# Hyperbolic isomorphisms in Banach spaces

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# Contents

# Introduction

The main goal of these notes is to generalize some results which are standard for linear isomorphisms of finite dimensional vector spaces to infinite dimensional spaces. In particular, we shall prove that there exists an invariant splitting for any hyperbolic bounded linear isomorphism on a Banach space and that such an isomorphism is structurally stable.

Let us start with some motivational considerations in the finite dimensional setting. Consider $A : E \to E$ an isomorphism of a vector space $E$ over $\mathbb{R}$ or $\mathbb{C}$ with finite dimension. It is well known that by a suitable choice of a basis in $E$ the isomorphism $A$ can be represented by a block diagonal matrix

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{pmatrix},$$

where each $J_i$ is a Jordan block. In the complex case, by a *Jordan block* we mean a square matrix of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}, \qquad (\star)$$

where each $\lambda_i$ is an eigenvalue of $A$. In the real case, a *Jordan block* can either be a matrix of the type $(\star)$, corresponding to a real eigenvalue $\lambda_i$ of $A$, or a matrix

$$J_i = \begin{pmatrix} C_i & I & & \\ & C_i & \ddots & \\ & & \ddots & I \\ & & & C_i \end{pmatrix},$$

where

$$C_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

corresponding to a complex eigenvalue $\lambda_i = a_i + ib_i$ with $b_i \neq 0$. In this last case, by an eigenvalue of $A$ we mean a root of the characteristic polynomial of $A$. We leave it as an exercise to show that:

1. if $e_1, \ldots, e_m$ are vectors in a basis associated to a Jordan block of an eigenvalue $\lambda$ of $A$ with $|\lambda| < 1$, then $A^n x \to 0$ as $n \to +\infty$ for all $x \in \operatorname{span}\{e_1, \ldots, e_m\}$;

2. if $f_1, \ldots, f_n$ are vectors in a basis associated to a Jordan block of an eigenvalue $\lambda$ of $A$ with $|\lambda| > 1$, then $A^{-n} x \to 0$ as $n \to +\infty$ for all $x \in \operatorname{span}\{f_1, \ldots, f_n\}$.

Hence, if $A$ has no eigenvalues in the unit circle, then there are subspaces $E^s$ and $E^u$ of $E$ (called the *stable* and *unstable* spaces of $A$, respectively) for which the following properties hold:

1. $E = E^s \oplus E^u$;
2. $A^n x \to 0$ as $n \to +\infty$ for all $x \in E^s$;
3. $A^{-n} x \to 0$ as $n \to +\infty$ for all $x \in E^u$.

As a linear isomorphism of an infinite dimensional Banach space may have no eigenvalues we need an alternative method to find the stable and unstable spaces. The idea is to consider the more general notion of spectrum of an operator and the projections associated to certain isolated parts of the spectrum.

We shall try to make these notes as self-contained as possible, in particular proving most of the results we need from Functional Analysis and Spectral Theory. In Sections 1 and 2 we consider some basic results on bounded linear operators. We introduce the notion of spectrum of a bounded linear operator and prove some of its basic properties. In Section 3 we prove a Spectral Decomposition Theorem for operators whose spectrum can be decomposed into two isolated parts.

The results in Sections 2 and 3 are specific for complex Banach spaces. In order to obtain our conclusions on the dynamics of hyperbolic linear isomorphisms in the real case as well, in Section 4 we recall the standard notion of complexification, both of a real vector space and of an operator defined on that real space. In particular, we shall introduce a suitable norm in the complexification space.

In Section 5 we apply the Spectral Decomposition Theorem to an operator whose spectrum is disjoint from the unit circle, both in the complex and real cases, and obtain for a linear isomorphisms of a Banach space the decomposition of that space into a direct sum of sable an unstable spaces. This depicts a fairly reasonable scenario for the dynamics of such hyperbolic isomorphisms.

Another important issue in Dynamical Systems Theory is the structural stability of a system, meaning that the dynamics is not affected under a small perturbation of the system. We shall also address this question for isomorphisms of infinite dimensional spaces. In Section 6 we prove that a hyperbolic linear isomorphism of a Banach space is topologically conjugate to any nearby isomorphism.

# 1 Bounded linear operators

Consider $E$ a vector space over $\mathbb{R}$ or $\mathbb{C}$ endowed with a norm $\| \ \|$. We say that $E$ is a *Banach space* if the space $E$ with the metric induced by $\| \ \|$ is a complete metric space. We say that a linear operator $T : E \to E$ is *bounded* if

$$\|T\| \equiv \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\| < \infty,$$

and define

$$L(E) = \{T : E \to E \mid \ T \text{ is linear and bounded}\}.$$

It is well known that $L(E)$ is a vector space and $\|T\|$ as defined above gives a norm in $L(E)$. Moreover, if $E$ is a Banach space, then $L(E)$ with that norm is a Banach space as well. We shall always consider this norm in the space $L(E)$.

In our first result we give some useful criteria for the existence of inverse for certain bounded operators in Banach spaces.

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**Lemma 1.1.** Let $E$ be a Banach space and $T \in L(E)$. Then

1. if $\sum_{n=0}^{\infty} T^n$ converges, then $I - T$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$;
2. if $\|T\| < 1$, then $I - T$ is invertible, $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ and $\|(I - T)^{-1}\| \leq 1/(1 - \|T\|)$.

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*Proof.* Assume first that the series $\sum_{n=0}^{\infty} T^n$ converges to $S \in L(E)$. Then we have $T^m \to 0$, when $m \to \infty$. A simple calculation gives that for each $m \in \mathbb{N}$

$$(I - T) \sum_{n=0}^{m} T^n = \left( \sum_{n=0}^{m} T^n \right) (I - T) = I - T^{m+1}.$$

Hence, using the continuity of the product in $L(E)$ and taking limit when $m \to \infty$, we obtain

$$(I - T)S = S(I - T) = I.$$

This gives that $I - T$ is invertible and $\sum_{n=0}^{\infty} T^n = (I - T)^{-1}$.

Assume now that $\|T\| < 1$. Then, as

$$\sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T\|^n$$

and this last series converges, we easily deduce that the partial sums of $\sum_{n=0}^{\infty} T^n$ form a Cauchy sequence in $L(E)$. Since $L(E)$ is a Banach space $\sum_{n=0}^{\infty} T^n$ converges. Now we use the first item and deduce that $I - T$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$. Moreover,

$$\|(I - T)^{-1}\| = \left\| \sum_{n=0}^{\infty} T^n \right\| \leq \sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1 - \|T\|}$$

for $\|T\| < 1$. $\square$

As illustrated in the introduction, the existence of stable and unstable spaces for an isomorphism on a finite dimensional space can be obtained knowing the eigenvalues of the isomorphism and the corresponding generalized eigenspaces. As eigenvalues of an isomorphism of an infinite dimensional space need not to exist in general, the idea is to consider the more general notion of spectrum of an operator and the projections associated to certain isolated parts of that spectrum.

**Definition 1.2.** We say that a bounded linear operator $P : E \to E$ of a vector space $E$ is a *projection* if $P^2 = P$.

In the next result we give some useful properties of projections. By $I : E \to E$ we mean the identity operator in the vector space $E$.

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**Proposition 1.3.** Let $P : E \to E$ be a projection. Then:

1. $I - P$ is a projection;
2. $P(E) = \ker(I - P)$;
3. $P(E)$ is a closed subspace of $E$;
4. $E = P(E) \oplus (I - P)(E)$.

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*Proof.* 1. A simple calculation gives

$$(I - P)^2 = (I - P)(I - P) = I - P - P + P^2 = I - P,$$

where this last equality follows form the fact that $P$ is a projection.

2. If $y = Px$ for some $x \in E$, then $y - Py = Px - P^2x = Px - Px = 0$. Hence, $y \in \ker(I - P)$. On the other hand, if $y \in \ker(I - P)$, then $y = Py$, and so $y \in P(E)$.

3. Since $P(E) = \ker(I - P)$ by the second item, then we have $P(E) = (I - P)^{-1}\{0\}$ where $\{0\}$ is a closed set and $I - P$ is continuous. Therefore $P(E)$ is a closed subspace.

4. Given $x \in E$, we may write $x = Px + (I - P)x$. It remains to prove that $P(E) \cap (I - P)(E) = \{0\}$. Given $x \in P(E) \cap (I - P)(E)$, there are $y, z \in E$ such that $x = Py$ and $x = z - Pz$. We have in particular $Py = z - Pz$. Applying $P$ to both sides, we obtain $P^2y = Pz - P^2z$. As $P$ is a projection, we have $Py = z - Pz = 0$ and this implies that $x = 0$. $\square$

## 2 Spectrum and resolvent

In this section we introduce the spectrum of a bounded linear operator on a complex Banach space and prove some of its basic properties.

**Definition 2.1.** Let $E$ be a complex Banach space. We define the *spectrum* of $T \in L(E)$ as $\sigma(T) = \{\lambda \in \mathbb{C} : \ \lambda I - T \text{ is not invertible}\}$ and the *resolvent set* of $T$ as $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Given $\lambda \in \rho(T)$, the *resolvent operator of $T$ at $\lambda$* is defined as $R_\lambda(T) = (\lambda I - T)^{-1}$.

It is worth noting that by the Open Mapping Theorem we have $R_\lambda(T)$ a bounded linear operator for any $T \in L(E)$ and $\lambda \in \rho(T)$. Also, if $\lambda$ is an eigenvalue of $T$, then $\lambda \in \sigma(T)$. Recalling that a linear operator from a finite dimensional space into itself is invertible if and only if the operator is injective, we have that the spectrum of a linear operator on a finite dimensional complex vector space coincides with the set of eigenvalues of that operator. However, in infinite dimensional spaces we may have elements in the spectrum of an operator which are not eigenvalues, as the next example illustrates.

**Example 2.2.** Consider the vector space of absolutely summable sequences in $\mathbb{C}$

$$\ell^1 = \left\{ (x_1, x_2, x_3, ...) : \ x_n \in \mathbb{C} \quad \text{and} \quad \sum_{n \geq 1} |x_n| < \infty \right\}.$$

It is well known that

$$\|(x_1, x_2, \dots)\| = \sum_{n \geq 1} |x_n|$$

defines a norm in $\ell^1$ with respect to which $\ell^1$ is a Banach space. Let $T : \ell^1 \to \ell^1$ be the shift operator given by

$$T(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

It is clear that $0I - T = -T$ is not invertible, and so $0 \in \sigma(T)$. However, $0$ is not an eigenvalue of $T$.

Next we give a result on the spectrum of inverse and conjugate operators, which is well known for isomorphisms of finite dimensional spaces.

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**Lemma 2.3.** Let $E$ be a complex Banach space and $A, B : E \to E$ be bounded linear isomorphisms. Then:

1. $\sigma\left(A^{-1}\right) = \{1/\lambda : \lambda \in \sigma(A)\}$;
2. $\sigma(BAB^{-1}) = \sigma(A)$.

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*Proof.* First of all observe that as $A$ is invertible, then $0 \notin \sigma(A)$. Furthermore, we have for $\lambda \neq 0$

$$\lambda I - A = -\lambda A \left( \frac{1}{\lambda} I - A^{-1} \right).$$

Hence, if $A \in GL(E)$, then $\lambda I - A$ is invertible if and only if $I/\lambda - A^{-1}$ is invertible. This proves the first assertion.

Given $B \in GL(E)$ and $\lambda \in \mathbb{C}$, we may write

$$\lambda I - BAB^{-1} = B(\lambda I)B^{-1} - BAB^{-1} = B(\lambda I - A)B^{-1}.$$

Hence, $\lambda I - BAB^{-1}$ is invertible if and only if $\lambda I - A$ is invertible. $\square$

In the next result we consider analytic functions defined on some open set of the complex plane and taking values in a complex Banach space. Essentially all the main definitions and results of the theory of complex valued functions, such as Contour Integrals, Residue Theorem and Liouville Theorem, remain true in this more general context; see e.g. [1].

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**Theorem 2.4.** Let $E$ be a complex Banach space and $T \in L(E)$. Then:

1. $\rho(T)$ is open in $\mathbb{C}$ and it contains $\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\}$;
2. $R_\lambda(T) = \dfrac{1}{\lambda} \displaystyle\sum_{n=0}^{\infty} \dfrac{T^n}{\lambda^n}$ for all $|\lambda| > \|T\|$;
3. $\rho(T) \ni \lambda \longmapsto R_\lambda(T) \in L(E)$ is an analytic function.

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*Proof.* If $|\lambda| > \|T\|$, then using Lema 1.1 we easily deduce that $\lambda I - T$ is invertible and

$$R_\lambda(T) = \frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{T^n}{\lambda^n}.$$

Thus we have proved that $\rho(T) \supset \{\lambda \in \mathbb{C} : |\lambda| > \|T\|\}$ and also the second item.

Given $\lambda_0 \in \rho(T)$, take $\lambda \in \mathbb{C}$ close enough to $\lambda_0$ so that

$$|\lambda - \lambda_0| \cdot \|R_{\lambda_0}(T)\| < 1. \tag{1}$$

Writing

$$\lambda I - T = \lambda_0 I - T + (\lambda - \lambda_0)I = \big[I - (\lambda - \lambda_0)R_{\lambda_0}(T)\big](\lambda_0 I - T),$$

then Lemma 1.1 gives that $\lambda \in \rho(T)$ for $\lambda$ satisfying (1), thus proving that $\rho(T)$ is an open set. Moreover,

$$R_\lambda(T) = R_{\lambda_0}(T)\left[I - (\lambda - \lambda_0)R_{\lambda_0}(T)\right]^{-1}.$$

Now Lemma 1.1 gives that

$$\left[I - (\lambda - \lambda_0)R_{\lambda_0}(T)\right]^{-1} = \sum_{n=0}^{\infty}(\lambda - \lambda_0)^n R_{\lambda_0}(T)^n,$$

and so

$$R_\lambda(T) = R_{\lambda_0}(T) \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(T)^n.$$

Thus we have written $R_\lambda(T)$ as a convergent power series around $\lambda_0$, thereby proving its analyticity. $\square$

This last theorem shows that $\sum_{n=1}^{\infty} T^{n-1}/\lambda^n$ is the Laurent series of the function $R_\lambda(T)$ in the annulus $\{\|T\| < |z| < \infty\}$. So we can use the theory of holomorphic functions to deduce many results about $R_\lambda(T)$ using this expansion. To start with, we consider a Jordan curve $\gamma$ oriented counterclockwise and containing the disk $\{|z| \le \|T\|\}$ in its interior. The Residue Theorem gives that

$$\frac{1}{2\pi i} \int_\gamma R_\lambda(T) d\lambda = I.$$

Such utilization of curves and contour integrals to separate different parts of the spectrum will be very useful in defining projection operators which will give a splitting into stable and unstable spaces of hyperbolic isomorphisms.

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**Theorem 2.5.** Let $E \neq \{0\}$ be a complex Banach space and $T \in L(E)$. Then $\sigma(T)$ is a nonempty compact set contained in $\{|z| \le \|T\|\}$.

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*Proof.* Noticing that $\sigma(T) = \mathbb{C} \setminus \rho(T)$, it immediately follows fromTheorem 2.4 that $\sigma(T)$ is a closed set and $\sigma(T) \subset \{z \in \mathbb{C} : |z| \le \|T\|\}$. It remains to see that $\sigma(T)$ is nonempty. Assume, by contradiction, that $\rho(T) = \emptyset$. This implies that $\lambda \mapsto R_\lambda(T)$ is an analytic function defined in the whole complex plane. Moreover, it follows from Lemma 1.1 that for $|\lambda| > \|T\|$ we have

$$\|R_\lambda(T)\| = \left\| \frac{1}{\lambda} \left( I - \frac{T}{\lambda} \right)^{-1} \right\| \le \frac{1}{|\lambda|(1 - \|T/\lambda\|)}.$$

Thus, $\lambda \mapsto R_\lambda(T)$ is a bounded function. Hence, by Liouville Theorem it must be constant. As $R_\lambda(T) \to 0$ when $\lambda \to \infty$, it follows that $R_\lambda(T) = 0$. This gives a contradiction for $E \neq \{0\}$. $\square$

**Definition 2.6.** Let $E$ be a complex vector space. We define the *spectral radius* of an opreator $T \in L(E)$ as

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

**Theorem 2.7.** Let $E$ be a complex Banach space and $T \in L(E)$. Then

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\geq 1} \|T^n\|^{1/n}.$$

*Proof.* We start by showing that $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists and coincides with

$$s(T) = \liminf_{n\geq 1} \|T^n\|^{1/n}.$$

For this, it is sufficient to show that

$$\limsup_{n\to+\infty} \|T^n\|^{1/n} \leq s(T). \tag{2}$$

Given an arbitrary $\epsilon > 0$, fix $p \in \mathbb{N}$ such that $\|T^p\|^{1/p} < s(T) + \epsilon$. Any $n \in \mathbb{N}$ can be written as $n = pq + r$ with $0 \leq r < p$. Then, using the fact that norm is submultiplicative, we obtain

$$\|T^n\|^{1/n} = \|T^{pq+r}\|^{1/n} \leq \|T^p\|^{q/n}\|T\|^{r/n} < (s(T)+\epsilon)^{pq/n}\|T\|^{r/n}.$$

Since $pq/n \to 1$ and $r/n \to 0$ when $n \to +\infty$, it follows that

$$\limsup_{n\to+\infty} \|T^n\|^{1/n} \leq s(T) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have proved (2).

Let us now prove that $r(T) \leq s(T)$, or equivalently, $\lambda \in \rho(T)$ for $|\lambda| > s(T)$. In fact, taking $c = |\lambda| - s(T) > 0$, we obtain $\|T^n\| < [s(T) + c/2]^n$ for sufficiently large $n$, since $\lim_{n\to+\infty} \|T^n\|^{1/n} = s(T)$. Then,

$$\frac{1}{|\lambda|^n}\|T^n\| \leq \left(\frac{s(T)+c/2}{s(T)+c}\right)^n,$$

which shows that the series $\sum_{n=0}^{\infty} T^n/\lambda^n$ converges in $L(E)$. Moreover, by Lemma 1.1 we can write

$$\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{T^n}{\lambda^n} = \frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = R_\lambda(T), \tag{3}$$

and so we have proved that $\lambda \in \rho(T)$ for $|\lambda| > s(T)$.

It remains to show that $r(T) \geq s(T)$. Theorem 2.4 gives that $r(T) \leq \|T\|$ and the function $\lambda \mapsto R_\lambda(T)$ is analytic for $|\lambda| > r(T)$. In particular, $R_\lambda(T)$ admits a Laurent series representation centered at 0 for $|\lambda| > r(T)$. Equation (3) shows that this series is necessarily $\sum_{n=1}^{\infty} T^{n-1}/\lambda^n$. Thus we have $\lim_{n\to\infty} \|\lambda^{-n}T^n\| = 0$ for any $|\lambda| > r(T)$. Hence, given an arbitrary $\epsilon > 0$, we have $\|T^n\| \leq [\epsilon + r(T)]^n$ for $n$ large. As $\epsilon > 0$ is arbitrary, we obtain

$$s(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \leq r(T).$$

Thus we have proved that $r(T) = s(T)$. $\square$

As $r(T)$ is obviously bounded by $\|T\|$, equation (3) shows that the formula obtained in Theorem 2.4 can be extended to the annulus $\{\lambda \in \mathbb{C} : r(T) < |\lambda| < \infty\}$.

# 3 Spectral decomposition

In the next result we show that if the spectrum of an operator on a complex Banac space can be split into two disjoint compact parts, then corresponding to such splitting in the spectrum there is a splitting in the space.

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**Theorem 3.1** (Spectral Decomposition)**.** Let $E$ be a complex Banach space and $T \in L(E)$. Assume that $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_1, \sigma_2$ are compact disjoint sets. Then there are closed spaces $E_1, E_2$ with:

1. $E = E_1 \oplus E_2$;
2. $T(E_1) \subset E_1$ and $T(E_2) \subset E_2$;
3. $\sigma(T|_{E_1}) = \sigma_1$ and $\sigma(T|_{E_2}) = \sigma_2$.

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*Proof.* Assume that $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_1, \sigma_2$ are compact and disjoint. Consider a Jordan curve $\gamma \subset \rho(T)$ oriented counterclockwise, containing $\sigma_1$ in its interior and $\sigma_2$ in its exterior. We define the operator in $L(E)$

$$P_1 = \frac{1}{2\pi i}\int_\gamma R_\lambda(T)d\lambda.$$

Let $\gamma'$ be another Jordan curve oriented counterclockwise, containing $\gamma$ in its interior and $\sigma_2$ in its exterior. As $\gamma$ and $\gamma'$ are homotopic in $\rho(T)$, we have

$$P_1 = \frac{1}{2\pi i}\int_\gamma R_\lambda(T)d\lambda = \frac{1}{2\pi i}\int_{\gamma'} R_{\lambda'}(T)d\lambda'.$$

Let us now prove that $P_1$ is a projection. First of all observe that for $\lambda, \lambda' \in \rho(T)$ we have

$$\begin{aligned} R_\lambda(T) &= R_\lambda(T)(\lambda' I - T)R_{\lambda'}(T) \\ &= R_\lambda(T)\big[\lambda I - T + (\lambda' - \lambda)I\big]R_{\lambda'}(T) \\ &= \big[I + (\lambda' - \lambda)R_\lambda(T)\big]R_{\lambda'}(T) \\ &= R_{\lambda'}(T) + (\lambda' - \lambda)R_\lambda(T)R_{\lambda'}(T). \end{aligned} \tag{4}$$

Now, using (4) in the second equality below we may write

$$\begin{aligned} P_1^2 &= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma'} R_\lambda(T)R_{\lambda'}(T)d\lambda' d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma'} \frac{R_\lambda(T) - R_{\lambda'}(T)}{\lambda' - \lambda}d\lambda' d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma'} \frac{R_\lambda(T)}{\lambda' - \lambda}d\lambda' d\lambda - \frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma'} \frac{R_{\lambda'}(T)}{\lambda' - \lambda}d\lambda' d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma R_\lambda(T)\frac{1}{2\pi i}\int_{\gamma'} \frac{d\lambda'}{\lambda' - \lambda}d\lambda + \frac{1}{2\pi i}\int_{\gamma'} R_{\lambda'}(T)\frac{1}{2\pi i}\int_\gamma \frac{d\lambda}{\lambda - \lambda'}d\lambda'. \end{aligned}$$

Noting that we are taking $\gamma$ inside $\gamma'$, we have for $\lambda \in \gamma$ and $\lambda' \in \gamma'$

$$\frac{1}{2\pi i}\int_{\gamma'} \frac{d\lambda'}{\lambda' - \lambda} = 1 \quad \text{and} \quad \frac{1}{2\pi i}\int_{\gamma} \frac{d\lambda}{\lambda - \lambda'} = 0,$$

from which we deduce that $P_1^2 = P_1$, and so $P_1$ is a projection.

Defining $P_2 = I - P_1$ and the spaces $E_1 = P_1(E)$ and $E_2 = P_2(E)$, we have by Proposition 1.3 that $E_1$ and $E_2$ are closed subspaces and

$$E = E_1 \oplus E_2.$$

Moreover, as $T$ commutes with $\lambda I - T$, we have that $T$ commutes with $R_\lambda(T)$ for $\lambda \in \rho(T)$. Integrating we get

$$P_1 T = T P_1.$$

Then we have

$$T(E_1) = T P_1(E) = P_1 T(E) \subset P_1(E) = E_1.$$

Similarly we see that $T(E_2) \subset E_2$, thus having proved the first two items of the theorem.

Now we are going to see that

$$\sigma(T) = \sigma(T|_{E_1}) \cup \sigma(T|_{E_2}). \tag{5}$$

It easily follows from (4) that for all $\lambda, \lambda' \in \rho(T)$ we have

$$R_{\lambda'}(T) R_\lambda(T) = R_\lambda(T) R_{\lambda'}(T),$$

and so, integrating with respect to $\lambda'$ along the curve $\gamma'$, this yields

$$P_1 R_\lambda(T) = R_\lambda(T) P_1.$$

Hence

$$(\lambda I - T) P_1 R_\lambda(T) = (\lambda I - T) R_\lambda(T) P_1 = P_1. \tag{6}$$

Also

$$R_\lambda(T)(\lambda I - T) P_1 = P_1. \tag{7}$$

Noticing that $P_1|_{E_1}$ is the identity, restricting equalities (6) and (7) to $E_1$ we deduce that $\lambda \in \rho(T|_{E_1})$ and $R_\lambda(T|_{E_1}) = R_\lambda(T)|_{E_1}$, thus proving that $\rho(T) \subset \rho(T|_{E_1})$. Using that $P_2$ commutes with $R_\lambda(T)$ as well, we also see that $\rho(T) \subset \rho(T|_{E_2})$. Hence we have

$$\rho(T) \subset \rho(T|_{E_1}) \cap \rho(T|_{E_2}). \tag{8}$$

Besides, as $T(E_1) \subset E_1$ and $T(E_2) \subset E_2$, we easily see that $R_\lambda(T) = R_\lambda(T|_{E_1}) \oplus R_\lambda(T|_{E_2})$ for $\lambda \in \rho(T|_{E_1}) \cap \rho(T|_{E_2})$. This shows that $\rho(T|_{E_1}) \cap \rho(T|_{E_2}) \subset \rho(T)$, which together with (8) gives (5).

Let us finally show that $\sigma(T|_{E_1}) = \sigma_1$ and $\sigma(T|_{E_2}) = \sigma_2$. Consider $D$ the set delimited by the Jordan curve $\gamma \subset \rho(T)$ used to define $P_1$, containing $\sigma_1$ in its interior and $\sigma_2$ in its exterior. We start by proving that

$$\mathbb{C} \setminus D \subset \rho(T|_{E_1}). \tag{9}$$

Given $\lambda \in \mathbb{C}$ and $\lambda' \in \rho(T)$ we may write

$$\begin{aligned}(\lambda I - T)R_{\lambda'}(T) &= \lambda R_{\lambda'}(T) - TR_{\lambda'}(T) \\ &= \lambda R_{\lambda'}(T) - \lambda' R_{\lambda'}(T) + \lambda' R_{\lambda'}(T) - TR_{\lambda'}(T) \\ &= (\lambda - \lambda')R_{\lambda'}(T) + I,\end{aligned}$$

Assuming that $\lambda \notin D$ we have

$$(\lambda I - T)\frac{1}{2\pi i}\int_\gamma \frac{R_{\lambda'}(T)}{\lambda - \lambda'}d\lambda' = \frac{1}{2\pi i}\int_\gamma R_{\lambda'}(T)d\lambda' + \frac{1}{2\pi i}\int_\gamma \frac{I}{\lambda - \lambda'}d\lambda'. \tag{10}$$

Noting this last integral is equal to zero for $\lambda \notin D$, we have

$$(\lambda I - T)\frac{1}{2\pi i}\int_\gamma \frac{R_{\lambda'}(T)}{\lambda - \lambda'}d\lambda' = P_1, \tag{11}$$

and observing that $\lambda I - T$ commutes with $R_{\lambda'}(T)$ we also have

$$\frac{1}{2\pi i}\int_\gamma \frac{R_{\lambda'}(T)}{\lambda - \lambda'}d\lambda' \, (\lambda I - T) = P_1. \tag{12}$$

Considering equations (11) and (12) restricted to $E_1$ we deduce that $\lambda \in \rho(T|_{E_1})$ for $\lambda \notin D$, and thus we have (9). It follows that $\sigma(T|_{E_1}) \subset D$, which together with (5) gives that

$$\sigma(T|_{E_1}) \subset \sigma(T) \cap D = \sigma_1. \tag{13}$$

Assume now that $\lambda$ is in the interior of $D$. In this case we have the last integral in (10) coinciding with $-I$, and so it follows that

$$(\lambda I - T)\frac{1}{2\pi i}\int_\gamma \frac{R_{\lambda'}(T)}{\lambda - \lambda'}d\lambda' = P_1 - I = -P_2.$$

Using the same kind of arguments that we used in the previous case we easily deduce that $\lambda \in \rho(T|_{E_2})$ for $\lambda$ in the interior of $D$. It follows that $\sigma(T|_{E_2}) \subset \mathbb{C} \setminus D$, which together with (5) gives

$$\sigma(T|_{E_2}) \subset \sigma(T) \cap (\mathbb{C} \setminus D) = \sigma_2. \tag{14}$$

Finally, noting that $\sigma(T) = \sigma_1 \cup \sigma_2$ it follows from (5), (13) and (14) that we necessarily have $\sigma(T|_{E_1}) = \sigma_1$ and $\sigma(T|_{E_2}) = \sigma_2$. $\square$

In the next result we state some useful characterization of contractions in complex Banach spaces.

**Proposition 3.2.** Let $E$ be a complex Banach space with a norm $\| \ \|$ and $T \in L(E)$. The following conditions are equivalent:

1. the spectral radius of $T$ is strictly smaller than 1;
2. there is a norm $| \ |$ equivalent to $\| \ \|$ for which $|T| < 1$;
3. there are $C > 0$ and $0 < \lambda < 1$ such that $\|T^n x\| \leq C\lambda^n \|x\|$ for all $x \in E$ and $n \in \mathbb{N}$.

*Proof.* Clearly, if condition 2 holds, then there is $0 < \lambda < 1$ such that $|T^n x| \leq \lambda^n |x|$ for all $x \in E$ and all $n \in \mathbb{N}$. Using that $| \ |$ is equivalent to $\| \ \|$ we easily deduce condition 3. From Theorem 2.7 we also deduce that condition 3 implies condition 1. It remains to check that condition 1 implies condition 2.

Assume that the spectral radius of $T$ is smaller than 1. It follows from Theorem 2.7 that if we fix $r(T) < s < 1$, then we have $\|T^n\|/s^n < 1$ for all $n$ sufficiently large. Hence, there exists $C > 0$ such that

$$\|T^n x\| \leq C s^n \|x\|. \tag{15}$$

for all $x \in E$ and all $n \in \mathbb{N}$. Fixing $N \in \mathbb{N}$ such that

$$C s^N < 1, \tag{16}$$

we define for $x \in E$

$$|x| = \sum_{n=0}^{N-1} \|T^n x\|.$$

Clearly, $| \ |$ is a norm in $E$ for which $\| \ \| \leq | \ |$. On the other hand, it follows from (15) and (16) that for all $x \in E$

$$|x| \leq \sum_{n=0}^{N-1} C s^n \|x\| \leq \frac{C}{1-s} \|x\|, \tag{17}$$

thus having shown that the norms $\| \ \|$ and $| \ |$ are equivalent. Let us now prove that $|T| < 1$. We have for each $x \in E$

$$|Tx| = \sum_{n=1}^{N} \|T^n x\| = |x| - \|x\| + \|T^N x\| \leq |x| - (1 - Cs^N)\|x\|,$$

which together with (17) gives

$$|Tx| \leq \left(1 - \frac{(1-s)(1-Cs^N)}{C}\right) |x|.$$

Observing that $1 - s > 0$ and $(1 - Cs^N) > 0$ we have $|T| < 1$. $\square$

# 4 Complexification

Here we recall the well-know process of complexification of a real vector space and introduce a useful norm in that complexification space. Given a real vector space $E$, define

$$E_{\mathbb{C}} = \{x + iy : x, y \in E\}.$$

We introduce in $E_{\mathbb{C}}$ a structure of complex vector space, defining the addition of $x + iy$ and $u + iv$ in $E_{\mathbb{C}}$ as

$$(x + iy) + (u + iv) = (x + u) + i(y + v)$$

and the multiplication of $\alpha + i\beta \in \mathbb{C}$ by $x + iy \in E_{\mathbb{C}}$ as

$$(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x).$$

We leave it as a straightforward exercise to verify that $E_{\mathbb{C}}$ endowed with these operations has a structure of complex vector space, called the *complexification* of $E$.

The problem of introducing a norm in $E_{\mathbb{C}}$ for which $E_{\mathbb{C}}$ inherits important properties from $E$ (being a Banach space, for instance) is more subtle than it may appear at first sight. An attempt to define a norm in the complexification space has been made by A. E. Taylor through

$$\|x + iy\|_{\mathbb{C}} = \sqrt{\|x\|^2 + \|y\|^2}$$

for each $x + iy \in E_{\mathbb{C}}$. The problem with $\| \ \|_{\mathbb{C}}$ is that in general it fails to have the homogeneous property over the complex numbers. A reasonable norm has been proposed later by A. E. Taylor, defining for each $x + iy \in E_{\mathbb{C}}$

$$\|x + iy\|_{\mathbb{C}} = \sup_{\theta \in [0, 2\pi]} \| \cos\theta\, x - \sin\theta\, y\|. \tag{18}$$

For many other considerations about possible choices of reasonable norms in complexification spaces see e.g. [6].

---

**Proposition 4.1.** If $E$ be a real vector space with a norm $\| \ \|$, then $\| \ \|_{\mathbb{C}}$ defined as in (18) gives a norm in $E_{\mathbb{C}}$ such that for all $x, y \in E$

$$\max\{\|x\|, \|y\|\} \le \|x + iy\|_{\mathbb{C}} \le \|x\| + \|y\|. \tag{19}$$

---

*Proof.* Let us see first that for all $x, y \in E$ we have (19). In fact, it easily follows from the definition of $\| \ \|_{\mathbb{C}}$ that for all $x, y \in E$

$$\|x\| = \| \cos 0\, x - \sin 0\, y\| \le \|x + iy\|_{\mathbb{C}}$$

and

$$\|y\| = \left\|\cos\frac{\pi}{2}\, x - \sin\frac{\pi}{2}\, y\right\| \le \|x + iy\|_{\mathbb{C}}.$$

Furthermore, for each $\theta \in [0, 2\pi]$ we have

$$\|\cos\theta x - \sin\theta y\| \leq |\cos\theta|\|x\| + |\sin\theta|\|y\| \leq \|x\| + \|y\|,$$

which finally gives (19).

Let us now see that $\| \ \|_{\mathbb{C}}$ defines a norm in $E_{\mathbb{C}}$. From (19) we easily deduce that

$$\|x + iy\|_{\mathbb{C}} = 0 \quad \Leftrightarrow \quad x = 0.$$

Moreover, given $x, y \in E$ and $\lambda \in \mathbb{C}$, choose $\alpha \in [0, 2\pi]$ such that $\lambda = |\lambda|(\cos\alpha + i\sin\alpha)$. We have

$$\begin{aligned}
\|\lambda(x+iy)\|_{\mathbb{C}} &= \||\lambda|(\cos\alpha + i\sin\alpha)(x+iy)\|_{\mathbb{C}} \\
&= \||\lambda|[(\cos\alpha\, x - \sin\alpha\, y) + i(\cos\alpha\, y + \sin\alpha\, x)]\|_{\mathbb{C}} \\
&= |\lambda| \sup_{\theta\in[0,2\pi]} \|\cos\theta(\cos\alpha\, x - \sin\alpha\, y) - \sin\theta(\cos\alpha\, y + \sin\alpha\, x)\| \\
&= |\lambda| \sup_{\theta\in[0,2\pi]} \|\cos(\theta+\alpha)x - \sin(\theta+\alpha)y\| \\
&= |\lambda|\, \|x+iy\|_{\mathbb{C}},
\end{aligned}$$

and so we have the homogeneous property. Let us finally prove the triangular inequality. For $u + iv, x + iy \in E_{\mathbb{C}}$ we have

$$\begin{aligned}
\|(u+iv) + (x+iy)\|_{\mathbb{C}} &= \|(u+x) + (v+y)i\|_{\mathbb{C}} \\
&= \sup_{\theta\in[0,2\pi]} \|\cos\theta\,(u+x) - \sin\theta\,(v+y)\| \\
&= \sup_{\theta\in[0,2\pi]} \|\cos\theta\, u - \sin\theta\, v + \cos\theta\, x - \sin\theta\, y\| \\
&\leq \sup_{\theta\in[0,2\pi]} \|\cos\theta\, u - \sin\theta\, v\| + \sup_{\theta\in[0,2\pi]} \|\cos\theta\, x - \sin\theta\, y\| \\
&= \|u+iv\|_{\mathbb{C}} + \|x+iy\|_{\mathbb{C}},
\end{aligned}$$

which gives the triangular inequality. □

From here on we shall always assume that $\| \ \|_{\mathbb{C}}$ related to $\| \ \|$ as in (18) is the norm in the complexification $E_{\mathbb{C}}$ of the real vector space $E$ with the norm $\| \ \|$. The next result shows that with this choice we do not loose completeness.

---

**Corollary 4.2.** If $E$ is a Banach space with a norm $\| \ \|$, then $E_{\mathbb{C}}$ is a Banach space with the norm $\| \ \|_{\mathbb{C}}$.

---

*Proof.* Observe that if $(x_n + iy_n)_n$ is a Cauchy sequence in $E_{\mathbb{C}}$, then $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences in $E$, by Proposition 4.1. Hence, both $(x_n)_n$ and $(y_n)_n$ must converge in $E$. The convergence of these sequences in $E$ implies the convergence of $(x_n + iy_n)_n$ in $E_{\mathbb{C}}$, again by Proposition 4.1. □

We are also interested in defining the complexification of an operator. Given a linear operator $T : E \to E$ of a real vector space $E$, we define the *complexification* of $T$ as

$$\begin{array}{rcrcl} T_{\mathbb{C}} & : & E_{\mathbb{C}} & \longrightarrow & E_{\mathbb{C}} \\ & & x & \longmapsto & Tx + iTy. \end{array}$$

We leave it as an exercise to verify that $T_{\mathbb{C}}$ is a linear operator on $E_{\mathbb{C}}$. In the next result we show that the norm of the complexification operator is preserved.

---

**Lemma 4.3.** Let $E$ be a real Banach space. If $T : E \to E$ is a bounded operator, then $T_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}$ is a bounded operator and $\|T_{\mathbb{C}}\|_{\mathbb{C}} = \|T\|$.

---

*Proof.* We have for each $x + iy \in E_{\mathbb{C}}$

$$\begin{aligned} \|T_{\mathbb{C}}(x+iy)\|_{\mathbb{C}} &= \|Tx + iTy\|_{\mathbb{C}} \\ &= \sup_{\theta\in[0,2\pi]} \|\cos\theta\, Tx - \sin\theta\, Ty\| \\ &= \sup_{\theta\in[0,2\pi]} \|T(\cos\theta\, x - \sin\theta\, y)\| \\ &\leq \|T\| \sup_{\theta\in[0,2\pi]} \|\cos\theta x - \sin\theta y\| \\ &= \|T\|\, \|x+iy\|_{\mathbb{C}}, \end{aligned}$$

thus having proved that $\|T_{\mathbb{C}}\| \leq \|T\|$. Moreover, noticing that for all $x \in E$ we have $\|x\|_{\mathbb{C}} = \|x\|$, it follows that

$$\|T_{\mathbb{C}}\|_{\mathbb{C}} \geq \sup_{x\neq 0} \frac{\|T_{\mathbb{C}}(x+i0)\|_{\mathbb{C}}}{\|x+i0\|_{\mathbb{C}}} = \sup_{x\neq 0} \frac{\|Tx\|_{\mathbb{C}}}{\|x\|_{\mathbb{C}}} = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \|T\|,$$

thus getting the other inequality. $\square$

The following result is an immediate consequence of the previous lemma.

---

**Corollary 4.4.** Let $E$ be a real Banach space. The mapping $L(E) \ni T \longmapsto T_{\mathbb{C}} \in L(E_{\mathbb{C}})$ is continuous.

---

Finally, given a bounded linear operator $T : E \to E$ of a real vector space $E$, we define its *spectrum* $\sigma(T)$ as the spectrum of its complexification, i.e.

$$\sigma(T) = \sigma(T_{\mathbb{C}}).$$

# 5 Hyperbolic isomorphisms

Given $E$ a vector space we denote by $GL(E)$ the set of elements in $L(E)$ which are invertible. Notice that under the assumption that $E$ a Banach space we have by the Open Mapping Theorem that $A^{-1} \in GL(E)$, whenever $A \in GL(E)$.

**Definition 5.1.** We say that $A \in GL(E)$ is *hyperbolic* if the spectrum of $A$ is disjoint from the unit circle $S^1 \subset \mathbb{C}$. We define $H(E) = \{A \in GL(E) : A \text{ is hyperbolic}\}$.

---

**Proposition 5.2.** Let $E$ be a Banach space. Then $H(E)$ is open in $GL(E)$.

---

*Proof.* We first see the case of $E$ being a complex vector space. Given $A \in H(E)$, we have $S^1 \subset \rho(A)$. We need to see that for $B \in GL(E)$ sufficiently close to $A$ we also have $S^1 \subset \rho(B)$, i.e. $\lambda I - B$ is invertible for all $\lambda \in S^1$. Given $\lambda \in S^1$ we may write

$$\lambda I - B = (\lambda I - A)\left[I + (\lambda I - A)^{-1}(A - B)\right].$$

Observing that $\lambda I - A$ is invertible for $\lambda \in S^1$, by Lema 1.1 it is enough to see that $\|(\lambda I - A)^{-1}(A - B)\| < 1$ for $B$ sufficiently close to $A$ (not depending on $\lambda$). Considering

$$M = \max_{\lambda \in S^1} \|(\lambda I - A)^{-1}\|$$

it is enough to take $\|B - A\| < M^{-1}$.

Let us assume now $E$ is a real space. Noticing that by definition one has $\sigma(A) = \sigma(A_{\mathbb{C}})$, it follows that $A \in GL(E)$ is hyperbolic if and only if $A_{\mathbb{C}} \in GL(E_{\mathbb{C}})$ is hyperbolic. The result in the real case then easily follows from Corollary 4.4 and the complex case. $\square$

We leave it as an exercise to prove that the set of hyperbolic isomorphisms $H(E)$ of a finite dimensional vector space $E$ forms a dense subset of $GL(E)$. To see this one can use the simple idea that, under a small perturbation of an operator on a finite dimensional space, we can get rid of the elements in the spectrum which lie on the unit circle. This does not necessarily hold in infinite dimensional spaces, as the next example[1] illustrates.

**Example 5.3.** Consider $D$ the disk of radius $1/2$ centered at the point $1 \in \mathbb{C}$ and let $E$ be the space of bounded analytic functions $f : D \to \mathbb{C}$. This is a Banach space with supremum norm. Let $A : E \to E$ be the bounded linear operator defined for each $f \in E$ as

$$Af(z) = zf(z), \quad \forall z \in D.$$

We leave it as an exercise to prove that $\sigma(A) = \overline{D}$. We claim that $1 \in \sigma(B)$ for any isomorphism $B$ sufficiently close to $A$ in the natural norm of $L(E)$. Indeed, given $\epsilon > 0$ and $B \in GL(E)$ with $\|A - B\| < \epsilon$ let us prove that, as long as $\epsilon > 0$ is sufficiently small,

[1]Thanks to Maurizio Monge for providing me with this example.

we have $1 \in \sigma(B)$. For this, it is enough to prove that $I-B$ is not surjective. Noticing that the constant function equal to 1 belongs to $E$ we shall actually prove that the equation $(I-B)f = 1$ has no solution $f \in E$. We have

$$(I-B)f = 1 \quad \Leftrightarrow \quad f - Af = 1 - (A-B)f.$$

Assume by contradiction that there is $f \in E$ for which

$$f - Af = 1 - (A-B)f. \tag{20}$$

Using the maximum modulus principle we get

$$\|f - Af\| = \frac{1}{2}\|f\| \geq,$$

which gives

$$\|(A-B)f\| \leq \epsilon\|f\| = 2\epsilon\|f - Af\|,$$

and so

$$\|f - Af\| \geq \frac{1}{2\epsilon}\|(A-B)f\|.$$

we obtain

$$\frac{1}{2\epsilon}\|(A-B)f\| \leq 1 + \|(A-B)f\|$$

which then yields

$$\|(A-B)f\| \leq \left(\frac{1}{2\epsilon} - 1\right)^{-1} < 1, \quad \text{for } \epsilon > 0 \text{ small.}$$

This in particular gives that the function $1-(A-B)f$ has no zeros in $D$. Noticing that $(f - Af)(z) = 0$ has a solution in $z = 1$ we have a contradiction with (20).

**Definition 5.4.** We define the *stable space* of $A \in GL(E)$ as

$$E^s = \left\{x \in E : \lim_{n\to+\infty} A^n x = 0\right\}$$

and the *unstable space* of $A$ as

$$E^u = \left\{x \in E : \lim_{n\to+\infty} A^{-n} x = 0\right\}.$$

We leave it as an exercise to verify that $E^s$ and $E^u$ are linear subspaces of $E$, which additionally are *invariant* under $A$, meaning that

$$A(E^s) = E^s \quad \text{and} \quad A(E^u) = E^u.$$

The next result shows that the stable and unstable spaces are complimentary when the isomorphism is hyperbolic. Clearly, this not necessarily true without the hyperbolicity assumption, even in fine dimension.

**Theorem 5.5.** Let $A$ be a Banach space and $A \in H(E)$. Then $E = E^s \oplus E^u$.

*Proof.* Consider first the case of $E$ being a complex space. Theorem 3.1 ensures that there is a spliting $E = E_1 \oplus E_2$ into closed subspaces which invariant under $A$ such that

$$\sigma(A|_{E_1}) = \sigma(A) \cap \{|z| < 1\} \quad \text{and} \quad \sigma(A|_{E_2}) = \sigma(A) \cap \{|z| > 1\}. \tag{21}$$

We are going to prove that $E^s = E_1$ and $E^u = E_2$. Observe that (21) gives in particular that

$$r(A|_{E_1}) < 1, \tag{22}$$

where $r(A|_{E_1})$ is the spectral radius of $A|_{E_1}$. Using Proposition 3.2 we easily get

$$E_1 \subset E^s, \tag{23}$$

and together with Lemma 2.3 we also get

$$E_2 \subset E^u. \tag{24}$$

Let us now prove the reverse inclusion of (23). More precisely, we shall prove that if $x \notin E_1$, then $A^n x$ cannot converge to 0 when $n \to +\infty$. Actually, if $x \notin E_1$, then by Theorem 3.1 there are $x_1 \in E_1$ and $0 \neq x_2 \in E_2$ such that $x = x_1 + x_2$. By (23) and (24) we have

$$\lim_{n \to +\infty} A^n x_1 = 0 \quad \text{and} \quad \lim_{n \to +\infty} A^{-n} x_2 = 0.$$

Now it is enough to verify that $A^n x_2$ does not converge to 0, when $n \to \infty$. Assuming by contradiction that $A^n x_2 \to 0$ when $n \to \infty$, then there is $n_0 \in \mathbb{N}$ such that $\|A^n x_2\| \leq 1$ for all $n \geq n_0$. Hence, we have for $n \geq n_0$

$$\|A^{-n}|_{E_2}\| \geq \|A^{-n} A^n x_2\| = \|x_2\| \neq 0.$$

This implies that

$$\|A^{-n}|_{E_2}\|^{1/n} \geq \|x_2\|^{1/n},$$

and so $r(A^{-1}|_{E^u}) \geq 1$, by Theorem 2.7. This gives a contradiction with (22), and so we have proved that $E_1 = E^s$. Noticing that the unstable space of $A$ coincides with the stable space of $A^{-1}$, then using Lemma 2.3 we also see that $E_2 = E^u$.

Let us now see that $E = E^s \oplus E^u$ in the real case. Consider $E_{\mathbb{C}}$ and $A_{\mathbb{C}}$ the complexifications of $E$ and $A$ respectively. We have by definition that $A_{\mathbb{C}}$ is hyperbolic and, by the complex case already proved, we have a decomposition $E_{\mathbb{C}} = E^s_{\mathbb{C}} \oplus E^u_{\mathbb{C}}$ invariant under $A_{\mathbb{C}}$. Defining

$$\begin{array}{rccc} \phi : & E & \longrightarrow & E_{\mathbb{C}} \\ & x & \longmapsto & x + i0 \end{array}$$

by Proposition 4.1 we have that $\phi$ is continuous. Hence, $\phi^{-1}(E^s_{\mathbb{C}})$ and $\phi^{-1}(E^u_{\mathbb{C}})$ are closed subspaces of $E$. Observe that

$$\begin{aligned} x \in \phi^{-1}(E^s_{\mathbb{C}}) &\Leftrightarrow x + i0 \in E^s_{\mathbb{C}} \\ &\Leftrightarrow \lim_{n\to+\infty} A^n_{\mathbb{C}}(x+i0) = 0 + i0 \\ &\Leftrightarrow \lim_{n\to+\infty} A^n x = 0, \end{aligned}$$

being this last equivalence a consequence of the definition of $A_{\mathbb{C}}$ and Proposition 4.1. Then we have $\phi^{-1}(E^s_{\mathbb{C}}) = E^s$, and so $E^s$ is a closed subspace of $E$. Noting that $(A_{\mathbb{C}})^{-1} = (A^{-1})_{\mathbb{C}}$ we easily deduce that $\phi^{-1}(E^u_{\mathbb{C}}) = E^u$ as well, thus proving that $E^u$ is a closed subspace of $E$. Moreover, observing that

$$\begin{aligned} x \in E^s \cap E^u &\Leftrightarrow \phi(x) \in E^s_{\mathbb{C}} \cap E^u_{\mathbb{C}} \\ &\Leftrightarrow x + i0 \in E^s_{\mathbb{C}} \cap E^u_{\mathbb{C}} \\ &\Leftrightarrow x = 0, \end{aligned}$$

we easily get that $E^s \cap E^u = \{0\}$. Finally, given $x \in E$, there are $x_s + iy_s \in E^s_{\mathbb{C}}$ and $x_u + iy_u \in E^u_{\mathbb{C}}$ such that

$$x + i0 = (x_s + iy_s) + (x_u + iy_u),$$

which then implies

$$x = x_s + x_u.$$

Noticing that

$$\begin{aligned} x_s + iy_s \in E^s_{\mathbb{C}} &\Leftrightarrow \lim_{n\to+\infty} A^n_{\mathbb{C}}(x_s + iy_s) = 0 + i0 \\ &\Rightarrow \lim_{n\to+\infty} A^n x_s = 0 \end{aligned}$$

by Proposition 4.1, we have $x_s \in E^s$. Using that $(A_{\mathbb{C}})^{-1} = (A^{-1})_{\mathbb{C}}$, we also get $x_u \in E^u$. $\square$

**Definition 5.6.** Let $E$ be a Banach and $A \in H(E)$. We call $E = E^s \oplus E^u$ the *hyperbolic splitting* of $A$, and denote $A^s = A|_{E^s}$ and $A^u = A|_{E^u}$.

**Definition 5.7.** Let $E$ with a norm $\| \ \|$ be a Banach space and $E = E^s \oplus E^u$ be the hyperbolic splitting of $A \in H(E)$. We say that a norm $| \ |$ in $E$ is *adapted* to $A$ if the following conditions hold:

1. $| \ |$ is equivalent to $\| \ \|$;
2. $|x^s + x^u| = \max\{|x^s|, |x^u|\}$, for all $x^s \in E^s$ and $x^u \in E^u$;
3. $|A^s| < 1$ and $|(A^u)^{-1}| < 1$.

**Theorem 5.8.** Let $E$ be a Banach space and $A \in H(E)$. Then $A$ has some adapted norm.

*Proof.* Consider first the case of $E$ being a complex space. Let $E = E^s \oplus E^u$ be the hyperbolic splitting of $A$. Considering the operators $A^s$ and $(A^u)^{-1}$, then by Proposition 3.2 and Lemma 2.3 there are norms $|\ \ |_s$ and $|\ \ |_u$ in the spaces $E^s$ and $E^u$, respectively, such that

$$|A^s|_s < 1 \quad \text{and} \quad |(A^u)^{-1}|_u < 1.$$

Furthermore, $|\ \ |_s$ and $|\ \ |_u$ are equivalent to $\|\ \ \|$ restricted to $E^s$ and $E^u$, respectively. This means that there is $C > 0$ such that for all $x^s \in E^s$ and $x^u \in E^u$ we have

$$\frac{1}{C}|x^s|_s \leq \|x^s\| \leq C|x^s|_s \quad \text{and} \quad \frac{1}{C}|x^u|_u \leq \|x^s\| \leq C|x^u|_u.$$

Define for $x^s \in E^s$ and $x^u \in E^u$

$$|x^s + x^u| = \max\left\{|x^s|_s, |x^u|_u\right\}.$$

To finish the complex case we just need to see that $|\ \ |$ is equivalent to $\|\ \ \|$ in $E$. Indeed, considering the projections $P^s$ and $P^u$ onto the subspaces $E^s$ and $E^u$, respectively, we have for $x^s + x^u \in E^s \oplus E^u$

$$\begin{aligned}
|x^s + x^u| &= \max\left\{|x^s|_s, |x^u|_u\right\} \\
&\leq |x^s|_s + |x^u|_u \\
&\leq C\|x^s\| + C\|x^u\| \\
&= C(\|P^s(x^s + x^u)\| + \|P^u(x^s + x^u)\|) \\
&\leq C(\|P^s\| + \|P^u\|)\|x^s + x^u\|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|x^s + x^u\| &\leq \|x^s\| + \|x^u\| \\
&\leq C|x^s|_s + C|x^u|_u \\
&\leq 2C\max\{|x^s|_s, |x^u|_u\} \\
&= 2C|x^s + x^u|.
\end{aligned}$$

This finishes the proof of the result in the complex case

Consider now the case of $E$ being a real space. Let $E_{\mathbb{C}}$ be the complexification of $E$ and $A_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}$ the complexification of $A$. Considering $E_{\mathbb{C}} = E^s_{\mathbb{C}} \oplus E^u_{\mathbb{C}}$ the hyperbolic splitting of $A_{\mathbb{C}} = A^s_{\mathbb{C}} \oplus A^u_{\mathbb{C}}$, we leave it as an easy exercise to verify that

$$E^s_{\mathbb{C}} = E^s + iE^s \quad \text{and} \quad E^u_{\mathbb{C}} = E^u + iE^u. \tag{25}$$

Moreover, for all $x, y \in E^s$ we have

$$A^s_{\mathbb{C}}(x + iy) = A^s x + iA^s y \tag{26}$$

and for all $x, y \in E^u$ we have

$$A^u_{\mathbb{C}}(x + iy) = A^u x + iA^u y. \tag{27}$$

Let $\| \ \|_{\mathbb{C}}$ be the norm in $E_{\mathbb{C}}$ introduced in (18). By the complex case already seen, there is some norm $| \ |_{\mathbb{C}}$ in $E_{\mathbb{C}}$ adapted to $A_{\mathbb{C}}$. In particular, there exists $C > 0$ such that for all $x, y \in E$

$$\frac{1}{C}\|x + iy\|_{\mathbb{C}} \leq |x + iy|_{\mathbb{C}} \leq C\|x + iy\|_{\mathbb{C}}. \tag{28}$$

Defining $|x| = |x + 0i|_{\mathbb{C}}$ for $x \in E$, from Proposition 4.1 and (28) we get

$$\frac{1}{C}\|x\| \leq |x| \leq C\|x\|,$$

which gives the equivalence of $| \ |$ and $\| \ \|$. Moreover, for each $x \in E$ we have

$$|x| = |x + i0|_{\mathbb{C}} = \max\{|x_s + i0|_{\mathbb{C}}, |x_u + i0|_{\mathbb{C}}\} = \max\{|x_s|, |x_u|\}.$$

Finally, it follows from (26), (27) and the the way we have defined $| \ |$ that if $|A^s_{\mathbb{C}}|_{\mathbb{C}} < 1$ and $|(A^u_{\mathbb{C}})^{-1}|_{\mathbb{C}} < 1$, then $|A^s| < 1$ and $|(A^u)^{-1}| < 1$ as well. □

Using that an adapted norm is, by definition, equivalent to the original norm, we easily get the following useful consequence of the previous result.

**Corollary 5.9.** Let $E$ be a Banach space with a norm $\| \ \|$ and $E = E^s \oplus E^u$ be the hyperbolic splitting of $A \in H(E)$. Then there are $C > 0$ and $0 < \lambda < 1$ such that:

1. $\|A^n x\| \leq C\lambda^n \|x\|$ for all $x \in E^s$;
2. $\|A^{-n} x\| \leq C\lambda^n \|x\|$ for all $x \in E^u$.

This gives in particular that points in the stable and unstable spaces converge to the zero vector exponentially fast under iterations by $A$ and $A^{-1}$ respectively.

# 6 Structural stability

In this section we prove that a hyperbolic linear isomorphism is structurally stable, meaning that it is topologically conjugate to any nearby linear isomorphism. We start with a well know result on the existence of fixed points for contractions on complete metric spaces, highlighting the maybe no so well-known fact that, for a continuous family of such contractions, the fixed point depends continuously on the parameter.

**Lemma 6.1.** Let $X$ be a complete metric space, $Y$ a topological space and $f : X \times Y \to X$ continuous for which there is $0 < \lambda < 1$ such that

$$\operatorname{dist}(f(x,y), f(x',y)) \leq \lambda \operatorname{dist}(x,x'), \quad \forall x, x' \in X \ \ \forall y \in Y. \tag{29}$$

Then there is a continuous function $p : Y \to X$ such that for each $y \in Y$:

1. $p(y)$ is the unique fixed point of $f_y : X \to X$ given by $f_y(x) = f(y,x)$ for $x \in X$;
2. $\lim_{n\to+\infty} f_y^n(x) = p(y)$ for all $x \in X$.

*Proof.* Given $x \in X$, define the sequence of points in $X$

$$x_n = f_y^n(x), \quad n \in \mathbb{N}.$$

Using (29) we can inductively see that for all $n \in \mathbb{N}$ we have

$$\operatorname{dist}(x_{n+1}, x_n) \leq \lambda^n \operatorname{dist}(x_1, x).$$

Then, for any $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} \operatorname{dist}(x_m, x_n) &\leq \operatorname{dist}(x_m, x_{m-1}) + \cdots + \operatorname{dist}(x_{n+1}, x_n) \\ &\leq \lambda^{m-1} \operatorname{dist}(x_1, x) + \cdots + \lambda^n \operatorname{dist}(x_1, x) \\ &\leq \frac{\lambda^n}{1-\lambda} \operatorname{dist}(x_1, x). \end{aligned}$$

Thus we have that $(x_n)_n$ is a Cauchy sequence in $X$, and so it must converge to some point $p(y) \in X$, by completeness of $X$. This means that

$$\lim_{n\to\infty} f_y^n(x) = p(y)$$

and using the continuity of $f_y$ we get

$$\lim_{n\to\infty} f_y^n(x) = \lim_{n\to\infty} f_y^{n+1}(x) = f_y(p(y)).$$

By uniqueness of the limit we necessarily have $f_y(p(y)) = p(y)$. Moreover, if $p'(y)$ is another possible fixed point of $f_y$ we have

$$\operatorname{dist}(p(y), p'(y)) = \operatorname{dist}(f_y(p(y), p'(y))) \leq \lambda \operatorname{dist}(p(y), p'(y)),$$

and this necessarily implies that $p'(y) = p(y)$.

Let us now prove that $p : Y \to X$ is continuous. Given $x \in X$ and $y \in Y$ we choose $n = n(x,y) \in \mathbb{N}$ such that

$$\operatorname{dist}(f_y^n(x), p(y)) \leq \frac{1}{1-\lambda} \operatorname{dist}(x, f_y(x)).$$

We have

$$\begin{aligned}
\operatorname{dist}(x, p(y)) &\leq \operatorname{dist}(x, f_y(x)) + \cdots + \operatorname{dist}(f_y^{n-1}(x), f_y^n(x)) + \operatorname{dist}(f_y^n(x), p(y)) \\
&\leq \operatorname{dist}(x, f_y(x)) + \cdots + \lambda^{n-1} \operatorname{dist}(x, f_y(x)) + \frac{1}{1-\lambda} \operatorname{dist}(x, f_y(x)) \\
&\leq \frac{2}{1-\lambda} \operatorname{dist}(x, f_y(x)).
\end{aligned}$$

Given $y_0 \in Y$ we apply this last inequality to $x = p(y_0)$, thus getting

$$\operatorname{dist}(p(y_0), p(y)) \leq \frac{2}{1-\lambda} \operatorname{dist}(p(y_0), f_y(p(y_0))) = \frac{2}{1-\lambda} \operatorname{dist}(f(p(y_0), y_0), f(p(y_0), y)).$$

Since $f$ is continuous, then $p$ is continuous. □

Given Banach spaces $E$ and $F$ we define

$$C_b^0(E, F) = \{\phi : E \to F \mid \phi \text{ is bounded and continuous}\}.$$

It is clear that $C_b^0(E, F)$ is a vector space, which moreover becomes a Banach space when endowed with the supremum norm. In the special case that $E = F$ we simply denote the space $C_b^0(E, F)$ by $C_b^0(E)$.

**Definition 6.2.** We say that $\phi : E \to E$ is *Lipschitz* if

$$\sup_{x \neq y} \frac{\|\phi(x) - \phi(y)\|}{\|x - y\|} < +\infty,$$

and denote this supremum by $\operatorname{Lip}(\phi)$.

---

**Lemma 6.3.** Let $E$ be a Banach space and $A \in GL(E)$. If $\phi \in C_b^0(E)$ is Lipschitz with $\operatorname{Lip}(\phi) < \|A^{-1}\|^{-1}$, then $A + \phi$ is a homeomorphism.

---

*Proof.* It is enough to show that for each $y \in E$ the equation $(A + \phi)(x) = y$ has a unique solution $x \in E$ depending continuously on $y$. Defining for each $x, y \in E$

$$f(x, y) = A^{-1}(y - \phi(x))$$

we have

$$(A + \phi)(x) = y \quad \Leftrightarrow \quad x = A^{-1}(y - \phi(x)) \quad \Leftrightarrow \quad f(x, y) = x.$$

For each $x, x', y \in E$ we have

$$\|f(x, y) - f(x', y)\| \leq \|A^{-1}\| \operatorname{Lip}(\phi) \|x - x'\|.$$

Thus, if $\operatorname{Lip}(\phi) < \|A^{-1}\|^{-1}$, we are in the conditions of Lemma 6.1. It follows that for each $y \in E$ the equation $(A + \phi)(x) = y$ has a unique solution in $x \in E$ which, morever, depends continuously on $y \in E$. □

**Definition 6.4.** We say that $f : E \to E$ and $g : E \to E$ are *topologically conjugate* if there is a homeomorphism $h : E \to E$ such that $h \circ f = g \circ h$.

---

**Proposition 6.5.** Let $E$ be a Banach space and $A \in H(E)$. There is $\epsilon > 0$ such that if $\phi_1, \phi_2 \in C_b^0(E)$ are Lipschitz with $\mathrm{Lip}(\phi_i) \leq \epsilon$, for $i = 1, 2$, then $A + \phi_1$ and $A + \phi_2$ are homeomorphisms topologically conjugate.

---

*Proof.* Let $E = E^s \oplus E^u$ be the hyperbolic splitting of $A \in H(E)$ and consider a norm $\| \ \|$ in $E$ adapted to $A$. Then there is $0 < a < 1$ such that

$$\|A^s\| \leq a \quad \text{and} \quad \|(A^u)^{-1}\| \leq a.$$

We start by choosing $\epsilon < \|A^{-1}\|^{-1}$. Lemma 6.3 ensures that if we take $\phi_1, \phi_2 \in C_b^0(E)$ with $\mathrm{Lip}(\phi_i) \leq \epsilon$ for $i = 1, 2$, then $A + \phi_1$ and $A + \phi_2$ are homeomorphisms. Our goal now is to prove that there is a homeomorphism $h : E \to E$ such that

$$(A + \phi_1)h = h(A + \phi_2). \tag{30}$$

We shall look for such an homeomorphism of the form $h = I + g$ with $g \in C_b^0(E)$. In these conditions, equation (30) can be written as

$$(A + \phi_1)(I + g) = (I + g)(A + \phi_2),$$

or equivalently

$$Ag - g(A + \phi_2) = \phi_2 - \phi_1(I + g). \tag{31}$$

This last equation motivates the linear operator

$$\begin{array}{rccl} L & : & C_b^0(E) & \longrightarrow & C_b^0(E) \\ & & g & \longmapsto & Ag - g(A + \phi_2). \end{array}$$

We claim that $L$ is a bounded linear isomorphism. Actually, we may write $L = ST$, where $S$ and $T$ are the linear operators in $C_b^0(E)$ given by

$$Sg = Ag \quad \text{and} \quad Tg = g - A^{-1}g(A + \phi_2).$$

We have that $S$ is a bounded linear isomorphism with

$$\|S\| = \|A\| \quad \text{and} \quad \|S^{-1}\| = \|A^{-1}\|.$$

Moreover, as $E^s$ and $E^u$ are invariant under $A^{-1}$, then $C_b^0(E, E^s)$ and $C_b^0(E, E^u)$ are invariant under $T$ as well. Hence we may write $T = T^s \oplus T^u$, where $T^s$ is the restriction of $T$ to $C_b^0(E, E^s)$ and $T^u$ is the restriction of $T$ to $C_b^0(E, E^u)$. Let us now see some useful properties on $T^s$ and $T^u$:

1. *$T^u$ is invertible and $\|(T^u)^{-1}\| \leq 1/(1-a)$.*

   Indeed, noticing that

$$\begin{array}{rcrcl} T^u & : & C_b^0(E, E^u) & \longrightarrow & C_b^0(E, E^u) \\ & & g & \longmapsto & g - (A^u)^{-1} g(A + \phi_2), \end{array}$$

   we may write $T^u = I + U$, where

$$\begin{array}{rcrcl} U & : & C_b^0(E, E^u) & \longrightarrow & C_b^0(E, E^u) \\ & & g & \longmapsto & -(A^u)^{-1} g(A + \phi_2). \end{array}$$

   Since $\|U\| \leq \|(A^u)^{-1}\| \leq a$, then Lemma 1.1 gives that $T^u$ is invertible and $\|(T^u)^{-1}\| \leq 1/(1-a)$.

2. *$T^s$ is invertible and $\|(T^s)^{-1}\| \leq a/(1-a)$.*

   In this case we have

$$\begin{array}{rcrcl} T^s & : & C_b^0(E, E^s) & \longrightarrow & C_b^0(E, E^s) \\ & & g & \longmapsto & g - (A^s)^{-1} g(A + \phi_2), \end{array}$$

   Since by Lemma 6.3 we have $A + \phi_2$ a homeomorphism, we may write $T^s = I + V = V(V^{-1} + I)$, where

$$\begin{array}{rcrcl} V^{-1} & : & C_b^0(E, E^u) & \longrightarrow & C_b^0(E, E^u) \\ & & g & \longmapsto & -A^s g(A + \phi_2)^{-1}. \end{array}$$

   Noticing that $\|V^{-1}\| \leq \|A^s\| \leq a < 1$, then Lemma 1.1 ensures that $T^s$ is invertible and $\|(T^s)^{-1}\| \leq a/(1-a)$.

It follows from the two properties above that $T = T^s \oplus T^u$ is invertible and

$$\|T^{-1}\| \leq \max\left\{\frac{1}{1-a}, \frac{a}{1-a}\right\} = \frac{1}{1-a}.$$

This gives that $L$ is a bounded linear isomorphism.

Now consider the operator

$$\begin{array}{rcrcl} R & : & C_b^0(E) & \longrightarrow & C_b^0(E) \\ & & g & \longmapsto & L^{-1}(\phi_2 - \phi_1(I + g)). \end{array}$$

A fixed point of $R$ clearly gives a solution to equation (31). By Lema 6.1 it is sufficient to see that $R$ is a contraction in the Banach space $C_b^0(E)$. We have for each $g_1, g_2 \in C_b^0(E)$

$$\begin{aligned} \|Rg_1 - Rg_2\| &= \|L^{-1}(-\phi_1(I + g_1) + \phi_1(I + g_2))\| \\ &\leq \|T^{-1}\| \|S^{-1}\| \operatorname{Lip}(\phi_1) \|g_1 - g_2\| \\ &\leq \frac{\|A^{-1}\|}{1-a} \operatorname{Lip}(\phi_1) \|g_1 - g_2\|. \end{aligned}$$

As we are assuming $\mathrm{Lip}(\phi_1) < \epsilon$, taking $\epsilon < (1-a)\|A^{-1}\|^{-1}$ we ensure that $R$ is a contraction in $C_b^0(E)$. Hence, there is a unique solution $g \in C_b^0(E)$ to equation (31) or, equivalently, there is a unique $g \in C_b^0(E)$ such that

$$(A+\phi_1)(I+g) = (I+g)(A+\phi_2). \tag{32}$$

It remains to see that $I+g$ is a homeomorphism. By symmetry on the roles of $\phi_1$ and $\phi_2$ we can also prove that there is a unique $\hat{g} \in C_b^0(E)$ such that

$$(I+\hat{g})(A+\phi_1) = (A+\phi_2)(I+\hat{g}).$$

Multiplying on the left by $I+g$, we obtain

$$(I+g)(I+\hat{g})(A+\phi_1) = (I+g)(A+\phi_2)(I+\hat{g}),$$

and using (32) we get

$$(I+g)(I+\hat{g})(A+\phi_1) = (A+\phi_1)(I+g)(I+\hat{g}).$$

Observing that

1. $(I+g)(I+\hat{g}) = I+g+\hat{g}+g\hat{g}$;
2. $g+\hat{g}+g\hat{g} \in C_b^0(E)$;
3. $I(A+\phi_1) = (A+\phi_1)I$;
4. $(I+g)(I+\hat{g})(A+\phi_1) = (A+\phi_1)(I+g)(I+\hat{g})$;

by the uniqueness of the solution of (32) in $C_b^0(E)$ we ensure that $(I+g)(I+\hat{g}) = I$. Similarly we prove that $(I+\hat{g})(I+g) = I$, thus having $I+g$ a homeomorphism. □

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**Lemma 6.6.** Let $E$ be a Banach space. If $\varphi : E \to E$ is Lipschitz, then $\varphi|_{B(0,1)}$ has some extension $\phi \in C_b^0(E)$ with $\mathrm{Lip}(\phi) \le 3\,\mathrm{Lip}(\varphi)$.

---

*Proof.* Take a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \eta(t) = 0, & \text{if } t \ge 2; \\ \eta(t) = 1, & \text{if } t \le 1; \\ |\eta'(t)| < 2, & \forall t \in \mathbb{R}, \end{cases}$$

and define for $x \in E$

$$\phi(x) = \varphi(0) + \eta\left(\|x\|\right)\left(\varphi(x) - \varphi(0)\right).$$

We clearly have $\phi|_{B(0,1)} = \varphi|_{B(0,1)}$ and $\phi \in C_b^0(E)$. For any $x_1, x_2 \in E$ we have

$$\begin{aligned} \|\phi(x_1) - \phi(x_2)\| &= \|\eta(\|x_1\|)\varphi(x_1) - \eta(\|x_2\|)\varphi(x_2)\| \\ &= \|[\eta(\|x_1\|) - \eta(\|x_2\|)]\,[\varphi(x_1) - \varphi(0)] + \eta(\|x_2\|)[\varphi(x_1) - \varphi(x_2)]\| \\ &\leq |\eta(\|x_1\|) - \eta(\|x_2\|)|\,\|\varphi(x_1) - \varphi(0)\| + \|\varphi(x_1) - \varphi(x_2)\|. \end{aligned} \tag{33}$$

Using the Mean Value Theorem for $\eta$ we deduce that if $x_1, x_2 \in B(0,1)$, then

$$\begin{aligned} \|\phi(x_1) - \phi(x_2)\| &\leq (2\|x_1 - x_2\|)\operatorname{Lip}(\varphi)\|x_1\| + \operatorname{Lip}(\varphi)\|x_1 - x_2\| \\ &\leq 3\operatorname{Lip}(\varphi)\|x_1 - x_2\|. \end{aligned}$$

We also deduce from (33) that if $x_1 \in B(0,1)$ and $x_2 \notin B(0,1)$, then

$$\|\phi(x_1) - \phi(x_2)\| \leq 2\operatorname{Lip}(\varphi)\|x_1 - x_2\|.$$

By symmetry on the roles of $x_1$ and $x_2$ we also get the conclusion for $x_1 \notin B(0,1)$ and $x_2 \in B(0,1)$. Finally, if $x_1, x_2 \notin B(0,1)$, then $\phi(x_1) = 0 = \phi(x_2)$, and this clearly gives what we need to prove also in this case. ◻

**Definition 6.7.** We say that $A \in GL(E)$ is *structurally stable* if there exists $\epsilon > 0$ such that for all $B \in GL(E)$ with $\|A - B\| < \epsilon$ we have $B$ topologically conjugate to $A$.

---

**Theorem 6.8.** Let $E$ be a Banach space and $A \in H(E)$. Then $A$ is structurally stable.

---

*Proof.* Take $A \in H(E)$. We need to check that for any $B \in GL(E)$ sufficiently close to $A$ there is a homeomorphism $h : E \to E$ such that $h \circ A = B \circ h$.

Let $\epsilon > 0$ be given by Proposition 6.5 and take $B \in GL(E)$ with $\|B - A\| < \epsilon/3$. It easily follows that $\operatorname{Lip}(B - A) \leq \epsilon/3$. Defining $V_0 = B(0,1)$ and $\varphi = B - A$, by Lemma 6.6 there exists an extension $\phi \in C_b^0(E)$ of $\varphi|_{V_0}$ with $\operatorname{Lip}(\phi) \leq 3\operatorname{Lip}(\varphi) < \epsilon$. Then, by Proposition 6.5, there is some homeomorphism $h : E \to E$ such that

$$h \circ A = (A + \phi) \circ h.$$

Since $(A + \phi)|_{V_0} = B|_{V_0}$, taking $U_0 = h^{-1}(V_0)$, we may write

$$h \circ A|_{U_0} = B \circ h|_{U_0}. \tag{34}$$

Our goal now is to extend this local conjugacy $h$ to a conjugacy between $A$ and $B$. Observe that by Proposition 5.2 we may suppose $B \in H(E)$. Let $E_A^s \oplus E_A^u$ and $E_B^s \oplus E_B^u$ be the hyperbolic splittings of $A$ and $B$ respectively. By Corollary 5.9 there are $C > 0$ and $0 < \lambda < 1$ such that

$$\|A^n x\| \leq C\lambda^n \|x\|, \quad \text{for all } x \in E_A^s$$

and

$$\|B^n x\| \le C\lambda^n \|x\|, \quad \text{for all } x \in E_B^s.$$

Then there are $V_1 \subset V_0$ and $U_1 \subset U_0$ small neighborhoods of 0 such that

$$A^n x \in U_0, \quad \forall\, x \in U_1 \cap E_A^s, \;\; \forall n \in \mathbb{N},$$

$$B^n x \in V_0, \quad \forall\, x \in V_1 \cap E_B^s, \;\; \forall n \in \mathbb{N}$$

and

$$h(U_1) = V_1.$$

Define

$$U_1{}^s = U_1 \cap E_A^s \quad \text{and} \quad V_1{}^s = V_1 \cap E_B^s.$$

It follows from (34) that for any $x \in U_1{}^s$ we have $h(A^n x) = B^n(h(x))$ for all $n \in \mathbb{N}$, and so by the continuity of $h$ we must have $B^n(h(x)) \to 0$, when $n \to \infty$. Recalling that $h(U_1) = V_1$ we obtain $h(U_1{}^s) \subset V_1{}^s$. From (34) we easily get

$$h^{-1} \circ B|_{V_1} = A \circ h^{-1}|_{V_1},$$

and so we similarly see that $h^{-1}(V_1{}^s) \subset U_1{}^s$. Hence we have proved that $h(U_1{}^s) = V_1{}^s$. This allows us to extend $h|_{U_1}$ to a homeomorphism $h^s : E_A^s \to E_B^s$ conjugating $A^s$ and $B^s$. Indeed, given $x^s \in E^s$, by Theorem 5.5 there is some $n = n(x^s) \ge 0$ such that $A^n x^s \in U_1^s$. We define

$$h^s(x^s) = B^{-n} \circ h \circ A^n(x).$$

We leave it as an exercise to prove that $h^s$ is well defined and is a homeomorphism conjugating $A|_{E_A^s}$ and $B|_{E_B^s}$.

It is not difficult to see that similar conclusion holds for the unstable spaces. Indeed, considering $U_2 = A(U_1)$ and $V_2 = B(V_1)$, it follows from (34) that $h^{-1}(V_2) = U_2$ and

$$h^{-1} \circ B^{-1}|_{V_2} = A^{-1} \circ h^{-1}|_{V_2}.$$

Using similar arguments to those above we can prove that there is also a homeomorphism $h^u : E_A^u \to E_B^u$ conjugating $A|_{E_A^u}$ and $B|_{E_B^u}$.

Finally, defining $h : E_A^s \oplus E_A^u \to E_B^s \oplus E_B^u$ by

$$h(x^s + x^u) = h^s(x^s) + h^u(x^u)$$

we have that $h$ is a homeomorphism conjugating $A$ and $B$. □

# 7 Open problem

We have proved in Theorem 6.8 that a hyperbolic isomorphism of a Banach space is structurally stable. We leave it as an exercise to show that in the finite dimensional case we have a converse to that result: any structurally stable linear isomorphism of a finite dimensional space is necessarily hyperbolic.

A possible strategy to prove the converse to Theorem 6.8 in the finite dimensional case can be based on the simple idea that given a non hyperbolic isomorphism in a finite dimensional space, under a small perturbation we can get rid of the eigenvalues on the unit circle. We can do that in such a way that arbitrarily close to the given non hyperbolic isomorphism there are two hyperbolic isomorphisms whose dimensions of stable and unstable spaces do not coincide. This is an obstruction to structural stability.

To the best of our knowledge, the necessity of hyperbolicity for structural stability in infinite dimensional Banach spaces is still an open problem.

**Problem.** *Let $E$ be a Banach space and let $A \in GL(E)$ be structurally stable. Is $A$ necessarily hyperbolic?*

# 8 Exercises

1. Let $E$ be a finite dimensional linear space and $A \in GL(E)$. Prove that:
   (a) if $\{e_1, \ldots, e_s\}$ is the basis associated to a Jordan block of an eigenvalue $\lambda$ of $A$ with $|\lambda| < 1$, then $A^n x \to 0$ as $n \to +\infty$ for all $x \in \operatorname{span}\{e_1, \ldots, e_r\}$;
   (b) if $\{f_1, \ldots, f_s\}$ is the basis associated to a Jordan block of an eigenvalue $\lambda$ of $A$ with $|\lambda| > 1$, then $A^{-n} x \to 0$ as $n \to +\infty$ for all $x \in \operatorname{span}\{f_1, \ldots, f_s\}$;
   (c) if $A$ has no eigenvalues in the unit circle, then there are subspaces $E^s$ and $E^u$ of $E$ such that:
      i. $E = E^s \oplus E^u$;
      ii. $A^n x \to 0$ as $n \to +\infty$ for all $x \in E^s$;
      iii. $A^{-n} x \to 0$ as $n \to +\infty$ for all $x \in E^u$.
2. Let $E$ be a vector space and $A : E \to E$ a linear isomorphism. Show that the stable and unstable spaces of $A$ are linear subspaces of $E$ invariant under $A$.
3. Given an example of a linear isomorphism of a Banach space $E$ for which the sum of its stable and unstable subspaces is not equal to $E$.
4. Let $A \in GL(E)$ and $A_{\mathbb{C}} \in L(E_{\mathbb{C}})$ be the complexification of $A$. Show that:
   (a) $E^s_{\mathbb{C}} = E^s + iE^s$ and $E^u_{\mathbb{C}} = E^u + iE^u$;
   (b) $A^s_{\mathbb{C}}(x + iy) = A^s x + iA^s y$ for each $x, y \in E^s$;
   (c) $A^u_{\mathbb{C}}(x + iy) = A^u x + iA^u y$ for each $x, y \in E^u$.

(d) $\|A^s_{\mathbb{C}}\|_{\mathbb{C}} = \|A^s\|$ and $\|A^u_{\mathbb{C}}\|_{\mathbb{C}} = \|A^u\|$.

5. Let $X = [1/3, 1/2] \cup [2, 3]$ and $E$ be the Banach space of bounded functions $f : X \to \mathbb{C}$ endowed with the sup norm. Define $A : E \to E$ by assigning to each $f \in E$ the function $Af \in E$, defined as

$$Af(x) = xf(x), \quad \forall x \in X.$$

   (a) Show that $A$ is a bounded linear isomorphism.

   (b) Show that $A$ is hyperbolic.

   (c) Determine the stable and unstable spaces of $A$.

6. Consider $D$ the disk of radius $1/2$ centered at the point $1 \in \mathbb{C}$ and let $E$ be the Banach space of bounded analytic functions $f : D \to \mathbb{C}$ with supremum norm. Let $A : E \to E$ be the bounded linear operator defined for each $f \in E$ as

$$Af(z) = zf(z), \quad \forall z \in D.$$

   Prove that $\sigma(A) = \overline{D}$.

7. Prove Corollary 5.9.

8. Let $A, B : \mathbb{R} \to \mathbb{R}$ be the linear isomorphisms given by $Ax = ax$ and $Bx = bx$, with $a, b > 1$. Show that $A$ and $B$ are topologically conjugate.

9. Let $A, B : E \to E$ be the hyperbolic isomorphisms of a Banach space $E$. Show that $A$ is topologically conjugate to $B$ if and only if both $A|_{E^s_A}$ is topologically conjugate to $B|_{E^s_B}$ and $A|_{E^u_A}$ is topologically conjugate to $B|_{E^u_B}$.

10. Let $E$ be a finite dimensional vector space. Show that $H(E)$ is dense in $GL(E)$.

11. Let $E$ be a finite dimensional vector space. Show that if $A \in GL(E)$ is structurally stable, then $A \in H(E)$.

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