DETERMINISTIC WALKS IN RANDOM ENVIRONMENT

ROMAIN AIMINO AND CARLANGELO LIVERANI

ABSTRACT. Motivated by the random Lorentz gas, we study deterministic walks in random environment and show that (in simple, yet relevant, cases) they can be reduced to a class of random walks in random environment where the jump probability depends (weakly) on the past. In addition, we prove few basic results (hopefully the germ of a general theory) on the latter, purely probabilistic, model.

1. Introduction

The motion of a point particle among periodically distributed elastically reflecting convex bodies has been intensively studied from many years. Both in the case of diluted obstacles, that can be treated with kinetic theory ideas (Boltzmann-Grad limit, see [20] and related work), and in the opposite case of high density (finite horizon) starting with the seminal work of Bunimovich, Sinai, Chernov [2] till the recent and much more precise results obtained in [7] using to the new standard pairs and martingale problem techniques introduced in the field by Dolgopyat (see [6] for an elementary introduction to such ideas and reference to the original works).

We are interested in the latter case, in particular to the statistical properties of the infinite system (recurrence, ergodicity, mixing, C.L.T. etc.).

While all the above results deal with the purely periodic case, in real situation any material is expected to have defects. It would hence be of paramount importance to obtain similar results for a situation in which the obstacles are distributed according to some random, translation invariant, stationary process. This encompasses a wide range of possibilities from small perturbations of a periodic array to obstacles distributed according to a Poisson process. The present work is motivated by variants of the former possibility, see section 2 for a more detailed description of a concrete example.

2000 *Mathematics Subject Classification.* 60J15, 37A25, 37C30.

Key words and phrases. Random Lorentz gas, Random walk in random environment, decay of correlations.

L.C. thanks Dmitry Dolgopyat and Marco Lenci for several discussions on random walks thru the years. Also we would like to thank Serge Troubetzkoy for pointing out the possibility that, in our setting, the ergodicity of the process as seen from the particle might hold under very weak extra assumptions. This work was partially supported by the European Advanced Grant Macroscopic Laws and Dynamical Systems (MALADY) (ERC AdG 246953). L.C. acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. This work started while R.A. was affiliated to Università di Roma (Tor Vergata). R.A. was partially supported by FCT fellowship SFRH/BPD/123630/2016, by FCT projects FAPESP/19805/2014 and PTDC/MAT-CAL/3884/2014, with national funds and also by CMUP (UID/MAT/00144/2013), which is funded by FCT with national (MCTES) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

Naively one could expect that the extra randomness coming from the distribution of the obstacles would simplify, rather than complicate, the problem. Unfortunately this seems not to be the case. Indeed, essentially no results are available in the non periodic case with the notable exception of [10, 11], where it is proven recurrence for special examples, and [8], where it is proven the CLT, for the situation in which the periodicity is broken only in a fixed finite region (hence translation invariance is violated). Note however that some results holds in full generality: [3, 23] establish criteria for recurrence and [10] shows that recurrent systems are ergodic. For example, the problems of recurrence and ergodicity are reduced, in the two dimensional case, to establishing a Central Limit Theorem, see [11] for details. Also, to establish recurrence for the case of a one dimensional arrays of obstacles (tubes) is substantially simpler and has been obtained in [25, 5, 4, 25, 13]. Nevertheless, even in the simpler one dimensional situation the study of rate of mixing and C.L.T. are wide open.

Part of the difficulties in studying the above problems stems from the fact that, on the one hand, one needs non trivial results in dynamical systems concerning the decay of correlations in order to show that the deterministic dynamics enjoy some type of *memory loss*, hence it is akin, in some precise technical sense, to a random process. On the other hand, one has to overcome the same type of hard obstacles that exists in analysing the problems of random walks in random environment (see [27] for a review on the subject).

In this article we aim at separating the above two difficulties, so they can be (hopefully) solved independently. To this end we investigate more general, and hence more flexible, models. First we describe a (purely deterministic) class of models for a deterministic random walk in random environments (see section 3), then a (purely probabilistic) class of models of random walks (with memory) in random environment (see section 4). The former includes, as special examples, many relevant cases of random Lorentz gases, in particular the ones discussed in section 2; while the latter contains, as a simple case, *persistent random walks* and, more in general, allows the transition probabilities to depend on all the past history of the particle, although in a precisely controlled way.

The introduction of these two class of models is motivated by the following conjecture: many relevant deterministic walks in random environment are equivalent to the above mentioned probabilistic models. Hence establishing, e.g., the CLT for the probabilistic model implies the CLT for the deterministic walk.

Of course, stated as above, the conjecture is rather vague. The rest of the paper is devoted to making it precise and proving it in some cases. We first illustrate explicitly such an equivalence in some unrealistically simple examples (see section 5) and then we prove the above mentioned equivalence for a much less trivial class of systems in Sections 6 and 7. In such systems the dynamics has properties similar to the billiard dynamics, yet, it is much simpler. This substantially reduces the technical difficulties, hence allowing to flash out the proposed strategy in a more transparent form. So, while the problem of proving the equivalence with a purely random model for the random Lorentz gas remains open, we believe we have convincingly shown that it is a very reasonable possibility.

Notation. *In the following we will use $C_\#$ to designate a generic constant that depends only on the parameters of the considered model. The actual value of $C_\#$ is* *immaterial. In particular, the value of $C_\#$ can change from one occurrence to the next.*

1.1. **Results and structure of the paper.** The paper is organised as follows: in the next section we recall briefly what is meant for Lorentz gases and we describe a subclass of examples that, although not fully general, provide a good starting point for the study of the general situation. In Section 3 we describe a general class of deterministic walks in random environment which encompass both the Lorentz gas and the example with a simpler dynamics that we will consider later.

In Section 4 we describe the class of random systems to which we hope to reduce the deterministic ones. We state precisely a set of properties for the random systems under which one can prove ergodicity of the process as seen from the particle. This show, in particular, that if one would succeed in reducing the Lorentz gas to such probabilistic model, then one would automatically recover all the known results. Of course, we believe that much more follows form such a reduction. For example, we do not make any use of an important property of the Lorentz gas: reversibility. Clearly, more work is needed to determine if our strategy can yield the hoped results (e.g. CLT of for some examples of the Lorentz gas), yet we believe that the present results justify pursuing further this line of thought.

In Section 5 we show that if one restricts the dynamics to a Markov one dimensional expanding map, admittedly an unreasonably excessive simplification, then the dynamical part of the problem can be completely obliterated and one is easily lead to a purely probabilistic problem, although not a trivial one.

Next, we explore the possibility of obliterating the deterministic dynamics in more realistic models. This is the content of Section 6 in which we illustrate precisely our simplified (but not absurdly so) model and we state precisely our results (stating exactly in which sense the deterministic dynamics can be disposed of and which kind of purely probabilistic model one can reduce to).

As the results stated in Section 6 holds under some technical condition, in Section 7 we show that such conditions are satisfied for a large set of one dimensional non-Markov expanding maps, whereby showing the they are not unreasonable. The following sections contains the proof of the statements in Section 6, they use dynamical systems techniques to show that the dynamics can be forgotten and the problem reduced to a purely probabilistic one.

2. The Random Lorentz gas

The random Lorentz gas consists of a distribution of convex, non overlapping, obstacles in $\mathbb{R}^d$, $d \geq 2$ and independent point particles that move of free motion and collides elastically with the obstacles. If one describes the particle density by a distribution, then the problem is reduced to studying the motion of one particle with an initial distribution given by a measure. If the obstacle distribution is described by some probability measure, then the goal is to study the dynamics of the particle for almost all obstacle distributions.

Of course, the problem of studying this situation depends rather heavily by the the type of measure describing the obstacle distribution. Two reasonable assumptions are that the distribution enjoys some type of stationarity and ergodicity with respect to some subgroup of the space translations. Also a key properties is the existence or not of trajectories that can spend a unboundedly long time without experiencing any collision (the presence of such trajectories adds an extra layer of complexity to the problem). Given the many possibilities, let us consider an explicit example that, while not being the most general case, is already very interesting: a small perturbation of a periodic array of discs on $\mathbb{Z}^2$. Note however that similar examples can be considered for $d = 1$ (Lorentz tubes [5, 4, 25]) and $d > 2$.

We start with an exactly periodic distribution on a square lattice, such a periodic array divides naturally $\mathbb{R}^2$ in cells. In each of these cells we assume that there is another disc which position is random in a small neighborhood of the center of the cell and the radius is large enough to prevent trajectory that can enter and leave the cell without experiencing a collision. The location of the central obstacle in different cell is independent and identically distributed. See figure 1 for a pictorial

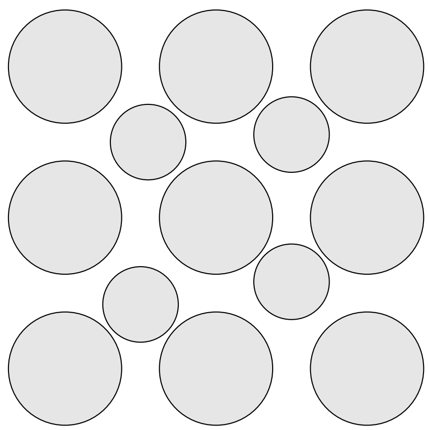

Figure 1. A random obstacle configuration for the random Lorentz gas

illustration of a region of such an obstacle distribution.

The above is a reasonable model for a material with a periodic structure and random impurities. The presence of the periodic structure makes a bit easier to describe the model as a simple dynamical systems. Indeed, we can partition $\mathbb{R}^2$ in cells, each one containing only one random obstacle. See Figure 2 for the picture of such a single cell. Then we can observe the system only when the point particle either hits one of the periodic obstacles or crosses from one cell to another. This is pictorially illustrated by the bold line (solid around the obstacle and dashed in the corridors among different cells) in Figure 2. Technically, this corresponds to defining a Poincaré section.

More precisely, one can suppose, without loss of generality, that the obstacles disposed periodically have centers at the points $(2k+1, 2j+1)$, $(k, j) \in \mathbb{Z}^2$. While the random obstacles have as center the points $2z + \omega(z)$, where $z \in \mathbb{Z}^2$ and $\omega(z)$ being a random variable on $\mathbb{Z}^2$ with values in $\mathbb{R}^2$, identically distributed, such that $\|\omega(z)\| \leq \delta$ for some δ small enough. The obstacle distribution is then described by a product measure $\mathbb{P}$, over $\Omega = \{\omega \in \mathbb{R}^2 \ : \ \|\omega\| \leq \delta\}^{\mathbb{Z}^2}$. On Ω are naturally defined the translations: for each $z \in \mathbb{Z}^2$, $\tau_z : \Omega \to \Omega$ is defined

$$\tau_z(\bar{\omega})_w = (\bar{\omega})_{w+z},$$

for all $\bar{\omega} \in \Omega$ ad $w \in \mathbb{Z}^2$.

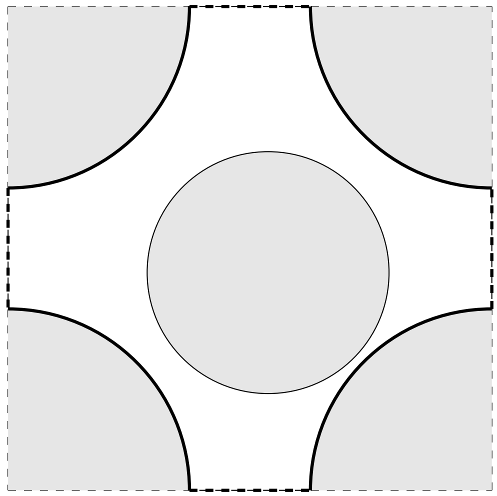

FIGURE 2. Billiard Cell. Poncaré section in bold

Next, let us consider the particle that moves among such an obstacle configuration. Note that, since the energy is conserved we can assume, without loss of generality that the particle moves with velocity one. Hence we can consider a particle at the position $q \in \mathbb{R}^2$ and with velocity p, $\|p\| = 1$. Note that the billiard has a natural invariant measure, Lebesgue. Then, given an obstacle configuration $\omega \in \Omega$, and an initial condition (q, p), we have that the position and velocity of the particle at time t are given by $(q(t), p(t)) = \phi_{\omega,t}(q, p)$, for some flow $\phi_{\omega,t}$ which depends on the obstacle configuration. We are interested in the long time behaviour of such a system. For example, if it has an asymptotic velocity (Law of large numbers), i.e. for almost all ω and initial conditions there exists

$$V = \lim_{t\to\infty} t^{-1}[q(t) - q(0)] \tag{2.1}$$

and if the CLT holds, i.e. when the initial conditions are distributed according to a measure absolutely continuous w.r.t. Lebesgue, then

$$\frac{1}{\sqrt{t}}[q(t) - q(0) - Vt] \tag{2.2}$$

converges in law to a Gaussian Random Variable for almost all ω (quenched Central Limit Theorem). Before discussing further such questions it is convenient to describe some alternative descriptions of the system.

2.1. **Poincaré section.** We can then consider a single cell centered around zero (see Figure 2) and the set $\mathcal{B} = \cup_{i=1}^4 \overline{C}_i \cup \overline{B}_i$ where $\overline{C}_i = C_i \times [-\pi/2, \pi/2]$, C_i being one of the arcs centered at the corners of the box and $[-\pi/2, \pi/2]$ being the angle that the post-collisional velocity forms with the external normal. While $\overline{B}_i = B_i \times [-\pi/2, \pi/2]$, B_i being one of the segments constituting the boundary of the box not contained in the obstacles and $[-\pi/2, \pi/2]$ being the angle that the velocity forms with the external normal to the box boundary. We can then consider the phase space $\mathbb{B} = \mathcal{B} \times \mathbb{Z}^2$ and, for each obstacle configuration $\bar{\omega}$, the dynamics

$$\mathbb{F}_{\bar{\omega}}(x, z) = (f_{\bar{\omega}_{z+e(\bar{\omega}_z,x)}}(x), z + e(\bar{\omega}_z, x)). \tag{2.3}$$

The coordinate z specifies in which cell is the particle, x specifies where the particle is on $\mathcal{B}$, and $\bar\omega_z$ specifies the position of the central obstacle in the cell of coordinate z. Given a particle with coordinates (x, z), one can follow its motion till it hits $\mathcal{B}$ again, this defines the map $f_{\bar\omega_z}(x)$. Note that, by convention, x and $f_{\bar\omega_z}(x)$ denote the pre-collisional position of the particle in $\mathcal{B}\times\{z\}$ and $\mathcal{B}\times\{z+e(\bar\omega_z,x)\}$ respectively. For future use let $\mathcal{W} = \{(0,0),(\pm 1,0),(0,\pm 1)\}$. Next, we divide $\cup_{i=1}^4 C_i \cup B_i$ into five parts (which we call *gates*): $G_{(0,0)} = \cup_{i=1}^4 C_i$, $G_{(1,0)}$ is the element of $\cup_{i=1}^4 B_i$ on the right, $G_{(-1,0)}$ is the element on the left, $G_{(0,1)}$ is the element on top and $G_{(0,-1)}$ is the element at the bottom (always referring to Figure 2), we define then

$$e(\bar\omega_z, x) = \sum_{w\in\mathcal{W}} \mathbb{1}_{G_w}(x) w,$$

where $\mathbb{1}_B$ is the characteristic function of the set B.

A moment thought shows that $\mathbb{F}$ is the Poincaré map associated to the the billiard flow. Namely, given a point (x, z), $\mathbb{F}_{\bar\omega}$ is obtained by first *crossing the Poincaré section* and updating the cell to which the particle belongs if it has crossed a gate, then following the particle till it gets to the Poincaré section again.

2.2. **The random process.** If we want to make a connection to the random walk problem, then we need to have some stochasticity in the system. Since the dynamics is deterministic, the stochasticity can be only in the initial conditions, this agrees with our previous discussion that the initial condition should be given by a probability measure. We can assume, w.l.o.g. that the particle starts from the zero cell with probability one, on the contrary we need to assume some regularity on the x distribution in order to hope for reasonable results. More precisely we assume

$$\mathbb{E}(\varphi) = \int_{\mathcal{B}} \varphi(0, x) h_0(x) dx$$

were h_0 is some smooth distribution on $\mathcal{B}$. Accordingly, we can consider

$$(x(\bar\omega, n), z(\bar\omega, n)) = \mathbb{F}^n_{\bar\omega}(x, z)$$

as random variables.[1]

We can then consider the path space $\mathbb{M} = \{(z(n)) \in (\mathbb{Z}^2)^{\mathbb{N}} \; : \; z(0) = 0, \; z(n+1) - z(n) \in \mathcal{W}\}$ and the random process on $\Omega_\star = \Omega \times \mathbb{M}$ defined, for each $\bar\omega \in \Omega$ by the probabilities,

$$\mathbb{P}_\star(\{z(1), \dots, z(n)\} \mid \bar\omega) = \int_{\mathcal{B}} \prod_{k=0}^{n-1} \mathbb{1}_{G_{w(k)}}(x(\bar\omega, k)) h_0(x) dx,$$

where $w(k) = z(k + 1) - z(k)$, while $\bar\omega$ is distributed according to $\mathbb{P}$.

We can write the above expression in a style more dynamical systems prone by introducing the transfer operators[2]

$$\mathcal{L}_{\bar\omega,z,w}\varphi(x) = \sum_{\{y \,:\, f_{\bar\omega_z}(y)=x\}} |\det \partial_y f_{\bar\omega_{z+w}}(y)|^{-1} \mathbb{1}_{G_w}(y)\varphi(y).$$

[1] Using the probabilistic usage we will often suppress the $\bar\omega$ dependency, when this does not create confusion.

[2] Note that in the present case the set on which we take the sum consists of only one element and the determinant of the Jacobian of the f_ω is always one. Yet, it is convenient to define it like this, also for consistency with our subsequent examples.

Then, changing variable repeatedly, yields

$$\mathbb{P}_\star(\{z(1),\dots,z(n)\}\mid\bar\omega)=\int_{\mathcal{B}}\mathcal{L}_{\bar\omega,z(n-1),w(n-1)}\cdots\mathcal{L}_{\bar\omega,z(0),w(0)}h_0. \tag{2.4}$$

The above formula reduces the problem of understanding the measure $\mathbb{P}_\star$ to the problem of studying products of transfer operators. Of course, the process $\mathbb{P}_\star$ is, in general, not Markov, however we expect it to exhibit a very strong loss of memory. More precisely, we conjecture:

Conjecture. *Under some appropriate technical conditions on $\mathbb{P}$ and the initial density h, there exist $\nu\in(0,1)$ and $C_\#>0$ such that for all $\bar\omega\in\Omega$*

$$\left|\mathbb{P}_\star(z(n)\mid z^n,\bar\omega)-\mathbb{P}_\star(\widehat z_m(n-m)\mid\widehat z^{n-m},\tau_{z(m)}\bar\omega)\right|\le C_\#\nu^{n-m},$$

where $z^k=z(1),\dots,z(k-1)$, $\widehat z^k=\widehat z_m(1),\dots,\widehat z_m(k-1)$ and $\widehat z_m$ is the path starting at 0 defined by $\widehat z_m(k)=z(m+k)-z(m)$.

If the above were true, then one could reduce the study of the dynamical system to a purely probabilistic problem (see section 4), albeit non necessarily an easy one. See however Section 4 for a beginning of a theory in some important cases.

In the rest of the paper we will try to convince the reader that this is a reasonable and fruitful point of view by first putting the described Lorentz gas models in a larger context and then working out simpler, but non trivial, classes of examples. In particular, we will prove the above conjecture for such examples, giving an idea of what *some appropriate technical conditions* might mean (see Theorem 6.1).

Remark 2.1. *Remark that the results obtained in section 4 do not apply in an obvious manner to the general determinist model discussed in section 6 due to the difficulty to verify condition (***Abs***) stated in section 4.2 (which may, in fact, be too strong). This is due to the fact that the models we analyse, due to their simplicity, typically cannot enjoy two important properties that hold for a vast class of Lorentz gasses (see however section 7.2 where we present a class of models that satisfy property (a)):*

a) all the maps f_ω share the same invariant measures and, at the same time, can have deterministic gates;
b) the dynamics is reversible (i.e., $\mathcal{W}$ is symmetric and there exists an involution i, $i^2(x,z)=(x,z)$, such that $i\circ\mathbb{F}_{\bar\omega}=\mathbb{F}_{\bar\omega}^{-1}\circ i$).

Indeed, we believe that the above two properties can play a major role in the study of the associated probabilist model making it more tractable than the general case.

We conclude the section with a brief discussion of the above properties and of their probabilistic meaning. In the case of Billiards the involution is given by $(q,p)\to(q,-p)$, which, at the level of the Poincaré map, writing $x\in\cup_{w\in\mathcal{W}}G_w$ as $x=(w,s,\theta)$[3], reads $i(w,s,\theta,z)=(-w,s,\theta,z+w)$.[4] Also, let $\pi(w,s,\theta,z)=(w,s,\theta)$ and $i_1(w,s,\theta)=(-w,s,\theta)$ so that $i_1\circ\pi=\pi\circ i$. Thus, choosing as initial measure

[3] The curvilinear coordinate s is chosen such that the segments $s\to(w,s)$ and $s\to(-w,s)$, $w\neq0$ go in the same direction.

[4] Recall that we are considering the case of deterministic gates, although the following consideration easily extend to the general case. In particular, e is a function of x only, and we have $e(w,s,\theta)=w$.

the common invariant measure of the Poincaré maps h_0 (hence $h_0 \circ i_1 = h_0$), we have

$$\begin{aligned}\mathbb{P}_\star(\{z(1),\dots,z(n)\} \mid \bar\omega) &= \int_{\mathcal{B}} \prod_{k=0}^{n-1} \mathbb{1}_{G_{w(k)}} \circ \pi \circ \mathbb{F}^k_{\bar\omega}(x,0) h_0(x)dx \\ &= \int_{\mathcal{B}} \prod_{k=0}^{n-1} \mathbb{1}_{G_{w(k)}} \circ \pi \circ \mathbb{F}^k_{\bar\omega} \circ i(i_1(x), e(x)) h_0(x)dx \\ &= \int_{\mathcal{B}} \prod_{k=0}^{n-1} \mathbb{1}_{G_{-w(k)}} \circ \pi \circ \mathbb{F}^{-k}_{\bar\omega}(x, -e(x)) h_0(x)dx \\ &= \int_{\mathcal{B}} \prod_{k=0}^{n-1} \mathbb{1}_{G_{-w(k)}}(f^{-1}_{\bar\omega_{z(k)}} \circ \cdots \circ f^{-1}_{\bar\omega_{z(1)}}(x)) h_0(x)dx,\end{aligned}$$

where, in the third line, we have used the invariance of the measure with respect to i_1 and the relation $e(i_1(x)) = -e(x)$. Next, using the invariance of the measure with respect to the maps f_σ,

$$\mathbb{P}_\star(\{z(1),\dots,z(n)\} \mid \bar\omega) = \int_{\mathcal{B}} \prod_{k=0}^{n-1} \mathbb{1}_{G_{-w(k)}}(f_{\bar\omega_{z(k+1)}} \circ \cdots \circ f_{\bar\omega_{z(n-1)}}(x)) h_0(x)dx.$$

Then, setting $\tilde w(k) = -w(n-1-k)$, $\tilde z(k) = \sum_{j=0}^{k-1} \tilde w(j)$ and $\widetilde\omega = \tau_{z(n)}\bar\omega$,

$$\begin{aligned}\mathbb{P}_\star(\{z(1),\dots,z(n)\} \mid \bar\omega) &= \int_{\mathcal{B}} \prod_{k=0}^{n-1} \mathbb{1}_{G_{\tilde w(k)}}(f_{\widetilde\omega_{\tilde z(k)}} \circ \cdots \circ f_{\widetilde\omega_{\tilde z(1)}}(x)) h_0(x)dx \\ &= \mathbb{P}_\star(\{\tilde z(n-1),\dots,\tilde z(0)\} \mid \tau_{z(n)}\bar\omega).\end{aligned} \tag{2.5}$$

Note that $\tilde z(k) = z(n-k) - z(n)$.

The above is just the reversibility of the random process and has important consequences (see Lemma 4.6 for a concrete example).

3. Deterministic walks in random environment

As already mentioned there is an extreme scarcity of results pertaining the random Lorentz Gas (apart from the low density regime, see [19] for a review of the Lorentz gas in the Boltzmann-Grad limit). It is then sensible to consider simpler models in which one can start to solve some of the outstanding difficulties. To this end, following Lenci [11], it is convenient to see the Lorentz gas as a special case of a more general class of models: *deterministic walks in random environment*. Even for such models very few results exist. Exceptions are [14] in which a zero-one law for systems with local dynamics which are markovian, but deterministic, is established and [26] which considers statistical properties for a related (simplified) model with local dynamics consisting of expanding linear maps of the circle or hyperbolic toral automorphisms.

The model can be stated in rather general terms, for simplicity let us restrict to the case of the $\mathbb{Z}^d$ lattice with bounded jumps $\mathcal{W} \subset \mathbb{Z}^d$, $\#\mathcal{W} < \infty$,[5] and local dynamics which live all on the same phase space $\mathcal{M}$.

Consider the set $\mathcal{A} = \{(f_\alpha, \mathcal{M}, \mathcal{P}_\alpha)\}_{\alpha\in A}$, where A is a finite index set, $f_\alpha : \mathcal{M} \to \mathcal{M}$ are maps, and $\mathcal{P}_\alpha = \{G_{\alpha,w}\}_{w\in\mathcal{W}}$ are partitions of $\mathcal{M}$. The environment is

[5] That is only jumps $w \in \mathcal{W}$ are allowed.

described by the probability space $\Omega = A^{\mathbb{Z}^d}$ equipped with a translation invariant probability $\mathbb{P}$. Also we assume that all the maps $f_\alpha : \mathcal{M} \to \mathcal{M}$ are nonsingular with respect to some reference measure m on $\mathcal{M}$. Then, for each realisation $\bar\omega \in \Omega$ we can define the dynamics $\mathbb{F}_{\bar\omega}(\cdot,\cdot) : \mathcal{M} \times \mathbb{Z}^d \to \mathcal{M} \times \mathbb{Z}^d$ by

$$\mathbb{F}_{\bar\omega}(x, z) = (f_{\bar\omega_{z+e(\bar\omega_z,x)}}(x), z + e(\bar\omega_z, x))$$
$$e(\alpha, x) = \sum_{w\in\mathcal{W}} \mathbb{1}_{G_{\alpha,w}}(x)w.$$

Finally, the randomness at fixed environment comes from the assumption that the initial condition is described by some probability measure μ absolutely continuous with respect to m.

In other words $z(0) = z_0$ while $x(0)$ is distributed according to μ. We will assume, w.l.o.g., $z(0) = 0$. Then the path $(z(n))_{n\in\mathbb{N}}$ is a random process in $\mathbb{M} = \{(z_n) \in (\mathbb{Z}^d)^{\mathbb{N}} : z_0 = 0, z_{n+1} - z_n \in \mathcal{W}\}$, which we call the space of *admissible paths*. Let $\mathbb{P}_\star$ be the law of the resulting process on $\Omega_\star = \Omega \times \mathbb{M}$.

Each map f_α admits a transfer operator $\mathcal{L}_{f_\alpha} : L^1(m) \to L^1(m)$ defined by

$$\int_{\mathcal{M}} (\mathcal{L}_{f_\alpha}\phi)\psi dm = \int_{\mathcal{M}} \phi \cdot \psi \circ f_\alpha dm$$

for all $\phi \in L^1(m)$ and $\psi \in L^\infty(m)$.

Let h_0 be the density of the initial condition ($d\mu = h_0 dm$). Then, setting $w(n) = z(n+1) - z(n)$, repeated changes of variables yield

$$\mathbb{P}_\star(z(1), \dots, z(n) \mid \bar\omega) = \int_{\mathcal{M}} \mathcal{L}_{\bar\omega,z(n-1),w(n-1)} \cdots \mathcal{L}_{\bar\omega,z(0),w(0)} h_0 dm \tag{3.1}$$

for each $\bar\omega \in \Omega$ and each admissible path $(z(n)) \in \mathbb{M}$, and with

$$\mathcal{L}_{\bar\omega,z,w}(\phi) = \mathcal{L}_{f_{\bar\omega_{z+w}}}(\mathbb{1}_{G_{\bar\omega_z,w}}\phi). \tag{3.2}$$

3.1. **The point of view of the particle.** An important process associated to our deterministic walk is the process of the environment as seen from the particle. This is the dynamical system defined on $\Omega \times \mathcal{M}$ by

$$\mathcal{F}(\bar\omega, x) = (\tau_{e(\bar\omega_0,x)}\bar\omega, f_{\bar\omega_{e(\bar\omega_0,x)}}(x)).$$

This is known to be a fruitful point of view, in particular if one knows the invariant measure. As noted by Lenci [10, 11], in an important subclass of deterministic random walks the invariant measure can be trivially computed.

Lemma 3.1. *If all the maps f_α have the same invariant measure λ and the set $\mathcal{P}_\alpha$ is deterministic (i.e., it does not depend on α), then the probability measure $\mathbb{P}_\star = \mathbb{P} \times \lambda$ is invariant for the map $\mathcal{F}$.*

Proof. Let $\mathbb{E}_\star$ be the expectation with respect to $\mathbb{P}_\star$. Since the set $\mathcal{P}_\alpha$ is deterministic, we can write $e(\alpha, x) = e(x)$ for all $\alpha \in A$ and $x \in \mathcal{M}$. For each bounded measurable function φ we have

$$\begin{aligned}\mathbb{E}_\star(\varphi \circ \mathcal{F}) &= \int \varphi(\tau_{e(x)}\bar{\omega}, f_{\bar{\omega}_{e(x)}}(x))\mathbb{P}(d\bar{\omega})\lambda(dx)\\ &= \int \varphi(\tau_{e(x)}\bar{\omega}, f_{(\tau_{e(x)}\bar{\omega})_0}(x))\mathbb{P}(d\bar{\omega})\lambda(dx)\\ &= \int \varphi(\bar{\omega}, f_{\bar{\omega}_0}(x))\mathbb{P}(d\bar{\omega})\lambda(dx)\\ &= \int \varphi(\bar{\omega}, x)\mathbb{P}(d\bar{\omega})\lambda(dx) = \mathbb{E}_\star(\varphi),\end{aligned}$$

where we have used first the invariance of $\mathbb{P}$ with respect to the translations and then the invariance of λ with respect to the maps f_α. □

Hence now we have a dynamical system with a finite measure.

Lemma 3.2. *In the hypotheses of Lemma 3.1 the limit*

$$V = \lim_{n\to\infty} \frac{1}{n} z(n)$$

exists $\mathbb{P}_\star$ almost surely. Moreover, if $(\Omega \times \mathcal{M}, \mathcal{F}, \mathbb{P}_\star)$ is ergodic, then $V = \lambda(e)$.

Proof. Note that, by setting $(\bar{\omega}(n), x(n)) = \mathcal{F}^n(\bar{\omega}, x)$, we have

$$\frac{1}{n} z(n) = \frac{1}{n} \sum_{k=0}^{n-1} e(x(k)).$$

Hence the existence of the limit follows from Birkhoff ergodic theorem for the dynamical system $(\Omega \times \mathcal{M}, \mathcal{F}, \mathbb{P}_\star)$ applied to $\varphi(\bar{\omega}, x) = e(\bar{\omega}_0, x) = e(x)$. By ergodicity, the limit equals the average of φ with respected to $\mathbb{P}_\star$ which is equal to $\mathbb{E}_\star(e)$. □

Just to emphasize that the above Lemmata are not devoid of applications, let us recall the following,

Lemma 3.3. *For the random Lorentz gas described in section 2 we have $V = 0$.*

Proof. In [11] is proven that $(\mathcal{F}, \mathbb{P}_\star)$ is ergodic. Then Lemma 3.2 implies that $V = \sum_{w \in \mathcal{W}} w\lambda(G_w)$ which average is zero due to the fact that λ is invariant for the change $p \to -p$. □

4. Gibbs random walks in random environment

We do not expect the process described by $\mathbb{P}_\star$ to be Markov, yet we expect that the jump rates have a weak dependence of the past. To be more precise, we conjecture that the process is a *random walk in random environment with weak memory*. In the probabilistic literature random walks with a finite memory are called *persistent*, here we expect the memory to be infinite although depending weakly on the far past, exactly like a potential of a Gibbs measure. Let us specify exactly what we mean by this.

Let A be a finite set and $\mathcal{W} \subset \mathbb{Z}^d$, $2 \le \sharp\mathcal{W} < \infty$. Consider the measurable space $\Omega = A^{\mathbb{Z}^d}$ and a translation invariant, ergodic, probability distribution $\mathbb{P}$ that describes the distribution of the environments $\bar{\omega} \in \Omega$. For each $n \in \mathbb{N}$ and

$(w_0,\dots,w_{n-1}) \in \mathcal{W}^n$, assume that are given compatible probabilities $p(\bar\omega, n, \cdot)$ on $\mathcal{W}^n$, i.e.

$$\sum_{(w_0,\dots,w_{n-1})\in\mathcal{W}^n} p(\bar\omega, n, w_0 \dots w_{n-1}) = 1,$$

and

$$p(\bar\omega, n, w_0 \dots w_{n-1}) = \sum_{w\in\mathcal{W}} p(\bar\omega, n+1, w_0 \dots w_{n-1}w)$$

for all $\bar\omega \in \Omega$, $n \geq 0$ and $(w_0,\dots,w_{n-1}) \in \mathcal{W}^n$. Assume also that all the maps $\bar\omega \mapsto p(\bar\omega, n, w_0 \dots w_{n-1})$ are measurable. We have then for each $\bar\omega \in \Omega$ a probability measure $\mathbb{P}^{\bar\omega}$ on the space $\mathcal{W}^{\mathbb{N}}$ by Kolmogorov extension theorem. By the monotone class theorem, the map $G \mapsto \mathbb{P}^{\bar\omega}(G)$ is measurable for any measurable set $G \subset \mathcal{W}^{\mathbb{N}}$, and we can thus define a probability measure $\mathbb{P}_\star$ on $\Omega \times \mathcal{W}^{\mathbb{N}}$ by

$$\mathbb{P}_\star(d\bar\omega, d\bar w) = \mathbb{P}(d\omega)\mathbb{P}^{\bar\omega}(d\bar w).$$

Remark 4.1. *The measure $\mathbb{P}_\star$ can be naturally identified with a measure on the space $\Omega_\star = \Omega \times \mathbb{M}$, where $\mathbb{M} = \{(z_n) \in (\mathbb{Z}^d)^{\mathbb{N}} : z_0 = 0, z_{n+1} - z_n \in \mathcal{W}, \forall n\}$ is the space of admissible paths starting at 0, since there is a 1-to-1 correspondance between elements of $\mathcal{W}^n$ and admissible paths of length n, via the relations $w_k = z_{k+1} - z_k$.*

4.1. **The weak memory requirement.** We find convenient, although not strictly necessary, to require the following assumption that ensures that all admissible paths have positive probability:

(Pos): for $\mathbb{P}$-a.e. $\bar\omega \in \Omega$, for all $n \geq 0$ and all $\bar w \in \mathcal{W}^{\mathbb{N}}$,

$$p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1}) > 0.$$

We write $\mathbb{P}_{\bar\omega}(\bar w_n \mid \bar w_0 \dots \bar w_{n-1})$ for the conditional probability $\frac{p(\bar\omega, n+1, \bar w_0 \dots \bar w_{n-1}\bar w_n)}{p(\bar\omega, n, \bar w_0 \dots \bar w n-1)}$. The requirement of weak memory is made precise by the following:

(Exp): there exist $C_\# > 0$ and $\nu \in (0,1)$ such that for $\mathbb{P}$-a.e. $\bar\omega \in \Omega$, all $n > m \geq 0$ and all $\bar w \in \mathcal{W}^{\mathbb{N}}$

$$\left|\mathbb{P}_{\bar\omega}(\bar w_n \mid \bar w_0 \dots \bar w_{m-1}\bar w_m \dots \bar w_{n-1}) - \mathbb{P}_{\tau_{z_m}\bar\omega}(\bar w_n \mid \bar w_m \dots \bar w_{n-1})\right| \leq C_\# \nu^{n-m}, \tag{4.1}$$

where $z_m = \sum_{k=0}^{m-1} \bar w_k$.

Remark 4.2. *This is the property we have conjectured to be true for the random Lorentz gas at the end of Section 2.*

Note that the above condition implies loss of memory:

Lemma 4.3. *If $\mathbb{P}_\star$ satisfies (4.1), then (using the notation of Remark 4.1),*

$$|\mathbb{P}_\star(z_{n+1} \mid z_{n-m+1}, \dots, z_n, \bar\omega) - \mathbb{P}_\star(z_{n+1} \mid z_1, \dots, z_n, \bar\omega)| \leq C_\# \nu^m.$$

Proof. Let $p(z_{n-m+1})$ be the set of admissible paths $(w_0,\dots,w_{n-m})$ that arrive in z_{n-m+1}, i.e. $\sum_{i=0}^{n-m} w_i = z_{n-m+1}$, and set $w_i = z_{i+1} - z_i$ for $i = n-m+1, \dots, n$.

Then we have, using **(Exp)**:

$$\begin{aligned}\mathbb{P}_\star(z_{n+1} \mid z_{n-m+1}, \dots, z_n, \bar\omega) &= \sum_{w \in p(z_{n-m+1})} \frac{p(\bar\omega, n+1, w_0 \dots w_n)}{\mathbb{P}_\star(z_{n-m+1}, \dots, z_n \mid \bar\omega)} \\ &= \sum_{w \in p(z_{n-m+1})} \mathbb{P}_{\bar\omega}(w_n \mid w_0 \dots w_{n-1}) \frac{p(\bar\omega, n, w_0 \dots w_{n-1})}{\mathbb{P}_\star(z_{n-m+1}, \dots, z_n \mid \bar\omega)} \\ &= \mathbb{P}_{\tau_{z_{n-m+1}}\bar\omega}(w_n \mid w_{n-m+1} \dots w_{n-1}) + \mathcal{O}(\nu^m),\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_\star(z_{n+1} \mid z_1, \dots, z_n, \bar\omega) &= \mathbb{P}_{\bar\omega}(w_n \mid w_0 \dots w_{n-1}) \\ &= \mathbb{P}_{\tau_{z_{n-m+1}}\bar\omega}(w_n \mid w_{n-m+1} \dots w_{n-1}) + \mathcal{O}(\nu^m).\end{aligned}$$

□

4.2. **The point of view of the particle and three further assumptions.** One can define the process of the environment as seen from the particle also in this context. It is given by the dynamical system on the space $\Omega_\star = \Omega \times \mathcal{W}^{\mathbb{N}}$ defined by

$$\mathcal{F}_\star(\bar\omega, \bar w) = (\tau_{\bar w_0}\bar\omega, \tau_\star \bar w),$$

where $\tau_\star : \mathcal{W}^{\mathbb{N}} \to \mathcal{W}^{\mathbb{N}}$ is the unilateral shift. Note that, in general, $\mathbb{P}_\star$ is not invariant for $\mathcal{F}_\star$. Next, in the attempt to convince the reader that the present class of systems is not completely unreasonable, let us show that some easy properties of the Markov case persists in the present context, under reasonable extra conditions.

Remark 4.4. *Note that in the probabilistic literature, see for instance [27], it is more usual to consider the random process $\bar\omega(n) = \tau_{z(n)}\bar\omega$ on $\overline{\Omega} = \Omega^{\mathbb{N}}$, and its law $\overline{\mathbb{P}}$ when $\bar\omega(0)$ is distributed according to $\mathbb{P}$ (as we did in Section 3.1). When the set of periodic environments has probability 0, these two points of view are equivalent, since the map which associates to each $(\bar\omega, \bar w) \in \Omega_\star$ the corresponding sequence $(\bar\omega(n)) \in \overline{\Omega}$ is invertible almost everywhere and realizes a conjugacy between the two dynamical systems $(\Omega_\star, \mathcal{F}_\star, \mathbb{P}_\star)$ and $(\overline{\Omega}, \bar\tau, \overline{\mathbb{P}})$, where $\bar\tau$ is the shift on $\overline{\Omega}$. The same comments holds also for the definition given in Section 3.1. See also Remark 4.24 for further comments.*

We are interested in the asymptotic properties for z_n. Note that if we define $\varphi(\bar\omega, \bar w) = \bar w_0$, then we have $z_n = \sum_{k=0}^{n-1} \varphi \circ \mathcal{F}_\star^k$.

The following assumption is useful to prove the existence of an interesting invariant measure for $\mathcal{F}_\star$:

- **(Abs):** There exists $C_\# > 0$ such that for $\mathbb{P}$-a.e. $\bar\omega \in \Omega$, all $n \geq 0$, all $k \geq 1$ and all $\bar w \in \mathcal{W}^{\mathbb{N}}$,

$$C_\#^{-1} \leq \sum_{(w_1, \dots, w_k) \in \mathcal{W}^k} \frac{p(\tau_{-(w_1 + \dots + w_k)}\bar\omega, n+k, w_1 \dots w_k \bar w_0 \dots \bar w_{n-1})}{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})} \leq C_\#.$$

To legitimate this assumption, we prove two relevant facts.

Lemma 4.5. *If there exists a probability measure $\mathbb{Q}_\star$ equivalent to $\mathbb{P}_\star$, invariant for $\mathcal{F}_\star$ and such that its density satisfies $c^{-1} \leq \frac{d\mathbb{Q}_\star}{d\mathbb{P}_\star} \leq c$ for some $c > 0$, then* **(Abs)** *holds. In particular,* **(Abs)** *holds if $\mathbb{P}_\star$ is $\mathcal{F}_\star$-invariant.*

Proof. For all measurable set $A \subset \Omega$ and all cylinder $[\bar{w}_0 \dots \bar{w}_{n-1}] \subset \mathcal{W}^{\mathbb{N}}$, the preimage $\mathcal{F}_\star^{-k}(A \times [\bar{w}_0, \dots, \bar{w}_{n-1}])$ is equal to the disjoint union

$$\bigcup_{(w^1,\dots,w^k)\in\mathcal{W}^k} \tau_{-(w^1+\dots+w^k)}(A) \times [w^1 \dots w^k \bar{w}_0 \dots \bar{w}_{n-1}].$$

Let $B = A \times [\bar{w}_0 \dots \bar{w}_{n-1}]$. By definition of $\mathbb{P}_\star$, we have

$$\mathbb{P}_\star(B) = \int_A p(\bar{\omega}, n, \bar{w}_0 \dots \bar{w}_{n-1}) \mathbb{P}(d\bar{\omega})$$

and

$$\begin{aligned}\mathbb{P}_\star(\mathcal{F}_\star^{-k} B) &= \sum_{(w^1,\dots,w^k)\in\mathcal{W}^k} \int_{\tau_{-(w^1+\dots+w^k)}A} p(\bar{\omega}, n+k, w^1 \dots w^k \bar{w}_0 \dots \bar{w}_{n-1}) \mathbb{P}(d\bar{\omega}) \\ &= \int_A \sum_{(w^1,\dots,w^k)\in\mathcal{W}^k} p(\tau_{w^1+\dots+w^k}\bar{\omega}, n+k, w^1 \dots w^k \bar{w}_0 \dots \bar{w}_{n-1}) \mathbb{P}(d\bar{\omega})\end{aligned}$$

thanks to the translation invariance of $\mathbb{P}$.

Since $\mathbb{P}_\star(B) \le c\mathbb{Q}_\star(B) = c\mathbb{Q}_\star(\mathcal{F}_\star^{-k}B) \le c^2 \mathbb{P}_\star(\mathcal{F}_\star^{-k}B)$, we have

$$\begin{aligned}&\int_A p(\bar{\omega}, n, \bar{w}_0 \dots \bar{w}_{n-1}) \mathbb{P}(d\bar{\omega}) \\ &\qquad \le c^2 \int_A \sum_{(w^1,\dots,w^k)\in\mathcal{W}^k} p(\tau_{w^1+\dots+w^k}\bar{\omega}, n+k, w^1 \dots w^k \bar{w}_0 \dots \bar{w}_{n-1}) \mathbb{P}(d\bar{\omega}),\end{aligned}$$

and similarly,

$$\begin{aligned}&\int_A \sum_{(w^1,\dots,w^k)\in\mathcal{W}^k} p(\tau_{w^1+\dots+w^k}\bar{\omega}, n+k, w^1 \dots w^k \bar{w}_0 \dots \bar{w}_{n-1}) \mathbb{P}(d\bar{\omega}) \\ &\qquad\qquad \le c^2 \int_A p(\bar{\omega}, n, \bar{w}_0 \dots \bar{w}_{n-1}) \mathbb{P}(d\bar{\omega}).\end{aligned}$$

Since it holds for all measurable sets $A \subset \Omega$, and that the set of cylinders is countable, this proves the lemma. □

We say thay the process is reversible if (see also (2.5) for the definition) $\mathcal{W}$ is symmetric (i.e. $-w \in \mathcal{W}$ for all $w \in \mathcal{W}$) and, for $\mathbb{P}$-a.e. $\bar{\omega} \in \Omega$, all $n \ge 0$ and all $\bar{w} \in \mathcal{W}^{\mathbb{N}}$:

$$p(\bar{\omega}, n, w_0 \dots w_{n-1}) = p(\tau_{(w_0+\dots+w_{n-1})}\bar{\omega}, n, -w_{n-1}, \dots, -w_0).$$

Lemma 4.6. *If the process is reversible, then $\mathbb{P}_\star$ is $\mathcal{F}_\star$-invariant. In particular,* **(Abs)** *is verified, by Lemma 4.5.*

Proof. It suffices to consider a measurable set B of the form $B = A\times[\bar w_0, \dots, \bar w_{n-1}]$. We have:

$$\begin{aligned}
\mathbb{P}_\star(\mathcal{F}_\star^{-1}B) &= \mathbb{P}_\star\left(\bigcup_{w\in\mathcal{W}} \tau_{-w}(A)\times[w,\bar w_0,\dots,\bar w_{n-1}]\right)\\
&= \sum_{w\in\mathcal{W}}\int_{\tau_{-w}(A)} p(\bar\omega, n+1, w, \bar w_0,\dots,\bar w_{n-1})\mathbb{P}(d\bar\omega)\\
&= \sum_{w\in\mathcal{W}}\int_A p(\tau_{-w}\bar\omega, n+1, w, \bar w_0,\dots,\bar w_{n-1})\mathbb{P}(d\bar\omega)\\
&= \sum_{w\in\mathcal{W}}\int_A p(\tau_{(\bar w_0+\dots+\bar w_{n-1})}\bar\omega, n+1, -\bar w_{n-1},\dots,-\bar w_0,-w)\mathbb{P}(d\bar\omega)\\
&= \int_A p(\tau_{(\bar w_0+\dots+\bar w_{n-1})}\bar\omega, n, -\bar w_{n-1},\dots,-\bar w_0)\mathbb{P}(d\bar\omega)\\
&= \int_A p(\bar\omega, n, \bar w_0\dots\bar w_{n-1})\mathbb{P}(d\bar\omega) = \mathbb{P}_\star(B).
\end{aligned}$$

□

Remark 4.7. *Lemma 4.5 suggest that (***Abs***) is too strong. Yet, there are simple models (e.g. Sinai walk, see Example 1 in Section 5.1) for which there does not exist an invariant probability measure absolutely continuous with respect to* $\mathbb{P}_\star$*. Hence some condition is necessary.*

As common for random walks, we require an ellipticity assumption:

(Ell): There exist $\gamma_0 > 0$ and $n_\star \geq 0$ such that for $\mathbb{P}_\star$-a.e. $\bar\omega\in\Omega$, all $n\geq n_\star$ and all $\bar w\in\mathcal{W}^{\mathbb{N}}$,

$$\mathbb{P}_{\bar\omega}(\bar w_n \mid \bar w_0\dots\bar w_{n-1}) \geq \gamma_0.$$

We finally formulate an assumption on the probability measure $\mathbb{P}$ that governs the environment distribution:

(Pro): Let $\mathcal{G}(\mathcal{W}) = \{z\in\mathbb{Z}^d \,:\, z = w_0+\dots+w_{n-1}, n\in\mathbb{N}, w_i\in\mathcal{W}\}$. We assume that $G = \mathcal{G}(\mathcal{W})$ is an additive group and that $\mathbb{P}$ is ergodic with respect to the action of G.

Remark 4.8. *It might not be necessary to assume that* $\mathcal{G}(\mathcal{W})$ *is a group, but we will not pursue in this direction, as the main example we have in mind, the Lorentz gas described in Section 2, satisfies the above assumption: indeed,* $\mathcal{G}(\mathcal{W})$ *is an additive group whenever* $\mathcal{W}$ *is symmetric. We leave to the interested reader possible weakening of property (***Pro***). We nevertheless mention that a closer look at its proof reveals that Theorem 4.9 remains valid if* $\mathbb{P}$ *is ergodic for each translation* τ_z*,* $z\in\mathbb{Z}^d$*,* $z\neq 0$ *(for instance if* $\mathbb{P}$ *is mixing when* $d=1$*, or if* $\mathbb{P}$ *is i.i.d. when* $d>1$*), without any extra assumption on* $\mathcal{W}$*.*

4.3. **A few basic results.**

The above assumptions are justified by the following Theorem. The rest of the section is devoted to its proof.

Theorem 4.9. *Suppose that the conditions* **(Pos)***,* **(Exp)***,* **(Abs)***,* **(Ell)** *and* **(Pro)** *hold. Then there exists a unique* $\mathcal{F}_\star$*-invariant probability measure* $\mathbb{Q}_\star$ *equivalent to* $\mathbb{P}_\star$ *and the dynamical system* $(\Omega_\star, \mathcal{F}_\star, \mathbb{Q}_\star)$ *is ergodic. In particular, we* *have*

$$\lim_{n\to\infty} \frac{z_n}{n} = V, \tag{4.2}$$

$\mathbb{P}_\star$*-a.s., with* $V = \int_{\Omega_\star} \varphi \, d\mathbb{Q}_\star \in \mathbb{R}^d$.

Remark 4.10. *If* $\mathbb{P}_\star$ *is invariant, then*

$$V = \int_{\Omega_\star} \varphi \, d\mathbb{P}_\star = \sum_{w\in\mathcal{W}} w \int_\Omega p(\bar\omega, 1, w) \, \mathbb{P}(d\bar\omega).$$

As an immediate but important consequence, we deduce the recurrence in 1-d:

Corollary 4.11. *Under the conditions of the above theorem, if* $d = 1$ *and* $V = 0$*, then the process* (z_n) *is recurrent:* $z_n = 0$ *infinitely often,* $\mathbb{P}_\star$*-a.s..*

Proof of Corollary 4.11. We refer to [24] for a nice survey on the recurrence of cocycles. Since $(\Omega_\star, \mathcal{F}_\star, \mathbb{Q}_\star)$ is ergodic by Theorem 4.9, part (1), when $d = 1$, the walk is recurrent if $V = 0$, see [1] or [24, Theorem 3]. Note that this result for recurrence of cocycles is stated for invertible dynamical systems, but it can be extended to non-invertible systems using the natural extension, see for instance [12, Appendix A.2]. □

Remark 4.12. *When* $d = 2$ *and* $V = 0$*, if* (z_n) *satisfied an annealed central limit theorem, i.e. if* $\frac{z_n}{\sqrt{n}}$ *converges in law to a Gaussian distribution under the probability measure* $\mathbb{Q}_\star$*, then the process* (z_n) *is recurrent by the results of Conze [3] and Schmidt [23].*

From now on and till the end of the section we will assume conditions **(Pos)**, **(Exp)**, **(Abs)**, **(Ell)** and **(Pro)** if not explicitly stated otherwise.

To prove Theorem 4.9, we will analyze the properties of the transfer operator $\mathcal{L}_\star$ associated to $\mathcal{F}_\star$ with respect to $\mathbb{P}_\star$. More precisely, we will show that the operator $\mathcal{L}_\star$ enjoys some regularization properties on a space of Hölder functions. We first define the usual separation time on $\mathcal{W}^{\mathbb{N}}$ by

$$s(\bar w, \bar w') = \inf\{n \ge 0 \,:\, \bar w_n \ne \bar w'_n\},$$

and for $0 < \theta < 1$, the metric $d_\theta(\bar w, \bar w') = \theta^{s(\bar w, \bar w')}$ on $\mathcal{W}^{\mathbb{N}}$.

For a measurable function $f : \Omega \times \mathcal{W}^{\mathbb{N}} \to \mathbb{C}$, we set:

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\bar\omega\in\Omega} \sup_{\bar w\in\mathcal{W}^{\mathbb{N}}} |f(\bar\omega, \bar w)|,$$

$$|f|_\theta = \operatorname*{ess\,sup}_{\bar\omega\in\Omega} \sup_{\bar w\ne\bar w'} \frac{|f(\bar\omega, \bar w) - f(\bar\omega, \bar w')|}{d_\theta(\bar w, \bar w')},$$

and define

$$\mathcal{H}_\infty = \{f : \Omega \times \mathcal{W}^{\mathbb{N}} \to \mathbb{C} \,:\, \|f\|_\infty < \infty\},$$

$$\mathcal{H}_\theta = \{f : \Omega \times \mathcal{W}^{\mathbb{N}} \to \mathbb{C} \,:\, \|f\|_\theta := \|f\|_\infty + |f|_\theta < \infty\}.$$

The space $\mathcal{H}_\theta$ is Banach algebra. The following result about density of $\mathcal{H}_\nu$ is based on very classical ideas, but we include it here for completeness:

Lemma 4.13. *For any function* $\varphi \in L^1(\mathbb{P}_\star)$*, there exists* $(\varphi_\epsilon)_\epsilon \subset \mathcal{H}_\nu$ *such that* $\varphi_\epsilon \to \varphi$ *in* $L^1(\mathbb{P}_\star)$*. Moreover, if* φ *is bounded,* $(\varphi_\epsilon)_\epsilon$ *can be chosen such that* $\sup_\epsilon \|\varphi_\epsilon\|_\infty < \infty$*.*

Proof. We first consider the case where $\varphi = \mathbb{1}_A$ is the indicator function of a measurable set $A \subset \Omega_\star$. We endow $\Omega_\star$ with the metric $d_\star((\bar\omega,\bar w),(\bar\omega',\bar w')) = d_\Omega(\bar\omega,\bar\omega') + d_\nu(\bar w,\bar w')$, where d_Ω is any metric defining the product topology on Ω. The metric $d_\star$ defines the product topology on $\Omega_\star$. For any open set $O \subset \Omega_\star$, we define

$$\varphi_{k,O}(\bar\omega,\bar w) = \inf\{k\, d_\star((\bar\omega,\bar w),\Omega_\star \setminus O), 1\}.$$

We clearly have $0 \le \varphi_{k,O} \le \varphi_{k+1,O} \le \mathbb{1}_O \le 1$, and $\lim_k \varphi_{k,O}(\bar\omega,\bar w) = \mathbb{1}_O(\bar\omega,\bar w)$ for all $(\bar\omega,\bar w) \in \Omega_\star$. The function $\varphi_{k,O}$ is clearly lipschitzian with respect to the metric $d_\star$, which also implies that $\varphi_{k,O} \in \mathcal{H}_\nu$. Since $\mathbb{P}_\star$ is a probability measure on the compact metric space $\Omega_\star$, it is outer regular, see [22, Theorem 2.17]: for any Borel set $A \subset \Omega_\star$ and any $\epsilon > 0$, there exists an open set $O_\epsilon \subset \Omega_\star$ such that $A \subset O_\epsilon$ and $\mathbb{P}_\star(O_\epsilon \setminus A) \le \epsilon$. By the dominated convergence theorem

$$\lim_{k\to\infty} \int_{\Omega_\star} |\varphi_{k,O_\epsilon} - \mathbb{1}_{O_\epsilon}|\, d\mathbb{P}_\star = 0.$$

We choose $k_\epsilon \ge 0$ such that $\int_{\Omega_\star} |\varphi_{k,O_\epsilon} - \mathbb{1}_{O_\epsilon}|\, d\mathbb{P}_\star \le \epsilon$ and set $\varphi_\epsilon = \varphi_{k_\epsilon,O_\epsilon}$. By the above arguments, we have $\varphi_\epsilon \in \mathcal{H}_\nu$, $\|\varphi_\epsilon\|_\infty \le 1$, and

$$\|\varphi_\epsilon - \mathbb{1}_A\|_{L^1(\mathbb{P}_\star)} \le \|\varphi_\epsilon - \mathbb{1}_{O_\epsilon}\|_{L^1(\mathbb{P}_\star)} + \mathbb{P}_\star(O_\epsilon \setminus A) \le 2\epsilon,$$

which proves the convergence in $L^1(\mathbb{P}_\star)$. Next, assume $\varphi \in L^\infty$. Without loss of generality we can assume $\varphi \ge 0$ and $\|\varphi\|_\infty = 2$. Let $A_1 = \{\xi \in \Omega_\star \;:\; \varphi(\xi) \ge 1\}$ and

$$A_k = \left\{ \xi \in \Omega_\star \;:\; \varphi(\xi) \ge 2^{-k} + \sum_{j=0}^{k-1} \mathbb{1}_{A_j} 2^{-j} \right\}.$$

By construction $\|\varphi - \sum_{j=0}^{k-1} \mathbb{1}_{A_j} 2^{-j}\|_\infty \le 2^{-k}$, hence we can use the above approximations of the characteristic functions to approximate φ in L^1 with a sequence with norm bounded by 2. The case $\varphi \in L^1$ can be obtained by approximation by bounded functions. □

Next, we state a useful technical lemma.

Lemma 4.14. *Under assumptions* **(Pos)**, **(Exp)** *and* **(Ell)** *there exists* $\gamma_* \in L^\infty(\Omega,\mathbb{P})$, $\gamma_\star > 0$ *such that, for all* $n > n_\star$ *and* $\bar w \in \mathcal{W}^{\mathbb{N}}$, *we have*

$$\gamma_\star(\omega) \le \frac{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})} \le \gamma_\star(\omega)^{-1}$$

Proof. For all $n > n_\star$, we have

$$\frac{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})} = \frac{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})}{p(\bar\omega, n-1, \bar w_0 \dots \bar w_{n-2})} \frac{p(\bar\omega, n-1, \bar w_0 \dots \bar w_{n-2})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})}$$
$$= \mathbb{P}_{\bar\omega}(\bar w_{n-1} \mid \bar w_0 \dots \bar w_{n-2}) \frac{p(\bar\omega, n-1, \bar w_0 \dots \bar w_{n-2})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})}$$
$$\geq \left[\mathbb{P}_{\tau_{\bar w_0}\bar\omega}(\bar w_{n-1} \mid \bar w_1 \dots \bar w_{n-2}) - C_\# \nu^n\right] \frac{p(\bar\omega, n-1, \bar w_0 \dots \bar w_{n-2})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})}$$
$$= \left[\frac{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega, n-2, \bar w_1 \dots \bar w_{n-2})} - C_\# \nu^n\right] \frac{p(\bar\omega, n-1, \bar w_0 \dots \bar w_{n-2})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})}$$
$$= \left[1 - C_\# \nu^n \frac{p(\tau_{\bar w_0}\bar\omega, n-2, \bar w_1 \dots \bar w_{n-2})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})}\right] \frac{p(\bar\omega, n-1, \bar w_0 \dots \bar w_{n-2})}{p(\tau_{\bar w_0}\bar\omega, n-2, \bar w_1 \dots \bar w_{n-2})}$$
$$\geq (1 - C_\# \gamma_0^{-1} \nu^n) \frac{p(\bar\omega, n-1, \bar w_0 \dots \bar w_{n-2})}{p(\tau_{\bar w_0}\bar\omega, n-2, \bar w_1 \dots \bar w_{n-2})}$$
$$\geq \prod_{j=n_\star+1}^{n} (1 - C_\# \gamma_0^{-1} \nu^j) \frac{p(\bar\omega, n_\star+1, \bar w_0 \dots \bar w_{n_\star})}{p(\tau_{\bar w_0}\bar\omega, n_\star, \bar w_1 \dots \bar w_{n_\star})},$$

where we have used **(Exp)** at the third line and **(Ell)** at the last line. The lower bound follows then by **(Pos)**, provided that $C_\# \gamma_0^{-1} \nu^{n_\star} < 1$, which we can always ensure by eventually redefining $n_\star$. The upper bound can be established similarly.[6] □

Remark 4.15. *Note that a slight strengthening of* **(Pos)** *would imply that $\gamma_\star$ can be chosen to be constant.*[7] *Then Lemma 4.14 would imply: for each $k \in \mathbb{N}$, there exists $C_k > 0$ such that for $\mathbb{P}$-a.e. $\bar\omega \in \Omega$, all $n \geq 0$ and all $\bar w \in \mathcal{W}^{\mathbb{N}}$,*

$$C_k^{-1} \leq \sum_{(w_1,\dots,w_k)\in\mathcal{W}^k} \frac{p(\tau_{-(w_1+\dots+w_k)}\bar\omega, n+k, w_1 \dots w_k \bar w_0 \dots \bar w_{n-1})}{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})} \leq C_k.$$

Hence the all point of assumption **(Abs)** *rests in the uniformity with respect to k.*

We can now define what will turn out to be the potential associated to $\mathcal{L}_\star$:

Lemma 4.16. *There exists a measurable function $J : \Omega \times \mathcal{W}^{\mathbb{N}} \to \mathbb{R}^+$ such that for for all $n \geq 0$*

$$\operatorname*{ess\,sup}_{\bar\omega\in\Omega} \sup_{\bar w\in\mathcal{W}^{\mathbb{N}}} \left| \frac{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})} - J(\bar\omega, \bar w) \right| \leq C_\# \nu^n. \tag{4.3}$$

Moreover, J belongs to $\mathcal{H}_\theta$ for all $\nu \leq \theta < 1$, and $J(\bar\omega, \bar w) > 0$ for $\mathbb{P}$-a.e $\bar\omega \in \Omega$ and all $\bar w \in \mathcal{W}^{\mathbb{N}}$.

[6] Note however that **(Ell)** is not needed to prove the upper bound.

[7] That is, one could ask, for all $n > n_\star$, $\inf_{\omega\in\Omega} p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1}) > 0$, which holds in all the example we have in mind.

Proof. Let $p_n(\bar\omega,\bar w)=\frac{p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega,n-1,\bar w_1\dots\bar w_{n-1})}$. Using **(Exp)**, we have

$$
\begin{aligned}
|p_n(\bar\omega,\bar w)-p_{n+1}(\bar\omega,\bar w)| &= \frac{p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega,n,\bar w_1\dots\bar w_n)}\\
&\quad\times\left|\frac{p(\tau_{\bar w_0}\bar\omega,n,\bar w_1\dots\bar w_n)}{p(\tau_{\bar w_0}\bar\omega,n-1,\bar w_1\dots\bar w_{n-1})}-\frac{p(\bar\omega,n+1,\bar w_0\dots\bar w_n)}{p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})}\right|\\
&= \frac{p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega,n,\bar w_1\dots\bar w_n)}\left|\mathbb{P}_{\tau_{\bar w_0}\bar\omega}(\bar w_n\mid\bar w_1\dots\bar w_{n-1})-\mathbb{P}_{\bar\omega}(\bar w_n\mid\bar w_0\dots\bar w_{n-1})\right|\\
&\le C_\#\nu^n\frac{p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega,n,\bar w_1\dots\bar w_n)}.
\end{aligned}
$$

From **(Abs)**, substituting $\tau_{\bar w_0}\bar\omega$ to $\bar\omega$, we have

$$p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})\le C_\# p(\tau_{\bar w_0}\bar\omega,n-1,\bar w_1\dots\bar w_{n-1}), \tag{4.4}$$

from which it follows, using **(Ell)**, for all $n\ge n_\star$,

$$\frac{p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega,n,\bar w_1\dots\bar w_n)}\le C_\#\frac{p(\tau_{\bar w_0}\bar\omega,n-1,\bar w_1\dots\bar w_{n-1})}{p(\tau_{\bar w_0}\bar\omega,n,\bar w_1\dots\bar w_n)}\le C_\#\gamma_0^{-1}.$$

We thus get for all $n\ge n_\star$

$$|p_{n+1}(\bar\omega,\bar w)-p_n(\bar\omega,\bar w)|\le C_\#\nu^n,$$

and so for any $m\ge 0$,

$$|p_{n+m}(\bar\omega,\bar w)-p_n(\bar\omega,\bar w)|\le C_\#\sum_{k=0}^{m-1}\nu^{n+k}\le C_\#\nu^n. \tag{4.5}$$

It follows that $(p_n(\bar\omega,\bar w))_n$ is a Cauchy sequence for $\mathbb{P}$-a.e. $\bar\omega\in\Omega$ and all $\bar w\in\mathcal{W}^{\mathbb{N}}$, and has thus a limit $J(\bar\omega,\bar w)$. Taking the limit $m\to\infty$ in (4.5), we obtain (4.3) for all $n\ge n_\star$. From (4.4), it follows that $\|J\|_\infty<\infty$, which also allows to deduce (4.3) for all $n\ge 0$. The fact that $|J|_\theta<\infty$ for all $\nu\le\theta<1$ is a direct consequence of (4.3).

The positivity of J follows then by Lemma 4.14. □

Accordingly, $\log J$ is Hölder with respect to the usual metric on the shift. Hence it can be seen as a potential of a Gibbs measure. Of course, such a Gibbs measure is random, depending on $\bar\omega$, and non translation invariant, but it is a natural generalisation of the usual random walk in random environment situation in which one has a random Markov chain on $\mathcal{W}^{\mathbb{N}}$.

The transfer operator $\mathcal{L}_\star$ has the following expression:

Lemma 4.17. *For any $f\in L^1(\mathbb{P}_\star)$, we have*

$$\mathcal{L}_\star f(\bar\omega,\bar w)=\sum_{w\in\mathcal{W}}J(\tau_{-w}\bar\omega,w\bar w)f(\tau_{-w}\bar\omega,w\bar w). \tag{4.6}$$

Proof. We have to prove that, for all $f\in L^1(\mathbb{P}_\star)$ and $g\in L^\infty(\mathbb{P}_\star)$,

$$\int_{\Omega_\star} f\,g\circ\mathcal{F}_\star\,d\mathbb{P}_\star=\int_{\Omega_\star}\mathcal{L}_\star f\,g\,d\mathbb{P}_\star, \tag{4.7}$$

where $\mathcal{L}_\star f$ is given by (4.6). We first assume that both f and g are bounded, and depend only on $(\bar\omega, \bar w_0, \dots, \bar w_{k-1})$ for some $k \ge 1$. For any $n \ge k$, we have

$$\int_{\Omega_\star} f\, g \circ \mathcal{F}_\star \, d\mathbb{P}_\star = \int_\Omega \int_{\mathcal{W}^{\mathbb{N}}} f(\bar\omega, \bar w_0, \dots, \bar w_{k-1})\, g(\tau_{\bar w_0}\bar\omega, \bar w_1, \dots, \bar w_k) \mathbb{P}_{\bar\omega}(d\bar w)\mathbb{P}(d\bar\omega)$$
$$= \int_\Omega \sum_{\bar w_0, \dots, \bar w_n \in \mathcal{W}} p(\bar\omega, n+1, \bar w_0 \dots \bar w_n) f(\bar\omega, \bar w_0, \dots, \bar w_{k-1}) g(\tau_{\bar w_0}\bar\omega, \bar w_1, \dots, \bar w_k)\mathbb{P}(d\bar\omega)$$
$$= \sum_{\bar w_0, \dots, \bar w_n \in \mathcal{W}} \int_\Omega p(\tau_{-\bar w_0}\bar\omega, n+1, \bar w_0 \dots \bar w_n) f(\tau_{-\bar w_0}\bar\omega, \bar w_0, \dots, \bar w_{k-1}) g(\bar\omega, \bar w_1, \dots, \bar w_k)\mathbb{P}(d\bar\omega)$$
$$= \int_\Omega \left(\sum_{w \in \mathcal{W}} \frac{p(\tau_{-w}\bar\omega, n+1, w\bar w_0 \dots \bar w_{n-1})}{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})} f(\tau_{-w}\bar\omega, w, \bar w_0, \dots) \right) g(\bar\omega, \bar w)\mathbb{P}_\star(d\bar\omega, d\bar w),$$

where we have used the translation invariance of $\mathbb{P}$ at the third line. Taking the limit as $n \to \infty$ and using Lemma 4.16, we obtain (4.7). The result for general f and g is obtained by approximation. □

Define for each $k \ge 1$,

$$J_k(\bar\omega, \bar w) = \prod_{i=0}^{k-1} J(\mathcal{F}_\star^i(\bar\omega, \bar w)) = \lim_{n\to\infty} \frac{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})}{p(\tau_{\bar w_0 + \dots \bar w_{k-1}}\bar\omega, n-k, \bar w_k \dots \bar w_{n-1})}.$$

It is immediate to verify that, for any $f \in L^1(\mathbb{P}_\star)$,

$$\mathcal{L}_\star^k f(\bar\omega, \bar w) = \sum_{w^k \in \mathcal{W}^k} J_k(\tau_{-w^k}\bar\omega, w^k \bar w) f(\tau_{-w^k}\bar\omega, w^k\bar w)$$

where, for $w^k = (w_0, \dots, w_{k-1})$, $\tau_{-w^k} = \tau_{-(w_0 + \dots + w_{k-1})}$.

We introduce, for $w^k = (w_0, \dots, w_{k-1}) \in \mathcal{W}^k$, the map

$$\psi_{w^k}(\bar\omega, \bar w) = (\tau_{-w^k}\bar\omega, w^k \bar w),$$

so that

$$\mathcal{L}_\star^k f = \sum_{w^k \in \mathcal{W}^k} J_k \circ \psi_{w^k} f \circ \psi_{w^k}.$$

Lemma 4.18. *J_k belong to $\mathcal{H}_\theta$, $\nu \le \theta < 1$, for all $k \ge 1$.*

Proof. Recall the notation $p_n(\bar\omega, \bar w) = \frac{p(\bar\omega, n, \bar w_0 \dots \bar w n-1)}{p(\tau_{\bar w_0}\bar\omega, n-1, \bar w_1 \dots \bar w_{n-1})}$, and set

$$p_{n,k}(\bar\omega, \bar w) := \frac{p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1})}{p(\tau_{\bar w_0 + \dots \bar w_{k-1}}\bar\omega, n-k, \bar w_k \dots \bar w_{n-1})} = \prod_{i=0}^{k-1} p_{n-i}(\mathcal{F}_\star^i(\bar\omega, \bar w)).$$

By Lemma 4.16, we have

$$\left| J(\mathcal{F}_\star^i(\bar\omega, \bar w)) - p_{n-i}(\mathcal{F}_\star^i(\bar\omega, \bar w)) \right| \le C_\# \nu^{n-i},$$

and, consequently, using the inequality

$$\left| \prod_{i=0}^{k-1} a_i - \prod_{i=0}^{k-1} b_i \right| \le \sum_{i=0}^{k-1} |a_i - b_i| \prod_{j \ne i} \max\{a_j, b_j\},$$

valid for all non-negative sequences $(a_i), (b_i)$, we obtain

$$\begin{aligned}|J_k(\bar\omega,\bar w)-p_{n,k}(\bar\omega,\bar w)| &\le C_\# \nu^n \sum_{i=0}^{k-1}\nu^{-i}\prod_{j\neq i}\max\{\|J\|_\infty,\|p_{n-i}\|_\infty\}\\ &= C_k\nu^n.\end{aligned}$$

where C_k depends only on k, since $\|J\|_\infty<\infty$ by Lemma 4.16 and $\sup_n\|p_n\|_\infty<\infty$ by **(Abs)**. The lemma follows immediately. □

Lemma 4.19. *There exists $C_\#>0$ such that $C_\#^{-1}\le \mathcal{L}_\star^k\mathbb{1}(\bar\omega,\bar w)\le C_\#$ for all $k\ge 0$, $\mathbb{P}$-a.e. $\bar\omega\in\Omega$ and all $\bar w\in\mathcal{W}^{\mathbb{N}}$.*

Proof. This is a simple reformulation of **(Abs)**, as

$$\begin{aligned}\mathcal{L}_\star^k\mathbb{1}(\bar\omega,\bar w) &= \sum_{(w_0,\dots,w_{k-1})\in\mathcal{W}^k} J_k(\tau_{-(w_0+\dots+w_{k-1})}\bar\omega, w_0\dots w_{k-1}\bar w)\\ &= \lim_{n\to\infty}\sum_{(w_0,\dots,w_{k-1})\in\mathcal{W}^k}\frac{p(\bar\omega,n,\bar w_0\dots\bar w_{n-1})}{p(\tau_{\bar w_0+\dots\bar w_{k-1}}\bar\omega,n-k,\bar w_k\dots\bar w_{n-1})}.\end{aligned}$$

□

Lemma 4.20. *There exist $C_\#>0$ and $\xi\in(0,1)$ such that for all $n\ge 0$ and all $f\in\mathcal{H}_\nu$,*

$$\begin{aligned}\|\mathcal{L}_\star^n f\|_\infty &\le C_\#\|f\|_\infty,\\ \|\mathcal{L}_\star^n f\|_\nu &\le C_\#\xi^n\|f\|_\nu + C_\#\|f\|_\infty.\end{aligned}$$

Proof. For $f\in\mathcal{H}_\nu$, we have

$$|\mathcal{L}_\star^n f|\le \sum_{w^n\in\mathcal{W}^n} J_n\circ\psi_{w^n}|f|\circ\psi_{w^n}\le \|f\|_\infty\mathcal{L}_\star^n\mathbb{1}\le C_\#\|f\|_\infty,$$

by Lemma 4.19. This proves that $\|\mathcal{L}_\star^n f\|_\infty\le C\|f\|_\infty$. We also have, setting $\eta=\psi_{w^n}(\bar\omega,\bar w)$ and $\eta'=\psi_{w^n}(\bar\omega,\bar w')$

$$\begin{aligned}|\mathcal{L}_\star^n f(\bar\omega,\bar w)-\mathcal{L}_\star^n f(\bar\omega,\bar w')| &\le \sum_{w^n\in\mathcal{W}^n} J_n(\eta)|f(\eta)-f(\eta')| + \sum_{w^n\in\mathcal{W}^n}|J_n(\eta)-J_n(\eta')||f(\eta')|\\ &\le \left(\sum_{w^n\in\mathcal{W}^n} J_n(\eta)|f|_\nu + \sum_{w^n\in\mathcal{W}^n}|J_n|_\nu\|f\|_\infty\right) d_\nu(w^n\bar w, w^n\bar w')\\ &\le \left(\mathcal{L}_\star^n\mathbb{1}(\bar\omega,\bar w)|f|_\nu + (\sharp\mathcal{W})^n|J_n|_\nu\|f\|_\infty\right)\nu^n d_\nu(\bar w,\bar w').\end{aligned}$$

By Lemma 4.19, this shows that, for all $n\ge 0$ and $f\in\mathcal{H}_\nu$,

$$\begin{aligned}\|\mathcal{L}_\star^n f\|_\nu &\le C_\#\nu^n|f|_\nu + (1+(\sharp\mathcal{W})^n\nu^n|J_n|_\nu)\|f\|_\infty\\ &\le C_\#\nu^n\|f\|_\nu + C_n\|f\|_\infty.\end{aligned}$$

In particular, $\mathcal{L}_\star:\mathcal{H}_\nu\to\mathcal{H}_\nu$ is a continuous operator.

Take $k\ge 0$ such that the term $\widetilde\nu := C_\#\nu^k$ in front of $\|f\|_\nu$ is strictly less than 1 and set $\xi=\widetilde\nu^{\frac1k}$. Writing $n=qk+r$, with $0\le r<k$, we have, by iterating the previous inequality,

$$\begin{aligned}\|\mathcal{L}_\star^n f\|_\nu &\le \widetilde\nu^q\|\mathcal{L}_\star^r f\|_\nu + C_\# C_k(1-\widetilde\nu^{-1})\|f\|_\infty\\ &\le \widetilde\nu^q\sup_{r<k}\|\mathcal{L}_\star^r\|_{\mathcal{H}_\nu\to\mathcal{H}_\nu}\|f\|_\nu + C_\#\|f\|_\infty\\ &\le C_\#\xi^n\|f\|_\nu + C_\#\|f\|_\infty.\end{aligned}$$

□

Lemma 4.21. *There exists a continuous projection* $\Pi : L^1(\mathbb{P}_\star) \to L^1(\mathbb{P}_\star)$ *with* $\Pi(L^1(\mathbb{P}_\star)) = \ker(\mathrm{id} - \mathcal{L}_\star)$ *such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\star^k \to \Pi$$

in the strong operator topology.

Proof. For $h \in \mathcal{H}_\nu$, Lemma 4.20 implies that the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\star^k h\right\}_{n\geq 1}$ is bounded in $L^\infty(\mathbb{P}_\star)$. By the Banach-Alaoglu theorem, since $L^\infty(\mathbb{P}_\star)$ is the dual of $L^1(\mathbb{P}_\star)$, the set $\left\{\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\star^k h\right\}_{n\geq 1}$ is weakly relatively compact in $L^1(\mathbb{P}_\star)$. This holds for all $h \in \mathcal{H}_\nu$, which is dense in $L^1(\mathbb{P}_\star)$ by Lemma 4.13, and so by the Kakutani-Yosida theorem [9, VIII.5.2, 5.3], the operators $\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\star^k$ converge in the strong operator topology to the projection Π with range the set of fixed points of $\mathcal{L}_\star$ in $L^1(\mathbb{P}_\star)$ and kernel the closure of $(\mathrm{id} - \mathcal{L}_\star)(L^1(\mathbb{P}_\star))$. □

Define

$$h_\star = \Pi \mathbb{1} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\star^k \mathbb{1},$$

in $L^1(\mathbb{P}_\star)$. By Lemma 4.21,we have $\mathcal{L}_\star h_\star = h_\star$. We clearly have $\int_{\Omega_\star} h_\star d\mathbb{P}_\star = 1$, and the fact that $C_\#^{-1} \leq h_\star \leq C_\#$, $\mathbb{P}_\star$-a.e., is an immediate consequence of Lemma 4.19. Consequently, the probability measure $\mathbb{Q}_\star$ defined by

$$d\mathbb{Q}_\star = h_\star d\mathbb{P}_\star \tag{4.8}$$

is $\mathcal{F}_\star$-invariant and equivalent to $\mathbb{P}_\star$.

Next, we show that $\Pi(L^1(\mathbb{P}_\star))$ is the one-dimensional subspace generated by $h_\star$.Firstly, we prove that it is included in $\mathcal{H}_\nu$.

Lemma 4.22. *If* $f \in L^\infty(\mathbb{P}_\star)$ *and* $\mathcal{L}_\star f = f$, *then* $f \in \mathcal{H}_\nu$.[8]

Proof. Let $(\varphi_\epsilon)_\epsilon \subset \mathcal{H}_\nu$ such that $\|f - \varphi_\epsilon\|_{L^1(\mathbb{P}_\star)} = \mathcal{O}(\epsilon)$ and $\|\varphi_\epsilon\|_\infty = \mathcal{O}(1)$ by Lemma 4.13. We have

$$\begin{aligned} f = \mathcal{L}_\star^n f &= \mathcal{L}_\star^n \varphi_\epsilon + \mathcal{L}_\star^n (f - \varphi_\epsilon) \\ &=: \widehat{\varphi}_\epsilon^{(n)} + \gamma_\epsilon^{(n)}. \end{aligned}$$

This decomposition satisfies

$$\|\gamma_\epsilon^{(n)}\|_{L^1(\mathbb{P}_\star)} = \|\mathcal{L}_\star^n (f - \varphi_\epsilon)\|_{L^1(\mathbb{P}_\star)} \leq \|f - \varphi_\epsilon\|_{L^1(\mathbb{P}_\star)} = \mathcal{O}(\epsilon),$$

and

$$\|\widehat{\varphi}_\epsilon^{(n)}\|_\nu = \|\mathcal{L}_\star^n \varphi_\epsilon\|_\nu \leq C_\# \xi^n \|\varphi_\epsilon\|_\nu + C_\# \|\varphi_\epsilon\|_\infty = \mathcal{O}(\xi^n \|\varphi_\epsilon\|_\nu + 1),$$

using Lemma 4.20. If we choose n_ϵ such that $\xi^{n_\epsilon}\|\varphi_\epsilon\|_\nu = \mathcal{O}(1)$ and set $\widehat{\varphi}_\epsilon = \widehat{\varphi}_\epsilon^{(n_\epsilon)}$ and $\gamma_\epsilon = \gamma_\epsilon^{(n_\epsilon)}$, we then have $f = \widehat{\varphi}_\epsilon + \gamma_\epsilon$ with $\|\gamma_\epsilon\|_{L^1(\mathbb{P}_\star)} = \mathcal{O}(\epsilon)$ and $\|\widehat{\varphi}_\epsilon\|_\nu = \mathcal{O}(1)$.

For $\delta > 0$, we define

$$B_{\epsilon,\delta} = \{|f - \widehat{\varphi}_\epsilon| > \delta\} = \{|\gamma_\epsilon| > \delta\},$$

which satisfies $\mathbb{P}_\star(B_{\epsilon,\delta}) \leq \delta^{-1}\|\gamma_\epsilon\|_{L^1(\mathbb{P}_\star)}$ by Markov's inequality.

[8] Of course, it means that there exists an element in the equivalence class of f that belongs to $\mathcal{H}_\nu$.

For $\mathbb{P}$-a.e. $\bar{\omega} \in \Omega$ and all $\bar{w}, \bar{w}' \in \mathcal{W}^{\mathbb{N}}$ such that both $(\bar{\omega}, \bar{w})$ and $(\bar{\omega}, \bar{w}')$ do not belong to $B_{\epsilon,\delta}$, we have

$$\begin{aligned}|f(\bar{\omega}, \bar{w}) - f(\bar{\omega}, \bar{w}')| &\leq |\widehat{\varphi}_\epsilon(\bar{\omega}, \bar{w}) - \widehat{\varphi}_\epsilon(\bar{\omega}, \bar{w}')| + |\gamma_\epsilon(\bar{\omega}, \bar{w}) - \gamma_\epsilon(\bar{\omega}, \bar{w}')| \\ &\leq |\widehat{\varphi}_\epsilon|_\nu d_\nu(\bar{w}, \bar{w}') + |\gamma_\epsilon(\bar{\omega}, \bar{w})| + |\gamma_\epsilon(\bar{\omega}, \bar{w}')| \\ &\leq C d_\nu(\bar{w}, \bar{w}') + 2\delta.\end{aligned}$$

We set $B_\delta = \bigcap_{k\geq 0} \bigcup_{j\geq k} B_{2^{-j},\delta}$, which satisfies $\mathbb{P}_\star(B_\delta) = 0$, since

$$\mathbb{P}_\star\left(\bigcup_{j\geq k} B_{2^{-j},\delta}\right) = \mathcal{O}\left(\sum_{j\geq k} \|\gamma_{2^{-j}}\|_{L^1(\mathbb{P}_\star)}\right) = \mathcal{O}\left(\sum_{j\geq k} 2^{-j}\right) = o(1).$$

Thus, $B = \cup_{n\in\mathbb{N}} B_{1/n}$ is also of zero measure and, eventually changing f on the zero measure set B, we have $f \in \mathcal{H}_\nu$. □

We can now prove the main theorem:

Proof of Theorem 4.9. The probability measure $\mathbb{Q}_\star$ defined by (4.8) is $\mathcal{F}_\star$-invariant and equivalent to $\mathbb{P}_\star$. If $A \subset \Omega_\star$ is a $\mathcal{F}_\star$-invariant set, we have

$$\mathcal{L}_\star(\mathbb{1}_A h_\star) = \mathcal{L}_\star((\mathbb{1}_A \circ \mathcal{F}_\star) h_\star) = \mathbb{1}_A \mathcal{L}_\star(h_\star) = \mathbb{1}_A h_\star,$$

and so $\mathbb{1}_A h_\star$ is a fixed point of $\mathcal{L}_\star$ in $L^\infty(\mathbb{P}_\star)$. By Lemma 4.22, we have $\mathbb{1}_A h_\star \in \mathcal{H}_\nu$, and so $\mathbb{1}_A = h_\star^{-1}(h_\star \mathbb{1}_A) \in \mathcal{H}_\nu$ [9]. This implies that there exists $N_A > 0$ such that $\mathbb{1}_A(\bar{\omega}, \bar{w}) = \mathbb{1}_A(\bar{\omega}, w_0, \dots, w_{N_A-1})$.

By the invariance of A it follows, for each $m \geq N_A$,

$$\begin{aligned}\mathbb{1}_A(\bar{\omega}, w_0, \dots, w_{N_A-1}) &= \mathbb{1}_A \circ \mathcal{F}_\star^m(\bar{\omega}, w_0, \dots, w_{N_A-1}) \\ &= \mathbb{1}_A(\tau_{w_0+\dots+w_{m-1}}\bar{\omega}, w_m, \dots, w_{N_A+m-1}).\end{aligned} \tag{4.9}$$

By **(Pro)** we can choose m and $w_{N_A}, \dots, w_{m-1}$ such that $w_0 + \cdots w_{m-1} = 0$. It follows that $\mathbb{1}_A(\bar{\omega}, \bar{w}) = \mathbb{1}_A(\bar{\omega})$. But then equation (4.9) implies $\tau_{w_0+\dots+w_{m-1}} A \subset A$ for all $(w_0, \dots, w_{m-1}) \in \mathcal{W}^m$. Accordingly, A is invariant for the group generated by $\mathcal{W}$ and, by **(Pro)** again, it is either of zero or full measure due to ergodicity of $\mathbb{P}$, which concludes the proof. □

Lemma 4.23. *For all $f \in L^1(\mathbb{P}_\star)$, we have*

$$\Pi f = \left(\int_{\Omega_\star} f \, d\mathbb{P}_\star\right) h_\star.$$

Proof. For each $\varphi \in L^\infty$ we have

$$\begin{aligned}\int_{\Omega_\star} \varphi \Pi f d\Pi_* &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\Omega_\star} \varphi \mathcal{L}_\star^k f d\mathbb{P}_* = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\Omega_\star} \varphi \circ \mathcal{F}_\star^k \cdot f d\mathbb{P}_\star \\ &= \left[\int_{\Omega_\star} \varphi h_\star d\mathbb{P}_\star\right] \left[\int_{\Omega_\star} f d\mathbb{P}_\star\right]\end{aligned}$$

where, in the second inequality, we have used Lebesgue dominated convergence Theorem and, in the second line, we have used the Birkhoff theorem and the ergodicity of $\mathbb{Q}_\star$ (and hence of $f\mathbb{P}_\star$) established in Theorem 4.9. □

[9] $h_\star$ belongs to $\mathcal{H}_\nu$ by Lemma 4.22 and so does $h_\star^{-1}$ since $\inf h_\star > 0$.

4.4. **Application to deterministic walks in random environment.** Deterministic walks in random environment, as presented in Section 3, naturally define random processes as described in the previous subsections. Indeed, if $\mathcal{A} = \{(f_\alpha, \mathcal{M}, \mathcal{P}_\alpha)\}_{\alpha\in A}$ is a deterministic walk in random environment, where all maps f_α are non-singular with respect to some reference measure m on $\mathcal{M}$, and the initial condition is given by an absolutely continuous probability measure $d\mu = h_0 dm$, then the probabilities $p(\bar\omega, n, w_0 \dots w_{n-1})$ are given by

$$p(\bar\omega, n, w_0 \dots w_{n-1}) = \int_{\mathcal{M}} \mathcal{L}_{\bar\omega, z_{n-1}, w_{n-1}} \dots \mathcal{L}_{\bar\omega, z_0, w_0} h_0 dm, \tag{4.10}$$

as we have seen in Section 3. Recall that $w_n = e(\bar\omega_{z_n}, x_n)$, where $(x_n, z_n) = \mathbb{F}^n_{\bar\omega}(x_0, z_0)$.

Remark 4.24. *Note that we have a priori defined two different notions of environment as seen from the particle, in subsections 3.1 and 4.2, but the map* $\Phi : \Omega\times\mathcal{M} \to \Omega\times\mathcal{W}^{\mathbb{N}}$ *defined by* $\Phi(\bar\omega, x) = (\bar\omega, \bar w)$ *with* $\bar w = (w_n)_n$, *is a semi-conjugacy between* $(\Omega\times\mathcal{M}, \mathcal{F})$ *and* $(\Omega\times\mathcal{W}^{\mathbb{N}}, \mathcal{F}_\star)$, *and if the maps* f_α *are expansive, it is invertible a.e.*

If we are able to check the assumptions **(Pos), (Exp), (Abs), (Ell)** and **(Pro)**, then Theorem 4.9 applies, and we deduce the existence of a deterministic drift V.

An particular situation, which we have already encountered in Lemma 3.1, occurs when all maps f_α preserve the same invariant measure $d\lambda = h_0 dm$, and the set $\mathcal{P}_\alpha$ is deterministic, i.e. $G_{\alpha,w} = G_w$ does not depend on $\alpha \in A$. In this case, the measure $\mathbb{P}_\star$ on $\Omega\times\mathcal{W}^{\mathbb{N}}$ is invariant under $\mathcal{F}_\star$, since it is the push-forward of $\mathbb{P}_\star = \mathbb{P}\times\lambda$, which is $\mathcal{F}$-invariant and the condition **(Abs)** is automatically satisfied by Lemma 4.5.

Remark 4.25. *In this situation, Lemma 3.2 and Remark 4.10 are in agreement, since*

$$\begin{aligned}\sum_{w\in\mathcal{W}} w \int_\Omega p(\bar\omega, 1, w)\mathbb{P}(d\bar\omega) &= \sum_{w\in\mathcal{W}} w \int_\Omega \int_{\mathcal{M}} \mathcal{L}_{\bar\omega,0,w} h_0 \, dm \, \mathbb{P}(d\bar\omega) \\ &= \sum_{w\in\mathcal{W}} w \int_\Omega \int_{\mathcal{M}} \mathcal{L}_{f_{\bar\omega_w}} \mathbb{1}_{G_w} h_0 \, dm \, \mathbb{P}(d\bar\omega) \\ &= \sum_{w\in\mathcal{W}} w \int_{G_w} h_0 dm\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_\star(e\circ\pi) &= \int_\Omega \int_{\mathcal{M}} e(\pi(\bar\omega, x)) h_0(x) m(dx) \mathbb{P}(d\bar\omega) \\ &= \int_\Omega \int_{\mathcal{M}} e(\bar\omega_0, x) h_0(x) m(dx) \mathbb{P}(d\bar\omega) \\ &= \sum_{w\in\mathcal{W}} w \int_{G_w} h_0 dm.\end{aligned}$$

5. Examples

It is an easy exercise to verify that the Random Lorentz gas presented in Section 2 is a special example of a deterministic walk in random environment. Note however that the Random Lorentz gas has several special features that could make much easier its study:

(1) the maps f_α have all the same invariant measure λ, the Lebesgue measure;
(2) the set $\mathcal{P}_\alpha$ is non random, i.e. it does not depend from α;
(3) the dynamics is reversible, in particular $\lambda(e) = 0$.

On the other hand it has a rather complex dynamics that is very hard to study. It is then reasonable to consider models where some of such properties do not hold, but the local dynamics is much simpler. It seems likely that if one is able to develop a sensible approach for such simpler models, then a similar line of attack could work also for billiards.

Interestingly, even super simple models yield a very rich set of probabilistic walks.

5.1. **Markovian models.** To try to get a better feeling for the difficulties involved in studying the above questions, let us try to invent a model stripped of all the technical difficulties present in the Lorentz gas dynamics. For simplicity let us discuss the case $d = 1$, although similar considerations hold in any higher dimensional lattice. To simplify the dynamics f in (2.3) let us suppose that it is a map from $[0, 1]$ to itself. Hence the map $\mathbb{F}_{\bar\omega}$ acts on $[0, 1]\times\mathbb{Z}$. Also, we assume that the environment is a random variable distributed according to a Bernoulli product measure over the space $\Omega = A^{\mathbb{Z}} = \{-1, 1\}^{\mathbb{Z}}$.

Example 1. The dynamics is defined by the map $f_\alpha(x) = 4x \mod 1$ for $\alpha \in A$, with $G_{-1,-1} = [0, 1/4]$, $G_{-1,+1} = [1/4, 1]$ and $G_{+1,-1} = [0, 3/4]$, $G_{+1,+1} = [3/4, 1]$.

Remark 5.1. *Here we are considering a more general situation than the one described for the Lorentz gas insofar also the gates are random. This is indeed the general case also for the Lorentz gas. We considered the case of deterministic gates only to simplify the exposition.*

Also, we consider the initial distribution $h_0 = 1$. Then an elementary computation shows that

$$\mathbb{P}_\star(z(n+1) - z(n) = \pm 1 \mid \bar\omega, z(n), \dots, z(0)) = \left|G_{\bar\omega_{z(n)},\pm 1}\right| = \frac{1}{2} \mp \frac{\bar\omega_{z(n)}}{4}.$$

This is an example of Sinai's walk, hence we do not have the classical CLT.

Example 2. Assume that $G_{\alpha,-1} = G_{-1} = [0, 1/2]$ and $G_{\alpha,+1} = G_{+1} = (1/2, 1]$ for any α and the maps are defined by

$$f_{-1}(x) = \begin{cases} 2x & x \in [0, 1/4] \\ 4x \mod 1 & x > 1/4 \end{cases}$$

$$f_{+1}(x) = \begin{cases} 4x \mod 1 & x \in [0, 3/4] \\ 2x - 1 & x > 3/4. \end{cases}$$

Again let us consider the initial distribution $h_0 = 1$. Denote by $\mathcal{L}_{\alpha,w}$ the operator $\mathcal{L}_{\alpha,w}(\phi) = \mathcal{L}_{f_\alpha}(\mathbb{1}_{G_w}\phi)$. The two dimensional vector space $\mathbb{V} = \{a_{-1}\mathbb{1}_{G_{-1}} + a_{+1}\mathbb{1}_{G_{+1}} : a_{-1}, a_{+1} \in \mathbb{R}\}$ is left invariant by the operators $\{\mathcal{L}_{\alpha,w}\}_{\alpha,w}$. Since $h_0 \in \mathbb{V}$, this allows to compute the transition probabilities by using formula (3.1).

If $\phi = a_{-1}\mathbb{1}_{G_{-1}} + a_{+1}\mathbb{1}_{G_{+1}}$, a direct computation shows that $\mathcal{L}_{\alpha,w}(\phi) = a_w \mathcal{L}_{\alpha,w}(\mathbb{1})$, and thus $\mathcal{L}_{\alpha',w'}\mathcal{L}_{\alpha,w}(\phi) = a_w \mathcal{L}_{\alpha',w'}\mathcal{L}_{\alpha,w}\mathbb{1}$. For any $\bar{\omega}$ and $z(1), \ldots, z(n), z(n+1)$, denote by $\alpha_k = \bar{\omega}_{z(k)+w(k)}$ and $w_k = w(k) = z(k+1) - z(k)$. We have

$$\mathbb{P}_\star(z(1), \ldots, z(n) \mid \bar{\omega}) = \int \mathcal{L}_{\alpha_{n-1},w_{n-1}} \ldots \mathcal{L}_{\alpha_0,w_0}\mathbb{1}.$$

Set $\phi = \mathcal{L}_{\alpha_{n-2},w_{n-2}} \ldots \mathcal{L}_{\alpha_0,w_0}\mathbb{1} = a_{-1}\mathbb{1}_{G_{-1}} + a_{+1}\mathbb{1}_{G_{+1}} \in \mathbb{V}$. We have

$$\mathbb{P}_\star(z(1), \ldots, z(n) \mid \omega) = \int \mathcal{L}_{\alpha_{n-1},w_{n-1}}\phi = a_{w_{n-1}} \int \mathcal{L}_{\alpha_{n-1},w_{n-1}}\mathbb{1}$$

and

$$\mathbb{P}_\star(z(1), \ldots, z(n), z(n+1) \mid \bar{\omega}) = a_{w_{n-1}} \int \mathcal{L}_{\alpha_n,w_n}\mathcal{L}_{\alpha_{n-1},w_{n-1}}\mathbb{1}.$$

It follows that

$$\mathbb{P}_\star(z(n+1) \mid z(1), \ldots, z(n), \bar{\omega}) = \frac{\int \mathcal{L}_{\alpha_n,w_n}\mathcal{L}_{\alpha_{n-1},w_{n-1}}\mathbb{1}}{\int \mathcal{L}_{\alpha_{n-1},w_{n-1}}\mathbb{1}}$$

which is a function of $z(n-1)$, $z(n)$ and $z(n+1)$ only. We have obtained a *persistent random walk*, that is a walk where the transition probability depends not only on the current position of the particle but also on its previous position.

Initial conditions. A natural question that arises at this point is what happens if one starts by a different initial measure. A moment thought shows that this is a non trivial issue. For instance, in the first example, there exists a Cantor set C (of zero Lebesgue measure) that corresponds to the coordinates $x(n)$ never belonging to $(1/4, 3/4)$. For such points $x \in C$, the set $\{x(n) \in G_{\alpha,w}\}$ does not depend on α, and so the process $(z(n))$ is completely unaffected by the environment. If we identify naturally the Cantor set C with $\{-1, +1\}^{\mathbb{N}}$ (in such a way that $x \in C$ is identified with the sequence (i_n) such that $x(n) \in I_{i_n}$ for all $n \geq 0$, where $I_{-1} = [0, 1/4]$ and $I_{+1} = [3/4, 1]$), then the initial distribution of x can be identified with a probability measure on $\{-1, +1\}^{\mathbb{N}}$ and this measure will be the distribution law of the random process $(w(n))$. In particular, if we consider the Bernoulli measure with equal probabilities on such a Cantor set as the initial distribution of x, then we obtain a standard random walk which has a very different behaviour than the Sinai's walk.

Without going to such extremes, one can (perhaps more naturally) start from a measure absolutely continuous with respect to Lebesgue and wonder which kind of process this will yield. We do not discuss this issue at present because is it part of the more general discussion that we will start in the next chapter.

Remark 5.2. *The above examples (among other obvious limitations) are unreasonable in one key aspect: their Markov structure. It is inevitable to ask what happens when the Markov structure is absent (as for billiards). The next section is devoted to investigating such a situation.*

6. Non-Markovian examples: general discussion

We now consider a model of d-dimensional deterministic random walk in random environment $\mathcal{A} = \{(f_\alpha, \mathcal{M}, \mathcal{P}_\alpha)\}_{\alpha \in A}$ for a finite set A, where $\mathcal{M} = [0, 1]$, all maps $f_\alpha : [0, 1] \to [0, 1]$ are piecewise C^2 and uniformly expanding (i.e. $|f'_\alpha| \geq \lambda > 1$), and the partitions $\mathcal{P}_\alpha = \{G_{\alpha,w}\}_{w \in \mathcal{W}}$ are made of subintervals of $[0, 1]$, for a given bounded subset $\mathcal{W} \subset \mathbb{Z}^d$.

Let $\mathbb{P}$ be a translation invariant probability on the set $\Omega = A^{\mathbb{Z}^d}$. For a given environment $\bar{\omega} \in \Omega$, we have the dynamics $\mathbb{F}_{\bar{\omega}}(\cdot, \cdot) : \mathcal{M} \times \mathbb{Z}^d \to \mathcal{M} \times \mathbb{Z}^d$ given by $\mathbb{F}_{\bar{\omega}}(x, z) = (f_{\bar{\omega}_{z+e(\bar{\omega}_z, x)}}(x), z + e(\bar{\omega}_z, x))$, where $e(\alpha, x) = \sum_{w \in \mathcal{W}} \mathbb{1}_{G_{\alpha,w}}(x) w$.

We are interested in the quenched evolution, $(x_n, z_n) = \mathbb{F}^n_{\bar{\omega}}(x_0, z_0)$, of such a system when the initial condition x_0 is distributed according to the measure

$$\mu(\varphi) = \int_0^1 \varphi(x, 0) h_0(x)\, dx$$

for some $h_0 \in \mathrm{BV}$, with $\inf h_0 > 0$.

Recall the definition of the probability measure $\mathbb{P}_\star$ and the dynamical system $(\Omega_\star, \mathcal{F}_\star)$ of the point of view of the particle, from Section 4.

Our goal is to reduce the study of this model to a situation as similar as possible to a conventional r.w.r.e. situation, as exposed in the previous sections. To this end we will need some technical conditions. We state them in the following subsection, and then we state the results.

6.1. Conditions *(C1), (C2), (C3).*

Let $\mathcal{T} = \{f_\alpha\}_{\alpha \in A}$ be the (finite) set of all the possible maps on $[0, 1]$, and $\mathcal{H}$ be the (finite) set of all the possible intervals of the partitions, i.e. $\mathcal{H} = \{G_{\alpha,w}\}_{w \in \mathcal{W}, \alpha \in A}$.

Notation. *The set $\mathcal{T} \times \mathcal{H}$ is canonically isomorphic to $A \times (A \times \mathcal{W})$, $\rho((\alpha, \beta)) = (f_\alpha, H_\beta)$ being the correspondence.*

From now on we will write $(T_\sigma, H_\sigma) = (f_{\pi_1 \circ \rho^{-1}(\sigma)}, H_{\pi_2 \circ \rho^{-1}(\sigma)})$, $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2(\alpha, \beta) = \beta$.

Set $\Sigma = \mathcal{T} \times \mathcal{H}$ and let $\tau : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the unilateral shift. For $\sigma \in \Sigma^{\mathbb{N}}$, we write $\sigma = (\sigma_1, \sigma_2, \ldots)$ with $\sigma_k = (T_{\sigma_k}, H_{\sigma_k})$. We denote $T^n_\sigma = T_{\sigma_n} \circ \ldots \circ T_{\sigma_1}$ and $H^n_\sigma = \bigcap_{j=0}^{n-1} (T^j_\sigma)^{-1}(H_{\sigma_{j+1}})$.

Let $\mathcal{L}_{\sigma_k}$ be the transfer operator of the map T_{σ_k} with respect to the Lebesgue measure, i.e.

$$\mathcal{L}_{\sigma_k} f(x) = \sum_{T_{\sigma_k} y = x} \frac{f(y)}{|T'_{\sigma_k}(y)|}.$$

We set $\widehat{\mathcal{L}}_{\sigma_k} f = \mathcal{L}_{\sigma_k}(f \mathbb{1}_{H_{\sigma_k}})$ and $\widehat{\mathcal{L}}^n_\sigma = \widehat{\mathcal{L}}_{\sigma_n} \circ \ldots \circ \widehat{\mathcal{L}}_{\sigma_1}$. We can write $\widehat{\mathcal{L}}^n_\sigma f(x) = \sum_{T^n_\sigma y = x} g^n_\sigma(y) f(y)$, where $g^n_\sigma = g_{\sigma_1} \times \ldots \times g_{\sigma_n} \circ T^{n-1}_\sigma$, with $g_{\sigma_k} = \mathbb{1}_{H_{\sigma_k}} \frac{1}{|T'_{\sigma_k}|}$. Let $D > 0$ and $0 < \Theta < 1$ be such that $\|g^n_\sigma\|_\infty \le D\Theta^n$ for all n and $\sigma \in \Sigma^{\mathbb{N}}$. Note that we can choose $D = 1$ and $\Theta^{-1} = \inf_{T \in \mathcal{T}} \inf_x |T'(x)|$.

We will only consider systems that satisfy

(C1): There exists $\delta > 0$ such that for all $\sigma_1 \in \mathcal{T} \times \mathcal{H}$, $\inf \widehat{\mathcal{L}}_{\sigma_1} \mathbb{1} \ge \delta$.

Observe that this condition is satisfied if, for any choice of T and H, T admits at least one full branch inside H. By iteration, we also have $\inf \widehat{\mathcal{L}}^n_\sigma \mathbb{1} \ge \delta^n$ for any $\sigma \in \Sigma^{\mathbb{N}}$ and $n \ge 1$.

Next, we define the functionals

$$\Lambda_\sigma(f) = \lim_{n \to \infty} \inf \frac{\widehat{\mathcal{L}}^n_\sigma f}{\widehat{\mathcal{L}}^n_\sigma \mathbb{1}}.$$

This limit is well defined as the limit of an increasing and bounded sequence. Indeed,

$$\inf \frac{\widehat{\mathcal{L}}_\sigma^{n+1} f}{\widehat{\mathcal{L}}_\sigma^{n+1}\mathbb{1}} \geq \inf \frac{\widehat{\mathcal{L}}_{\sigma_{n+1}}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1} \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}})}{\widehat{\mathcal{L}}_\sigma^{n+1}\mathbb{1}}$$
$$\geq \inf \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} \inf \frac{\widehat{\mathcal{L}}_{\sigma_{n+1}}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})}{\widehat{\mathcal{L}}_\sigma^{n+1}\mathbb{1}} = \inf \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}};$$

and $-\|f\|_\infty \leq \inf \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} \leq \|f\|_\infty$.

The above functional satisfies the following properties:

- $\Lambda_\sigma(\mathbb{1}) = 1$;
- $|\Lambda_\sigma(f)| \leq \|f\|_\infty$;
- $f \geq g$ implies $\Lambda_\sigma(f) \geq \Lambda_\sigma(g)$ (monotonicity);
- $\Lambda_\sigma(\lambda f) = \lambda \Lambda_\sigma(f)$, for $\lambda > 0$ (positive homogeneity);
- $\Lambda_\sigma(f+g) \geq \Lambda_\sigma(f) + \Lambda_\sigma(g)$ (super-additivity);
- $\Lambda_\sigma(f+b) = \Lambda_\sigma(f) + b$ for all $b \in \mathbb{R}$.

All the above follows immediately from the definition. Note that it is not clear at the moment if Λ_σ is linear or not.

We set

$$\rho_\sigma = \Lambda_{\tau\sigma}(\widehat{\mathcal{L}}_{\sigma_1}\mathbb{1})\ ; \quad \rho = \inf_{\sigma \in \Sigma^{\mathbb{N}}} \rho_\sigma. \tag{6.1}$$

Let $\mathcal{Z}_\sigma^n$ be the partition of smoothness intervals of T_σ^n, and $\widehat{\mathcal{Z}}_\sigma^n$ be the coarsest partition which is finer than $\mathcal{Z}_\sigma^n$ and enjoying the property that the elements of the partition are either disjoint or contained in H_σ^n.

Let us define the collections of intervals

$$\begin{aligned} \mathcal{Z}_{\sigma,\star}^n &= \{Z \in \widehat{\mathcal{Z}}_\sigma^n \,|\, Z \subset H_\sigma^n\}, \\ \mathcal{Z}_{\sigma,b}^n &= \{Z \in \mathcal{Z}_{\sigma,\star}^n \,|\, \Lambda_\sigma(\mathbb{1}_Z) = 0\} \\ \mathcal{Z}_{\sigma,g}^n &= \{Z \in \mathcal{Z}_{\sigma,\star}^n \,|\, \Lambda_\sigma(\mathbb{1}_Z) > 0\}. \end{aligned} \tag{6.2}$$

Definition 1. *We will call contiguous two elements of $\mathcal{Z}_{\sigma,\star}^n$ that are either contiguous in the usual sense, or separated by a connected component of $(H_\sigma^n)^c = \bigcup_{j=0}^{n-1}(T_\sigma^j)^{-1}(H_{\sigma_{j+1}}^c)$.*

We can now introduce the other conditions needed to state our results.

(C2): there exist constants $K \geq 0$ and $\xi \geq 1$ such that for any n and $\sigma \in \Sigma^{\mathbb{N}}$, at most $K\xi^n$ elements of $\mathcal{Z}_{\sigma,b}^n$ are contiguous. In addition, $\theta := \xi\Theta < \rho$. In particular, $\rho > 0$.

(C3(N, N')): there exists $\epsilon_n > 0$ such that $\inf \frac{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}_Z}{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}} \geq \epsilon_n$ for all $n \leq N$ and $Z \in \mathcal{Z}_{\sigma,g}^n$, and for any sequence $\sigma \in \Sigma^{\mathbb{N}}$. In particular, $\Lambda_\sigma(\mathbb{1}_Z) \geq \epsilon_n$.

6.2. **The results.**

We are finally able to state our main result.

Theorem 6.1. *There exists an integer $n_2 \geq 1$ depending on the classes $\mathcal{T}$ and $\mathcal{H}$, explicitly computable (see Remark 9.10), such that if* (C1), (C2) *and* (C3(n_2, n_3)) *hold for some $n_3 \geq n_2$, then the condition* **(Exp)** *holds. In particular, the property of loss memory from Lemma 4.3 is verified.*

As we already pointed out in Section 5, the choice of the initial condition might play an important role. The following result states that if we restrict ourselves to initial conditions absolutely continuous to Lebesgue, with density in BV bounded uniformly away from 0, this difference is not so important in the sense that for large times, the transition probabilities are exponentially close. For two different initial densities $h_0, h_0' \in \mathrm{BV}$ with $\inf h_0 > 0$ and $\inf h_0' > 0$, we denote by $\mathbb{P}_\star$ and $\mathbb{P}_\star'$ the probability measures corresponding to h_0 and h_0' respectively.

Theorem 6.2. *Under the assumptions of Theorem 6.1, we have for all realisation of the environment $\bar\omega \in \Omega$, $n \geq 0$ and all densities h_0, h_0' as above:*

$$|\mathbb{P}_\star(z(n) \mid z(1), \dots, z(n-1), \bar\omega) - \mathbb{P}_\star'(z(n) \mid z(1), \dots, z(n-1), \bar\omega)| \leq C_{h_0,h_0'} \nu^n,$$

where $C_{h_0,h_0'} > 0$ depends only on the densities h_0 and h_0'.

Next, we consider the situation where all maps f_α preserve a common density $h_0 \in \mathrm{BV}$ such that $\inf h_0 > 0$, and when the partitions $\mathcal{P}_\alpha$ are deterministic, i.e. $G_{\alpha,w} = G_w$ does not depend on $\alpha \in A$. In this situation, the dynamical system $(\Omega_\star, \mathcal{F}_\star, \mathbb{P}_\star)$ is measure-preserving, and so condition **(Abs)** holds by Lemma 4.5.

Theorem 6.3. *Under the assumptions of Theorem 6.1, if the maps f_α preserve a common density, the partitions $\mathcal{P}_\alpha$ are deterministic, and if furthermore condition* **(Pro)** *of Section 4.2 holds, then the dynamical system $(\Omega_\star, \mathcal{F}_\star, \mathbb{P}_\star)$ is ergodic. In particular, for $\mathbb{P}$-a.e. $\bar\omega \in \Omega$ and Lebesgue-almost every x_0,*

$$\lim_{n\to\infty} \frac{1}{n} z(n) = \sum_{w \in \mathcal{W}} w \int_{G_w} h_0 dm.$$

Remark 6.4. *The assumption that the maps all preserve a common measure and that the partitions are deterministic is only used to check the validity of condition* **(Abs)** *thanks to Lemma 4.5, and to have an explicit formula for the drift. If for a concrete example, one is able to check* **(Abs)** *by any other mean, then Theorem 4.9 applies and there exists $V \in \mathbb{R}^d$ such that $\lim_{n\to\infty} \frac{1}{n} z(n) = V$ for $\mathbb{P}$-a.e. $\bar\omega \in \Omega$ and Lebesgue-almost every x_0.*

The proofs of Theorems 6.1, 6.2 and 6.3 will be provided in Section 8.

7. Existence of Non-Markovian examples: β-maps

The conditions under which Theorem 6.1 holds look rather convoluted, so the reader might wonder if examples that satisfy them exist at all. We acknowledge that the conditions are a bit contrived, yet they are checkable: they pertain only the properties of a finite set of maps and gates. Of course, to check them in a specific situation might be laborious, nevertheless to ensure that they are not empty it suffices to verify them in some limiting regime, for example when the dynamics has a lot of expansion. This is the aim of the present section. To further simplify things we will limit ourselves to β maps, a rather popular class of maps in the field of dynamical systems.

7.1. General β-maps.

More precisely, we consider the situation where the class of maps is $\mathcal{T} = \{T_{\beta_1}, T_{\beta_2}\}$ for $\beta_2 > \beta_1 > 1$, with $T_\beta(x) = \beta x \mod 1$ and the partitions $\mathcal{P}_\alpha$ are such that $\mathcal{H} = \{[0, \frac{1}{2}], (\frac{1}{2}, 1]\}$. Note that $\mathcal{W}$ is not specified, and

$\mathcal{P}_\alpha$ can be random [10]. For $\sigma \in \Sigma^{\mathbb{N}} = (\mathcal{T} \times \mathcal{H})^{\mathbb{N}}$, we denote by β_{σ_i} the value of β such that $T_{\sigma_i} = T_{\beta_{\sigma_i}}$. For simplicity, we will fix $\varpi > 1$ and assume that $\beta_1 = \beta$ and $\beta_2 = \varpi\beta$. We will show that assumptions *(C1)*, *(C2)* and *(C3)* are verified for a large set of β. When needed, we will denote by $\mathcal{Z}^n_\sigma(\beta)$ the partition of smoothness intervals of T^n_σ, to emphasize the dependence on β, and similarly for the objects defined in equation (6.2). We will do the same with the subsets of this partition introduced in Section 6.

Proposition 7.1. *There exists a set $\mathcal{B} \subset (1, \infty)$, with $\mathrm{Leb}(\mathcal{B} \cap (1, t)) = \mathcal{O}(\log t)$ as $t \to \infty$, such that the model described above satisfies the assumptions of Theorem 6.1 when $\beta \notin \mathcal{B}$.*

The rest of the section is devoted to the proof of Proposition 7.1. If suffices to check conditions *(C1), (C2)* and *(C3).*

7.1.1. *Condition* (C1). This condition is satisfied if every map $T \in \mathcal{T}$ admits at least one full branch inside any interval $H \in \mathcal{H}$. This is the case whenever $\beta \geq 3$.

7.1.2. *Condition* (C2). We first give a general criterion to check this condition. Let $\mathcal{Z}^n_{\sigma,f}$ be the collection of elements Z in $\mathcal{Z}^n_{\sigma,\star}$ such that $T^n_\sigma Z = [0, 1]$, and let $\mathcal{Z}^n_{\sigma,u} = \mathcal{Z}^n_{\sigma,\star} \setminus \mathcal{Z}^n_{\sigma,f}$.

We will say the system is ξ-full branched, with $\xi > 0$, if there exists $K > 0$ such that for all $\sigma \in \Sigma$ and all n, the number of contiguous elements in $\mathcal{Z}^n_{\sigma,u}$ does not exceed $K\xi^n$.

Clearly, a system ξ-full branched satisfies the condition *(C2)* with the same $\xi > 0$.

Lemma 7.2. Calling C^n_σ the maximal number of contiguous elements in $Z^n_{\sigma,u}$, holds

$$C^n_\sigma \leq 2 \sum_{i=0}^{n-1} (C^{(1)} + 2)^i C^{(1)},$$

where $C^{(1)}$ is the supremum over all σ of C^1_σ.

Proof. The proof is by induction on n. Clearly it is true for $n = 1$. Let us suppose it true for n. The elements of the partition $\mathcal{Z}^{n+1}_{\sigma,\star}$ are formed by $\{T^{-1}_{\sigma_1} Z \cap Z_1\}$ where $Z \in \mathcal{Z}^n_{\tau\sigma,\star}$ and $Z_1 \in \mathcal{Z}^1_{\sigma,\star}$. Now, if $Z_1 \in \mathcal{Z}^1_{\sigma,f}$, the elements maintain the same nature, i.e. if $Z \in \mathcal{Z}^n_{\tau\sigma,f}$ (resp. $\mathcal{Z}^n_{\tau\sigma,u}$) then $T^{-1}_{\sigma_1} Z \cap Z_1 \in \mathcal{Z}^{n+1}_{\sigma,f}$ (resp. $\mathcal{Z}^{n+1}_{\sigma,u}$). So we have in Z_1 at most $C^n_{\tau\sigma}$ contiguous elements of $\mathcal{Z}^{n+1}_{\sigma,u}$. The only problem can arise when a block of contiguous elements ends at the boundary of Z_1 since in such a case it can still be contiguous to others elements of $\mathcal{Z}^{n+1}_{\sigma,u}$. Yet, if the contiguous elements of Z_1 are in $\mathcal{Z}^1_{\sigma,f}$, then there can be at most a block of length $2C^n_{\tau\sigma}$. One must then analyze what can happen if $Z_1 \in \mathcal{Z}^1_{\sigma,u}$. In this case, a set of contiguous elements can either have only partial preimage in Z_1, hence we get a shorter group of contiguous elements, or all the group can have preimage. In this last case, the worst case scenario is when the elements contiguous to the groups (that must belong to $\mathcal{Z}^n_{\tau\sigma,f}$) are cut while taking preimages. This means that at most two new contiguous elements can be generated, but in this case the group must end at the boundary of Z_1. Since there are at most C^1_σ contiguous elements in $\mathcal{Z}^1_{\sigma,u}$ in this way we can generate at most $C^1_\sigma(C^n_{\tau\sigma} + 2)$ contiguous elements that,

[10] For instance, $\mathcal{W} = \{-1, +1\}$ and whether $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ correspond to $-1, +1$ or $+1, -1$ respectively is random.

again in the worst case scenario, can be contiguous to two blocks belonging to the neighboring elements in $\mathcal{Z}^1_{\sigma,f}$. Accordingly,

$$C^{n+1}_\sigma \le C^1_\sigma(C^n_{\tau\sigma}+2)+2C^n_{\tau\sigma} = (C^1_\sigma+2)C^n_{\tau\sigma}+2C^1_\sigma \le 2\sum_{i=0}^{n}(C^{(1)}+2)^iC^{(1)},$$

where we have used the induction hypothesis. □

This lemma implies that any system is ξ-full branched with $\xi = C^{(1)}+2$. We can also set $K = \frac{2C^{(1)}}{C^{(1)}+1}$ in condition *(C2)*.

In order to check condition *(C2)* one has also to estimate ρ. We can first remark that $\rho_\sigma \ge \inf \widehat{\mathcal{L}}_{\sigma_1}\mathbb{1}$. Letting N to be the minimal number of onto branches of T inside H for any $T \in \mathcal{T}$ and $H \in \mathcal{H}$, the following holds:

$$\widehat{\mathcal{L}}_{\sigma_1}\mathbb{1}(x) = \sum_{T_{\sigma_1}y=x}\mathbb{1}_{H_{\sigma_1}}(y)\frac{1}{|T'_{\sigma_1}(y)|} \ge \frac{N}{M},$$

with $M = \sup_T |T'(x)|$. Thus, we have $\rho \ge \frac{N}{M}$, and condition *(C2)* is satisfied if $(C^{(1)}+2)\Theta < \frac{N}{M}$.

For our example with β transformations, we have $C^{(1)} = 2$ and thus $\xi = 4$ and $K = \frac{4}{3}$. We also have $\Theta = \beta^{-1}$, $M = \varpi\beta$ and $N \ge \lfloor\frac{\beta}{2}\rfloor - 1$. Condition *(C2)* is then satisfied if $4\varpi\beta < \beta(\lfloor\frac{\beta}{2}\rfloor - 1)$, which is the case if $\beta \ge 8\varpi + 4$.

7.1.3. *Condition* (C3). We need the following:

Lemma 7.3. *For each $m \ge 1$, there exists a set $\mathcal{B}_m \subset (1,\infty)$ with $\mathrm{Leb}(\mathcal{B}_m \cap (1,t)) = \mathcal{O}(\log t)$ as $t \to \infty$, such that for any $\beta > 1$ with $\beta \notin \mathcal{B}_n$,*

(7.1) $\forall n \le m,\ \forall\sigma \in \Sigma^{\mathbb{N}},\ \forall Z \in \mathcal{Z}^n_{\sigma,\star}(\beta),\ T^{n+1}_\sigma(Z\cap H^{n+1}_\sigma) = [0,1]$ *or* $T^n_\sigma Z \subset H^c_{\sigma_{n+1}}$.

This lemma establishes a strong dichotomy between good and bad elements of $\mathcal{Z}^n_{\sigma,\star}(\beta)$ when $\beta \notin \mathcal{B}_m$: either $T^n_\sigma Z \cap H_{\sigma_{n+1}}$ is large enough to cover $[0,1]$ after one more iteration if Z is good, or it is empty if Z is bad.

Lemma 7.3 implies that if $\beta \notin \bigcup_{k\le n}\mathcal{B}_k$ then *(C3(n, n+1))* is satisfied with $\epsilon_n = \frac{1}{(2\varpi\beta)^{n+1}}$. Indeed, for $Z \in \mathcal{Z}^k_{\sigma,g}(\beta)$, $k \le n$, we have $T^{n+1}_\sigma(Z\cap H^{n+1}_\sigma) = T^{n-k}_{\tau^{k+1}\sigma}(T^{k+1}_\sigma(Z\cap H^{k+1}_\sigma)\cap H^{n-k}_{\tau^{k+1}\sigma}) = [0,1]$ and then

$$\widehat{\mathcal{L}}^{n+1}_\sigma\mathbb{1}_Z(x) = \sum_{T^{n+1}_\sigma y=x}\frac{\mathbb{1}_Z(y)\mathbb{1}_{H^{n+1}_\sigma}(y)}{|(T^{n+1}_\sigma)'(y)|} \ge \frac{1}{\sup|(T^{n+1}_\sigma)'|} \ge \frac{1}{(\varpi\beta)^{n+1}},$$

and $\widehat{\mathcal{L}}^{n+1}_\sigma\mathbb{1}(x) \le \mathcal{L}^{n+1}_\sigma\mathbb{1}(x) \le 2^{n+1}$, since $\mathcal{L}_{T_{\beta_i}}\mathbb{1}(x) \le 2$.

The above facts suffice to prove Proposition 7.1:

Proof of Proposition 7.1. We can choose $\mathcal{B} = (1,\beta_0)\cup\bigcup_{k=1}^n\mathcal{B}_k$ for $\beta_0 > 1$ and $n \ge 1$ large enough. From our analysis of conditions *(C1)* and *(C2)*, we must choose $\beta_0 > 8\varpi + 4$. We use Remark 9.10 to determine the value of n_2 given in Theorem 6.1.

We first show that when β is large enough, we can choose $n_0 = 1$. By Remark 9.4, the constant $C_\star$ can be chosen so that $D(3C+5)+(3C+2)2K\xi^n) \le C_\star\xi^n$ for all n. In our situation, all maps are piecewise linear, and then $C = 0$. Since $D = 1$, $K = \frac{4}{3}$ and $\xi = 4$, we thus need $(5 + \frac{16}{3}4^n) \le C_\star 4^n$. This is verified for $C_\star = \frac{31}{3}$. To have $n_0 = 1$, by Remark 9.10, we have to be able to choose $\eta \in (\theta\rho^{-1}, 1)$ so that

$C_\star^{\frac{1}{n}}\theta\rho^{-1} \leq \eta$ and $C_\star\eta^n \leq \frac{1}{4}$ for all $n \geq 1$. Since $C_\star > 1$ and $\eta < 1$, it is sufficient to check it for $n = 1$, i.e. we need

$$C_\star\theta\rho^{-1} \leq \eta \leq \frac{1}{4C_\star} \tag{7.2}$$

and $\theta\rho^{-1} < \eta < 1$. Since $\theta = \xi\Theta = 4\beta^{-1}$ and, by the analysis in subsection 7.1.2, $\rho^{-1} \leq \frac{\varpi\beta}{\lfloor\frac{\beta}{2}\rfloor - 1} = \mathcal{O}(1)$ as $\beta \to \infty$, we have $C_\star\theta\rho^{-1} = \mathcal{O}(\beta^{-1})$. Since $\frac{1}{4C_\star}$ is a constant independent of β, the set of η satisfying (7.2) is non empty for all β large enough.

Now, if $n_0 = 1$ and $\beta \notin \mathcal{B}_1$ then condition *(C3(1,2))* is satisfied with $\epsilon_1 = \frac{1}{(2\varpi\beta)^2}$ and then $C_1 = 2(2K\xi^1 + 1)\Theta^1\epsilon_1^{-1} = \mathcal{O}(\beta)$ by Remark 9.4. Consequently, $a_0 = 8C_1\rho^{-1} + \frac{C_1}{C_\star\theta^1} = \mathcal{O}(\beta^2)$, $a = \mathcal{O}(\beta^2)$ and $B = 1 + 2aC_\star = \mathcal{O}(\beta^2)$. We have $\delta^{-1} = 4aB(1 + 2C_1\rho^{-1}) = \mathcal{O}(\beta^5)$. Since $D = 1$, $\theta\rho^{-1} = \mathcal{O}(\beta^{-1})$ and $\delta^{-1} = \mathcal{O}(\beta^5)$, we see by Remark 9.10 that $n_2 = n_1(\delta) = 6$ is sufficient for β large enough, say for $\beta \geq \beta_0$. We can then set $\mathcal{B} = (1, \beta_0) \cup \bigcup_{k=1}^6 \mathcal{B}_k$ to conclude the proof, as for each $\beta \notin \mathcal{B}$, we have $n_2 = 6$ and $C(n_2, n_2 + 1)$ is satisfied. □

The rest of the section is devoted to the proof of Lemma 7.3. Due to the form of the elements of $\widehat{\mathcal{Z}}_\sigma^n(\beta)$, we have to discard β when the n-th iterates of elements of $\partial\widehat{Z}_\sigma^n(\beta)$ come too close to $\{0, \frac{1}{2}, 1\}$ as it will be made precise later. We have:

Lemma 7.4. *For all $\sigma \in \Sigma^\mathbb{N}$ and $n \geq 1$, one has $T_\sigma^n(\partial\widehat{\mathcal{Z}}_\sigma^n(\beta)) \subset \mathcal{Q}_n(\beta) := \{0, 1\} \cup \{T_{\sigma'}^i(1), T_{\sigma'}^i(\frac{1}{2}) \,/\, i = 1, \dots, n,\ \sigma' \in \Sigma^\mathbb{N}\}$.*

Proof. We proceed by induction, the result being clearly true for $n = 1$. Note that $\mathcal{Q}_n(\beta) \cup T_{\tau^n\sigma}^1(\mathcal{Q}_n(\beta)) \subset \mathcal{Q}_{n+1}(\beta)$. Since every $Z \in \widehat{\mathcal{Z}}_\sigma^{n+1}(\beta)$ is of the form $Z = Z' \cap (T_\sigma^n)^{-1}(Z'')$ for $Z' \in \widehat{\mathcal{Z}}_\sigma^n(\beta)$ and $Z'' \in \widehat{\mathcal{Z}}_{\tau^n\sigma}^1(\beta)$, if $a \in \partial\widehat{\mathcal{Z}}_\sigma^{n+1}(\beta)$, then either $a \in \partial\widehat{\mathcal{Z}}_\sigma^n(\beta)$ or $T_\sigma^n a \in \partial\widehat{\mathcal{Z}}_{\tau^n\sigma}^1(\beta)$. In the first case, $T_\sigma^{n+1}a \in T_{\tau^n\sigma}^1(T_\sigma^n(\widehat{\mathcal{Z}}_\sigma^n(\beta))) \subset \mathcal{Q}_{n+1}(\beta)$, and in the second case, $T_\sigma^{n+1}a \in T_{\tau^n\sigma}^1(\partial\widehat{\mathcal{Z}}_{\tau^n\sigma}^1(\beta)) \subset \mathcal{Q}_1(\beta) \subset \mathcal{Q}_{n+1}(\beta)$. □

We thus see that we need to control all the possible orbits of 1 and $\frac{1}{2}$. For this purpose, for $x \in (0, 1]$[11] and $\sigma \in \Sigma^\mathbb{N}$, we introduce the map $\phi_n^{x,\sigma} : (1, \infty) \to [0, 1]$ defined by $\phi_n^{x,\sigma}(\beta) = T_\sigma^n(x)$. For non negative integers $i_1, \dots, i_n$, we define the intervals

$$I_{i_1,\dots,i_n}^{x,\sigma} = \left\{\beta \in (1, \infty) \,|\, \lfloor\beta_{\sigma_k}\phi_{k-1}^{x,\sigma}(\beta)\rfloor = i_k,\ k = 1, \dots, n\right\}.$$

Note that $I_{i_1}^{x,\sigma} = [\frac{i_1}{x}, \frac{i_1+1}{x})$ if $\beta_{\sigma_1} = \beta_1$, $I_{i_1}^{x,\sigma} = [\frac{i_1}{\varpi x}, \frac{i_1+1}{\varpi x})$ if $\beta_{\sigma_1} = \beta_2$, and that in both cases, $I_{i_1}^{x,\sigma} \subset [\frac{i_1}{\varpi x}, \frac{i_1+1}{x})$. The family $\{I_{i_1,\dots,i_n,i_{n+1}}^{x,\sigma}\}_{i_{n+1}\geq 0}$ forms a partition into finitely many (at most $\lfloor\varpi\frac{i_1+1}{x}\rfloor + 1$) intervals of $I_{i_1,\dots,i_n}^{x,\sigma}$. From the relation $\phi_{n+1}^{x,\sigma}(\beta) = \beta_{\sigma_{n+1}}\phi_n^{x,\sigma}(\beta) \mod 1$, we deduce easily by induction:

Lemma 7.5. *The map $\phi_n^{x,\sigma}$ is $\mathcal{C}^1$ and stricly increasing on each interval $I_{i_1,\dots,i_n}^{x,\sigma}$, and verifies $(\phi_n^{x,\sigma})'(\beta) \geq \beta^{n-1}x$.*

Lemma 7.6. *For each $x \in (0, 1]$ and $n \geq 1$, there exists a set $\mathcal{B}_{n,x} \subset (1, \infty)$, with $\mathrm{Leb}(\mathcal{B}_{n,x} \cap (1, t)) = \mathcal{O}(\log t)$ as $t \to \infty$, such that, if $\beta \notin \mathcal{B}_{n,x}$,*

$$\forall\sigma \in \Sigma^\mathbb{N},\ \ d(\phi_n^{x,\sigma}(\beta), \{0, 1/2, 1\}) > 3\beta^{-1}.$$

[11] Note that we will only consider $x = \frac{1}{2}$ and $x = 1$ in the following.

Proof. Fix $n \geq 1$ and $\sigma \in \Sigma$, and consider $\beta \in I^{x,\sigma}_{i_1,\dots,i_n}$. We have $\beta^{-1} \leq \varpi x i_1^{-1}$ and thus $d(\phi^{x,\sigma}_n(\beta), \{0, 1/2, 1\}) > 3\beta^{-1}$ whenever

$$d(\phi^{x,\sigma}_n(\beta), \{0, 1/2, 1\}) > 3\varpi x i_1^{-1}. \tag{7.3}$$

Define $\mathcal{B}^{x,\sigma}_{n,i_1}$ to be the set of β in $I^{x,\sigma}_{i_1}$ which do not satisfy (7.3). By Lemma 7.5, the Lebesgue measure of $\mathcal{B}^{x,\sigma}_{n,i_1} \cap I^{x,\sigma}_{i_1,\dots,i_n}$ is less than[12]

$$3\left(\inf_{I^{x,\sigma}_{i_1,\dots,i_n}} |(\phi^{x,\sigma}_n)'|\right)^{-1} 3\varpi x i_1^{-1} \leq C_{\varpi,x,n} i_1^{-n}.$$

As $\{I^{x,\sigma}_{i_1,\dots,i_n}\}_{i_2,\dots,i_n}$ forms a partition of $I^{x,\sigma}_{i_1}$ into at most $C_{\varpi,x,n} i_1^{n-1}$ elements, the set $\mathcal{B}^{x,\sigma}_{n,i_1}$ has a measure less than $C_{\varpi,x,n} i_1^{-1}$. For $j = 1, 2$, set $\mathcal{B}^{x,j}_{n,i_1} = \bigcup_{\{\sigma \,|\, \beta_{\sigma_1} = \beta_j\}} \mathcal{B}^{x,\sigma}_{n,i_1}$. For n fixed, the condition (7.3) depends on σ only through its n first terms, and thus $\mathcal{B}^{x,j}_{n,i_1}$ is of measure less than $C_{\varpi,x,n} i_1^{-1}$. The set $\mathcal{B}_{n,x} = \bigcup_{j=1,2} \bigcup_{i_1 \geq 1} \mathcal{B}^{x,j}_{n,i_1}$ then satisfies the conclusion of the lemma. □

We can now conclude the proof:

Proof of Lemma 7.3. We set $\mathcal{B}_m = \cup_{1 \leq n \leq m} \mathcal{B}_{n,1/2} \cup \mathcal{B}_{n,1}$ and we proceed by induction over $n \leq m$. Note that if $Z \in \mathcal{Z}^n_{\sigma,\star}(\beta)$, then $Z \cap H^k_\sigma = Z$ for all $k \leq n$.

If $Z \in \mathcal{Z}^1_{\sigma,\star}(\beta)$, then either Z is a full interval of $\mathcal{Z}^1_{\sigma,\star}(\beta)$ and so $T^2_\sigma(Z \cap H^2_\sigma) = T_{\sigma_2}([0,1] \cap H_{\sigma_2}) = [0,1]$, or one of the endpoints of Z is $\frac{1}{2}$ or 1. In the latter case, the other endpoint of Z must be sent after one iteration to 0 or 1 and so if $T^1_\sigma Z \cap H^1_{\tau\sigma}$ is not empty, then $|T^1_\sigma Z \cap H^1_{\tau\sigma}| > 3\beta^{-1}$ by Lemma 7.6. The interval $T^1_\sigma Z \cap H^1_{\tau\sigma}$ therefore contains at least one full interval of $\mathcal{Z}^1_{\tau\sigma,\star}(\beta)$, which implies $T^2_\sigma(Z \cap H^2_\sigma) = [0,1]$.

Now, we suppose that (7.1) holds for n and we will prove it still holds for $n+1$, if $n+1 \leq m$.

Any $Z \in \mathcal{Z}^{n+1}_{\sigma,\star}(\beta)$ is of the form $Z = Z' \cap (T^n_\sigma)^{-1}(Z'')$ with $Z' \in \mathcal{Z}^n_{\sigma,\star}(\beta)$ and $Z'' \in \mathcal{Z}^1_{\tau^n\sigma,\star}(\beta)$. If both endpoints of Z belong to the interior of Z', then $T^n_\sigma Z = Z'' \in \mathcal{Z}^1_{\tau^n\sigma,\star}(\beta)$ and we have

$$T^{n+2}_\sigma(Z \cap H^{n+2}_\sigma) = T^2_{\tau^n\sigma}(T^n_\sigma Z \cap H^2_{\tau^n\sigma}) = T^2_{\tau^n\sigma}(Z'' \cap H^2_{\tau^n\sigma}) = [0,1],$$

or

$$T^{n+1}_\sigma Z = T^1_{\tau^n\sigma}(T^n_\sigma Z) = T^1_{\tau^n\sigma} Z'' \subset H^c_{\sigma_{n+1}},$$

according to whether Z'' is a good a or bad element of $\mathcal{Z}^1_{\tau^n\sigma,\star}(\beta)$ respectively.

If one endpoint of Z is also an endpoint of Z', then $T^n_\sigma Z = [T^n_\sigma a, b]$[13] with $a \in \partial\mathcal{Z}^n_{\sigma,\star}(\beta)$ and $b \in \partial\mathcal{Z}^1_{\tau^n\sigma,\star}(\beta)$. By Lemma 7.4, $T^n_\sigma a = T^i_{\sigma'} x$, with $\sigma' \in \Sigma$, $0 \leq i \leq n$ and $x \in \left\{\frac{1}{2}, 1\right\}$.[14] We consider two subcases: either $b \notin \left\{\frac{1}{2}, 1\right\}$ or $b \in \left\{\frac{1}{2}, 1\right\}$.

In the first subcase, we have $y := T_{\sigma_{n+1}} b \in \{0, 1\}$. Therefore, $T^{n+1}_\sigma Z = [T_{\sigma_{n+1}} T^i_{\sigma'} x, y]$. Since $i+1 \leq n+1 \leq m$, $\beta \notin \mathcal{B}_{n+1,y}$ and one of the endpoints of $T^{n+1}_\sigma Z$ belongs to $\{0,1\}$, it follows that $T^{n+1}_\sigma Z \cap H_{\sigma_{n+2}}$, if it is non empty, is

[12]In the following, $C_{\varpi,x,n}$ will denote a constant, the value of which may change from one line to another, depending on ϖ, x and n, but not on i_1.

[13]We write [x,y] to denote the interval joining x and y, disregarding whether $x \leq y$ or $x \geq y$.

[14]Note that if $T^n_\sigma a = 0$, then we are reduced to the previous situation.

an interval of length strictly larger than $3\beta^{-1}$ by Lemma 7.6 and thus contains at least one full interval of $\mathcal{Z}^1_{\tau^{n+1}\sigma,\star}(\beta)$. Consequently, $T^{n+2}_\sigma(Z \cap H^{n+2}_\sigma) = [0,1]$.

In the second subcase, $T^n_\sigma Z = [T^i_{\sigma'}x, b]$ has one endpoint belonging to $\{0, \frac{1}{2}, 1\}$, and so, as above, we obtain that $T^n_\sigma Z \cap H_{\sigma_n}$, if it is non empty, is of length striclty larger than $3\beta^{-1}$ by Lemma 7.6. We deduce that $T^n_\sigma Z \cap H_{\sigma_n}$ contains at least one full interval of $\mathcal{Z}^1_{\tau^n\sigma,\star}(\beta)$, which implies that $T^{n+2}_\sigma(Z \cap H^{n+2}_\sigma) = [0,1]$.

Finally, if both endpoints of Z are also endpoints of Z', then $Z = Z' \in \mathcal{Z}^n_{\sigma,\star}(\beta)$. Since $Z \in \mathcal{Z}^{n+1}_{\sigma,\star}(\beta)$, we have $T^n_\sigma Z \subset H_{\sigma_{n+1}}$ and so by our induction hypothesis, we can only have $T^{n+1}_\sigma Z = T^{n+1}_\sigma(Z' \cap H^{n+1}_\sigma) = [0,1]$. This implies $T^{n+2}_\sigma(Z \cap H^{n+2}_\sigma) = [0,1]$ and concludes the induction. □

7.2. **Markov maps with non-Markov gates.** The reader might wonder if it is possible to produce an example more similar to the Lorenz gas. In particular, one in which the invariant measures of the maps is always the same and the gates are deterministic, so that one knows explicitly the invariant measure of the process of the environment as seen from the particle.

This is indeed possible, as we discuss briefly. We consider now the situation where the class of maps is $\mathcal{T} = \{T_{\beta_1}, T_{\beta_2}\}$, $\beta_2 > \beta_1$ are both integers, and the partitions $\mathcal{P}_\alpha = \mathcal{P}$ are deterministic with $\mathcal{W} = \{-1, 0, +1\}$ and $G_{-1} = [0, y]$, $G_0 = (y, 1-y]$ and $G_{+1} = (1-y, 1]$, for $0 < y < \frac{1}{2}$. Proceeding similarly to the previous section, but considering this time y as a parameter, instead of β, for fixed β_1 and β_2, we can show the following:

Proposition 7.7. *Let $\varpi > 1$. Then for each $1 < \beta_1 < \beta_2 < \varpi\beta_1$ integers, there exists a measurable set $\mathcal{B}_{\beta_1,\beta_2} \subset (0, \frac{1}{2})$, with $\mathrm{Leb}(\mathcal{B}_{\beta_1,\beta_2}) = \mathcal{O}(\beta_1^{-1})$ as $\beta_1 \to \infty$, such that the model described above, with $y \notin \mathcal{B}_{\beta_1,\beta_2}$, satisfies the assumptions of Theorems 6.1.*

In particular, this class of models is non-empty when β_1 is large enough. Since all maps in $\mathcal{T}$ preserves the Lebesgue measure and all gates are deterministic, Theorem 6.3 also applies[15] and we therefore have $\frac{1}{n}z(n) \to 0$ a.e., since the drift is equal to

$$V = \sum_{w \in \mathcal{W}} w \int_{G_w} h_0 dm = \sum_{w \in \{-1,0,+1\}} w|G_w| = 0.$$

By corollary 4.11, the walk (z_n) is then recurrent.

8. Equivalence with a Gibbs random walk

In this section, we prove Theorem 6.1. The proof will rely on a property of exponential loss of memory for compositions of the operators $\widehat{\mathcal{L}}_\sigma$. More precisely, we will investigate the properties of compositions of the form $\widehat{\mathcal{L}}^n_\sigma f$, in order to understand better the asymptotics of the probabilities $p(\bar{\omega}, n, w_0 \dots w_{n-1})$. For convenience, we will consider bi-infinite sequences $\sigma \in \Sigma^{\mathbb{Z}} = (\mathcal{T} \times \mathcal{H})^{\mathbb{Z}}$, with $\tau : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ the bilateral shift. We can extend the definitions of $\widehat{\mathcal{L}}^n_\sigma$, ρ_σ, C^a_σ and Λ_σ to the case $\sigma \in \Sigma^{\mathbb{Z}}$ in a straightforward way. We will prove that for n large, $\widehat{\mathcal{L}}^n_\sigma f$ is exponentially close in the L^∞-norm to $\rho_\sigma \dots \rho_{\tau^{n-1}\sigma}\Lambda_\sigma(f)h_{\tau^n\sigma}$, where $\{h_\sigma\}$ is a family of positive functions in BV satisfying $\widehat{\mathcal{L}}^1_\sigma h_\sigma = \rho_\sigma h_{\tau\sigma}$. This is summarized in the following result, whose proof is contained in Section 9.

[15] Condition **(Pro)** is satisfied here, since $\mathcal{W}$ is symmetric.

Proposition 8.1. *There exists an integer $n_2 \geq 1$ depending on the classes $\mathcal{T}$ and $\mathcal{H}$, explicitely computable, such that if* (C1)*,* (C2) *and* (C3(n_2, n_3)) *hold for some $n_3 \geq n_2$, then there exist $\nu \in (0,1)$, a family of positive numbers $\{\rho_\sigma\}_{\sigma\in\Sigma^{\mathbb{Z}}}$ and a family of positive functions $\{h_\sigma\}_{\sigma\in\Sigma^{\mathbb{Z}}}$ in* BV *such that $\Lambda_\sigma(h_\sigma) = 1$ and for all $\sigma \in \Sigma^{\mathbb{Z}}$, $f \in \mathrm{BV}$ and $n \geq 0$:*

$$\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} - \Lambda_\sigma(f) h_{\tau^n\sigma} \right\|_\infty \leq C_\# \nu^n \|f\|_{\mathrm{BV}}. \tag{8.1}$$

Remark 8.2. *The statement of the above result is similar to the one in* [21]*. Note however that here the setting is very different insofar in* [21] *only small holes and near by maps are considered. The upgrade of the result to large holes and arbitrary maps (as we inescapably need) turns out to be highly non trivial.*

To prove Proposition 8.1, we will adapt the strategy of [17]. More precisely, we will show that the family of cones

$$C_\sigma^a = \{h \in \mathrm{BV} \mid h \neq 0, h \geq 0, \bigvee h \leq a\Lambda_\sigma(h)\}$$

is stricly invariant under compositions of large enough length of transfer operators (i.e. $\widehat{\mathcal{L}}_\sigma^n C_\sigma^a \subset C_{\tau^n\sigma}^{a/2}$ for all $n \geq n_0$) for a suitable $a > 0$, see Lemma 9.6. From this, we will deduce that $\widehat{\mathcal{L}}_\sigma^n C_\sigma^a$ has uniform finite diameter in $C_{\tau^n\sigma}^a$ for the corresponding Hilbert metric (Lemma 9.13), which will imply that $\widehat{\mathcal{L}}_\sigma^n$ is a strict contraction for the Hilbert metric, and then enjoys exponential loss of memory.

Remark 8.3. *To deduce Theorem 6.1 from Proposition 8.1, we will need some technical facts which will be proved throughout Section 9 and that we list below:*

1. *for all $n \geq 0$ and $\sigma \in \Sigma^{\mathbb{N}}$, $\inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1} \geq \varrho' \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})$ (Remark 9.12);*
2. *for all $n \geq 0$ and $\sigma \in \Sigma^{\mathbb{N}}$, $\|\widehat{\mathcal{L}}_\sigma^n \mathbb{1}\|_\infty \leq B\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})$ (proof of Lemma 9.7);*
3. *for all $n \geq 0$ and $\sigma \in \Sigma^{\mathbb{N}}$, $\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1}) = \rho_\sigma \ldots \rho_{\tau^{n-1}\sigma}$ (Lemma 9.16);*
4. *for all $\sigma \in \Sigma^{\mathbb{Z}}$, $\inf h_\sigma \geq \varrho$ and $\|h_\sigma\|_\infty \leq 1 + a$ (Lemma 9.14).*

We also note that, since $\mathcal{T} \times \mathcal{H}$ is finite, $\|\widehat{\mathcal{L}}_\sigma^1 \mathbb{1}\|_\infty < \infty$ and $\inf \widehat{\mathcal{L}}_\sigma^1 \mathbb{1} > 0$, there exist $\delta > 0$ and $M < \infty$ such that $\|\widehat{\mathcal{L}}_\sigma^n \mathbb{1}\|_\infty \leq M^n$ and $\inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1} \geq \delta^n$ for all $n \geq 0$ and $\sigma \in \Sigma^{\mathbb{N}}$. Also, $\rho_\sigma \leq \|\widehat{\mathcal{L}}_\sigma^1 \mathbb{1}\|_\infty \leq M$, and so $\sup_\sigma \rho_\sigma < \infty$.

Recall from Section 4 that the transition probabilities are given by (4.10). For $(\bar{\omega}, \bar{w}) \in \Omega_\star$, we define $\sigma = \sigma(\bar{\omega}, \bar{w}) \in \Sigma^{\mathbb{N}}$ such that $\widehat{\mathcal{L}}_{\sigma_n} = \mathcal{L}_{\bar{\omega}, z_{n-1}, \bar{w}_{n-1}}$, so that

$$p(\bar{\omega}, n, \bar{w}_0 \ldots \bar{w}_{n-1}) = \int \widehat{\mathcal{L}}_\sigma^n h_0 dm.$$

We will still denote by $\sigma \in \Sigma^{\mathbb{Z}}$ an arbitrary element of $\Sigma^{\mathbb{Z}}$ which coincides with σ for future components (for instance, given an arbitrary $\sigma_\star \in \Sigma$, we identify $\sigma \in \Sigma^{\mathbb{N}}$ with the element $\widetilde{\sigma} \in \Sigma^{\mathbb{Z}}$ defined by $\widetilde{\sigma}_i = \sigma_i$ if $i \geq 1$ and $\widetilde{\sigma}_i = (\sigma_\star)_i$ if $i \leq 0$).

We are now ready to prove the announced results.

Proof of Theorem 6.1. We have

$$\mathbb{P}_{\bar{\omega}}(\bar{w}_n \mid \bar{w}_0 \ldots \bar{w}_{n-1}) = \frac{p(\bar{\omega}, n+1, \bar{w}_0 \ldots \bar{w}_n)}{p(\bar{\omega}, n, \bar{w}_0 \ldots \bar{w}_{n-1})} = \frac{\int \widehat{\mathcal{L}}_\sigma^{n+1} h_0 dm}{\int \widehat{\mathcal{L}}_\sigma^n h_0 dm} \tag{8.2}$$

and

$$\begin{aligned}\mathbb{P}_{\tau_{z_m}\bar{\omega}}(\bar{w}_n \mid \bar{w}_m \dots \bar{w}_{n-1}) &= \frac{p(\tau_{z_m}\bar{\omega}, n-m+1, \bar{w}_m \dots \bar{w}_n)}{p(\tau_{z_m}\bar{\omega}, n-m, \bar{w}_m \dots \bar{w}_{n-1})} \\ &= \frac{\int \widehat{\mathcal{L}}^{n-m+1}_{\tau_{z_m}\sigma} h_0 dm}{\int \widehat{\mathcal{L}}^{n-m}_{\tau_{z_m}\sigma} h_0 dm}.\end{aligned} \tag{8.3}$$

We can then write

$$\frac{\int \widehat{\mathcal{L}}^{n+1}_\sigma h_0 dm}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm} = \rho_{\tau^n\sigma} \frac{\int \widehat{\mathcal{L}}^{n+1}_\sigma d_0 dm}{\rho_\sigma \dots \rho_{\tau^n\sigma}} \frac{\rho_\sigma \dots \rho_{\tau^{n-1}\sigma}}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm}.$$

Using Proposition 8.1, we have

$$\left| \frac{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} - \Lambda_\sigma(h_0) \int h_{\tau^n\sigma} dm \right| \leq C_\# \nu^n \|h_0\|_{\mathrm{BV}}, \tag{8.4}$$

which also implies

$$\left| \frac{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm} - \left(\Lambda_\sigma(h_0) \int h_{\tau^n\sigma} dm \right)^{-1} \right| \leq C_\# \nu^n \frac{\|h_0\|_{\mathrm{BV}}}{(\inf h_0)^2}, \tag{8.5}$$

since

$$\frac{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} \geq (\inf h_0) \frac{\inf \widehat{\mathcal{L}}^n_\sigma \mathbb{1}}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} \geq (\inf h_0) \frac{\varrho' \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1})}{\rho_\sigma \dots \rho_{\tau^{n-1}\sigma}} = \varrho(\inf h_0),$$

and $\Lambda_\sigma(h_0) \int h_{\tau^n\sigma} dm \geq \varrho \inf h_0$.

Note that we also have

$$\frac{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} \leq \|h_0\|_\infty \frac{\|\widehat{\mathcal{L}}^n_\sigma \mathbb{1}\|_\infty}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} \leq B\|h_0\|_\infty.$$

Consequently, using (8.4) with n replaced by $n+1$, (8.5) and the above inequalities:

$$\begin{aligned}\left| \frac{\int \widehat{\mathcal{L}}^{n+1}_\sigma h_0 dm}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm} - \rho_{\tau^n\sigma} \frac{\int h_{\tau^{n+1}\sigma} dm}{\int h_{\tau^n\sigma} dm} \right| &\leq \left[\left| \frac{\int \widehat{\mathcal{L}}^{n+1}_\sigma h_0 dm}{\rho_\sigma \cdots \rho_{\tau^n\sigma}} - \Lambda_\sigma(h_0) \int h_{\tau^{n+1}\sigma} dm \right| \frac{\rho_\sigma \cdots \rho_{\tau^n\sigma}}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm} \right. \\ &\left. + \Lambda_\sigma(h_0) \int h_{\tau^{n+1}\sigma} dm \left| \frac{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm} - \left(\Lambda_\sigma(h_0) \int h_{\tau^n\sigma} dm \right)^{-1} \right| \right] \\ &\leq C_\# \nu^n \left(\frac{\|h_0\|_{\mathrm{BV}}}{\inf h_0} + \frac{\|h_0\|^2_{\mathrm{BV}}}{(\inf h_0)^2} \right).\end{aligned}$$

Hence, for all $n \geq 0$,

$$\sup_{\sigma \in \Sigma} \left| \frac{\int \widehat{\mathcal{L}}^{n+1}_\sigma h_0 dm}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm} - \rho_{\tau^n\sigma} \frac{\int h_{\tau^{n+1}\sigma} dm}{\int h_{\tau^n\sigma} dm} \right| \leq C_{h_0} \nu^n, \tag{8.6}$$

where C_{h_0} depends only on the density h_0, and, by (8.2) and (8.3), this implies **(Exp)**. □

Proof of Theorem 6.2. By (8.6), we have

$$\sup_{\sigma \in \Sigma} \left| \frac{\int \widehat{\mathcal{L}}^{n+1}_\sigma h_0 dm}{\int \widehat{\mathcal{L}}^n_\sigma h_0 dm} - \frac{\int \widehat{\mathcal{L}}^{n+1}_\sigma h'_0 dm}{\int \widehat{\mathcal{L}}^n_\sigma h'_0 dm} \right| \leq (C_{h_0} + C_{h'_0}) \nu^n,$$

for all $n \geq 0$, and the theorem follows with $C_{h_0,h'_0} = C_{h_0} + C_{h'_0}$. □

Proof of Theorem 6.3. According to Theorem 4.9 and the discussion in Section 4.4, and since condition **(Exp)** already holds by Theorem 6.1, it is enough to show that assumptions **(Pos)** and **(Ell)** are satisfied for the probabilities defined by (4.10).

Verification of **(Pos)***.* We have, by Proposition 8.1

$$
\begin{aligned}
p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1}) = \int \widehat{\mathcal{L}}_\sigma^n h_0 dm = \rho_\sigma \cdots \rho_{\tau^{n-1}\sigma} \frac{\int \widehat{\mathcal{L}}_\sigma^n h_0 dm}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} \\
\geq \rho^n \left(\Lambda_\sigma(h_0) \int h_{\tau^n\sigma} - C_\# \nu^n \|h_0\|_{\mathrm{BV}} \right) \\
\geq \rho^n \left(\varrho \inf h_0 - C_\# \nu^n \|h_0\|_{\mathrm{BV}} \right).
\end{aligned}
$$

Consequently, $p(\bar\omega, n, \bar w_0 \dots \bar w_{n-1}) > 0$ for all n large enough, and since this quantity is non-increasing, this proves the positivity for all $n \geq 0$.

Verification of **(Ell)***.* By (8.6), we have for all $n \geq 0$,

$$
\begin{aligned}
\mathbb{P}_{\bar\omega}(\bar w_n \mid \bar w_{n-1} \dots \bar w_0) &= \frac{\int \widehat{\mathcal{L}}_\sigma^{n+1} h_0 dm}{\int \widehat{\mathcal{L}}_\sigma^n h_0 dm} \\
&\geq \rho_{\tau^n\sigma} \frac{\int h_{\tau^{n+1}\sigma} dm}{\int h_{\tau^n\sigma} dm} - C_{h_0} \nu^n \\
&\geq \rho \frac{\varrho}{1+a} - C_{h_0} \nu^n,
\end{aligned}
$$

which proves **(Ell)**. □

9. Loss of Memory

To prove Proposition 8.1, we will adapt the strategy of [17] to our non-stationary case, and employ the theory of Hilbert metrics, that we recall below.

Definition 2. *Let $\mathcal{V}$ be a vector space. We will call convex cone a subset $\mathcal{C} \subset \mathcal{V}$ which enjoys the following properties:*

(i) $\mathcal{C} \cap -\mathcal{C} = \emptyset$.
(ii) $\forall \lambda > 0$, $\lambda\mathcal{C} = \mathcal{C}$.
(iii) *$\mathcal{C}$ is a convex set.*
(iv) $\forall f, g \in \mathcal{C}$, $\forall \alpha_n \in \mathbb{R}$ $\alpha_n \to \alpha$, $g - \alpha_n f \in \mathcal{C} \Rightarrow g - \alpha f \in \mathcal{C} \cup \{0\}$.

We now define the Hilbert metric on $\mathcal{C}$:

Definition 3. *The distance $d_{\mathcal{C}}(f, g)$ between two points f, g in $\mathcal{C}$ is given by*

$$
\begin{aligned}
\alpha(f, g) &= \sup\{\lambda > 0 \mid g - \lambda f \in \mathcal{C}\}, \\
\beta(f, g) &= \inf\{\mu > 0 \mid \mu f - g \in \mathcal{C}\}, \\
d_{\mathcal{C}}(f, g) &= \log \frac{\beta(f, g)}{\alpha(f, g)},
\end{aligned}
$$

where we take $\alpha = 0$ or $\beta = \infty$ when the corresponding sets are empty.

The next theorem shows that every positive linear operator is a contraction, provided that the diameter of the image is finite.

Theorem 9.1 ([15, Theorem 1.1])**.** *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vector spaces, $\mathcal{C}_1 \subset \mathcal{V}_1$ and $\mathcal{C}_2 \subset \mathcal{V}_2$ two convex cones and $L : \mathcal{V}_1 \to \mathcal{V}_2$ a positive linear operator (which implies $L(\mathcal{C}_1) \subset \mathcal{C}_2$). If we denote*

$$\Delta = \sup_{f,g \in L(\mathcal{C}_1)} d_{\mathcal{C}_2}(f,g),$$

then

$$d_{\mathcal{C}_2}(Lf, Lg) \leq \tanh\left(\frac{\Delta}{4}\right) d_{\mathcal{C}_1}(f,g) \ \forall f,g \in \mathcal{C}_1.$$

The following lemma links the Hilbert metric to suitable norms on $\mathcal{V}$:

Lemma 9.2 ([18, Lemma 2.2])**.** *Let $\|\cdot\|$ be a norm on $\mathcal{V}$ such that*

$$\forall f,g \in \mathcal{V} \ g-f, g+f \in \mathcal{C} \Rightarrow \|f\| \leq \|g\|$$

and let $\ell : \mathcal{C} \to \mathbb{R}^+$ be a homogeneous and order preserving function, i.e.

$$\begin{array}{ll} \forall f \in \mathcal{C}, \forall \lambda \in \mathbb{R}^+ & \ell(\lambda f) = \lambda \ell(f), \\ \forall f,g, \in \mathcal{C} & g-f \in \mathcal{C} \Rightarrow \ell(f) \leq \ell(g), \end{array}$$

then

$$\forall f,g \in \mathcal{C} \ \ell(f) = \ell(g) > 0 \Rightarrow \|f-g\| \leq (e^{d_{\mathcal{C}}(f,g)} - 1)\min(\|f\|, \|g\|).$$

From now on, we will always assume that conditions *(C1)* and *(C2)* hold. Our main tool will be the following Lasota-Yorke type inequality:

Lemma 9.3. *If condition* $(\mathrm{C3}(N,N'))$ *holds, then for any $n \leq N$, for any $\sigma \in \Sigma^{\mathbb{N}}$ and $h \in \mathrm{BV}$, we have*

$$\bigvee \widehat{\mathcal{L}}_\sigma^n h \leq C_\star (\xi\Theta)^n \bigvee h + C_N \Lambda_\sigma(|h|),$$

where $C_\star$ and C_N do not depend on h and σ.

Proof. First notice that $\widehat{\mathcal{L}}_\sigma^n(h\mathbb{1}_Z) = 0$ if $Z \in \widehat{\mathcal{Z}}_\sigma^n \setminus \mathcal{Z}_{\sigma,\star}^n$. We can then write

$$\widehat{\mathcal{L}}_\sigma^n h = \sum_{Z \in \mathcal{Z}_{\sigma,\star}^n} \widehat{\mathcal{L}}_\sigma^n(\mathbb{1}_Z h) = \sum_{Z \in \mathcal{Z}_{\sigma,\star}^n} (\mathbb{1}_Z g_\sigma^n h) \circ (T_{\sigma,Z}^n)^{-1},$$

where $(T_{\sigma,Z}^n)^{-1}$ is the inverse branch of T_σ^n restricted to Z.

Accordingly,

$$\bigvee \widehat{\mathcal{L}}_\sigma^n h \leq \sum_{Z \in \mathcal{Z}_{\sigma,\star}^n} \bigvee \mathbb{1}_{T_\sigma^n Z}(g_\sigma^n h) \circ (T_{\sigma,Z}^n)^{-1}.$$

We estimate each term of the sum separately.

$$\begin{aligned} \bigvee \mathbb{1}_{T_\sigma^n Z}(g_\sigma^n h) \circ (T_{\sigma,Z}^n)^{-1} &\leq \bigvee_Z h g_\sigma^n + 2\sup_Z |h g_\sigma^n| \\ &\leq 3\bigvee_Z h g_\sigma^n + 2\inf_Z |h g_\sigma^n| \\ &\leq 3\|g_\sigma^n\|_\infty \bigvee_Z h + 3\sup_Z |h| \bigvee_Z g_\sigma^n + 2\inf_Z |h g_\sigma^n| \\ &\leq 5\|g_\sigma^n\|_\infty \bigvee_Z h + 3C\sup_Z |h| \|g_\sigma^n\|_\infty + 2\|g_\sigma^n\|_\infty \inf_Z |h| \\ &\leq (3C+5)\|g_\sigma^n\|_\infty \bigvee_Z h + (3C+2)\|g_\sigma^n\|_\infty \inf_Z |h|, \end{aligned}$$

where we have used the fact there exists a constant C such that $\bigvee_Z g_\sigma^n \le C\|g_\sigma^n\|_\infty$ for all n, σ and $Z \in \mathcal{Z}_{\sigma,\star}^n$ by bounded distortion.

By assumption *(C3(N, N'))*, we have for each $x \in [0,1]$,

$$\inf_{Z\in\mathcal{Z}_{\sigma,g}^n} \frac{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}_Z(x)}{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}(x)} \ge \epsilon_N.$$

Accordingly, for each $x \in [0,1]$, $h \in \mathrm{BV}$ and $Z \in \mathcal{Z}_{\sigma,g}^n$ holds

$$\widehat{\mathcal{L}}_\sigma^{N'}(|h|\mathbb{1}_Z)(x) \ge \inf_Z |h|\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}_Z(x) \ge \inf_Z |h|\epsilon_N\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}(x).$$

To deal with elements in $\mathcal{Z}_{\sigma,b}^n$, we use condition *(C2)*. Note that elements of $Z_{\sigma,g}^n$ can be separated by at most $K\xi^n$ elements of $\mathcal{Z}_{\sigma,b}^n$. For each $Z \in \mathcal{Z}_{\sigma,b}^n$, let $I_\pm(Z)$ be the union of the contiguous elements of $\mathcal{Z}_{\sigma,b}^n$ on the left and on the right of Z respectively. Clearly, for each $Z' \subset I_\pm(Z)$, holds

$$\inf_{Z'} |h| \le \inf_Z |h| + \bigvee_{I_\pm(Z)} h.$$

Accordingly,

$$\sum_{Z\in\mathcal{Z}_{\sigma,b}^n} \inf_Z |h| \le 2K\xi^n \left[\sum_{Z\in\mathcal{Z}_{\sigma,g}^n} \inf_Z |h| + \bigvee h\right].$$

For all x, we thus have

$$\begin{aligned}\sum_{Z\in\mathcal{Z}_{\sigma,\star}^n} \inf_Z |h| &\le (2K\xi^n+1)\sum_{Z\in\mathcal{Z}_{\sigma,g}^n} \inf_Z |h| + 2K\xi^n\bigvee h\\ &\le (2K\xi^n+1)\frac{1}{\epsilon_N}\sum_{Z\in\mathcal{Z}_{\sigma,g}^n}\frac{\widehat{\mathcal{L}}_\sigma^{N'}(|h|\mathbb{1}_Z)(x)}{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}(x)} + 2K\xi^n\bigvee h\\ &\le (2K\xi^n+1)\frac{1}{\epsilon_N}\frac{\widehat{\mathcal{L}}_\sigma^{N'}(|h|)(x)}{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}(x)} + 2K\xi^n\bigvee h.\end{aligned}$$

We can then conclude

$$\begin{aligned}\bigvee \widehat{\mathcal{L}}_\sigma^n h \le &((3C+5)+(3C+2)2K\xi^n)\|g_\sigma^n\|_\infty\bigvee h\\ &+ (3C+2)(2K\xi^n+1)\|g_\sigma^n\|_\infty\frac{1}{\epsilon_N}\frac{\widehat{\mathcal{L}}_\sigma^{N'}(|h|)(x)}{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}(x)}.\end{aligned}$$

Taking the inf over x in the previous expression and recalling that $\|g_\sigma^n\|_\infty \le D\Theta^n$ and $\inf_x \frac{\widehat{\mathcal{L}}_\sigma^{N'}(|h|)(x)}{\widehat{\mathcal{L}}_\sigma^{N'}\mathbb{1}(x)} \le \Lambda_\sigma(|h|)$, we obtain the result. □

Remark 9.4. *The constants $C_\star$ and C_N in the Lasota-Yorke inequality can be chosen so that $D((3C+5)+(3C+2)2K\xi^n) \le C_\star\xi^n$ and $(3C+2)(2K\xi^n+1)\Theta^n\frac{1}{\epsilon_N} \le C_N$ for all $n \le N$, where C is such that $\bigvee_Z g_\sigma^n \le C\|g_\sigma^n\|_\infty$ for all n, σ and $Z \in \mathcal{Z}_{\sigma,\star}^n$. Note that if all the maps in $\mathcal{T}$ are piecewise linear, we can take $C = 0$.*

We will show that the family of cones

$$\mathcal{C}_\sigma^a = \{h \in \mathrm{BV} \mid h \ne 0, h \ge 0, \bigvee h \le a\Lambda_\sigma(h)\}$$

is strictly invariant under the transfer operators defined above.

Recall that, under assumption *(C2)*, $\theta = \xi\Theta < \rho$.

Lemma 9.5. *For all σ and all $g \in \mathrm{BV}$, $g \geq 0$, we have $\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}_{\sigma_1} g) \geq \rho_\sigma \Lambda_\sigma(g)$. In particular, $\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma g) \geq \rho^n \Lambda_\sigma(g)$ and $\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1}) \geq \rho^n$.*

Proof. We just need to prove the first part of the statement, the second part follows by iteration, since $\rho_\sigma \geq \rho$ for all σ. For each $g \in \mathrm{BV}$, $g \geq 0$ and $x \in [0,1]$, holds

$$\frac{\widehat{\mathcal{L}}^n_{\tau\sigma}\widehat{\mathcal{L}}_{\sigma_1} g(x)}{\widehat{\mathcal{L}}^n_{\tau\sigma}\mathbb{1}(x)} \geq \frac{\widehat{\mathcal{L}}_{\sigma_{n+1}}\left[\frac{\widehat{\mathcal{L}}^n_\sigma g}{\widehat{\mathcal{L}}^n_\sigma \mathbb{1}}\widehat{\mathcal{L}}^n_\sigma \mathbb{1}\right](x)}{\widehat{\mathcal{L}}^n_{\tau\sigma}\mathbb{1}(x)} \geq \frac{\widehat{\mathcal{L}}^n_{\tau\sigma}(\widehat{\mathcal{L}}_{\sigma_1}\mathbb{1})(x)}{\widehat{\mathcal{L}}^n_{\tau\sigma}\mathbb{1}(x)} \inf \frac{\widehat{\mathcal{L}}^n_\sigma g}{\widehat{\mathcal{L}}^n_\sigma \mathbb{1}}$$

and taking the inf on x and the limit $n \to \infty$, we get the result. □

Lemma 9.6. *There exist $n_0 \in \mathbb{N}$ and $a_0 > 0$ such that if $\mathrm{C3}(n_0, N')$ holds, then for all $a \geq a_0$ and σ, we have*

$$\widehat{\mathcal{L}}^n_\sigma \mathcal{C}^a_\sigma \subset \mathcal{C}^{a/2}_{\tau^n\sigma} \ \forall n \geq n_0 \quad \textit{and} \quad \widehat{\mathcal{L}}^n_\sigma \mathcal{C}^a_\sigma \subset C^{2aC_\star}_{\tau^n\sigma} \ \forall n \geq 0.$$

Proof. Let $n_0 \in \mathbb{N}$ which will be chosen later. Let $h \in \mathcal{C}^a_\sigma$, then we can write each n as $n = kn_0 + m$, $m < n_0$, and by Lemma 9.3, we have

(9.1)
$$\begin{aligned}
\bigvee \widehat{\mathcal{L}}^n_\sigma h &\leq C_\star \theta^{n_0} \bigvee \widehat{\mathcal{L}}^{(k-1)n_0+m}_\sigma h + C_{n_0}\Lambda_{\tau^{(k-1)n_0+m}\sigma}(\widehat{\mathcal{L}}^{(k-1)n_0+m}_\sigma h) \\
&\leq C^k_\star \theta^{kn_0} \bigvee \widehat{\mathcal{L}}^m_\sigma h + \sum_{i=0}^{k-1} C_{n_0}(C_\star\theta^{n_0})^i \Lambda_{\tau^{(k-i-1)n_0+m}\sigma}(\widehat{\mathcal{L}}^{(k-i-1)n_0+m}_\sigma h) \\
&\leq C^{k+1}_\star \theta^n \bigvee h + \sum_{i=0}^{k-1} C_{n_0}(C_\star\theta^{n_0})^i \Lambda_{\tau^{(k-i-1)n_0+m}\sigma}(\widehat{\mathcal{L}}^{(k-i-1)n_0+m}_\sigma h) \\
&\quad + C_m(C_\star\theta^{n_0})^k\Lambda_\sigma(h).
\end{aligned}$$

Using Lemma 9.5, we obtain

$$\bigvee \widehat{\mathcal{L}}^n_\sigma h \leq \left[\left(a + \frac{C_m}{C_\star\theta^m}\right)\frac{C^{k+1}_\star\theta^n}{\rho^n} + \frac{C_{n_0}}{\rho^{n_0}}\sum_{i=0}^{k-1}\left(\frac{C_\star\theta^{n_0}}{\rho^{n_0}}\right)^i\right]\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h).$$

If $k = 0$, for $a_0 \geq \max_{i \leq n_0} \frac{C_i}{C_\star\theta^i}$, we have

$$\bigvee \widehat{\mathcal{L}}^n_\sigma h \leq 2aC_\star\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h).$$

When $k > 0$ instead

$$\bigvee \widehat{\mathcal{L}}^n_\sigma h \leq \left[\frac{1}{4}\left(a + \frac{C_m}{C_\star\theta^m}\right) + 2C_{n_0}\rho^{-n_0}\right]\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h)$$

provided n_0 is such that $\frac{C^{k+1}_\star\theta^n}{\rho^n} \leq \frac{1}{4}$ for all $n \geq n_0$. This can be achieved by choosing first $\eta \in (\frac{\theta}{\rho}, 1)$ and $n_0^\star$ such that $C_\star^{1/n}\theta\rho^{-1} \leq \eta$ for all $n \geq n_0^\star$. If we choose $n_0 \geq n_0^\star$ such that $C_\star\eta^n \leq \frac{1}{4}$ for all $n \geq n_0$, then we have $\frac{C^{k+1}_\star\theta^n}{\rho^n} \leq C_\star\left(\frac{C_\star^{1/n_0}\theta}{\rho}\right)^n \leq C_\star\eta^n \leq \frac{1}{4}$ for all $n \geq n_0$.

Hence, for all $n \geq n_0$ and $a \geq a_0 = 8C_{n_0}\rho^{-n_0} + \max_{i \leq n_0}\frac{C_i}{C_\star\theta^i}$,

$$\bigvee \widehat{\mathcal{L}}^n_\sigma h \leq \frac{a}{2}\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h).$$

□

Lemma 9.7. *If* (C3(n_0, N')) *holds, there exists* $B > 0$ *such that for each* $h \in \mathrm{BV}$, $h \geq 0$, $n \in \mathbb{N}$ *and* $\sigma \in \Sigma^{\mathbb{N}}$,

$$\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})\Lambda_\sigma(h) \leq \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n h) \leq B\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})\Lambda_\sigma(h).$$

Proof. For $x \in [0,1]$, we have

$$\frac{\widehat{\mathcal{L}}^m_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h)(x)}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\mathbb{1}(x)} \geq \frac{\widehat{\mathcal{L}}^n_{\tau^m\sigma}\left[\frac{\widehat{\mathcal{L}}^m_\sigma h}{\widehat{\mathcal{L}}^m_\sigma \mathbb{1}}\widehat{\mathcal{L}}^m_\sigma \mathbb{1}\right](x)}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\mathbb{1}(x)} \geq \frac{\widehat{\mathcal{L}}^m_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1})(x)}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\mathbb{1}(x)} \inf \frac{\widehat{\mathcal{L}}^m_\sigma h}{\widehat{\mathcal{L}}^m_\sigma \mathbb{1}}$$

where we have used twice the fact that $\widehat{\mathcal{L}}^m_{\sigma^n\sigma}\widehat{\mathcal{L}}^n_\sigma = \widehat{\mathcal{L}}^n_{\sigma^m\sigma}\widehat{\mathcal{L}}^m_\sigma$. Taking the inf on x and the limit $m \to \infty$, we get the first inequality.

For the second, for $x \in [0,1]$, we have

$$\frac{\widehat{\mathcal{L}}^m_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h)(x)}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\mathbb{1}(x)} = \frac{\widehat{\mathcal{L}}^m_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h)(x)}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1})(x)}\frac{\widehat{\mathcal{L}}^m_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1})(x)}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\mathbb{1}(x)} \leq \frac{\widehat{\mathcal{L}}^{n+m}_\sigma h(x)}{\widehat{\mathcal{L}}^{n+m}_\sigma \mathbb{1}(x)}\|\widehat{\mathcal{L}}^n_\sigma \mathbb{1}\|_\infty,$$

which, by taking the inf on x and the limit $m \to \infty$, yields

$$\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma h) \leq \|\widehat{\mathcal{L}}^n_\sigma \mathbb{1}\|_\infty \Lambda_\sigma(h).$$

By applying Lemma 9.6 to $\mathbb{1} \in \mathcal{C}^a_\sigma$, we obtain $\bigvee \widehat{\mathcal{L}}^n_\sigma \mathbb{1} \leq 2aC_\star \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1})$. Thus

$$\|\widehat{\mathcal{L}}^n_\sigma \mathbb{1}\|_\infty \leq \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1}) + \bigvee \widehat{\mathcal{L}}^n_\sigma \mathbb{1} \leq (1 + 2aC_\star)\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1}),$$

from which the result follows with $B = 1 + 2aC_\star$. □

Lemma 9.8. *For each* $\delta > 0$, *there exists* $n_1 = n_1(\delta)$ *such that for each* $n \geq n_1$, *the partition* $\widehat{\mathcal{Z}}^n_\sigma$ *has the property that*

$$\sup_{Z \in \widehat{\mathcal{Z}}^n_\sigma} \Lambda_\sigma(\mathbb{1}_Z) \leq \delta.$$

Proof. Choose $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $D\theta^n\rho^{-n} \leq \delta$, which is possible due to condition *(C2)*. Then, for $Z \in \widehat{\mathcal{Z}}^n_1$,

$$\widehat{\mathcal{L}}^n_\sigma \mathbb{1}_Z(x) = \sum_{T^n_\sigma y = x} g^n_\sigma(y)\mathbb{1}_Z(y) \leq \|g^n_\sigma\|_\infty \leq D\theta^n.$$

Accordingly, for each $x \in [0,1]$,

$$\frac{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\widehat{\mathcal{L}}^n_\sigma \mathbb{1}_Z(x)}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\widehat{\mathcal{L}}^n_\sigma \mathbb{1}(x)} \leq D\theta^n \frac{1}{\inf \frac{\widehat{\mathcal{L}}^m_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1})}{\widehat{\mathcal{L}}^m_{\tau^n\sigma}\mathbb{1}}}.$$

Taking the inf on x and the limit $m \to \infty$, this yields

$$\Lambda_\sigma(\mathbb{1}_Z) \leq D\theta^n \frac{1}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}^n_\sigma \mathbb{1})} \leq D\theta^n\rho^{-n} \leq \delta$$

where we have used Lemma 9.5. □

Lemma 9.9. *If* (C3(n_0, N')) *holds, then for each* $a \geq a_0$, *there exists* $n_2 \in \mathbb{N}$ *such that for each* $n \geq n_2$ *and* $h \in \mathcal{C}^a_\sigma$ *there exists* $Z \in \mathcal{Z}^n_{\sigma,g}$ *with*

$$\inf_Z h \geq \frac{1}{4}\Lambda_\sigma(h).$$

Proof. For each n, m with $n < m$, we can write

$$\widehat{\mathcal{L}}_\sigma^m h(x) = \sum_{Z \in \widehat{\mathcal{Z}}_\sigma^n} \widehat{\mathcal{L}}_\sigma^m (h\mathbb{1}_Z)(x) = \sum_{Z \in \mathcal{Z}_{\sigma,\star}^n} \widehat{\mathcal{L}}_\sigma^m (h\mathbb{1}_Z)(x).$$

Suppose the lemma is not true. Then, we have

$$\begin{aligned}\widehat{\mathcal{L}}_\sigma^m h(x) &= \sum_{Z \in \mathcal{Z}_{\sigma,g}^n} \widehat{\mathcal{L}}_\sigma^m (h\mathbb{1}_Z)(x) + \sum_{Z \in \mathcal{Z}_{\sigma,b}^n} \widehat{\mathcal{L}}_\sigma^m (h\mathbb{1}_Z)(x) \\ &\le \sum_{Z \in \mathcal{Z}_{\sigma,g}^n} \widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z(x) \frac{\Lambda_\sigma(h)}{4} + \sum_{Z \in \mathcal{Z}_{\sigma,g}^n} \widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z(x) \bigvee_Z h + \|h\|_\infty \sum_{Z \in \mathcal{Z}_{\sigma,b}^n} \widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z(x) \\ &\le \widehat{\mathcal{L}}_\sigma^m \mathbb{1}(x) \frac{\Lambda_\sigma(h)}{4} + \sum_{Z \in \mathcal{Z}_{\sigma,g}^n} \left[\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z) + \bigvee \widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z\right] \bigvee_Z h \\ &\quad + \|h\|_\infty \sum_{Z \in \mathcal{Z}_{\sigma,b}^n} \widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z(x).\end{aligned}$$

If $Z \in \mathcal{Z}_{\sigma,b}^n$, by Lemma 9.7, we have $\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z) \le B\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1})\Lambda_\sigma(\mathbb{1}_Z) = 0$, which implies

$$\widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z(x) \le \bigvee \widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z \le 2C_\star^{m/n_0+1}\theta^m \le 2C_\star(C_\star^{1/n_0}\theta\rho^{-1})^m \Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}),$$

by inequality (9.1) and Lemma 9.5.

If $Z \in \mathcal{Z}_{\sigma,g}^n$, the same argument gives

$$\begin{aligned}\bigvee \widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z &\le 2C_\star^{m/n_0+1}\theta^m + 2C_{n_0}\rho^{-n_0}\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}_Z) \\ &\le \left[2C_\star(C_\star^{1/n_0}\theta\rho^{-1})^m + 2C_{n_0}\rho^{-n_0}B\Lambda_\sigma(\mathbb{1}_Z)\right]\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}).\end{aligned}$$

Setting $\kappa = C_\star^{1/n_0}\theta\rho^{-1} \le 4^{-1/n_0}$, we have

$$\begin{aligned}\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m h) \le &\frac{\Lambda_\sigma(h)}{4}\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}) \\ &+ \sum_{Z \in \mathcal{Z}_{\sigma,g}^n} \Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}) \bigvee_Z h \left[B(1 + 2C_{n_0}\rho^{-n_0})\Lambda_\sigma(\mathbb{1}_Z) + 2C_\star\kappa^m\right] \\ &+ \|h\|_\infty \sum_{Z \in \mathcal{Z}_{\sigma,b}^n} 2C_\star\kappa^m \Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1}).\end{aligned}$$

Dividing this inequality by $\Lambda_{\tau^m\sigma}(\widehat{\mathcal{L}}_\sigma^m \mathbb{1})$ and taking the limit $m \to \infty$ yields

$$\begin{aligned}\Lambda_\sigma(h) &\le \frac{\Lambda_\sigma(h)}{4} + B(1 + 2C_{n_0}\rho^{-n_0}) \bigvee h \sup_{Z \in \mathcal{Z}_{\sigma,g}^n} \Lambda_\sigma(\mathbb{1}_Z) \\ &\le \left[\frac{1}{4} + aB(1 + 2C_{n_0}\rho^{-n_0}) \sup_{Z \in \mathcal{Z}_{\sigma,g}^n} \Lambda_\sigma(\mathbb{1}_Z)\right]\Lambda_\sigma(h) \le \frac{1}{2}\Lambda_\sigma(h)\end{aligned}$$

where we have chosen n large enough and applied Lemma 9.8. This yields the announced contradiction. □

Remark 9.10. *Note that n_2 is, at least in principle, computable for any given pair $(\mathcal{T}, \mathcal{H})$. Let us explain how: recall the values of $C_\star$ and C_n from Remark 9.4, and the definitions of ρ, θ and D from Section 6.1.*

(1) *We choose $\eta \in (\frac{\theta}{\rho}, 1)$ and $n_0^\star$ such that $C_\star^{1/n}\theta\rho^{-1} \leq \eta$ for all $n \geq n_0^\star$ and then define $n_0 \geq n_0^\star$ so that $C_\star\eta^n \leq \frac{1}{4}$ for all $n \geq n_0$, see the proof of Lemma 9.6.*
(2) *We have $a_0 = 8C_{n_0}\rho^{-n_0} + \max_{i\leq n_0}\frac{C_i}{C_\star\theta^i}$, see the proof of Lemma 9.6. We then set $a = \max\{a_0, 1\}$.*
(3) *We have $B = 1 + 2aC_\star$, see the proof of Lemma 9.7.*
(4) *For $\delta > 0$, $n_1(\delta)$ is such that $D\theta^n\rho^{-n} \leq \delta$ for all $n \geq n_1(\delta)$, see the proof of Lemma 9.8.*
(5) *We set $\delta = \frac{1}{4}(aB(1 + 2C_{n_0}\rho^{-n_0}))^{-1}$ and choose $n_2 = n_1(\delta)$, see the proof of Lemma 9.9.*

Lemma 9.11. *If (C3(n_2, n_3)) holds for some $n_3 \geq n_2$, then there exists $\varrho > 0$ such that*

$$\inf \widehat{\mathcal{L}}_\sigma^n f \geq \varrho\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f),$$

for all $n \geq n_3$, all $\sigma \in \Sigma^{\mathbb{N}}$ and all $f \in C_\sigma^a$.

Proof. We first prove the result when $n = n_3$ and then extend it to all $n \geq n_3$. By Lemma 9.9, for each $f \in C_\sigma^a$, there exists $Z \in \mathcal{Z}_{\sigma,g}^{n_2}$ such that

$$\inf_Z f \geq \frac{1}{4}\Lambda_\sigma(f).$$

Consequently, for any n, we have

$$\inf \widehat{\mathcal{L}}_\sigma^n f \geq \frac{1}{4}\Lambda_\sigma(f) \inf \frac{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}_Z}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} \inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1}.$$

Since condition *(C3(n_2, n_3))* holds, one has

$$\inf \frac{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}_Z}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} \geq \epsilon_{n_2}$$

for all $\sigma \in \Sigma^{\mathbb{N}}$, $Z \in \mathcal{Z}_{\sigma,g}^{n_2}$ and $n \geq n_3$, and we thus get

$$\inf \widehat{\mathcal{L}}_\sigma^n f \geq \frac{\epsilon_{n_2}}{4}\Lambda_\sigma(f) \inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1}. \tag{9.2}$$

By Lemma 9.7,

$$\Lambda_\sigma(f) \geq B^{-1}\frac{\Lambda_{\tau^{n_3}\sigma}(\widehat{\mathcal{L}}_\sigma^{n_3} f)}{\Lambda_{\tau^{n_3}\sigma}(\widehat{\mathcal{L}}_\sigma^{n_3}\mathbb{1})} \geq B^{-1}\frac{\Lambda_{\tau^{n_3}\sigma}(\widehat{\mathcal{L}}_\sigma^{n_3} f)}{\sup \widehat{\mathcal{L}}_\sigma^{n_3}\mathbb{1}}.$$

By assumption *(C1)*, we have $\inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1} \geq \delta^n$ for any σ and n. Since $\mathcal{T}$ and $\mathcal{H}$ are finite, we can find a constant $M > 1$ such that $\|\widehat{\mathcal{L}}_\sigma^n \mathbb{1}\|_\infty \leq M^n$ for any σ and n.

We obtain

$$\inf \widehat{\mathcal{L}}_\sigma^{n_3} f \geq \varrho'\Lambda_{\tau^{n_3}\sigma}(\widehat{\mathcal{L}}_\sigma^{n_3} f),$$

with $\varrho' = B^{-1}\frac{\epsilon_{n_2}}{4}(\frac{\delta}{M})^{n_3}$.

By Lemma 9.6, $\widehat{\mathcal{L}}_\sigma^{n_0} C_\sigma^a \subset C_{\tau^{n_0}\sigma}^a$ for all σ. We thus have

$$\inf \widehat{\mathcal{L}}_\sigma^{n_3+kn_0} f \geq \varrho'\Lambda_{\tau^{n_3+kn_0}\sigma}(\widehat{\mathcal{L}}_\sigma^{n_3+kn_0} f),$$

for all $k \geq 0$, $\sigma \in \Sigma^{\mathbb{N}}$ and $f \in C_\sigma^a$.

Let now $n \geq n_3$. We write $n = kn_0 + n_3 + r = n' + r$ with $r < n_0$. We have

$$\begin{aligned}\inf \widehat{\mathcal{L}}_\sigma^n f = \widehat{\mathcal{L}}_{\tau^{n'}\sigma}^r \widehat{\mathcal{L}}_\sigma^{n'} f &\geq \varrho' \Lambda_{\tau^{n'}\sigma}(\widehat{\mathcal{L}}_\sigma^{n'} f) \inf \widehat{\mathcal{L}}_{\tau^{n'}\sigma}^r \mathbb{1} \\ &\geq \varrho' \delta^r \Lambda_{\tau^{n'}\sigma}(\widehat{\mathcal{L}}_\sigma^{n'} f) \\ &\geq \varrho' \delta^{n_0} \Lambda_{\tau^{n'}\sigma}(\widehat{\mathcal{L}}_\sigma^{n'} f).\end{aligned}$$

But,

$$\begin{aligned}\Lambda_{\tau^{n'}\sigma}(\widehat{\mathcal{L}}_\sigma^{n'} f) &= \lim_{k\to\infty} \inf \frac{\widehat{\mathcal{L}}_{\tau^{n'+r}\sigma}^k \widehat{\mathcal{L}}_{\tau^{n'}\sigma}^r \widehat{\mathcal{L}}_\sigma^{n'} f}{\widehat{\mathcal{L}}_{\tau^{n'+r}\sigma}^k \widehat{\mathcal{L}}_{\tau^{n'}\sigma}^r \mathbb{1}} \\ &\geq M^{-r} \lim_{k\to\infty} \inf \frac{\widehat{\mathcal{L}}_{\tau^{n'+r}\sigma}^k \widehat{\mathcal{L}}_\sigma^{n'+r} f}{\widehat{\mathcal{L}}_{\tau^{n'+r}\sigma}^k \mathbb{1}} \\ &= M^{-r} \Lambda_{\tau^{n'+r}\sigma}(\widehat{\mathcal{L}}_\sigma^{n'+r} f) \\ &\geq M^{-n_0} \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f).\end{aligned}$$

We have thus proved the result with $\varrho = \varrho'(\frac{\delta}{M})^{n_0}$.

□

Remark 9.12. *Since* $\|\widehat{\mathcal{L}}_\sigma^n \mathbb{1}\|_\infty \leq M^n$ *and* $\inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1} \geq \delta^n$ *for all* $n \geq 0$ *and* $\sigma \in \Sigma^{\mathbb{N}}$, *we have*

$$\inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1} \geq \varrho' \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1}),$$

with $\varrho' = \min\{\varrho, 1, \frac{\delta}{M}, \dots, (\frac{\delta}{M})^{n_3-1}\}$.

Lemma 9.13. *If* (C3(n_2, n_3)) *holds for some* $n_3 \geq n_2$, *then there exists* $a > 0$ *such that for all* $\sigma \in \Sigma^{\mathbb{N}}$ *and* $n \geq n_3$, *one has*

$$\widehat{\mathcal{L}}_\sigma^n \mathcal{C}_\sigma^a \subset C_{\tau^n\sigma}^a$$

with uniform finite diameter less than $\Delta_n < \infty$.

Proof. By (9.2), we have

$$\inf \widehat{\mathcal{L}}_\sigma^n h \geq \frac{\epsilon_{n_2}}{4} \Lambda_\sigma(h) \inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1},$$

for any $\sigma \in \Sigma^{\mathbb{N}}$, $h \in C_\sigma^a$ and $n \geq n_3$ and, using Lemma 9.7,

$$\sup \widehat{\mathcal{L}}_\sigma^n h \leq \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n h) + \bigvee \widehat{\mathcal{L}}_\sigma^n h \leq \left(1 + \frac{a}{2}\right) B \Lambda_\sigma(h) \Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1}).$$

Since $\|\widehat{\mathcal{L}}_\sigma^n \mathbb{1}\|_\infty \leq M^n$ and $\inf \widehat{\mathcal{L}}_\sigma^n \mathbb{1} \geq \delta^n$ for any σ and n, we obtain:

$$\mathrm{diam}_{\mathcal{C}_{\tau^n\sigma}^a} \widehat{\mathcal{L}}_\sigma^n(\mathcal{C}_\sigma^a) \leq 2\log\left[\frac{\max\left\{\frac{3}{2}, BM^n(1+\frac{a}{2})\right\}}{\min\left\{\frac{1}{2}, \frac{\epsilon_{n_2}\delta^n}{4}\right\}}\right] =: \Delta_n < \infty$$

for any $\sigma \in \Sigma^{\mathbb{N}}$.

Indeed, by a simple adaptation of the proof of [16, Lemma 3.1], one has

$$d_{\mathcal{C}_\sigma^a}(g, \mathbb{1}) \leq \log\left[\frac{\max\{(1+\nu)\Lambda_\sigma(g), \sup g\}}{\min\{(1-\nu)\Lambda_\sigma(g), \inf g\}}\right]$$

for any $g \in \mathcal{C}_\sigma^{\nu a}$ with $0 < \nu < 1$. Using this with $\nu = \frac{1}{2}$, we obtain the desired bound on the diameter, since $\widehat{\mathcal{L}}_\sigma^n(\mathcal{C}_\sigma^a) \subset \mathcal{C}_{\tau^n\sigma}^{a/2}$ by Lemma 9.6. □

Since we are interested in functions of the form $\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f$, we will need to consider functions f that belong to the intersections of all the cones C^a_σ, $\sigma \in \Sigma^{\mathbb{Z}}$. For this purpose, we introduce the family of cones

$$C^a_{\text{inf}} = \{f \in \text{BV} \,:\, f \neq 0,\ f \geq 0,\ \bigvee f \leq a \inf f\}.$$

We clearly have $C^a_{\text{inf}} \subset C^a_\sigma$ for any $\sigma \in \Sigma^{\mathbb{Z}}$, and thus $d_{C^a_\sigma} \leq d_{C^a_{\text{inf}}}$ by Theorem 9.1.

Lemma 9.14. *There exist $\nu \in (0,1)$ and a family of positive functions $\{h_\sigma\}_{\sigma \in \Sigma^{\mathbb{Z}}}$ in* BV *such that for all $f \in C^a_{\text{inf}}$, $\sigma \in \Sigma^{\mathbb{Z}}$ and $n \geq 0$:*

$$\left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} - h_\sigma \right\|_\infty \leq C_\# \nu^n,$$

and

$$\left\| \frac{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)}{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f} - h_\sigma^{-1} \right\|_\infty \leq C_\# \nu^n.$$

Furthermore, $h_\sigma \in C^{a/2}_\sigma$, $\|h_\sigma\|_\infty \leq 1 + a$ and $\inf h_\sigma \geq \varrho > 0$ for all $\sigma \in \Sigma^{\mathbb{Z}}$, where ϱ is defined in Lemma 9.11.

Proof. Writing $n \geq 2n_3$ as $n = kn_3 + r$, with $k \geq 2$ and $r < n_3$, for all $f \in C^a_{\text{inf}}$, $\sigma \in \Sigma^{\mathbb{Z}}$ and $m \geq 0$, we have by Theorem 9.1 and Lemma 9.13:

$$d_{C^a_\sigma}(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f, \widehat{\mathcal{L}}^{n+m}_{\tau^{-(n+m)}\sigma} f) \leq \gamma^{k-2} d_{C^a_{\tau^{-(k-2)n_3}\sigma}}(\widehat{\mathcal{L}}^{2n_3+r}_{\tau^{-n}\sigma} f, \widehat{\mathcal{L}}^{2n_3+r+m}_{\tau^{-(n+m)}\sigma} f),$$

with $\gamma = \tanh\left(\frac{\Delta_{n_3}}{4}\right) < 1$.

Since both $\widehat{\mathcal{L}}^{n_3+r}_{\tau^{-n}\sigma} f$ and $\widehat{\mathcal{L}}^{n_3+r+m}_{\tau^{-(n+m)}\sigma} f$ belong to $C^a_{\tau^{-(k-1)n_3}\sigma}$ by Lemma 9.13 again, as $f \in C^a_{\text{inf}} \subset C^a_{\tau^{-n}\sigma} \cap C^a_{\tau^{-(n+m)}\sigma}$, we have, using Lemma 9.13 one more time:

$$d_{C^a_{\tau^{-(k-2)n_3}\sigma}}(\widehat{\mathcal{L}}^{2n_3+r}_{\tau^{-n}\sigma} f, \widehat{\mathcal{L}}^{2n_3+r+m}_{\tau^{-(n+m)}\sigma} f) \leq \Delta_{n_3}.$$

Consequently, for all $n \geq 2n_3$, $m \geq 0$, $\sigma \in \Sigma^{\mathbb{Z}}$ and $f \in C^a_{\text{inf}}$,

$$d_{C^a_\sigma}(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f, \widehat{\mathcal{L}}^{n+m}_{\tau^{-(n+m)}\sigma} f) \leq C_\# \nu^n,$$

with $\nu = \gamma^{\frac{1}{n_3}}$.

Using Lemma 9.2 with $\|\cdot\| = \|\cdot\|_\infty$ and $\ell(\cdot) = \Lambda_\sigma(\cdot)$, we get

$$\left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} - \frac{\widehat{\mathcal{L}}^{n+m}_{\tau^{-(n+m)}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^{n+m}_{\tau^{-(n+m)}\sigma} f)} \right\|_\infty \leq (e^{d_{C^a_\sigma}(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f, \widehat{\mathcal{L}}^{n+m}_{\tau^{-(n+m)}\sigma} f)} - 1) \left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} \right\|_\infty.$$

Since $\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f \in C^a_\sigma$, we have

$$\|\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f\|_\infty \leq \Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f) + \bigvee \widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f \leq (1+a)\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f), \tag{9.3}$$

and we deduce that

$$\left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} - \frac{\widehat{\mathcal{L}}^{n+m}_{\tau^{-(n+m)}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^{n+m}_{\tau^{-(n+m)}\sigma} f)} \right\|_\infty \leq C_\# \nu^n.$$

This implies that $\frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)}$ is a Cauchy sequence in L^∞, and thus converges to a function $h_\sigma \in L^\infty$. Since $C^{a/2}_\sigma$ is closed in L^∞ and $\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f \in C^{a/2}_\sigma$ for $n \geq n_0$

by Lemma 9.6, we have $h_\sigma \in C_\sigma^{a/2}$. Passing to the limit $m \to \infty$ in the previous relation, we obtain

$$\left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} - h_\sigma \right\|_\infty \le C_\# \nu^n,$$

for all $n \ge 2n_3$, and all $\sigma \in \Sigma^{\mathbb{Z}}$. Using the same reasoning, we have for any pair $f, f' \in C^a_{\tau^{-n}\sigma}$,

$$\left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} - \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f'}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f')} \right\|_\infty \le C_\# \nu^n, \tag{9.4}$$

which proves that the limit h_σ does not depend on the choice of $f \in C^a_{\inf}$. By Lemma 9.11, $\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f \ge \varrho \Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)$ for all $n \ge n_3$, whence we obtain $\inf h_\sigma \ge \varrho$. Remark that (9.3) implies $\|h_\sigma\|_\infty \le 1 + a$. When $n < 2n_3$, we have

$$\left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} - h_\sigma \right\|_\infty \le \frac{M^{2n_3}}{\delta^{2n_3}} \frac{\sup f}{\inf f} + (1+a) \le (1+a)\left(\frac{M^{2n_3}}{\delta^{2n_3}} + 1 \right) \le C_\# \nu^n,$$

where $M > 1$ and $0 < \delta < 1$ are such that $\|\widehat{\mathcal{L}}^n_\sigma \mathbb{1}\|_\infty \le M^n$ and $\inf \widehat{\mathcal{L}}^n_\sigma \mathbb{1} \ge \delta^n$ for all $n \ge 0$ and σ. For $n \ge n_3$ and $f \in C^a_{\inf}$, since $\inf h_\sigma \ge \varrho$ and $\inf \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} \ge \varrho$ by Lemma 9.11, we have

$$\left\| \frac{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)}{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f} - h_\sigma^{-1} \right\|_\infty \le \varrho^{-2} \left\| \frac{\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} f)} - h_\sigma \right\|_\infty \le C_\# \nu^n.$$

We handle the case $n < n_3$ as previously, since $\|h_\sigma\|_\infty \le 1 + a$. □

Lemma 9.15. *For all $\sigma \in \Sigma^{\mathbb{Z}}$, there exists $\lambda_\sigma \ge \rho_\sigma$ such that $\widehat{\mathcal{L}}^1_\sigma h_\sigma = \lambda_\sigma h_{\tau\sigma}$.*

Proof. Applying Lemma 9.7 with $h = \widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} \mathbb{1}$, we have by definition of $\rho_\sigma = \Lambda_{\tau\sigma}(\widehat{\mathcal{L}}^1_\sigma \mathbb{1})$:

$$\rho_\sigma \le \frac{\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}^{n+1}_{\tau^{-n}\sigma} \mathbb{1})}{\Lambda_\sigma(\widehat{\mathcal{L}}^n_{\tau^{-n}\sigma} \mathbb{1})} \le B\rho_\sigma.$$

Consequently, there exist a subsequence $\{n_j\}$ and $\lambda_\sigma \in [\rho_\sigma, B\rho_\sigma]$ such that

$$\lambda_\sigma = \lim_j \frac{\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}^{n_j+1}_{\tau^{-n_j}\sigma} \mathbb{1})}{\Lambda_\sigma(\widehat{\mathcal{L}}^{n_j}_{\tau^{-n_j}\sigma} \mathbb{1})}.$$

We can now compute

$$\begin{aligned}
\widehat{\mathcal{L}}^1_\sigma h_\sigma = \lim_j \widehat{\mathcal{L}}^1_\sigma \frac{\widehat{\mathcal{L}}^{n_j}_{\tau^{-n_j}\sigma} \mathbb{1}}{\Lambda_\sigma(\widehat{\mathcal{L}}^{n_j}_{\tau^{-n_j}\sigma} \mathbb{1})} &= \lim_j \frac{\widehat{\mathcal{L}}^{n_j+1}_{\tau^{-n_j}\sigma} \mathbb{1}}{\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}^{n_j+1}_{\tau^{-n_j}\sigma} \mathbb{1})} \frac{\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}^{n_j+1}_{\tau^{-n_j}\sigma} \mathbb{1})}{\Lambda_\sigma(\widehat{\mathcal{L}}^{n_j}_{\tau^{-n_j}\sigma} \mathbb{1})} \\
&= \lim_j \frac{\widehat{\mathcal{L}}^{n_j+1}_{\tau^{-(n_j+1)}\tau\sigma} \mathbb{1}}{\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}^{n_j+1}_{\tau^{-(n_j+1)}\tau\sigma} \mathbb{1})} \frac{\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}^{n_j+1}_{\tau^{-n_j}\sigma} \mathbb{1})}{\Lambda_\sigma(\widehat{\mathcal{L}}^{n_j}_{\tau^{-n_j}\sigma} \mathbb{1})} \\
&= \lambda_\sigma h_{\tau\sigma}.
\end{aligned}$$

□

Lemma 9.16. *For all $\sigma \in \Sigma^{\mathbb{N}}$, the functional Λ_σ (restricted to* BV*) is linear, positive, and enjoys the property $\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}_\sigma^1 f) = \rho_\sigma \Lambda_\sigma(f)$ for all $f \in$ BV. Moreover, $\lambda_\sigma = \rho_\sigma$ and $\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - \Lambda_\sigma(f) \right\|_\infty \leq C_\# \nu^n \|f\|_{\mathrm{BV}}$ for all $f \in$ BV.*

Proof. For $f \in C_{\inf}^a$, we can write

$$\frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} = \frac{\widehat{\mathcal{L}}_\sigma^n f}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)} \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})} \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}}.$$

So

$$\begin{aligned}\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})} \right\|_\infty &= \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})} \left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)} \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - 1 \right\|_\infty \\ \leq \|f\|_\infty \left(\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)} - h_\sigma \right\|_\infty \left\| \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} \right\|_\infty \right. & \left. + \|h_\sigma\|_\infty \left\| \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - h_\sigma^{-1} \right\|_\infty \right).\end{aligned}$$

Since $\left\| \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} \right\|_\infty \leq \varrho^{-1}$ for $n \geq n_3$ by Lemma 9.11, we get, using Lemma 9.14, for all $f \in C_{\inf}^a$ and $n \geq n_3$:

$$\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})} \right\|_\infty \leq C_\# \nu^n \|f\|_\infty. \tag{9.5}$$

But, $\Lambda_\sigma(f) = \lim_{n\to\infty} \inf \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}}$ by definition, and, since $\frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})}$ are constants, we deduce that $\lim_{n\to\infty} \frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})} = \Lambda_\sigma(f)$ and

$$\lim_{n\to\infty} \left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - \Lambda_\sigma(f) \right\|_\infty = 0.$$

Now, if $f \in$ BV, we have $f + c \in C_{\inf}^a$ for $c = (1 + a^{-1})\|f\|_{\mathrm{BV}}$, so we get that $\Lambda_\sigma(f) = \lim_{n\to\infty} \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}}$ in L^∞ for all $f \in$ BV, since $\Lambda_\sigma(f + c) = \Lambda_\sigma(f) + c$. The linearity of Λ follows from the linearity of the limit.

Next, as $\widehat{\mathcal{L}}_\sigma^1 f \in$ BV, we know that

$$\Lambda_{\tau\sigma}(\widehat{\mathcal{L}}_\sigma^1 f) = \lim_{n\to\infty} \frac{\widehat{\mathcal{L}}_\sigma^{n+1} f}{\widehat{\mathcal{L}}_{\tau\sigma}^n \mathbb{1}} = \lim_{n\to\infty} \frac{\widehat{\mathcal{L}}_\sigma^{n+1} f}{\widehat{\mathcal{L}}_\sigma^{n+1} \mathbb{1}} \frac{\widehat{\mathcal{L}}_\sigma^{n+1} \mathbb{1}}{\widehat{\mathcal{L}}_{\tau\sigma}^n \mathbb{1}} = \Lambda_\sigma(f) \Lambda_{\tau\sigma}(\widehat{\mathcal{L}}_\sigma^1 \mathbb{1}) = \rho_\sigma \Lambda_\sigma(f).$$

But then $\rho_\sigma = \lambda_\sigma$ is obtained by taking $f = h_\sigma$, since $\Lambda_\sigma(h_\sigma) = 1$. In particular, we have $\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f) = \rho_\sigma \cdots \rho_{\tau^{n-1}\sigma} \Lambda_\sigma(f)$ for all $f \in$ BV, so $\frac{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n f)}{\Lambda_{\tau^n\sigma}(\widehat{\mathcal{L}}_\sigma^n \mathbb{1})} = \Lambda_\sigma(f)$. But if we look back to (9.5), the above implies that for all $f \in C_{\inf}^a$ and $n \geq n_3$:

$$\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - \Lambda_\sigma(f) \right\|_\infty \leq C_\# \nu^n \|f\|_\infty.$$

This can be easily extended to all $n \geq 0$, since $\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - \Lambda_\sigma(f) \right\|_\infty \leq 2\|f\|_\infty$. We can again cover the general case $f \in$ BV using the fact that $f + c \in C_{\inf}^a$ for $c = (1 + a^{-1})\|f\|_{\mathrm{BV}}$, which finally implies

$$\left\| \frac{\widehat{\mathcal{L}}_\sigma^n f}{\widehat{\mathcal{L}}_\sigma^n \mathbb{1}} - \Lambda_\sigma(f) \right\|_\infty \leq C_\# \nu^n \|f + c\|_\infty \leq C_\# \nu^n \|f\|_{\mathrm{BV}}.$$

□

Remark 9.17. *Following the ideas of* [17]*, it is possible to prove that* Λ_σ *can be interpreted as a non-atomic measure* μ_σ*, i.e.* $\Lambda_\sigma(f) = \int f \, d\mu_\sigma$ *for all* $f \in \mathrm{BV}$*, and that the measure* ν_σ *defined by* $d\nu_\sigma = h_\sigma d\mu_\sigma$ *satisfies* $(T^1_\sigma)_\star \nu_\sigma = \nu_{\tau\sigma}$*. Since we will not make use of these facts, we leave their proofs to the interested reader.*

The main properties of Λ_σ being proved, we can now improve Lemma 9.14 by extending it to general functions in BV. This allows to deduce Proposition 8.1:

Proof of Proposition 8.1. By (9.4) with $f' = h_\sigma$, for any $f \in C^a_\sigma$, we get using Lemma 9.16

$$\left\| \frac{\widehat{\mathcal{L}}^n_\sigma f}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} - \Lambda_\sigma(f) h_{\tau^n\sigma} \right\|_\infty \le C_\# \nu^n \Lambda_\sigma(f) \le C_\# \nu^n \|f\|_\infty.$$

Now, if $f \in \mathrm{BV}$, we have $f + ch_\sigma \in C^a_\sigma$ for all $\sigma \in \Sigma^{\mathbb{Z}}$ with $c = 2(1+a^{-1})\|f\|_{\mathrm{BV}}$. Indeed, since $\bigvee h_\sigma \le \frac{a}{2}\Lambda_\sigma(h_\sigma) = \frac{a}{2}$ by Lemma 9.14, we have

$$\bigvee (f + ch_\sigma) \le \bigvee f + c \bigvee h_\sigma \le \bigvee f + \frac{ac}{2},$$

and

$$\Lambda_\sigma(f + ch_\sigma) = \Lambda_\sigma(f) + c \ge \inf f + c.$$

So $f + ch_\sigma \in C^a_\sigma$ if $c \ge 2(a^{-1}\bigvee f - \inf f)$, which is the case for our particular choice of c. Consequently, we have

$$\left\| \frac{\widehat{\mathcal{L}}^n_\sigma (f + ch_\sigma)}{\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma}} - \Lambda_\sigma(f + ch_\sigma) h_{\tau^n\sigma} \right\|_\infty \le C_\# \nu^n \|f + ch_\sigma\|_\infty \le C_\# \nu^n \|f\|_{\mathrm{BV}},$$

which leads to (8.1) after simplifications, since $\widehat{\mathcal{L}}^n_\sigma(f + ch_\sigma) = \widehat{\mathcal{L}}^n_\sigma f + c\rho_\sigma \cdots \rho_{\tau^{n-1}\sigma} h_{\tau^n\sigma}$ and $\Lambda_\sigma(f + ch_\sigma) = \Lambda_\sigma(f) + c$. □

References

[1] G. Atkinson. Recurrence of co-cycles and random walks. *Journal of the London Mathematical Society*, s2-13(3):486–488, 1976.

[2] L. A. Bunimovich, Y. G. Sinaĭ, and N. I. Chernov. Statistical properties of two-dimensional hyperbolic billiards. *Uspekhi Mat. Nauk*, 46(4(280)):43–92, 192, 1991.

[3] J.-P. Conze. Sur un critère de récurrence en dimension 2 pour les marches stationnaires, applications. *Ergodic Theory Dynam. Systems*, 19(5):1233–1245, 1999.

[4] G. Cristadoro, M. Lenci, and M. Seri. Erratum: "Recurrence for quenched random Lorentz tubes" [Chaos 20, 023115 (2010)] [mr2741899]. *Chaos*, 20(4):049903, 1, 2010.

[5] G. Cristadoro, M. Lenci, and M. Seri. Recurrence for quenched random Lorentz tubes. *Chaos*, 20(2):023115, 7, 2010.

[6] J. De Simoi and C. Liverani. The martingale approach after Varadhan and Dolgopyat. In *Hyperbolic dynamics, fluctuations and large deviations*, volume 89 of *Proc. Sympos. Pure Math.*, pages 311–339. Amer. Math. Soc., Providence, RI, 2015.

[7] D. Dolgopyat, D. Szász, and T. Varjú. Recurrence properties of planar Lorentz process. *Duke Math. J.*, 142(2):241–281, 2008.

[8] D. Dolgopyat, D. Szász, and T. Varjú. Limit theorems for locally perturbed planar Lorentz processes. *Duke Math. J.*, 148(3):459–499, 2009.

[9] N. Dunford, J. Schwartz, W. Bade, and R. Bartle. *Linear Operators: General theory. Part. I.* Linear Operators. Interscience Publishers, 1988.

[10] M. Lenci. Aperiodic Lorentz gas: recurrence and ergodicity. *Ergodic Theory Dynam. Systems*, 23(3):869–883, 2003.

[11] M. Lenci. Typicality of recurrence for Lorentz gases. *Ergodic Theory Dynam. Systems*, 26(3):799–820, 2006.

[12] M. Lenci. Central limit theorem and recurrence for random walks in bistochastic random environments. *Journal of Mathematical Physics*, 49(12):125213, 2008.
[13] M. Lenci and S. Troubetzkoy. Infinite-horizon Lorentz tubes and gases: recurrence and ergodic properties. *Phys. D*, 240(19):1510–1515, 2011.
[14] C. Little. Deterministically driven random walks in a random environment on $\mathbb{Z}$. *Discrete and Continuous Dynamical Systems - A*, 36:5555–5578, 2016.
[15] C. Liverani. Decay of correlations. *Ann. of Math. (2)*, 142(2):239–301, 1995.
[16] C. Liverani. Decay of correlations for piecewise expanding maps. *Journal of Statistical Physics*, 78(3):1111–1129, 1995.
[17] C. Liverani and V. Maume-Deschamps. Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(3):385–412, 2003.
[18] C. Liverani, B. Saussol, and S. Vaienti. Conformal measure and decay of correlation for covering weighted systems. *Ergodic Theory Dynam. Systems*, 18(6):1399–1420, 1998.
[19] J. Marklof. The low-density limit of the lorentz gas: periodic, aperiodic and random. In *Proceedings of the ICM 2014, Seoul, Vol. III*, pages 623–646. 2004.
[20] J. Marklof and A. Strömbergsson. The Boltzmann-Grad limit of the periodic Lorentz gas. *Ann. of Math. (2)*, 174(1):225–298, 2011.
[21] A. Mohapatra and W. Ott. Memory loss for nonequilibrium open dynamical systems. *Discrete Contin. Dyn. Syst.*, 34(9):3747–3759, 2014.
[22] W. Rudin. *Real and complex analysis*. Mathematics series. McGraw-Hill, 1987.
[23] K. Schmidt. On joint recurrence. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(9):837–842, 1998.
[24] K. Schmidt. Recurrence of cocycles and stationary random walks. In *Dynamics & stochastics*, volume 48 of *IMS Lecture Notes Monogr. Ser.*, pages 78–84. Inst. Math. Statist., Beachwood, OH, 2006.
[25] M. Seri, M. Lenci, M. degli Esposti, and G. Cristadoro. Recurrence and higher ergodic properties for quenched random Lorentz tubes in dimension bigger than two. *J. Stat. Phys.*, 144(1):124–138, 2011.
[26] T. Simula and M. Stenlund. Deterministic walks in quenched random environments of chaotic maps. *Journal of Physics A: Mathematical and Theoretical*, 42(24):245101, 2009.
[27] O. Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin, 2004.

Romain Aimino, Departamento de Matemética, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal.
E-mail address: romain.aimino@fc.up.pt

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: liverani@mat.uniroma2.it