Inducing and deducing

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http://www.fc.up.pt/pessoas/jfalves/UFRJ.pdf
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Physical measures

Physical measures

Let M be a compact Riemannian manifold and $f\colon M \to M$. An f-invariant probability measure μ on the Borel sets of M is called a physical measure if, for a positive Lebesgue measure set of points $x \in M$, we have

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \xrightarrow[n\to\infty]{w^*} \mu, \qquad (*)$$

or equivalently, for all continuous $\varphi : M \to \mathbb{R}$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu.$$

We define the basin of μ as

$$\mathcal{B}(\mu) = \{x \in M : \ (*) \text{ holds}\}$$

Singular vs. absolutely continuous

- The average of Dirac measures supported on an attracting periodic orbit is a physical measure.
- Any ergodic absolutely continuous (wrt Lebesgue measure) invariant probability measure is a physical measure.

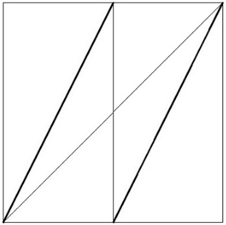
Exercise

Prove the second statement above.
Hint for the second one: use Birkhoff's Ergodic Theorem and the fact that $C^0(M)$ has a countable dense subset.

Toy model I: Doubling map

Consider $f : S^1 \to S^1$ given by

$$f(x) = 2x \pmod 1.$$



It is clear that f preserves the length of intervals, and so (...) f preserves the Lebesgue (length) measure m on the Borel sets.

Exercise

Show that m is ergodic.
Hint 1: Use Fourier series and the fact that f is ergodic iff for all $\varphi \in L^2(m)$

$$\varphi \circ f = \varphi \implies \varphi = \text{const.}$$

Hint 2: Use that any interval becomes the whole interval after a finite number iterates, the fact that f preserves proportions and Lebesgue Density Theorem.

Toy model II: Solenoid attractor

Consider the unit disk $D \subset \mathbb{C}$, the map $F : S^1 \times D \to S^1 \times D$ given by

$$F(t,z) = \left(2t \,(\mathrm{mod}\ 1), \frac{z}{4} + \frac{1}{2} e^{2\pi i t} \right),$$

and the attractor

$$A = \bigcap_{n \geq 0} F^n(S^1 \times D).$$

Some well-known facts:

1. A is a hyperbolic attractor.
2. Each $x \in A$ has a stable leaf $\gamma^s(x)$ and an unstable leaf $\gamma^u(x)$.
3. The unstable leaves of points in A are contained in A.
4. There exists a unique F-invariant ergodic probability measure μ such that $\pi_* \mu = m$.

$$\begin{array}{ccc} A & \xrightarrow{F} & A \\ \pi \downarrow & & \downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Some more facts:

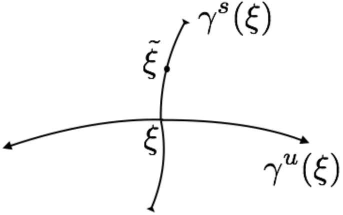
(1) A is foliated by unstable leaves.

(2) The conditional measures of μ on unstable leaves are equivalent to the conditionals of Lebesgue measure on those leaves.

(3) For any continuous $\phi : A \to \mathbb{R}$ and any $\tilde{\xi} \in \gamma^s(\xi)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(F^j(\tilde{\xi})) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(F^j(\xi)).$$

(4) The stable foliation is absolutely continuous.



μ ergodic supported on A

$\Downarrow$ (1)

μ_{γ^u} almost every point in an unstable leaf γ^u belongs in $B(\mu)$

$\Downarrow$ (2)

m_{γ^u} almost every point in an unstable leaf γ^u belongs in $B(\mu)$

$\Downarrow$ (3)+(4)

μ is a physical measure

Lyapunov exponents

Let $f : M \to M$ be a diffeomorphism of a smooth manifold M.
Given $x \in M$ and $v \in T_x M$, define

$$\lambda(x, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x)v\|,$$

if these limits exist and coincide.

Theorem (Oseledec 1968)

Assume that f preserves an invariant probability measure μ. There exist measurable functions λ_i and a Df-invariant splitting $T_x M = \oplus_i E_i(x)$ with $\lambda(x, v) = \lambda_i(x)$ for μ almost every $x \in M$ and every $v \in E_i(x)$. Moreover, if μ is ergodic, then λ_i and $\dim(E_i)$ are constant μ almost everywhere.

Each λ_i is called a Lyapunov exponent of f (with respect to μ).
We define the regular set $\mathcal{R} \subset M$ as the set of points for which the Lyapunov exponents are defined.

SRB measures

Theorem (Pesin 1976)

If $x \in \mathcal{R}$ has at least one positive Lyapunov exponent, then there is a small disk $\gamma^u(x) \subset M$ tangent to $\oplus_{\lambda_i > 0} E_i(x)$ such that for all $y \in \gamma^u(x)$

$$\limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(y), f^{-n}(x)) < 0.$$

$\gamma^u(x)$ is called the local unstable manifold of $x \in \mathcal{R}$.
A local stable manifold $\gamma^s(x)$ can be obtained similarly for a point $x \in \mathcal{R}$ with at least one negative Lyapunov exponent.

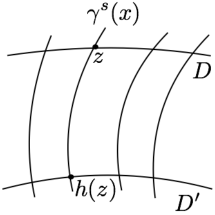
The measure μ is called an Sinai-Ruelle-Bowen (SRB) measure if it has at least one positive Lyapunov exponent and the conditionals $\{\mu_{\gamma^u}\}$ of the *Rokhlin decomposition* of μ on local unstable manifolds are absolutely continuous with respect to the Lebesgue conditionals $\{m_{\gamma^u}\}$.

Absolute continuity of the stable foliation

Given $D, D' \subset M$ embedded disks intersecting transversally a set $\{\gamma^s(x)\}_x$ of stable leaves, define the holonomy map

$$h : \bigcup_x \gamma^s(x) \cap D \longrightarrow \bigcup_x \gamma^s(x) \cap D'$$

assigning to $z \in \gamma^s(x) \cap D$ the unique point in $\gamma^s(x) \cap D'$. The stable foliation is absolutely continuous if for any $A \subset \bigcup_x \gamma^s(x) \cap D$, we have $m_D(A) > 0$ iff $m_{D'}(h(A)) > 0$.



Theorem (Pesin 1976)

Let $f : M \to M$ be a C^2 diffeomorphism having all Lyapunov exponents nonzero with respect to an ergodic invariant probability measure μ. Then the stable foliation is absolutely continuous.

Corollary

Every ergodic SRB measure with non-zero Lyapunov exponents is a physical measure.

Decay of correlations

The correlation of observables $\varphi, \psi \colon M \to \mathbb{R}$ is defined as

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

Taking φ and ψ characteristic functions of Borel sets, we obtain the usual notion of mixing. when $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \to 0$. We are interested in specific rates (polynomial, exponential,...) for the convergence of $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n)$ to zero. For this, we usually need to impose some regularity of the observables and assume that (at least) φ is Hölder continuous.

Remark

In many cases, μ is equivalent to the Lebesgue measure m. Assuming m normalized and taking $\varphi = dm/d\mu$, we have

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \left| \int \psi f_*^n dm - \int \psi d\mu \right|.$$

So, the decay of $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n)$ gives information on the speed at which the push-forwards $f_*^n m$ approach the physical measure μ.

Other statistical properties

Here we will be focused on Decay of Correlations. Under the same approach (inducing schemes) several other statistical properties of SRB measures can be deduced.

- [Young 1998; Young 1999]: Central Limit Theorem;
- [Melbourne and Nicol 2008; Melbourne 2009]: Large Deviations;
- [Melbourne and Nicol 2005; Melbourne and Nicol 2009]: Almost Sure Invariance Principle;
- [Gouëzel 2005]: Local Limit Theorem;
- [Gouëzel 2005]: Berry-Esseen Theorem.

Entropy

Let μ be a probability measure on M and $\mathcal{P}$ a countable μ mod 0 partition of M into measurable subsets of M. We define the entropy of $\mathcal{P}$ with respect to μ as

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P).$$

Given $f : M \to M$ preserving μ and $n \in \mathbb{N}$, consider the dynamically generated partition

$$\mathcal{P}^n = \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P}).$$

The entropy of f and the partition $\mathcal{P}$ with respect to μ is given by

$$h_\mu(f, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{P}^n).$$

The entropy of f with respect to μ is

$$h_\mu(f) = \sup_{\mathcal{P}} h_\mu(f, \mathcal{P}),$$

where the supremum is taken over all partitions of M as above.

Entropy formulas

- [Ruelle 1978]: upper bound for the entropy of an invariant measure in terms of the positive **Lyapunov exponents**;
- [Pesin 1977]: equality in Ruelle's result if the measure is **absolutely continuous** w.r.t. Lebesgue measure;
- [Ledrappier and Young 1985]: characterization of measures satisfying Pesin's entropy formula;
- [Ledrappier and Strelcyn 1982]: entropy formulas for certain systems with singularities (inspired by **billiards**);
- [Liu 1998]: Pesin's entropy formula for **endomorphisms**;
- [Alves, Oliveira, and Tahzibi 2006; Alves, Carvalho, and Freitas 2010a]: use entropy formulas to obtain the **continuous variation of entropy** for SRB measures in certain families of systems.

Systems with expanding structures

Gibbs-Markov maps

Let $(\Delta_0, \mathcal{A}, m)$ be a finite measure space. We say that $F : \Delta_0 \to \Delta_0$ is Gibbs-Markov if there exists an m mod 0 countable partition $\mathcal{P}$ into measurable subsets of Δ_0 such that:

1. **Markov**: F maps each $\omega \in \mathcal{P}$ bijectively to Δ_0.
2. **Nonsingular**: $\exists\, J_F > 0$ such that for each $A \subset \omega \in \mathcal{P}$
$$m(F(A)) = \int_A J_F \, dm.$$
3. **Separation**: for all $x, y \in \Delta_0$ there is
$$s(x, y) = \min\left\{n \geq 0 : F^n(x),\ F^n(y) \text{ lie in distinct elements of } \mathcal{P}\right\}.$$
4. **Bounded distortion**: $\exists\, K > 0$ and $0 < \beta < 1$ s.t. for all $x, y \in \omega \in \mathcal{P}$
$$\log \frac{J_F(x)}{J_F(y)} \leq K \beta^{s(F(x), F(y))}.$$

Consider the space

$$\mathcal{F}_\beta(\Delta_0) = \left\{ \varphi : \Delta_0 \to \mathbb{R} \text{ s.t. } |\varphi|_\beta \equiv \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\beta^{s(x,y)}} < \infty \right\}.$$

endowed with the norm

$$|\varphi|_\beta + \|\varphi\|_\infty,$$

and

$$\mathcal{F}_\beta^+(\Delta_0) = \{\varphi \in \mathcal{F}_\beta(\Delta_0) : \varphi \geq c \text{ for some } c > 0\}.$$

Lemma 2.1

$\mathcal{F}_\beta(\Delta_0)$ is relatively compact in $L^1(\Delta_0)$.

Exercise

Prove this lemma.

Hint: mimic the proof of Ascoli-Arzela Theorem in Wikipedia.

An F-invariant probability measure is exact ($\Rightarrow$ mixing $\Rightarrow$ ergodic) if

$$A \in \bigcap_{n\geq 0} F^{-n}(\mathcal{A}) \quad \text{and} \quad \nu(A) > 0 \quad \Longrightarrow \quad \nu(A) = 1.$$

Theorem 2.2

Any Gibbs-Markov map has a unique exact absolutely continuous invariant probability measure ν. Moreover, $d\nu/dm$ belongs in $\mathcal{F}_\beta(\Delta_0)$ and there is $K > 0$ such that

$$\frac{1}{K} \leq \frac{d\nu}{dm} \leq K.$$

The idea is to prove that the sequence of densities of the measures

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} F_*^j m.$$

is bounded in $\mathcal{F}_\beta(\Delta_0)$. By Lemma 2.1, it has an accumulation point in $L^1(\Delta_0)$, which is the density of an F-invariant measure.

Inducing schemes

Consider m a measure on M and $f : M \to M$. Given $\Delta_0 \subset M$ with $m(\Delta_0) < \infty$ we say that a Gibbs-Markov $F : \Delta_0 \to \Delta_0$ is an induced map for f if there is $R : \Delta_0 \to \mathbb{N}$ constant on each $\omega \in \mathcal{P}$ such that

$$F|_\omega = f^{R(\omega)}|_\omega.$$

Proposition 2.3

If ν_0 is the ergodic f^R-invariant probability measure $\ll m|_{\Delta_0}$, then

1. $\mu' = \sum_{j=0}^{\infty} f_*^j(\nu_0|\{R > j\})$ is an ergodic f-invariant measure;
2. μ' finite $\iff R \in L^1(m|_{\Delta_0}) \iff \sum_{j=0}^{\infty} m\{R > j\} < \infty$;
3. f nonsingular with respect to $m \implies \mu' \ll m$;
4. if μ' is finite, then $\mu = \mu'/\mu'(M)$ is the unique ergodic f-invariant probability measure with $\mu \ll m$ and $\mu(\Delta_0) > 0$.

We usually denote the induced map F by f^R and say that μ is liftable to ν.

Decay of correlations

Consider now the case of a smooth map $f : M \to M$, where M is a Riemannian manifold and m is Lebesgue measure on the Borel sets, and $\mathcal{H}$ the space of Hölder continuous functions from M to $\mathbb{R}$.

Theorem (Young 1999)

Assume that f has an induced Gibbs-Markov map f^R with $R \in L^1(m)$. Then f has some (liftable) ergodic invariant probability measure $\mu \ll m$. Moreover, if $\gcd\{R\} = 1$, then for all $\varphi \in \mathcal{H}$ and $\psi \in L^\infty(m)$

1. if $m\{R > n\} \lesssim n^{-\alpha}$ for some $\alpha > 0$, then $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \lesssim n^{-\alpha+1}$;
2. if $m\{R > n\} \lesssim e^{-cn^\theta}$ for some $c > 0$ and $0 < \theta \le 1$, then $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \lesssim e^{-c' n^\theta}$ for some $c' > 0$.

If $\gcd\{R\} = k$, the same conclusion holds for f^k.

The optimal estimate in the stretched exponential case is due to [Gouëzel 2006].

Entropy formula

Theorem 2.4 (Alves and Mesquita 2020)

Let $f : M \to M$ be a measurable map admitting a strictly positive Jacobian J_f with respect to some finite reference measure. If f has an induced Gibbs-Markov map with integrable recurrence time, then for any liftable f-invariant probability measure μ we have

$$h_\mu(f) = \int \log J_f \, d\mu.$$

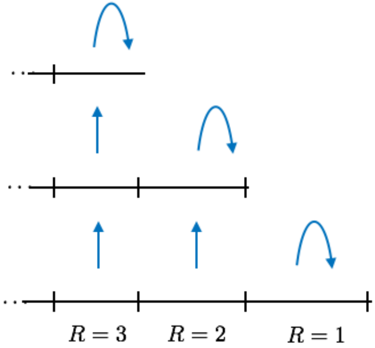
Tower extension

Consider the partition $\mathcal{P} = \{\Delta_{0,i}\}_i$ associated to an induced Gibbs-Markov map $f^R : \Delta_0 \to \Delta_0$. Define the tower over Δ_0 as

$$\Delta = \{(x,\ell) : x \in \Delta_0 \text{ and } 0 \le \ell < R(x)\},$$

and the tower map $T : \Delta \to \Delta$ as

$$T(x,\ell) = \begin{cases} (x,\ell+1), & \text{if } \ell < R(x)-1; \\ (f^R(x),0), & \text{if } \ell = R(x)-1. \end{cases}$$



The map

$$\begin{array}{rccc} \pi : & \Delta & \longrightarrow & M \\ & (x,\ell) & \longmapsto & f^\ell(x) \end{array} \qquad (1)$$

satisfies $f \circ \pi = \pi \circ T$.

The ℓ^{th} level of the tower is the set

$$\Delta_\ell = \{(x, \ell) \in \Delta\}.$$

The 0^{th} level is naturally identified with the set $\Delta_0 \subset M$. Under this identification we have that $T^R = f^R : \Delta_0 \longrightarrow \Delta_0$ is a Gibbs-Markov map. The ℓ^{th} level of the tower is a copy of $\{R > \ell\} \subset \Delta_0$. This allows us to extend the σ-algebra $\mathcal{A}$ and the reference measure m to the tower Δ. We also extend $\mathcal{P}$ to an m mod 0 partition of Δ

$$\mathcal{Q} = \{\Delta_{\ell,i}\}.$$

Finally we extend the separation time to $\Delta \times \Delta$, defining $s(x, y)$ for $x, y \in \Delta$ in the following way: if $x, y \in \Delta_\ell$, then there exist unique $x_0, y_0 \in \Delta_0$ such that $x = T^\ell(x_0)$ and $y = T^\ell(y_0)$. Set

$$s(x, y) = s(x_0, y_0).$$

Define $s(x, y) = 0$ for all other points $x, y \in \Delta$.

We consider as before

$$\mathcal{F}_\beta(\Delta) = \left\{\varphi : \Delta \to \mathbb{R} \,|\, \exists C > 0 : |\varphi(x) - \varphi(y)| \leq C\beta^{s(x,y)}, \forall x, y \in \Delta\right\}$$

and

$$\mathcal{F}_\beta^+(\Delta) = \{\varphi \in \mathcal{F}_\beta(\Delta) \,|\, \exists c > 0 : \varphi \geq c\}.$$

Theorem 2.5

If $R \in L^1(m)$, then the tower map $T : \Delta \to \Delta$ has a unique ergodic invariant probability measure ν which is equivalent to m. Moreover, $d\nu/dm \in \mathcal{F}_\beta^+(\Delta)$ and (T, ν) is exact if $\gcd\{R\} = 1$.

Note that $\gcd\{R\} > 1 \implies (T, \nu)$ is not mixing.

Existence and uniqueness of ν follows from Proposition 2.3. (Recall that $T^R = f^R$ is a Gibbs-Markov induced map for T).

Back to the original dynamics

Define $\mu = \pi_*\nu$, where $\pi : \Delta \to M$ is the projection given by (1).
Recalling that $f \circ \pi = \pi \circ T$, we have that

(T, ν) is an **extension** of (f, μ).

Lemma 2.6

1. $\nu|\Delta_0 = \nu_0$ (the f^R-invariant measure in Proposition 2.3);
2. $$\mu = \frac{1}{\sum_{j=0}^{\infty} \nu_0\{R > j\}} \sum_{j=0}^{\infty} f_*^j(\nu_0|\{R > j\});$$
3. $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathrm{Cor}_\nu(\varphi \circ \pi, \psi \circ \pi \circ T^n)$ for all φ, ψ and $n \geq 1$;
4. for each $\varphi \in \mathcal{H}$, there is $\beta > 0$ such that $\varphi \circ \pi \in \mathcal{F}_\beta(\Delta)$;
5. $$\int \log J_f \, d\mu = \int \log J_T \, d\nu;$$
6. $h_\mu(f) = h_\nu(T)$.

By the first two items $\mu = \pi_*\nu$ is the measure given by Proposition 2.3.
The last item has been proved in [Buzzi 1999] for general extensions.

Decay of correlations for tower maps

Young Theorem is then a consequence of

Theorem 2.7

Assume that $\gcd\{R\} = 1$. *For all* $\varphi \in \mathcal{F}_\beta(\Delta)$ *and all* $\psi \in L^\infty(m)$

1. *if* $m\{R > n\} \lesssim n^{-\alpha}$ *for some* $\alpha > 0$, *then* $\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \lesssim n^{-\alpha+1}$;
2. *if* $m\{R > n\} \lesssim e^{-cn^\theta}$ *for some* $c > 0$ *and* $0 < \theta \leq 1$, *then* $\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \lesssim e^{-c' n^\theta}$ *for some* $c' > 0$.

Below we show how $\mathrm{Cor}_\nu(\varphi, \psi \circ T^n)$ can be controlled in terms of the total variation of a certain sequence of signed measures.

Given $\varphi \in L^\infty(m)$ with $\varphi \neq 0$, define

$$\varphi^* = \frac{1}{\int (\varphi + 2\|\varphi\|_\infty) d\nu} (\varphi + 2\|\varphi\|_\infty). \tag{2}$$

Note that φ^* is strictly positive and its integral with respect to ν is 1.

Lemma 2.8

For all $\varphi \in \mathcal{F}_\beta(\Delta)$ with $\varphi \neq 0$ we have

1. $\varphi^* \in \mathcal{F}_\beta^+(\Delta)$ and $1/3 \leq \varphi^* \leq 3$;
2. $\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \leq 3\|\varphi\|_\infty \|\psi\|_\infty |T_*^n \lambda - \nu|$ for all $\psi \in L^\infty(m)$, where λ is the probability measure on Δ such that $d\lambda/d\nu = \varphi^*$.

We have

$$\|\varphi\|_\infty \leq \varphi + 2\|\varphi\|_\infty \leq 3\|\varphi\|_\infty. \tag{3}$$

Since ν is a probability measure, we get

$$\frac{1}{3\|\varphi\|_\infty} \leq \frac{1}{\int (\varphi + 2\|\varphi\|_\infty) d\nu} \leq \frac{1}{\|\varphi\|_\infty}. \tag{4}$$

From (3) and (4) we get $1/3 \leq \varphi^* \leq 3$.

For all $x, y \in \Delta$ we have

$$\frac{\varphi^*(x) - \varphi^*(y)}{\beta^{s(x,y)}} = \frac{1}{\int(\varphi + 2\|\varphi\|_\infty)d\nu} \cdot \frac{\varphi(x) - \varphi(y)}{\beta^{s(x,y)}}. \tag{5}$$

Since $\varphi^* \geq 1/3$,

$$\varphi \in \mathcal{F}_\beta(\Delta) \implies \varphi^* \in \mathcal{F}_\beta^+(\Delta).$$

Setting $a = \int(\varphi + 2\|\varphi\|_\infty)d\nu$, we may write

$$\begin{aligned}
\mathrm{Cor}_\mu(\varphi, \psi \circ T^n) &= \left| \int \varphi(\psi \circ T^n) d\nu - \int \varphi d\nu \int \psi d\nu \right| \\
&= a \left| \int \varphi^*(\psi \circ T^n)\, d\nu - \int \varphi^* d\nu \int \psi\, d\nu \right| \\
&= a \left| \int (\psi \circ T^n)\, d\lambda - \int \psi\, d\nu \right| \\
&= a \left| \int \psi\, dT_*^n \lambda - \int \psi\, d\nu \right| \\
&\leq a \|\psi\|_\infty |T_*^n \lambda - \nu|.
\end{aligned}$$

Observing that $a \leq 3\|\varphi\|_\infty$, we obtain Lemma 2.8.
The proof of Theorem 2.7 is then reduced to estimate $|T_*^n \lambda - \nu|$.

Convergence to the equilibrium

Theorem 2.9 (Young 1999; Gouëzel 2006)

Assume that $\gcd\{R\} = 1$. Given any measure λ such that $\varphi = d\lambda/dm$ belongs in $\mathcal{F}_\beta^+(\Delta)$ we have:

1. if $m\{R > n\} \leq Cn^{-\zeta}$ for some $C > 0$ and $\zeta > 1$, then $|T_*^n\lambda - \nu| \leq C'n^{-\zeta+1}$ for some $C' > 0$;
2. if $m\{R > n\} \leq Ce^{-cn^\eta}$ for some $C, c > 0$ and $0 < \eta \leq 1$, then $|T_*^n\lambda - \nu| \leq C'e^{-c'n^\eta}$ for some $C', c' > 0$;

Moreover, c' does not depend on φ and C' depends only on C_φ^+.

The proof of this result uses a probabilistic **coupling argument**, based on a careful study of returns to the base of the tower.

Entropy formula for the tower map

Proposition 2.10

$$h_\nu(T) = \int \log J_T \, d\nu.$$

Entropy formula for Gibbs-Markov maps can be deduced using (in an importnt way) the Markov property; see e.g.

(1) [Denker, Keller, and Urbański 1990]
(2) [Alves, Oliveira, and Tahzibi 2006]
(3) [Alves and Pumariño 2018]

For tower maps, we follow ideas from (3), using a quasi-Markov property: there is $\eta > 0$ such that for m-almost every $(x, \ell) \in \Delta$ there are infinitely many values $n \in \mathbb{N}$ for which

$$m(T^n(\mathcal{Q}_n(x, \ell))) \geq \eta > 0, \tag{6}$$

where for each $n \geq 1$

$$\mathcal{Q}_n = \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{Q}).$$

Given $(x,\ell) \in \Delta$, let $\mathcal{M}(x,\ell)$ be the set of $k \in \mathbb{N}$ for which (6) holds.

Volume Lemma

There exists $C > 0$ such that for all $(x,\ell) \in \Delta$ and $k \in \mathcal{M}(x,\ell)$,

$$C^{-1} \le m(\mathcal{Q}_k(x,\ell)) \cdot J_T^k(x,\ell) \le C.$$

Bounded distortion gives $C_0 > 0$ such that for all $k \ge 0$ and all $(x,\ell),(y,\ell) \in \Delta$ belonging in the same element of $\mathcal{Q}_k$, we have

$$C_0^{-1} \le \frac{J_T^k(x,\ell)}{J_T^k(y,\ell)} \le C_0. \tag{7}$$

Using the Jacobian, it follows that

$$\begin{aligned} m(T^k(\mathcal{Q}_k(x,\ell))) &= \int_{\mathcal{Q}_k(x,\ell)} J_T^k(y,\ell)\ dm(y,\ell) \\ &= \int_{\mathcal{Q}_k(x,\ell)} \frac{J_T^k(y,\ell)}{J_T^k(x,\ell)} J_T^k(x,\ell)\ dm(y,\ell). \end{aligned}$$

Now, on the one hand, using the bounded distortion, we deduce that

$$m(\Delta) \geq m(T^k(\mathcal{Q}_k(x,\ell))) \geq C_0^{-1} \cdot J_T^k(x,\ell) \cdot m(\mathcal{Q}_k(x,\ell))$$

and consequently, for all $k \in \mathbb{N}$,

$$J_T^k(x,\ell) \cdot m(\mathcal{Q}_k(x,\ell)) \leq m(\Delta) \cdot C_0.$$

On the other hand, for all $k \in \mathcal{M}(x,\ell)$,

$$\eta \leq m(T^k(\mathcal{Q}_k(x,\ell))) \leq C_0 \cdot J_T^k(x,\ell) \cdot m(\mathcal{Q}_k(x,\ell)).$$

This gives the Volume Lemma.

Using that

1. $\mathcal{Q}$ is a generating partition
2. Shannon-McMillan-Breiman Theorem
3. ν is ergodic and equivalent to m
4. the Volume Lemma
5. the chain rule for the Jacobian
6. Birkhoff's Ergodic Theorem

we get

$$
\begin{aligned}
h_\nu(T) = h_\nu(T,\mathcal{Q}) &= \lim_{n\to\infty} -\frac{1}{n}\log \nu(\mathcal{Q}_n(x,\ell)) = \lim_{n\to\infty} -\frac{1}{n}\log m(\mathcal{Q}_n(x,\ell)) \\
&= \lim_{\substack{k\to\infty \\ k\in\mathcal{M}(x,\ell)}} -\frac{1}{k}\log m(\mathcal{Q}_k(x,\ell)) = \lim_{\substack{k\to\infty \\ k\in\mathcal{M}(x,\ell)}} \frac{1}{k}\log J_T^k(x,\ell) \\
&= \lim_{\substack{k\to\infty \\ k\in\mathcal{M}(x,\ell)}} \frac{1}{k}\sum_{i=0}^{k-1}\log J_T(T^i(x,\ell)) = \int \log J_T \; d\nu
\end{aligned},
$$

thus proving Proposition 2.10.

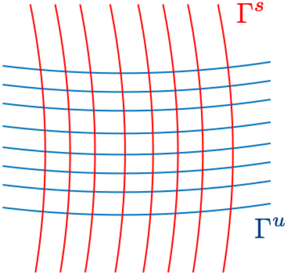
Systems with hyperbolic structures

Young structures

Let M be a Riemannian manifold and $f : M \setminus \mathcal{S} \to M$ a diffeomorphism onto its image. We say that a compact set $\Lambda \subset f^{-n}(M \setminus \mathcal{S})$ has a product structure if there exist a family $\Gamma^s = \{\gamma^s\}$ of stable disks and a family $\Gamma^u = \{\gamma^u\}$ of unstable disks in $M \setminus \mathcal{S}$ such that

- $\Lambda = (\cup \gamma^u) \cap (\cup \gamma^s)$;
- $\dim \gamma^u + \dim \gamma^s = \dim M$;
- each γ^s and γ^u meet in exactly one point;

Given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$.



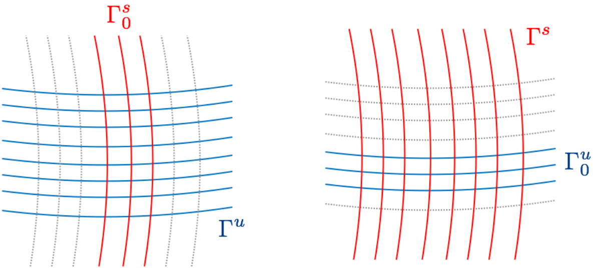
Given disks $\gamma, \gamma' \in \Gamma^u$, define $\Theta_{\gamma,\gamma'} : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ by

$$\Theta_{\gamma,\gamma'}(x) = \gamma^s(x) \cap \gamma, \tag{8}$$

and $\Theta_\gamma : \Lambda \to \gamma \cap \Lambda$ by

$$\Theta_\gamma(x) = \Theta_{\gamma^u(x),\gamma}(x). \tag{9}$$

We say that the hyperbolic product structure is measurable if the maps $\Theta_{\gamma,\gamma'}$ and Θ_γ are measurable, for all $\gamma, \gamma \in \Gamma^u$.



$\Lambda_0 \subset \Lambda$ is called an s-subset if $\Lambda_0 = \Gamma_0^s \cap \Gamma^u$ for some $\Gamma_0^s \subset \Gamma^s$.

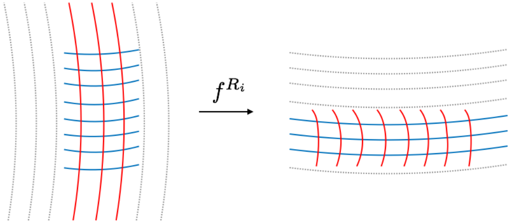
$\Lambda_0 \subset \Lambda$ is called a u-subset if $\Lambda_0 = \Gamma_0^u \cap \Gamma^s$ for some $\Gamma_0^u \subset \Gamma^u$.

A set Λ with a measurable product structure for which (Y_1)-(Y_5) below hold will be called a Young structure.

(Y_1) **Markov:** $\exists$ pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \cdots \subset \Lambda$ such that

- $m_\gamma(\Lambda \cap \gamma) > 0$ and $m_\gamma(\Lambda \setminus \cup_i \Lambda_i) \cap \gamma) = 0$ for all $\gamma \in \Gamma^u$;
- $\forall i \in \mathbb{N}\ \exists R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is a u-subset and for all $x \in \Lambda_i$

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$



We define the recurrence time $R : \Lambda \to \mathbb{N}$ and the return map $f^R : \Lambda \to \Lambda$

$$R|_{\Lambda_i} = R_i \quad \text{and} \quad f^R|_{\Lambda_i} = f^{R_i}.$$

The separation time for $s(x, y)$ for $x, y \in \Lambda$ is the smallest $n \geq 0$ such that $(f^R)^n(x)$ and $(f^R)^n(y)$ lie in distinct Λ_i's.

Consider $C > 0$ and $0 < \beta < 1$ constants depending only on f and Λ .

(Y_2) **Contraction on stable disks**: for all $\gamma \in \Gamma^s$ and $x, y \in \gamma$

- $\mathrm{dist}(f^R(y), f^R(x)) \leq \beta\, \mathrm{dist}(x, y)$;
- $\mathrm{dist}(f^j(y), f^j(x)) \leq C\, \mathrm{dist}(x, y)$, for all $1 \leq j < R(x)$.

(Y_3) **Expansion on unstable disks**: for all $\gamma \in \Gamma^u$, all Λ_i and $x, y \in \gamma \cap \Lambda_i$

- $\mathrm{dist}(x, y) \leq \beta\, \mathrm{dist}(f^R(y), f^R(x))$;
- $\mathrm{dist}(f^j(y), f^j(x)) \leq C\, \mathrm{dist}(f^R(x), f^R(y))$, for all $1 \leq j < R(x)$.

(Y_4) **Absolute continuity of Γ^s**: for all $\gamma, \gamma' \in \Gamma^u$, the map $\Theta_{\gamma,\gamma'}$ is absolutely continuous; moreover, letting $\xi_{\gamma,\gamma'}$ denote the density of $(\Theta_{\gamma,\gamma'})_* m_\gamma$ with respect to $m_{\gamma'}$, we have for all $x, y \in \gamma' \cap \Lambda$

$$\frac{1}{C} \leq \xi_{\gamma,\gamma'}(x) \leq C \quad \text{and} \quad \log \frac{\xi_{\gamma,\gamma'}(x)}{\xi_{\gamma,\gamma'}(y)} \leq C\beta^{s(x,y)}.$$

(Y_5) **Bounded distortion**: $\exists \gamma_0 \in \Gamma^u$ such that for all Λ_i and $x, y \in \gamma_0 \cap \Lambda_i$

$$\log \frac{\det Df^R | T_x\gamma_0}{\det Df^R | T_y\gamma_0} \leq C\beta^{s(f^R(x), f^R(y))}.$$

The structure has integrable recurrence time if for some (hence all) $\gamma \in \Gamma^u$

$$\int_{\gamma\cap\Lambda} R\, dm_\gamma < \infty.$$

SRB measures

Theorem 3.1

The return map f^R of a Young structure has a unique ergodic SRB measure ν. Moreover, the densities of its conditionals with respect to Lebesgue on unstable disks are bounded above and below by constants.

Proof similar to Theorem 2.2, controlling the densities of the measures

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1}(f^R)_*^j m_{\gamma^u}, \quad \text{some } \gamma \in \Gamma^u.$$

Theorem 3.2

If f has a Young structure Λ with integrable recurrence times, then f has a unique ergodic SRB measure with $\mu(\Lambda) > 0$.

$$\mu = \frac{1}{\sum_{j=0}^{\infty}\nu\{R > j\}}\sum_{j=0}^{\infty} f_*^j(\nu|\{R > j\}). \tag{10}$$

Decay of Correlations

Let $\mathcal{H}$ be the space of Hölder continuous functions from M to $\mathbb{R}$.

Theorem 3.3 (Young 1998)

Let f have a Young structure Λ with integrable recurrence time R and μ be the unique ergodic SRB measure of f with $\mu(\Lambda) > 0$. If $\gcd(R) = 1$, then

1. if $m_\gamma\{R > n\} \le Cn^{-a}$ for some $\gamma \in \Gamma^u$ and $C > 0, a > 1$, then for all $\varphi, \psi \in \mathcal{H}$ there exists $C' > 0$ such that $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \le C' n^{-a+1}$;
2. if $m_\gamma\{R > n\} \le Ce^{-cn^a}$ for some $\gamma \in \Gamma^u$ and constants $C, c > 0$ and $0 < a \le 1$, then for all $\varphi, \psi \in \mathcal{H}$ there exists $C' > 0$ such that $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \le C' e^{-c' n^a}$.

If $\gcd\{R\} = k$, the same conclusion holds f^k.

See also the contribution of [Korepanov, Kosloff, and Melbourne 2019][1] in the present (simplified) setting of Young structures.

[1] allegedly based on an oral communication by Gouëzel

Entropy formula

Consider

$$J_f^u = |\det f^u|,$$

where f^u is the restriction of f to unstable disks. Note that if μ an SRB measure, then J_f^u is defined μ almost everywhere.

Theorem 3.4 (Alves and Mesquita 2020)

Let f have a Young structure Λ with integrable recurrence time R and μ be the unique ergodic SRB measure of f with $\mu(\Lambda) > 0$. Then

$$h_\mu(f) = \int \log J_f^u \, d\mu.$$

Tower extension

Let $f : M \to M$ have a Young structure Λ with recurrence time $R : \Lambda \to \mathbb{N}$. As before, we define a tower

$$\hat{\Delta} = \{(x, \ell) \colon x \in \Lambda \text{ and } 0 \le \ell < R(x)\},$$

and a tower map $\hat{T} : \hat{\Delta} \to \hat{\Delta}$ as

$$\hat{T}(x, \ell) = \begin{cases} (x, \ell + 1), & \text{if } \ell + 1 < R(x); \\ (f^R(x), 0), & \text{if } \ell + 1 = R(x). \end{cases}$$

The ℓ-level of the tower is

$$\hat{\Delta}_\ell = \{(x, \ell) \in \hat{\Delta}\}.$$

The 0-level of the tower $\hat{\Delta}_0$ is naturally identified with Λ. We have a partition of $\hat{\Delta}_0$ into subsets $\hat{\Delta}_{0,i} = \Lambda_i$. This gives a partition $\{\hat{\Delta}_{\ell,i}\}_i$ on each level ℓ. Collecting all these sets we obtain a partition $\hat{\mathcal{Q}} = \{\hat{\Delta}_{\ell,i}\}_{\ell,i}$ of $\hat{\Delta}$.

Setting

$$\begin{array}{cccc} \pi & : & \hat{\Delta} & \longrightarrow & M \\ & & (x,\ell) & \longmapsto & f^{\ell}(x) \end{array}$$

we have $f \circ \pi = \pi \circ \hat{T}$.

Theorem 3.5

Let f^R be the return map and $\hat{T}$ the tower map of a Young structure Λ with integrable recurrence time R. If ν is the SRB measure of f^R, then

$$\hat{\nu} = \frac{1}{\sum_{j=0}^{\infty} \nu\{R > j\}} \sum_{j=0}^{\infty} \hat{T}_*^j(\nu|\{R > j\})$$

is the unique ergodic SRB measure of $\hat{T}$. Moreover, $\mu = \pi_*\hat{\nu}$ is the unique ergodic SRB measure of f with $\mu(\Lambda) > 0$.

$\pi_*\hat{\nu}$ gives precisely the formula in (10).

Quotient return map

Given $\gamma_0 \in \Gamma^u$ as in (Y_5), we define the quotient map of f^R on $\gamma_0 \cap \Lambda$

$$\begin{array}{rccl} F : & \gamma_0 \cap \Lambda & \longrightarrow & \gamma_0 \cap \Lambda \\ & x & \longmapsto & \Theta_{\gamma,\gamma_0} \circ f^R(x), \end{array}$$

where $\gamma = \gamma^u(f^R(x))$.

Proposition 3.6

F is Gibbs-Markov with respect to the m_{γ_0} mod 0 partition $\mathcal{P} = \{\gamma_0 \cap \Lambda_1, \gamma_0 \cap \Lambda_2, \dots\}$ of $\gamma_0 \cap \Lambda$.

Lemma 3.7

Let $F : \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda$ be the quotient map of $f^R : \Lambda \to \Lambda$.
If ν is an SRB measure of f^R, then $\nu_0 = (\Theta_{\gamma_0})_* \nu$ is the F-invariant probability measure such that $\nu_0 \ll m_{\gamma_0}$.

Quotient tower

Fix $\gamma_0 \in \Gamma^u$ as in (Y_5), and the quotient map

$$F : \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda.$$

Consider the tower map $T : \Delta \to \Delta$ of F with recurrence time R.
Notice that for all $i \geq 1$

$$R|_{\gamma_0 \cap \Lambda_i} = R|_{\Lambda_i} = R_i.$$

Since $\gamma_0 \cap \Lambda \subset \Lambda$, it easily follows that for all $\ell \geq 0$ we have

$$\Delta_\ell \subset \hat{\Delta}_\ell \quad \text{and} \quad T = \hat{T}|_\Delta. \tag{11}$$

Moreover, $\hat{T} \circ \Theta = \Theta \circ T$, where

$$\begin{array}{cccccc} \Theta & : & \hat{\Delta} & \longrightarrow & \Delta \\ & & (x,\ell) & \longmapsto & (\Theta_{\gamma_0}(x),\ell)\,. \end{array} \tag{12}$$

Proposition 3.8

If $\hat{\nu}$ is the ergodic SRB measure of $\hat{T}$, then $\Theta_*\hat{\nu}$ is the unique ergodic T-invariant probability measure absolutely continuous with respect to m_{γ_0}.

Decay of correlations

We have

$$\pi \circ \hat{T} = f \circ \pi \quad \text{and} \quad \Theta \circ \hat{T} = T \circ \Theta. \tag{13}$$

Let

- $\hat{\nu}$ be the unique ergodic SRB measure of $\hat{T}$;
- μ be the unique ergodic SRB measure of f with $\mu(\Lambda) > 0$;
- ν be the unique ergodic T-invariant measure such that $\nu \ll m_{\gamma_0}$.

By Theorem 3.5 and Proposition 3.8, we have

$$\mu = \pi_* \hat{\nu} \quad \text{and} \quad \nu = \Theta_* \hat{\nu}. \tag{14}$$

Given $\varphi, \psi \in \mathcal{H}$, define

$$\hat{\psi} = \psi \circ \pi \quad \text{and} \quad \hat{\varphi} = \varphi \circ \pi. \tag{15}$$

Lemma 3.9

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathrm{Cor}_{\hat{\nu}}(\hat{\varphi}, \hat{\psi} \circ \hat{T}^n).$$

It is enough to obtain estimates for $\mathrm{Cor}_\nu(\hat{\varphi}, \hat{\psi} \circ \hat{T}^n)$. The idea is to reduce to a problem on the quotient tower $T : \Delta \to \Delta$, and apply Theorem 2.9.

Given $k \geq 1$, define

$$\hat{\mathcal{Q}}_k = \bigvee_{j=0}^{k-1} \hat{T}^{-j}\hat{\mathcal{Q}}. \tag{16}$$

Define the *discretisation* $\varphi_k : \hat{\Delta} \to \mathbb{R}$ of $\hat{\varphi}$, setting for each $Q \in \hat{\mathcal{Q}}_{2k}$

$$\varphi_k|_Q = \inf\{\hat{\varphi} \circ \hat{T}^k(x) \colon x \in Q\}. \tag{17}$$

φ_k may as well be thought of as function on Δ.

Proposition 3.10

For all $\varphi, \psi \in \mathcal{H}$ and $1 \leq k \leq n$,

$$\mathrm{Cor}_{\hat{\nu}}(\hat{\varphi}, \hat{\psi} \circ \hat{T}^n) \leq \mathrm{Cor}_{\nu}(\varphi_k, \psi_k \circ T^n) + 2\|\varphi\|_0 \|\hat{\psi} \circ \hat{T}^k - \psi_k\|_1 + 2\|\psi\|_0 \|\hat{\varphi} \circ \hat{T}^k - \varphi_k\|_1.$$

$\| \; \|_1$ is the L^1-norm with respect to the probability measure $\hat{\nu}$ on $\hat{\Delta}$.

We are left to estimate the L^1-norms in Proposition 3.10.

Define for $x \in \hat{\Delta}$ and $k \geq 1$

$$b_k(x) = \#\{1 \leq j \leq k : \hat{T}^j(x) \in \hat{\Delta}_0\}.$$

Since (11) holds, we may use the same notation as for the tower T of the quotient map F. Recall that each b_k is constant on stable disks.

Lemma 3.11

For every Hölder continuous $\varphi : M \to \mathbb{R}$ there are $C > 0$ and $0 < \sigma < 1$ such that for all $k \geq 1$ and $x \in \Delta$ we have

$$|\hat{\varphi} \circ \hat{T}^k(x) - \varphi_k(x)| \leq C \left(\sigma^{b_k(x)} + \sigma^{b_k(\hat{T}^k(x))} \right).$$

Define $R_k = \sum_{j=0}^{k-1} R \circ F^j$, for each $k \geq 1$

Proposition 3.12

Given $0 < \sigma < 1$, *there exists* $C > 0$ *such that for all* $k \geq 1$ *we have*

$$\int \sigma^{b_k} d\nu \leq C \sum_{\ell \geq k/3} m_{\gamma_0}\{R \geq \ell\} + Ck \sum_{\ell \geq 1} \sigma^{\ell} m_{\gamma_0}\left\{R_\ell > \frac{k}{3}\right\}.$$

As a consequence of Lemma 3.11 and Proposition 3.12.

Corollary 3.13

For every Hölder continuous $\varphi : M \to \mathbb{R}$ and $k \geq 1$

$$\|\hat{\varphi} \circ \hat{T}^k - \varphi_k\|_1 \leq C \sum_{\ell \geq k/3} m_{\gamma_0}\{R \geq k\} + Ck \sum_{\ell \geq 1} \sigma^{\ell} m_{\gamma_0}\left\{R_\ell > \frac{k}{3}\right\}.$$

This enables us to deduce the desired estimates in the polynomial and (stretched) exponential cases.

Entropy formula

The **natural extension** of the system $(\hat{\Delta}, \hat{T}, \hat{\mathcal{B}}, \hat{\nu})$ is a new measure preserving system $(\hat{\Delta}^{\#}, \hat{T}^{\#}, \hat{\mathcal{B}}^{\#}, \hat{\nu}^{\#})$ defined as

- $\hat{\Delta}^{\#} = \{(\dots, (x_{-1}, l_{-1}), (x_0, l_0)) \in \prod_{-\infty}^{i=0} \hat{\Delta} \mid \hat{T}(x_n, l_n) = (x_{n+1}, l_{n+1}) \ \forall n < 0\}$
- $\hat{T}^{\#}(\dots, (x_{-1}, l_{-1}), (x_0, l_0)) = (\dots, (x_{-1}, l_{-1}), (x_0, l_0), \hat{T}(x_0, l_0))$.
- $\hat{\mathcal{B}}^{\#}$ σ-algebra generated by **cylinder sets** of the form

$$[A_k, \dots, A_0] = \{(x_n, l_n)_{n \le 0} \in \hat{\Delta}^{\#} \mid (x_i, l_i) \in A_i \text{ for all } i = k, \dots, 0\},$$

  where $A_i \in \hat{\mathcal{B}}$ for all $i = k, \dots, 0$.
- $\hat{\nu}^{\#}([A_k, \dots, A_0]) := \hat{\nu}(A_k \cap \hat{T}^{-1}(A_{k-1}) \cap \dots \cap \hat{T}^{-k}(A_0))$.

Similar for $(\Delta, T, \mathcal{B}, \nu)$.

Proposition (Demers, Wright, and Young 2012)

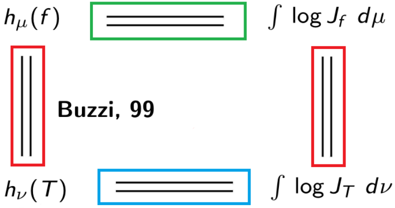
$(\hat{\Delta}^{\#}, \hat{\mathcal{B}}^{\#}, \hat{\nu}^{\#}, \hat{T}^{\#}) \simeq (\Delta^{\#}, \mathcal{B}^{\#}, \nu^{\#}, T^{\#})$

Proposition (Rohlin 1967)

$h_{\hat{\nu}}(\hat{T}) = h_{\hat{\nu}^{\#}}(\hat{T}^{\#})$ and $h_{\nu}(T) = h_{\nu^{\#}}(T^{\#})$

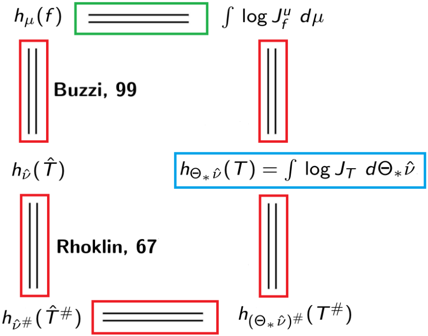
Entropy relations

Systems with Gibbs-Markov structures



Proposition 2.10

Systems with Young structures



DWY 2012

Recall that by (14) we have

$$\mu = \pi_* \hat\nu \quad \text{and} \quad \nu = \Theta_* \hat\nu.$$

Applications

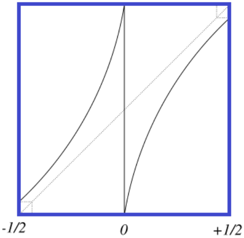
Since the appearance of [Young 1999; Young 1998] many results have been obtained using induced Gibbs-Markov maps or Young structures. This led to a fairly complete theory of *non-uniformly expanding maps*, *partially hyperbolic attractors* and *Hénon attractors*.

1. Existence of SRB measures [Alves, Dias, Luzzatto, and Pinheiro 2017]
2. Decay of Correlations [Benedicks and Young 2000; Gouëzel 2006; Alves, Luzzatto, and Pinheiro 2005; Alves and Li 2015]
3. Large deviations [Melbourne and Nicol 2008; Melbourne 2009]
4. Statistical stability [Alves 2004; Freitas 2005; Alves, Carvalho, and Freitas 2010b; Alves and Soufi 2012]
5. Continuity of entropy [Alves, Oliveira, and Tahzibi 2006; Alves, Carvalho, and Freitas 2010a]

Below we present some examples of systems with discontinuities where entropy formula can be obtained, using Theorem 2.4 or Theorem 3.4.

Lorenz maps

$\{f_X\}_{X\in\mathcal{X}}$, where $\mathcal{X}$ is the family of **geometric Lorenz vector fields**



- $f_X : I_X \to I_X$ transitive C^{1+} local diffeomorphism;
- $I_X = [-r_X, r_X]$, $r_X \sim 1/2$;
- $\mathcal{S}_X = \{s_X \sim 0\}$ singular set **(unbounded derivative)**;
- unique ergodic SRB measure.

Theorem (Alves and Mesquita 2020)

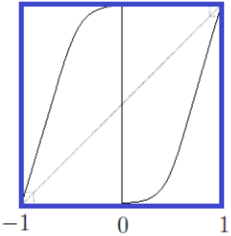
Each f_X has and induced Gibbs-Markov map with exponential tail of recurrence times.

Corollary

Entropy formula holds for the SRB measure of f_X.

Rovella maps

$\{f_a\}_{a\in\mathcal{R}}$, with $\mathcal{R} \subseteq [0,1]$ satisfying $\lim_{\varepsilon\to 0} \frac{|\mathcal{R}\cap[0,\varepsilon]|}{\varepsilon} = 1$.



- $I = [-1, 1]$;
- $f_a : I \setminus \{0\} \to I$ transitive C^{1+} local diffeomorphism;
- $\{0\}$ is a critical/singular set;
- unique ergodic SRB measure.

Theorem (Alves and Soufi 2012)

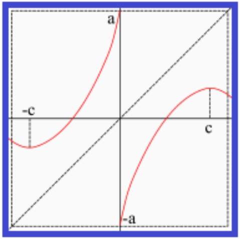
Each f_a with $a \in \mathcal{R}$ has and induced Gibbs-Markov map with exponential tail of recurrence times.

Corollary

Entropy formula holds for the SRB measure of f_a with $a \in \mathcal{R}$.

Luzzatto-Viana maps

$\{f_a\}_{a \in \mathcal{LV}}$, with $\mathcal{LV} \subseteq \mathbb{R}^+_{\geq c}$ such that $\lim_{\epsilon \to 0} \frac{|\mathcal{LV} \cap [c, c+\epsilon]|}{\epsilon} = 1.$



- $I_a = [-a, a]$, $a \geq c > 0$.
- $f_a : I_a \to I_a$ topologically mixing C^{1+} local diffeomorphism;
- $\{0, \pm c\}$ is the critical/singular set;
- unique ergodic SRB measure.

Theorem (Alves and Gama 2019)

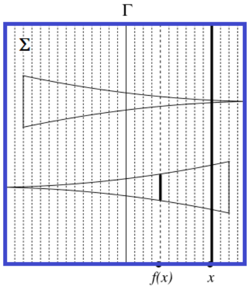
Each f_a with $a \in \mathcal{LV}$ has and induced Gibbs-Markov map with exponential tail of recurrence times.

Corollary

Entropy formula holds for the unique SRB measure of f_a with $a \in \mathcal{LV}$.

Poincaré map for geometric Lorenz attractor

$\{P_X\}_{X\in\mathcal{X}}$, where $\mathcal{X}$ is the family of **geometric Lorenz vector fields**



- $P_X : \Sigma\backslash\Gamma_X \to \Sigma$ diffeomorphism
- Γ_X nearly vertical singular line;
- nearly horizontal **unstable direction**;
- nearly vertical **stable direction**;
- unique ergodic SRB measure.

Theorem (Alves and Mesquita 2020)

Each P_X has and induced Gibbs-Markov map with exponential tail of recurrence times.

Corollary

Entropy formula holds for the SRB measure of P_X.

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