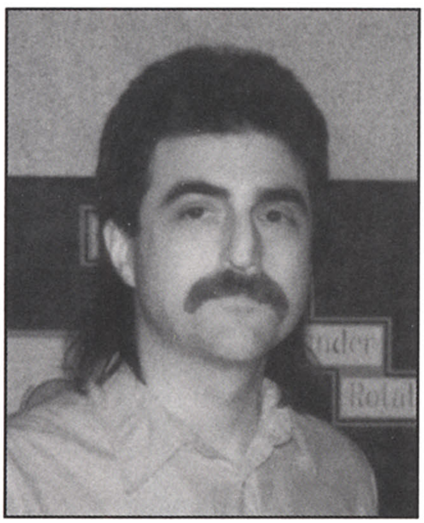
The Greater-Than-g Acceleration of a Bungee Jumper

By David Kagan and Alan Kott



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As teachers we have all hoped that some of our students would jump off a bridge. While discussing this possibility with one of my students (Kott), we decided that not only would it be a good idea to be tied to the bridge with an elastic cord, but that we might learn some very interesting physics by videotaping the jump. Originally we thought that we would study the elastic properties of the cord, but discovered that a recent article in *The Physics Teacher*[1] describes these properties in great detail. In addition, it contains a brief history of the intriguing sport of bungee jumping.

For lack of anything better to do with our video data, we decided to use it to measure the acceleration due to gravity. After studying several jumps we were deeply troubled. When a second-order polynomial fit was applied to the observed position versus time data, the apparent acceleration was about 1.5 g! This is contrary to our usual experience with freely falling objects and to the usual assumptions made when modeling bungee jumping.

Of course our first thought was that there were problems with the data collection process. After we satisfied ourselves that this was not the case, we began to address the possibility that the downward acceleration could really be greater than g. Because it is so hard to believe that a bungee jumper falls with an acceleration greater than 9.8 m/s^2, further experiments were needed to support such a conjecture. Alas, bungee jumping is expensive and the possibility of losing a student is always disturbing (except perhaps during a lecture). We decided to search for preliminary evidence in the laboratory.

Using strobe photography we compared the motion of a freely falling ball with the motion of the free end of a rope supported by its other end. The ball and one end of the rope were released at the same time. Figure 1 shows the ball starting off with a very slight lead. It is, however, the latter part of the fall that tells the tale. Near the bottom of the motion, the displacement of the tip of the rope visibly exceeds that of the freely falling ball!

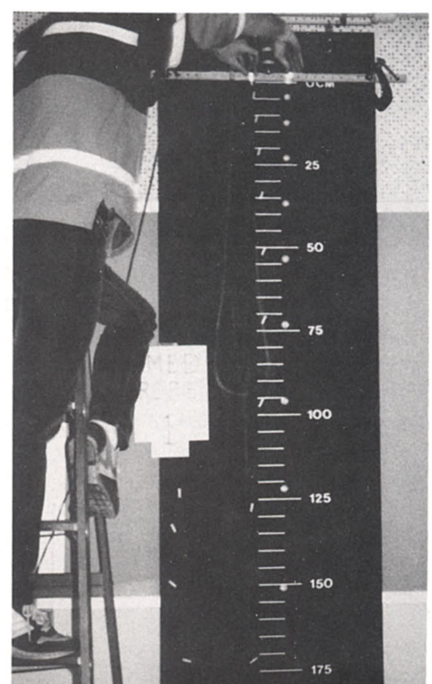


Fig. 1. Flash photograph of a freely falling ball and the free end of a rope released simultaneously. The end of the rope accelerates much faster than the ball.

We will explain this anomalous acceleration using energy considerations. Then we will present experimental results using several different types of rope and a chain. These experiments clearly establish the fact that the acceleration of a bungee jumper can be greater than g.

Theory

The motion of the bungee cord and jumper is similar to that of a whip. When a whip is cracked, energy from the entire whip is transferred via internal forces to the tip in such a way that initial speeds of a few meters per second become supersonic at the tip. Since our knowledge of the internal forces in the cord is severely limited, we decided to approach the problem from the point of view of energy.

Initially, one end of the bungee cord is attached to the bridge and the other end is attached to the jumper. The remainder of the cord hangs freely as shown in Fig. 2. We will examine only the "free fall" portion of the motion, that is, the motion before the jumper has fallen a distance equal to the unstretched length of the cord. The total mass of the cord is often of the order of the mass of the jumper, so it is not negligible. The mass of the bungee cord actually plays a key role in understanding the anomalous acceleration, so we will include not only the energy of the jumper, but the energy of the bungee cord as well. In this model we will ignore air resistance, all horizontal motion of the system, and any energy retained in the cord in forms such as the elastic stretching or thermal energy. In doing so, we must keep in mind that the results must now be considered as upward limits.

Referring to Fig. 2, the initial potential energy of the center of mass of the cord referenced to the bridge is

$$E_{pi} = -mg\frac{L}{4} \tag{1}$$

where m is the mass of the cord and L is its unstretched length. The initial potential energy of the jumper is zero, as is the initial kinetic energy of the cord and jumper. Therefore, the total initial energy is just the initial potential energy of the cord,

$$E_i = -mg\frac{L}{4} \tag{2}$$

After falling a distance y, the kinetic energy of the cord-jumper system is

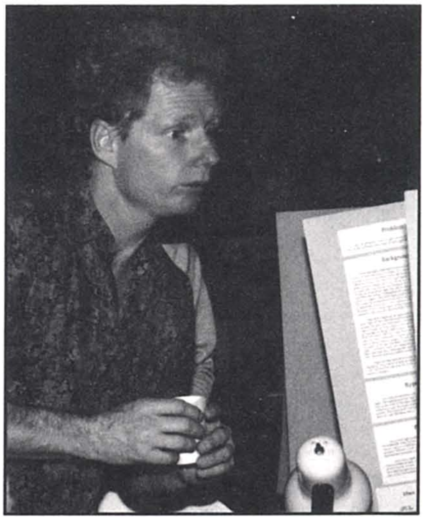
$$E_k = \tfrac{1}{2}m\frac{L-y}{2L}v^2 + \tfrac{1}{2}Mv^2 \tag{3}$$

where M is the mass of the jumper and v is the speed of the jumper, which must also equal the speed of the falling portion of the cord. The fraction after the mass of the cord represents the portion of the cord that is in motion. The potential energy of the falling portion of the cord is

$$E_{p,\,moving} = -m\frac{L-y}{2L}g\left(y+\frac{L-y}{4}\right) \tag{4}$$

The quantity after g is the location of the center of mass of this part of the cord. The potential energy of the still portion of the cord is

$$E_{p,\,still} = -m\frac{L+y}{2L}g\frac{L+y}{4} \tag{5}$$



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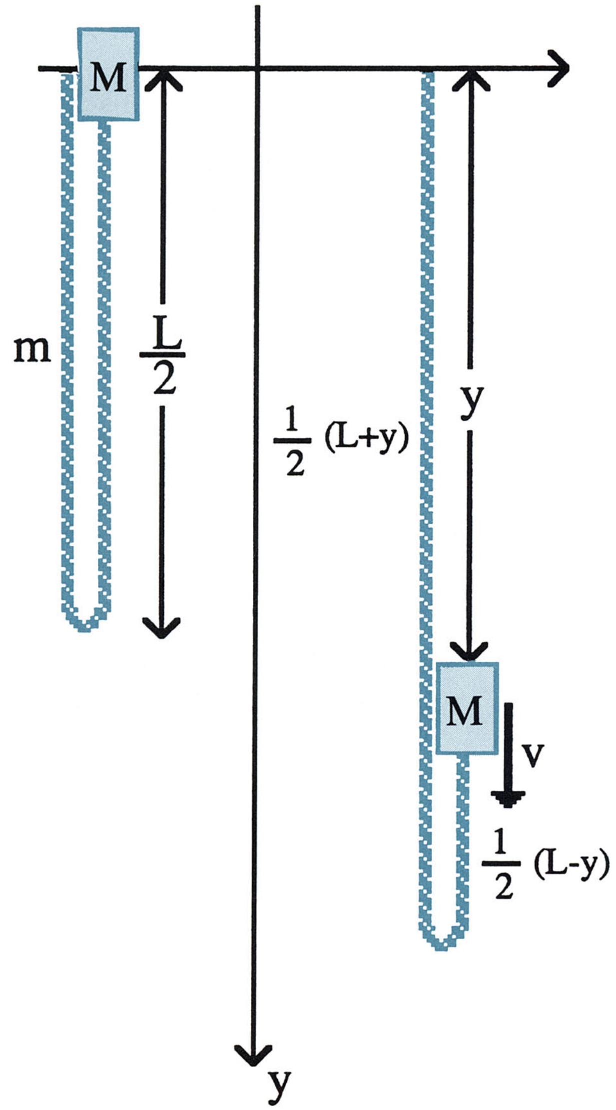

Fig. 2. Initially a jumper of mass *M* attached to a cord of length *L* is at rest at the top of the bridge (left). Later the jumper has fallen a distance *y* and is traveling at a speed *v* (right).

The quantity after the mass of the cord represents the fraction of the cord that is at rest; the fraction after g is the location of the center of mass of this part of the cord.

Using Eqs. (3), (4), and (5), the total final energy of the cord and jumper is now

$$E_f = \left(\frac{1}{2} m \frac{L-y}{2L} v^2 + \frac{1}{2} M v^2\right) - Mgy - m\frac{L-y}{2L} g \left(y + \frac{L-y}{4}\right) - m \frac{L+y}{2L} g \frac{L+y}{4} \tag{6}$$

The initial energy of Eq. (2) can be set equal to the final energy of Eq. (6). Solving for speed,

$$v^2 = gy \frac{(4ML + 2mL - my)}{(mL - my + 2ML)} \tag{7}$$

Note that this result agrees with the proper solution for a falling chain.[2] This solution has in fact been experimentally verified by Calkin and March.[3]

We see that the final speed at $y = L$ is

$$v^2 = 2gL + \frac{1}{2}\frac{m}{M} gL \tag{8}$$

The first term is the standard result for free fall. The second term, which tends to increase the final speed, is a result of the potential energy loss of the falling bungee cord. It is this second term that yields an average acceleration greater than g. As expected, this term goes to zero when the mass of the cord is small compared with the mass of the jumper.

The acceleration can be found by differentiating Eq. (7) with respect to time:

$$2v\frac{dv}{dt} = g\frac{dy}{dt}\left[\frac{(mL - my + 2ML)(4ML + 2mL - 2my) + my(4ML + 2mL - my)}{(mL - my + 2ML)^2}\right] \tag{9}$$

Using a for the acceleration instead of $\frac{dv}{dt}$, canceling v with $\frac{dy}{dt}$, and simplifying further,

$$a = g\left[1 + \frac{my(4ML + 2mL - my)}{2(mL - my + 2ML)^2}\right] \tag{10}$$

Notice that as the mass of the bungee becomes negligible (m tends to zero) the acceleration of the jumper becomes g as expected. When the jumper leaves the bridge ($y = 0$) the initial acceleration is g, as might also be expected. Performing the straightforward, but tedious differentiation of Eq. (10) with respect to the mass of the bungee yields

$$\frac{da}{dm} = MgyL\frac{m(L-y) + 2(M+m)L}{[2ML + m(L-y)]^3} \tag{11}$$

Since this quantity is always positive, the acceleration must get larger as the mass of the bungee cord increases.

To examine the change in acceleration as the fall progresses, we must look at the differentiation of Eq. (10) with respect to the distance:

$$\frac{da}{dy} = mg\frac{(2M+m)^2 L^2}{[2ML + m(L-y)]^3} \tag{12}$$

Since this is positive for all values of $y < L$, the acceleration grows as the jumper falls. Any measurement of the average acceleration will therefore give a value greater than g. The maximum acceleration must occur when $y = L$:

$$a_{y=L} = g\left[1+\frac{m(4M+m)}{8M^2}\right] \tag{13}$$

This result is more conveniently expressed in terms of the ratio of the mass of the cord to the mass of the jumper:

$$\mu \equiv \frac{m}{M} \tag{14}$$

Now we can write

$$a_{y=L} = g\left[1+\frac{\mu(4+\mu)}{8}\right] \tag{15}$$

For a bungee cord that has the same mass as the jumper, the maximum acceleration is approximately 1.6 g. This effect is large enough for a jumper to notice, providing the jumper is noticing such things at a time like this.

This result adds an interesting twist to the problem posed by Peter Palffy-Muhoray in the *American Journal of Physics*.[4] In that problem, the bungee cord is assumed to be massless. He goes on to show that the jumper's maximum acceleration is roughly 3g's and it occurs when the cord is at its maximum stretch. Yet, Eq. (15) suggests that an acceleration of 3g's will occur before the cord even stretches, if the mass of the bungee is at least two and a half times the mass of the jumper. It should be noted that in the solution to the problem, Palffy-Muhoray assumes that the jumper's acceleration is g during the "free fall." He should be forgiven however, because as Calkin and March[3] demonstrate, this problem has been solved incorrectly for over 100 years!

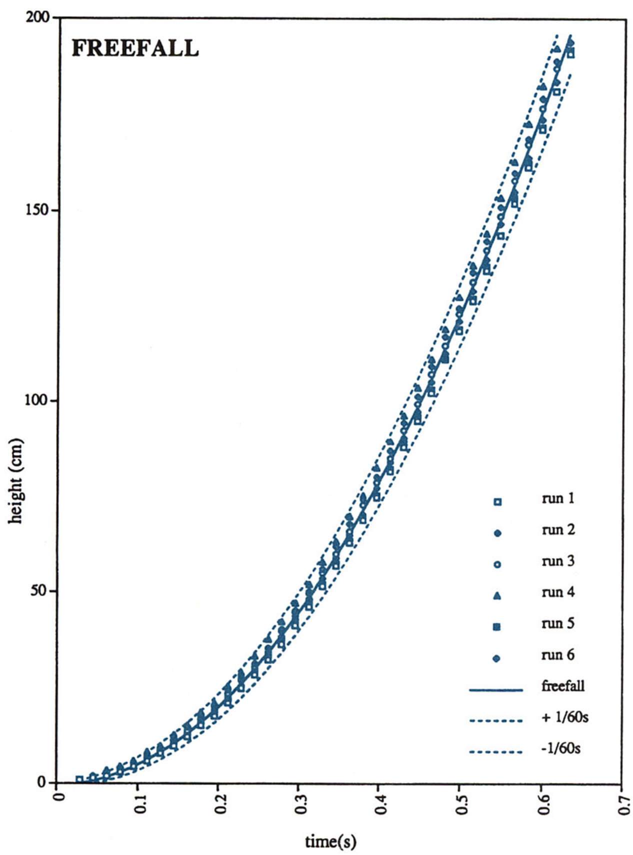

Fig. 3. Six free-fall trials including the theoretical curve (solid line). All data fall within a band ±1/60 s from the theory curve, as might be expected when using a video camera.

While the energy model does clearly predict an acceleration greater than g, it is just not as intuitively satisfying as a force model. The forces responsible for the motion are gravity, the contact force exerted by the bridge on the cord, and the shear and compressional (tension) forces within the cord. To fully understand the behavior of the system from a force perspective would require a complete finite element analysis of the cord. While this analysis would be interesting, it would not lead to deep intuitive understanding. Some insight can be gleaned from thinking about the fact that the portion of cord at the bottom of the loop is coming to rest. Since it was moving downward, an upward force is required on this part of the system. Since this upward force is an internal force, and Newton's Third Law must be satisfied, there must be an equal and opposite downward force somewhere within the system. The surprise is that a portion of this reaction force acts on the falling portion of the cord, causing it to accelerate. A portion also acts on the stationary part of the cord.

If the falling portion of the cord were accelerating at g, then the downward reaction force on it would not exist. All the force required to stop the cord would come from the stationary side. Therefore the tension in the cord on this side

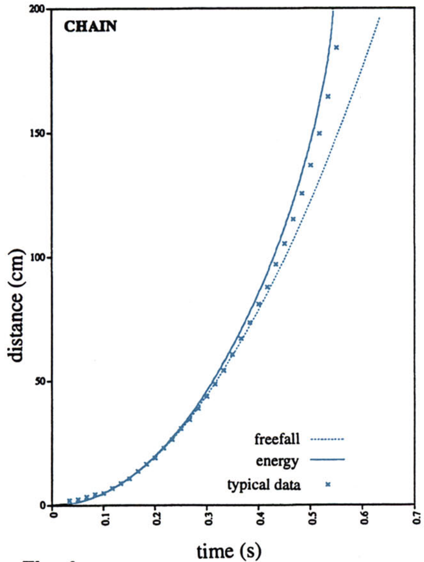

Fig. 4a.

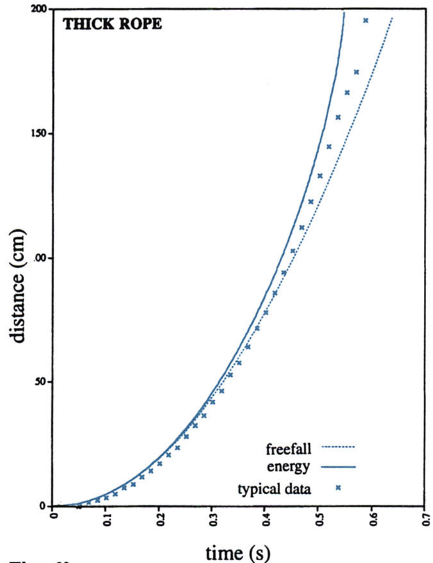

Fig. 4b.

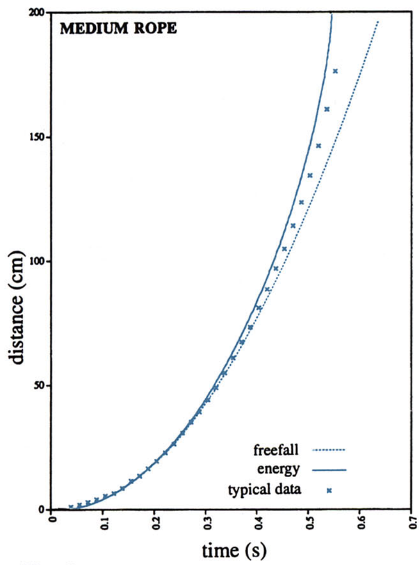

Fig. 4c.

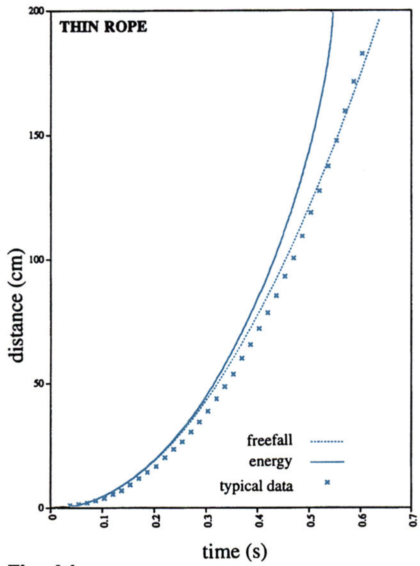

Fig. 4d.

Fig. 4. Typical data sets from one of six trials using a) chain, b) thick rope, c) medium rope, and d) thin rope. The solid line is the energy model; the dashed line is free fall.

would be larger than in the energy conservation model where some of the force comes from the moving side. It is this tension exerted on the bridge that was measured by Calkin and March.[3] They found that for the case of chains the tension was in fact smaller and very consistent with the energy model described here. In addition, Calkin and March state that their flash photographs also reveal accelerations greater than g at the tip of the chain. Unfortunately, these photographs do not appear in their paper.

In a way, the falling portion of the cord can be thought of as a rocket moving downward. It is "ejecting" cord as its "fuel" in such a way that the ejected cord is always at rest. So this ejected cord is "fired" upward, producing a downward "thrust" on the remaining portion of moving cord.

Experiment

To reproduce the effects of a massive bungee cord in a more controllable environment, we opted to reduce the scale of the experiment. Using items common in any instructional physics laboratory, we set up the experiment in miniature. Using three 2-m ropes of different thicknesses and a 2-m linked chain, we constructed a variety of bungee systems with massive cords and "massless" jumpers. The massless jumper maximizes the difference between the energy model and free fall. The purpose of using ropes of different thicknesses was to compare any effects that the stiffness of the ropes might introduce. The chain was used because of its minimal stiffness.

A video camera recorded the entire fall of our laboratory bungees. The camera rested on a tripod 2 m from the cords, which were set up to fall in front of a dark backboard. The backboard was marked with increments of distance as can be seen in Fig. 1. First we established a control by recording data for a freely falling ball. We carried out six trials and plotted the data on a graph (Fig. 3) against a theoretical curve for true free fall. The data fit well within $\pm 1/60$ of a second to the free-fall curve. This spread represents the frame speed of the video camera and the resulting uncertainty of the time measurements.

The next step was to construct the upper-limit curve of position of the free end of the bungee versus time for the energy model. Since Eq. (10) cannot be solved directly for the position as a function of time, a numerical method, such as a spreadsheet, is required.[5] This model provides an excellent example of a "real-life" physics problem that can't be solved analytically but is tractable with a spreadsheet.

Six sets of data were recorded for each type of bungee. These data show the same types of variations that the free-fall data exhibit. The resulting plots are shown in Figs. 4a–4d.

Results

As expected, data for all four laboratory bungees fell between the true free-fall curve and the upper limit set by the energy model. We suspect that this is due to the energy we didn't account for in the horizontal motion of the system, thermal and elastic energy in the rope, and perhaps air resistance. The shape of the curves of the actual data more closely resembles the energy model than the true free-fall parabola. The chain's motion most closely approaches the curve for the energy model. The thick and medium ropes seem to experience additional energy loss and fall somewhere between the two curves. Of the three ropes, the thin one most closely exhibits free-fall characteristics, although the shape is similar in form to the energy curve.

Conclusions

The uncertainties in the chain and rope data are comparable in size to those illustrated in the free-fall data of Fig. 3. Since these uncertainties are too small to explain the deviations of the rope and chain data from the free-fall curves, we believe that our experiments clearly show an acceleration that exceeds g. Also, the data may indicate some differences between the behavior of the various ropes and the chain, but these differences are much less certain. Perhaps these differences can be explained by the different "stiffness" of each rope. By stiffness, we mean the ability to support compression as opposed to extension. The lighter rope seemed to be stiffer than the heavier ropes. This notion is supported by the steeper velocity curve for the chain which, because of its construction, is not stiff at all.

If indeed the energy model we have constructed is a good approximation to the real world, and the data supports this view, then there are some additional concerns for bungee jumpers. As shown earlier, when the jumper has fallen the length of the cord, the acceleration is purely a function of the ratio of the mass of the bungee cord to the mass of the jumper. If the mass of the bungee is five times that of the jumper, according to Eq. (15), the final acceleration would be about 6.6 g! Accelerations on this order can cause physiological problems for humans.

There is some very interesting physics illustrated by the fact that the laboratory bungees do not strictly obey either the energy curve or the free-fall curve. Real ropes are much more complex than usually assumed in basic physics classes. Perhaps that finite element analysis might be worth doing after all. Besides, work at a computer keyboard is much safer than jumping from a bridge tied to a very massive bungee cord.

Perhaps the moral of this tale is that we should be more willing to tell our students to jump off a bridge. After all, look at the wonderfully interesting physics to be learned!

Acknowledgments

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5. See Ref. 2 for a solution in terms of elliptic integrals.