Higher Order Gravity Tensors

In four dimensions, field equations for gravity are taken to be

$$G_{ab} + \Lambda g_{ab} = 0$$

since this is the most general tensor which

1. is symmetric
2. depends only on the metric and its first two derivatives
3. is divergence free

(4. is linear in second derivatives of the metric)

But in five dimensions, the second order Lovelock tensor

$$H_{ab} = RR_{ab} - 2R_{ac}R^{c}{}_{b} - 2R^{cd}R_{acbd} + R_{a}{}^{cde}R_{bcde} - \frac{1}{4}g_{ab}\mathcal{L}_{\mathrm{GB}}$$

which can be obtained from variation of action containing Gauss-Bonnet term

$$\mathcal{L}_{\mathrm{GB}} = R^2 - 4R_{ab}R^{ab} + R^{abcd}R_{abcd}$$

also satisfies required conditions.

Thus should take 5-d field equations to be

$$G_{ab} + 2\alpha H_{ab} + \Lambda g_{ab} = 0$$

Should also consider extra term since $\mathcal{L}_{\mathrm{GB}}$ appears as quantum gravity correction to string theory tree level effective action.

Particularly relevant for brane world models, since they are motivated by string theory.

If dilaton is included, general second order term is instead

$$\mathcal{L}_2 = c_1\mathcal{L}_{\text{GB}} - 16c_2 G_{ab}\nabla^a\phi\nabla^b\phi + 16c_3(\nabla\phi)^2\nabla^2\phi - 16c_4(\nabla\phi)^4$$

Not all coefficients fixed by string theory.

In fact low energy string theory action also suggests other possibilities:

$$R^2 \ , \quad R_{ab}R^{ab} \ , \quad R(\nabla\phi)^2 \ , \quad R\nabla^2\phi$$

However these all give ghosts at high energy (but not actually ruled out for string theory since dealing with low energy effective action).

General bulk action

$$S_{\text{Bulk}} = \frac{M^3}{2}\int d^5x\sqrt{-g}e^{-2\phi}\left\{R - 4\omega(\nabla\phi)^2 + M^{-2}\mathcal{L}_2 - 2\Lambda\right\}$$

Coefficients can be determined from origin of ϕ.
For toroidal compactification of

$$\int d^{5+N}x\sqrt{-g_{(5+N)}}\left(R^{(5+N)} + c_1 M^{-2}\mathcal{L}_{\text{GB}}^{(5+N)} - 2\Lambda\right)$$

with $ds^2_{5+N} = g_{ab}(x)dx^a dx^b + e^{-4\phi(x)/N}\eta_{AB}dX^A dX^B$

Obtain $\omega = -1 + N^{-1}$ and

$$c_2 = -\omega c_1 \ , \quad c_3 = (2\omega+1)\omega c_1 \ , \quad c_4 = -(2\omega+1)\omega^2 c_1$$

$\omega = -1$ corresponds to dilaton (with extra symmetries).

Define $\alpha = c_1/M^2$ for later convenience.
In string theory context $\alpha \propto \alpha'$, hence small.

As with Einstein-Hilbert action, need boundary term to get well defined action [Myers]

$$\mathcal{L}^{(b)}_{\rm GB} = 4KK_{ac}K^{ac} - \frac{8}{3}K_{ac}K^{cb}K^a{}_b - \frac{4}{3}K^3 - 8G^{(4)}_{ab}K^{ab}$$

With higher order scalar field terms as well, brane contribution to action is

$$S_{\rm brane} = -M^3 \int d^4x\sqrt{-h}e^{-2\phi}\left\{2K + M^{-2}\mathcal{L}^{(b)}_2 + T\right\}$$

with

$$\begin{aligned}\mathcal{L}^{(b)}_2 &= c_1\mathcal{L}^{(b)}_{\rm GB} - 16c_2(K_{ab} - Kh_{ab})D^a\phi D^b\phi \\ &\quad -16c_3(n\cdot\nabla\phi)\left(\frac{1}{3}(n\cdot\nabla\phi)^2 + (D\phi)^2\right)\end{aligned}$$

Variation of action gives generalised Israel junction conditions (as before). These do not depend on brane thickness (not true for other second order terms).

Can also derive by treating brane as δ-function contribution to energy-momentum tensor.

E.g. for $ds_5^2 = e^{2A}ds_4^2 + dz^2$ have

$$A'' \propto 2\delta(z)\ , \quad A' \propto \text{sign}(z)$$

Field equations give

$$-3A'' + 12\alpha A'^2A'' + \cdots = T\delta(z)$$

So for $\alpha = 0$, $T = -6A'$.

If $\alpha \neq 0$, not clear, since value of A'^2A'' is ambiguous.

Literature contains conflicting results.

Linearised Brane World Gravity

Consider brane world solutions with metric $ds^2 = e^{2A} dx^\nu dx_\nu + dz^2$ where $A = -|z|/\ell$, and

$$\frac{\phi'}{A'} = u = \text{constant}$$

(simplest extension of RS to include a scalar field)

Consider positive warp factor ($\ell > 0$) solutions (needed for localised gravity).

Also take $u\ell > 0$ to avoid naked curvature singularities in bulk.

Consider general perturbation of Randall-Sundrum-like brane world with scalar field

$$ds^2 = e^{2A}(\eta_{\mu\nu} + \gamma_{\mu\nu}) dx^\mu dx^\nu + 2 v_\mu dx^\mu dz + (1 + \psi) dz^2$$

$$\phi = -u|z|/\ell + \varphi$$

where $\gamma_{\mu\nu}$, v_μ, ψ and φ are small.

For effective 4-d gravity need to worry about perturbations of brane position as well as bulk metric.

Can address this by using gauge in which brane stays where it is: Solve Einstein equations perpendicular to brane to obtain lapse function ψ and shift vector v_μ.

Graviton modes

Useful to split $\gamma_{\mu\nu}$ into tensor and scalar parts

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + \frac{1}{4}\gamma\eta_{\mu\nu} + \frac{4}{3}c_\chi(u)\left(\frac{1}{4}\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)\chi$$

where $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu}$, and $\partial^\mu\bar{\gamma}_{\mu\nu} = 0$.

$\bar{\gamma}_{\mu\nu}$ corresponds to graviton modes.

$\chi \propto 8\varphi - u\gamma$ is scalar perturbation (absent in RS model).

Bulk graviton equation

$$\mu_\gamma(u)\left(\partial_z^2 - 2(2-u)\ell^{-1}\partial_z + f_\gamma^2(u)e^{-2A}\Box_4\right)\bar{\gamma}_{\mu\nu} = 0$$

Note no 3rd or 4th order derivatives.

Compare with

$$\left(\partial_z^2 - 4\ell^{-1}\partial_z + e^{-2A}\Box_4\right)\bar{\gamma}_{\mu\nu} = {}^{(5)}\nabla^2\bar{\gamma}_{\mu\nu} = 0$$

for RS model.

If higher order gravity terms absent, $\mu_\gamma = f_\gamma^2 = 1$.

Otherwise, if $\mu_\gamma(u) < 0$ or $f_\gamma^2(u) < 0$, kinetic term for $\bar{\gamma}_{\mu\nu}$ will have wrong sign in effective action, so bulk will have ghosts.

See that f_γ rescales momentum dependence, which changes effective 4d gravity too.

4d gravity

Brane junction conditions imply

$$\mu_\gamma \ell \partial_z \bar{\gamma}_{\mu\nu} + 4c_1[1-2u]\Box_4 \bar{\gamma}_{\mu\nu} \propto -\left\{ S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box_4}\right) S \right\}$$

($S_{\mu\nu}$ is brane energy momentum tensor, $M = 1$)

Compare with $\ell \partial_z \bar{\gamma}_{\mu\nu} \propto -\{S_{\mu\nu} - \cdots\}$ for RS.

If $4c_1[1-2u] < 0$, then either $M_{\rm Pl}^2 < 0$ or vacuum has non-trivial solution with spacelike momenta, i.e. tachyons, so solution will be unstable.

Bulk solution (Fourier space, spacelike p):

$$\bar{\gamma}_{\mu\nu} \propto e^{-ip\cdot x} e^{-(2-u)A} K_{2-u}\left(f_\gamma \ell p e^{-A}\right)$$

For $1/p \gg \ell f_\gamma$ (larger distances)

$$\partial_z \bar{\gamma}_{\mu\nu} \approx \frac{\ell f_\gamma^2}{2(1-u)} \Box_4 \bar{\gamma}_{\mu\nu} \qquad \text{if} \quad u = \phi'/A' < 1$$

So at large distances have $\Box_4 \bar{\gamma}_{\mu\nu} \propto -\{S_{\mu\nu} - \cdots\}$ as in 4d linearised gravity.
(similar to Randall Sundrum scenario).

Extra $\Box_4 \bar{\gamma}_{\mu\nu}$ term in junction conditions gives 4d gravity at short distances too
– hence higher order gravity leads to weaker constraints.

Analysis of scalar modes is qualitatively similar.

Example: $\omega = -1$ and $c_i = \alpha$ (dilaton)

For Einstein gravity ($\alpha = 0$), one solution, with $u = \infty$:

Similar to $\ell < 0$ solution, so does not give localised gravity.

When higher order terms are included, develops tachyon.

Two extra solutions appear when $\alpha \neq 0$

$$\frac{\phi'}{A'} = u = \frac{3}{2} \pm \sqrt{\frac{3}{4} + \frac{\ell^2}{8\alpha}}$$

$(+)$: always unstable. $(-)$: stable if $u < 1/2$.

Scalar (χ) perturbations are qualitatively similar to graviton equations (ghosts and tachyons possible, effective 4d gravity possible at all scales).

However corresponding coefficients are different, so degeneracy between scalar and tensor modes is broken by higher order gravity.

In particular can have

$$f_\gamma^2 = \frac{1-2u}{1-u} \ll f_\chi^2 = \frac{3}{3-2u}$$

if $u \approx 1/2$ for above solution.

Brane junction conditions give effective 4d gravity. Obtain (to leading order) Brans-Dicke gravity at all scales.
(4d graviton and scalar mass scales are $M_{\rm Pl}$ and M_ϕ)

For $1/p \gg \ell f_\chi$ (large distances), find

$$M_{\rm Pl}^2 = 8M^3\alpha\ell^{-1}(2-u)f_\gamma^2$$

$$M_\phi^2 = 8M^3\alpha\ell^{-1}(3-2u)$$

so $M_\phi \gg M_{\rm Pl}$ if solution fine-tuned to have $f_\gamma \ll 1$.

Hence can potentially avoid conflict with constraints from solar system. This is despite the fact that for underlying 5d theory $M_{\rm Pl}^{(5)} \sim M_\phi^{(5)}$.

At medium ($\ell f_\gamma \ll 1/p \ll \ell f_\chi$) and short ($1/p \ll \ell f_\gamma$) distance scales, find $M_\phi^2 \leq 3M_{\rm Pl}^2$. But not a problem, since short distance constraints are weak.

(Above fine-tuning of parameters is in addition to usual brane world fine-tuning of cosmological constants Λ and T)

Modified large distance gravity

At large distance scales ($1/p \gg \ell f_\gamma$) have

$$\partial_z \bar{\gamma}_{\mu\nu} \approx -\bar{\gamma}_{\mu\nu} \times \begin{cases} \dfrac{\ell\, f_\gamma^2\, p^2}{2(1-u)} & \text{if} \quad u < 1 \\[2ex] \dfrac{2\,\Gamma(u-1)}{\ell\,\Gamma(2-u)} \left(\dfrac{p\ell}{2}\right)^{4-2u} & \text{if} \quad 1 < u < 2 \end{cases}$$

with similar behaviour for scalar modes.

Recall that $[\mu_\gamma \ell \partial_z + 4c_1(1-2u)\Box_4]\, \bar{\gamma}_{\mu\nu} \propto -\{S_{\mu\nu} - \cdots\}$

Corresponding Newton potentials behave as

$$-\frac{1}{r} \quad (u < 1) \quad \text{and} \quad -\frac{1}{r^{2u-1}} \quad (1 < u < 2)$$

Hence if effects of scalar field are greater than warping of space time ($\phi'/A' = u > 1$), large distance gravity is modified.

No longer have graviton (and scalar) zero mode confined to brane, but still get 4d gravity at short distances from $\Box_4$ terms in junction conditions.

Existence of solutions with $1 < u < 2$ seems to require "wrong" choice of sign for coefficients (c_i) of higher order terms. Suggests theory has ghosts (for Minkowski space), but this is not relevant for our warped space solutions.

Summary

- In 5-d gravity (e.g. brane world) natural to include second order Gauss-Bonnet curvature term.
- Israel junction conditions generalise to higher order without problems.
- Linearised gravity with scalar and higher order terms qualitatively similar to Einstein case.
- Several new types of instabilities can occur.
- Field equations have more solutions.
- Higher order terms can effect graviton and scalar perturbations differently – can produce hierarchy between scalar and graviton couplings.
- Weaker constraints from gravity experiments.
- Modified large distance gravity possible.