Static Screening: Elementary Treatment

Lectures on Condensed Matter Physics

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An elementary calculation of screening in electron gas.

1 A charge in an electron gas

Suppose you have a uniform gas and stick a positive charge into it (in addition to the neutralizing background). You expect of course that it will attract electrons, increasing the density of negative charge in the vicinity of the extra charge. This in turn will change the force that charge exerts on another positive charge placed at a distance. The extra negative charge will (partially, completely?) cancel the repulsion between these extra charges. This is what we mean by *screening*. On the other hand, electrons themselves are charged; an electron at some position will tend to repel nearby electrons, and the electron-electron interaction will also be reduced by this effect. Can we calculate it?

It must be recognized that what we are dealing with is with the response of the electron gas to an external potential; in the examples mentioned, the Coulomb potential of the charged particles. What we need to compute is the response of the charge density to an external potential. We start by looking at the system ignoring the electron-electron interaction.

2 Density response of non-interacting gas

$$\hat{H}_0 = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \tag{1}$$

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The perturbation of an external potential

$$\hat{V} = \int d^d\mathbf{r} V(\mathbf{r})\hat{\rho}(\mathbf{r}) = \frac{1}{\Omega}\int d^r\mathbf{r} V(\mathbf{r}) \sum_{\mathbf{k}\mathbf{k}'\sigma} e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} c^\dagger_{\mathbf{k}'\sigma} c_{\mathbf{k}\sigma}$$
$$= \frac{1}{\Omega}\sum_{\mathbf{k}\mathbf{q}\sigma}\left[\int d^d\mathbf{r} V(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}}\right] c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma}$$
$$= \frac{1}{\Omega}\sum_{\mathbf{k}\mathbf{q}\sigma} \tilde{V}(\mathbf{q})\, c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} = \frac{1}{\Omega}\sum_{\mathbf{q}\sigma}\tilde{V}(\mathbf{q})\,\hat{\rho}_{-\mathbf{q}} \tag{2}$$

The Fourier transform of the Density

$$\hat{\rho}_{\mathbf{q}} = \int d^d\mathbf{r}\hat{\rho}(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} = \frac{1}{\Omega}\int d^d\mathbf{r} \sum_{\mathbf{k}\mathbf{k}'\sigma} e^{-i(\mathbf{k}'-\mathbf{k}+\mathbf{q})\cdot\mathbf{r}} c^\dagger_{\mathbf{k}'\sigma} c_{\mathbf{k}\sigma}$$
$$= \sum_{\mathbf{k}\mathbf{k}'\sigma} c^\dagger_{\mathbf{k}'\sigma} c_{\mathbf{k}\sigma} \frac{1}{\Omega}\int d^d\mathbf{r} e^{-i(\mathbf{k}'-\mathbf{k}+\mathbf{q})\cdot\mathbf{r}} = \sum_{\mathbf{k}\mathbf{k}'\sigma} c^\dagger_{\mathbf{k}'\sigma} c_{\mathbf{k}\sigma}\delta_{\mathbf{k}'+\mathbf{q}-\mathbf{k},\mathbf{0}}$$
$$= \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}-\mathbf{q}\sigma} c_{\mathbf{k}\sigma} \tag{3}$$

Use a general Statistical Physics results

$$\langle\hat{A}\rangle = \frac{1}{\mathsf{Tr}\left[e^{-\beta\hat{H}_0-\beta\lambda\hat{A}}\right]}\mathsf{Tr}\left[\hat{A} e^{-\beta\left(\hat{H}_0+\lambda\hat{A}\right)}\right] =$$
$$-\beta^{-1}\frac{\partial}{\partial\lambda}\log\left[\mathsf{Tr}\left(e^{-\beta\hat{H}_0-\beta\lambda\hat{A}}\right)\right] = \frac{\partial\mathcal{F}}{\partial\lambda} \tag{4}$$

In the present case, $T\to 0$, $\mathcal{F}\to E_G$ (ground state energy), $\lambda\to\tilde{V}(-\mathbf{q})$, and $\hat{A}\to\hat{\rho}_{\mathbf{q}}$

$$\frac{\partial E_G}{\partial\tilde{V}(-\mathbf{q})} = \frac{1}{\Omega}\langle\hat{\rho}_{\mathbf{q}}\rangle \tag{5}$$

$\langle\ldots\rangle$ is thermal average. To first order in $\tilde{V}(-\mathbf{q})$ we need to compute this a $V=0$

$$E_G = E_G^0 + \left.\frac{\partial E_G}{\partial\tilde{V}(-\mathbf{q})}\right|_{V=0}\tilde{V}(-\mathbf{q}) = \frac{1}{\Omega}\langle\hat{\rho}_{\mathbf{q}}\rangle\tilde{V}(-\mathbf{q}) \tag{6}$$

In first order perturbation theory and at zero T $\langle\ldots\rangle := \langle\psi_G|\ldots|\psi_G\rangle$

$$\left.\frac{\partial E_G}{\partial\tilde{V}(-\mathbf{q})}\right|_{V=0}\tilde{V}(-\mathbf{q}) = \langle\psi_G|\int d^d\mathbf{r} V(\mathbf{r})\hat{\rho}(\mathbf{r})\,|\psi_G\rangle = \int d^d\mathbf{r} V(\mathbf{r})\,\langle\psi_G|\,\hat{\rho}(\mathbf{r})\,|\psi_G\rangle = \int d^d\mathbf{r} V(\mathbf{r})$$
$$= \frac{N_e}{\Omega}\tilde{V}(\mathbf{q}=0)$$

From which

$$\langle\hat{\rho}_{\mathbf{q}}\rangle_0 = \frac{N_e}{\Omega}\delta_{\mathbf{q},0} \tag{7}$$

as expected for the uniform gas. We need to compute the derivative of E_G at non-zero $\tilde{V}(-\mathbf{q})$ to find a change in density. If we compute E_G to second order we will get the linear response for $\langle \hat{\rho}_{\mathbf{q}} \rangle_0$,

$$\langle \hat{\rho}_{\mathbf{q}} \rangle = \Omega \frac{\partial E_G^{(2)}}{\partial \tilde{V}(-\mathbf{q})} \tag{8}$$

In 2nd order perturbation theory

$$\begin{aligned} E_G^{(2)} &= \sum_{n(\neq 0)} \frac{\langle \psi_0 | \hat{V} | \psi_n \rangle \langle \psi_n | \hat{V} | \psi_0 \rangle}{E_0 - E_n} \\ &= \frac{1}{\Omega^2} \sum_{\mathbf{q},\mathbf{q}'} \tilde{V}(\mathbf{q}')\tilde{V}(\mathbf{q}) \sum_{n(\neq 0)} \frac{\langle \psi_0 | \hat{\rho}_{-\mathbf{q}'} | \psi_n \rangle \langle \psi_n | \hat{\rho}_{-\mathbf{q}} | \psi_0 \rangle}{E_0 - E_n} \end{aligned} \tag{9}$$

Note that $\hat{\rho}_{-\mathbf{q}} |\psi\rangle = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} |\psi\rangle$ so $\hat{\rho}_{-\mathbf{q}}$ changes the momentum of $|\psi\rangle$ by $\mathbf{q}$. The GS momentum is zero so the product $\langle \psi_0 | \hat{\rho}_{-\mathbf{q}'} | \psi_n \rangle \langle \psi_n | \hat{\rho}_{-\mathbf{q}} | \psi_0 \rangle = 0$, unless $\mathbf{q} + \mathbf{q}'=0$;

$$E_G^{(2)} = \frac{1}{\Omega^2} \sum_{\mathbf{q}} \tilde{V}(-\mathbf{q})\tilde{V}(\mathbf{q}) \sum_{n(\neq 0)} \frac{\langle \psi_0 | \hat{\rho}_{\mathbf{q}} | \psi_n \rangle \langle \psi_n | \hat{\rho}_{-\mathbf{q}} | \psi_0 \rangle}{E_0 - E_n} \tag{10}$$

$$\begin{aligned} \langle \hat{\rho}_{\mathbf{q}} \rangle = \Omega \frac{\partial E_G^{(2)}}{\partial \tilde{V}(-\mathbf{q})} = \frac{1}{\Omega} \Bigg[&\sum_{n(\neq 0)} \frac{\langle \psi_0 | \hat{\rho}_{\mathbf{q}} | \psi_n \rangle \langle \psi_n | \hat{\rho}_{-\mathbf{q}} | \psi_0 \rangle}{E_0 - E_n} \\ &+ \sum_{n(\neq 0)} \frac{\langle \psi_0 | \hat{\rho}_{-\mathbf{q}} | \psi_n \rangle \langle \psi_n | \hat{\rho}_{\mathbf{q}} | \psi_0 \rangle}{E_0 - E_n} \Bigg] \tilde{V}(\mathbf{q}) \end{aligned} \tag{11}$$

In other words the linear response

$$\langle \hat{\rho}_{\mathbf{q}} \rangle = \chi_0(\mathbf{q}) \tilde{V}(\mathbf{q}) \tag{12}$$

where the susceptibility

$$\chi_0(\mathbf{q}) = \frac{1}{\Omega} \sum_{n(\neq 0)} \frac{\langle \psi_0 | \hat{\rho}_{\mathbf{q}} | \psi_n \rangle \langle \psi_n | \hat{\rho}_{-\mathbf{q}} | \psi_0 \rangle}{E_0 - E_n} + (\text{term with } \mathbf{q} \longleftrightarrow -\mathbf{q}) \tag{13}$$

For non-interacting system if $\langle \psi_n | \hat{\rho}_{-\mathbf{q}} | \psi_0 \rangle = \sum_{\mathbf{k}\sigma} \langle \psi_n | c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} | \psi_0 \rangle$ and if this is to be non-zero one must have $E_n = E_0 + \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}$

$$\sum_{n(\neq 0)} \frac{\langle \psi_0 | \hat{\rho}_{\mathbf{q}} | \psi_n \rangle \langle \psi_n | \hat{\rho}_{-\mathbf{q}} | \psi_0 \rangle}{E_0 - E_n} = \sum_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} \frac{-1}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}} \sum_{n(\neq 0)} \langle \psi_0 | c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} c_{\mathbf{k}'\sigma'} | \psi_n \rangle \langle \psi_n | c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} | \psi_0 \rangle$$

When $\mathbf{q} \neq 0$

$$\langle \psi_0 | c^\dagger_{\mathbf{k}+\mathbf{q}\sigma} c_{\mathbf{k}\sigma} | \psi_0 \rangle = 0$$

and we can remove the restriction not to sum over $n = 0$

$$\sum_n \langle\psi_0|\,\hat\rho_{\mathbf{q}}\,|\psi_n\rangle\,\langle\psi_n|\,\hat\rho_{-\mathbf{q}}\,|\psi_0\rangle = \sum_{\mathbf{k}\sigma,\mathbf{k}',\sigma'} \langle\psi_0|\,c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}c_{\mathbf{k}'\sigma'}c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}c_{\mathbf{k}\sigma}\,|\psi_0\rangle \tag{14}$$

To compute this average we use Wick's Theorem

$$\begin{aligned}\langle\psi_0|\,c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}c_{\mathbf{k}'\sigma'}c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}c_{\mathbf{k}\sigma}\,|\psi_0\rangle &= \langle c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}c_{\mathbf{k}'\sigma'}\rangle\langle c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma}c_{\mathbf{k}'\sigma}\rangle + \langle c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma}c_{\mathbf{k}'\sigma}\rangle\langle c_{\mathbf{k}'\sigma}c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}\rangle \\ &= \langle c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}c_{\mathbf{k}\sigma}\rangle\langle c_{\mathbf{k}'\sigma'}c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}\rangle \end{aligned}\tag{15}$$

because the first term is zero for $\mathbf{q} \neq 0$. The second

$$\langle c^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'}c_{\mathbf{k}\sigma}\rangle = f_{\mathbf{k}}\delta_{\sigma\sigma'}\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}}\delta_{\sigma\sigma'} \tag{16}$$

$$\langle c_{\mathbf{k}'\sigma'}c^\dagger_{\mathbf{k}+\mathbf{q}\sigma}\rangle = \left(1 - f_{\mathbf{k}+\mathbf{q}}\right)\delta_{\sigma\sigma'}\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}} \tag{17}$$

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$$\chi_0(\mathbf{q}) = \frac{1}{\Omega}\sum_{\mathbf{k},\sigma}\frac{f_{\mathbf{k}}\left(1 - f_{\mathbf{k}+\mathbf{q}}\right)}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}} + \frac{f_{\mathbf{k}}\left(1 - f_{\mathbf{k}-\mathbf{q}}\right)}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}-\mathbf{q}}} \tag{18}$$

A simple change of variable in second term

$$\begin{aligned}\chi_0(\mathbf{q}) &= \frac{1}{\Omega}\sum_{\mathbf{k},\sigma}\frac{f_{\mathbf{k}}\left(1 - f_{\mathbf{k}+\mathbf{q}}\right)}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}} + \frac{f_{\mathbf{k}+\mathbf{q}}\left(1 - f_{\mathbf{k}}\right)}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}} \\ &= \frac{1}{\Omega}\sum_{\mathbf{k},\sigma}\frac{f_{\mathbf{k}}\left(1 - f_{\mathbf{k}+\mathbf{q}}\right) - f_{\mathbf{k}+\mathbf{q}}\left(1 - f_{\mathbf{k}}\right)}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}} \\ &= \frac{1}{\Omega}\sum_{\mathbf{k},\sigma}\frac{f_{\mathbf{k}} - f_{\mathbf{k}+\mathbf{q}}}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}}\end{aligned} \tag{19}$$

So this is the static density density response function a non-interacting electron gas, with translational invariance

$$\chi_0(\mathbf{q}) = \frac{2}{\Omega}\sum_{\mathbf{k}}\frac{f_{\mathbf{k}} - f_{\mathbf{k}+\mathbf{q}}}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}} \tag{20}$$

3 Screening of Coulomb Interaction

3.1 Interacting density response in RPA

Consider the electric potential created by charge density $-e\rho(\mathbf{r})$

$$-\nabla^2\phi(\mathbf{r}) = \frac{-e\rho(\mathbf{r})}{\epsilon_0} \tag{21}$$

Fourier transforming (FT)

$$q^2\tilde\phi(\mathbf{q}) = -\frac{e}{\epsilon_0}\rho_{\mathbf{q}} \tag{22}$$

or

$$-e\tilde{\phi}(\mathbf{q}) = \frac{e^2}{\epsilon_0 q^2}\rho_{\mathbf{q}} \tag{23}$$

This is just the FT of

$$-e\phi(\mathbf{r}) = \int d^3 r v_c(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r})$$

with the Coulomb interaction

$$v_c(\mathbf{r}-\mathbf{r}') = \frac{e^2}{4\pi\epsilon_0 \left|\mathbf{r}-\mathbf{r}'\right]} \tag{24}$$

$$\tilde{v}_c(\mathbf{q}) = \frac{e^2}{\epsilon_0 q^2} \tag{25}$$

Suppose then we apply a weak potential to the electronic system. If the electrons were not charged we would induce a density response

$$\langle\hat{\rho}_{\mathbf{q}}\rangle = \chi_0(\mathbf{q})\tilde{V}(\mathbf{q}) \tag{26}$$

But in a charged system there is an extra potential energy coming from the charge modulation

$$-e\tilde{\phi}(\mathbf{q}) = \frac{e^2}{\epsilon_0 q^2}\langle\hat{\rho}_{\mathbf{q}}\rangle \tag{27}$$

Hence we take the electron-electron interactions in a mean-field philosophy as the non-interacting response to the total potential, external plus induced. This is a way to include the Coulomb interactions

$$\begin{aligned}\langle\hat{\rho}_{\mathbf{q}}\rangle &= \chi_0(\mathbf{q})\left[\tilde{V}(\mathbf{q}) + (-e)\,\tilde{\phi}(\mathbf{q})\right] \\ &= \chi_0(\mathbf{q})\left[\tilde{V}(\mathbf{q}) + \frac{e^2}{\epsilon_0 q^2}\langle\hat{\rho}_{\mathbf{q}}\rangle\right]\end{aligned} \tag{28}$$

or

$$\langle\hat{\rho}_{\mathbf{q}}\rangle = \frac{\chi_0(\mathbf{q})}{1-\tilde{v}_c(\mathbf{q})\chi_0(\mathbf{q})}\tilde{V}(\mathbf{q}) \tag{29}$$

Including the response to the induced potential thus leads to

$$\chi(\mathbf{q}) = \frac{\chi_0(\mathbf{q})}{1-\tilde{v}_c(\mathbf{q})\chi_0(\mathbf{q})} \tag{30}$$

This is the RPA approximation to the static density–density response function.

3.2 The screened Coulomb interaction

Another way of looking at it, the total potential, external plus induced

$$\begin{aligned}\tilde{U}(\mathbf{q}) &= \tilde{V}(\mathbf{q}) + \frac{e^2}{\epsilon_0 q^2}\langle\hat{\rho}_{\mathbf{q}}\rangle \\ \tilde{U}(\mathbf{q}) &= \tilde{V}(\mathbf{q}) + \frac{e^2}{\epsilon_0 q^2}\chi_0(\mathbf{q})\tilde{U}(\mathbf{q}) \\ &= \frac{\tilde{V}(\mathbf{q})}{(1-\tilde{v}_c(\mathbf{q})\chi_0(\mathbf{q}))}\end{aligned}$$

By definition of dielectric constant

$$\tilde{V}(\mathbf{q}) = \epsilon(\mathbf{q})\tilde{U}(\mathbf{q})$$

so

$$\epsilon(\mathbf{q}) = 1 - \tilde{v}_c(\mathbf{q})\chi_0(\mathbf{q})$$

Consider now the case where the external potential is the one created by a charge

$$\tilde{V}(\mathbf{q}) = \tilde{v}_c(\mathbf{q}) = \frac{e^2}{\epsilon_0 q^2}$$

and the total field felt by the charges in the system is

$$\begin{aligned}\tilde{U}_c(\mathbf{q}) &= \frac{e^2}{\epsilon_0 q^2}\frac{1}{1-\frac{e^2}{\epsilon_0 q^2}\chi_0(\mathbf{q})} \\ &= \frac{1}{\epsilon_0}\frac{e^2}{q^2 - e^2\chi_0(\mathbf{q})/\epsilon_0}\end{aligned}$$

This is the screened Coulomb interaction.

4 Calculation of $\chi_0(\mathbf{q})$

$$\chi_0(\mathbf{q}) = \frac{1}{\Omega}\sum_{\mathbf{k},\sigma}\frac{f_{\mathbf{k}} - f_{\mathbf{k}+\mathbf{q}}}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}} \tag{31}$$

If we want the large distance properties of the screened interaction we must look at small $\mathbf{q}$. In particular, if $q \ll k_F$ the states $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ are at a distance in momentum space much smaller than the radius of the Fermi surface; on the other hand their contribution to $\chi_0(\mathbf{q})$ is zero unless they are on different sides of the Fermi sphere (otherwise $f_{\mathbf{k}} = f_{\mathbf{k}+\mathbf{q}}$).

That means that, for $q \ll k_F$, only states with $k \sim k_F$ contribute to $\chi_0(\mathbf{q})$.

$$\begin{aligned}
\chi_0(\mathbf{q}) &= \frac{1}{\Omega} \sum_{\mathbf{k},\sigma} \frac{f(\epsilon_\mathbf{k}) - f(\epsilon_{\mathbf{k}+\mathbf{q}})}{\epsilon_\mathbf{k} - \epsilon_{\mathbf{k}+\mathbf{q}}} \\
&= \frac{2}{\Omega} \sum_{\mathbf{k},\sigma} \frac{\Theta(\mu - \epsilon_\mathbf{k}) - \Theta(\mu - \epsilon_{\mathbf{k}+\mathbf{q}})}{\epsilon_\mathbf{k} - \epsilon_{\mathbf{k}+\mathbf{q}}} \\
&= 2 \int \frac{d^3k}{(2\pi)^3} \frac{\Theta(\mu - \epsilon_\mathbf{k}) - \Theta(\mu - \epsilon_\mathbf{k} - \mathbf{v}_\mathbf{k} \cdot \mathbf{q})}{-\mathbf{v}_\mathbf{k} \cdot \mathbf{q}} \\
&- 2 \int d\epsilon \rho(\epsilon) \int \frac{d\Omega}{4\pi} \frac{\Theta(\mu - \epsilon) - \Theta(\mu - \epsilon - v_F q \cos\theta)}{-v_F q \cos\theta}
\end{aligned} \tag{32}$$

The integral over ϵ in limited to the range $|\mu - \epsilon| < |v_F q \cos\theta|$ where we can assume $\rho(\epsilon) \approx \rho_F$; The integration over energy gives gives $\rho_F \times v_F q \cos\theta$

$$\chi_0(\mathbf{q}) = -2\rho_F \int \frac{d\Omega}{4\pi} \frac{v_F q \cos\theta}{v_F q \cos\theta} = -2\rho_F; \qquad \text{for } q \ll k_F \tag{33}$$

The screened Coulomb interaction becomes

$$\tilde{U}_c(\mathbf{q}) = \frac{1}{\epsilon_0} \frac{e^2}{q^2 - e^2 \chi_0(\mathbf{q})/\epsilon_0} \tag{34}$$

$$= \frac{1}{\epsilon_0} \frac{e^2}{q^2 + 2\rho_F e^2/\epsilon_0} = \frac{1}{\epsilon_0} \frac{e^2}{q^2 + q_{TF}^2} \tag{35}$$

with

$$q_{TF}^2 = \frac{2\rho_F e^2}{\epsilon_0}$$

The $q \to 0$ singularity has gone. In real space the interaction has a finite integral

$$\int d^3r U_c(r) = \tilde{U}_c(\mathbf{q} = 0) = \frac{e^2}{\epsilon_0 q_{TF}^2} \tag{36}$$

One can show

$$U_c(r) = \frac{e^2}{4\pi\epsilon_0} \frac{e^{-q_{TF} r}}{r} \tag{37}$$

The charge if completely screened and the effective interaction becomes short range. We can estimate the decay length

$$q_{TF}^2 = \frac{2m_e k_F}{2\pi^2 \hbar^2} \frac{e^2}{\epsilon_0} \tag{38}$$

$$\frac{q_{TF}^2}{k_F^2} = \frac{2}{\pi} \frac{m_e e^2}{4\pi\epsilon_0 \hbar^2} \frac{1}{k_F} = \frac{2}{\pi} \frac{1}{k_F a_0} \tag{39}$$

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}; \qquad \text{Bohr radius} \tag{40}$$

For usual metals $k_F a_0 \sim 1$ and $q_{TF} \sim k_F$. The screening length is of the order of inter-electron distances.