ν. We consider a class of non square-integrable observables ϕ, mostly of form $\phi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$ where x_0 is non-periodic point satisfying some other genericity conditions, and more generally regularly varying observables with index $\alpha \in (0,2)$. The two types of maps we concatenate are a class of piecewise C^2 expanding maps, and a class of intermittent maps possessing an indifferent fixed point at the origin. Under conditions on the dynamics and α we establish Poisson limit laws, convergence of scaled Birkhoff sums to a stable limit law and functional stable limit laws, in both the annealed and quenched case. The scaling constants for the limit laws for almost every quenched realization are the same as those of the annealed case and determined by ν. This is in contrast to the scalings in quenched central limit theorems where the centering constants depend in a critical way upon the realization and are not the same for almost every realization.

Contents

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1. Introduction

In this paper we consider non square-integrable observables $\phi : [0,1] \to \mathbb{R}$ on two simple classes of random dynamical system. One consists of randomly choosing in an iid manner from a finite set of maps which are strictly polynomially mixing with an indifferent fixed point at the origin, the other consisting of randomly choosing from a finite set of maps which are uniformly expanding and exponentially mixing. The main type of observable we consider is of the form $\phi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$, $\alpha \in (0, 2)$ which in the IID case lies in the domain of attraction of a stable law of index α. For certain results the point x_0 has to satisfy some nongenericity conditions and in particular not be a periodic point for almost every realization of the random system (see Definition 6.1). Some of our results, particularly those involving convergence to exponential and Poisson laws hold for general observables that are regularly varying with index α.

Our setup is to consider a finite set of m maps of the unit interval and choose from the set $\{T_i\}_{i=1}^m$ in an iid fashion according to a probability vector $(p_1, \ldots, p_m)$.

Let $\Omega := \{1, \ldots, m\}^{\mathbb{Z}}$, and define on Ω the product measure $\mathbb{P} := \{(p_1, \ldots, p_m)\}^{\mathbb{Z}}$. The left shift σ on Ω preserves $\mathbb{P}$. We write $\omega \in \Omega$ as $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots, \omega_n)$ and denote

$$T_\omega^n := T_{\sigma^{n-1}\omega} \circ \ldots \circ T_{\omega_0} = T_{\omega_{n-1}} \circ \ldots \circ T_{\omega_0} \tag{1.1}$$

Fixing $\omega \in \Omega$ we form the quenched discrete time Birkhoff sum

$$S_n^\omega := \frac{1}{b_n} \sum_{j=0}^{n-1} \phi \circ T_\omega^j - c_n, \tag{1.2}$$

for some sequence of positive scaling constants b_n, c_n, and the quenched continuous time process

$$X_n^\omega(t) := \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor -1} \phi \circ T_\omega^j - t c_n, \quad t \geq 0, \tag{1.3}$$

Amongst other results, we describe the convergence of this quenched Birkhoff sum (1.2) to a stable law and of the corresponding quenched continuous time process (1.3) to a Lévy process (in the J_1 Skorohod topology) for $\mathbb{P}$-a.e. $\omega \in \Omega$ under various assumptions. A key result is that the scaling constants b_n, c_n are the same for $\mathbb{P}$-a.e. ω and are the same as for the annealed system. Our proofs are based on a Poisson process approach developed for dynamical systems by Marta Tyran-Kaminska [TK10a, TK10b]. Our main results are given in detail in Section 6.

2. Probabilistic tools

In this section, we review some topics from Probability Theory.

2.1. **Regularly varying functions and domains of attraction.**

We briefly describe here the relations between domains of attraction of stable laws and regularly varying functions; we refer to Feller [Fel71] or Bingham, Goldie and Teugels [BGT87] for more details.

Definition 2.1 (slow variation)**.** *A measurable function $L : (0, \infty) \to (0, \infty)$ is slowly varying if for all $\lambda > 0$,*

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1.$$

Let (Y, ν) be a probability space.

Definition 2.2 (regular variation)**.** *We say that $\phi : Y \to \mathbb{R}$ is regularly varying with index $\alpha > 0$ (with respect to ν) if there exists $p \in [0, 1]$ and a slowly varying function L such that*

$$\lim_{x \to \infty} \frac{\nu(\phi > x)}{\nu(|\phi| > x)} = p \quad \textit{and} \quad \lim_{x \to \infty} \frac{\nu(|\phi| > x)}{x^{-\alpha} L(x)} = 1. \tag{2.1}$$

Definition 2.3 (scaling constants)**.** *We consider a sequence $(b_n)_{n \geq 1}$ of positive real numbers such that*

$$\lim_{n \to \infty} n \nu(|\phi| > b_n) = 1. \tag{2.2}$$

The sequence b_n plays the role of scaling constants in stable limit theorems. For ϕ regularly varying such a sequence can be written as $b_n = n^{\frac{1}{\alpha}} \widetilde{L}(n)$ for a slowly varying function $\widetilde{L}$. As a consequence, with L the slowly varying function corresponding to ϕ,

$$\lim_{n \to \infty} \frac{n L(b_n)}{b_n^\alpha} = 1,$$

and for every $\lambda > 0$,

$$\lim_{n \to \infty} n \, \nu(|\phi| > \lambda b_n) = \lambda^{-\alpha}. \tag{2.3}$$

Definition 2.4 (centering constants)**.** *Let $\phi : Y \to \mathbb{R}$ be a measurable function such that $\nu(\phi \neq 0) = 1$. We assume that ϕ is regularity varying with index $\alpha \in (0,2)$. Let (b_n) be a sequence of positive numbers as in* (2.2)*.*

We define the centering sequence $(c_n)_{n\geq 1}$ by

$$c_n = \begin{cases} 0 & \text{if } \alpha \in (0,1) \\ \frac{n}{b_n}\mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi|\leq b_n\}}) & \text{if } \alpha = 1 \\ \frac{n}{b_n}\mathbb{E}_\nu(\phi) & \text{if } \alpha \in (1,2) \end{cases}.$$

Remark 2.5. *When $\alpha \in (0,1)$ then ϕ is not integrable and one can choose the centering sequence (c_n) to be identically 0. When $\alpha = 1$, it might happen that ϕ is not integrable, and it is then necessary to define c_n with suitably truncated moments as above. If ϕ is integrable then center by $c_n = nb_n^{-1}\mathbb{E}_\nu(\phi)$.*

We will use the following asymptotics for truncated moments, which can be deduced from Karamata's results concerning the tail behavior of regularly varying functions:

Proposition 2.6 (Karamata)**.** *Let ϕ be regularly varying with index $\alpha \in (0,2)$. Then, setting $\beta := 2p-1$ and, for $\varepsilon > 0$,*

$$c_\alpha(\varepsilon) := \begin{cases} 0 & \text{if } \alpha \in (0,1) \\ -\beta \log \varepsilon & \text{if } \alpha = 1 \\ \varepsilon^{1-\alpha}\beta\alpha/(\alpha-1) & \text{if } \alpha \in (1,2) \end{cases} \tag{2.4}$$

the following hold for all $\varepsilon > 0$:

(a) $\mathbb{E}_\nu(|\phi|^2 \mathbf{1}_{\{|\phi|\leq \varepsilon b_n\}}) \sim \dfrac{\alpha}{2-\alpha}(\varepsilon b_n)^2 \nu(|\phi| > \varepsilon b_n)$,

(b) if $\alpha \in (0,1)$,

$$\mathbb{E}_\nu(|\phi| \mathbf{1}_{\{|\phi|\leq \varepsilon b_n\}}) \sim \frac{\alpha}{1-\alpha}\varepsilon b_n \nu(|\phi| > \varepsilon b_n),$$

(c) if $\alpha \in (1,2)$,

$$\lim_{n\to\infty} \frac{n}{b_n}\mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) = c_\alpha(\varepsilon),$$

(d) if $\alpha = 1$,

$$\lim_{n\to\infty} \frac{n}{b_n}\mathbb{E}_\nu(\phi \mathbf{1}_{\{\varepsilon b_n < |\phi| \leq b_n\}}) = c_\alpha(\varepsilon),$$

(e) if $\alpha = 1$,

$$\frac{n}{b_n}\mathbb{E}_\nu(|\phi| \mathbf{1}_{\{|\phi|\leq \varepsilon b_n\}}) \sim \widetilde{L}(n),$$

for a slowly varying function $\widetilde{L}$,

In our results for concreteness we will consider the observable

$$\phi_{x_0}(x) = d(x,x_0)^{-\frac{1}{\alpha}} \tag{2.5}$$

where x_0 is a point in $[0,1]$, the phase space of the dynamical systems we consider.

2.2. **Lévy α-stable processes.** A helpful and more detailed discussion can be found, e.g., in [TK10a, TK10b].

$X(t)$ is a Lévy stable process if $X(0) = 0$, X has stationary independent increments and $X(1)$ has an α-stable distribution.

The Lévy-Khintchine representation for the characteristic function of an α-stable random variable $X_{\alpha,\beta}$ with index $\alpha \in (0, 2)$ and parameter $\beta \in [-1, 1]$ has the form:

$$\mathbb{E}[e^{itX}] = \exp\left[ita_\alpha + \int (e^{itx} - 1 - itx1_{[-1,1]}(x))\Pi_\alpha(dx)\right]$$

where

- $a_\alpha = \begin{cases} \beta\frac{\alpha}{1-\alpha} & \alpha \neq 1 \\ 0 & \alpha = 1 \end{cases}$,
- Π_α is a Lévy measure given by
$$d\Pi_\alpha = \alpha(p1_{(0,\infty)}(x) + (1-p)1_{(-\infty,0)}(x))|x|^{-\alpha-1}dx$$
- $p = \dfrac{\beta+1}{2}$.

Note that p and β may equally serve as parameters for $X_{\alpha,\beta}$. We will drop the β from $X_{\alpha,\beta}$, as is common in the literature, for simplicity of notation and when it plays no essential role.

2.3. **Poisson point processes.** Let $(T_n)_{n\geq 1}$ be a sequence of measurable transformations on a probability space $(Y, \mathcal{B}, \mu)$. For $n \geq 1$ we denote

$$T_1^n := T_n \circ \ldots \circ T_1. \tag{2.6}$$

Given $\phi : Y \to \mathbb{R}$ measurable, recall that we define the scaled Birkhoff sum by

$$S_n := \frac{1}{b_n}\sum_{j=0}^{n-1} \phi \circ T_1^j - c_n, \tag{2.7}$$

for some real constants $b_n > 0$, c_n and the scaled random process $X_n(t)$, $n \geq 1$, by

$$X_n(t) := \frac{1}{b_n}\sum_{j=0}^{\lfloor nt\rfloor -1} \phi \circ T_1^j - tc_n,\ t \geq 0, \tag{2.8}$$

For $X_\alpha(t)$ a Lévy α-stable process and $B \in \mathcal{B}((0,\infty) \times (\mathbb{R}/\{0\}))$ define

$$N_{(\alpha)}(B) := \#\{s > 0 : (s, \Delta X_\alpha(s)) \in B\}$$

where $\Delta X_\alpha(t) := X_\alpha(t) - X_\alpha(t^-)$.

The random variable $N_{(\alpha)}(B)$, which counts the jumps (and their time) of the Lévy process that lie in B, is finite a.s. if and only if $(m \times \Pi_\alpha)(B) < \infty$. In that case $N_{(\alpha)}(B)$ has a Poisson distribution with mean $(m \times \Pi_\alpha)(B)$.

Similarly, for the process given by (2.8), define

$$N_n(B) := \#\left\{j \geq 1\ :\ \left(\frac{j}{n}, \frac{\phi \circ T_1^{j-1}}{b_n}\right) \in B\right\},\ n \geq 1,$$

$N_n(B)$ counts the jumps of the process (2.8) that lie in B. When a realization $\omega \in \Omega$ is fixed we define

$$N_n^\omega(B) := \#\left\{ j \geq 1 \,:\, \left(\frac{j}{n}, \frac{\phi \circ T_\omega^{j-1}}{b_n} \right) \in B \right\},\ n \geq 1.$$

Definition 2.7. *We say N_n converges in distribution to $N_{(\alpha)}$ and write*

$$N_n \xrightarrow{d} N_{(\alpha)}$$

if and only if $N_n(B) \xrightarrow{d} N_{(\alpha)}(B)$ for all $B \in B((0,\infty) \times (\mathbb{R} \setminus \{0\}))$ with $(m \times \Pi_\alpha)(B) < \infty$ and $(m \times \Pi_\alpha)(\partial B) = 0$.

2.4. **Skorohod J_1 topology.** Let $\mathbb{D}[0,\infty)$ be the space of all $\mathbb{R}$-valued *cádlág* functions ψ on $[0,\infty)$, that is functions which are right continuous and have finite left hand limits for all $t > 0$.

Let Λ denote the set of strictly increasing continuous maps λ of $[0,\infty]$ to $[0,\infty]$ such that $\lambda(0) = 0$ and $\lambda(\infty) = \infty$.

The Skorohod J_1 topology on $\mathbb{D}[0,\infty)$ is defined as follows: ϕ_n converges to ψ in the J_1 topology if and only if there exists a sequence $\{\lambda_n\} \subset \Lambda$ such that

$$\sup_s |\lambda_n(s) - s| \to 0$$

and

$$\sup_{s \leq m} |\psi_n(\lambda_n(s)) - \psi(s)| \to 0$$

for all positive integers m.

3. Modes of Convergence

Consider the process X_α determined by the observable ϕ (that is, an iid version of ϕ which regularly varying with the same index α and parameter p). We are interested the following limits:

(A) **Poisson point process convergence.**

$$N_n^\omega \xrightarrow{d} N_{(\alpha)}$$

with respect to ν^ω for P a.e. ω where $N_{(\alpha)}$ is the Poisson point process of an α-stable process with parameter determined by ν, the annealed measure.

(B) **Stable law convergence.**

$$S_n^\omega := \frac{1}{b_n} \sum_{j=0}^{n-1} \phi \circ T_\omega^j - c_n \xrightarrow{d} X_\alpha(1)$$

for $\mathbb{P}$-a.e. ω, with respect to ν^ω, for ϕ regularly varying with index α and $X_\alpha(t)$ the corresponding α-stable process, for suitable scaling and centering constants b_n and c_n. See (1.1) for the definition of T_ω^j.

(C) **Functional stable law convergence.**

$$X_n^\omega(t) := \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi \circ T_\omega^j - t c_n \xrightarrow{d} X_\alpha(t)$$

in $\mathbb{D}[0,\infty)$ in the J_1 topology $\mathbb{P}$-a.e. ω, with respect to ν^ω for ϕ regularly varying with index α and $X_\alpha(t)$ the corresponding α-stable process.

For the cases we are considering, the scaling constants b_n are given by (2.2) in Definition 2.3, and the centering constants c_n are given in Definition 2.4 (see also Remark 2.5).

Remark 3.1 (Convergence with respect to Lebesgue measure)**.** *We state our limiting theorems with respect to the fiberwise measures ν^ω but by general results of Eagleson* [Eag76]*(see also* [Zwe07]*) the convergence holds with respect to any measure μ for which $\mu \ll \nu^\omega$, in particular our convergence results hold with respect to Lebesgue measure m. Further details are given in the Appendix.*

Remark 3.2 (IID random variables)**.** *An iid sequence of random variables ϕ_n with distribution function ϕ satisfies a stable limit law iff it satisfies a functional stable limit law. In that case the limit is an α-stable random variable, respectively a Lévy process with index α, and $\alpha \in (0,2]$. The case $\alpha = 2$ corresponds to Brownian Motion.*

In particular, if ϕ is regularly varying with index $\alpha \in (0,2)$, then the iid sequence is in the (generalized) domain of attraction of a stable law with index α.

Remark 3.3. *In the limit laws for quenched systems that we obtain of type (B) and (C), the centering sequence c_n* does not depend on the realization *ω. This is in contrast to the case of the CLT, where a random centering is necessary; see* [AA16, Theorem 9] *and* [NPT21, Theorem 5.3].

4. Stationary dynamical systems

There have been many results on stable laws for ergodic dynamical systems (T, X, μ) with some degree of hyperbolicity. The two main scenarios are: (1) the observable $\phi \notin L^2(\mu)$ [1]; (2) slow decay of correlations of (T, X, μ) for a class of regular observables on a Riemannian manifold X.

Example of (1): Gouëzel [Gou, Theorem 2.1] showed that if $T : [0,1] \to [0,1]$ is the doubling map $T(x) = 2x \pmod 1$ with Lebesgue as invariant measure, and $\phi(x) = x^{-\frac{1}{\alpha}}$, $\alpha \in (0,2)$ then there exists a sequence c_n such that

$$\frac{2^{\frac{1}{\alpha}} - 1}{n^{\frac{1}{\alpha}}} \sum_{j=0}^{n-1} \phi \circ T^j - c_n \xrightarrow{d} X_{\alpha,1}(1)$$

Example of (2): The Liverani-Saussol-Vaienti map, a form of Manneville-Pomeau map modeling intermittency, is defined for $\gamma \in (0,1)$ by

$$T_\gamma : [0,1] \to [0,1], \qquad T_\gamma(x) := \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } 0 \le x \le \frac{1}{2}; \\ 2x - 1 & \text{if } \frac{1}{2} < x < 1. \end{cases} \tag{4.1}$$

The map T_γ has a unique absolutely continuous invariant probability measure μ_γ.

Gouëzel [Gou04, Theorem 1.3] showed that if $\gamma > \frac{1}{2}$ and $\phi : [0,1] \to \mathbb{R}$ is Hölder continuous with $\phi(0) \neq 0$, $E_{\mu_\gamma}(\phi) = 0$ then for $\alpha = \frac{1}{\gamma}$

$$\frac{1}{bn^{\frac{1}{\alpha}}} \sum_{j=0}^{n-1} \phi \circ T^j \xrightarrow{d} X_{\alpha,\beta}(1)$$

(β has a complicated expression).

[1] Or, cases where the CLT does not hold because the observable is not sufficiently regular.

5. A Poisson Point Process Approach to random and sequential dynamical systems

Our results are based on the Poisson point process approach developed by Marta Tyran-Kamińska [TK10a, TK10b] adapted to our random setting (see Theorems 5.1 and 5.3). Namely, convergence to a stable law or a Lévy process follows from the convergence of the corresponding (Poisson) jump processes, and control of the small jumps.

A key role is played by Kallenberg's Theorem [Kal76, Theorem 4.7] to check convergence of the Poisson point processes, $N_n \xrightarrow{d} N_{(\alpha)}$. Kallenberg's theorem does not assume stationarity and hence we may use it in our setting.

In this section, we provide general conditions ensuring weak convergence to Lévy stable processes for non-stationary dynamical systems, following closely the approach of Tyran-Kamińska [TK10b]. We start from the very general setting of non-autonomous sequential dynamics and then specialize to the case of quenched random dynamical systems, which will be useful to treat i.i.d. random compositions in the later sections.

5.1. Sequential transformations.

Recall the notations introduced in Section 2.3. $(T_n)_{n\geq 1}$ is a sequence of measurable transformations on a probability space $(Y, \mathcal{B}, \mu)$. For $n \geq 1$, we define

$$T_1^n = T_n \circ \ldots \circ T_1.$$

For a measurable $\phi : Y \to \mathbb{R}$, we define the random process X_n, $n \geq 1$, by

$$X_n(t) = \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi \circ T_1^j - tc_n, \ t \geq 0,$$

for some constants $b_n > 0$, $c_n \in \mathbb{R}$.

For $B \in \mathcal{B}((0,\infty) \times (\mathbb{R} \setminus \{0\}))$ we define

$$N_n(B) := \# \left\{ j \geq 1 \ : \ \left(\frac{j}{n}, \frac{\phi \circ T_1^{j-1}}{b_n} \right) \in B \right\}, \ n \geq 1,$$

and write

$$N_n \xrightarrow{d} N_{(\alpha)}$$

if and only if $N_n(B) \xrightarrow{d} N_{(\alpha)}(B)$ for all $B \in \mathcal{B}((0,\infty) \times (\mathbb{R} \setminus \{0\}))$ with $(\mathrm{Leb} \times \Pi_\alpha)(B) < \infty$ and $(\mathrm{Leb} \times \Pi_\alpha)(\partial B) = 0$.

The proof of the following statement is essentially the same as the proof of [TK10b, Theorem 1.1].

Note that the measure μ does not have to be invariant. Moreover (see [TK10b, Remark 2.1]), the convergence $X_n \xrightarrow{d} X_{(\alpha)}$ holds even without the condition $\mu(\phi \circ T_1^j \neq 0) = 1$, which is used only for the converse implication of the "if and only if".

Theorem 5.1 (Functional stable limit law, [TK10b, Theorem 1.1])**.** *Let $\alpha \in (0,2)$ and suppose that $\mu(\phi \circ T_1^j \neq 0) = 1$ for all $j \geq 0$. Then $X_n \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ under the probability measure μ for some constants $b_n > 0$ and c_n if and only if*

- $N_n \xrightarrow{d} N_{(\alpha)}$ *and*

- *for all $\delta > 0$, $\ell \geq 1$, with $c_\alpha(\varepsilon)$ given by (2.4),*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mu\left(\sup_{0\leq t\leq \ell}\left|\frac{1}{b_n}\sum_{j=0}^{\lfloor nt\rfloor-1}\phi\circ T_1^j\mathbf{1}_{\{|\phi\circ T_1^j|\leq\varepsilon b_n\}}-t(c_n-c_\alpha(\varepsilon))\right|\geq\delta\right)=0 \tag{5.1}$$

Remark 5.2. *In some cases the convergence $N_n \xrightarrow{d} N_{(\alpha)}$ does not hold, but one has convergence of the marginals, $N_n((0,1]\times\cdot)\xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot)$. In this case, although unable to obtain a functional stable law convergence of type (C), we can in some settings prove the convergence to a stable law for the Birkhoff sums (convergence of type (B)).*

In particular, we are unable to prove $N_n^\omega \xrightarrow{d} N_{(\alpha)}$ for the case of random intermittent maps, Example 5.12. On the other hand, in the setting of random uniformly expanding maps of Example 5.10, we use the spectral gap to show that $N_n^\omega \xrightarrow{d} N_{(\alpha)}$, and then obtain the functional stable limit law.

The next statement is [TK10b, Lemma 2.2, part (2)], which follows from [TK10a, Theorem 3.2]. Again, the measure does not have to be invariant.

Theorem 5.3 (Stable limit law, [TK10b, Lemma 2.2])**.** *For $\alpha\in(0,2)$, consider an observable ϕ on the probability measure μ, and $c_\alpha(\varepsilon)$ given by (2.4).*

If

$$N_n((0,1]\times\cdot)\xrightarrow{d} N_{(\alpha)}((0,1]\times\cdot)$$

and, for all $\delta>0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mu\left(\left|\frac{1}{b_n}\sum_{j=0}^{n-1}\phi\circ T_1^j\mathbf{1}_{\{|\phi\circ T_1^j|\leq\varepsilon b_n\}}-(c_n-c_\alpha(\varepsilon))\right|\geq\delta\right)=0 \tag{5.2}$$

then

$$\frac{1}{b_n}\sum_{j=0}^{n-1}\phi\circ T_1^j - c_n\xrightarrow{d} X_{(\alpha)}(1)$$

under the probability measure μ.

5.2. **Random dynamical systems.** We will be considering the following set-up, with (Ω,σ) the full two-sided shift on finitely many symbols, and $Y=[0,1]$.

Let $\sigma:\Omega\to\Omega$ be an invertible ergodic measure-preserving transformation on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. For a measurable space $(Y,\mathcal{B})$, let

$$\begin{array}{rccc} F: & \Omega\times Y & \to & \Omega\times Y\\ & (\omega,x) & \mapsto & (\sigma\omega,T_\omega(x))\end{array}$$

preserving a probability measure ν_F on $\Omega\times Y$. We assume that ν_F admits a disintegration given by $\nu_F(d\omega,dx)=\mathbb{P}(d\omega)\nu^\omega(dx)$. For all $n\geq 1$, we have

$$F^n(\omega,x)=(\sigma^n\omega,T_\omega^n x),$$

where, as in (1.1),

$$T_\omega^n=T_{\sigma^{n-1}\omega}\circ\ldots\circ T_\omega,$$

which satisfies the equivariance relations $(T_\omega^n)_*\nu^\omega=\nu^{\sigma^n\omega}$ for $\mathbb{P}$-a.e. $\omega\in\Omega$.

Let $\phi : Y \to \mathbb{R}$ be a measurable function such that $\nu^\omega(\phi \neq 0) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. As in (1.3), we define for every $\omega \in \Omega$ the random process $X_n^\omega(t)$, $n \geq 1$, by

$$X_n^\omega(t) = \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi \circ T_\omega^j - tc_n,\ t \geq 0$$

for some constants $b_n > 0$, $c_n \in \mathbb{R}$.

As in Section 5.1, for $B \in \mathcal{B}((0,\infty) \times (\mathbb{R} \setminus \{0\}))$ we define

$$N_n^\omega(B) := \# \left\{ j \geq 1 \,:\, \left(\frac{j}{n}, \frac{\phi \circ T_\omega^{j-1}}{b_n} \right) \in B \right\},\ n \geq 1.$$

Proposition 5.4 ([TK10b, proof of Theorem 1.2])**.**
Let $\alpha \in (0,1)$. With b_n as in Definition 2.3 and $c_n = 0$, suppose that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{b_n} \sum_{j=0}^{n\ell - 1} \mathbb{E}_{\nu^{\sigma^j \omega}}(|\phi| \mathbf{1}_{\{|\phi| \leq \varepsilon b_n\}}) = 0 \quad \textit{for all } \ell \geq 1, \tag{5.3}$$

and

$$N_n^\omega \xrightarrow{d} N_{(\alpha)}.$$

Then $X_n^\omega \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ under the probability measure ν^ω for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proof. We will check that the hypothesis of Theorem 5.1 are met for $\mathbb{P}$-a.e. ω with $T_n = T_{\sigma^{n-1}\omega}$, $\mu = \nu^\omega$. Recall that $c_n = c_\alpha(\varepsilon) = 0$ when $\alpha \in (0,1)$. Using [KW69, Theorem 1] (see Theorem 5.6) and the equivariance of the family of measures $\{\nu^\omega\}_{\omega \in \Omega}$, we have

$$\nu^\omega \left(\sup_{0 \leq t \leq \ell} \left| \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi \circ T_\omega^j \mathbf{1}_{\{|\phi \circ T_\omega^j| \leq \varepsilon b_n\}} \right| \geq \delta \right) \leq \frac{1}{\delta b_n} \sum_{j=0}^{n\ell - 1} \mathbb{E}_{\nu^{\sigma^j \omega}}(|\phi| \mathbf{1}_{\{|\phi| \leq \varepsilon b_n\}})$$

which shows that condition (5.3) implies condition (5.1) for all $\delta > 0$ and $\ell \geq 1$. □

Remark 5.5. *One could replace condition* (5.3) *by one similar to* (5.5)*, and use the argument in the proof of Proposition 5.7.*

Theorem 5.6 (Kounias and Weng [KW69, special case of Theorem 1 therein])**.**
Assume the random variables X_k are in $L^1(\mu)$. Then

$$\mu \left(\max_{1 \leq k \leq n} \left| \sum_{\ell=1}^k X_\ell \right| \geq \delta \right) \leq \frac{1}{\delta} \sum_{k=1}^n \mathbb{E}_\mu(|X_k|).$$

Proposition 5.7. *Let $\alpha \in [1,2)$.*
With b_n and c_n as in Definitions 2.3 and 2.4, and $c_\alpha(\varepsilon)$ as in (2.4)*, suppose that for all $\varepsilon > 0$ and all $\ell \geq 1$,*

$$\lim_{n \to \infty} \sup_{0 \leq t \leq \ell} \left| \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor - 1} \mathbb{E}_{\nu^{\sigma^j \omega}}(\phi \mathbf{1}_{\{|\phi| \leq \varepsilon b_n\}}) - t(c_n - c_\alpha(\varepsilon)) \right| = 0 \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega, \tag{5.4}$$

and that for all $\delta > 0$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\operatorname*{esssup}_{\omega\in\Omega}\nu^\omega\Big(\max_{1\le k\le n}\Big|\frac{1}{b_n}\sum_{j=0}^{k-1}\big[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\big]\Big|\ge\delta\Big)=0. \tag{5.5}$$

If $N_n^\omega \xrightarrow{d} N_{(\alpha)}$ for $\mathbb{P}$-a.e. $\omega\in\Omega$, then $X_n^\omega \xrightarrow{d} X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ under the probability measure ν^ω for $\mathbb{P}$-a.e. $\omega\in\Omega$.

Proof. As in the proof of Proposition 5.4, we check the hypothesis of Theorem 5.1 with $T_n = T_{\sigma^{n-1}\omega}$, $\mu=\nu^\omega$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. We will see that (5.1) follows from (5.4) and (5.5).

Using the equivariance of $\{\nu^\omega\}_{\omega\in\Omega}$, we see that condition (5.1) is implied by (5.4) and (5.6) below:

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu^\omega\left(\sup_{1\le k\le n\ell}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\Big[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\Big]\right|\ge\delta\right)=0. \tag{5.6}$$

We next show that condition (5.5) implies (5.6).

Since

$$\left\{\sup_{1\le k\le n\ell}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\Big[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\Big]\right|\ge\delta\right\}$$
$$\subset\bigcup_{i=0}^{\ell-1}\left\{\sup_{in<k\le(i+1)n}\left|\frac{1}{b_n}\sum_{j=in}^{k-1}\Big[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\Big]\right|\ge\frac{\delta}{\ell}\right\},$$

we obtain that, using again the equivariance, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\nu^\omega\left(\sup_{1\le k\le n\ell}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\Big[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\Big]\right|\ge\delta\right)$$
$$\le\sum_{i=0}^{\ell-1}\nu^{\sigma^{in}\omega}\left(\sup_{1\le k\le n}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\Big[\phi\circ T_{\sigma^{in}\omega}^j\mathbf{1}_{\{|\phi\circ T_{\sigma^{in}\omega}^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j(\sigma^{in}\omega)}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\Big]\right|\ge\frac{\delta}{\ell}\right)$$
$$\le\ell\cdot\operatorname*{esssup}_{\omega'\in\Omega}\nu^{\omega'}\left(\max_{1\le k\le n}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\Big[\phi\circ T_{\omega'}^j\mathbf{1}_{\{|\phi\circ T_{\omega'}^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega'}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\Big]\right|\ge\frac{\delta}{\ell}\right).$$

Thus, condition (5.5) implies (5.6), which concludes the proof. □

The analogue for the convergence to a stable law is the following.

Proposition 5.8. *Suppose that for $\mathbb{P}$-a.e. $\omega\in\Omega$, we have*

$$N_n^\omega((0,1]\times\cdot)\xrightarrow{d}N_{(\alpha)}((0,1]\times\cdot).$$

If $\alpha \in (0,1)$ (so $c_n = 0$), we require in addition that

$$\lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{b_n} \sum_{j=0}^{n-1} \mathbb{E}_{\nu^{\sigma^j \omega}}(|\phi| \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}}) = 0 \tag{5.7}$$

If $\alpha \in [1,2)$, we require instead of (5.7) that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \left| \frac{1}{b_n} \sum_{j=0}^{n-1} \mathbb{E}_{\nu^{\sigma^j \omega}}(\phi \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}}) - (c_n - c_\alpha(\varepsilon)) \right| = 0$$

and

$$\lim_{\varepsilon \to 0} \limsup_{n\to\infty} \nu^\omega \left(\left| \frac{1}{b_n} \sum_{j=0}^{n-1} \left[\phi \circ T_\omega^j \mathbf{1}_{\{|\phi \circ T_\omega^j| \le \varepsilon b_n\}} - \mathbb{E}_{\nu^{\sigma^j \omega}}(\phi \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}}) \right] \right| \ge \delta \right) = 0.$$

Then

$$\frac{1}{b_n} \sum_{j=0}^{n-1} \phi \circ T_\omega^j - c_n \xrightarrow{d} X_{(\alpha)}(1)$$

under the probability measure ν^ω for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proof. We check the conditions of Theorem 5.3.

The proof for $\alpha \in (0,1)$ is similar to the proof of Proposition 5.4, the proof of the case $\alpha \in [1,2)$ is similar to the proof of Proposition 5.7. □

5.2.1. *Annealed transfer operator.* We assume that the random dynamical system $F : \Omega \times [0,1] \to \Omega \times [0,1]$, which can also be viewed as a Markov process on $[0,1]$, has a stationary measure ν with density h. The map $F : \Omega \times [0,1] \to \Omega \times [0,1]$ will preserve $\mathbb{P} \times \nu$. Recall that $\mathbb{P} := \{(p_1, \dots, p_m)\}^{\mathbb{Z}}$.

We use the notation $P_{\mu,i}$ for the transfer operator of $T_i : [0,1] \to [0,1]$ with respect to a measure μ on $[0,1]$, i.e.

$$\int f \cdot g \circ T_i d\mu = \int (P_{\mu,i} f) g d\mu, \text{ for all } f \in L^1(\mu),\ g \in L^\infty(\mu).$$

The annealed transfer operator is defined by

$$P_\mu(f) := \sum_{i=1}^m p_i P_{\mu,i}(f)$$

with adjoint

$$U(f) := \sum_{i=1}^m p_i f \circ T_i$$

which satisfies the duality relation

$$\int f(g \circ U) d\mu = \int (P_\mu f) g d\mu, \text{ for all } f \in L^1(\mu),\ g \in L^\infty(\mu).$$

As above, we assume there are sample measures $d\nu^\omega = h_\omega dx$ on each fiber $[0,1]$ of the skew product such that

$$P_\omega h_\omega = h_{\sigma\omega}$$

where P_ω is the transfer operator of T_{ω_0} with respect to the Lebesgue measure

Therefore

$$\nu(A) = \int_\Omega [\int_A h_\omega dx] d\mathbb{P}(\omega)$$

for all Borel sets $A \subset [0,1]$.

Remark 5.9. *We emphasize that in the limit laws we obtain for quenched realizations that the scaling and centering constants, b_n and c_n, are the same for $\mathbb{P}$-a.e. $\omega \in \Omega$, and determined by the annealed measure ν.*

5.3. **Examples.** We now introduce two systems for which we are able to establish stable limit laws of various types. The first RDS is uniformly expanding while the second is strictly polynomially mixing.

Example 5.10 (β-transformations)**.** *A simple example is to take m β-maps of the unit interval, $T_{\beta_i}(x) = \beta_i x$ (mod 1). We suppose $\beta_i > 1 + a$, $a > 0$, for all β_i, $i = 1, \ldots, m$.*

By results of [CR07] *(see also* [ANV15]*) in this setting the stationary measure ν on $[0,1]$ has a density h bounded away from zero and bounded above. In fact h is of bounded variation (BV).*

The functions h_ω, h (recall that $d\nu^\omega = h_\omega dx$) are BV, uniformly bounded in BV norm and uniformly bounded away from zero.

Remark 5.11. *Example 5.10 fits in a larger class of uniformly expanding random maps, see Section 5.5.*

Example 5.12 (intermittent maps)**.** *Liverani, Saussol and Vaienti* [LSV99] *introduced the map* (4.1) *as a simple model for intermittent dynamics:*

$$T_\gamma : [0,1] \to [0,1], \qquad T_\gamma(x) := \begin{cases} (2^\gamma x^\gamma + 1)x & \text{if } 0 \le x < \frac{1}{2}; \\ 2x - 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

If $0 \le \gamma < 1$ then T_γ has an absolutely continuous invariant measure μ_γ with density h_γ bounded away from zero and satisfying $h_\gamma(x) \sim Cx^{-\gamma}$ for x near zero.

We form a random dynamical system by selecting $\gamma_i \in (0,1)$, $i = 1, \ldots, m$ and setting $T_i := T_{\gamma_i}$. The associated Markov process on $[0,1]$ has a stationary invariant measure ν which is absolutely continuous, with density h bounded away from zero.

We denote $\gamma_{max} := \max_{1 \le i \le m}\{\gamma_i\}$ and $\gamma_{min} := \min_{1 \le i \le m}\{\gamma_i\}$.

5.4. **Decay of correlations.** We now consider the decay of correlations properties of the annealed systems associated to Example 5.10 and Example 5.12.

By [ANV15, Proposition 3.1] in the setting of Example 5.10, we have exponential decay in BV against L^1: there are $C > 0$, $0 < \lambda < 1$ such that

$$\left| \int fg \circ U^n d\nu - \int f d\nu \int g d\nu \right| \le C\lambda^n \|f\|_{BV} \|g\|_{L^1(\nu)}$$

In the setting of Example 5.12, by [BB16, Theorem 1.2], we have polynomial decay in Hölder against L^∞: there exists $C > 0$ such that

$$\left| \int fg \circ U^n d\nu - \int f d\nu \int g d\nu \right| \le Cn^{1 - \frac{1}{\gamma_{min}}} \|f\|_{\text{Hölder}} \|g\|_{L^\infty(\nu)}.$$

5.4.1. *The sample measures h_ω.* The regularity properties of the sample measures h_ω, both as functions of ω and as functions of x on $[0,1]$ play a key role in our estimates. We will first recall how the sample measures are constructed. Suppose $\omega := (\dots, \omega_{-1}, \omega_0, \omega_1, \dots, \omega_n, \dots,)$ and define $h_n(\omega) = P_{\omega_{-1}} \dots P_{\omega_{-n}} 1$ as a sequence of functions on the fiber I above ω. In the setting both of Example 5.10 and Example 5.12 $\{h_n(\omega)\}$ is a Cauchy sequence and has a limit h_ω.

In the setting of Example 5.10 (β maps), h_ω is uniformly BV in ω as

$$\|h_n(\omega) - h_{n+1}(\omega)\|_{BV} \le \|P_{\omega_{-1}} P_{\omega_{-2}} \dots P_{\omega_{-n}} (1 - P_{\omega_{-n-1}} 1)\|_{BV} \le C\lambda^n.$$

In the setting of Example 5.12 (intermittent maps), with $\gamma_{max} = \max_{1 \le i \le m}\{\gamma_i\}$, the densities h_ω lie in the cone

$$\begin{aligned} L := \big\{ f \in \mathcal{C}^0((0,1]) \cap L^1(m), \quad & f \ge 0,\ f \text{ non-increasing}, \\ & X^{\gamma_{max}+1} f \text{ increasing}, f(x) \le a x^{-\gamma_{max}} m(f) \big\} \end{aligned}$$

where $X(x) = x$ is the identity function and $m(f)$ is the integral of f with respect to m. In [AHN$^+$15] it is proven that for a fixed value of $\gamma_{max} \in (0,1)$, provided that the constant a is big enough, the cone L is invariant under the action of all transfer operators P_{γ_i} with $0 < \gamma_i \le \gamma_{max}$ and so (see e.g. [NPT21, Proposition 3.3], which summarizes results of [NTV18])

$$\begin{aligned} \|h_n(\omega) - h_{n+k}(\omega)\|_{L^1(m)} &\le \|P_{\omega_{-1}} P_{\omega_{-2}} \dots P_{\omega_{-n}} (1 - P_{\omega_{-n-1}} \dots P_{\omega_{-n-k}} 1)\|_{L^1(m)} \\ &\le C_{\gamma_{max}} n^{1 - \frac{1}{\gamma_{max}}} (\log n)^{\frac{1}{\gamma_{max}}} \end{aligned}$$

whence $h_\omega \in L^1(m)$. In later arguments we will use the approximation

$$\|h_n(\omega) - h_\omega\|_{L^1(m)} \le C_{\gamma_{max}} n^{1 - \frac{1}{\gamma_{max}}} (\log n)^{\frac{1}{\gamma_{max}}}. \tag{5.8}$$

We mention also the recent paper [KL21] where the logarithm term in Equation (5.8) is shown to be unnecessary and moment estimates are given.

We now show that h_ω is a Hölder function of ω on (Ω, d_θ) in the setting of Example 5.10.

For $\theta \in (0,1)$, we introduce on Ω the symbolic metric

$$d_\theta(\omega, \omega') = \theta^{s(\omega,\omega')}$$

where $s(\omega, \omega') = \inf\{k \ge 0 : \omega_\ell \ne \omega'_\ell \text{ for some } |\ell| \le k\}$.

Suppose ω, ω' agree in coordinates $|k| \le n$ (i.e. backwards and forwards in time) so that $d_\theta(\omega, \omega') \le \theta^n$ in the symbolic metric on Ω. Then

$$\begin{aligned} \|h_\omega - h_{\omega'}\|_{\mathrm{BV}} &\le \|P_{\omega_{-1}} P_{\omega_1} \dots P_{\omega_{-n+1}} (h_{(\sigma^{-n+1}\omega)} - h_{(\sigma^{-n+1}\omega')})\|_{\mathrm{BV}} \\ &\le C\lambda^{n-1} = C' d_\theta(\omega, \omega')^{\log_\theta \lambda} \end{aligned}$$

Recall that $\|f\|_\infty \le C\|f\|_{\mathrm{BV}}$, see e.g. [BG97, Lemma 2.3.1].

That is, Condition U (see Definition 5.13) holds for Example 5.10, see Remark 5.15.

The map $\omega \mapsto h_\omega$ is not Hölder in the setting of Example 5.12; in several arguments we will use the regularity properties of the approximation $h_n(\omega)$ for h_ω.

However, on intervals that stay away from zero, all functions in the cone L are comparable to their mean. Therefore, on sets that are uniformly away from zero, all the above densities/measures ($d\nu = h dx$, h_ω, $h_n(\omega)$) are still comparable.

Namely,

$$\begin{gathered} \text{for any } \delta \in (0,1) \text{ there is } C_\delta > 0 \text{ such that} \\ h \in L \implies 1/C_\delta < h(x)/m(h) < C_\delta \text{ for } x \in [\delta, 1] \end{gathered} \tag{5.9}$$

Indeed, $h/m(h)$ is bounded below by [LSV99, Lemma 2.4], and the upper bound follows from the definition of the cone.

5.5. **Random uniformly expanding maps of** $[0,1]$**.** We now consider a slightly more general set-up than Example 5.10. As before, we assume the existence of an invariant probability measure ν for the Markov chain associated to the random system, with disintegration along the fibers given by $d\nu^\omega = h_\omega dm$.

Definition 5.13 (**Condition U**)**.** *We assume that almost each ν^ω is absolutely continuous with respect to the Lebesgue measure m, and*

$$\text{for some } C > 0, \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega \implies C^{-1} \le h_\omega := \frac{d\nu^\omega}{d\nu} \le C, \ m\text{-a.e.} \tag{5.10}$$

$$\text{the map } \omega \in \Omega \mapsto h_\omega \in L^\infty(m) \text{ is Hölder continuous.} \tag{5.11}$$

Consequently, the stationary measure ν is also absolutely continuous with respect to m, with density $h \in L^\infty(m)$ given by $h(x) = \int_\Omega h_\omega(x)\mathbb{P}(d\omega)$ and satisfying (5.10)*.*

We consider random i.i.d. compositions described in Section 5.2 with additional assumptions of uniform expansion. Let $\mathcal{S}$ be a finite collection of m piecewise C^2 uniformly expanding maps of the unit interval $[0,1]$. More precisely, we assume that for each $T \in \mathcal{S}$, there exist a finite partition $\mathcal{A}_T$ of $[0,1]$ into intervals, such that for each $I \in \mathcal{A}_T$, T can be continuously extended as a strictly monotonic C^2 function on $\bar{I}$ and

$$\lambda := \inf_{I \in \mathcal{A}_T} \inf_{x \in \bar{I}} |T'(x)| > 1.$$

As in Section 5.2, the maps T_ω (determined by the 0-th coordinate of ω) are chosen from $\mathcal{S}$ in an i.i.d. fashion according to a Bernoulli probability measure $\mathbb{P}$ on $\Omega := \{1, \dots, m\}^{\mathbb{Z}}$. We will denote by $\mathcal{A}_\omega$ the partition of monotonicity of T_ω, and by $\mathcal{A}_\omega^n = \vee_{k=0}^{n-1} (T_\omega^k)^{-1}(\mathcal{A}_{\sigma^k\omega})$ the partition associated to T_ω^n. We introduce

$$\mathcal{D} = \cup_{n \ge 0} \cup_{\omega \in \Omega} \partial \mathcal{A}_\omega^n$$

the set of discontinuities of all the maps T_ω^n. Note that $\mathcal{D}$ is at most a countable set.

For each $\omega \in \Omega$, we denote the transfer operator P_ω of T_ω with respect to the Lebesgue measure m: for all $\phi \in L^\infty(m)$ and $\psi \in L^1(m)$,

$$\int_{[0,1]} (\phi \circ T_\omega) \cdot \psi \, dm = \int_{[0,1]} \phi \cdot P_\omega \psi \, dm.$$

We can then form, for $\omega \in \Omega$ and $n \ge 1$, the cocycle

$$P_\omega^n = P_{\sigma^{n-1}\omega} \circ \ldots \circ P_\omega.$$

These operators are contractions on $L^1(m)$: $\|P_\omega^n f\|_{L^1(m)} \le \|f\|_{L^1(m)}$, and, from the duality relation, it easily follows that

$$P_\omega^n(f \cdot g \circ T_\omega^n) = P_\omega^n(f) g.$$

We will let them act on the space BV of functions of bounded variation on $[0,1]$, whose norm is given by

$$\|f\|_{\mathrm{BV}} = \|f\|_{L^1(m)} + \mathrm{Var}(f),$$

where $\mathrm{Var}(f)$ is, as usual, the infimum, over all functions g with $f = g$ m-a.e., of the total variation of g on $[0,1]$. The space BV is continuously embedded in $L^\infty(m)$, as $\|f\|_{L^\infty(m)} \leq \|f\|_{\mathrm{BV}}$, and, furthermore, for all $f, g \in \mathrm{BV}$, we have

$$\mathrm{Var}(fg) \leq \mathrm{Var}(f)\|g\|_{L^\infty(m)} + \|f\|_{L^\infty}\mathrm{Var}(g),$$

see for instance [BG97].

We assume that the class of transfer operators $\{P_\omega\}_{\omega\in\Omega}$ satisfies a uniform Lasota-Yorke inequality on the space BV:

(LY): there exist $r \geq 1$, $M > 0$ and $D > 0$ and $\rho \in (0,1)$ such that for all $\omega \in \Omega$ and all $f \in \mathrm{BV}$,

$$\|P_\omega f\|_{\mathrm{BV}} \leq M\|f\|_{\mathrm{BV}},$$

and

$$\mathrm{Var}(P_\omega^r f) \leq \rho \mathrm{Var}(f) + D\|f\|_{L^1(m)}.$$

Iterating these two inequalities, we obtain that there exists $\lambda \in (0,1)$ and $C > 0$ such that

$$\|P_\omega^n f\|_{\mathrm{BV}} \leq C\lambda^n \|f\|_{\mathrm{BV}} + C\|f\|_{L^1(m)}.$$

In particular, $\|P_\omega^n f\|_{\mathrm{BV}} \leq C\|f\|_{\mathrm{BV}}$.

We also assume the following two other conditions from Conze and Raugi [CR07]:

(Dec): there exists $C > 0$ and $\theta \in (0,1)$ such that for all $n \geq 1$, all $\omega \in \Omega$ and all $f \in \mathrm{BV}$ with $\mathbb{E}_m(f) = 0$:

$$\|P_\omega^n f\|_{\mathrm{BV}} \leq C\theta^n \|f\|_{\mathrm{BV}}$$

and

(Min): there exists $c > 0$ such that for all $n \geq 1$ and all $\omega \in \Omega$,

$$\inf_{x\in[0,1]} (P_\omega^n \mathbf{1})(x) \geq c > 0.$$

Lemma 5.14. *Properties* **(LY)**, **(Min)** *and* **(Dec)** *imply Condition U. Namely, there exists a unique Hölder map $\omega \in \Omega \mapsto h_\omega \in \mathrm{BV}$ such that $P_\omega h_\omega = h_{\sigma\omega}$ and* (5.10), (5.11) *are satisfied* [ANV15].

Proof. By **(Dec)**, and as all the operators P_ω are Markov with respect to m, we have

$$\|P^{n+k}_{\sigma^{-(n+k)}\omega}\mathbf{1} - P^n_{\sigma^{-n}\omega}\mathbf{1}\|_{\mathrm{BV}} \leq C\kappa^n \|\mathbf{1} - P^k_{\sigma^{-(n+k)}\omega}\mathbf{1}\|_{\mathrm{BV}} \leq C\kappa^n, \tag{5.12}$$

which proves that $(P^n_{\sigma^{-n}\omega}\mathbf{1})_{n\geq 0}$ is a Cauchy sequence in BV converging to a unique limit $h_\omega \in BV$ satisfying $P_\omega h_\omega = h_{\sigma\omega}$ for all ω. The lower bound in (5.10) follows from the condition **(Min)**, while the upper bound is a consequence of the uniform Lasota-Yorke inequality **(LY)**, as actually the family $\{h_\omega\}_{\omega\in\Omega}$ is bounded in BV. To prove the Hölder continuity of $\omega \mapsto h_\omega$ with respect to the distance d_θ, we remark that if ω and ω' agree in coordinates $|k| \leq n$, then

$$\|h_\omega - h_{\omega'}\|_{\mathrm{BV}} = \|P^k_{\sigma^{-k}\omega}(h_{\sigma^{-k}\omega} - h_{\sigma^{-k}\omega'})\|_{\mathrm{BV}} \leq C\theta^n \leq Cd_\theta(\omega,\omega').$$

Note that the density h of the stationary measure ν also belongs to BV and is uniformly bounded from above and below, as the average of h_ω over Ω. ∎

Remark 5.15. *As mentioned above, a class of maps satisfying these assumptions is given by the β-transformations of Example 5.10: if all maps $T \in \mathcal{S}$, with $\mathcal{S}$ finite, are of the form $T : x \mapsto \beta x$* mod 1, *with $\beta > 1 + a$, $a > 0$, then* **(LY)**, **(Dec)** *and* **(Min)** *are satisfied. We refer to* [CR07] *for a proof, and for a more in-depth treatment of these assumptions.*

6. Main Results

6.1. **A simple class of unbounded observables.** Let $x_0 \in [0,1]$, and, for $\alpha \in (0,2)$, consider the function

$$\phi_{x_0}(x) = |x - x_0|^{-\frac{1}{\alpha}},\ x \in [0,1].$$

Then, we have for t small enough

$$\nu(|\phi_{x_0}| > t) = \int_{|x-x_0|<t^{-\alpha}} h(x)dx.$$

This proves that ϕ_{x_0} is regularity varying with index α, having a constant slowly varying function L given by $L(t) \equiv b$ and scaling sequence $b_n = b^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}}$, where

$$b := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{|x-x_0|<\varepsilon} h(x)dx.$$

Under the setting of condition U, the limit exists for all $x_0 \in [0,1]$ since $h \in \mathrm{BV}$ and $b > 0$ as h is bounded from below. If h admits a continuous version at x_0 and $x_0 \in (0,1)$, then $b = 2h(x_0)$. In the setting of Example 5.12, the limit exists for all $x_0 \in (0,1]$ since the density h belongs to the cone L and is thus Lipschitz on any interval $[\delta, 1]$, $\delta > 0$.

As the observable ϕ_{x_0} is positive, the parameter $p \in [0,1]$ such that

$$\lim_{t\to\infty} \frac{\nu(\phi_{x_0} > t)}{\nu(|\phi_{x_0}| > t)} = p$$

is equal to 1.

Recall that the (deterministic) centering is defined in Definition 2.4 by

$$c_n = \begin{cases} 0 & \text{if } \alpha \in (0,1) \\ \frac{n}{b_n}\mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi| \le b_n\}}) & \text{if } \alpha = 1 \\ \frac{n}{b_n}\mathbb{E}_\nu(\phi) & \text{if } \alpha \in (1,2) \end{cases}.$$

Definition 6.1. *We say that x_0 is periodic if there exist $\omega \in \Omega$ and $n \ge 1$ such that $T_\omega^n(x_0) = x_0$.*

Remark 6.2. *For the sake of concreteness, we restricted ourselves to observables of the form $\phi_{x_0}(x) = |x - x_0|^{-\frac{1}{\alpha}}$, but it is possible to consider more general regularly varying observables ϕ which are piecewise monotonic with finitely many branches, see for instance* [TK10b, Section 4.2] *in the deterministic case.*

6.2. **Exponential law and point process results.** We denote by $\mathcal{J}$ the family of all finite unions of intervals of the form $(x,y]$, where $-\infty \le x < y \le \infty$ and $0 \notin [x,y]$.

For a measurable subset $U \subset [0,1]$, we define the hitting time of $(\omega, x) \in \Omega \times [0,1]$ to U by

$$R_U(\omega)(x) := \inf\left\{k \ge 1 \,:\, T_\omega^k(x) \in U\right\}. \tag{6.1}$$

Recall that $\phi_{x_0}(x) := d(x, x_0)^{-\frac{1}{\alpha}}$ depends on the choice of $x_0 \in [0,1]$.

Theorem 6.3. *In the setting of Section 5.5, assume* **(LY)**, **(Min)** *and* **(Dec)**. *If $x_0 \notin \mathcal{D}$ is not periodic, then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $0 \le s < t$,*

$$\lim_{n\to\infty} \nu^{\sigma^{\lfloor ns \rfloor}\omega}\left(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor n(t-s) \rfloor\right) = e^{-(t-s)\Pi_\alpha(J)}.$$

where $A_n := \phi_{x_0}^{-1}(b_n J)$, $J \in \mathcal{J}$.

Theorem 6.4. *Assume the conditions of Example 5.12 and that $\gamma_{max} < \frac{1}{3}$. Then for m-a.e. x_0 for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $0 \le s < t$,*

$$\lim_{n\to\infty} \nu^{\sigma^{\lfloor ns \rfloor}\omega}\left(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor n(t-s) \rfloor\right) = e^{-(t-s)\Pi_\alpha(J)}.$$

where $A_n := \phi_{x_0}^{-1}(b_n J)$, $J \in \mathcal{J}$.

Theorem 6.5. *In the setting of Section 5.5, assume* **(LY)**, **(Min)** *and* **(Dec)**. *If $x_0 \notin \mathcal{D}$ is not periodic, then for $\mathbb{P}$-a.e. $\omega \in \Omega$, then*

$$N_n^\omega \xrightarrow{d} N_{(\alpha)},$$

under the probability ν^ω.

Theorem 6.6. *In the setting of Example 5.12 for m-a.e. x_0 for $\mathbb{P}$-a.e. ω,*

$$N_n^\omega((0,1] \times \cdot) \xrightarrow{d} N_{(\alpha)}((0,1] \times \cdot)$$

6.3. Limit theorems.

Theorem 6.7. *In the setting of Section 5.5, assume* **(LY)**, **(Min)** *and* **(Dec)**. *Suppose that $x_0 \notin \mathcal{D}$ is not periodic for $\mathbb{P}$-a.e. $\omega \in \Omega$ and consider the observable ϕ_{x_0}.*

If $\alpha \in (0,1)$ then for $\mathbb{P}$-a.e. $\omega \in \Omega$, the Functional Stable Limit holds:

$$X_n^\omega(t) := \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi_{x_0} \circ T_\omega^j - tc_n \xrightarrow{d} X_{(\alpha)}(t) \quad \text{in} \quad \mathbb{D}[0,\infty)$$

in the J_1 topology under the probability measure ν^ω, where $X_{(\alpha)}(t)$ is the α-stable process with Lévy measure $\Pi_\alpha(dx) = \alpha|x|^{-(\alpha+1)}$ on $[0,\infty)$.

If $\alpha \in [1,2)$ then the same result holds for m-a.e. x_0.

Remark 6.8. *It would be possible to remove the assumption that $x_0 \notin \mathcal{D}$ by doubling the discontinuity points, see* [AFV15, Section 3.3] *for the deterministic case. In the case of β-transformations of Example 5.10, we can consider each map as a map of the unit circle, by identifying* 0 *and* 1, *in which case the only discontinuity point is* 1, *and thus the assumption $x_0 \notin \mathcal{D}$ reduces in assuming that all the random orbits of x_0 never hit* 1.

Theorem 6.9. *In the setting of Example 5.12 suppose $\alpha \in (0,1)$ and $\gamma < \frac{1}{3}$. Then, for m-a.e. x_0 $\frac{1}{b_n}\sum_{j=0}^{n-1} \phi_{x_0} \circ T_\omega^j \xrightarrow{d} X_{(\alpha)}(1)$ under the probability measure ν^ω for $\mathbb{P}$-a.e. ω (recall that $c_n = 0$ for $\alpha \in (0,1)$).*

7. Scheme of proofs

7.1. **Two useful lemmas.** We now proceed to the proofs of the main results. We will use the following technical propositions which are a form of spatial ergodic theorem which allows us to prove exponential and Poisson limit laws.

Lemma 7.1. *Under the setting of Condition U, let $\chi_n : Y \to \mathbb{R}$ be a sequence of functions in $L^1(m)$ such that $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1}\widetilde{L}(n))$ for some slowly varying function $\widetilde{L}$. Then, for $\mathbb{P}$-a.e.*

$\omega \in \Omega$ and for all $\ell \geq 1$,

$$\lim_{n\to\infty} \sup_{0\leq k\leq \ell} \left| \sum_{j=0}^{kn-1} \left(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n) - \mathbb{E}_{\nu}(\chi_n)\right) \right| = 0.$$

Therefore, given $(s,t] \subset [0,\infty)$ and $\varepsilon > 0$, for $\mathbb{P}$-a.e. ω there exists $N(\omega)$ such that

$$\left| \sum_{r=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n) - \mathbb{E}_{\nu}(\chi_n)\right) \right| \leq \varepsilon$$

for all $n \geq N(\omega)$.

Proof. We obtain the second claim by taking the difference between two values of ℓ in the first claim.

Fix $\ell \geq 1$. For $\delta > 0$, let

$$U_k^n(\delta) = \left\{ \omega \in \Omega \,:\, \left| \sum_{j=0}^{kn-1} \left(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n) - \mathbb{E}_{\nu}(\chi_n)\right) \right| \geq \delta \right\},$$

and

$$B^n(\delta) = \left\{ \omega \in \Omega \,:\, \sup_{0\leq k\leq \ell} \left| \sum_{j=0}^{kn-1} \left(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n) - \mathbb{E}_{\nu}(\chi_n)\right) \right| \geq \delta \right\}.$$

Note that

$$B^n(\delta) = \bigcup_{k=0}^{\ell} U_k^n(\delta).$$

We define $f_n(\omega) = \mathbb{E}_{\nu^\omega}(\chi_n)$ and $\overline{f}_n = \mathbb{E}_{\mathbb{P}}(f_n)$. We claim that $f_n : \Omega \to \mathbb{R}$ is Hölder with norm $\|f_n\|_\theta = \mathcal{O}(n^{-1}\widetilde{L}(n))$. Indeed, for $\omega \in \Omega$, we have

$$|f_n(\omega)| = \left| \int_Y \chi_n(x) d\nu^\omega(x) \right| \leq \|h_\omega\|_{L_m^\infty} \|\chi_n\|_{L_m^1} \leq \frac{C}{n}\widetilde{L}(n),$$

and for $\omega, \omega' \in \Omega$, we have

$$\begin{aligned}
|f_n(\omega) - f_n(\omega')| &= \left| \int_Y \chi_n(x) d\nu^\omega(x) - \int_Y \chi_n(x) d\nu^{\omega'}(x) \right| \\
&\leq \int_Y |\chi_n(x)| \cdot |h_\omega(x) - h_{\omega'}(x)| dm(x) \\
&\leq \|h_\omega - h_{\omega'}\|_{L_m^\infty} \|\chi_n\|_{L_m^1} \\
&\leq \frac{C}{n}\widetilde{L}(n) d_\theta(\omega, \omega'),
\end{aligned}$$

since $\omega \in \Omega \mapsto h_\omega \in L^\infty(m)$ is Hölder continuous. In particular, we also have that $\overline{f}_n = \mathcal{O}(n^{-1}\widetilde{L}(n))$.

We have, using Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}(U_k^n(\delta)) &= \mathbb{P}\left(\left\{\omega\in\Omega\,:\,\left|\sum_{j=0}^{kn-1}\left(f_n\circ\sigma^j-\overline{f}_n\right)\right|\geq\delta\right\}\right)\\ &\leq \frac{1}{\delta^2}\mathbb{E}_{\mathbb{P}}\left(\left(\sum_{j=0}^{kn-1}\left(f_n\circ\sigma^j-\overline{f}_n\right)\right)^2\right)\\ &\leq \frac{1}{\delta^2}\left[\sum_{j=0}^{kn-1}(\mathbb{E}_{\mathbb{P}}|f_n\circ\sigma^j-\overline{f}_n|^2+2\sum_{0\leq i<j\leq kn-1}\mathbb{E}_{\mathbb{P}}((f_n\circ\sigma^i-\overline{f}_n)(f_n\circ\sigma^j-\overline{f}_n))\right].\end{aligned}$$

By the σ-invariance of $\mathbb{P}$, we have

$$\mathbb{E}_{\mathbb{P}}|f_n\circ\sigma^j-\overline{f}_n|^2=\mathbb{E}_{\mathbb{P}}|f_n-\overline{f}_n|^2,$$

and, since $(\Omega,\mathbb{P},\sigma)$ admits exponential decay of correlations for Hölder observables, there exist $\lambda\in(0,1)$ and $C>0$ such that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}((f_n\circ\sigma^i-\overline{f}_n)(f_n\circ\sigma^j-\overline{f}_n)) &= \mathbb{E}_{\mathbb{P}}((f_n-\overline{f}_n)(f_n\circ\sigma^{j-i}-\overline{f}_n))\\ &\leq C\lambda^{j-i}\|f_n-\overline{f}_n\|_\theta^2.\end{aligned}$$

We then obtain that

$$\begin{aligned}\mathbb{P}(U_k^n(\delta)) &\leq \frac{C}{\delta^2}\left[kn\|f_n-\overline{f}_n\|_{L^2_m}^2+2\sum_{0\leq i<j\leq kn-1}\lambda^{i-j}\|f_n-\overline{f}_n\|_\theta^2\right]\\ &\leq C\frac{nk}{\delta^2}\|f_n\|_\theta^2\\ &\leq C\frac{k}{n\delta^2}(\widetilde{L}(n))^2,\end{aligned}$$

which implies that

$$\mathbb{P}(B^n(\delta))\leq C\frac{\ell^2}{n\delta^2}(\widetilde{L}(n))^2.$$

Let $\eta>0$. By the Borel-Cantelli lemma, it follows that for $\mathbb{P}$-a.e. $\omega\in\Omega$, there exists $N(\omega,\delta)\geq 1$ such that $\omega\notin B^{\lfloor p^{1+\eta}\rfloor}(\delta)$ for all $p\geq N(\omega,\delta)$.

Let now $P:=\lfloor p^{1+\eta}\rfloor<n\leq P'=\lfloor (p+1)^{1+\eta}\rfloor$ for p large enough. Let $0\leq k\leq\ell$. Then, since $\|f_n\|_\infty=\mathcal{O}(n^{-1}\widetilde{L}(n))$,

$$\begin{aligned}\left|\sum_{j=0}^{kP-1}\left(f_n(\sigma^j\omega)-\overline{f}_n\right)-\sum_{j=0}^{kn-1}\left(f_n(\sigma^j\omega)-\overline{f}_n\right)\right| &\leq \sum_{j=kP}^{kn-1}\left|f_n(\sigma^j\omega)-\overline{f}_n\right|\\ &\leq C\frac{P'-P}{P}\widetilde{L}(n)\leq C\frac{\widetilde{L}(p^{1+\eta})}{p},\end{aligned}$$

because on the one hand

$$\frac{P'-P}{P}=\frac{\lfloor (p+1)^{1+\eta}\rfloor-\lfloor p^{1+\eta}\rfloor}{\lfloor p^{1+\eta}\rfloor}=\mathcal{O}\left(\frac{1}{p}\right),$$

and on the other hand, by Potter's bounds, for $\tau > 0$,

$$\widetilde{L}(n) \le C\widetilde{L}(P)\left(\frac{n}{P}\right)^{\tau} \le C\widetilde{L}(P)\left(\frac{P'}{P}\right)^{\tau} \le C\widetilde{L}(P).$$

Since

$$\left|\sum_{j=0}^{kP-1}\left(f_n(\sigma^j\omega) - \overline{f}_n\right)\right| < \delta$$

for all $0 \le k \le \ell$, it follows that for $\mathbb{P}$-a.e. ω, there exists $N(\omega,\delta)$ such that $\omega \notin B^n(2\delta)$ for all $n \ge N(\omega,\delta)$, which concludes the proof. □

We now consider a corresponding result to Lemma 7.1 in the setting of Example 5.12.

Lemma 7.2. *In the setting of Example 5.12, assume that $\gamma_{max} < 1/2$, and that $\chi_n \in L^1(m)$ is such that $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1})$, $\|\chi_n\|_\infty = \mathcal{O}(1)$ and there is $\delta > 0$ such that* $\mathrm{supp}(\chi_n) \subset [\delta, 1]$ *for all n.*

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for all $\ell \ge 1$,

$$\lim_{n\to\infty}\sup_{0\le k\le\ell}\left|\sum_{j=0}^{kn-1}\left(\mathbb{E}_{\nu^{\sigma^j\omega}}(\chi_n) - \mathbb{E}_{\nu}(\chi_n)\right)\right| = 0.$$

Proof. In the setting of Example 5.12 we must modify the argument of Lemma 7.1 slightly as h_ω is not a Hölder function of ω. Instead, we consider $h^i_\omega = P^i_{\sigma^{-i}\omega}\mathbf{1}$. and use that, by (5.8),

$$\|h^i_\omega - h_\omega\|_{L^1(m)} \le Ci^{1-\frac{1}{\gamma_{max}}} \qquad \text{(leaving out the log term).} \tag{7.1}$$

Note that h^i_ω is the i-th approximate to h_ω in the pullback construction of h_ω. Let ν^i_ω be the measure such that $\frac{d\nu^i_\omega}{dm} = h^i_\omega$.

Consider

$$f^i_n(\omega) = \mathbb{E}_{\nu^i_\omega}(\chi_n)\,, \qquad f_n(\omega) = \mathbb{E}_{\nu^\omega}(\chi_n)$$
$$\overline{f}^i_n = \mathbb{E}_{\mathbb{P}}(f^i_n)\,, \qquad \overline{f}_n = \mathbb{E}_{\mathbb{P}}(f_n).$$

By (5.9), on the set $[\delta,1]$ the densities involved ($h^k_\omega, h_\omega, h = d\nu/dm$) are uniformly bounded above and away from zero. Thus $\|f^i_n\|_\infty = \mathcal{O}(n^{-1})$.

Pick $0 < a < 1$ is such that $\beta := (\frac{1}{\gamma_{max}} - 1)a - 1 > 0$.

For a given n take $i = i_n = n^a$. By (7.1), for all ω, n and $i = n^a$

$$|f^i_n(\omega) - f_n(\omega)| \le \|h^i_\omega - h_\omega\|_{L^1(m)}\|\chi_n\|_{L^\infty(m)} = \mathcal{O}(n^{-(\beta+1)}).$$

Then

$$|\overline{f}^i_n - \overline{f}_n| = \mathcal{O}(n^{-(\beta+1)})$$

and

$$\left|\sum_{r=0}^{kn-1}[f^i_n(\sigma^r\omega) - f_n(\sigma^r\omega)]\right| \le C\ell n^{-\beta}.$$

Given ε, choose n large enough that for all $0 \le k \le \ell$,

$$\left\{\omega\in\Omega\,:\,\left|\sum_{r=0}^{kn-1}(f_n(\sigma^r\omega) - \overline{f}_n)\right| > \varepsilon\right\} \subset \left\{\omega\in\Omega\,:\,\left|\sum_{r=0}^{kn-1}(f^i_n(\sigma^r\omega) - \overline{f}^i_n)\right| > \frac{\varepsilon}{2}\right\}.$$

By Chebyshev

$$\mathbb{P}\left(\left|\sum_{r=0}^{kn-1}(f_n^i\circ\sigma^r-\overline{f}_n^i)\right|>\frac{\varepsilon}{2}\right)\le\frac{4}{\varepsilon^2}\sum_{r=0}^{kn-1}\mathbb{E}_{\mathbb{P}}\left(\left[f_n^i\circ\sigma^r-\overline{f}_n^i\right]^2\right)$$
$$+\frac{4}{\varepsilon^2}\left[2\sum_{r=0}^{kn-1}\sum_{u=r+1}^{kn-1}\left|\mathbb{E}_{\mathbb{P}}[(f_n^i\circ\sigma^r-\overline{f}_n^i)(f_n^i\circ\sigma^u-\overline{f}_n^i)]\right|\right]$$

We bound

$$\sum_{r=0}^{kn-1}\mathbb{E}_{\mathbb{P}}\left(\left[f_n^i-\overline{f}_n^i\right]^2\right)\le C\sum_{r=0}^{kn-1}\|f_n^i\|_\infty^2\le\frac{C\ell}{n}$$

and note that if $|r-u|>n^a$ then by independence

$$\mathbb{E}_{\mathbb{P}}\left[(f_n^i\circ\sigma^r-\overline{f}_n^i)(f_n\circ\sigma^u-\overline{f}_n^i)\right]=\mathbb{E}_{\mathbb{P}}\left[f_n^i\circ\sigma^r-\overline{f}_n^i\right]\mathbb{E}_{\mathbb{P}}\left[f_n^i\circ\sigma^u-\overline{f}_n^i\right]=0$$

and hence we may bound

$$\sum_{r=0}^{kn-1}\sum_{u=r+1}^{kn-1}\left|\mathbb{E}_{\mathbb{P}}[(f_n^i\circ\sigma^r-\overline{f}_n^i)(f_n^i\circ\sigma^u-\overline{f}_n^i)]\right|\le\frac{C\ell}{n^{1-a}}.$$

Thus, for n large enough,

$$\mathbb{P}\left(\left\{\omega\in\Omega\,:\,\left|\sum_{r=0}^{kn-1}[f_n(\sigma^r\omega)-\overline{f}_n]\right|>\varepsilon\right\}\right)\le\frac{C\ell}{n^{1-a}\varepsilon^2}.$$

The rest of the argument proceeds as in the case of Lemma 7.1 using a speedup along a sequence $n=p^{1+\eta}$ where $\eta>\frac{a}{1-a}$, since $\|f_n\|_\infty=\mathcal{O}(n^{-1})$ still holds. □

7.2. **Criteria for stable laws and functional limit laws.** The next theorem shows that for regularly varying observables, Poisson convergence and Condition U imply convergence in the J_1 topology if $\alpha\in(0,1)$ and gives an additional condition to be verified in the case $\alpha\in[1,2)$.

Note that (7.2) is essentially condition (5.5) of Proposition 5.7.

Theorem 7.3. *Assume ϕ is regularly varying, Condition U holds and that*

$$N_n^\omega\xrightarrow{d}N_{(\alpha)}$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$.

If $\alpha\in[1,2)$, assume furthermore that for all $\delta>0$, and $\mathbb{P}$-a.e. $\omega\in\Omega$

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\nu\left(\max_{1\le k\le n}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\left[\phi\circ T_\omega^j\mathbf{1}_{\{|\phi\circ T_\omega^j|\le\varepsilon b_n\}}-\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})\right]\right|\ge\delta\right)=0.\tag{7.2}$$

Then $X_n^\omega\xrightarrow{d}X_{(\alpha)}$ in $\mathbb{D}[0,\infty)$ under the probability measure ν^ω for $\mathbb{P}$-a.e. $\omega\in\Omega$.

Remark 7.4. *From* (5.10) *and Theorem 5.1, it follows that the convergence of X_n^ω also happens under the probability measure ν.*

Proof of Theorem 7.3. When $\alpha \in (0,1)$, we check the hypothesis of Proposition 5.4. Using (5.10), we have

$$\left| \frac{1}{b_n} \sum_{j=0}^{n\ell-1} \mathbb{E}_{\nu^{\sigma^j\omega}}(|\phi| \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}}) \right| \le C \frac{n\ell}{b_n} \mathbb{E}_{\nu}(|\phi| \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}})$$

Using (2.3) and Proposition 2.6, we see that condition (5.3) is satisfied since $\alpha < 1$, thus proving the theorem in this case.

When $\alpha \in [1,2)$, we consider instead Proposition 5.7. Firstly, we remark that condition (5.5) is implied by (7.2) and (5.10). It remains to check condition (5.4), which constitutes the rest of the proof.

If $\alpha \in (1,2)$, since $c_n = nb_n^{-1}\mathbb{E}_\nu(\phi)$, we have

$$\left| \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor -1} \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi \mathbf{1}_{\{|\phi| \le \varepsilon b_n\}}) - t(c_n - c_\alpha(\varepsilon)) \right| \le A_n^\omega(t) + B_{n,\varepsilon}^\omega(t) + C_{n,\varepsilon}^\omega(t) \tag{7.3}$$

with

$$A_n^\omega(t) = \left| \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor -1} \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi) - tc_n \right|,$$

$$B_{n,\varepsilon}^\omega(t) = \left| \frac{1}{b_n} \sum_{j=0}^{\lfloor nt \rfloor -1} \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) - \frac{nt}{b_n} \mathbb{E}_{\nu}(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) \right|$$

and

$$C_{n,\varepsilon}^\omega(t) = \left| \frac{nt}{b_n} \mathbb{E}_{\nu}(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) - tc_\alpha(\varepsilon) \right|.$$

Since ϕ is regularity varying with index $\alpha > 1$, it is integrable and the function $\omega \mapsto \mathbb{E}_{\nu^\omega}(\phi)$ is Hölder. Hence, it satisfies the law of the iterated logarithm, and we have for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\left| \frac{1}{k} \sum_{j=0}^{k-1} \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi) - \mathbb{E}_\nu(\phi) \right| = \mathcal{O}\left(\frac{\sqrt{\log \log k}}{\sqrt{k}} \right).$$

Thus, we have

$$\sup_{0 \le t \le \ell} A_n^\omega(t) = \mathcal{O}\left(\frac{\sqrt{n\ell}\sqrt{\log\log(n\ell)}}{b_n} \right).$$

As a consequence, we can deduce that $\lim_{n\to\infty} \sup_{0 \le t \le \ell} A_n^\omega(t) = 0$ since $b_n = n^{\frac{1}{\alpha}} \widetilde{L}(n)$ for a slowly varying function $\widetilde{L}$, with $\alpha < 2$.

By Proposition 2.6, we also have

$$\lim_{n\to\infty} nb_n^{-1} \mathbb{E}_\nu(\phi \mathbf{1}_{\{|\phi| > \varepsilon b_n\}}) = c_\alpha(\varepsilon).$$

In particular, we have

$$\lim_{n\to\infty} \sup_{0 \le t \le \ell} C_{n,\varepsilon}^\omega(t) = 0.$$

This also implies that $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1})$ if we define $\chi_n = b_n^{-1}\phi\mathbf{1}_{\{|\phi|>\varepsilon b_n\}}$. From Lemma 7.1, it follows that $\lim_{n\to\infty}\sup_{0\le t\le \ell} B^\omega_{n,\varepsilon}(t) = 0$.

Putting all these estimates together concludes the proof when $\alpha \in (1,2)$.

When $\alpha = 1$, we estimate (7.3) by $A^\omega_{n,\varepsilon}(t) + B^\omega_{n,\varepsilon}(t)$ with

$$A^\omega_{n,\varepsilon}(t) = \left| \frac{1}{b_n} \sum_{j=0}^{\lfloor nt\rfloor -1} \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}) - \frac{nt}{b_n}\mathbb{E}_\nu(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}) \right|$$

and

$$B^\omega_{n,\varepsilon}(t) = \left| \frac{nt}{b_n}\mathbb{E}_\nu(\phi\mathbf{1}_{\{\varepsilon b_n<|\phi|\le b_n\}}) - tc_\alpha(\varepsilon) \right|.$$

instead, since $c_n = nb_n^{-1}\mathbb{E}_\nu(\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}})$.

We define $\chi_n = b_n^{-1}\phi\mathbf{1}_{\{|\phi|\le\varepsilon b_n\}}$. By Proposition 2.6, we have $\mathbb{E}_m(|\chi_n|) = \mathcal{O}(n^{-1}\widetilde{L}(n))$ for some slowly varying function $\widetilde{L}$, and so by Lemma 7.1,

$$\lim_{n\to\infty}\sup_{0\le t\le\ell} A^\omega_{n,\varepsilon}(t) = 0.$$

On the other hand, by Proposition 2.6, we have

$$\lim_{n\to\infty} nb_n^{-1}\mathbb{E}_\nu(\phi\mathbf{1}_{\{\varepsilon b_n<|\phi|\le\varepsilon b_n\}}) = c_\alpha(\varepsilon)$$

and so $\lim_{n\to\infty}\sup_{0\le t\le\ell} B^\omega_{n,\varepsilon}(t) = 0$ which completes the proof. $\square$

8. An exponential law

8.1. **General considerations.** We denote by $\mathcal{J}$ the family of all finite unions of intervals of the form $(x,y]$, where $-\infty \le x < y \le \infty$ and $0 \notin [x,y]$. For $J \in \mathcal{J}$, we will establish a quenched exponential law for the sequence of sets $A_n = {\phi_{x_0}}^{-1}(b_nJ)$. Similar results were obtained in [CF20, FFV17, HRY20, RSV14, RT15].

Since ϕ is regularly varying, it is easy to verify that

$$\lim_{n\to\infty} n\nu(A_n) = \Pi_\alpha(J).$$

In particular, $m(A_n) = \mathcal{O}(n^{-1})$.

Lemma 8.1. *Assume Condition U and that ϕ is regularly varying with index α.*

If $A_n \subset [0,1]$ is a sequence of measurable subsets such that $m(A_n) = \mathcal{O}(n^{-1})$, then for all $0 \le s < t$,

$$\lim_{n\to\infty}\left(\left[\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n) \right] - n(t-s)\nu(A_n) \right) = 0.$$

The same result holds in the setting of Example 5.12 if $A_n \subset [\delta,1]$ for some $\delta > 0$ with $m(A_n) = \mathcal{O}(n^{-1})$. In particular, if $A_n = \phi_{x_0}^{-1}(b_nJ)$ for $J \in \mathcal{J}$, then for all $0 \le s < t$.

$$\lim_{n\to\infty} \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n) = (t-s)\Pi_\alpha(J).$$

Proof. For the first statement, it suffices to apply Lemma 7.1 or Lemma 7.2 with $\chi_n = \mathbf{1}_{A_n}$. The second statement immediately follows since $\lim_n n\nu(A_n) = \Pi_\alpha(J)$. □

Corollary 8.2. *Assume the hypothesis of Lemma 8.1.*
Let $J \in \mathcal{J}$, and set $A_n = \phi^{-1}(b_n J)$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, and all $0 \le s < t$,

$$\lim_{n\to\infty} \prod_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \Big(1 - \nu^{\sigma^j\omega}(A_n)\Big) = e^{-(t-s)\Pi_\alpha(J)}.$$

Proof. Since $\nu^\omega(A_n)$ is of order at most n^{-1} uniformly in $\omega \in \Omega$, it follows that

$$\log\left[\prod_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \Big(1 - \nu^{\sigma^j\omega}(A_n)\Big)\right] = -\left(\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n)\right) + \mathcal{O}(n^{-1}).$$

By Lemma 8.1,

$$\lim_{n\to\infty} \sum_{j=\lfloor ns\rfloor}^{\lfloor nt\rfloor-1} \nu^{\sigma^j\omega}(A_n) = (t-s)\Pi_\alpha(J),$$

which yields the conclusion. □

Definition 8.3. *For a measurable subset $U \subset Y = [0,1]$, we define the hitting time of $(\omega, x) \in \Omega\times Y$ to U by*

$$R_U(\omega)(x) := \inf\Big\{k \ge 1 \,:\, T_\omega^k(x) \in U\Big\}.$$

and the induced measure by ν on U by

$$\nu_U(A) := \frac{\nu(A\cap U)}{\nu(U)}$$

In order to establish our exponential law, we will first obtain a few estimates, based on the proof of [HSV99, Theorem 2.1], to relate $\nu^\omega(R_{A_n}(\omega) > \lfloor nt\rfloor)$ to $\sum_{j=0}^{\lfloor nt\rfloor-1} \nu^{\sigma^j\omega}(A_n)$ so that we are able to invoke Corollary 8.2.

The next lemma is basically [RSV14, Lemma 6].

Lemma 8.4. *For every measurable set $U \subset [0,1]$, we have the bound*

$$\begin{aligned}\left|\nu^\omega(R_U(\omega) > k) - \prod_{j=1}^{k}(1-\nu^{\sigma^j\omega}(U))\right| &\le \sum_{j=1}^{k} \nu^{\sigma^j\omega}(U)\; c_{\sigma^j\omega}(k-j, U) \prod_{i=1}^{j-1}(1-\nu^{\sigma^i\omega}(U)) \\ &\le \sum_{j=1}^{k} \nu^{\sigma^j\omega}(U)\; c_{\sigma^j\omega}(U)\end{aligned}$$

where

$$c_\omega(k, U) := |\nu_U^\omega(R_U(\omega) > k) - \nu^\omega(R_U(\omega) > k)|$$

and

$$c_\omega(U) := \sup_{k\ge 0} c_\omega(k, U).$$

Proof. Note that $\{R_U(\omega) > k\} = [T^1_\omega]^{-1}(U^c \cap \{R_U(\sigma\omega) > k-1\})$ and so, using the equivariance of $\{\nu^\omega\}_{\omega\in\Omega}$,

$$\nu^\omega(R_U(\omega) > k) = \nu^{\sigma\omega}(U^c \cap \{R_U(\sigma\omega) > k-1\}).$$

Hence

$$\nu^\omega(R_U(\omega) > k) = \nu^{\sigma\omega}(R_U(\sigma\omega) > k-1) - \nu^{\sigma\omega}(U \cap \{R_U(\sigma\omega) > k-1\}).$$

We note that

$$\begin{aligned}\nu^\omega(R_U(\omega) > k) &= \nu^{\sigma\omega}(R_U(\sigma\omega) > k-1) - \nu^{\sigma\omega}(U)[\nu^{\sigma\omega}(R_U(\sigma\omega) > k-1) + c_{\sigma\omega}(k-1, U)]\\ &= (1-\nu^{\sigma\omega}(U))\nu^{\sigma\omega}(R_U(\sigma\omega) > k-1) - \nu^{\sigma\omega}(U)c_{\sigma\omega}(k-1,U).\end{aligned}$$

Iterating we obtain, using the fact that for $\mathbb{P}$-a.e. ω, $\nu^\omega(R_U(\omega) \geq 1) = 1$,

$$\nu^\omega(R_U(\omega) > k) = \prod_{j=1}^{k}(1-\nu^{\sigma^j\omega}(U)) - \sum_{j=1}^{k}\nu^{\sigma^j\omega}(U)c_{\sigma^j\omega}(k-j,U)\prod_{i=1}^{j-1}(1-\nu^{\sigma^i\omega}(U))$$

which yields the conclusion. □

We will estimate now the coefficients $c_\omega(U)$.

Lemma 8.5. *Fix N. Then, for any measurable subset $U \subset Y$ such that $\mathbf{1}_U \in \mathrm{BV}$, we have*

$$c_\omega(U) \leq \nu^\omega_U(R_U(\omega) \leq N) + \nu^\omega(R_U(\omega) \leq N) + \frac{1}{\nu^\omega(U)}\left\|P^N_\omega([\mathbf{1}_U - \nu^\omega(U)]h_\omega)\right\|_{L^1(m)} \tag{8.1}$$

with C independent of N and

$$\nu^\omega_U(R_U(\omega) \leq N) \leq \frac{1}{\nu^\omega(U)}\nu^\omega(R_U(\omega) \leq N), \qquad \nu^\omega(R_U(\omega) \leq N) \leq \sum_{i=1}^{N}\nu^{\sigma^i\omega}(U) \tag{8.2}$$

Proof. The estimates (8.2) follow from

$$\{R_U(\omega) \leq N\} = \bigcup_{i=1}^{N}(T^i_\omega)^{-1}(U).$$

and therefore

$$\nu^\omega(R_U(\omega) \leq N) \leq \sum_{i=1}^{N}\nu^{\sigma^i\omega}(U)$$

For (8.1), note that

$$c_\omega(U) = |\nu^\omega_U(R_U(\omega) \leq j) - \nu^\omega(R_U(\omega) \leq j)|$$

If $j \leq N$ then

$$c_\omega(U) \leq \nu^\omega_U(R(\omega) \leq N) + \nu^\omega(R(\omega) \leq N)$$

If $j > N$ we write

$$\begin{aligned}\nu^\omega_U(R_U(\omega) \leq j) - \nu^\omega(R_U(\omega) \leq j) =\ & \nu^\omega_U(R_U(\omega) \leq j) - \nu^\omega_U(T^{-N}_\omega(R_U(\sigma^N\omega) \leq j-N))\\ +\ & \nu^\omega_U(T^{-N}_\omega(R_U(\sigma^N\omega) \leq j-N)) - \nu^\omega(T^{-N}_\omega(R_U(\sigma^N\omega) \leq j-N))\\ +\ & \nu^\omega(T^{-N}_\omega(R_U(\sigma^N\omega) \leq j-N)) - \nu^\omega(R_U(\omega) \leq j)\\ =\ & (a) + (b) + (c).\end{aligned}$$

To bound (a) and (c) note that

$$\{R_U(\omega) \leq j\} = \{R_U(\omega) \leq N\} \cup T_\omega^{-N}(\{R_U(\sigma^N\omega) \leq j - N)\})$$

so

$$|\nu^\omega(R_U(\omega) \leq j) - \nu^\omega(T_\omega^{-N}(R_U(\sigma^N\omega) \leq j - N))| \leq \nu^\omega(R_U(\omega) \leq N) \tag{8.3}$$

and similarly for ν_U^ω.

To bound (b) we use the decay of P_ω^k. Setting $V = \left\{R_U(\sigma^N\omega) \leq j - N\right\}$, we have

$$\begin{aligned} |\nu_U^\omega(T_\omega^{-N}(V)) - \nu^\omega(T_\omega^{-N}(V))| &= \frac{1}{\nu^\omega(U)} \left| \int_Y \mathbf{1}_U \mathbf{1}_V \circ T_\omega^N h_\omega dm - \nu^\omega(U) \int_Y \mathbf{1}_V \circ T_\omega^N h_\omega dm \right| \\ &= \frac{1}{\nu^\omega(U)} \left| \int_Y \mathbf{1}_V P_\omega^N([\mathbf{1}_U - \nu^\omega(U)]h_\omega) dm \right| \\ &\leq \frac{1}{\nu^\omega(U)} \left\| P_\omega^N([\mathbf{1}_U - \nu^\omega(U)]h_\omega) \right\|_{L^1(m)}. \end{aligned}$$

□

8.2. **Uniformly expanding maps.** We can now prove the exponential law for $A_n = \phi^{-1}(b_n J)$, $J \in \mathcal{J}$.

Proof of Theorem 6.3. Due to rounding errors when taking the integer parts, we have

$$\begin{aligned} &\left| \nu^{\sigma^{\lfloor ns \rfloor}\omega} \left(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor n(t-s) \rfloor \right) - \nu^{\sigma^{\lfloor ns \rfloor}\omega} \left(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor nt \rfloor - \lfloor ns \rfloor \right) \right| \\ &\qquad\qquad \leq \nu^{\sigma^{\lfloor nt \rfloor}\omega}(A_n) \leq Cm(A_n) \to 0, \end{aligned}$$

and it is thus enough to prove the convergence of $\nu^{\sigma^{\lfloor ns \rfloor}\omega} \left(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor nt \rfloor - \lfloor ns \rfloor \right)$.

By Lemmas 8.4 and 8.5, for all $N \geq 1$, we have

$$\left| \nu^{\sigma^{\lfloor ns \rfloor}\omega} \left(R_{A_n}(\sigma^{\lfloor ns \rfloor}\omega) > \lfloor nt \rfloor - \lfloor ns \rfloor \right) - \prod_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} (1 - \nu^{\sigma^j\omega}(A_n)) \right| \leq \text{(I)} + \text{(II)} + \text{(III)}, \tag{8.4}$$

with

$$\text{(I)} = \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \nu^{\sigma^j\omega} \left(A_n \cap \left\{ R_{A_n}(\sigma^j\omega) \leq N \right\} \right),$$

$$\text{(II)} = \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \nu^{\sigma^j\omega}(A_n) \nu^{\sigma^j\omega}(R_{A_n}(\sigma^j\omega) \leq N)$$

and

$$\text{(III)} = \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \left\| P_{\sigma^j\omega}^N \left(\left[\mathbf{1}_{A_n} - \nu^{\sigma^j\omega}(A_n) \right] h_{\sigma^j\omega} \right) \right\|_{L^1(m)}.$$

To estimate (I), we choose $\varepsilon > 0$ such that $J \subset \{|x| > \varepsilon\}$ and we introduce $V_n = \{|\phi| > \varepsilon b_n\}$. For a measurable subset $V \subset Y$, we also define the shortest return to V by

$$r_\omega(V) = \inf_{x \in V} R_V(\omega)(x),$$

and we set

$$r(V) = \inf_{\omega \in \Omega} r_\omega(V).$$

We have

$$\begin{aligned}
\nu^{\sigma^j\omega}\left(A_n \cap \left\{R_{A_n}(\sigma^j\omega) \le N\right\}\right) &\le \nu^{\sigma^j\omega}\left(V_n \cap \left\{R_{V_n}(\sigma^j\omega) \le N\right\}\right) \\
&\le \sum_{i=r_{\sigma^j\omega}(V_n)}^{N} \nu^{\sigma^j\omega}\left(V_n \cap (T^i_{\sigma^j\omega})^{-1}(V_n)\right) \\
&\le \sum_{i=r_{\sigma^j\omega}(V_n)}^{N} \int_Y \mathbf{1}_{V_n} P^i_{\sigma^j\omega}(\mathbf{1}_{V_n} h_{\sigma^j\omega}) dm.
\end{aligned}$$

It follows from **(Dec)** that

$$\begin{aligned}
\left| \int_Y \mathbf{1}_{V_n} P^i_{\sigma^j\omega}(\mathbf{1}_{V_n} h_{\sigma^j\omega}) dm - \nu^{\sigma^j\omega}(V_n)\nu^{\sigma^{i+j}\omega}(V_n) \right| &\le \|\mathbf{1}_{V_n}\|_{L^1_m} \left\| P^i_{\sigma^j\omega}\left(\left[\mathbf{1}_{V_n} - \nu^{\sigma^j\omega}(V_n)\right] h_{\sigma^j\omega}\right)\right\|_{L^\infty_m} \\
&\le C\theta^i m(V_n) \left\| \left[\mathbf{1}_{V_n} - \nu^{\sigma^j\omega}(V_n)\right] h_{\sigma^j\omega}\right\|_{\mathrm{BV}} \\
&\le C\theta^i m(V_n),
\end{aligned}$$

as BV is a Banach algebra, and both $\|\mathbf{1}_{V_n}\|_{\mathrm{BV}}$ and $\|h_{\sigma^j\omega}\|_{\mathrm{BV}}$ are uniformly bounded. [2].

Consequently,

$$\begin{aligned}
\text{(I)} &\le \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \sum_{i=r_{\sigma^j\omega}(V_n)}^{N} \left[\nu^{\sigma^j\omega}(V_n)\nu^{\sigma^{i+j}\omega}(V_n) + \mathcal{O}\left(\theta^i m(V_n)\right)\right] \\
&\le C\left(m(V_n)^2 nN + m(V_n) n\theta^{r(V_n)}\right).
\end{aligned}$$

On the other hand, we have by (8.2),

$$\begin{aligned}
\text{(II)} &\le \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n) \sum_{i=1}^{N} \nu^{\sigma^{i+j}\omega}(A_n) \\
&\le CnNm(A_n)^2,
\end{aligned}$$

and it follows from **(Dec)** that

$$\begin{aligned}
\text{(III)} &\le C\theta^N \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left\| \left[\mathbf{1}_{A_n} - \nu^{\sigma^j\omega}(A_n)\right] h_{\sigma^j\omega}\right\|_{\mathrm{BV}} \\
&\le Cn\theta^N,
\end{aligned}$$

[2]Recall that, from the definition of ϕ, it follows that V_n is an open interval, and thus $\mathbf{1}_{V_n}$ has a uniformly bounded BV norm.

since $\{h_\omega\}_{\omega\in\Omega}$ is a bounded family in BV, A_n is the union of at most two intervals and thus $\|\mathbf{1}_{A_n}\|_{\mathrm{BV}}$ is uniformly bounded. We can thus bound (8.4) by

$$C\left(m(V_n)^2nN + m(V_n)n\theta^{r(V_n)} + m(A_n)^2nN + n\theta^N\right) \le C\left(n^{-1}N + \theta^{r(V_n)} + n\theta^N\right),$$

and, assuming for the moment that $r(V_n) \to +\infty$, we obtain the conclusion by choosing $N = N(n) = 2\log n$ and letting $n \to \infty$.

It thus remains to show that $r(V_n) \to +\infty$. Recall that V_n is the ball of centre x_0 and radius $b^{-1}\varepsilon^{-\alpha}n^{-1}$. Let $R \ge 1$ be a positive integer. Since x_0 is assumed to be non-periodic, and that the collection of maps T_ω^j for $\omega \in \Omega$ and $0 < j < R$ is finite, we have that

$$\delta_R := \inf_{\omega\in\Omega}\inf_{0<j<R}|T_\omega^j(x_0) - x_0| > 0$$

is positive. Since all the maps T_ω^j are continuous at x_0 by assumption, there exists $n_R \ge 1$ such that for all $n \ge n_R$, $j < R$ and $\omega \in \Omega$,

$$x \in V_n \Longrightarrow |T_\omega^j(x) - T_\omega^j(x_0)| < \frac{\delta_R}{2}.$$

Increasing n_R if necessary, we can assume that $b^{-1}\varepsilon^{-\alpha}n^{-1} < \frac{\delta_R}{2}$ for all $n \ge n_R$.

Then, for all $n \ge n_R$, $\omega \in \Omega$, $j < R$ and $x \in V_n$, we have

$$|T_\omega^j(x) - x_0| \ge |T_\omega^j(x_0) - x_0| - |T_\omega^j(x) - T_\omega^j(x_0)| > \frac{\delta_R}{2} > b^{-1}\varepsilon^{-\alpha}n^{-1},$$

and thus $T_\omega^j(x) \notin V_n$.

This implies that $r(V_n) > R$ for all $n \ge n_R$, which concludes the proof as R is arbitrary. □

Remark 8.6. *A quenched exponential law for random piecewise expanding maps of the interval is proved in Theorem 7.1* [HRY20, Section 7.1]. *Our proof follows the same standard approach. We are able to specify that Theorem 6.3 holds for non-periodic x_0, since our assumptions imply decay of correlations against L^1 observables, which is known to be necessary for this purpose, see* [AFV15, Section 3.1]. *Our proof is shorter, as we consider the simpler setting of finitely many maps, which are all uniformly expanding. In addition we use the exponential law in the intermittent case of Theorem 7.2* [HRY20, Section 7.2] *to establish the short returns condition of Lemma 8.7 below.*

8.3. **Intermittent maps.** In order to prove the exponential law in the intermittent setting, Theorem 6.4, we need a genericity condition on the point x_0 in the definition (2.5) of ϕ_{x_0}.

Lemma 8.7. *In the setting of Example 5.12, with $\gamma_{max} < \frac{1}{3}$, for m-a.e. x_0 and for $\mathbb{P}$-a.e. $\omega \in \Omega$*

$$\lim_{n\to\infty}\sum_{j=\lfloor sn\rfloor+1}^{\lfloor tn\rfloor} m\left(B_{cn^{-1}}(x_0) \cap \left\{R^{\sigma^j\omega}_{B_{cn^{-1}}(x_0)} \le \lfloor n(\log n)^{-1}\rfloor\right\}\right) = 0.$$

for all $c > 0$ and all $0 \le s < t$.

Proof. Let $N = \lfloor n(\log n)^{-1}\rfloor$ an $V_n = B_{cn^{-1}}(x_0)$. First, we remark that for m-a.e. x_0 and $\mathbb{P}$-a.e. ω,

$$m\left(V_n \cap \{R_{V_n}(\omega) \le N\}\right) = o(n^{-1}). \tag{8.5}$$

This is a consequence of [HRY20, Theorem 7.2]. Their result is stated for two intermittent LSV maps both with $\gamma < \frac{1}{3}$ but generalizes immediately to a finite collection of maps with a uniform bound of $\gamma_{max} < \frac{1}{3}$. The exponential law for return times to nested balls imples that for a fixed t, for m-a.e x_0 and $\mathbb{P}$-a.e. ω

$$\lim_{n\to\infty} \frac{1}{\nu^\omega(V_n)} \nu^\omega \left(V_n \cap \{R_{V_n}(\omega) \le nt\} \right) = 1 - e^{-t}.$$

which shows in particular, since $\{R_{V_n}(\omega) \le N\} \subset \{R_{V_n}(\omega) \le nt\}$ for all n large enough, that for all $t > 0$, m-a.e x_0 and $\mathbb{P}$-a.e. ω

$$\limsup_{n\to\infty} \frac{1}{\nu^\omega(V_n)} \nu^\omega \left(V_n \cap \{R_{V_n}(\omega) \le N\} \right) \le 1 - e^{-t}. \tag{8.6}$$

Using (5.9), taking the limit $t \to 0$ proves (8.5). Note that, even though the set of full measure of x_0 and ω such that (8.6) holds may depend on t, it is enough to consider only a sequence $t_k \to 0$.

Now, for $k \ge 0$ and $n_0 \ge 1$, we introduce the set

$$\Omega_k^{n_0} = \left\{ \omega \in \Omega \,:\, m\left(V_n \cap \{R_{V_n}(\omega) \le N\} \right) \le \frac{2^{-k}}{n} \text{ for all } n \ge n_0 \right\}.$$

According to (8.5), we have for all $k \ge 0$,

$$\lim_{n_0\to\infty} \mathbb{P}(\Omega_k^{n_0}) = \mathbb{P}\left(\bigcup_{n_0 \ge 1} \Omega_k^{n_0} \right) = 1.$$

By the Birkhoff ergodic theorem, for al $k \ge 0$, $n_0 \ge 1$ and $\mathbb{P}$-a.e. ω,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_{\Omega_k^{n_0}}(\sigma^j \omega) = \mathbb{P}(\Omega_k^{n_0}),$$

which implies that for all $0 \le s < t$,

$$\lim_{n\to\infty} \frac{1}{(\lfloor nt \rfloor - \lfloor ns \rfloor)} \sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \mathbf{1}_{\Omega_k^{n_0}}(\sigma^j \omega) = \mathbb{P}(\Omega_k^{n_0}).$$

Let $n_0 = n_0(\omega, k)$ such that $\mathbb{P}(\Omega_k^{n_0}) \ge 1 - 2^{-k}$, and for all $n \ge n_0$,

$$\frac{1}{(\lfloor nt \rfloor - \lfloor ns \rfloor)} \sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \mathbf{1}_{\Omega_k^{n_0}}(\sigma^j \omega) \ge \mathbb{P}(\Omega_k^{n_0}) - 2^{-k}.$$

Then, for all $n \ge n_0(\omega, k)$ we have

$$\frac{1}{(\lfloor nt \rfloor - \lfloor ns \rfloor)} \sum_{j=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \mathbf{1}_{(\Omega_k^{n_0})^c}(\sigma^j \omega) \le 2^{-(k-1)}.$$

Consequently,

$$\sum_{\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} m\left(V_n \cap \{R_{V_n}(\omega) \le N\} \right) \le (\lfloor nt \rfloor - \lfloor ns \rfloor) \frac{2^{-k}}{n} + (\lfloor nt \rfloor - \lfloor ns \rfloor)\, 2^{-(k-1)} m(V_n).$$

This proves that

$$\limsup_{n\to\infty} \sum_{\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} m\left(V_n \cap \{R_{V_n}(\omega) \le N\}\right) \le C\, 2^{-k},$$

and the result follows by taking the limit $k \to \infty$.

Note that the set of x_0 and ω for which the lemma holds depends a priori on $c > 0$, but it is enough to consider a countable and dense set of c, since for $c < c'$,

$$\left\{B_{cn^{-1}}(x_0) \cap \left\{R^{\omega}_{B_{cn^{-1}}(x_0)} \le N\right\}\right\} \subset \left\{B_{c'n^{-1}}(x_0) \cap \left\{R^{\omega}_{B_{c'n^{-1}}(x_0)} \le N\right\}\right\}.$$

□

The exponential law for random intermittent maps follows from Lemma 8.7:

Proof of Theorem 6.4. We consider the three terms in (8.4) with $N = \lfloor n(\log n)^{-1}\rfloor$.

Let $V_n = \{|\phi| > \varepsilon b_n\}$ where $\varepsilon > 0$ is such that $A_n \subset V_n$ for all $n \ge 1$. Since V_n is a ball of centre x_0 and radius $b^{-1}\varepsilon^{-\alpha}n^{-1}$, and since $V_n \subset [\delta, 1]$, the term

$$\text{(I)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}\left(A_n \cap \left\{R_{A_n}(\sigma^j\omega) \le N\right\}\right) \le C \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} m\left(V_n \cap \left\{R_{V_n}(\sigma^j\omega) \le N\right\}\right)$$

tends to zero by Lemma 8.7 for m-a.e x_0.

The term

$$\text{(II)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \nu^{\sigma^j\omega}(A_n)\nu^{\sigma^j\omega}(R_{A_n}(\sigma^j\omega) \le N) \le CnNm(A_n)^2$$

also tends to zero since $N = o(n)$. Lastly we consider

$$\text{(III)} = \sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left\| P^N_{\sigma^j\omega}\left(\left[\mathbf{1}_{A_n} - \nu^{\sigma^j\omega}(A_n)\right] h_{\sigma^j\omega}\right)\right\|_{L^1(m)}.$$

We approximate $\mathbf{1}_{A_n}$ by a C^1 function g such that $\|g\|_{C^1} \le n^{\tau}$, $g = \mathbf{1}_{A_n}$ on A_n and $\|g - \mathbf{1}_{A_n}\|_{L^1} \le n^{-\tau}$ (recall A_n is two intervals of length roughly $\frac{1}{n}$ so a simple smoothing at the endpoints of the intervals allows us to find such a function g). Later we will specify $\tau > 1$ will suffice. By [NPT21, Lemma 3.4] with $h = h_\omega$ and $\varphi = g - m(gh_\omega)$, for all ω,

$$\begin{aligned}\left\|P^N_\omega([g - m(gh_\omega)]h_\omega)\right\|_{L^1} &\le Cn^{\tau}N^{1-\frac{1}{\gamma_{max}}}(\log N)^{\frac{1}{\gamma_{max}}}\\ &\le Cn^{\tau+1-\frac{1}{\gamma_{max}}}(\log n)^{\frac{2}{\gamma_{max}}-1}.\end{aligned}$$

Using the decomposition $\mathbf{1}_{A_n} - \nu^\omega(A_n) = (\mathbf{1}_{A_n} - g) - (\nu^\omega(A_n) - m(gh_\omega)) + (g - m(gh_\omega))$ we estimate, leaving out the log term,

$$\text{(III)} \le C\left[n^{1-\tau} + n^{\tau+2-\frac{1}{\gamma_{max}}}\right]$$

where the value of C may change line to line. Taking $\gamma_{max} < \frac{1}{3}$ and $1 < \tau < \frac{1}{\gamma_{max}} - 2$ suffices. □

9. Point process results

We now proceed to the proof of the Poisson convergence. In Section 11 we will consider an annealed version of our results.

Recall that the counting point processes N_n^ω are defined by

$$N_n^\omega(B) := \#\left\{j \geq 1 \,:\, \left(\frac{j}{n}, \frac{\phi \circ T_\omega^{j-1}}{b_n}\right) \in B\right\},\ n \geq 1,$$

for $B \in \mathcal{B}((0,\infty) \times (\mathbb{R} \setminus \{0\}))$.

9.1. **Uniformly expanding maps.**

Recall Theorem 6.5: under the conditions of Section 5.5, in particular **(LY)**, **(Min)** and **(Dec)**, if $x_0 \notin \mathcal{D}$ is not periodic, then for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$N_n^\omega \xrightarrow{d} N_{(\alpha)}$$

under the probability measure ν^ω.

Our proof of Theorem 6.5 uses the existence of a spectral gap for the associated transfer operators P_ω^n, and breaks down in the setting of Example 5.12. The use of the spectral gap is encapsulated in the following lemma.

Lemma 9.1. *Assume* **(LY)**. *Then there exists $C > 0$ such that for all $\omega \in \Omega$, all $f, f_n \in \mathrm{BV}$ with*

$$\sup_{j\geq 1} \|f_j\|_{L^\infty(m)} \leq 1 \ \ \textit{and} \ \ \sup_{j\geq 1} \|f_j\|_{\mathrm{BV}} < \infty,$$

we have

$$\sup_{n\geq 0} \left\| P_\omega^n \left(f \cdot \prod_{j=1}^n f_j \circ T_\omega^j \right) \right\|_{\mathrm{BV}} \leq C \|f\|_{\mathrm{BV}} \left(\sup_{j\geq 1} \|f_j\|_{\mathrm{BV}} \right)$$

Proof. We proceed in four steps.

Step 1. We define

$$g_\omega^n = \prod_{j=0}^n f_j \circ T_\omega^j,$$

where we have set $f_0 = \mathbf{1}$. We observe that for all $n \geq 0$, there exists $C_n > 0$ such that for all $\omega \in \Omega$,

$$\|g_\omega^n\|_{L^\infty(m)} \leq \left(\sup_{j\geq 1} \|f_j\|_{L^\infty(m)} \right)^{n+1} \leq 1 \ \text{ and } \ \|g_\omega^n\|_{\mathrm{BV}} \leq C_n \left(\sup_{j\geq 1} \|f_j\|_{\mathrm{BV}} \right). \tag{9.1}$$

The first estimate is immediate, and the second follows, because

$$\begin{aligned}
\mathrm{Var}(g_\omega^{n+1}) &\le \mathrm{Var}(g_\omega^n)\|f_{n+1}\circ T_\omega^{n+1}\|_{L^\infty(m)} + \|g_\omega^n\|_{L^\infty(m)}\mathrm{Var}(f_{n+1}\circ T_\omega^{n+1}) \\
&\le \mathrm{Var}(g_\omega^n) + \mathrm{Var}(f_{n+1}\circ T_\omega^{n+1}) \\
&= \mathrm{Var}(g_\omega^n) + \sum_{I\in\mathcal{A}_\omega^{n+1}} \mathrm{Var}_I(f_{n+1}\circ T_\omega^{n+1}) \\
&= \mathrm{Var}(g_\omega^n) + \sum_{I\in\mathcal{A}_\omega^{n+1}} \mathrm{Var}_{T_\omega^{n+1}(I)}(f_{n+1}) \\
&\le \mathrm{Var}(g_\omega^n) + \left(\#\mathcal{A}_\omega^{n+1}\right)\mathrm{Var}(f_{n+1}),
\end{aligned}$$

and so we can define by induction $C_{n+1} = C_n + \sup_{\omega\in\Omega}\#\mathcal{A}_\omega^{n+1}$ which is finite, as there are only finitely many maps in $\mathcal{S}$.

Step 2. We first prove the lemma in the case where $r = 1$ in the condition **(LY)**. Before, we claim that for $f \in \mathrm{BV}$ and sequences $(f_j) \subset \mathrm{BV}$ as in the statement, we have

$$\begin{aligned}
\mathrm{Var}\left(P_\omega^n(fg_\omega^n)\right) \le \sum_{j=0}^n \rho^j \|P_\omega^{n-j}(fg_\omega^{n-j-1})\|_{L^\infty(m)}\|f_{n-j}\|_{\mathrm{BV}} \\
+ D\sum_{j=0}^{n-1}\rho^j\|P_\omega^{n-1-j}(fg_\omega^{n-1-j})\|_{L^1(m)}\|f_{n-j}\|_{L^\infty(m)}.
\end{aligned} \tag{9.2}$$

This implies the lemma when $r = 1$, since

$$\|P_\omega^{n-j}(fg_\omega^{n-j-1})\|_{L^\infty(m)} \le \|g_\omega^{n-j-1}\|_{L^\infty(m)}\|P_\omega^{n-j}|f|\|_{L^\infty(m)} \le C\|f\|_{\mathrm{BV}},$$

and

$$\|P_\omega^{n-j}(fg_\omega^{n-j})\|_{L^1(m)} \le \|fg_\omega^{n-j}\|_{L^1(m)} \le \|f\|_{L^\infty(m)}\|g_\omega^{n-j}\|_{L^1(m)} \le \|f\|_{\mathrm{BV}}.$$

We prove the claim by induction on $n \ge 0$. It is immediate for $n = 0$, and for the induction step, we have, using **(LY)**,

$$\begin{aligned}
&\mathrm{Var}(P_\omega^{n+1}(fg_\omega^{n+1})) \\
&\quad= \mathrm{Var}(P_\omega^{n+1}(fg_\omega^n f_{n+1}\circ T_\omega^{n+1})) = \mathrm{Var}(P_\omega^{n+1}(fg_\omega^n)f_{n+1}) \\
&\quad\le \mathrm{Var}(P_\omega^{n+1}(fg_\omega^n))\|f_{n+1}\|_{L^\infty(m)} + \|P_\omega^{n+1}(fg_\omega^n)\|_{L^\infty(m)}\mathrm{Var}(f_{n+1}) \\
&\quad\le \left(\rho\mathrm{Var}(P_\omega^n(fg_\omega^n)) + D\|P_\omega^n(fg_\omega^n)\|_{L^1(m)}\right)\|f_{n+1}\|_{L^\infty(m)} + \|P_\omega^{n+1}(fg_\omega^n)\|_{L^\infty(m)}\mathrm{Var}(f_{n+1}) \\
&\quad\le \rho\mathrm{Var}(P_\omega^n(fg_\omega^n)) + D\|P_\omega^n(fg_\omega^n)\|_{L^1(m)}\|f_{n+1}\|_{L^\infty(m)} + \|P_\omega^{n+1}(fg_\omega^n)\|_{L^\infty(m)}\|f_{n+1}\|_{\mathrm{BV}},
\end{aligned}$$

which proves (9.2) for $n+1$, assuming it holds for n.

Step 3. Now, we consider the general case $r \ge 1$ and we assume that n is of the particular form $n = pr$, with $p \ge 0$. We note that the random system defined with $\mathcal{T} = \{T_\omega^r\}_{\omega\in\Omega}$ satisfies the condition **(LY)** with $r = 1$. Consequently, by the second step and (9.1), we have

$$\|P_\omega^n(fg_\omega^n)\|_{\mathrm{BV}} = \left\| P^r_{\sigma^{r-1}\omega} \circ \ldots \circ P^r_\omega \left(f \prod_{j=1}^{p} g^r_{\sigma^{jr}\omega} \circ T_\omega^{jr} \right) \right\|_{\mathrm{BV}}$$
$$\leq C\|f\|_{\mathrm{BV}} \left(\sup_{j\geq 1} \|g^r_{\sigma^{jr}\omega}\|_{\mathrm{BV}} \right) \leq CC_r \|f\|_{\mathrm{BV}} \left(\sup_{j\geq 1} \|f_j\|_{\mathrm{BV}} \right).$$

Step 4. Finally, if $n = pr + q$, with $p \geq 0$ and $q \in \{0, \ldots, r-1\}$, as an immediate consequence of **(LY)**, we obtain

$$\begin{aligned}\|P_\omega^n(fg_\omega^n)\|_{\mathrm{BV}} &= \|P^q_{\sigma^{pr}\omega} P^{pr}_\omega (f g^{pr}_\omega g^q_{\sigma^{pr}\omega} \circ T^{pr}_\omega)\|_{\mathrm{BV}} \\ &= \|P^q_{\sigma^{pr}\omega}(P^{pr}_\omega(fg^{pr}_\omega) g^q_{\sigma^{pr}\omega})\|_{\mathrm{BV}} \leq C\|P^{pr}_\omega(fg^{pr}_\omega) g^q_{\sigma^{pr}\omega}\|_{\mathrm{BV}}.\end{aligned}$$

But, from Step 3, we have

$$\begin{aligned}\|P^{pr}_\omega(fg^{pr}_\omega) g^q_{\sigma^{pr}\omega}\|_{L^1(m)} &\leq \|g^q_{\sigma^{pr}\omega}\|_{L^\infty(m)} \|P^{pr}_\omega(fg^{pr}_\omega)\|_{L^1(m)} \\ &\leq \|P^{pr}_\omega(fg^{pr}_\omega)\|_{L^1(m)} \leq C\|f\|_{\mathrm{BV}} \left(\sup_{j\geq 1} \|f_j\|_{\mathrm{BV}} \right),\end{aligned}$$

and, using (9.1),

$$\begin{aligned}\mathrm{Var}(P^{pr}_\omega(fg^{pr}_\omega) g^q_{\sigma^{pr}\omega}) &\leq \|P^{pr}_\omega(fg^{pr}_\omega)\|_{L^\infty(m)} \mathrm{Var}(g^q_{\sigma^{pr}\omega}) + \mathrm{Var}(P^{pr}_\omega(fg^{pr}_\omega)) \|g^q_{\sigma^{pr}\omega}\|_{L^\infty(m)} \\ &\leq \left[C_q \|g^{pr}_\omega\|_{L^\infty(m)} \|P^{pr}_\omega |f|\|_{L^\infty(m)} + C\|f\|_{\mathrm{BV}} \right] \left(\sup_{j\geq 1} \|f_j\|_{\mathrm{BV}} \right) \\ &\leq C \left(1 + \max_{q=0,\ldots,r-1} C_q \right) \|f\|_{\mathrm{BV}} \left(\sup_{j\geq 1} \|f_j\|_{\mathrm{BV}} \right),\end{aligned}$$

which concludes the proof of the lemma. □

Proof of Theorem 6.5. We denote by $\mathcal{R}$ the family of finite unions of rectangles R of the form $R = (s,t] \times J$ with $J \in \mathcal{J}$. By Kallenberg's theorem, see [Kal76, Theorem 4.7] or [Res87, Proposition 3.22], $N_n^\omega \xrightarrow{d} N_{(\alpha)}$ if for any $R \in \mathcal{R}$,

$$(a) \quad \lim_{n\to\infty} \nu^\omega(N_n^\omega(R) = 0) = \mathbb{P}(N_{(\alpha)}(R) = 0),$$

and

$$(b) \quad \lim_{n\to\infty} \mathbb{E}_{\nu^\omega} N_n^\omega(R) = \mathbb{E} N_{(\alpha)}(R).$$

We first prove (b). We write

$$R = \bigcup_{i=1}^{k} R_i,$$

with $R_i = (s_i, t_i] \times J_i$ disjoint.

Then

$$\mathbb{E}N_{(\alpha)}(R)=\sum_{i=1}^{k}(t_i-s_i)\Pi_\alpha(J_i)$$

and

$$\begin{aligned}\mathbb{E}_{\nu^\omega}N_n^\omega(R)=\sum_{i=1}^{k}\mathbb{E}_{\nu^\omega}N_n^\omega((s_i,t_i]\times J_i)&=\sum_{i=1}^{k}\sum_{ns_i<j\le nt_i}\mathbb{E}_{\nu^\omega}(\mathbf{1}_{\phi_{x_0}^{-1}(b_nJ_i)}\circ T_\omega^{j-1})\\&=\sum_{i=1}^{k}\sum_{ns_i<j\le nt_i}\nu^{\sigma^{j-1}\omega}(\phi_{x_0}^{-1}(b_nJ_i))\\&=\sum_{i=1}^{k}\sum_{j=\lfloor ns_i\rfloor}^{\lfloor nt_i\rfloor-1}\nu^{\sigma^j\omega}(\phi_{x_0}^{-1}(b_nJ_i)).\end{aligned}$$

By Lemma 8.1, for $\mathbb{P}$-a.e. $\omega\in\Omega$, we have

$$\lim_{n\to\infty}\sum_{i=1}^{k}\sum_{j=\lfloor ns_i\rfloor}^{\lfloor nt_i\rfloor-1}\nu^{\sigma^j\omega}(\phi_{x_0}^{-1}(b_nJ_i))=(t_i-s_i)\Pi_\alpha(J_i),$$

which proves (b).

We next establish (a). We will use induction on the number of "time" intervals $(s_i,t_i]\subset(0,\infty]$. Let $R=(s_1,t_1]\times J_1$ where $J_1\in\mathcal{J}$. Define

$$A_n=\phi_{x_0}^{-1}(b_nJ_1).$$

Since

$$\begin{aligned}\{N_n^\omega(R)=0\}&=\{x:T_\omega^j(x)\notin A_n,ns_1<j+1\le nt_1\}\\&=\left\{1_{A_n^c}\circ T_\omega^{\lfloor ns_1\rfloor}\cdot 1_{A_n^c}\circ T_\omega^{\lfloor ns_1\rfloor+1}\cdot\ldots\cdot 1_{A_n^c}\circ T_\omega^{\lfloor nt_1\rfloor-1}\neq 0\right\}\\&=\left\{x:\left(\prod_{j=0}^{\lfloor nt_1\rfloor-1-\lfloor ns_1\rfloor}1_{A_n^c}\circ T_{\sigma^{\lfloor ns_1\rfloor}\omega}^j\right)\circ T_\omega^{\lfloor ns_1\rfloor}(x)\neq 0\right\},\end{aligned}$$

we have that,

$$\begin{aligned}\left|\nu^\omega(N_n^\omega(R)=0)-\nu^{\sigma^{\lfloor ns_1\rfloor}\omega}\left(R_{A_n}(\sigma^{\lfloor ns_1\rfloor}\omega)>\lfloor n(t_1-s_1)\rfloor\right)\right|&\\\le\nu^{\sigma^{\lfloor ns_1\rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns_1\rfloor}\omega)=0)=\nu^{\sigma^{\lfloor ns_1\rfloor}\omega}(A_n)\le Cm(A_n)\to 0,&\end{aligned}\tag{9.3}$$

because, due to rounding when taking integer parts, $\lfloor nt_1\rfloor-\lfloor ns_1\rfloor-1$ is either equal to $\lfloor n(t_1-s_1)\rfloor-1$ or to $\lfloor n(t_1-s_1)\rfloor$. By Theorem 6.3,

$$\nu^{\sigma^{\lfloor ns_1\rfloor}\omega}(R_{A_n}(\sigma^{\lfloor ns_1\rfloor}\omega)>\lfloor n(t_1-s_1)\rfloor)\to e^{-(t_1-s_1)\Pi_\alpha(J)}$$

as desired.

Now let $R=\cup_{j=1}^k(s_i,t_i]\times J_i$ with $0\le s_1<t_1<\ldots<s_k<t_k$ and $J_i\in\mathcal{J}$. Furthermore, define $s_i'=s_i-s_1$ and $t_i'=t_i-s_1$.

Observe that, accounting for the rounding errors when taking integer parts as for (9.3), we get

$$\left|\nu^{\omega}\left(N_n^{\omega}\left(\bigcup_{i=1}^{k}(s_i,t_i]\times J_i\right)=0\right)-\nu^{\sigma^{\lfloor ns_1\rfloor}\omega}\left(N_n^{\sigma^{\lfloor ns_1\rfloor}\omega}\left(\bigcup_{i=1}^{k}(s_i',t_i']\times J_i\right)=0\right)\right| \le 2C\sum_{i=1}^{k} m(\phi_{x_0}^{-1}(b_nJ_i))\to 0 \tag{9.4}$$

so, after replacing ω by $\sigma^{\lfloor ns_1\rfloor}\omega$, we can assume that $s_1=0$. Let

$$R_1=(0,t_1]\times J_1$$
$$R_2=\bigcup_{i=2}^{k}(s_i,t_i]\times J_i$$
$$R_2'=\bigcup_{i=2}^{k}(s_i-s_2,t_i-s_2]\times J_i$$

Then, with $A_n=\phi_{x_0}^{-1}(b_nJ_1)$,

$$\left|\nu^{\eta}\left(N_n^{\eta}\left(R_1\cup R_2\right)=0\right)-\nu^{\eta}\left[\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}\cap T_{\eta}^{-\lfloor ns_2\rfloor}\left(N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0\right)\right]\right|\to 0 \tag{9.5}$$

as $n\to\infty$, uniformly in $\eta\in\Omega$, as in (9.4). Moreover, as we check below,

$$\left|\nu^{\eta}\left[\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}\cap T_{\eta}^{-\lfloor ns_2\rfloor}\left(N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0\right)\right] -\nu^{\eta}(R_{A_n}(\eta)>\lfloor nt_1\rfloor)\cdot\nu^{\eta}(N_n^{\eta}(R_2)=0)\right|\to 0 \tag{9.6}$$

as $n\to\infty$, uniformly in $\eta\in\Omega$. Therefore, setting $\eta=\sigma^{\lfloor ns_2\rfloor}\omega$ in (9.5) and (9.6), we have, by Theorem 6.3,

$$\lim_{n\to\infty}\left|\nu^{\sigma^{\lfloor ns_2\rfloor}\omega}(N_n^{\sigma^{\lfloor ns_2\rfloor}\omega}(R_1\cup R_2)=0)-e^{-t_1\Pi_\alpha(J_1)}\nu^{\sigma^{\lfloor ns_2\rfloor}\omega}(N_n^{\sigma^{\lfloor ns_2\rfloor}\omega}(R_2)=0)\right|=0$$

which gives the induction step in the proof of (a).

We prove now (9.6). Our proof uses the spectral gap for P_ω^n and breaks down for random intermittent maps.

Similarly to (9.4),

$$\left|\nu^{\eta}(N_n^{\eta}(R_2)=0)-\nu^{\eta}(T_{\eta}^{-\lfloor ns_2\rfloor}(N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0))\right|\to 0 \text{ as } n\to\infty, \text{ uniformly in } \eta.$$

We have, using the notation

$$U=\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}, \qquad V=\left\{N_n^{\sigma^{\lfloor ns_2\rfloor}\eta}(R_2')=0\right\},$$

that

$$
\begin{aligned}
\Big|\nu^\eta\left(U\cap T_\eta^{-\lfloor ns_2\rfloor}(V)\right) - \nu^\eta(U)\nu^\eta\left(T_\eta^{-\lfloor ns_2\rfloor}(V)\right)\Big| \\
&= \left|\int P_\eta^{\lfloor ns_2\rfloor}\left((\mathbf{1}_U-\nu^\eta(U))h_\eta\right)\mathbf{1}_V\,dm\right| \\
&\le C\left\|P_\eta^{\lfloor ns_2\rfloor}\left((\mathbf{1}_U-\nu^\eta(U))h_\eta\right)\right\|_{BV} \\
&= \left\|P_{\sigma^{\lfloor nt_1\rfloor}\eta}^{\lfloor ns_2\rfloor-\lfloor nt_1\rfloor}P_\eta^{\lfloor nt_1\rfloor}\left((\mathbf{1}_U-\nu^\eta(U))h_\eta\right)\right\|_{\mathrm{BV}} \\
&\le C\theta^{\lfloor ns_2\rfloor-\lfloor nt_1\rfloor}\left\|P_\eta^{\lfloor nt_1\rfloor}\left((\mathbf{1}_U-\nu^\eta(U))h_\eta\right)\right\|_{\mathrm{BV}}
\end{aligned}
$$

where the last inequality follows from the decay, uniform in η, of $\{P_\eta^k\}_k$ in BV (condition **(Dec)**).
But

$$
\sup_\eta\sup_n\left\|P_\eta^{\lfloor nt_1\rfloor}\left((\mathbf{1}_{\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}}-\nu^\eta(R_{A_n}(\eta)>\lfloor nt_1\rfloor))h_\eta\right)\right\|_{\mathrm{BV}}<\infty, \tag{9.7}
$$

which proves (9.6). This follows from Lemma 9.1 below applied to $f=h_\eta$ and $f_j=\mathbf{1}_{A_n^c}$, because

$$
\mathbf{1}_{\{R_{A_n}(\eta)>\lfloor nt_1\rfloor\}}=\prod_{j=1}^{\lfloor nt_1\rfloor}\mathbf{1}_{A_n^c}\circ T_\eta^j,
$$

and both $\|h_\eta\|_{\mathrm{BV}}$ and $\|\mathbf{1}_{A_n^c}\|_{\mathrm{BV}}$ are uniformly bounded. Note that for the stationary case the estimate (9.7) is used in the proof of [TK10b, Theorem 4.4], which refers to [ADSZ04, Proposition 4]. ☐

9.2. **Intermittent maps.** We prove a weaker form of convergence in the setting of Example 5.12, which suffices to establish stable limit laws but not functional limit laws.

In the setting of Example 5.12, we will show that for $\mathbb{P}$-a.e. ω,

$$
N_n^\omega((0,1]\times\cdot)\xrightarrow{d}N_{(\alpha)}((0,1]\times\cdot)
$$

Proof of Theorem 6.6. We will show that for $\mathbb{P}$-a.e. $\omega\in\Omega$, the assumptions of Kallenberg's theorem [Kal76, Theorem 4.7] hold.

Recall that $\mathcal{J}$ denotes the set of all finite unions of intervals of the form $(x,y]$ where $x<y$ and $0\notin[x,y]$.

By Kallenberg's theorem [Kal76, Theorem 4.7], $N_n^\omega[(0,1]\times\cdot)\to^d N_{(\alpha)}((0,1]\times\cdot)$ if for all $J\in\mathcal{J}$,

$$
(a)\lim_{n\to\infty}\nu^\omega(N_n^\omega((0,1]\times J)=0)=\mathbb{P}(N_{(\alpha)}((0,1]\times J)=0)
$$

and

$$
(b)\lim_{n\to\infty}\mathbb{E}_{\nu^\omega}N_n^\omega((0,1]\times J)=\mathbb{E}[N_{(\alpha)}((0,1]\times J)]
$$

We prove first (b) following [TK10b, page 12]. Write

$$
J=\bigcup_{i=1}^k J_i
$$

with $J_i=(x_i,y_i]$ disjoint.

Then

$$\mathbb{E}N_{(\alpha)}((0,1]\times J)=\sum_{i=1}^{k}\Pi_\alpha(J_i)=\Pi_\alpha(J)$$

and

$$\mathbb{E}_{\nu^\omega}N_n^\omega((0,1]\times J)=\sum_{i=1}^{k}\sum_{j=1}^{n}\mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_nJ_i))}\circ T_\omega^{j-1}]=\sum_{j=1}^{n}\mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_nJ))}\circ T_\omega^{j-1}]$$

We check that

$$\lim_{n\to\infty}\sum_{j=1}^{n}\mathbb{E}_{\nu^\omega}\left(\mathbf{1}_{\{\phi_{x_0}^{-1}(b_nJ)\}}\circ T_\omega^j\right)=\Pi_\alpha(J)$$

for $J=\cup_{i=1}^k J_i$.

Write $A_n:=\phi_{x_0}^{-1}(b_nJ)$. Then

$$\mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_nJ))}\circ T_\omega^j]=\nu^{\sigma^j\omega}(A_n)$$

hence

$$\lim_{n\to\infty}\sum_{j=1}^{n}\mathbb{E}_{\nu^\omega}[\mathbf{1}_{(\phi_{x_0}^{-1}(b_nJ_i))}\circ T_\omega^j(x)]=\Pi_\alpha(J)$$

by Lemma 7.2.

Now we prove (a), i.e.

$$\lim_{n\to\infty}\nu^\omega(N_n^\omega((0,1]\times J)=0)=P(N_{(\alpha)}((0,1]\times J)=0)$$

for all $J\in\mathcal{J}$.

Let $J\in\mathcal{J}$ and denote as above $A_n:=\phi_{x_0}^{-1}(b_nJ)\subset X=[0,1]$. Then

$$\{N_n^\omega((0,1]\times J)=0\}=\{x:T_\omega^j(x)\notin A_n, 0<j+1\le n\}=\{R_{A_n}(\omega)>n-1\}\cap A_n^c$$

Hence

$$|\nu^\omega(N_n^\omega((0,1]\times J)=0)-\nu^\omega(R_{A_n}(\omega)>n)|\le Cm(A_n)\to 0$$

and by Theorem 6.4, for m-a.e. x_0

$$\nu^\omega(R_{A_n}(\omega)>n)\to e^{-\Pi_\alpha(J)}.$$

This proves (a). □

10. Stable laws and functional limit laws

10.1. **Uniformly expanding maps.** In this section, we prove Theorem 6.7, under the conditions given in Section 5.5, in particular **(LY)**, **(Dec)** and **(Min)**.

For this purpose, we consider first some technical lemmas regarding short returns. For $\omega\in\Omega$, $n\ge 1$ and $\varepsilon>0$, let

$$\mathcal{E}_n^\omega(\varepsilon)=\{x\in[0,1]\,:\,|T_\omega^n(x)-x|\le\varepsilon\}.$$

Lemma 10.1. *There exists $C>0$ such that for all $\omega\in\Omega$, $n\ge 1$ and $\varepsilon>0$,*

$$m(\mathcal{E}_n^\omega(\varepsilon))\le C\varepsilon.$$

Proof. We follow the proof of [HNT12, Lemma 3.4], conveniently adapted to our setting of random non-Markov maps. Recall that $\mathcal{A}_\omega^n$ is the partition of monotonicity associated to the map T_ω^n. Consider $I \in \mathcal{A}_\omega^n$. Since $\inf_I |(T_\omega^n)'| \geq \lambda^n > 1$, there exists at most one solution $x_I^\pm \in I$ to the equation

$$T_\omega^n(x_I^\pm) = x_I^\pm \pm \varepsilon, \tag{10.1}$$

and since there is no sign change of $(T_\omega^n)'$ on I, we have

$$\mathcal{E}_n^\omega(\varepsilon) \cap I \subset [x_I^-, x_I^+]. \tag{10.2}$$

We have

$$T_\omega^n(x_I^+) - T_\omega^n(x_I^-) = x_I^+ - x_I^- + 2\varepsilon,$$

and by the mean value theorem,

$$\left|T_\omega^n(x_I^+) - T_\omega^n(x_I^-)\right| = \left|(T_\omega^n)'(c)\right| \left|x_I^+ - x_I^-\right|, \text{ for some } c \in I.$$

Consequently,

$$\left|x_I^+ - x_I^-\right| \leq \left(\sup_I \frac{1}{|(T_\omega^n)'|}\right) \left[\left|x_I^+ - x_I^-\right| + 2\varepsilon\right] \leq \lambda^{-n} \left|x_I^+ - x_I^-\right| + 2\varepsilon \sup_I \frac{1}{|(T_\omega^n)'|}. \tag{10.3}$$

Note that if there is no solutions to (10.1), then the estimate (10.3) is actually improved. Rearranging (10.3) and summing over $I \in \mathcal{A}_\omega^n$, we obtain thanks to (10.2)

$$m(\mathcal{E}_n^\omega(\varepsilon)) \leq \sum_{I \in \mathcal{A}_\omega^n} \left|x_I^+ - x_I^-\right| \leq \frac{2\varepsilon}{1 - \lambda^{-n}} \sum_{I \in \mathcal{A}_\omega^n} \sup_I \frac{1}{|(T_\omega^n)'|} \leq C\varepsilon.$$

The fact that

$$\sum_{I \in \mathcal{A}_\omega^n} \sup_I \frac{1}{|(T_\omega^n)'|} \leq C \tag{10.4}$$

for a constant $C > 0$ independent from ω and n follows from a standard distortion argument for one-dimensional maps that can be found in the proof of part 3 of [ANV15, Lemma 8.5] (see also [AR16, Lemma 7]), where finitely many piecewise C^2 uniformly expanding maps with finitely many discontinuities are also considered. Since it follows from **(LY)** that $\|P_\omega^n f\|_{\mathrm{BV}} \leq C\|f\|_{\mathrm{BV}}$ for some uniform $C > 0$, we do not have to average (10.4) over ω as in [ANV15], but instead we can simply have an estimate that holds uniformly in ω. □

Recall that, for a measurable subset U, $R_U^\omega(x) \geq 1$ is the hitting time of (ω, x) to U defined by (6.1).

Lemma 10.2. *Let $a > 0$, $\frac{2}{3} < \psi < 1$ and $0 < \kappa < 3\psi - 2$. Then there exist sequences $(\gamma_1(n))_{n\geq 1}$ and $(\gamma_2(n))_{n\geq 1}$ with $\gamma_1(n) = \mathcal{O}(n^{-\kappa})$ and $\gamma_2(n) = o(1)$, and for all $\omega \in \Omega$, a sequence of measurable subsets $(A_n^\omega)_{n\geq 1}$ of $[0,1]$ with $m(A_n^\omega) \leq \gamma_1(n)$ and such that for all $x_0 \notin A_n^\omega$,*

$$(\log n) \sum_{i=0}^{n-1} m\left(B_{n^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \leq \lfloor a \log n \rfloor\right\}\right) \leq \gamma_2(n).$$

Proof. Let

$$E_n^\omega = \left\{ x \in [0,1] \, : \, |T_\omega^j(x) - x| \leq 2n^{-\psi} \text{ for some } 0 < j \leq \lfloor a \log n \rfloor \right\}.$$

Since $B_{n^{-\psi}}(x_0) \cap \left\{ R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \leq \lfloor a \log n \rfloor \right\} \subset B_{n^{-\psi}}(x_0) \cap E_n^{\sigma^i\omega}$, it is enough to consider

$$(\log n) \sum_{i=0}^{n-1} m\left(B_{n^{-\psi}}(x_0) \cap E_n^{\sigma^i\omega} \right).$$

According to Lemma 10.1, we have

$$m(E_n^\omega) \leq \sum_{j=1}^{\lfloor a \log n \rfloor} m\left(\mathcal{E}_j^\omega(2n^{-\psi}) \right) \leq C \frac{\log n}{n^\psi}.$$

We introduce the maximal function

$$M_n^\omega(x_0) = \sup_{t>0} \frac{1}{2t} \int_{x_0-t}^{x_0+t} \left(\sum_{i=0}^{n-1} \mathbf{1}_{E_n^{\sigma^i\omega}}(z) \right) dz = \sup_{t>0} \frac{1}{2t} \sum_{i=0}^{n-1} m\left(B_t(x_0) \cap E_n^{\sigma^i\omega} \right)$$

By [Rud87, Equation (5) page 138], for all $\lambda > 0$, we have

$$m(M_n^\omega > \lambda) \leq \frac{C}{\lambda} \left\| \sum_{i=0}^{n-1} \mathbf{1}_{E_n^{\sigma^i\omega}} \right\|_{L^1_m} \leq \frac{C}{\lambda} \sum_{i=0}^{n-1} m(E_n^{\sigma^i\omega}) \leq \frac{C}{\lambda} \frac{\log n}{n^{\psi-1}} \tag{10.5}$$

Let $\rho > 0$ and $\xi > 0$ to be determined later. We define

$$F_n^\omega = \left\{ x_0 \in [0,1] \, : \, m\left(B_{n^{-\psi}}(x_0) \cap E_n^\omega \right) \geq 2n^{-\psi(1+\rho)} \right\},$$

so that we have

$$\sum_{i=0}^{n-1} m\left(B_{n^{-\psi}}(x_0) \cap E_n^{\sigma^i\omega} \right) \geq \left(\sum_{i=0}^{n-1} \mathbf{1}_{F_n^{\sigma^i\omega}}(x_0) \right) 2n^{-\psi(1+\rho)}.$$

By definition of the maximal function M_n^ω, this implies that

$$M_n^\omega(x_0) \geq n^{-\psi\rho} \left(\sum_{i=0}^{n-1} \mathbf{1}_{F_n^{\sigma^i\omega}}(x_0) \right),$$

from which it follows, by (10.5) with $\lambda = (\log n) n^{\xi - \psi\rho}$,

$$m\left(A_n^\omega \right) \leq m\left(M_n^\omega > (\log n) n^{\xi-\psi\rho} \right) \leq C n^{-(\xi + (1-\rho)\psi - 1)} =: \gamma_1(n),$$

where

$$A_n^\omega = \left\{ \left(\sum_{i=0}^{n-1} \mathbf{1}_{F_n^{\sigma^i\omega}} \right) > (\log n) n^\xi \right\}.$$

If $x_0 \notin A_n^\omega$, then

$$\begin{aligned}(\log n) \sum_{i=0}^{n-1} m\left(B_{n^{-\psi}}(x_0) \cap E_n^{\sigma^i\omega} \right) &\leq (\log n) \left(\sum_{i=0}^{n-1} \mathbf{1}_{F_n^{\sigma^i\omega}}(x_0) \right) m(B_{n^{-\psi}}(x_0)) + 2(\log n) n^{1-\psi(1+\rho)} \\ &\leq C(\log n) \left((\log n) n^{-(\psi-\xi)} + n^{-(\psi(1+\rho)-1)} \right) =: \gamma_2(n).\end{aligned}$$

Since $\frac{2}{3} < \psi < 1$ and $0 < \kappa < 3\psi - 2$, it is possible to choose $\rho > 0$ and $\xi > 0$ such that $\kappa = \xi + (1-\rho)\psi - 1$, $\psi > \xi$ and $\psi(1+\rho) > 1$ [3], which concludes the proof. ☐

Lemma 10.3. *Suppose that $a > 0$ and $\frac{3}{4} < \psi < 1$. Then for m-a.e. $x_0 \in [0,1]$ and $\mathbb{P}$-a.e. $\omega \in \Omega$ and , we have*

$$\lim_{n\to\infty} (\log n) \sum_{i=0}^{n-1} m\left(B_{n^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\right\}\right) = 0.$$

Proof. Let $0 < \kappa < 3\psi - 2$ to be determined later. Consider the sets $(A_n^\omega)_{n\ge1}$ given by Lemma 10.2, with $m(A_n^\omega) \le \gamma_1(n) = \mathcal{O}(n^{-\kappa})$. Since $\kappa < 1$, we need to consider a subsequence $(n_k)_{k\ge1}$ such that $\sum_{k\ge1} \gamma_1(n_k) < \infty$. For such a subsequence, by the Borel-Cantelli lemma, for m-a.e. x_0, there exists $K = K(x_0,\omega)$ such that for all $k \ge K$, $x_0 \notin A^\omega_{n_k}$. Since $\lim_{k\to\infty} \gamma_2(n_k) = 0$, this implies

$$\lim_{k\to\infty} (\log n_k) \sum_{i=0}^{n_k-1} m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \le \lfloor a \log n_k \rfloor\right\}\right) = 0.$$

We take $n_k = \lfloor k^\zeta \rfloor$, for some $\zeta > 0$ to be determined later. In order to have $\sum_{k\ge1} \gamma_1(n_k) < \infty$, we need to require that $\kappa\zeta > 1$. Set $U_n^\omega(x_0) = B_{n^{-\psi}}(x_0) \cap \left\{R^\omega_{B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\right\}$. To obtain the convergence to 0 of the whole sequence, we need to prove that

$$\lim_{k\to\infty} \sup_{n_k \le n < n_{k+1}} \left|(\log n) \sum_{i=0}^{n-1} m(U_n^{\sigma^i\omega}(x_0)) - (\log n_k) \sum_{i=0}^{n_k-1} m(U_{n_k}^{\sigma^i\omega}(x_0))\right| = 0. \tag{10.6}$$

For this purpose, we estimate

$$\left|(\log n) \sum_{i=0}^{n-1} m(U_n^{\sigma^i\omega}(x_0)) - (\log n_k) \sum_{i=0}^{n_k-1} m(U_{n_k}^{\sigma^i\omega}(x_0))\right| \le \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)}.$$

where

$$\text{(I)} = |\log n - \log n_k| \sum_{i=0}^{n-1} m(U_n^{\sigma^i\omega}(x_0)), \;\; \text{(II)} = (\log n_k) \sum_{i=n_k}^{n-1} m(U_n^{\sigma^i\omega}(x_0)),$$

$$\text{(III)} = (\log n_k) \sum_{i=0}^{n_k-1} \left| m\left(B_{n^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\right\}\right) - m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\right\}\right)\right|,$$

$$\text{(IV)} = (\log n_k) \sum_{i=0}^{n_k-1} \left| m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\right\}\right) - m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\right\}\right)\right|,$$

$$\text{(V)} = (\log n_k) \sum_{i=0}^{n_k-1} \left| m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor\right\}\right) - m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \le \lfloor a \log n_k \rfloor\right\}\right)\right|.$$

Before proceeding to estimate each term, we note that $|n_{k+1} - n_k| = \mathcal{O}(k^{-(1-\zeta)})$, $|n_{k+1}^{-\psi} - n_k^{-\psi}| = \mathcal{O}(k^{-(1+\zeta\psi)})$, $|\log n_{k+1} - \log n_k| = \mathcal{O}(k^{-1})$ and $m(U_n^\omega(x_0)) \le m(B_{n^{-\psi}}(x_0)) = \mathcal{O}(k^{-\zeta\psi})$.

[3] For instance, take $\xi = \psi - \delta$ and $\rho = \psi^{-1} - 1 + \delta\psi^{-1}$ with $\delta = \frac{3\psi - 2 - \kappa}{2}$.

From these observations, it follows

$$(\mathrm{I}) \le C\left|\log n_{k+1} - \log n_k\right| n_{k+1} k^{-\zeta\psi} \le C k^{-(1-(1-\psi)\zeta)},$$

$$(\mathrm{II}) \le C(\log n_k)|n_{k+1} - n_k| k^{-\zeta\psi} \le C(\log k) k^{-(1-(1-\psi)\zeta)},$$

$$(\mathrm{III}) \le C(\log n_k) n_k m(B_{n_k^{-\psi}}(x_0) \setminus B_{n^{-\psi}}(x_0)) \le C(\log n_k) n_k |n_{k+1}^{-\psi} - n_k^{-\psi}| \le C(\log k) k^{-(1-(1-\psi)\zeta)},$$

$$\begin{aligned}(\mathrm{IV}) &\le C(\log n_k) \sum_{i=0}^{n_k-1} m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{ R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)\setminus B_{n^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor \right\}\right) \\ &\le C(\log n_k) \sum_{i=0}^{n_k-1} a(\log n) m\left(B_{n_k^{-\psi}}(x_0) \setminus B_{n^{-\psi}}(x_0)\right) \\ &\le C(\log k)^2 k^{-(1-(1-\psi)\zeta)}\end{aligned}$$

and

$$\begin{aligned}(\mathrm{V}) &\le C(\log n_k) \sum_{i=0}^{n_k-1} m\left(B_{n_k^{-\psi}}(x_0) \cap \left\{ \lfloor a \log n_k \rfloor < R^{\sigma^i\omega}_{B_{n_k^{-\psi}}(x_0)} \le \lfloor a \log n \rfloor \right\}\right) \\ &\le C(\log n_k) \sum_{i=0}^{n_k-1} a\left|\log n_{k+1} - \log n_k\right| m(B_{n_k^{-\psi}}(x_0)) \\ &\le C(\log k) k^{-(1-(1-\psi)\zeta)}.\end{aligned}$$

To obtain (10.6), it is thus sufficient to choose $\kappa > 0$ and $\zeta > 0$ such that $\kappa < 3\psi - 2$, $\kappa\zeta > 1$ and $(1-\psi)\zeta < 1$, which is possible if $\psi > \frac{3}{4}$. □

We can now prove the functional convergence to a Lévy stable process for i.i.d. uniformly expanding maps.

Proof of Theorem 6.7. We apply Theorem 7.3. By Theorem 6.5, we have $N_n^\omega \xrightarrow{d} N_{(\alpha)}$ under the probability ν^ω for $\mathbb{P}$-a.e. $\omega \in \Omega$. It thus remains to check that equation (7.2) holds for m-a.e. x_0 when $\alpha \in [1,2)$ to complete the proof. For this purpose, we will use a reverse martingale argument from [NTV18] (see also [AR16, Proposition 13]). Because of (5.10), it is enough to work on the probability space $([0,1], \nu^\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Let $\mathcal{B}$ denote the σ-algebra of Borel sets on $[0,1]$ and

$$\mathcal{B}_{\omega,k} = (T_\omega^k)^{-1}(\mathcal{B})$$

To simplify notation a bit let

$$f_{\omega,j,n}(x) = \phi_{x_0}(x) \mathbf{1}_{\left\{|\phi_{x_0}| \le \varepsilon b_n\right\}}(x) - \mathbb{E}_{\nu^{\sigma^j\omega}}(\phi_{x_0} \mathbf{1}_{\left\{|\phi_{x_0}| \le \varepsilon b_n\right\}}).$$

From (5.10), it follows that $\mathbb{E}_m(|f_{\omega,j,n}|) \le C\varepsilon b_n$, and from the explicit definition of ϕ, we can estimate the total variation of $f_{\omega,j,n}$ and obtain the existence of $C > 0$, independent of ω, ε, n and j, such that

$$\|f_{\omega,j,n}\|_{\mathrm{BV}} \le C\varepsilon b_n. \tag{10.7}$$

We define

$$S_{\omega,k,n} := \sum_{j=0}^{k-1} f_{\omega,j,n} \circ T_\omega^j$$

and

$$H_{\omega,k,n} \circ T_\omega^n := \mathbb{E}_{\nu^\omega}(S_{\omega,k,n} | \mathcal{B}_{\omega,k}) \tag{10.8}$$

Hence $H_{\omega,1,n} = 0$ and an explicit formula for $H_{\omega,k,n}$ is

$$H_{\omega,k,n} = \frac{1}{h_{\sigma^k\omega}} \sum_{j=0}^{k-1} P_{\sigma^j\omega}^{k-j}(f_{\omega,j,n} h_{\sigma^j\omega}).$$

From the explicit formula, the exponential decay in the BV norm of $P_{\sigma^j\omega}^{n-j}$ from **(Dec)**, (5.10) and (10.7), we see that $\|H_{\omega,k,n}\|_{\mathrm{BV}} \leq C\varepsilon b_n$, where the constant C may be taken as constant over $\omega \in \Omega$. If we define

$$M_{\omega,k,n} = S_{\omega,k,n} - H_{\omega,k,n} \circ T_\omega^k$$

then the sequence $\{M_{\omega,k,n}\}_{k\geq 1}$ is a reverse martingale difference for the decreasing filtration $\mathcal{B}_{\omega,k} = (T_\omega^n)^{-1}(\mathcal{B})$ as

$$\mathbb{E}_{\nu^\omega}(M_{\omega,k,n} | \mathcal{B}_{\omega,k}) = 0$$

The martingale reverse differences are

$$M_{\omega,k+1,n} - M_{\omega,k,n} = \psi_{\omega,k,n} \circ T_\omega^k$$

where

$$\psi_{\omega,k,n} := f_{\omega,k,n} + H_{\omega,k,n} - H_{\omega,k+1,n} \circ T_{\sigma^{k+1}\omega}.$$

We see from the L^∞ bounds on $\|H_{\omega,k,n}\|_\infty \leq C b_n \varepsilon$ and the telescoping sum that

$$\left| \sum_{j=0}^{k-1} \psi_{\omega,j,n} \circ T_\omega^j - \sum_{j=0}^{k-1} f_{\omega,j,n} \circ T_\omega^j \right| \leq C\varepsilon b_n. \tag{10.9}$$

By Doob's martingale maximal inequality

$$\nu^\omega \left\{ \max_{1\leq k\leq n} \left| \sum_{j=0}^{k-1} \psi_{\omega,j,n} \circ T_\omega^j \right| \geq b_n \delta \right\} \leq \frac{1}{b_n^2 \delta^2} \mathbb{E}_{\nu^\omega} \left| \sum_{j=0}^{n-1} \psi_{\omega,j,n} \circ T_\omega^j \right|^2.$$

Note that

$$\sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}\left[\psi_{\omega,j,n}^2 \circ T_\omega^j\right] = \mathbb{E}_{\nu^\omega}\left[\sum_{j=0}^{n-1} \psi_{\omega,j,n} \circ T_\omega^j\right]^2$$

by pairwise orthogonality of martingale reverse differences.

As in [HNTV17, Lemma 6]

$$\mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2] = \sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}[\psi_{\omega,j,n}^2 \circ T_\omega^j] + \mathbb{E}_{\nu^\omega}[H_{\omega,1,n}^2] - \mathbb{E}_{\nu^\omega}[H_{\omega,n,n}^2 \circ T_\omega^n].$$

So we see that

$$\nu^\omega \left\{ \max_{1\le k\le n} \left| \sum_{j=0}^{k-1} \psi_{\omega,j,n} \circ T_\omega^j \right| \ge b_n\delta \right\} \le \frac{1}{b_n^2\delta^2} \mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2] + 2\frac{C^2\varepsilon^2}{\delta^2} \tag{10.10}$$

where we have used $\|H^2_{\omega,j,n}\|_\infty \le C^2 b_n^2 \varepsilon^2$.

Now we estimate

$$\mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2] \le \sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}[f^2_{\omega,j,n} \circ T^j_\omega] + 2\sum_{i=0}^{n-1}\sum_{i<j} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega]. \tag{10.11}$$

Using the equivariance of the measures $\{\nu^\omega\}_{\omega\in\Omega}$ and (5.10), we have

$$\sum_{j=0}^{n-1} \mathbb{E}_{\nu^\omega}[f^2_{\omega,j,n}\circ T^j_\omega] \le Cn\mathbb{E}_\nu(\phi^2_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le \varepsilon b_n\}}) \sim C\varepsilon^{2-\alpha} b_n^2, \tag{10.12}$$

by Proposition 2.6 and (2.3).

On the other hand, we are going to show that for m-a.e. x_0

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{b_n^2} \sum_{i=0}^{n-1}\sum_{i<j} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] = 0. \tag{10.13}$$

The first observation is that, due to condition **(Dec)**,

$$\mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] \le C\theta^{j-i}\|f_{\omega,i,n}\|_{\mathrm{BV}}\|f_{\omega,j,n}\|_{L^1_m} \le C\varepsilon^2 b_n^2 \theta^{j-i}$$

where $\theta < 1$. Hence there exists $a > 0$ independently of n and ε such that

$$\sum_{j-i>\lfloor a\log n\rfloor} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] \le C\varepsilon^2 n^{-2} b_n^2$$

and it is enough to prove that for $\varepsilon > 0$,

$$\sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor} \mathbb{E}_{\nu^\omega}[f_{\omega,j,n}\circ T^j_\omega \cdot f_{\omega,i,n}\circ T^i_\omega] = o(b_n^2) = o(n^{\frac{2}{\alpha}}).$$

By construction, the term $\mathbb{E}_{\nu^\omega}[f_{\omega,i,n}\circ T^i_\omega \cdot f_{\omega,j,n}\circ T^j_\omega]$ is a covariance, and since ϕ is positive, we can bound this quantity by $\mathbb{E}_{\nu^\omega}[f_n\circ T^i_\omega \cdot f_n\circ T^j_\omega] = \mathbb{E}_{\nu^{\sigma^i\omega}}[f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}]$ where $f_n = \phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}}$. Then, since the densities are uniformly bounded by (5.10), we are left to estimate

$$\sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor} \mathbb{E}_m[f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}]. \tag{10.14}$$

Let $\frac{3}{4} < \psi < 1$ and $U_n = B_{n^{-\psi}}(x_0)$. We bound (10.14) by (I) + (II) + (III), where

$$\text{(I)} = \sum_{i=0}^{n-1}\sum_{j=i+1}^{i+\lfloor a\log n\rfloor} \int_{U_n\cap(T^{j-i}_{\sigma^i\omega})^{-1}(U_n)} f_n\cdot f_n\circ T^{j-i}_{\sigma^i\omega}\,dm,$$

$$\text{(II)} = \sum_{i=0}^{n-1} \sum_{j=i+1}^{i+\lfloor a \log n \rfloor} \int_{U_n \cap (T_{\sigma^i\omega}^{j-i})^{-1}(U_n^c)} f_n \cdot f_n \circ T_{\sigma^i\omega}^{j-i} dm$$

and

$$\text{(III)} = \sum_{i=0}^{n-1} \sum_{j=i+1}^{i+\lfloor a \log n \rfloor} \int_{U_n^c} f_n \cdot f_n \circ T_{\sigma^i\omega}^{j-i} dm.$$

Since $\|f_n\|_\infty \le \varepsilon b_n$, it follows that

$$\begin{aligned}\text{(I)} &\le \varepsilon^2 b_n^2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{i+\lfloor a \log n \rfloor} m\left(U_n \cap (T_{\sigma^i\omega}^{j-i})^{-1}(U_n)\right) \\ &\le a\varepsilon^2 b_n^2 (\log n) \sum_{i=0}^{n-1} m\left(U_n \cap \left\{R_{U_n}^{\sigma^i\omega} \le a \log n\right\}\right),\end{aligned}$$

which by Lemma 10.3 is a $o(b_n^2)$ as $n \to \infty$ for m-a.e. x_0.

To estimate (II) and (III), we will use Hölder's inequality. We first observe by a direct computation that

$$\int_{U_n^c} \phi_{x_0}^2 dm = \mathcal{O}(n^{\psi(\frac{2}{\alpha}-1)}). \tag{10.15}$$

We consider (III) first. Let $A = U_n^c$. We have

$$\int_{U_n^c} f_n \cdot f_n \circ T_{\sigma^i\omega}^{j-i} dm \le \int_A \phi_{x_0} \cdot f_n \circ T_{\sigma^i\omega}^{j-i} dm \le \left(\int_A \phi_{x_0}^2 dm\right)^{\frac{1}{2}} \left(\int f_n^2 \circ T_{\sigma^i\omega}^{j-i} dm\right)^{\frac{1}{2}} \tag{10.16}$$

$$\le C\left(\int_A \phi_{x_0}^2 dm\right)^{\frac{1}{2}} \left(\int f_n^2 dm\right)^{\frac{1}{2}}. \tag{10.17}$$

By (10.15), $\left(\int_A \phi_{x_0}^2 dm\right)^{\frac{1}{2}} \le Cn^{\frac{\psi}{2}(\frac{2}{\alpha}-1)}$ and by Proposition 2.6, $\left(\int f_n^2 dm\right)^{\frac{1}{2}} \le Cn^{\frac{1}{\alpha}-\frac{1}{2}}$. Hence we may bound (10.16) by $Cn^{(1+\psi)(\frac{1}{\alpha}-\frac{1}{2})}$.

To bound (II), let $B = U_n \cap (T_{\sigma^i\omega}^{j-i})^{-1}(U_n^c)$. Then,

$$\int_{U_n \cap (T_{\sigma^i\omega}^{j-i})^{-1}(U_n^c)} f_n \cdot f_n \circ T_{\sigma^i\omega}^{j-i} dm \le \int_B f_n \cdot \phi_{x_0} \circ T_{\sigma^i\omega}^{j-i} dm \le \left(\int f_n^2 dm\right)^{\frac{1}{2}} \left(\int_B \phi_{x_0}^2 \circ T_{\sigma^i\omega}^{j-i} dm\right)^{\frac{1}{2}}. \tag{10.18}$$

As before $\left(\int f_n^2 dm\right)^{\frac{1}{2}} \le Cn^{\frac{1}{\alpha}-\frac{1}{2}}$ and

$$\left(\int_B \phi_{x_0}^2 \circ T_{\sigma^i\omega}^{j-i} dm\right)^{\frac{1}{2}} \le \left(\int \phi_{x_0}^2 \circ T_{\sigma^i\omega}^{j-i} \mathbf{1}_{(T_{\sigma^i\omega}^{j-i})^{-1}(U_n^c)} dm\right)^{\frac{1}{2}} \le C\left(\int_{U_n^c} \phi_{x_0}^2 dm\right)^{\frac{1}{2}} \le Cn^{\frac{\psi}{2}(\frac{2}{\alpha}-1)}$$

by (10.15), and so (10.18) is bounded by $Cn^{(1+\psi)(\frac{1}{\alpha}-\frac{1}{2})}$.

It follows that (II) + (III) $\le C(\log n) n^{1+(1+\psi)(\frac{1}{\alpha}-\frac{1}{2})} = o(n^{\frac{2}{\alpha}})$, since $\psi < 1$. This proves that (10.14) is a $o(b_n^2)$ and concludes the proof of (10.13).

Finally, from (10.11), (10.12) and (10.13), we obtain

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{b_n^2}\mathbb{E}_{\nu^\omega}[(S_{\omega,n,n})^2]=0, \tag{10.19}$$

which gives the result by taking the limit first in n and then in ε in (10.10). □

10.2. **Intermittent maps.** We prove convergence to a stable law in the setting of Example 5.12 when $\alpha\in(0,1)$.

Proof of Theorem 6.9. We apply Proposition 5.8. By Theorem 6.6, it remains to prove (5.7), since $\alpha\in(0,1)$. We will need an estimate for $\mathbb{E}_{\nu^\omega}(|\phi_{x_0}|\mathbf{1}_{\{\phi_{x_0}\le\varepsilon b_n\}})$ which is independent of ω. For this purpose, we introduce the absolutely continuous probability measure $\nu_{\max}$ whose density is given by $h_{\max}(x)=\kappa x^{-\gamma_{\max}}$. Since all densities h_ω belong to the cone L, we have that $h_\omega\le\frac{a}{\kappa}h_{\max}$ for all ω. Thus,

$$\frac{1}{b_n}\sum_{j=0}^{n-1}\mathbb{E}_{\nu^{\sigma^j\omega}}(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}})\le\frac{n}{b_n}\frac{a}{\kappa}\mathbb{E}_{\nu_{\max}}(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}}).$$

We can easily verify that ϕ_{x_0} is regularly varying of index α with respect to $\nu_{\max}$, with scaling sequence equal to $(b_n)_{n\ge1}$ up to a multiplicative constant factor. Consequently, by Proposition 2.6, we have that, for some constant $c>0$,

$$\mathbb{E}_{\nu_{\max}}(\phi_{x_0}\mathbf{1}_{\{|\phi_{x_0}|\le\varepsilon b_n\}})\sim c\varepsilon^{1-\alpha}n^{\frac{1}{\alpha}-1},$$

which implies (5.7). □

11. The annealed case

In this section, we consider the annealed counterparts of our results. Even though the annealed versions do not seem to follow immediately from the quenched version, it is easy to obtain them from our proofs in the quenched case. We take $\phi_{x_0}(x)=d(x,x_0)^{-\frac{1}{\alpha}}$ as before we consider the convergence on the measure space $\Omega\times[0,1]$ with respect to $\nu_F(d\omega,dx)=\mathbb{P}(d\omega)\nu^\omega(dx)$. We give precise annealed results in the case of Theorems 6.7 and 6.9, where we consider

$$X_n^a(\omega,x)(t):=\frac{1}{b_n}\sum_{j=0}^{\lfloor nt\rfloor-1}\phi_{x_0}(T_\omega^j x)-tc_n,\ t\ge0,$$

viewed as a random process defined on the probability space $(\Omega\times[0,1],\nu_F)$.

Theorem 11.1. *Under the same assumptions as Theorem 6.7, the random process $X_n^a(t)$ converges in the J_1 topology to the Lévy α-stable process $X_{(\alpha)}(t)$ under the probability measure ν_F.*

Proof. We apply [TK10b, Theorem 1.2] to the skew-product system $(\Omega\times[0,1],F,\nu_F)$ and the observable ϕ_{x_0} naturally extended to $\Omega\times[0,1]$. Recall that ν_F is given by the disintegration $\nu_F(d\omega,dx)=\mathbb{P}(d\omega)\nu^\omega(dx)$.

We have to prove that

(a) $N_n\xrightarrow{d}N_{(\alpha)}$,

(b) if $\alpha \in [1,2)$, for all $\delta > 0$,

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\nu_F\left((\omega,x)\ :\ \max_{1\le k\le n}\left|\frac{1}{b_n}\sum_{j=0}^{k-1}\left[\phi_{x_0}(T_\omega^j x)\mathbf{1}_{\left\{|\phi_{x_0}\circ T_\omega^j|\le\varepsilon b_n\right\}}(x)-\mathbb{E}_\nu(\phi_{x_0}\mathbf{1}_{\left\{|\phi_{x_0}|\le\varepsilon b_n\right\}})\right]\right|\ge\delta\right)=0,$$

where

$$N_n(\omega,x)(B) := N_n^\omega(x)(B) = \#\left\{j\ge 1\ :\ \left(\frac{j}{n},\frac{\phi_{x_0}(T_\omega^{j-1}(x))}{b_n}\right)\in B\right\},\ n\ge 1.$$

To prove (a), we take $f \in C_K^+((0,\infty)\times(\mathbb{R}\setminus\{0\}))$ arbitrary. Then, by Theorem 6.5, we have for $\mathbb{P}$-a.e. ω

$$\lim_{n\to\infty}\mathbb{E}_{\nu^\omega}(e^{-N_n^\omega(f)}) = \mathbb{E}(e^{-N(f)}).$$

Integrating with respect to $\mathbb{P}$ and using the dominated convergence theorem yields

$$\lim_{n\to\infty}\mathbb{E}_{\nu_F}(e^{-N_n(f)}) = \mathbb{E}(e^{-N(f)}),$$

which proves (a).

To prove (b), we simply have to integrate with respect to $\mathbb{P}$ in the estimates in the proof of Theorem 6.7, which hold uniformly in $\omega\in\Omega$, and then to take the limits as $n\to\infty$ and $\varepsilon\to 0$. □

Similarly, we have:

Theorem 11.2. *Under the same assumptions as Theorem 6.9, $X_n^a(1)\xrightarrow{d}X_{(\alpha)}(1)$ under the probability measure ν_F.*

Proof. We can proceed as for Theorem 11.1 in order to check the assumptions of [TK10b, Theorem 1.3] for the skew-product system $(\Omega\times[0,1],F,\nu_F)$ and the observable ϕ_{x_0}. □

12. Appendix

The observation that our distributional limit theorems hold for any measures $\mu \ll \nu^\omega$ follows from Theorem 1, Corollary 1 and Corollary 3 of Zweimüller's work [Zwe07].

Let

$$S_n(x) = \frac{1}{b_n}[\sum_{j=0}^{n-1}\phi\circ T_\omega^j(x) - a_n].$$

and suppose

$$S_n \to_{\nu_\omega} Y$$

where Y is a Lévy random variable.

We consider first the setup of Example 5.12. We will show that for any measure ν with density h i.e. $d\nu = hdm$ in the cone L of Example 5.12, in particular Lebesgue measure m with $h=1$,

$$S_n \to_\nu Y$$

We focus on m. According to [Zwe07, Theorem 1] it is enough to show that

$$\int\psi(S_n)d\nu_\omega - \int\psi(S_n)dm \to 0.$$

for any $\psi:\mathbb{R}\to\mathbb{R}$ which is bounded and uniformly Lipschitz.

Fix such a ψ and consider

$$\int \psi(\frac{1}{b_n}[\sum_{j=0}^{n-1} \phi \circ T_\omega^j(x) - a_n])(h_\omega - 1)dm$$
$$\leq \int \psi(\frac{1}{b_n}[\sum_{j=0}^{n-1} \phi \circ T_{\sigma^k\omega}^j(x) - a_n])P_\omega^k(h_\omega - 1)dm$$
$$\leq \|\psi\|_\infty \|P_\omega^k(h_\omega - 1)\|_{L^1(m)}.$$

Since $\|P_\omega^k(h_\omega - 1)\|_{L^1_m} \to 0$ in case of Example 5.12 and maps satisfying (LY), (Dec) and (Min) the assertion is proved. By [Zwe07, Corollary 3], the proof for continuous time distributional limits follows immediately.

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