On Dual Goldie Dimension

Diplomarbeit

von

Christian Lomp

aus

Düsseldorf

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der Heinrich-Heine Universität Düsseldorf

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Preface for revisted version 2000

Since many people asked me for copies of my Diplomarbeit/MSc Thesis I have revised it partly. I included a new module theoretic proof of implication $(j) \Rightarrow (a)$ of Theorem 3.3.7 (Camps and Dicks Theorem) that shortens Camps and Dicks's original proof. I gave a shorter proof for Lemma 3.4.2. The original proof was due to Garcia Hernandez and Gomez Pardo. I corrected a mistake in the origianl version of Proposition 5.1.2 that relates the trace ideal of a non-small projective with the class of all non-small modules.

Moreover I would like to make some comments on chapter 4.2. "Lifting modules with chain conditions". The main theorem of this small section was to prove that a lifting module whose radical has ACC on direct summands is the direct sum of a semisimple module and a finite direct sum of hollow modules (Theorem 4.2.2). The following Corollary 4.2.3 appeared also as Theorem 4.7 in a paper published in 1997: *K.A. Brown and M.H. Wright, "Decompositions of D1 modules" in Advances in ring theory, 49-64, Birkhaeuser, Trends in Mathematics, 1997.* It states that a lifting module with ACC(DCC) on small modules is a direct sum of a semisimple and a noetherian(artinian) module. In their proofs Brown and Wright re-prove Al-Khazzi and Smith's Theorem in [2] that the radical of a module M is noetherian(artinian) if and only if M has ACC(DCC)on small modules. Just after having read Brown and Wright's paper I recognized that part of Corollary 4.2.3 followed from a more general theorem in Al-Khazzi and P.Smith's second paper from 1995 *"Classes of Modules with many direct summands", J. Austral. Math. Soc (Ser. A) 59, (1995), 8-19.* In this paper we find:

Theorem 3.5 For any ring R and ordinal $\alpha \geq 0 : d^*\mathcal{K}_\alpha = \mathcal{C} \oplus \mathcal{K}_\alpha$.

Their notation is $\mathcal{C}$=class of semisimple right R-modules; $\mathcal{K}_\alpha$=class of right R-modules with Krull dimension at most α and $d^*\mathcal{X}$ = class of right R-modules M such that for every submodule N there exists a direct summand D of M with $D \subseteq N$ and $N/D \in \mathcal{X}$ where $\mathcal{X}$ is a class of right R-modules closed under isomorphisms and containing the zero module.

Hence we can use Al-Khazzi and Smith's Theorem in the following way: If every small submodule of a lifting module M has Krull dimension at most α, then M

belongs to $d^*\mathcal{K}_\alpha$ and therefore by Al-Khazzi and Smith's Theorem M is a direct sum of a semisimple and a lifting module with Krull dimension at most α. This of course generalizes the artinian part of my and of Brown and Wright's result.

One could also set $\mathcal{X}$ to be the class of all small R-modules. Then the d^*-concept generalizes the notion of lifting modules. In chapter 4 of her PhD thesis (Hacettepe Univ, 1999) A.Cigdem Ozcan used the d^*-concept to generalize lifting modules by studying $d^*\mathcal{X}$ where $\mathcal{X}$ = cosingular modules (=sums of small modules). Here she calls those modules in $d^*\mathcal{X}$ "Modules with property (S*)".

Acknowledgments

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Last but not least I would like to thank my parents for all their crucial support.

Summary

This dissertation reviews attempts of dualizing the Goldie dimension. Moreover we choose one of these attempts as the dualization of Goldie dimension and study modules with this finiteness condition under various aspects.

Chapter 1 defines basic ideas as dualizations of well-known notions. Small submodules, hollow modules, small covers, supplements, coclosed submodules and coindependent families of submodules are introduced as dual concepts of essential submodules, uniform modules, essential extensions, complements, closed submodules and independent families of submodules.

In Chapter 2 existing attempts of dualizing the Goldie dimension are reviewed and compared. Section 2.1 is devoted to the earliest approach while in section 2.2 three equivalent approaches are considered. Section 2.3 states a general lattice theoretical approach equivalent to the approaches in 2.2.

The core of this dissertation is formed by Chapter 3. In Section 3.1 we choose one of the approaches as dualization of Goldie dimension and call it *hollow dimension*. The main characterizations and properties are stated. Dimension formulas as for vector spaces are considered in Section 3.2. We show in Section 3.3 that rings with finite hollow dimension are exactly the semilocal rings. The situation when the hollow dimension of a module coincides with the hollow dimension of the endomorphism ring is studied in Section 3.4. Here we study properties of modules with semilocal endomorphism rings as well. Relationships of certain chain conditions and hollow dimension are stated in Section 3.5. In Section 3.6 modules with property $AB5^*$ whose submodules have finite hollow dimension are considered.

The dual concept of extending (or CS) modules namely *lifting modules* is introduced in Chapter 4 and their relation to hollow modules is studied. Basic definitions of lifting modules and a decomposition of lifting modules with finite hollow dimension are given in Section 4.1. The structure of lifting modules with certain chain conditions on the radical is given in Section 4.2.

In the last chapter of this thesis, Chapter 5, we study dualizations of singular and polyform modules in connection with Goldie's Theorem. The notion of M-small and non-M-small modules are introduced in Section 5.1 as dual concepts of M-singular and non-M-singular modules. Eventually co-rational submodules and copolyform modules are defined in Section 5.2 as dual notions of rational submodules and polyform modules.

Contents

Introduction

The title of this dissertation suggests falsely that there exists just one dual Goldie dimension. In fact there are at least four different definitions of the notions of a dual Goldie dimension. So the correct title should probably be "On the dualization of the Goldie dimension". But since, as we will see, most of these attempts are equivalent to each other, we keep this title.

Uniform modules, essential extension and independent families of submodules play an important role in Goldie's work. An R-module M with finite uniform dimension (Goldie dimension) can be characterized by one of the equivalent statements:

(U1) M contains no infinite direct sum of non-zero submodules.

(U2) M contains an essential submodule, which is a finite direct sum of uniform submodules of M.

(U3) for every ascending chain of submodules $N_1 \subset N_2 \subset N_3 \subset \cdots$ there is an integer n, such that N_n is essential in N_k for every $k \geq n$.

One of the earliest attempts to define a dual Goldie dimension was done by P.Fleury [13] in 1974. He called his dual Goldie dimension *spanning dimension* and introduced the notion of *hollow* modules, as the dual concept of uniform modules that appeared in Goldie's work. Fleury's spanning dimension dualizes chain condition (U3). Modules with finite spanning dimension are closely related to artinian, respectively hollow, modules and the rings with finite spanning dimension are exactly the artinian, respectively local, rings. T.Takeuchi [59] introduced coindependent families of submodules as a dual notion of independent families of submodules in 1975. With the help of this notion he dualized (U1) and called his dual Goldie dimension *cofinite-dimension*. Actually his definition was based on an early paper by Y.Miyashita [38] in 1966, where he introduces a dimension notion for modular lattices. In 1979 K.Varadarajan approached the dualization of the Goldie dimension in a more categorical way by dualizing (U2). He called his dual Goldie dimension *corank*. Comparing his definition with Fleury's he showed that "finite spanning dimension" implies "finite corank". Varadarajan's corank was probably the most often used definition of the dual Goldie dimension in the past. E.Reiter [49] gave a definition of a dual Goldie dimension in 1981. He called his dual Goldie dimension *codimension* and dualized property (U3) as Fleury did. It is quite easy to see that Reiter's and Takeuchi's definition are equivalent to each other. In 1984

P.Grezeszcuk and E.R.Puczyłowski [20] compared all four named approaches and showed that Takeuchi's, Varadarajan's and Reiter's definitions are equivalent to each other. Their approach was lattice-theoretical by defining the Goldie dimension of a modular lattice and by setting the dual Goldie dimension of a modular lattice to the Goldie dimension of the dual lattice. Eventually they applied these notions to the lattice of submodules of a module. Moreover they showed that modules with finite spanning dimension satisfy their definition of dual Goldie dimension as well. Since hollow modules play an important role in our study we call the dual Goldie dimension of a module M *hollow dimension* (analogously to calling the Goldie dimension of M *uniform dimension*) if M satisfies Takeuchi's, Reiter's or Varadarajan's definition. We will state a result by S.Page [44] in Section 3.1, that the hollow dimension of a module ${}_RM$ can be computed by the uniform dimension of the dual module $\mathrm{Hom}_R(M, Q)_T$, where ${}_RQ_T$ is an injective cogenerator in $R-$Mod and $T := \mathrm{End}_R(Q)$.

The existence of complements plays an important role in the study of modules with uniform dimension. The dual concept of complements is the notion of supplements. Following Zöschinger [74] we introduce weak supplements and weakly supplemented modules (i.e. every submodule has a weak supplement). In Section 1.3 we show that semilocal rings are exactly the rings that are weakly supplemented as left or right modules over itself. Moreover we will show in Section 3.3 that a finitely generated module M has finite hollow dimension if and only if it is weakly supplemented if and only if $M/\mathrm{Rad}(M)$ is semisimple. This shows that rings with finite hollow dimension are exactly the semilocal rings. This fact was first shown by Varadarajan in [62].

The relation between the hollow dimension of a module and the hollow dimension of its endomorphism ring is studied in Section 3.4. Using a result by J.L. Garcia Hernandez and J.L. Gomez Pardo [15], T.Takeuchi [60] showed that the hollow dimension of a module is invariant under equivalences. Moreover he showed that a self-projective module has finite hollow dimension if and only if it has a semilocal endomorphism ring. Since modules with semilocal endomorphism ring have interesting properties we state some results by D.Herbera and A.Shamsuddin [29], A.Facchini et al. [11] as well as K.R.Fuller and W.A.Shutters [14].

Modules with finite uniform dimension can be characterized by ACC (respectively DCC) on complements. If we assume the existence of amply supplements then a dual characterization in terms of supplements is also possible for hollow dimension. This was observed by T.Takeuchi [59] and K.Varadarajan [62] and we will state this in Section 3.5. A result by V.P.Camillo [6] characterizes modules whose factor modules have finite uniform dimension. We examine a dual version of Camillo's theorem and consider modules whose submodules have finite hollow dimension. This leads to a characterization of artinian modules in terms of hollow dimension.

Since the uniform dimension of a torsionfree abelian groups coincides with the ordinary rank, K.Varadarajan [62] as well as Hanna and Shamsuddin [24] studied the hollow dimension of abelian groups. They showed that hollow $\mathbb{Z}$-modules are exactly the modules $\mathbb{Z}_{p^k}$ with p prime and $1 \leq k \leq \infty$ and that a $\mathbb{Z}$-module with finite hollow dimension is a finite direct sum of hollow modules and hence artinian.

Inspired by a question in [2] E.R.Puczyłowski asked in [46] if the radical of a module has Krull dimension if every small submodule has Krull dimension. E.R.Puczyłowski himself showed in the same paper that the answer to this question is in general negative but we are able to give a positive answer if the module has property $AB5^*$.

In Section 3.6 modules with property $AB5^*$ whose submodules have finite hollow dimension are considered. P.N.Ánh et al.[3] as well as G.Brodskii [5] showed that those modules are lattice anti-isomorphic to linearly compact modules.

As the concept of extending (or CS) modules can be seen as a generalization of injective modules their dual concept, *lifting modules* can be seen as a generalization of semiperfect modules. This class of modules is introduced and considered in Chapter 4.

We state a decomposition theorem of lifting modules with finite hollow dimension and obtain some results from S.H.Mohamed and B.J.Müller [39] as well as R.Wisbauer [67]. The structure of lifting modules with certain chain conditions on their radical is given in Section 4.2.

In the last chapter of this thesis, Chapter 5, we study dualizations of M-singular and polyform modules in connection with Goldie's Theorem. We define M-small modules and non-M-small modules as dualizations of M-singular and non-M-singular modules as they appear in [10]. These modules form torsion theories and are related to dual polyform modules. The concept of polyform modules appeared first in Zelmanowitz' work [70]. A slightly improved version of Goldie's theorem [10] states that the endomorphism ring of the M-injective cover of a module M is semisimple artinian and is the classical quotient ring of End (M) if and only if M is polyform with finite uniform dimension. The aim of this section was to prove a dual result in terms of hollow dimension. For that reason *co-rational* and *co-polyform* modules are introduced in Section 5.2 as dual notions of rational and polyform modules. Co-rational submodules and extensions appeared first in R.C. Courter's work [9]. Finally as a partial result we can prove that if a module M has a projective cover P then End (P) is semisimple artinian if and only if M is co-polyform with finite hollow dimension.

Since the purpose of this dissertation is a review of existing knowledge most of the results stated here are known and are indicated by one or more references. Apart from some corollaries and lemmas the following results are due to the author: 1.2.1, Section 1.3, 2.1.6, 3.1.6, 3.1.12, 3.4.13, 3.5.6, 3.5.18, 3.5.20, 3.5.21, 4.1.4 - 4.1.7, Section 4.2 and all results in Chapter 5 except from 5.1.2 and 5.2.1.

For the reader's sake an effort was made to avoid citations of papers and to include most of the proofs in a way that they fit in the context of this dissertation. As main reference R.Wisbauer's text book [67] is most cited. Further the language of $\sigma[M]$ is used to indicate that some results only depend on properties of the given module M and not on properties of the ring.

Christian Lomp, Glasgow, October 1996

Notation

R	associative ring with unit
$R-$Mod	category of left R-modules
Mod$-R$	category of right R-modules
Jac (R)	Jacobson radical of R
$E(M)$	injective hull of a module M
$\mathcal{L}$	a complete modular lattice
$\mathcal{L}(M)$	the lattice of submodules of a module M
$\sigma\,[M]$	subcategory of $R-$Mod subgenerated by a module M
$\sigma_{\mathrm{f}}\,[M]$	subcategory of $\sigma\,[M]$ of all submodules of finitely M-generated modules
$\widehat{N}$	M-injective hull of a module $N \in \sigma[M]$
$\mathrm{Hom}_R\,(M,N)$	R-homomorphisms from M to N
End (M)	endomorphism ring of M
Im (f)	image of a map f
Ker (f)	kernel of a map f
Coke (f)	cokernel of a map f
$Tr(M,N)$	trace of a module M in N
$Re(M,N)$	reject of a module N in M
$K \trianglelefteq M$	K is an essential submodule of M
$K \ll M$	K is a superfluous submodule of M
Soc (M)	socle of M
Rad (M)	radical of M
$udim(M)$	the uniform dimension of M
$hdim(M)$	the hollow dimension of M
$lg(M)$	the length of M
$sd(M)$	the spanning dimension of M
$Ke(X)$	$\bigcap_{f\in X}\mathrm{Ker}\,(f)$, for $X \subseteq \mathrm{Hom}\,(M,N)$
$An(K)$	$\{f \in \mathrm{Hom}\,(M,N) \mid (K)f = 0\}$ for $K \subseteq M$
$\varprojlim M_i$	inverse limit of modules M_i
$\delta_{i,j}$	Kronecker symbol
$\mathcal{T}_M^*$	class of M-small modules
$\mathcal{F}_M^*$	class of non-M-small modules

Chapter 1

Basic notions

In what follows R always means an associative ring with identity. We will denote the full category of left R-modules by $R-\mathrm{Mod}$ and the full category of right R-modules by $\mathrm{Mod}-R$. Unless mentioned otherwise by an R-module we mean a unitary left R-module. Let M and N be R-modules. Arguments of module homomorphisms are written on the same side as scalars, i.e. write $(x)f$ for a left R-module homomorphism $f : M \to N$ and $x \in M$. N is called *generated by* M or *M-generated* if there exist an index set Λ and an epimorphism $M^{(\Lambda)} \to N$. N is called *subgenerated by* M if it is isomorphic to a submodule of a M-generated module, i.e. there exist an index set Λ, an R-module X, an epimorphism $g : M^{(\Lambda)} \to X$ and a monomorphism $f : N \to X$.

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow{\scriptstyle f} & & \\ M^{(\Lambda)} & \xrightarrow{g} & X & \longrightarrow & 0 \end{array}$$

We denote by $\sigma[M]$ the full subcategory of $R-\mathrm{Mod}$ whose objects are all R-modules subgenerated by M. For basic properties of $\sigma[M]$ we will refer to [67].

1.1 Small modules

Let M be an R-module. A submodule K of M is *essential* or *large* in M provided for all non-zero submodules $L \subseteq M$, $K \cap L \neq 0$ holds. We will denote essential submodules by $K \trianglelefteq M$ and M is called an *essential extension* of K. Let N be an R-module and $f : N \to M$ a monomorphism. Then f is called an *essential monomorphism* if $\mathrm{Im}\,(f) \trianglelefteq M$. Hence N is an essential extension of a submodule K if and only if the inclusion map $K \to N$ is an essential monomorphism. If N is a submodule of a module M then we say N is an essential extension of K in M.

We will introduce dual definitions for *essential* submodules and *essential extensions*.

Definition. Let M be an R-module. A submodule K of M is *small* in M provided for all proper submodules L of M, $L + K \neq M$ holds. We will denote small submodules by $K \ll M$ and M is called a *small cover* of M/K. An epimorphism $f : M \to L$ is called *small* if $\mathrm{Ker}\,(f)$ is small in M. Hence M is a *small cover* of M/N if and only if the canonical projection $M \to M/N$ is a small epimorphism. Dual to an essential extension N of K in M, we say N *lies above* K in M if M/K is a small cover of M/N, i.e. $N/K \ll M/K$. Clearly a submodule N is small in M if and only if N lies above 0 or equivalently if M is a small cover of M/N.

Remarks: Let M be an R-module and $K \subseteq N$ submodules of M. In [59] Takeuchi calls K a *coessential extension* of N in M if N lies above K.

Before we list some properties of *lying above*, let us state an easy, but useful lemma:

Lemma 1.1.1. *([49, Lemma 2.2]) Let K,L,N be submodules of M. If $K + L = M$ and $(K \cap L) + N = M$ hold, then $K + (L \cap N) = L + (K \cap N) = M$.*

Proof: $K + (L \cap N) = K + (L \cap K) + (L \cap N) = K + (L \cap ((L \cap K) + N)) = K + (L \cap M) = K + L = M$. Applying the same argument to $L + (K \cap N)$ we get $L + (K \cap N) = M$. $\square$

1.1.2. Properties of "lying above". *([59, 1.1,1.2,1.6], [32, Lemma 2])*

For submodules $L \subseteq N$ of M the following properties hold:

1. *N lies above L in M if and only if $L + K = M$ holds for all $K \subseteq M$ with $N + K = M$.*
   In this case, $N \cap K$ lies above $L \cap K$, for all $K \subseteq M$ with $N + K = M$.

2. *$N \ll M$ if and only if N lies above L and $L \ll M$.*

3. *For submodules $K \subseteq L \subseteq N$ of M, N lies above K if and only if N lies above L and L lies above K.*

4. *Let $G \subseteq H$ be submodules of M. If N lies above L and H lies above G and $N + H = M$, then $L + G = M$ and $N \cap H$ lies above $L \cap G$.*

Proof: (1) Suppose that N lies above L in M. If $N + K = M$, then

$$M/L = (N + K)/L = N/L + (K + L)/L = (K + L)/L.$$

Hence $K + L = M$. Conversely, suppose that $L + K = M$ for all $K \subseteq M$ with $N + K = M$. If there is a $K \subseteq M$ containing L such that $N/L + K/L = M/L$, then $M = N + K$ yields $M = L + K = K$, so N lies above L. Furthermore, let K be a submodule of M, such that $N + K = M$. If there is a submodule X containing $(L \cap K)$, such that

$$M/(L \cap K) = (N \cap K)/(L \cap K) + X/(L \cap K),$$

then $(N \cap K) + X = M$. By applying Lemma 1.1.1 twice we get

$$M = N + (K \cap X) = L + (K \cap X) = (L \cap K) + X = X.$$

Thus $N \cap K$ lies above $L \cap K$.
(2) Easy check using (3); (3) Easy check using (1);

(4) Applying (1) twice we get $M = N + H = L + H = L + G$ and $N \cap H$ lies above $L \cap H$ and $L \cap H$ lies above $L \cap G$. So by (3) we have $N \cap H$ lies above $L \cap G$.
$\square$

A non-zero R-module M is called *uniform* if every non-zero submodule of M is essential in M. Dual to the concept of uniform modules, Fleury defined the notion of *hollow* modules in [13].

Definition. An R-module M is called *hollow* if $M \neq 0$ and every proper submodule N of M is small in M. M is called *local* if it has exactly one maximal submodule that contains all proper submodules.

Remarks:

1. Miyashita calls hollow R-modules *R-sum-irreducible* (see [38]).

2. Hollow modules are indecomposable modules and every factor module of a hollow module is hollow.

3. Clearly a local module is hollow and the unique maximal submodule has to be the radical. Examples of hollow modules are simple or uniserial modules, e.g. $\mathbb{Z}_{p^\infty}$ or $\mathbb{Z}_{p^k}$ with p prime and $k \in \mathbb{N}$.

1.1.3. Properties of hollow modules.
Let M be an R-module.

1. *M is hollow if and only if every factor module of M is indecomposable.*
2. *The following statements are equivalent:*
   (a) *M is local;*
   (b) *M is hollow and cyclic (or finitely generated);*
   (c) *M is hollow and* $\mathrm{Rad}\,(M) \neq M$.
3. *If M is self-projective then the following statements are equivalent:*
   (a) *M is hollow;*
   (b) $\mathrm{End}\,(M)$ *is a local ring.*

Proof: See [67, 41.4] for (1) and (2) and [47, Proposition 2.6] for (3). □

1.2 Coclosed submodules

A closed submodule N of a module M has no proper essential extension in M. Let us consider a dual notion of closed submodules.

Definition (Golan). Following Golan [16], we will call a submodule N of M *coclosed* in M if and only if N has no proper submodule K such that N lies above K (or N has no proper coessential extension).

A submodule N of an R-module M is called a *complement* of a submodule L in M if it is maximal with respect to $N \cap L = 0$. By applying *Zorn's Lemma* there exists always for every submodule L of M a complement N of L. Moreover a submodule is a complement in M if and only if it is closed in M (see [10, pp. 6]).

Dual to the concept of complements we define the notion of supplements.

Definition. Let N and L be submodules of M, then we call N a *supplement* of L if N is minimal with respect to $N + L = M$. Equivalently N is a supplement of L if and only if $N + L = M$ and $N \cap L \ll N$. A submodule N of M is called a supplement if there is a submodule L of M and N is a supplement of L. Following Zöschinger [74] we call N a *weak supplement* of L in M if and only if $N + L = M$ and $N \cap L \ll M$. N is called a weak supplement in M if there exists a submodule L such that N is a weak supplement of L in M. Clearly any supplement is a weak supplement.

Remarks:

1. Complements always exist but supplements do not. For example no proper submodule in ${}_{\mathbb{Z}}\mathbb{Z}$ has a supplement in $\mathbb{Z}$. To see this assume that a proper submodule N of $\mathbb{Z}$ has a supplement L in $\mathbb{Z}$. Then $N \cap L \ll \mathbb{Z}$ holds implying $N \cap L = 0$ since $\mathrm{Jac}\,(\mathbb{Z}) = 0$. But since $\mathbb{Z}$ is uniform we have that N or L is equal to zero.

2. Let H be a hollow submodule of an R-module M. If H is not small in M then there exists a proper submodule $K \subset M$ with $H + K = M$. Since H is hollow, $H \cap K \ll H$. Thus H is a supplement in M (see also [32, Proposition 6]).

3. Let $L \subseteq N \subseteq M$. By 1.1.2, N lies above L if and only if $N + K = M$ implies $L + K = M$ for all $K \subseteq M$. If N is minimal with respect to $N + K = M$ for some K, then there cannot be a submodule L of N such that N lies above K. Thus N is coclosed.

The classes of complements and closed submodules are the same. We now determine the relation between supplements and coclosed submodules in the following Proposition.

Proposition 1.2.1. *Let N be a submodule of M. Consider the following statements:*

(i) N is a supplement in M;

(ii) N is coclosed in M;

(iii) for all $K \subseteq N$, $K \ll M$ implies $K \ll N$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) holds and if N is a weak supplement in M, then (iii) $\Rightarrow$ (i) holds.

Proof: (i) $\Rightarrow$ (ii) Assume that N is a supplement of $L \subseteq M$. For all submodules $K \subseteq N$ such that N lies above K, we have that $N + L = M$ implies $K + L = M$ (see 1.1.2(1)). By the minimality of N with respect to this property we get $K = N$. Hence N is coclosed.
(ii) $\Rightarrow$ (iii) Let $K \ll M$ and $K \subseteq N$. Assume $N = K + X$ for $X \subseteq N$; then for every $Y \subseteq M$ with $N + Y = M$ we get $M = X + Y$ since $K \ll M$. By 1.1.2(1) N lies above X. By the coclosure of N we get $X = N$ and thus $K \ll N$.
Assume N to be a weak supplement of $L \subseteq M$. (iii) $\Rightarrow$ (i) N is a weak supplement

of L, so $N \cap L \ll M$. By assumption $N \cap L \ll N$. Thus N is a supplement of L in M. $\square$

Remarks: The equivalence between (i) and (ii) appeared in [59, 2.6] and [32, Proposition 3] in the following form: if N has a supplement K in M and N is coclosed in M then N is a supplement in M. In 1.2.1 we showed that a coclosed submodule N of M having a weak supplement in M is a supplement in M.

Definition. An R-module M is called *supplemented* if every submodule has a supplement in M. M is called *amply supplemented* if for every submodules N and L of M with $N + L = M$, N contains a supplement of L in M. Clearly every amply supplemented module is supplemented.

The next proposition is dual to [10, 1.10] and states some properties of coclosed submodules.

Proposition 1.2.2. *Let M be an R-module with submodules $K \subseteq L$ and N.*

1. *If M is amply supplemented then every submodule of M that is not small in M lies above a supplement in M.*

2. *If L is coclosed in M, then L/K is coclosed in M/K.*

3. *Assume that L is a supplement in M. Then K is coclosed in L if and only if K is coclosed in M.*

Proof: (1) Let $M = N + X$ with X a supplement of N; then N contains a supplement Y of X in M. Hence $N \cap X \ll X$ implies

$$(N \cap X)/(Y \cap X) \ll X/(Y \cap X).$$

Since $(N \cap X)/(Y \cap X) \simeq N/Y$ and $X/(Y \cap X) \simeq M/Y$ we get $N/Y \ll M/Y$. Thus N lies above Y in M.
(2) Since L is coclosed in M, for every proper submodule N/K of L/K, $(L/K)/(N/K) \simeq L/N$ is not small in $M/N \simeq (M/K)/(N/K)$.
(3) Let L be a supplement of $X \subset M$. Assume K is coclosed in M then it is coclosed in L since whenever $K/N \ll L/N$ we get $K/N \ll M/N$ as $L/N \subseteq M/N$. Now assume that K is coclosed in L and that K lies above a proper submodule $H \subset K$ in M. Since K is coclosed in L, K does not lie above H in L. Hence there exists a proper submodule G of L containing H such that $K/H + G/H = L/H$ holds. Hence

$M = L + X = K + G + X$ implies $M = H + G + X = G + X$ since K lies above H in M. But since L is a supplement of X in M we get $G = L$; a contradiction to G being a proper submodule of L. Hence K is coclosed in M. □

Definition. Let M be an R-module and $N \in \sigma[M]$. A projective module P in $\sigma[M]$ together with a small epimorhpism $\pi : P \to N$ is called a *projective cover* of N in $\sigma[M]$. We will write (P, π) or just P for a projective cover P. If $\sigma[M] = R-$Mod we call P a projective cover of N. A module $N \in \sigma[M]$ is called *semiperfect* in $\sigma[M]$ if every factor module of N has a projective cover in $\sigma[M]$. A ring R is called semiperfect if R is semiperfect as a left (right) R-module (see [67, 42.6]).

Note the following important fact: A projective module P in $\sigma[M]$ is semiperfect if and only if it is (amply) supplemented (see [67, 42.3]).

1.3 Weak supplements

Definition. Following Zőschinger [74] we say that M is called *weakly supplemented* if every submodule N of M has a weak supplement.

Remarks: Applying 1.2.1 we see, that in a weakly supplemented module, supplements and coclosed submodules are the same.

It is well-known that the rings that are supplemented as a left (right) module over themselves are exactly the semiperfect rings (see [67, 42.6]). The notion of weak supplements generalizes the notion of supplements and we will discover that the rings that are weakly supplemented as left (right) module over themselves are exactly the semilocal rings (see 1.3.4). Moreover we will see that modules with finite dual Goldie dimension are weakly supplemented modules and that a finitely generated module has finite dual Goldie dimension if and only if it is weakly supplemented. Before we give a summarizing list of properties of weakly supplemented modules, we will state a general result:

Proposition 1.3.1. *Let M be an R-module and N a proper submodule of M. The following statements are equivalent:*

(a) M/N is semisimple;

(b) for every $L \subseteq M$ there exists a submodule $K \subseteq M$ such that $L + K = M$ and $L \cap K \subseteq N$;

(c) *there exists a decomposition* $M = M_1 \oplus M_2$ *such that* M_1 *is semisimple,* $N \trianglelefteq M_2$ *and* M_2/N *is semisimple.*

Proof: (a) $\Rightarrow$(c) Let M_1 be a complement of N. Then $M_1 \oplus N$ is essential in M. $M_1 = (M_1 \oplus N)/N$ is a direct summand in M/N, hence semisimple and there is a semisimple submodule M_2/N such that $(M_1 \oplus M_2)/N = M/N$. Hence $M = M_1 + M_2$ and $M_1 \cap M_2 \subseteq N \cap M_2 = 0$. Thus $M = M_1 \oplus M_2$. Because M_1 is a complement, N is essential in M_2.
(c) $\Rightarrow$(a) Clear, since $M/N \simeq (M_1 \oplus M_2/N)$.
(a) $\Rightarrow$(b) Clear, since $(L + N)/N$ is a direct summand in M/N.
(b) $\Rightarrow$(a) Let $L/N \subseteq M/N$; then there exists a submodule $K \subseteq M$ such that $L + K = M$ and $L \cap K \subseteq N$. Thus $L/N \oplus K/N = M/N$. Hence every submodule of M/N is a direct summand. $\square$

1.3.2. Properties of weakly supplemented modules.
Let M *be an* R*-module.*

1. *If* M *is weakly supplemented then the following properties hold:*

   (i) $M/\mathrm{Rad}\,(M)$ *is semisimple;*

   (ii) $M = M_1 \oplus M_2$ *with* M_1 *semisimple and* $\mathrm{Rad}\,(M) \trianglelefteq M_2$*;*

   (iii) *every factor module of* M *is weakly supplemented;*

   (iv) *if* N *is a small cover of* M*, then* N *is weakly supplemented;*

   (v) *every supplement in* M *and every direct summand of* M *is weakly supplemented.*

2. *Let* K *and* M_1 *be submodules of* M *such that* M_1 *is weakly supplemented and* $M_1 + K$ *has a weak supplement in* M*, then* K *has a weak supplement in* M*.*

3. *If* $M = M_1 + M_2$*, with* M_1 *and* M_2 *weakly supplemented, then* M *is weakly supplemented.*

Proof: (1)(i),(ii) follows from 1.3.1 since for every $L \subseteq M$ there exists a weak supplement $K \subseteq M$ such that $L + K = M$ and $L \cap K \subseteq \mathrm{Rad}\,(M)$.

(iii) Let $K \subseteq M$ and $N/K \subseteq M/K$. Then $N + L = M$ and $N \cap L \ll M$ for a submodule $L \subseteq M$. Hence $N/K + (L + K)/K = M/K$ and $N/K \cap (L + K)/K = ((N \cap L) + K)/K \ll M/K$ holds.

(iv) Let $M \simeq N/K$ for some $K \ll N$. Then for every submodule $L \subseteq N$, $(L + K)/K$ has a weak supplement X/K in N/K, with $((L + K) \cap X)/K \ll N/K$.

By 1.1.2(ii) $(L+K)\cap X$ is small in N. Thus $L\cap X \subseteq (L\cap X)+K = (L+K)\cap X \ll N$ and $L + X = N$. Hence X is a weak supplement of L in N.

(v) If $N \subseteq M$ is a supplement of M, then $N + K = M$ for some $K \subseteq M$ and $K \cap N \ll N$. By (iii) $M/K \simeq N/(N \cap K)$ is weakly supplemented and by (iv) N is weakly supplemented. Direct summands are supplements and hence weakly supplemented.

(2) By assumption $M_1 + K$ has a weak supplement $N \subseteq M$, such that $M_1 + K + N = M$ and $(M_1+K)\cap N \ll M$. Because M_1 is weakly supplemented, $(K+N)\cap M_1$ has a weak supplement $L \subseteq M_1$. So

$$M = M_1 + K + N = L + ((K + N) \cap M_1) + K + N = K + (L + N)$$

and

$$K\cap(L+N) \subseteq ((K+L)\cap N)+((K+N)\cap L) \subseteq ((K+M_1)\cap N)+((K+N)\cap L) \ll M.$$

This means that N+L is a weak supplement of K in M.

(3) For every submodule $N \subseteq M$, $M_1 + (M_2 + N)$ has a trivial weak supplement and by (2) $M_2 + N$ has one. Applying (2) again we get a weak supplement for N. $\square$

We get the following corollary from 1.3.2(1)(ii).

Corollary 1.3.3. *An R-module M with* $\mathrm{Rad}(M) = 0$ *is weakly supplemented if and only if it is semisimple.*

For modules M with small radical (e.g. finitely generated modules) we see by 1.3.2(1)(iv) and the previous corollary, that it is equivalent for M to be weakly supplemented or $M/\mathrm{Rad}\,(M)$ to be semisimple:

Corollary 1.3.4. *Let M be an R-module with* $\mathrm{Rad}\,(M) \ll M$.
Then M is weakly supplemented if and only if $M/\mathrm{Rad}\,(M)$ *is semisimple.*

Definition. A ring R is called *semilocal* if $R/\mathrm{Jac}\,(R)$ is semisimple.

Remarks:

1. We see, that a ring R is semilocal if and only if it is weakly supplemented as a left (or right) R-module.

2. Recall that a ring is semiperfect if and only if it is supplemented as a left (or right) R-module. Moreover a ring is semiperfect if and only if it is semilocal

and idempotents in $R/\mathrm{Jac}(R)$ can be lifted to R (see [67, 42.6]). Since the class of semilocal rings is strictly larger than the class of semiperfect modules there are modules that are weakly supplemented but not supplemented. Consider, for example, a semilocal commutative domain with two maximal ideals. Then there exists a non-trivial idempotent in $R/\mathrm{Jac}\,(R)$ that cannot be lifted to R. Take for example $\mathbb{Z}_{p,q} := \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, p \nmid b \text{ and } q \nmid b\}$, where p and q are primes. Then $\mathbb{Z}_{p,q}$ is a semilocal noetherian domain with two maximal ideals.

1.3.5. Endomorphism rings of weakly supplemented modules.
Let M be a self-projective, finitely generated, weakly supplemented R-module. Then $\mathrm{End}\,(M)$ *is semilocal.*

Proof: Since $M/\mathrm{Rad}\ (M)$ is semisimple and finitely generated we get that $\mathrm{End}\ (M/\mathrm{Rad}\ (M))$ is a semisimple ring. By [67, 22.2] we have $\mathrm{End}\,(M)/\mathrm{Jac}\,(\mathrm{End}\,(M)) \simeq \mathrm{End}\,(M/\mathrm{Rad}\,(M))$. Thus $\mathrm{End}\,(M)$ is semilocal. $\square$

1.4 Coindependent families of submodules

A non-empty family $\{N_\lambda\}_\Lambda$ of non-zero submodules of a module is called *independent* if for every $\lambda \in \Lambda$ and subset $F \subseteq \Lambda \setminus \{\lambda\}$ the following holds:

$$N_\lambda \cap \sum_{i \in F} N_i = 0,$$

with the convention that the summation with an empty index set is zero.

As a dualization of independent families we define the notion of coindependent families of submodules.

Definition. Let M be an R-module. A non-empty family $\{N_\lambda\}_\Lambda$ of proper submodules of M is called *coindependent* if for any $\lambda \in \Lambda$ and any finite subset $F \subseteq \Lambda \setminus \{\lambda\}$

$$N_\lambda + \bigcap_{i \in F} N_i = M,$$

with the convention, that the intersection with an empty index set is set to be M.

Remarks:

1. Miyashita in [38] calls a coindependent family *d-independent*, Zelinsky in [69] *independent*.

2. A coindependent family $\{N_\lambda\}_\Lambda$ that contains more than one submodule is a set of comaximal submodules of M, i.e. $N_\lambda + N_\mu = M$ for all $\mu, \lambda \in \Lambda$ with $\mu \neq \lambda$, but the converse is not true. For example consider the two dimensional real vector space $\mathbb{R}^2$ over $\mathbb{R}$. Then $\{\mathbb{R}(1,0), \mathbb{R}(0,1), \mathbb{R}(1,1)\}$ is a set of comaximal submodules of $\mathbb{R}^2$, but $\mathbb{R}(1,0) \cap \mathbb{R}(0,1) = (0,0)$ yields $\mathbb{R}(1,1) + (\mathbb{R}(1,0) \cap \mathbb{R}(0,1)) \neq \mathbb{R}^2$.

3. Clearly $\{N\}$ is a coindependent family for every proper $N \subset M$ by the convention that the intersection with an empty index set is set to be M.

4. A module is hollow if and only if every coindependent family of submodules has exactly one element.

1.4.1. Properties of coindependent families. *([59, 1.3,1.6])*
Let $\{K_\lambda\}_\Lambda$ be a coindependent family of submodules of M. The following properties hold:

1. *Every subfamily $\{K_\gamma\}_\Gamma$ with non-empty subset $\Gamma \subseteq \Lambda$ is coindependent.*

2. *Let $\{N_\lambda\}_\Lambda$ be a family of proper submodules of M, such that for every $\lambda \in \Lambda$, $K_\lambda \subseteq N_\lambda$. Then $\{N_\lambda\}_\Lambda$ is coindependent.*

3. *Let $L \subset M$, such that $\bigcap_F K_\lambda + L = M$ for every finite subset F of Λ. Then $\{K_\lambda\}_\Lambda \cup \{L\}$ is coindependent.*

4. *Let $\{L_\lambda\}_\Lambda$ be a family of submodules of M, such that K_λ lies above L_λ for every $\lambda \in \Lambda$, then for every finite subset F of Λ,*

   (i) $\{K_\lambda\}_{\Lambda\setminus F} \cup \{L_i\}_F$ is coindependent;

   (ii) $\bigcap_F K_i$ lies above $\bigcap_F L_i$.

   Moreover $\{L_\lambda\}_\Lambda$ is coindependent.

Proof: (1)Clear; (2) since

$$M = K_\lambda + \bigcap_{i\in F} K_i \subseteq N_\lambda + \bigcap_{i\in F} N_i \subseteq M$$

for every $\lambda \in \Lambda$ and finite subset $F \subset \Lambda \setminus \{\lambda\}$ holds.
(3) Let F be a finite subset of Λ and $\mu \in \Lambda \setminus F$. Let $K = \bigcap_F K_i$, then by hypothesis $K + K_\mu = M$ and $(K \cap K_\mu) + L = M$. Hence by Lemma 1.1.1 we get $M =$

$(K \cap L) + K_\mu$ as $K + L = M$. This means that $\{K_\lambda\}_\Lambda \cup \{L\}$ is a coindependent family of submodules of M.
(4) (i) By induction on the cardinality of F and applying (3). Hence $\{L_\lambda\}_\Lambda$ is coindependent since for every finite subset F and $\lambda \in \Lambda$ we get by (i) and (1) that $\{L_i\}_F \cup \{L_\lambda\}$ is coindependent and thus $M = L_\lambda + \bigcap_F L_i$. By induction and 1.1.2(4) it is easy to see that (ii) holds. □

1.4.2. Characterization of coindependent families. *([24, Lemma 7])*
Let $\{L_i\}_{\mathbb{N}}$ be a family of proper submodules of M. Then the following statements are equivalent:

(a) $\{L_i\}_{\mathbb{N}}$ is a coindependent family;

(b) $\{L_1, \cdots, L_n\}$ is a coindependent family, for every $n \in \mathbb{N}$;

(c) $L_n + (L_1 \cap \cdots \cap L_{n-1}) = M$ holds, for every $n > 1$;

(d) $\bigcap_I L_i + \bigcap_J L_j = M$ holds, for all disjoint finite subsets $I, J \subset \mathbb{N}$.

Proof: (a) ⇒(c) Clear.
(c) ⇒(b) By induction on n and 1.4.1(3).
(b) ⇒ (a) Let $F \subset \mathbb{N}$ be a finite subset and $i \in \mathbb{N} \setminus F$. Let $n := max\{i\} \cup F$, then $\{L_1, \ldots, L_n\}$ is coindependent and hence $L_i + \bigcap_F L_j = M$.
(d) ⇒(c) Clear; and (c) ⇒ (d) by induction on the cardinality of I and 1.1.1. Let J be a finite subset of $\mathbb{N}$ and $n := |I|$, for $n = 1$ our claim is clear. Assume that for all finite subsets $I \subset \mathbb{N}$ with cardinality n and $I \cap J = \emptyset$,

$$\bigcap_I L_i + \bigcap_J L_j = M$$

holds. Let $|I| = n + 1$ and $i \in I$. By the coindependency of $\{L_i\}_{\mathbb{N}}$,

$$L_i + \left(\bigcap_J L_j \cap \bigcap_{I \setminus \{i\}} L_i \right) = M$$

holds. By 1.1.1 we get $\bigcap_J L_j + \bigcap_I L_i = M$. □

Lemma 1.4.3. Chinese Remainder Theorem.
Let M be an R-module. For any coindependent family of submodules $\{K_i\}_I$ with I finite $M / \bigcap_{i \in I} K_i \simeq \oplus_{i \in I} M/K_i$ holds.

Proof: Let us prove this by induction on $n := |I|$. For $n = 1$ our claim is trivial. Let $n > 1$ and suppose that our claim holds for all coindependent families $\{L_1, \dots, L_{n-1}\}$ of submodules of M. Let $\{K_1, \dots, K_n\}$ be a coindependent family of cardinality n; then $\{K_1, \dots, K_{n-1}\}$ is a coindependent family. Set $K := \bigcap_{i=1}^{n-1} K_i$. By induction we have $M/K \simeq \oplus_{i=1}^{n-1} M/K_i$. Further $K + K_n = M$, so

$$\begin{aligned} M/\bigcap_{i=1}^{n} K_i &= M/(K \cap K_n) \\ &= K/(K \cap K_n) \oplus K_n/(K \cap K_n) \\ &\simeq M/K_n \oplus M/K \\ &\simeq \oplus_{i=1}^{n} M/K_i. \end{aligned}$$

□

Definition. Let M be an R-module and $\{N_\lambda\}_\Lambda$ a family of proper submodules. Then $\{N_\lambda\}_\Lambda$ is called *completely coindependent* if for every $\lambda \in \Lambda$:

$$N_\lambda + \bigcap_{\mu \neq \lambda} N_\mu = M$$

holds.

Remarks:

1. Oshiro defines a family of proper submodules $\{N_\lambda\}_\Lambda$ of M to be *coindependent* if it is completely coindependent and $M/\bigcap_\Lambda N_\lambda \simeq \bigoplus_\Lambda M/N_\lambda$ (see [41, pp. 361]).

2. Completely coindependent families of submodules are coindependent, but the converse is not true in general. For example, the collection of submodules $\mathbb{Z}p$ where p runs through the primes in $\mathbb{Z}$, is coindependent but not completely coindependent.

Definition. An R-module M has property $AB5^*$ if and only if for every submodule N and inverse systems $\{M_i\}_{i \in I}$ of submodules of M the following holds:

$$N + \bigcap_{i \in I} M_i = \bigcap_{i \in I}(N + M_i)$$

Examples for modules having $AB5^*$ are artinian or more generally linearly compact modules (see [67, 29.8]).

Lemma 1.4.4. *Every coindependent family of submodules of a module with property $AB5^*$ is completely coindependent.*

Proof: Let M be an R-module with the property $AB5^*$ and $\{N_\lambda\}_\Lambda$ a coindependent family of submodules of M. Define

- $\Omega := \{J \subseteq \Lambda | J \text{ is finite }\}$;
- $M_J := \bigcap_{j \in J} N_j$, for every $J \in \Omega$;
- $\Omega_\lambda := \{J \in \Omega | \lambda \notin J\}$, for every $\lambda \in \Lambda$.

Clearly $\{M_J\}_{\Omega_\lambda}$ forms an inverse system and $N_\lambda + M_J = M$ holds for all $\lambda \in \Lambda$ and $J \in \Omega_\lambda$. Thus we get for each $\lambda \in \Lambda$:

$$N_\lambda + \bigcap_{\mu \neq \lambda} N_\mu = N_\lambda + \bigcap_{J \in \Omega_\lambda} M_J = \bigcap_{J \in \Omega_\lambda} (N_\lambda + M_J) = M.$$

Thus $\{N_\lambda\}_\Lambda$ is completely coindependent. $\square$

Now we are able to extend 1.4.1(4).

Lemma 1.4.5. *Let M be an R-module with $AB5^*$, $\{L_\lambda\}_\Lambda$ a coindependent family of submodules such that for each $\lambda \in \Lambda$ there exists a submodule $N_\lambda \subseteq L_\lambda$ such that L_λ lies above N_λ in M. Then $\bigcap_\Lambda L_\lambda$ lies above $\bigcap_\Lambda N_\lambda$ in M.*

Proof: Using the same notation as in Lemma 1.4.4, Ω denotes the set of all finite subsets of Λ. Define for all $J \in \Omega$

$$A_J := \bigcap_{j \in J} L_j \text{ and } B_J := \bigcap_{j \in J} N_J.$$

By 1.4.1(4) A_J lies above B_J for all $J \in \Omega$. Since $\{A_J\}_{J \in \Omega}$ and $\{B_J\}_{J \in \Omega}$ are inverse systems, we get for a submodule $K \subset M$:

$$\begin{aligned} M &= K + \bigcap_{\lambda \in \Lambda} L_\lambda = K + \bigcap_{J \in \Omega} A_J = \bigcap_{J \in \Omega} (K + A_J) = \bigcap_{J \in \Omega} (K + B_J) \\ &= K + \bigcap_{J \in \Omega} B_J = K + \bigcap_{\lambda \in \Lambda} N_\lambda. \end{aligned}$$

$\square$

1.4.6. Weak Chinese Remainder Theorem.
Let M be an R-module, $\{N_\lambda\}_\Lambda$ a family of non-zero R-modules and $\{f_\lambda : M \to N_\lambda\}_\Lambda$ a family of epimorphisms. Write $K_\lambda :=$ Ker (f_λ) for every $\lambda \in \Lambda$. Then there is a homomorphism $f : M \to \prod_\Lambda N_\lambda$ and the following holds:

1. $\mathrm{Ker}\,(f) = \bigcap_\Lambda K_\lambda$.
2. *If f is epimorph, then $\{K_\lambda\}_\Lambda$ is a completely coindependent family.*
3. *If Λ is finite and $\{K_\lambda\}_\Lambda$ is a coindependent family, then f is epimorph.*

Proof: By the universal property of the product, there is a homomorphism $f : M \to \prod_\Lambda N_\lambda$ such that $f_\lambda = f\pi_\lambda$, where $\pi_\lambda : \prod_\Lambda N_\lambda \to N_\lambda$ is the canonical projection. Hence we get $(m)f = \{(m)f_\lambda\}_\Lambda$ for all $m \in M$.
(1)

$$(x)f = 0 \Leftrightarrow (x)f_\lambda = 0 \text{ for all } \lambda \in \Lambda \Leftrightarrow x \in \bigcap_{\lambda\in\Lambda} K_\lambda.$$

Hence $\mathrm{Ker}\,(f) = \bigcap_\Lambda K_\lambda$.
(2) Let $\lambda \in \Lambda$. We prove, that

$$M = K_\lambda + \bigcap_{\mu\in\Lambda\setminus\{\lambda\}} K_\mu.$$

Let $m \in M$. If $m \notin K_\lambda$, then $(m)f_\lambda \neq 0$. Hence $(\delta_{\mu\lambda}(m)f_\lambda)_{\mu\in\Lambda}$ is an element of $\prod_\Lambda N_\lambda$, where $\delta_{\mu\lambda}$ denotes the Kronecker symbol

$$\delta_{\mu\lambda} = \begin{cases} 1_R & \text{if } \mu = \lambda \\ 0_R & \text{if } \mu \neq \lambda \end{cases}$$

for every $\mu, \lambda \in \Lambda$. Since f is epimorph, there is an element $m_\lambda \in M$ such that $(m_\lambda)f = (\delta_{\mu\lambda}(m)f_\lambda)_{\mu\in\Lambda}$. Thus for all $\mu \in \Lambda$:

$$(m_\lambda)f_\mu = \delta_{\mu\lambda}(m)f_\lambda.$$

Hence for all $\mu \neq \lambda$, $m_\lambda \in K_\mu$ yields $m_\lambda \in \bigcap_{\mu\in\Lambda\setminus\{\lambda\}} K_\mu$. And for $\mu = \lambda$, $(m_\lambda)f_\lambda = (m)f_\lambda$ yields $(m - m_\lambda) \in K_\lambda$. Eventually we get

$$m = m - m_\lambda + m_\lambda \in K_\lambda + \textstyle\bigcap_{\mu\in\Lambda\setminus\{\lambda\}} K_\mu.$$

(3) Apply Lemma 1.4.3. □

Chapter 2

Approaches to dual Goldie dimension

Several attempts have been made to dualize the Goldie dimension. One of the earliest of these was done by Patrick Fleury in [13], but his definition of the dual Goldie dimension turned out to be restrictive. After that three other definitions were given by Varadarajan [62], Takeuchi [59] and Reiter[49] and fortunately they were all equivalent to each other. A general lattice theoretical definition of the dual Goldie dimension was given by Grezeszczuk and Pucyłowski in [20] and by applying this definition to the lattice of submodules of a module it was shown that their definition corresponds to Varadarajan's (Takeuchi's, Reiter's) definition.

An R-module M with finite Goldie dimension or finite uniform dimension can be characterized as follows (see [10, 5.9]):

(U1) M contains no infinite direct sum of non-zero submodules.

(U2) M contains an essential submodule, which is a finite direct sum of uniform submodules of M.

(U3) For every ascending chain of submodules $N_1 \subset N_2 \subset N_3 \subset \cdots$ there exists an integer n, such that N_n is essential in N_k for every $k \geq n$.

2.1 Fleury's approach: Finite spanning dimension

Fleury dualized property (U3) of the above characterization of modules with finite uniform dimension. His definition was:

Definition. (**Fleury**, [13, Definition 1.1]) An R-module M has *finite spanning dimension* if for every descending chain of submodules $N_1 \supset N_2 \supset N_3 \supset \cdots$ there is a number k such that $N_i \ll M$ for all $i \geq k$.

Examples for such modules are obviously artinian and hollow modules. We will see that these are the only examples in the class of self-projective modules.

Proposition 2.1.1. *Every supplement of an R-module with finite spanning dimension has finite spanning dimension.*

Proof: Let L be a supplement in M and $N_1 \supset N_2 \supset N_3 \supset \cdots$ be a descending chain of submodules of L. Then there exists a number k such that $N_i \ll M$. By 1.2.1 $N_i \ll L$. Hence L has finite spanning dimension. □

Remarks: We can refer to a module M with finite spanning dimension as a module that has DCC on submodules that are not small in M.
In [47] Rangaswamy recalls Fleury's definition incorrectly. He defines finite spanning dimension for a module M as follows: for every descending chain of submodules $N_1 \supset N_2 \supset N_3 \supset \cdots$ there is a number k such that $N_i = N_k$ or $N_i \ll N_k$ for all $i > k$. The next example will show that Rangaswamy's and Fleury's definitions do not match.

Example 2.1.2. *Let K be a field and V an infinite dimensional K-vector space; define*

$$R := \begin{pmatrix} K & V \\ 0 & K \end{pmatrix}, \ M := \begin{pmatrix} 0 & V \\ 0 & K \end{pmatrix}, \ N := \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

Then R is a ring by standard matrix addition and matrix multiplication and M and N are left R-modules. ${}_RM$ is a local module, hence it has finite spanning dimension, and $\mathrm{Rad}\,({}_RM) \simeq {}_KV$. ${}_RN$ is simple and ${}_RR = {}_RN \oplus {}_RM$. Since V is an infinite K-vector space, there are infinitely many independent subspaces V_i such that $\bigoplus_{i=1}^{\infty} V_i \subseteq V$. Let

$$L_j := \begin{pmatrix} K & \bigoplus_{i=j}^{\infty} V_i \\ 0 & 0 \end{pmatrix},$$

then $R = L_j + M$ holds for all $j \in \mathbb{N}$. Thus we get an infinite descending chain of submodules of R that are not small in ${}_RR$:

$$L_1 \supset L_2 \supset L_3 \supset \cdots$$

Thus $_RR$ can be expressed as a direct sum of two R-modules with finite spanning dimension, but does not have finite spanning dimension. Although M has finite spanning dimension it does not satisfy Rangaswamy's definition. Consider

$$N_j := \begin{pmatrix} 0 & \bigoplus_{i=j}^{\infty} V_i \\ 0 & 0 \end{pmatrix},$$

for all $j \in \mathbb{N}$, then all N_j are small in M, but $N_k \not\ll N_j$ for all numbers $k \leq j$. Thus

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$$

is a proper descending chain not having Rangaswamy's property but having Fleury's.

Proposition 2.1.3. *([13, Lemma 2.4]) Every R-module with finite spanning dimension is amply supplemented.*

Proof: Let M be an R-module with finite spanning dimension and N, L submodules of M with $N + L = M$ and $L \neq M$. Assume that N does not contain a supplement of L; then there exists a strictly descending chain

$$N = N_1 \supset N_2 \supset N_3 \supset \ldots$$

of submodules of N with $N_i + L = M$. This is a contradiction to the finite spanning dimension of M. Hence N must contain a supplement of L in M. Thus M is amply supplemented. $\square$

Remarks: An R-module P that is supplemented and projective in $\sigma[M]$ is semiperfect in $\sigma[M]$ and by [67, 42.4] a direct sum of local modules. Hence a projective module P in $\sigma[M]$ with finite spanning dimension is a finite direct sum of local modules.

The following collection of properties of modules with finite spanning dimension was obtained from [13] and [47].

2.1.4. Properties of modules with finite spanning dimension.
Let M be an R-module with finite spanning dimension. Then the following statements hold:

1. *every factor module of M has finite spanning dimension;*
2. *if $N \not\ll M$ then M/N is artinian;*
3. *M is indecomposable or artinian;*

4. *if* Rad (M) *is not essential in* M*, then* M *is artinian;*

5. M *has ACC and DCC on supplements;*

6. $M/\mathrm{Rad}\,(M)$ *is semisimple finitely generated.*

Proof: (1)+(2) Let M/N be a factor module of M. For every strictly descending chain of submodules

$$L_1/N \supset L_2/N \supset L_3/N \supset \cdots$$

there is an index k such that $L_k \ll M$ implying $L_k/N \ll M/N$ and $N \ll M$ by 1.1.2(2). Thus M/N has finite spanning dimension. If N was not small then $L_k = N$ must hold. Hence in this case M/N is artinian.

(3) If M is not indecomposable, then there exists a decomposition $M = M_1 \oplus M_2$. By (2) M_1 and M_2 are artinian.

(4) By (6) $M/\mathrm{Rad}\,(M)$ is semisimple. If $\mathrm{Rad}\,(M)$ is not essential in M, we can get a simple submodule S with $S \cap \mathrm{Rad}\,(M) = 0$ and S not small in M. By (2) M/S is artinian and so is M.

(5) Since every supplement submodule is not small, every strictly descending chain of supplements has to stop. Let

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

be a strictly ascending chain of supplements in M. Since M is amply supplemented, we will get a supplement L_1 of N_1. Clearly $N_2 + L_1 = M$ and we can get a supplement $L_2 \subseteq L_1$ of N_2. If $L_1 = L_2$, then $N_2 = N_1 + (N_2 \cap L_1)$ with $N_2 \cap L_1 \ll M$. This implies N_2 lies above N_1 in M contradicting that N_2 is coclosed. Hence $L_1 \supset L_2$. Getting supplements L_i in the same way for every N_i leads to a strictly descending chain of supplements, that has to stop.

(6) Since M is supplemented by 2.1.3, $M/\mathrm{Rad}\,(M)$ is semisimple by 1.3.2. By (5) $M/\mathrm{Rad}\,(M)$ has ACC on supplements and so on direct summands. Hence $M/\mathrm{Rad}\,(M)$ is a finite direct sum of simple modules. $\square$

The next definition is due to Zöschinger (see [74]).

Definition. An R-module M is called a *Minimax-module* if there exists an exact sequence

$$0 \longrightarrow F \longrightarrow M \longrightarrow A \longrightarrow 0$$

with F finitely generated and A artinian.

Remarks: Zöschinger proved in [73, 1.7] that every linearly compact module over a commutative noetherian ring is a Minimax-module. Moreover his student Rudlof showed in [52] that a module M over a commutative noetherian ring is a Minimax-module if and only if every decomposition of a homomorphic image of M is finite.

Corollary 2.1.5. *Let M be an R-module with finite spanning dimension. Then M is a Minimax-module or an indecomposable module with* $\mathrm{Rad}\,(M) = M$.

Proof: If M is not indecomposable then M is artinian by 2.1.4(3) and hence a Minimax-module. Assume $\mathrm{Rad}\,(M) \neq M$ and let $0 \neq x \in M \setminus \mathrm{Rad}\,(M)$. Then $Rx \not\ll M$ and the following sequence is exact:

$$0 \longrightarrow Rx \longrightarrow M \longrightarrow M/Rx \longrightarrow 0$$

with Rx cyclic and M/Rx artinian by 2.1.4(2). Hence M is a Minimax-module. □

Applying 2.1.4(3) we can easily prove a slightly modified version of a result by Rangaswamy [47], saying that modules with finite spanning dimension are either hollow or artinian if they satisfy a certain generalized projectivity condition.

Proposition 2.1.6. *Let M be an R-module such that every supplement is a direct summand. Then M has finite spanning dimension if and only if it is hollow or artinian.*

Proof: The sufficiency is clear. Assume that M is not hollow. Then there exists a submodule N that is not small in M. By 2.1.3, M is amply supplemented. So N has a supplement K in M. By hypothesis K is a direct summand. Hence there exists a decomposition $M = K \oplus L$ holds and by 2.1.4 M is artinian. □

Remarks:

1. We will call amply supplemented modules with the property that every supplement is a direct summand *lifting modules* in Chapter 4. Thus we showed that a lifting module has finite spanning dimension if and only if it is hollow or artinian.

2. Rangaswamy in [47, Proposition 3.5] proved the previous result for self-projective modules. Self-projective modules always satisfies the condition that the intersection of mutual supplements is zero. Hence an amply supplemented self-projective module satisfies the condition that every supplement is a direct

summand, since for each supplement N in M we can find a supplement K of N such that N and K are mutual supplements. Thus a projective R-module (e.g. R itself) has finite spanning dimension if and only if it is local or artinian.

3. We will show in Chapter 3.2 that one can assign a unique "dimension" number to a module having finite spanning dimension. This number is an invariant of the module.

4. More on finite spanning dimension can be found in Satyanarayana's papers [54], [55] and [56].

2.2 Reiter's, Takeuchi's and Varadarajan's approach

Takeuchi's approach to dual Goldie dimension was by dualizing (U1):

Definition. (**Takeuchi**, [59, Definition 4.7])
An R-module M is *cofinite-dimensional* if M contains no infinite coindependent family of submodules.

Reiter dualized chain condition (U3) as Fleury, but in a stricter way. His definition of finite dual Goldie dimension was:

Definition. (**Reiter**, [49, Definition 1.2])
An R-module has finite *codimension* if there is no infinite descending chain of intersections

$$U_1 \supset U_1 \cap U_2 \supset U_1 \cap U_2 \cap U_3 \supset \dots$$

of submodules $U_i \subset M$ such that for all $n \in \mathbb{N}$, $\{U_1, \dots, U_n\}$ is a coindependent family.

Remarks:

1. Reiter called an intersection of submodule $U_1 \cap \cdots \cap U_n$ in [49] to be a *direct intersection* if $\{U_1, \dots, U_n\}$ forms a coindependent family of proper submodules.

2. If M admits an infinite coindependent family $\{U_i\}_{i \in \mathbb{N}}$ of proper submodules of M, then M admits an infinite descending chain of direct intersections $U_1 \cap$

$\cdots \cap U_n$. Hence M does not have finite codimension. If M admits an infinite descending chain of direct intersections $U_1 \cap \cdots \cap U_n$ then $\{U_i\}_{i\in\mathbb{N}}$ forms an infinite coindependent family of proper submodules of M by 1.4.2. Hence we see that Reiter's and Takeuchi's definitions are equivalent.

2.2.1. Descending chain condition for finite codimension.
Let M be an R-module. M has finite codimension if and only if for every descending chain of submodules $N_1 \supset N_2 \supset N_3 \supset \cdots$ there exists an integer n such that N_n lies above N_k for all $k \geq n$.

Proof: For the proof we refer to 3.1.2 $(a) \Leftrightarrow (d)$ or [49, Theorem 2.5]. □

Remarks:

1. Comparing condition (U3) for finite uniform dimension to Reiter's descending chain condition we see, that the property "N_n is essential in N_k" was dualized to "N_n lies above N_k" (or in Takeuchi's words "N_n is a coessential extension of N_k").

2. One can easily see, that if a module satisfies Fleury's chain condition, it also satisfies Reiter's chain condition, because if N_n is small in M, than $N_n/N_k \ll M/N_k$ for every submodule N_k of N_n. (see 1.1.2)

Varadarajan proceeded in a more categorical way to dualize the Goldie dimension.

Definition. (**Varadarajan**, [62, Definition 1.8])
An R-module M has $corank(M) = k$ if there exists an epimorphism from M to a product of k non-zero factor modules, but there is no epimorphism from M to a product of $k+1$ non-zero factor modules.

Remarks:

1. If $corank(M) = k$ then by 1.4.6 there exists no coindependent family with more than k submodules. Hence a module with finite corank is cofinite-dimensional. We will show that the converse is also true.

2. Varadarajan defined the notion of *weak corank*: A module M has weak corank k if there is a small epimorphism from M to a direct sum of k non-zero, hollow

modules. Sarath and Varadarajan proved that an R-module M has finite corank if and only if there is a small epimorphism from M to a finite direct sum of hollow modules (see [53, Theorem 1.8]). This can be seen as the dual property of (U2).

2.3 A lattice theoretical approach

In [20], Grzeszczuk and Puczyłowski gave a lattice theoretical definition of the dual Goldie dimension. In this section we will state their results and give a dualized proof of their main theorem for Goldie dimension. Let us recall basic notions for lattices:

Definition. For a complete lattice $\mathcal{L} =< L; \vee, \wedge, 1, 0 >$ with $0 \neq 1$ we say:

- An element $a \in L$ with $a \neq 1$ is *small* in $\mathcal{L}$ if for any element $x \in L$ with $x \neq 1$, $a \vee x \neq 1$ holds.
- A lattice $\mathcal{L}$ is *hollow* if every element $a \in L \setminus \{1\}$ is small in $\mathcal{L}$.
- A subset I of $L \setminus \{1\}$ is *meet-independent* if for any finite subset X of I and $x \in I \setminus X$ we have $(\bigwedge X) \vee x = 1$.

These definitions correspond obviously to the definition of small submodules, hollow modules and coindependent families of submodules.

Remarks:

1. It is easy to see, that $\{a_i\}_{i \in \mathbb{N}}$ is a meet-independent set of elements of L if and only if for all $k > 1$ $(a_1 \wedge \cdots \wedge a_{k-1}) \vee a_k = 1$ holds (see also the characterization for coindependent submodules 1.4.2).
2. The set $\mathcal{M} := \{I \subseteq L |$ I is meet-independent $\}$ is partially ordered by set-theoretical inclusion. Moreover $\bigcup_{\lambda \in \Lambda} I_\lambda$ is again a meet-independent set for a chain $\{I_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{M}$, because we have to 'test' meet-independence only for finite subsets. Hence $\mathcal{M}$ has a maximal member by Zorn's lemma.

The next lemma was proved for submodules in [49, 3.1] and [24, 7.3].

Lemma 2.3.1. *Let $\mathcal{L}$ be a complete modular lattice that does not contain an infinite meet-independent set. Then for every element $1 \neq b \in L$ there exists an element $b \leq c \neq 1$ in L such that $[c, 1]$ is hollow.*

Proof: Assume that there is no element $b \leq c \neq 1$ in L such that $[c, 1]$ is hollow, then by induction we construct a sequence $c_1, c_2, \ldots$ of elements of $L \setminus \{1\}$ such that the set $\{c_1, c_2, \ldots\}$ is meet-independent and, for any k, $c_1 \wedge \cdots \wedge c_k$ is not small in $[b, 1]$. For $k = 1$ the construction is clear, since $[b, 1]$ is not hollow. Hence there exists an element $c_1 \geq b_1$ such that c_1 is not small in $[b, 1]$. Now let us assume that we have constructed elements $c_1, \ldots, c_{k-1}$. Since $c_1 \wedge \cdots \wedge c_{k-1}$ is not small in $[b, 1]$, there exists $b \leq d \neq 1$ such that $(c_1 \wedge \cdots \wedge c_{k-1}) \vee d = 1$. By assumption the lattice $[d, 1]$ is not hollow. Hence there exist $d \leq d_1, d_2 \neq 1$ with $d_1 \vee d_2 = 1$. Put $c_k := d_1$. Clearly $\{c_1, \cdots, c_k\}$ is meet-independent (see above remark (1)) and $c_1 \wedge \cdots \wedge c_k$ is not small in $[b, 1]$ as $(c_1 \wedge \cdots \wedge c_k) \vee d_2 = 1$ and $d_2 \neq 1$ (see 1.1.1). Thus we will get an infinite meet-independent set of elements of L. This contradicts our hypothesis. Thus there must exist an element $b \leq c \neq 1$ such that $[c, 1]$ is hollow. □

Note that the terminology N *lies above* K in M for submodules N and K of a module M is exactly the same as N is a *small element* in the lattice $[K, M]$.

The next lemma is the dual version of [20, Corollary 4].

Lemma 2.3.2. *Let $\mathcal{L}$ be a complete modular lattice with elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of L such that $\{b_1, \ldots, b_n\}$ is a meet-independent set and a_i is small in $[b_i, 1]$ for all i. Then $a_1 \wedge \cdots \wedge a_n$ is small in $[b_1 \wedge \cdots \wedge b_n, 1]$.*

Proof: The proof is the same as in 1.4.1(4). □

In [49, Lemma 3.5] Reiter proved the following result for modules.

Lemma 2.3.3. *Let $\mathcal{L}$ be a complete modular lattice. Assume there exists a meet-independent set $\{a_1, \ldots, a_n\}$ in L such that $[a_i, 1]$ is hollow for all i and $a_1 \wedge \cdots \wedge a_n$ is small in $\mathcal{L}$. Then an element $b \in L$ is small in $\mathcal{L}$ if and only if $a_i \vee b \neq 1$ holds for every $i \in \{1, \ldots, n\}$.*

Proof: The necessity is clear. Assume $a_i \vee b \neq 1$ for all $i \in \{1, \ldots, n\}$. Then $a_i \vee b$ is small in $[a_i, 1]$ as $[a_i, 1]$ is hollow. By Lemma 2.3.2 $(a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$ is small in $[a_1 \wedge \cdots \wedge a_n, 1]$. Since $a_1 \wedge \cdots \wedge a_n$ is small in $\mathcal{L}$, we get that $(a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$ is small in $\mathcal{L}$ (see also 1.1.2). Hence $b \leq (a_1 \vee b) \wedge \cdots \wedge (a_n \vee b)$ is small in $\mathcal{L}$. □

Now we are able to state a dualized proof of Grzeszczuk and Puczyłowski's main theorem.

2.3.4. Modular lattices with finite hollow dimension.

For a complete modular lattice $\mathcal{L}$ the following are equivalent:

(a) $\mathcal{L}$ does not contain infinite meet-independent sets.

(b) $\mathcal{L}$ contains a finite meet-independent set $\{a_1, \ldots, a_n\}$ such that $a_1 \wedge \cdots \wedge a_n$ is small in $\mathcal{L}$ and the lattices $[a_i, 1]$ are hollow for $1 \leq i \leq n$.

(c) $sup\{k|\mathcal{L}$ contains a meet-independent subset of cardinality $k\} = n < \infty$.

(d) For any descending chain $a_1 > a_2 > \cdots$ of elements of L there exists j such that for all $k \geq j$, a_j is small in $[a_k, 1]$.

Proof: (a) $\Rightarrow$ (b) As in above remark (2) the set

$$\mathcal{M}_h := \{I \in \mathcal{M} | \text{for all } a \in I : [a,1] \text{ is hollow}\} \subseteq \mathcal{M}$$

is partially ordered by set-inclusion where $\mathcal{M}$ is the set of all meet-independent subsets of L. Let $X \in \mathcal{M}_h$ be a maximal meet-independent subset of L such that the lattice $[x, 1]$ is hollow for all $x \in X$. By (a) X is finite, say $X = \{x_1, \ldots, x_n\}$. We claim that $x_1 \wedge \cdots \wedge x_n$ is small in $\mathcal{L}$. Assume that $(x_1 \wedge \cdots \wedge x_n) \vee a = 1$ for some $1 \neq a \in L$. By Lemma 2.3.1 there exists an element $1 \neq c \geq a$ such that the lattice $[c, 1]$ is hollow. Obviously the set $\{x_1, \ldots, x_n, c\}$ is meet-independent. This contradicts the maximality of X.

(b) $\Rightarrow$ (c). Assume that L contains a meet-independent set $\{b_1, \ldots, b_k\}$ with $k > n$. We show by induction that by rearranging $a_1, \ldots, a_n$, if necessary

(∗) for any $0 \leq j \leq n$ the set $\{a_1, \ldots, a_j, b_{j+1}, \ldots, b_k\}$ is meet-independent.

For $j = 0$ (∗) is clear. Now let $j > 0$ and $c := a_1 \wedge \cdots \wedge a_{j-1} \wedge b_{j+1} \wedge \cdots \wedge b_k$. As $c \vee b_j = 1$, c is not small in $\mathcal{L}$. By Lemma 2.3.3 $c \vee a_s = 1$ holds for some $1 \leq s \leq n$. Clearly $s \geq j$ otherwise $a_s = 1$. By sorting $\{a_j, \ldots, a_n\}$ we put $j = s$ and obtain that the set $\{a_1, \ldots, a_j, b_{j+1}, \ldots, b_n\}$ is meet-independent. Thus (∗) holds.
In particular (∗) implies that the set $\{a_1, \ldots, a_n, b_{n+1}, \ldots, b_k\}$ is meet-independent. This is impossible as $a_1 \wedge \ldots \wedge a_n$ is small in $\mathcal{L}$. Thus every meet-independent set of L has at most n elements.
(c) $\Rightarrow$(d). If (d) is not satisfied, then there exists a chain $1 \neq a_1 > a_2 > \cdots$ of elements of L such that for any $j \geq 1$ there exists a number $k(j) > j$ such that a_j is not small in $[a_{k(j)}, 1]$. Let $\{j_m\}_{m \in \mathbb{N}}$ be a sequence of indices defined as follows: $j_1 := 1$ and $j_m := k(j_{m-1})$ for all $m > 1$. By the foregoing there exist elements a'_{j_m} such that $a_{j_{m+1}} \leq a'_{j_m} \neq 1$ with $a_{j_m} \vee a'_{j_m} = 1$ for all m. Thus

$$(a_1 \wedge a'_1 \wedge \cdots \wedge a'_{j_{m-1}}) \vee a'_{j_m} \geq a_{j_m} \vee a'_{j_m} = 1$$

for all $m > 1$. Then by the above remark (1) we get that $\{a_1, a'_{j_1}, a'_{j_2}, \ldots, a'_{j_m}, \ldots\}$ is meet-independent. This contradicts (c).

(d) $\Rightarrow$ (a). If (a) is not satisfied, then L contains an infinite meet-independent set $\{a_1, a_2, \ldots\}$. Then $a_1 > a_1 \wedge a_2 > a_1 \wedge a_2 \wedge a_3 > \cdots$ and for any $k \in \mathbb{N}$, $(a_1 \wedge \cdots \wedge a_k) \vee a_{k+1} = 1$ implies that $a_1 \wedge \cdots \wedge a_k$ is not small in $[a_1 \wedge \cdots \wedge a_l, 1]$ for all $l > k$. This contradicts (d). $\square$

Remarks: Looking at the proof it is obvious that the numbers n from (b) and from (c) must be the same and unique.

Let $\mathcal{L} =< L; \vee, \wedge, 0, 1 >$ be a complete modular lattice with $0 \neq 1$. The dual lattice $\mathcal{L}^0 =< L; \wedge, \vee, 1, 0 >$ is modular as well. By the Duality Principle we know, that a lattice has a property if and only if the dual lattice has the dual property. Exchanging $\vee$ and $\wedge$ we get dual definitions and a dual theorem:

Definition. Let $\mathcal{L}$ be a lattice:

- An element $a \in L \setminus \{0\}$ is *essential* in $\mathcal{L}$ if for any element $x \in L \setminus \{0\}$, $a \wedge x \neq 0$.
- A lattice is *uniform* if every element $a \in L \setminus \{0\}$ is essential in $\mathcal{L}$.
- A subset I of $L \setminus \{0\}$ is *join-independent* if for any finite subset X of I and $x \in I \setminus X$ we have $(\bigvee X) \wedge x = 0$.

2.3.5. Modular lattices with finite uniform dimension.
For a complete modular lattice $\mathcal{L}$ the following are equivalent:

(a) $\mathcal{L}$ does not contain infinite join-independent sets.

(b) $\mathcal{L}$ contains a finite join-independent set $\{a_1, \ldots, a_n\}$ such that $a_1 \vee \cdots \vee a_n$ is essential in $\mathcal{L}$ and the lattices $[0, a_i]$ are uniform for $1 \leq i \leq n$.

(c) sup$\{k | \mathcal{L}$ contains a join-independent subset of cardinality equal to $k\} = n < \infty$

(d) For any ascending chain $a_1 < a_2 < \cdots$ of elements of L there exists j such that for all $k \geq j$, a_j is essential in $[0, a_k]$.

Let $\mathcal{L}$ be the lattice of submodules of a module. Then the above theorem is a well-known characterization of modules having finite Goldie dimension. Hence it is convenient to define the Goldie and dual Goldie dimension of a modular lattice.

Definition. If $\mathcal{L}$ satisfies one of the equivalent conditions (a)-(d) of Theorem 2.3.5, then the *Goldie dimension* of a modular lattice $udim(\mathcal{L})$ of L is equal to n. If $\mathcal{L}$ does not satisfy the conditions, we put $udim(\mathcal{L}) = \infty$.

Definition. If $\mathcal{L}$ satisfies one of the equivalent conditions (a)-(d) of Theorem 2.3.4, then *the dual Goldie dimension* $hdim(\mathcal{L})$ of L is equal to n. If $\mathcal{L}$ does not satisfy these conditions, we put $hdim(\mathcal{L}) = \infty$. Obviously we have $hdim(\mathcal{L}) = udim(\mathcal{L}^0)$ and $udim(\mathcal{L}) = hdim(\mathcal{L}^0)$.

Chapter 3

Hollow dimension

3.1 Finite hollow dimension

Since the lattice $\mathcal{L}(M)$ of all submodules of a module M is complete and modular, we can apply the results from Chapter 2.3 to the lattice of submodules of a module.

One can easily see that the notions of essential (small) submodules, uniform (hollow) modules and independent (coindependent) families of submodules match with the notions of essential (small) elements, uniform (hollow) lattices and join-independent (meet-independent) sets of sublattices.

By 2.3.5 we get the following well known result:

3.1.1. Modules with finite uniform dimension.
For a non-zero module M the following are equivalent:

(a) M does not contain an infinite independent set of submodules.

(b) M contains a finite independent set of submodules $\{N_1, \ldots, N_n\}$ such that $\bigoplus_{i=1}^{n} N_i \trianglelefteq M$ and N_i is a uniform submodule for every $1 \leq i \leq n$.

(c) sup$\{k|M$ contains an independent family of submodules of cardinality $k\} = n < \infty$.

(d) For any ascending chain $N_1 \subset N_2 \subset \cdots$ of submodules of M there exists j such that for all $k \geq j$, $N_j \trianglelefteq N_k$.

Definition. An R-module M is said to have *finite uniform dimension* if it satisfies one of the conditions in 3.1.1. Let $udim(M)$ denote the number n from 3.1.1.

Note that if N is a submodule of M, then the sublattice $[N, M]$ of the lattice $\mathcal{L}(M)$ is isomorphic to $\mathcal{L}(M/N)$. Now we can apply 2.3.4.

3.1.2. Modules with finite hollow dimension.
For a non-zero module M the following are equivalent:

(a) M does not contain an infinite coindependent family of submodules.

(b) M contains a finite coindependent family of submodules $\{N_1, \ldots, N_n\}$ such that $\bigcap_{i=i}^n N_i$ is small in M and M/N_i is a hollow module for every $1 \leq i \leq n$.

(c) sup$\{k|M$ contains a coindependent family of submodules of cardinality equal to $k\} = n < \infty$.

(d) For any descending chain $N_1 \supset N_2 \supset \cdots$ of submodules of M there exists j such that for all $k \geq j$, N_j lies above N_k in M.

(e) There exists a small epimorphism from M to a finite direct sum of n hollow factor modules.

Proof: $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$ follow by 2.3.4. $(b) \Leftrightarrow (e)$ follows by the Chinese Remainder Theorem 1.4.3. □

Definition. An R-module M is said to have *finite hollow dimension* if it satisfies one of the conditions in 3.1.2. Let $hdim(M)$ denote the number n from 3.1.2. If $M = 0$ we write $hdim(M) = 0$ and if M does not have finite hollow dimension we write $hdim(M) = \infty$.

Remarks:

1. Obviously every artinian module has finite hollow dimension. A module is hollow if and only if it has hollow dimension 1.

2. In 3.1.2 (a) corresponds to Takeuchi's definition and (d) to Reiter's Theorem 2.2.1. Applying the Chinese Remainder Theorem 1.4.1, we see that (c) states, that there cannot be an epimorphism from M to a finite direct sum of more then n summands. Hence condition (c) is equivalent to Varadarajan's definition of corank. The equivalence between Varadarajan's corank condition and (e) was proved in [53, Theorem 1.8].

3. Since modules with finite spanning dimension satisfy the chain condition (d) we get that these modules have finite hollow dimension.

4. In [17] Golan and Wu pointed out that since the lattice of subobjects of an object in a Grothendieck category is also a modular lattice, one can define the Goldie and dual Goldie dimension of such objects using Grzeszczuk and Puczyłowski's definition.

5. Using the same arguments Page in [44] as well as Park and Rim in [45] defined the dual Goldie dimension relative to a torsion theory.

The following results are analogue to chapter 5 in [10]. Let us consider a technical, but useful lemma first.

Lemma 3.1.3. *([50, Theorem 5]) Let M be an R-module. Assume M has a proper ascending chain of submodules $0 =: N_0 \subset N_1 \subset N_2 \subset N_3 \subset \cdots$, such that for all $k \geq 1$, N_k does not lie above N_{k-1} in M. Then M contains an infinite coindependent family of submodules.*

Proof: By assumption N_k/N_{k-1} is not small in M/N_{k-1} for every $k \geq 1$. For every $k \geq 1$ there is a proper submodule L_k of M such that $N_{k-1} \subset L_k$ and $L_k + N_k = M$ holds.

Claim: $L_k = N_{k-1} + (L_1 \cap \cdots \cap L_k)$ holds for all $k \geq 1$.

We will prove this by induction on k:
for $k = 1$ this is clear;
$k \to k+1$: $M = N_k + L_k$ implies

$$\begin{aligned} L_{k+1} &= N_k + (L_{k+1} \cap L_k) \\ &= N_k + L_{k+1} \cap (N_{k-1} + (L_1 \cap \cdots \cap L_k)) \\ &= N_k + N_{k-1} + (L_1 \cap \cdots \cap L_{k+1}) \\ &= N_k + (L_1 \cap \cdots \cap L_{k+1}). \end{aligned}$$

Thus for every $k > 1$ we get

$$M = N_{k-1} + L_{k-1} = N_{k-1} + N_{k-2} + (L_1 \cap \cdots \cap L_{k-1}) \subseteq L_k + (L_1 \cap \cdots \cap L_{k-1}) \subseteq M.$$

Hence by 1.4.2 $\{L_i\}_{\mathbb{N}}$ is an infinite coindependent family of proper submodules. □

3.1.4. Modules with hollow factor modules. *([49, 3.1], [45, 11])*
Let M be a non-zero R-module such that every coindependent family of submodules is finite. Then M has a hollow factor module.

Proof: The proof is the same as in 2.3.1. On the other hand Lemma 3.1.3 allows us to prove it quickly: Assume M is not hollow and has no hollow factor module. Then we can construct an ascending chain of proper submodules $N_1 \subset N_2 \subset N_3 \subset \dots$ such that for no $k \geq 1$, N_k lies above N_{k-1} as follows: for each $k \in \mathbb{N}$, M/N_k is not hollow and there exists a submodule $N_{k+1}/N_k \not\ll M/N_k$. By 3.1.3 M has an infinite coindependent family of submodules. This contradiction shows, that M must have a hollow factor module. □

Remarks: With the same argument as in the proof of 3.1.4 we get that every non-zero factor module M/N has a hollow factor module.

Definition. An R-module M is called *conoetherian* if every finitely cogenerated module in $\sigma[M]$ is artinian (see [67, 31.6]).

Corollary 3.1.5.

1. *Any non-zero artinian module has a hollow factor module.*
2. *Let M be a locally artinian module. Then any non-zero module in $\sigma[M]$ has a hollow subfactor.*
3. *Let M be a conoetherian module. Then any non-zero module in $\sigma[M]$ has a hollow factor module.*
4. *Let R be a left conoetherian ring, then any non-zero R-module has a hollow factor module.*

Proof: (1) Clear by 3.1.2 and 3.1.4;
(2) any finitely generated module in $\sigma[M]$ is artinian. Thus any non-zero cyclic submodule of a module in $\sigma[M]$ has a hollow factor module.
(3) Every module $N \in \sigma[M]$ has a non-zero finitely cogenerated factor module L. By hypothesis L is artinian and by (1) it has a hollow factor module.
(4) Set $M := R$ and apply (3). □

3.1.6. Small submodules and hollow factor modules.
Let M be a non-zero R-module such that every non-zero factor module has a hollow factor module. Then M contains a coindependent family $\{K_\lambda\}_\Lambda$ of submodules such that M/K_λ is hollow for every $\lambda \in \Lambda$ and $\bigcap_\Lambda K_\lambda$ is small in M.

Proof: Let $\mathcal{M}$ denote the set consisting of all non-empty coindependent families of submodules K of M with M/K hollow, i.e.

$$\mathcal{M} = \{\{K_\omega\}_\Omega | \{K_\omega\}_\Omega \text{ coindependent, } \Omega \neq \emptyset,\ M/K_\omega \text{ hollow for all } \omega \in \Omega\}.$$

$\mathcal{M}$ is partially ordered by set-theoretical inclusion: $\{K_\omega\}_\Omega \subseteq \{L_\lambda\}_\Lambda$ if for every $\omega \in \Omega$ there is a $\lambda \in \Lambda$ such that $K_\omega = L_\lambda$. Let

$$\{K_{\omega_1}\}_{\Omega_1} \subset \{K_{\omega_2}\}_{\Omega_2} \subset \{K_{\omega_3}\}_{\Omega_3} \subset \cdots$$

be a chain of elements of $\mathcal{M}$. Then we have to show, that

$$\mathcal{U} = \bigcup_{i\in I} \{K_{\omega_i}\}_{\Omega_i} = \{K_{\omega_i}\}_\Omega \text{ where } \Omega := \bigcup_{i\in I} \Omega_i$$

is a coindependent family of submodules. Consider submodule $K_{i_1} \in \mathcal{U}$ and a finite number of submodules $\{K_{i_2}, \cdots, K_{i_n}\} \subset \mathcal{U} \setminus \{K_{i_1}\}$. Then there must be an element $\{K_{\omega_i}\}_{\Omega_i}$ such that $\{K_{i_1}, \dots, K_{i_n}\} \subseteq \{K_{\omega_i}\}_{\Omega_i}$. Since $\{K_{\omega_i}\}_{\Omega_i}$ is coindependent, $K_{i_1} + (K_{i_2} \cap \cdots \cap K_{i_n}) = M$ holds.

Hence we can apply Zorn's Lemma. So $\mathcal{M}$ has a maximal member $\{K_\lambda\}_\Lambda$ that is a coindependent family of proper submodules, such that M/K_λ is hollow for every $\lambda \in \Lambda$. Let $K = \bigcap_\Lambda K_\lambda$ denote the intersection of this family. If K is not small in M, then there is a proper submodule L of M such that $K + L = M$. By hypothesis M/L has a hollow factor module M/N. So $L \subseteq N$ and $K + N = M$ holds. By 1.4.1(2) $\{K_\lambda\}_\Lambda \cup \{N\}$ is a coindependent family and hence it is an element of $\mathcal{M}$. But this is a contradiction to the maximality of $\{K_\lambda\}_\Lambda$. Hence K must be small. □

Corollary 3.1.7. *Let M be a conoetherian module, then every non-zero module $N \in \sigma[M]$ contains a maximal coindependent family $\{K_\lambda\}_\Lambda$ of submodules such that N/K_λ is hollow for every $\lambda \in \Lambda$ and $\bigcap_\Lambda K_\lambda$ is small in N.*

Proof: By 3.1.5 and 3.1.6. □

Together with 3.1.4 and 3.1.6 we are able to prove 3.1.2$(a) \Leftrightarrow (e)$ without using the lattice-theoretical result 2.3.4.

3.1.8. Finiteness condition and hollow modules. *([45, 12])*
Let M be a non-zero module that contains no infinite coindependent family of proper submodules. Then there is a small epimorphism from M to a finite direct sum of hollow modules.

Proof: By 3.1.4, 3.1.6 and the Chinese Remainder Theorem 1.4.6. □

The next theorem restates 2.3.3 for submodules of a module.

3.1.9. Small submodules and hollow factor modules. *([49, 3.5])*
Let M be an R-module, N a submodule of M and $f : M \to \bigoplus_{i=1}^{n} H_i$ a small epimorphism, with $H_i \simeq M/K_i$ hollow factor modules of M and K_i submodules of M. Then

$$N \text{ is small in } M \text{ if and only if } N + K_i \neq M \text{ for every } 1 \leq i \leq n.$$

Proof: The proof is the same as in 2.3.3. □

Let $N \subset M$ and $\pi : M \to M/N$ be the canonical projection. Let $g : M/N \to \bigoplus_{i=1}^{k} N_i$ be an epimorphism with $N_i \neq 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{\pi} & M/N & \longrightarrow & 0 \\ & & & & & & {\scriptstyle g}\big\downarrow & & \\ & & & & & & \bigoplus_{i=1}^{k} N_i & & \end{array}$$

Then there exists an epimorphism from M to a direct sum of k non-zero modules. Hence $hdim(M) \geq hdim(M/N)$. This shows that the hollow dimension of a factor module M/N is always smaller than the hollow dimension of M. If $hdim(M/N) = \infty$ then $hdim(M) = \infty$. Assume $hdim(M/N) = k$ and $N \ll M$. Then Ker $(g) \ll M/N$ holds and πg is a small epimorphism. Hence $hdim(M) = k = hdim(M/N)$. Thus $hdim(M) = hdim(M/N)$ whenever $N \ll M$.

3.1.10. Finite hollow dimension.
Let N and K be submodules of an R-module M.

1. *If $M = M_1 \oplus \cdots \oplus M_k$, then $hdim(M) = hdim(M_1) + \cdots + hdim(M_k)$.*

2. *If $N \ll M$, then $hdim(M) = hdim(M/N)$.*
   Conversely, if M has finite hollow dimension and $hdim(M) = hdim(M/N)$, then $N \ll M$.

3. *Assume N is a weak supplement of K in M. Then $hdim(M) = hdim(M/N) + hdim(M/K)$ holds.*

4. *Any module with finite hollow dimension is weakly supplemented.*

5. *Assume both N and M/N have finite hollow dimension. Then M has finite hollow dimension.*

6. *Assume the following sequence is exact:*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0.$$

Then the following holds: $hdim(L) \leq hdim(M) \leq hdim(L) + hdim(N)$.

7. *Assume M has finite hollow dimension, then any epimorphism $f : M \to M$ is small. If M is self-projective, then f is an isomorphism.*

Proof: (1) $hdim(M) \geq hdim(M_i)$ holds by above remark. Thus if $hdim(M_i) = \infty$ for any direct summand M_i, then $hdim(M) = \infty$. Assume that for all $i \in \{1, \dots, k\}$ $hdim(M_i) = n_i < \infty$ and there exists a small epimorphisms $f_i : M_i \to \bigoplus_{j=1}^{n_i} H_{ij}$ with H_{ij} hollow for all $1 \leq j \leq n_i$. Then we get a small epimorphism $f = (f_1, \dots, f_k)$

$$M \xrightarrow{\ f\ } \bigoplus_{i=1}^{k} \left(\bigoplus_{j=1}^{n_i} H_{ij} \right) \longrightarrow 0.$$

Thus $hdim(M) = hdim(M_1) + \cdots + hdim(M_k)$.

(2) clear by above remark. Assume $hdim(M) = n < \infty$, $f : M \to \bigoplus_{i=1}^{n} M/K_i$ a small epimorphism and $N \not\ll M$. Then by 3.1.9 there exists an index i such that $N + K_i = M$. Thus $M/(N \cap K_i) \simeq M/N \oplus M/K_i$. By (1) $hdim(M) \geq hdim(M/N) + hdim(M/K_i) > hdim(M/N)$ holds but this is a contradiction to $hdim(M) = hdim(M/N)$. Hence $N \ll M$.

(3) By assumption, $K + N = M$ and $K \cap N \ll M$ yields:

$$\begin{aligned} hdim(M) &= hdim(M/(K \cap N)), \text{ by (2)} \\ &= hdim(K/(K \cap N) \oplus N/(K \cap N)) \\ &= hdim(K/(K \cap N)) + hdim(N/(K \cap N)), \text{ by (1)} \\ &= hdim(M/N) + hdim(M/K), \text{ by (2).} \end{aligned}$$

(4) By 3.1.8 M is a small cover of a finite direct sum of hollow modules. Since hollow modules are (weakly) supplemented, M is weakly supplemented by 1.3.2.

(5) Suppose, to the contrary, that M does not have finite hollow dimension and let $\{K_i\}_{\mathbb{N}}$ be an infinite coindependent family of submodules of M. Then let

$$L_1 = K_1, L_2 = K_2 \cap K_3, \cdots, L_n = K_{t+1} \cap \cdots \cap K_{t+n},$$

with integer $t = n(n+1)/2$. For every $n \in \mathbb{N}$ we have $M/L_n \simeq \bigoplus_{i=1}^{n} M/K_{t+i}$ as $\{K_{t+1}/L_n, \dots, K_{t+n}/L_n\}$ is coindependent. Hence $n \leq hdim(M/L_n)$. $\{L_i\}_{\mathbb{N}}$ is again an infinite coindependent family of submodules of M (see 1.4.2). Since $hdim(M/N)$ is finite $\{N + L_i\}_{\mathbb{N}}$ is not coindependent and so $N + L_n = M$ for almost all n. Choose n such that $n > hdim(N)$ and $N + L_n = M$. Then $M/L_n \simeq N/(N \cap L_n)$ is a factor module of N. Thus $n \leq hdim(M/L_n) \leq hdim(N) < n$ yields a contradiction. Hence M cannot contain an infinite coindependent family of proper submodules. Thus it has finite hollow dimension.

(6) Clearly $hdim(L) \leq hdim(M)$ is always true for a factor module L of M by above reamrk and if $hdim(M)$ is not finite, then the equation is clear by (5). Let

M have finite hollow dimension, then every submodule N has a weak supplement K. By (3) we get:

$$hdim(M) = hdim(M/N) + hdim(M/K) \leq hdim(L) + hdim(N),$$

since $M/N \simeq L$ and $M/K \simeq N/(N \cap K)$.

(7) Since M has finite hollow dimension and $hdim(M) = hdim(\mathrm{Im}\,(f)) = hdim(M/\mathrm{Ker}\,(f))$ we get by applying (1) that $\mathrm{Ker}\,(f) \ll M$. If M is self-projective, then $\mathrm{Ker}\,(f)$ is a direct summand and hence 0. □

Remarks:

1. The properties (1), (2) and (5) appeared in various papers: e.g. (1) [38, 5.13], (2) and (5) [23] and [45]. For (3) see [49, Theorem 4.1]; for (4) [24].

2. Let $M = \sum_\Lambda M_\lambda$ with $M_\lambda \neq 0$ for all $\lambda \in \Lambda$ and consider

$$\bigoplus_\Lambda M_\lambda \xrightarrow{\ f\ } M \longrightarrow 0$$

with $(\{m_\lambda\}_\Lambda)f = \sum_\Lambda m_\lambda$. By above remark we have $hdim(M) \leq hdim(\bigoplus_\Lambda M_\lambda)$. If $|\Lambda| = \infty$ then clearly $hdim(\bigoplus_\Lambda M_\lambda) = \infty = \sum_\Lambda hdim(M_\lambda)$. If $|\Lambda| < \infty$ then by (1) we get $hdim(\bigoplus_\Lambda M_\lambda) = \sum_\Lambda hdim(M_\lambda)$. Thus we get $hdim(M) \leq \sum_\Lambda hdim(M_\lambda)$.

3. If $hdim(M) = n$ is finite, $\{K_1, \ldots, K_n\}$ a maximal coindependent family and N a small submodule of M. Then we get by 3.1.9 that $L_i := N + K_i$ is a proper submodule of M for all $1 \leq i \leq n$. Hence $\{L_1, \ldots, L_n\}$ is a maximal coindependent family in M and $N \subseteq L_1 \cap \cdots \cap L_n$ holds. Thus every small submodule N of a module M with finite hollow dimension is contained in an intersection $L_1 \cap \cdots \cap L_n \ll M$ such that $\{L_1, \ldots, L_n\}$ form a coindependent family of submodules in M (see [49]).

A further characterization of a module M with finite uniform dimension is that the M-injective hull $\widehat{M}$ of M is isomorphic to a finite direct sum of uniform modules and the endomorphism ring of $\widehat{M}$ is semiperfect. As an analogue we get the following result:

3.1.11. Projective covers with finite hollow dimension.
Let M be an R-module with a projective cover P in $\sigma[M]$. Then the following statements are equivalent.

(a) M has finite hollow dimension and is semiperfect in $\sigma[M]$;

(b) $P = \bigoplus_{i=1}^{n} L_i$ with L_i non-zero local modules;

(c) End(P) is semiperfect;

Proof: $(a) \Rightarrow (b)$ Assume that M has finite hollow dimension and consider the following diagram:

$$\begin{array}{ccccc} P & \xrightarrow{\simeq} & \bigoplus_{i=1}^{n} P_i & & \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{f} & \bigoplus_{i=1}^{n} H_i & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

where f is a small epimorphism to a finite direct sum of hollow modules H_i. Since M is semiperfect, there exist projective covers P_i in $\sigma[M]$ for each H_i that are hollow and by [67, 19.7] local. By [67, 19.5] $\bigoplus_{i=1}^{n} P_i$ forms a projective cover for $\bigoplus_{i=1}^{n} H_i$ in $\sigma[M]$ and $P \simeq \bigoplus_{i=1}^{n} P_i$. Each P_i is isomorphic to a direct summand L_i of P. Thus $P = \bigoplus_{i=1}^{n} L_i$.

$(b) \Rightarrow (a)$ By [67, 42.3(3)] M is semiperfect in $\sigma[M]$ and by 3.1.10 $hdim(M) = hdim(P) < \infty$.

$(b) \Leftrightarrow (c)$ By [67, 42.4(1)] P is equal to a finite direct sum of local modules (projective covers of simple modules) if and only if P is finitely generated and semiperfect in $\sigma[M]$. Since P is finitely generated and self-projective it is projective in $\sigma[P]$ by [67, 18.3] and hence semiperfect in $\sigma[P]$. Thus by applying [67, 42.12]: P is finitely generated and semiperfect in $\sigma[P]$ if and only if End$\,(P)$ is semiperfect. $\square$

The next result is due to Page [44] and shows the duality between hollow and uniform dimension. For that we have to introduce some notation of *annihilator conditions in M* and Hom$\,(M, Q)$ for an *injective cogenerator* Q in $\sigma[M]$.

Assume ${}_RM$ to be an R-module, ${}_RQ$ to be an injective cogenerator in $\sigma[M]$. Let $T := \mathrm{End}_R\,(Q)$, $N \in \sigma[M]$ and $N^* := \mathrm{Hom}_R\,(N, Q)_T$ a right T-module. Define for any R-submodule $K \subseteq N$ and T-submodule $X \subseteq N^*$:

$$An(K) := \{f \in N^* | (K)f = 0\} \subseteq N^*,$$

$$Ke(X) := \bigcap \{\mathrm{Ker}\,(g) | g \in X\} \subseteq N$$

By definition $An(K_1 + K_2) = An(K_1) \cap An(K_2)$ holds for all $K_1, K_2 \subseteq N$.

By [67, 28.1] the following conditions hold since ${}_RQ$ is an injective cogenerator in $\sigma[M]$:

(AC1) $Ke(An(K)) = K$ for all $K \subseteq N$;

(AC2) $An(Ke(X)) = X$ for every finitely generated T-submodule $X \subseteq N^*$;

(AC3) $An(K_1 \cap K_2) = An(K_1) + An(K_2)$ for all $K_1, K_2 \subseteq N$.

3.1.12. Hollow dimension and duality. *([44, Proposition 1])*
Let M be an R-module and ${}_RQ$ an injective cogenerator in $\sigma[M]$, $T :=$ End $({}_RQ)$. For any module $N \in \sigma[M]$ set $N^* :=$ Hom $(N, Q)_T$. Then $hdim({}_RN) = udim(N^*_T)$ holds.

Proof: Assume N admits the following exact sequence, with H_i non-zero factor modules of N:

$$N \longrightarrow \bigoplus_{i=1}^k H_i \longrightarrow 0.$$

Since Q is N-injective, $\mathrm{Hom}_R(-, Q)$ is exact in $\sigma[M]$ (see [67, 16.3]) and by applying this functor we get the exact sequence:

$$0 \longrightarrow \bigoplus_{i=1}^k \mathrm{Hom}\,(H_i, Q) \longrightarrow N^*$$

where all Hom (H_i, Q) are non-zero submodules of N^*, since the H_i were non-zero and Q a cogenerator in $\sigma[M]$. Hence N^* contains a direct sum of k submodules. Thus $hdim({}_RN) \leq udim(N^*_T)$.
On the other hand, assume that N^* contains a submodule X which is a direct sum of k non-zero submodules. Without loss of generality suppose this sum is a sum of cyclic submodules, so take $X = f_1T \oplus \cdots \oplus f_kT$ with $0 \neq f_i \in N^*$. Obviously $Ke(f_iT) =$ Ker (f_i) is a proper submodule of N for every $1 \leq i \leq k$.
Next we will show, that $\{$Ker $(f_1), \ldots,$ Ker $(f_k)\}$ is a coindependent family of proper

submodules of N. Applying $(AC1)-(AC3)$ we get for all $1 \leq i \leq k$ the following:

$$\begin{aligned}
0 &= f_iT \cap \sum_{j\neq i} f_jT \\
&= An(Ke(f_iT)) \cap An(Ke(\sum_{j\neq i} f_jT)) \text{ ,by } (AC2) \\
&= An(\text{Ker}\,(f_i) + Ke(\sum_{j\neq i} f_jT)) \\
&= An(\text{Ker}\,(f_i) + Ke(\sum_{j\neq i} An(\text{Ker}\,(f_j)))) \text{ ,by } (AC2) \\
&= An(\text{Ker}\,(f_i) + Ke(An(\bigcap_{j\neq i} \text{Ker}\,(f_j)))) \text{ ,by } (AC3) \\
&= An(\text{Ker}\,(f_i) + \bigcap_{j\neq i} \text{Ker}\,(f_j)) \text{ ,by } (AC1)
\end{aligned}$$

Applying $(AC1)$ yields

$$N = Ke(0) = Ke(An(\text{Ker}\,(f_i) + \bigcap_{j\neq i} \text{Ker}\,(f_j))) = \text{Ker}\,(f_i) + \bigcap_{j\neq i} \text{Ker}\,(f_j).$$

Hence $\{\text{Ker}\,(f_1), \ldots, \text{Ker}\,(f_k)\}$ is coindependent. Thus $udim(N_T^*) \leq hdim({}_RN)$. □

Remarks: Since there exists always an injective cogenerator ${}_RQ$ in $\sigma[M]$ we are able to express the hollow dimension of a module $N \in \sigma[M]$ in terms of uniform dimension.

Denote by $\sigma_f[M]$ the full subcategory of $\sigma[M]$ whose objects are submodules of finitely M-generated modules. Note that $\sigma_f[R]$ just consists of submodules of finitely generated R-modules. For the definition and characterization of dualities we refer to [67, Chapter 47]. Page's result gives us the following corollary.

Corollary 3.1.13. *Let U be a left R-module and $S := \text{End}\,({}_RU)$ and assume*

$$\text{Hom}_R\,(-,U) : \sigma_f[{}_RU] \to \sigma_f[S_S]$$

to be a duality. Then for all $N \in \sigma_f[{}_RU]$ the following hold:

$$hdim(N) = udim(N^*) \text{ and } udim(N) = hdim(N^*).$$

Remarks: Since $\text{Hom}_R\,(-,U)$ is a duality between $\sigma_f[{}_RU]$ and $\sigma_f[S_S]$ every module in $\sigma_f[{}_RU]$ is linearly compact (see [67, 47.3]). Hence every module in $\sigma_f[{}_RU]$ has finite uniform dimension, finite hollow dimension and a semilocal endomorphism ring as we will see in Section 3.5.

3.2 Dimension formulas

In [7] Camillo and Zelmanowitz have pointed out that the Goldie dimension does not satisfy the familiar formulas for vector space dimension:

(1) $dim(M) = dim(M/N) + dim(N)$;

(2) $dim(N + L) = dim(N) + dim(L) - dim(N \cap L)$;

for subspaces $N, L \subseteq M$, and have found the corrections required (see [7, Lemma 3 and Theorem 4]):

(1) If N is essential in L and L a complement in M, then

$$udim(M) = udim(M/N) + udim(N) - udim(L/N)$$

(2) If N and L are submodules of M, f a maximal monic extension of the identity map $1_{N\cap L}$ considered as a homomorphism from N to L, and $K = Domain(f)$, then

$$udim(N + L) = udim(N) + udim(L) - udim(K) + udim(K/(N \cap L))$$

These formulas are called the first and second *Camillo-Zelmanowitz formulas*. In [22] Haack showed, that the duals of the *Camillo-Zelmanowitz* formulas hold for hollow dimension if there are enough supplements.

3.2.1. First dual Camillo-Zelmanowitz formula.*([22, Theorem 5])*
Let M be an R-module and N and L submodules of M. If N lies above a supplement L in M then $hdim(M) = hdim(M/N) + hdim(N) - hdim(N/L)$.

Proof: Assume L is a supplement of a submodule K of M and N lies above L. Then $N \cap K$ lies above $L \cap K$ by 1.1.2 and since $L \cap K \ll M$ we get $N \cap K \ll M$ by 1.1.2. Hence N is a weak supplement of K in M. By 3.1.10(3)

$$hdim(M) = hdim(M/N) + hdim(M/K).$$

Further $N \cap K$ is a weak supplement of L in N since by modularity

$$N = N \cap (L + K) = L + (N \cap K)$$

and $(N \cap K) \cap L = L \cap K \ll L \subseteq N$ holds. Applying 3.1.10(3) again, we get

$$hdim(N) = hdim(N/(N \cap K)) + hdim(N/L) = hdim(M/K) + hdim(N/L).$$

Subtracting these two dimension formulas we get the result:

$$hdim(M) = hdim(M/N) + hdim(N) - hdim(N/L).$$

□

Remarks:

1. Haack's original assumption on the submodule N were: N has a weak supplement K in M such that there exists a supplement $L \subset N$ of K in M. From this follows, that N lies above L, because whenever $N + X = M$ holds for a proper submodule $X \subset M$, then $M = N + X = L + (N \cap K) + X = L + X$ is satisfied since $N \cap K \ll M$. Thus by 1.1.2 N lies above L in M. On the other hand assume that N lies above a supplement L of a submodule K. Clearly $N + K = M$ holds and by 1.1.2 $N \cap K$ lies above $L \cap K$ and since $L \cap K$ is small in M this implies $N \cap K \ll M$. Hence N is a weak supplement of K.

2. If M is amply supplemented then every submodule N of M lies above a supplement L (see 1.2.2). Hence the formula in 3.2.1 holds for every submodule N of M (independent from the supplement L).

Corollary 3.2.2. *([23, 7.8], [45, Lemma 19]) Let M be an R-module and N a supplement in M, then*

$$hdim(M) = hdim(M/N) + hdim(N).$$

Corollary 3.2.3. *Let M be an R-module with finite hollow dimension and N a submodule of M. Then the following holds:*

$$N \textit{ is a supplement in } M \Leftrightarrow hdim(M) = hdim(M/N) + hdim(N).$$

Proof: If N is a supplement, then the formula holds by the previous corollary. Assume that the above formula holds for a submodule N of M. Since M has finite hollow dimension N has a weak supplement K, by 3.1.10(4), such that by applying 3.1.10(3)

$$hdim(M) = hdim(M/N) + hdim(M/K).$$

Thus $hdim(N) = hdim(M/K) = hdim(N/(N \cap K))$ holds and $hdim(N)$ is finite. Applying 3.1.10 we get $N \cap K \ll N$, but this means N is a supplement of K in M.
□

Let *lg(M)* denote the length of a module M.

Corollary 3.2.4. *Let M be an R-module then the following statements are equivalent:*

(a) M is semisimple;

(b) $hdim(M) = hdim(M/N) + hdim(N)$ holds for every $N \subseteq M$ and M is weakly supplemented;

(c) $udim(M) = udim(M/N) + udim(N)$ holds for every $N \subseteq M$.

In this case $hdim(M) = udim(M) = lg(M)$.

Proof: $(a) \Rightarrow (b), (c)$ Obvious, since every submodule is a direct summand and the dimension notions *hdim* and *udim* are additive with respect to decompositions.
$(b) \Rightarrow (a)$ For every small submodule K of M, $hdim(M/K) = hdim(M)$ holds by 3.1.10 and implies $hdim(K) = 0$. Hence $K = 0$ and so $Rad(M) = 0$. Since M is weakly supplemented, it is semisimple by 1.3.3.
$(c) \Rightarrow (a)$ For every essential submodule K of M, $udim(K) = udim(M)$ holds and implies $udim(M/K) = 0$. Hence $K = M$ and $Soc(M) = M$.

In the case that M is semisimple, then $M = \bigoplus_{\Lambda} E_{\lambda}$ with E_{λ} simple. Hence $|\Lambda| = lg(M) = udim(M) = hdim(M)$ holds. □

A supplemented module with finite hollow dimension can be written as an irredundant sum of hollow submodules. This was first shown by Fleury in [13] and also by Varadarajan in [62].

Definition. A sum $M = \sum_{\Lambda} M_{\lambda}$ of non-zero modules M_{λ} is called *irredundant* if for all $\lambda \in \Lambda : \sum_{\mu \neq \lambda} M_{\mu} \neq M$.

The next theorem was obtained from several papers (see [23, Theorem 7.10], [20, Theorem 14], [50, Lemma 1]).

3.2.5. Supplemented modules with finite hollow dimension.
Let M be an R-module.

1. *If $M = \sum_{i=1}^{n} H_i$ is an irredundant sum of hollow modules. Then $hdim(M) = n$.*

2. *If M is supplemented and $hdim(M) = n$. Then there are hollow submodules H_i of M such that $M = \sum_{i=1}^{n} H_i$ is an irredundant sum.*

Proof: (1) Consider the following epimorphism

$$f : H_1 \oplus \cdots \oplus H_n \to M$$

$$(h_1, \ldots, h_n) \mapsto h_1 + \cdots + h_n.$$

Then Ker $(f) = K_1 \oplus \cdots \oplus K_n$ with $K_i := H_i \cap (H_1 + \cdots + H_{i-1} + H_{i+1} + \cdots + H_n)$. Since $K_i \ll H_i$ as the given sum was irredundant and H_i hollow, we get that Ker $(f) \ll H_1 \oplus \cdots \oplus H_n$. Thus $hdim(M) = hdim(H_1 \oplus \cdots \oplus H_n) = n$.

(2) We will prove this by induction on n. For $n = 1$, M is hollow. Let $n > 1$ and assume that all modules with hollow dimension $n-1$ can be written as an irredundant sum of $n-1$ hollow modules. Since M has finite hollow dimension there exists a non-zero hollow factor module M/N by 3.1.4. Since M is supplemented N has a supplement H_1 in M. Since

$$hdim(H_1) = hdim(H_1/(H_1 \cap N)) = hdim(M/N) = 1,$$

we get that H_1 is hollow. Let H' be a supplement of H_1 in M. Since H_1 is hollow, H_1 is a supplement of H' as well. By 3.2.2 we have

$$\begin{aligned} hdim(M) &= hdim(H') + hdim(M/H') = hdim(H') + hdim(H_1/(H_1 \cap H')) \\ &= hdim(H') + 1. \end{aligned}$$

Thus $hdim(H') = n - 1$. By assumption $H' = \sum_{i=2}^{n} H_i$ is an irredundant sum of hollow modules. Thus $M = \sum_{i=1}^{n} H_i$ is irredundant as H' and H_1 are mutual supplements. □

Remarks:

1. Whenever $hdim(M) = n < \infty$ and $M = \sum_{i=1}^{m} L_i$ an irredundant sum of hollow modules L_i then $m = n$ as $hdim(M)$ is an invariant number.

2. Modules M with finite spanning dimension have finite hollow dimension (see 3.1.2) and are (amply) supplemented (see 2.1.3). Thus Fleury denoted the unique number of summands of this irredundant sum by $sd(M) = n$ and set $sd(M) = \infty$ for modules without finite spanning dimension. We see that $sd(M) = hdim(M)$ holds, but as example 2.1.2 showed there are modules having finite hollow dimension but not finite spanning dimension.

3. If M is a supplemented module with finite hollow dimension such that every supplement is a direct summand then M is a finite direct sum of hollow modules (see 4.1.6).

4. As a module M with finite spanning dimension is amply supplemented, every submodule N of M that is not small in M lies above a non-zero supplement L in M (see 1.2.2). Based on this fact Satyanarayana defined in [56] a new notion of the dimension of a module M with $sd(M) < \infty$: For every $N \subseteq M$ set
$$Sd_M(N) = \begin{cases} 0 & \text{if } N \ll M \\ sd(L) & \text{for a supplement } L \subseteq N \text{ in } M \\ & \text{and } N \text{ lying above } L \text{ in } M \end{cases}$$
Applying 3.2.2 it is easy to show, that $Sd_M(N)$ is well-defined. By definition and 3.2.2 Sd_M satisfies the ordinary vector space formula $Sd_M(M) = Sd_M(N) + Sd_M(M/N)$.

Recall that for the dimension notion of vector spaces A, B the following holds: $dim(A+B) = dim(A) + dim(B) - dim(A \cap B)$. There have been two approaches to prove a second dual Camillo-Zelmanowitz formula; one by Xin in [68] and the other one by Haack in [22].

3.2.6. Xin's Second dual Camillo-Zelmanowitz formula.
Let M be an R-module and N, L proper submodules of M. Consider $K := M/(N \cap L)$ as a submodule of $M/N \oplus M/L$ under the canonical monomorphism. If K lies above a supplement K' in $M/N \oplus M/L$ then the following formula holds:

$$hdim(M/(N+L)) = hdim(M/N) + hdim(M/L) - hdim(M/(N \cap L)) + hdim(K/K').$$

Proof: Consider the homomorphism:

$$g : M/N \oplus M/L \to M/(N+L),$$

$$(x+N, y+L) \mapsto x - y + N + L.$$

Then g is an epimorphism. Clearly $K \subseteq \text{Ker}\,(g)$. Let $(x+N, y+L) \in \text{Ker}\,(g)$. Then $x - y \in N + L$ implies $x = y + l + n$ for some $l \in L$ and $n \in N$. Hence we get $(x+N, y+L) = (z+N, z+L)$ for $z = y + l$. Thus $\text{Ker}\,(g) = K = M/(N \cap L)$ and the following sequence is exact:

$$0 \longrightarrow M/(N \cap L) \longrightarrow M/N \oplus M/L \xrightarrow{g} M/(N+L) \longrightarrow 0$$

Since K lies above a supplement K' we may apply the first dual Camillo-Zelmanowitz formula 3.2.1 and get:
$hdim(M/N) + hdim(M/L)$

$= hdim((M/N \oplus M/L)/K) + hdim(K) - hdim(K/K')$
$= hdim(M/(N+L)) + hdim(M/(N \cap L)) - hdim(K/K')$. □

Remarks: Since every factor module of an amply supplemented module M is again amply supplemented (see [67, 41.7]) we get that the above formula holds for all submodules N, L of M.

Corollary 3.2.7. *Let M be an R-module with $M/Rad(M)$ semisimple. Then for all submodules N, L of M that contain $Rad(M)$ the following holds:*

$$hdim(M/(N \cap L)) + hdim(M/(N+L)) = hdim(M/N) + hdim(M/L).$$

We will state Haack's version of the second dual Camillo-Zelmanowitz formula without a proof because it would be too technical.

3.2.8. Haack's second dual Camillo-Zelmanowitz formula.
Let M be an R-module and N, L submodules of M. Assume there is a submodule K of M minimal with respect to $N \subseteq K \subseteq N+L$ and the property that there is an epimorphism $g : M/L \to M/K$ with $g\eta^K = \eta^L$, where $\eta^X : M/X \to M/(N+L)$ denotes the canonical projection for all $X \subseteq N+L$.
Assume further that there are weak supplements for

$$\{(m_1 + N, m_2 + L) : m_1 + K = (m_2 + L)g\} \subset M/N \oplus M/L,$$

$$\{m + N \cap L : m + K = (m + L)g\} \subset M/(N \cap L).$$

Then the following holds:

$$\begin{aligned} hdim(M/(N \cap L)) &= hdim(M/N) + hdim(M/L) \\ &+ hdim((N+L)/K) - hdim(M/K) \end{aligned}$$

Proof: For the proof we refer to [22]. □

3.3 Semilocal rings

We have seen in 3.1.10, that modules having finite hollow dimension are weakly supplemented. By 1.3.2 every weakly supplemented module is a direct sum of a semisimple submodule and a submodule with essential radical. By 3.1.10 both summands have finite hollow dimension. A semisimple module having finite hollow dimension is obviously finitely generated.

Corollary 3.3.1. *An R-module M with finite hollow dimension is a direct sum of a finitely generated semisimple module and a module having finite hollow dimension and having an essential radical.*

Corollary 3.3.2. *([53, 1.10]) An R-module M with* $\mathrm{Rad}\,(M) = 0$ *has finite hollow dimension if and only if it is finitely generated semisimple.*
In this case $hdim(M) = lg(M) = udim(M)$.

Corollary 3.3.3. *([53, 1.11], [23, 7.14]) If M has finite hollow dimension, then* $M/\mathrm{Rad}\,(M)$ *is finitely generated semisimple.*

Proof: If M has finite hollow dimension so has the factor module $M/\mathrm{Rad}\,(M)$. Since $\mathrm{Rad}\,(M/\mathrm{Rad}\,(M)) = 0$ the result follows by the above corollary. □

Remarks: The converse of the last corollary is in general false. For example consider ${}_{\mathbb{Z}}\mathbb{Q}$. Since $\mathrm{Rad}\,(\mathbb{Q}) = \mathbb{Q}$, we have that $\mathbb{Q}/\mathrm{Rad}\,(\mathbb{Q}) = 0$ is trivially finitely generated semisimple. But $\mathbb{Q}/\mathbb{Z}$ has infinite hollow dimension and hence so too does $\mathbb{Q}$.

3.3.4. Hollow dimension and small radical.

Let M be an R-module with $\mathrm{Rad}\,(M) \ll M$. *Then the following statements are equivalent:*

(a) M has finite hollow dimension;

(b) M is weakly supplemented and finitely generated;

(c) $M/\mathrm{Rad}\,(M)$ *is finitely generated semisimple;*

(d) $M/\mathrm{Rad}\,(M)$ *is finitely cogenerated.*

In this case $hdim(M) = lg(M/\mathrm{Rad}\,(M))$ *holds.*

Proof: $(a) \Rightarrow (c)$ by 3.3.2; $(c) \Rightarrow (a)$ by 3.1.10; $(a) \Rightarrow (b)$ since M is finitely generated if and only if $\mathrm{Rad}\,(M) \ll M$ and $M/\mathrm{Rad}\,(M)$ is finitely generated. By 3.1.10 M is weakly supplemented.
$(b) \Rightarrow (a)$ by 1.3.2 $M/\mathrm{Rad}\,(M)$ is semisimple and since M is finitely generated, $M/\mathrm{Rad}\,(M)$ is semisimple and finitely generated. Hence by 3.1.10 M has finite hollow dimension.
$(c) \Leftrightarrow (d)$ is a well-known fact (see [67, 21.6]). □

Remarks: The equivalence between (a) and (c) appeared in various papers, e.g. [38], [53] and [23].

The last corollary can be applied to rings. Recall that a ring is called *semilocal* if $R/\mathrm{Jac}\,(R)$ is semisimple.

Corollary 3.3.5. *For a ring R the following statements are equivalent:*

(a) $_RR$ *has finite hollow dimension;*

(b) $_RR$ *is weakly supplemented;*

(c) R *is semilocal;*

(d) R_R *is weakly supplemented;*

(e) R_R *has finite hollow dimension.*

In this case $hdim(_RR) = lg(R/\mathrm{Jac}\,(R)) = hdim(R_R)$.

Proof: Follows from 3.3.4 and the fact that (c) is left-right symmetric. $\square$

Remarks:

1. The equivalence between (a) and (c) appeared also in [53, 1.14].
2. The last corollary shows, that semilocal rings and rings with finite hollow dimension are exactly the same. Furthermore the hollow dimension of a ring is left-right symmetric and we can set $hdim(R) := hdim(_RR) = hdim(R_R) = lg(R/\mathrm{Jac}\,(R))$ for any ring R.

Before we state a summarizing characterization of semilocal rings, we will give a characterization in terms of hollow dimension:

3.3.6. Characterization of semilocal rings by hollow dimension.
For a ring R the following statements are equivalent:

(a) R *is semilocal;*

(b) R *has finite hollow dimension as a left* R*-module;*

(c) every finitely generated left R*-module has finite hollow dimension;*

(d) every finitely generated left R*-module is weakly supplemented;*

(e) every finitely generated, self-projective, left R*-module has semilocal endomorphism ring;*

(f) *any injective cogenerator* $_RQ$ *of* $R-$Mod *has finite uniform dimension as a right* T*-module, where* $T := End(_RQ)$*;*

(g) *the left-right duals of the statements above.*

In this case $hdim(R) = lg(R/\mathrm{Jac}\,(R)) = udim(Q_T))$*.*

Proof: (a) $\Leftrightarrow$ (b) $\Leftrightarrow$(c) clear by 3.1.10 and 3.3.5; (c) $\Leftrightarrow$ (d) by 3.3.4; (c) $\Leftrightarrow$ (e) by 3.4.6 and (a) $\Leftrightarrow$(f) by 3.1.12. □

For the next characterization, we have to define some notions:

Definition. An R-module M is called *extending* if every submodule is an essential submodule of a direct summand of M (see [10].
A submodule N of an R-module M is called *pure* in M if $X \otimes N \to X \otimes M$ is monic for all right R-modules X (see [67, 34.5]). An R-module M is called *regular* if every finitely generated submodule of M is pure in M. A ring R is *von Neumann* regular if and only if it is regular as left (right) module over itself (see [67, 37.6]).
Let $_RM$ be a left R-module. For every $s \in R$ denote

$$r.ann_M(s) := \{m \in M | sm = 0\}.$$

Moreover write $\bar{R} := R/\mathrm{Jac}\,(R)$ and for every element $r \in R$ write $\bar{r} := r + \mathrm{Jac}\,(R) \in \bar{R}$.

3.3.7. Characterization of semilocal rings.
For a ring R *the following are equivalent:*

(a) *R is semilocal;*

(b) $R/\mathrm{Jac}\,(R)$ *is finitely cogenerated;*

(c) *every product of simple left R-modules is semisimple;*

(d) *for every left R-module M,* $Soc(M) = \{m \in M : \mathrm{Jac}\,(R)m = 0\}$*;*

(e) $R/\mathrm{Jac}\,(R)$ *is regular and every regular left R-module is semisimple;*

(f) *every left R-module M with* $\mathrm{Rad}\,(M) = 0$ *is an extending module;*

(g) *every left R-module M with* $\mathrm{Rad}\,(M) = 0$ *is self-injective;*

(h) *there exists a ring S and an $R-S$ bimodule M, such that $udim(M_S)$ is finite and $r.ann_M(r) \neq 0$ for all non-units $r \in R$;*

(i) *there exists an integer n and a function $d : R \to \{0, \cdots, n\}$ such that for all $s, t \in R$*

(1) *$d(s - sts) = d(s) + d(1 - ts)$ and*

(2) *if $d(s) = 0$ then s is a unit in R;*

(j) *there exists a partial order $\geq$ on R satisfying the minimum condition, such that for all $s, t \in R$, if $1 - ts$ is not invertible in R, then $s > s - sts$;*

(k) *the left-right duals of the statements above.*

Then $hdim(R) \leq n$, where n is the integer in (i) and $hdim(R) \leq udim(M_S)$ where M_S is the module in (h).

Proof: (a) $\Leftrightarrow$ (b) Clear by [67, 21.6] since a module M is finitely generated and semisimple if and only if M is finitely cogenerated and Rad $(M) = 0$.
(a) $\Rightarrow$ (d) Denote $An_M(\text{Jac}(R)) := \{m \in M : \text{Jac}(R)m = 0\}$ for every $M \in R-$Mod. Since Jac $(R)An_M(\text{Jac}\,(R)) = 0$ holds $An_M(\text{Jac}\,(R))$ is a $R/\text{Jac}\,(R)$-module, hence semisimple and contained in Soc (M). On the other hand it is well-known that Jac (R)Soc $(M) = 0$ holds for all R-modules M (see [67, 21.12]). Thus Soc $(M) = An_M(\text{Jac}\,(R))$.
(d) $\Rightarrow$ (c) If M is a product of simple R-modules, then Jac$\,(R)m = 0$ for all elements $m \in M$ and by (d) we get Soc $(M) = M$, i.e. M is semisimple.
(c) $\Rightarrow$ (a) $R/\text{Jac}\,(R)$ is a submodule of a product of simple R-modules. By (c) this product is semisimple and hence $R/\text{Jac}\,(R)$ is semisimple.
(a) $\Rightarrow$ (e) Clearly $R/\text{Jac}\,(R)$ is regular. Let M be a regular left R-module and N a finitely generated submodule of Jac$\,(R)M$. N is pure and so $N = \text{Jac}\,(R)N$ (see [67, 34.9]). By Nakayama's lemma we have $N = 0$ implying Jac $(R)M = 0$. Thus M is a left $R/\text{Jac}\,(R)$-module and hence semisimple, and semisimple as a left R-module.
(e) $\Rightarrow$(a) If $R/\text{Jac}\,(R)$ is regular, then it is regular as an R-module and hence semisimple.
(a) $\Rightarrow$ (g) Let M be an R-module with Rad $(M) = 0$. Then Jac $(R)M = 0$, hence M is also a left $R/\text{Jac}\,(R)$-module. Thus M is semisimple as an $R/\text{Jac}\,(R)$-module and also as an R-module. By [67, 23.2] M is self-injective.
(g) $\Rightarrow$ (f) Clear (see, for example, [10, 7.2]);
(f) $\Rightarrow$ (a) Put $\bar{R} := R/\text{Jac}\,(R)$. Then ${}_R\bar{R}$ and ${}_{\bar{R}}\bar{R}$ are semiprimitive and hence extending modules. Hence for each set Λ, ${}_{\bar{R}}\bar{R}^{(\Lambda)}$ is a left extending $\bar{R}$-module. Thus ${}_{\bar{R}}\bar{R}$ is a Σ-extending $\bar{R}$-module. Applying [10, 11.13] this yields, that $\bar{R}$ is semiperfect and hence semisimple as Jac $(\bar{R}) = 0$.

(a) $\Rightarrow$ (h) Let $S := R/\text{Jac}\,(R)$ and note that the image of a non-unit in R is a non-unit in S. Consider $M := S$ as an $R-S-$bimodule. Then M_S is semisimple and $udim(M_S)$ is finite. Since $\bar{r}$ is a non-unit in S whenever $r \in R$ is a non-unit and hence $\bar{r}$ a left zero divisor in S. Thus $r.ann_M(r) \neq 0$.
(h) $\Rightarrow$ (i) Note that $r.ann_M(t)$ is a right S-module for all $t \in R$. Set $n := udim(M_S)$ and define $d : R \to \{0, \cdots, n\}$ by $d(r) := udim(r.ann_M(r)_S)$ for all $r \in R$. Then $d(r) = 0 \Rightarrow r.ann_M(r) = 0 \Rightarrow r$ is a unit in R. For every $s, t \in R, r.ann_M(s) \oplus r.ann_M(1 - ts) = r.ann_M(s - sts)$ holds. Thus $d(s - sts) = d(s) + d(1 - ts)$.
(i) $\Rightarrow$ (j) For every $s, t \in R$ set $s > t$ if $d(s) < d(t)$. This implies for $s, t \in R$ and $1 - ts$ a non-unit, $d(1 - ts) \neq 0$ and hence $d(s - sts) > d(s)$. Thus $s - sts < s$ holds.
(j) $\Rightarrow$ (a) Assume that there exists a left ideal $I \subset R$ which has no weak supplement. Then we can construct an infinite strictly descending chain of elements

$$1 > b_1 > b_2 > \cdots > b_n > \cdots$$

such that for all $n \in \mathbb{N}$ we have $I + Rb_n = R$. Since $(R, \leq)$ is artinian - this is a contradiction, hence I must have a weak supplement in R. Thus R is semilocal. We construct the chain as follows: Let $n = 1$. Since $I \not\ll R$ there is an $a \in I$ such that $1 - a$ is not a unit in R. Hence $1 > 1 - a =: b_1$ and $I + Rb_1 = R$ holds. Now assume that we constructed a chain $1 > b_1 > b_2 > \cdots > b_n$ for all $n \geq 1$ with $I + Rb_n = R$. By assumption $I \cap Rb_n \not\ll R$ implies that there is an $r \in R$ such that $rb_n \in I$ and $x := 1 - rb_n$ is not a unit in R. Hence

$$b_n > b_n(1 - rb_n) = b_n x =: b_{n+1}.$$

An easy calculation shows that $Rb_{n+1} = Rb_n \cap Rx$. We have $b_n = b_n - b_n x + b_n x = b_n(1 - x) + b_{n+1} = b_n r b_n + b_{n+1} \in I + Rb_{n+1}$. Thus $Rb_n \subseteq I + Rb_{n+1}$, but as $Rb_{n+1} \subset Rb_n$ we get by the modularity law $Rb_n = (I \cap Rb_n) + Rb_{n+1}$. Together we get $R = I + Rb_n = I + Rb_{n+1}$. $\square$

Remarks: (a)-(d) were taken from [67, 21.15], (e) was considered in Fieldhouse [12], (f) and (g) were considered in Hirano et al. [30] and (h) - (k) were obtained by Camps and Dicks [8].

Corollary 3.3.8. *Let R,S be rings and $f : R \to S$ be a ring homomorphism such that non-units $r \in R$ are carried to non-units $f(r) \in S$. If S is semilocal then R is semilocal and $hdim(R) \leq hdim(S)$.*

Proof: The canonical projection $\pi : S \to S/\text{Jac}\,(S)$ is a ring homomorphism such that non-units of S are carried to non-units of $S/\text{Jac}(S)$. Hence $\pi f : R \to S/\text{Jac}(S)$

is such a ring homomorphism as well. If S is semilocal then $S/\mathrm{Jac}\,(S)$ is semismple artinian. So let us assume that S is semisimple artinian and that there exist a ring homomorphism from $f : R \to S$ such that non-units of R are carried to non-units of S. We will apply 3.3.7(h) $\Rightarrow$ (a)To show that R is semilocal. Clearly S is a left R-module by the multiplication $r * s := f(r)s$. Let $M := S$ and as M_S is semisimple artinian we get that $udim(M_S) = lg(M_S)$ is finite. It remains to show that $r.ann_M(r) \neq 0$ for all non-units $r \in R$. Let r be a non-unit in R then $f(r)$ is a non-unit in S. Consider the descending sequence $f(r)S \supseteq f(r)^2S \supseteq f(r)^3S \supseteq \cdots$. Since S is artinian there must be a number $n \in \mathbb{N}$ and an element $s \in S$ such that $f(r)^n = f(r)^{n+1}s$ and so $f(r)^n(1 - f(r)s) = 0$ holds. Since $f(r)$ is not invertible we get that $1 - f(r)s \neq 0$. It is easy to see that there must be a number $k < n$ such that $f(r)f(r)^k(1 - f(r)s) = 0$ with $f(r)^k(1 - f(r)s) \neq 0$. Thus $f(r)^k(1 - f(r)s) \in r.ann_M(r)$. By 3.3.7 we get that R is semilocal and that $hdim(R) \leq udim(M_S) = hdim(S)$ holds. □

Remarks: As a consequence from the last corollary we get that if G is a group and R a ring such that the group ring RG is semilocal. Then for every subgroup H of G, RH is semilocal and $hdim(RH) \leq hdim(RG)$.

3.4 Endomorphism rings and hollow dimension

In the following we will discuss the relation between the hollow dimension of a module and the hollow dimension of its endomorphism ring.

The next theorem was obtained from Herbera & Shamsuddin in [29] and uses Camps & Dicks characterization of semilocal rings (see 3.3.7).

3.4.1. Semilocal endomorphism ring. *([29, Theorem 3])*
Let M be an R-module and $S := End(M)$.

1. *If M has finite hollow dimension and every epimorphism $f \in S$ is an isomorphism, then S is semilocal and $hdim(S) \leq hdim(M)$.*

2. *If M has finite uniform dimension and every monomorphism $f \in S$ is an isomorphism, then S is semilocal and $hdim(S) \leq udim(M)$.*

3. *If M has finite uniform and hollow dimension, then S is semilocal and $hdim(S) \leq hdim(M) + udim(M)$.*

Proof: Let $f, g \in S$; then clearly $\text{Ker}\,(f) \cap \text{Ker}\,(1 - fg) = 0$ and $\text{Ker}\,(f - fgf) = \text{Ker}\,(f) + \text{Ker}\,(1 - fg)$ since for all $x \in \text{Ker}\,(f - fgf)$, $x = (x)(fg + 1 - fg)$, where $(x)fg \in \text{Ker}\,(1 - fg)$ and $(x)(1 - fg) \in \text{Ker}\,(f)$. Thus

$$\text{Ker}\,(f - fgf) = \text{Ker}\,(f) \oplus \text{Ker}\,(1 - fg).$$

Dually, let $\text{Coke}\,(f) := M/\text{Im}\,(f)$; then $M = \text{Im}\,(gf) + \text{Im}\,(1 - gf) = \text{Im}\,(f) + \text{Im}\,(1 - gf)$ and $\text{Im}\,(f - fgf) = \text{Im}\,(f) \cap \text{Im}\,(1 - gf)$ implies

$$\text{Coke}\,(f - fgf) \simeq \text{Coke}\,(f) \oplus \text{Coke}\,(1 - gf).$$

(1) Let $n_1 := hdim(M)$; define

$$d_1 : S \to \{0, 1, \ldots, n_1\},$$

$$f \mapsto hdim(\text{Coke}\,(f)).$$

Then for all $f, g \in S$, $d_1(f - fgf) = d_1(f) + d_1(1 - gf)$ holds and whenever $0 = d_1(f) = hdim(\text{Coke}\,(f))$, then $\text{Im}\,(f) = M$ implies f is an epimorphism and by assumption an isomorphism. By 3.3.7(i) S is semilocal and $hdim(S) \leq n_1 = hdim(M)$.

(2) Let $n_2 := udim(M)$; define

$$d_2 : S \to \{0, 1, \ldots, n_2\},$$

$$f \mapsto udim(\text{Ker}\,(f)).$$

Since for every $f, g \in S$, f gives an isomorphism between $\text{Ker}\,(1 - fg)$ and $\text{Ker}\,(1 - gf)$, we get $d_2(1 - fg) = d_2(1 - gf)$. Hence $d_2(f - fgf) = d_2(f) + d_2(1 - gf)$ and whenever $0 = d_2(f) = udim(\text{Ker}\,(f))$, then $\text{Ker}\,(f) = 0$ implies f is a monomorphism and by assumption an isomorphism. By 3.3.7(i) S is semilocal and $hdim(S) \leq n_2 = udim(M)$.

(3) Define

$$d = d_1 + d_2 : S \to \{0, 1, \ldots, n_1 + n_2\}.$$

For every $f, g \in S$, $d(f - fgf) = d(f) + d(1 - gf)$ holds. Assume $d(f) = 0$, then $d_1(f) = 0$ implies $\text{Ker}\,(f) = 0$ and $d_2(f) = 0$ implies $\text{Im}\,(f) = M$. Hence f is an isomorphism. Again by 3.3.7(i) S is semilocal and $hdim(S) \leq udim(M) + hdim(M)$. □

Remarks: Since a self-projective module with finite hollow dimension has the property that every epimorphism is an isomorphism (see 3.1.10), we get $hdim(End(M)) \leq hdim(M)$ as a corollary of the above theorem. We will show that more generally $hdim(M) = hdim(\mathrm{Hom}\,(P,M))$ holds for a self-projective module P and a finitely P-generated module M.

The next lemma is due to Garcia Hernandez and Gomez Pardo; it will allow us to prove Proposition 3.4.3 below.

Lemma 3.4.2. *Let M be a finitely generated R-module and $\{N_1, \ldots, N_m\}$ a coindependent family of proper submodules. Then there exist finitely generated submodules $L_i \subseteq N_i$ for each $i \in \{1, \ldots, m\}$ such that $\{L_1, \ldots, L_m\}$ forms a coindependent family of M.*

Proof: Since M is finitely generated, for each $1 \leq i \leq m$, there exist finitely generated submodules $X_i \subseteq N_i$ and $Y_i \subseteq \bigcap_{j \neq i} N_j$ such that $X_i + Y_i = M$. Let $L_i := X_i + \sum_{j \neq i} Y_j \subseteq N_i$. As $L_i + \bigcap_{j \neq i} L_j \supseteq X_i + Y_i = M$ holds the result follows. $\square$

Definition. Let M and P denote left R-modules and let $S = End(P)$. For every S-submodule $X \subseteq_S \mathrm{Hom}_R\,(P,M)$ set

$$(P)X := \sum_{f \in X} \mathrm{Im}\,(f).$$

Remarks:

(1) $(P)\mathrm{Hom}\,(P,N) = Tr(P,N) \subseteq N$ holds for every $N \subseteq M$.

(2) Assume that P generates M, then $M = Tr(P,M)$ holds by [67, 13.5]. Let $N \subseteq M$ and assume $Hom(P,N) = Hom(P,M)$. Then

$$M = Tr(P,M) = (P)\mathrm{Hom}\,(P,M) = (P)\mathrm{Hom}\,(P,N) = Tr(P,N) \subseteq N.$$

implies $N = M$. Hence N is a proper submodule of M if and only if $\mathrm{Hom}(P,N)$ is a proper submodule of $\mathrm{Hom}\,(P,M)$.

(3) Assume P to be self-projective and X a finitely generated S-submodule of ${}_S\mathrm{Hom}_R\,(P,M)$. Applying [1, Proposition 4.9] we get: $X = \mathrm{Hom}\,(P,(P)X)$. (The notation in [1] is : $l'_S(N) = \mathrm{Hom}\,(P,N)$ and $r'_M(X) = (P)X$.)

Proposition 3.4.3. *([15, Theorem 4.2])*
Let P be a self-projective R-module, $S := \mathrm{End}\,(P)$ and M a P-generated R-module with ${}_S\mathrm{Hom}\,(P,M)$ finitely generated as an S-module. Then the following statement holds:

$$hdim({}_S\mathrm{Hom}_R\,(P,M)) \leq hdim({}_RM).$$

Proof: Assume that $hdim({}_S\text{Hom}\,(P,M)) \geq m$. Then there exists a coindependent family $\{N_1,\ldots,N_m\}$ of proper submodules of $\text{Hom}\,(P,M)$. By Lemma 3.4.2 there exist finitely generated submodules $L_i \subseteq N_i$ such that $\{L_1,\ldots,L_m\}$ form a coindependent family of submodules of $\text{Hom}\,(P,M)$.

Define $K_i := (P)L_i$ for every $1 \leq i \leq m$. Since L_i is finitely generated we get by above remark (3) that $L_i = \text{Hom}\,(P,K_i)$ holds. Hence K_i is a proper submodule as L_i is proper by above remark (2). ¿From the coindependency of the $L_i's$ it follows, that:

$$\begin{aligned}\text{Hom}\,(P,M) &= \text{Hom}\,(P,K_i) + \bigcap_{j\neq i}\text{Hom}\,(P,K_j)\\ &= \text{Hom}\,(P,K_i) + \text{Hom}\,(P,\bigcap_{j\neq i}K_j)\\ &\subseteq \text{Hom}\,(P,K_i + \bigcap_{j\neq i}K_j).\end{aligned}$$

Thus by above remark (2), $M = K_i + \bigcap_{j\neq i}K_j$ holds for every $1 \leq i \leq m$. We conclude that $\{K_1,\ldots,K_m\}$ is a coindependent family of proper submodules and that $hdim(M) \geq m$. $\square$

Remarks: ${}_S\text{Hom}_R\,(P,M)$ is finitely generated as an S-module if for example M is isomorphic to a finite direct sum of copies of P or more generally if M is finitely P-generated. In this case there exists an exact sequence

$$P^k \longrightarrow M \longrightarrow 0.$$

Since P is self-projective, the covariant functor $\text{Hom}\,(P,-)$ is exact with respect to this sequence (see [67, pp. 148]). Thus we get the exact sequence

$$\text{Hom}\,(P,P^k) \longrightarrow \text{Hom}\,(P,M) \longrightarrow 0.$$

Hence $\text{Hom}\,(P,M)$ is finitely generated as an S-module because $S^k \simeq \text{Hom}\,(P,P^k)$.

The next definition is due to Takeuchi [60].

Definition. An R-module P is called *cofinitely M-projective* if P is projective for every exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

with N finitely cogenerated, i.e. for every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow f & & \\ M & \xrightarrow[h]{} & N & \longrightarrow & 0 \end{array}$$

with N finitely cogenerated and exact row, there exists a homomorphism g from P to M, such that $gh = f$.

A similar definition can be found in Hiremath [31].

We will need a technical lemma to prove a theorem due to Takeuchi.

Lemma 3.4.4. *([60]) Let P be cofinitely M-projective and $\{N_1, \cdots, N_n\}$ a coindependent family of proper non-zero submodules of M, such that M/N_i is finitely cogenerated for every $1 \leq i \leq n$. Then for any homomorphisms $f_1, \cdots, f_n$ in* $\mathrm{Hom}\,(P, M)$ *there exists a homomorphism $g \in \mathrm{Hom}\,(P, M)$ such that $g - f_i \in \mathrm{Hom}\,(P, N_i)$ for every $1 \leq i \leq n$.*

Proof: Define $f : P \to \bigoplus_{i=1}^{n} M/N_i$ by

$$p \mapsto ((p)f_1 + N_1, \cdots, (p)f_n + N_n)$$

for every $p \in P$ and consider the following diagram:

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow[\pi]{} & \bigoplus_{i=1}^{n} M/N_i & \longrightarrow & 0. \end{array}$$

where π denotes the canonical projection. Since $\{N_1, \cdots, N_n\}$ is coindependent π is epimorph and there is a homomorphism $g : P \to M$ such that $g\pi = f$ as P is cofinitely M-projective. Let $\pi_i : M \to M/N_i$ for all i. Then $(g - f_i)\pi_i = 0$ and therefore we have $(P)(g - f_i) \subseteq N_i$ for every i and $g - f_i \in \mathrm{Hom}\,(P, N_i)$. $\square$

Proposition 3.4.5. *([60, Proposition 3]) Let P be a cofinitely M-projective R-module and M be a P-generated R-module. Then the following statement holds:*

$$hdim({}_RM) \leq hdim({}_S\mathrm{Hom}\,(P, M)).$$

Proof: Let L be a proper submodule of M. Then L is contained in a proper submodule N of M, such that M/N is finitely cogenerated. Assume $\{L_1, \cdots, L_n\}$ is a coindependent set of proper submodules of M. Then every submodule L_i is contained in a proper submodule N_i, such that $\{N_1, \cdots, N_n\}$ is a coindependent set of submodules. Since P generates M and N_i is proper in M, we have that $\mathrm{Hom}\,(P, N_i)$ is a proper submodule of $\mathrm{Hom}\,(P, M)$ (see remark (2) before 3.4.3). Let $f \in \mathrm{Hom}\,(P, M)$; then by the preceding lemma, for every i there exists a

$g_i \in \text{Hom}\,(P, M)$ such that $g_i - f \in \text{Hom}\,(P, N_i)$ and $g_i - 0 \in \text{Hom}\,(P, N_j)$ for every $j \neq i$. Thus

$$\text{Hom}\,(P, N_i) + \bigcap_{j \neq i} \text{Hom}\,(P, N_j) = \text{Hom}\,(P, M)$$

for every i and hence $\{\text{Hom}\,(P, N_1), \cdots, \text{Hom}\,(P, N_n)\}$ is a coindependent set of proper non-zero submodules of Hom (P, M). This yields $hdim({}_RM) \leq hdim({}_S\text{Hom}\,(P, M))$. □

As a corollary to 3.4.3 and 3.4.5 we get the following.

Corollary 3.4.6. *(see [60])*

1. *If P is self-projective and M finitely P-generated, then*

$$hdim(M) = hdim({}_S\text{Hom}_R\,(P, M)).$$

2. *If M is a self-projective R-module, then $hdim(M) = hdim(End(M))$.*

Proof: (1) Since P is self-projective and M finitely P-generated we get that Hom (P, M) is finitely generated as an S-module (see remarks after the proof of 3.4.3). Hence we can apply 3.4.3. On the other hand since M is finitely P-generated, there exists an integer k and an epimorphism from P^k to M. Since P is self-projective it is P^k-projective and hence M-projective (see [67, 18.2]). Thus it is cofinitely M-projective as well and we can apply 3.4.5;
(2) follows from (1). □

Remarks:

1. A self-projective module has finite hollow dimension if and only if its endomorphism ring is semilocal (see 3.3.5 and 3.4.6).

2. Gupta and Varadarajan proved in [21, 4.22] that if P is a finitely generated self-projective R-module and M a P-generated module such that P is M-projective. Then $hdim(M) = hdim(\text{Hom}\,(P, M))$ holds.

Takeuchi's result shows, that hollow dimension is invariant under equivalences. We show this next: let M be an R-module and S a ring. By [67, 46.2], $\sigma[M]$ is equivalent to $S-$Mod if and only if there exists a finitely generated projective generator P in $\sigma[M]$ with End $(P) \simeq S$. Moreover the equivalence is given by the functor ${}_S\text{Hom}_R\,(P, -)$ and the inverses $P \otimes_S -$. A finitely generated projective generator in $\sigma[M]$ is called a *progenerator*.

Corollary 3.4.7. *Let M be an R-module and S a ring such that $\sigma[M]$ is equivalent to $S-$Mod with progenerator P in $\sigma[M]$ and $\mathrm{End}\,(P) \simeq S$. Then we have for every finitely generated R-module N in $\sigma[M]$: $hdim(N) = hdim({}_S\mathrm{Hom}_R\,(P, N))$ and for every finitely generated S-module T: $hdim(T) = hdim(P \otimes_S T)$.*

Together with the characterization of semilocal rings by Camps and Dicks see 3.3.7$(a) \Leftrightarrow (i)$ and Takeuchi's result we can generalize Herbera and Shamsuddin's Theorem [29, Theorem 1].

Corollary 3.4.8. *Let M be a self-projective R-module and let $S := End(M)$. Then the following statements are equivalent.*

(a) M has finite hollow dimension.

(b) There exists an integer n and a function $d : S \to \{0, \cdots, n\}$ such that for all $f, g \in S$

(i) $d(f - fgf) = d(f) + d(1 - gf)$ and

(ii) if $d(f) = 0$ then f is an isomorphism.

We will consider properties of modules with semilocal endomorphism ring. We have seen that examples of such modules are modules with finite hollow dimension whose surjective endomorphisms are bijective or modules with finite uniform dimension whose injective endomorphisms are bijective (e.g. artinian modules).

3.4.9. Bass' Theorem.
Let R be a semilocal ring, $a \in R$ and I a right ideal of R. If $aR + I = R$, then there exists an $r \in I$ such that $a + r$ is a unit.

Proof: (The proof we will give is due to Swan and was obtained from [34].) Since an element $r \in R$ is a unit in R if and only if $\bar{r}$ is a unit in $R/\mathrm{Jac}\,(R)$ we may replace R by $R/\mathrm{Jac}\,(R)$ and assume that R is semisimple. Since R is semisimple we are able to find a right ideal J in I such that $I = (aR \cap I) \oplus J$. Thus $R = aR \oplus J$. Consider the exact sequence:

$$0 \longrightarrow K \longrightarrow R \xrightarrow{f} aR \longrightarrow 0$$

with $f(r) := ar$ for all $r \in R$ and $K := \mathrm{Ker}\,(f)$. Since aR is a direct summand of R and projective, the sequence above splits. Hence there is an $h : aR \to R$ such that $R = \mathrm{Im}\,(h) \oplus K$. Let $g : R \to K$ be the canonical projection onto K. Thus

$$(f, g) : R \to aR \oplus K$$

is an isomorphism. Since $R = aR \oplus J$, there exists an isomorphism $\gamma : K \to J$. Consider the composition

$$R@>(f,g)>>aR \oplus K@>(1,\gamma)>>aR \oplus J = R$$

mapping an element $s \in R$ to $as + \gamma g(s)$. Since this composition is an isomorphism, the image of $1 \in R$ is invertible in R. Thus

$$a1 + \gamma g(1) = a + r$$

is a unit, with $r := \gamma g(1) \in J \subseteq I$. □

Remarks: Clearly Bass's Theorem holds also for left ideals I of R as the property semilocal is left-right-symmetrical.

Definition. A ring R is said to have *right stable range 1* if, whenever $aR + bR = R$ for elements $a, b \in R$, there exists an element $r \in R$ such that $a + br$ is a unit.

By Bass' Theorem, a semilocal ring has right (left) stable range 1.

Definition. An R-module is said to *cancel from direct sums* if whenever $M \oplus N \simeq M \oplus L$ for R-modules N and L then $N \simeq L$ holds.

The next theorem is due to Evans and was obtained from [34].

3.4.10. Cancellation Theorem. *([34, 20.11])*
Let M be a left R-module such that $End(M)$ has right stable range 1. Then M cancels from direct sums.

Proof: Assume $M \oplus N \simeq M \oplus L$ holds for left R-modules N and L. Then we get a splitting epimorphism $h = (f, g) : M \oplus N \to M$ with $\mathrm{Ker}\,(h) \simeq L$. Since h splits there exists a homomorphism $h' = (f', g') : M \to M \oplus N$ such that

$$id_M = h'h = f'f + g'g$$

holds. Thus $S = f'S + g'gS$ with $S := \mathrm{End}\,(M)$. Since S has right stable range 1 there exists an element $e \in S$ such that

$$u := f' + (g'g)e$$

is invertible in S. Define $k : M \oplus N \to M$ by $k := (1, ge)$. Then

$$h'k = (f', g')(1, ge) = f' + (g'g)e = u.$$

Thus the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\,(h) & \longrightarrow & M \oplus N & \underset{h}{\longrightarrow} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & \mathrm{Ker}\,(k) & \longrightarrow & M \oplus N & \xrightarrow{k} & M & \longrightarrow & 0
\end{array}$$

Since the splitting homomorphism for h is h' and k is $u^{-1}h'$, we get

$$\mathrm{Ker}\,(k) \simeq (M \oplus N)/\mathrm{Im}\,(h') \simeq \mathrm{Ker}\,(h) \simeq L.$$

On the other hand the mapping $n \mapsto (-(n)ge, n) \in M \oplus N$ for all $n \in N$ gives an isomorphism between N and Ker (k). We conclude $N \simeq L$. □

Lemma 3.4.11. *([11, Lemma 1.4]) Let M be an R-module and $S := End(M)$. Then there exists a bijection α between the set of all finite direct-sum decompositions of ${}_RM$ and finite direct-sum decompositions of ${}_SS$:*

$$\alpha : \{M_i\}_I \mapsto \{Se_i\}_I$$

where $M = \bigoplus_I M_i$, I a finite set and $e_i = \pi_i \epsilon_i$ is an idempotent ($\pi_i : M \to M_i$ and $\epsilon_i : M_i \to M$ denote the canonical projection, respectively inclusion). The inverse mapping $\alpha^{-1}$is given by

$$\{S_i\}_I \mapsto \{(M)S_i\}_I$$

where $S = \bigoplus_I S_i$ and I a finite set. Then the following holds for all decompositions $M = \bigoplus_I M_i$ and $i, j \in I$:

(1) M_i is indecomposable if and only if S_i is indecomposable;

(2) $M_i \simeq M_j$ as R-modules if and only if $S_i \simeq S_j$ as S-modules.

Proof: Clearly ${}_SS = \bigoplus_I {}_SSe_i$ holds whenever $M = \bigoplus_I M_i$ and $M = \bigoplus_I (M)S_i$ holds whenever $S = \bigoplus_I S_i$. Further we have $\alpha^{-1}(\alpha(M_i)) = \alpha^{-1}(Se_i) = (M)Se_i = M_i$ and $\alpha(\alpha^{-1}(S_i)) = \alpha((M)S_i) = S_i$ since $S_i = Se_i$ for an idempotent $e_i \in S$. (1) and (2) are easy to check. □

3.4.12. The n^{th} root uniqueness property. *([11, Proposition 2.1])*
Let M and N be left R-modules such that $End(M)$ and $End(N)$ are semilocal. Then for any $n \in \mathbb{N}$ the following holds:

$$M^n \simeq N^n \Rightarrow M \simeq N \ (n^{th} \text{ root uniqueness}).$$

Proof: Let $L := \bigoplus_{i=1}^{n} M_i = \bigoplus_{i=1}^{n} N_i$ with $M_i \simeq M$ and $N_i \simeq N$ for all $i \in \{1, \dots, n\}$. By Lemma 3.4.11 we get two decompositions of the semilocal endomorphism ring $S = End(L)$. Write $S = \bigoplus_{i=1}^{n} Se_i = \bigoplus_{i=1}^{n} Sf_i$ where $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are orthogonal idempotents such that $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} f_i = 1_S$, $\mathrm{End}\,(M) \simeq Se_i$ and $\mathrm{End}\,(N) \simeq Sf_i$ for all $1 \leq i \leq n$.

Let $\bar{S} := S/\mathrm{Jac}(S)$. For all idempotents $e, f \in S$ the following holds: $Se \simeq Sf \Leftrightarrow \bar{S}\bar{e} \simeq \bar{S}\bar{f}$. (see [67, 21.17(3)]). Thus we get two decompositions $\bar{S} = \bigoplus_{i=1}^{n} \bar{S}\bar{e}_i = \bigoplus_{i=1}^{n} \bar{S}\bar{f}_i$ in which every $\bar{S}\bar{e}_i \simeq \bar{S}\bar{e}_j$ and $\bar{S}\bar{f}_i \simeq \bar{S}\bar{f}_j$. But since S is semilocal, the ring $\bar{S}$ is semisimple artinian and therefore $\bar{S}\bar{e}_1 \simeq \bar{S}\bar{f}_1$. Thus $End(M) \simeq Se_1 \simeq Sf_1 \simeq End(N)$ yields $M \simeq N$ by 3.4.11(2). □

The number of isomorphism classes of direct summands of a self-projective module M is bounded if the module has finite hollow dimension. As a generalization of [11, Proposition 2.1(ii)] we get the following theorem.

3.4.13. Projective direct summands.
Let M be an R-module with finite hollow dimension and small radical. Then the number of non-isomorphic M-projective direct summands of M is bound by 2^k with $k := hdim(M)$.

Proof: If M has finite hollow dimension, then $M/\mathrm{Rad}\,(M)$ is finitely generated semisimple (see 3.3.3). Let $M/\mathrm{Rad}\,(M) = E_1 \oplus \cdots \oplus E_k$ with E_i simple for all $1 \leq i \leq k$ and $k = lg(M/\mathrm{Rad}\,(M)) \leq hdim(M)$. Let P and Q be two M-projective direct summands of M. Since M has small radical P and Q have small radical. Then

$$P/\mathrm{Rad}\,(P) \simeq E_1^{(x_1)} \oplus \cdots \oplus E_k^{(x_k)} \text{ and } Q/\mathrm{Rad}\,(Q) \simeq E_1^{(y_1)} \oplus \cdots \oplus E_k^{(y_k)}$$

where $x_i, y_i \in \{0, 1\}$ for all $i \in \{1, \dots, k\}$. If $x_i = y_i$ for all i, then P maps epimorphically onto Q as P is Q-projective and $\mathrm{Rad}\,(Q) \ll Q$. Since Q is P-projective; Q is isomorphic to a direct summand of P. On the other hand, applying the same argument, P is isomorphic to a direct summand of Q.

Hence $hdim(Q) \leq hdim(P)$ and $hdim(P) \leq hdim(Q)$ implies $hdim(P) = hdim(Q)$. Assume $P \simeq Q \oplus X$. Then $hdim(P) = hdim(Q) + hdim(X)$ implies $hdim(X) = 0$ and $X = 0$, because $hdim(P)$ is finite. Hence $P \simeq Q$.

Thus we get: $P \not\simeq Q$ implies that there exists an index $i \in \{1, \dots, k\}$ such that $x_i \neq y_i$ holds. There are at most 2^k distinct n-tuples $(x_1, \dots, x_k)$ with $x_i \in \{0, 1\}$. Thus there are at most 2^k non-isomorphic M-projective direct summands of M. □

As a corollary of the above theorem we get a result by A.Facchini, et al. (see [11, Proposition 2.1]).

Corollary 3.4.14. *Let M be an R-module such that $S := End(M)$ is semilocal and $k := hdim(M)$. Then M has at most 2^k isomorphism classes of direct summands. Moreover if M is artinian then $k \leq udim(M)$.*

Proof: The number of non-isomorphic direct summands of M is equal to the number of non-isomorphic direct summands of $S = End(M)$ by Lemma 3.4.11. By Theorem 3.4.13 this number is finite and at most 2^k where $k = lg(S/\text{Jac}\,(S)) = hdim(S)$. If M is artinian, then we have $hdim(S) \leq udim(M)$ by 3.4.1(2). □

With the same proof as in [14] we are able to generalize slightly a theorem by Fuller and Shutters.

3.4.15. Finitely generated indecomposable projective modules in $\sigma[M]$. *Let M be an R-module with finite hollow dimension and small radical. Then there are only finitely many isomorphism classes of finitely generated indecomposable projective modules in $\sigma[M]$.*

Proof: (see [14, Theorem 9]) By 3.3.4 M is finitely generated and $M/\text{Rad}\,(M)$ is semisimple. Let $M/\text{Rad}\,(M) \simeq E_1 \oplus \cdots \oplus E_n$ with E_i simple for all $1 \leq i \leq n$ and $n := hdim(M)$. Let P and Q be non-zero finitely generated indecomposable projective modules in $\sigma[M]$. Hence there exist positive integers k and l such that P is a direct summand of M^k and Q is a direct summand of M^l.

$$P/\text{Rad}\,(P) \simeq E_1^{(x_1)} \oplus \cdots \oplus E_n^{(x_n)} \text{ and } Q/\text{Rad}\,(Q) \simeq E_1^{(y_1)} \oplus \cdots \oplus E_n^{(y_n)}$$

where x_i and y_i are non-negative integers. If $x_i \geq y_i$ for all $i \in \{1, \ldots, n\}$ then $P/\text{Rad}\,(P)$ maps epimorphically onto $Q/\text{Rad}\,(Q)$ and since P is Q-projective P maps onto Q. As the canonical projection $Q \to Q/\text{Rad}\,(Q)$ is a small epimorphism P maps epimorphically onto Q (see [67, 19.2]). On the other hand, Q is P-projective implies that Q is isomorphic to a direct summand of P and hence $P \simeq Q$ as P is indecomposable. Thus we have:

$$(*)\ P \simeq Q \Leftrightarrow x_i = y_i, \text{ for all } i \in \{1, \ldots, n\} \Leftrightarrow x_i \geq y_i, \text{ for all } i \in \{1, \ldots, n\}.$$

The next argument is of a combinatorical nature: Let X denote the set of all n-tuples $(x_1, \ldots, x_n)$ that correspond to the isomorphism classes of finitely generated indecomposable projective modules in $\sigma[M]$. Assume X is infinite, then it must be unbounded in at least one component. Renumbering $E_1, \ldots, E_n$ we may assume X is unbounded in the first component to obtain an infinite sequence in X:

$$((x_{1_i}, x_{2_i}, \ldots, x_{n_i}))_{i\in\mathbb{N}}$$

with

$$x_{1_1} < x_{1_2} < x_{1_3} < \cdots$$

By $(*)$ all $n-1$-tuples $(x_{2_i}, \ldots, x_{n_i})$ must be distinct. Otherwise assume that there are two equal $n-1$-tuples $(x_{2_i}, \ldots, x_{n_i})$ and $(x_{2_j}, \ldots, x_{n_j})$ and let $x_{1_i} < x_{1_j}$ then $x_{k_i} \leq x_{k_j}$ for all k. Thus by $(*)$ these n-tuples must be equal - a contradiction. Thus by renumbering $E_2, \ldots, E_n$ we can find a subsequence

$$((x_{1_{i_j}}, x_{2_{i_j}}, \ldots, x_{n_{i_j}}))_{j\in\mathbb{N}}$$

with

$$x_{1_{i_1}} < x_{1_{i_2}} < x_{1_{i_3}} < \cdots \text{ and } x_{2_{i_1}} < x_{2_{i_2}} < x_{2_{i_3}} < \cdots$$

Continuing this process n times we see that X must be unbounded in every component. Hence we obtain two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ with $x_i > y_i$ for all $i \in \{1, \ldots, n\}$. But this contradicts $(*)$. Hence X must be finite. $\square$

As a consequence we get Fuller and Shutter's original version of above theorem as a corollary.

Corollary 3.4.16. *A semilocal ring has only finitely many isomorphism classes of finitely generated indecomposable projective modules.*

Remarks: Summarizing we have seen, that modules with semilocal endomorphism ring cancel from direct sums, have the n^{th} root property and have only a finite number of non-isomorphic direct summands. Moreover there are only finitely many non-isomorphic finitely generated indecomposable projective module in $\sigma[M]$ if M has finite hollow dimension and a small radical.

Not every module with semilocal endomorphism ring has finite hollow dimension. This is shown by the next example taken from [29, Example 10].

Example 3.4.17. *(1) Let R be a ring that can be embedded in a local ring S, then R can be realized as the endomorphism ring of a local module.*
(2) There exists a cyclic module with infinite hollow dimension whose endomorphism ring is semilocal.

Proof: (1) Let $R \subseteq S$ and consider the (S,R)-bimodule $M :=_S \mathrm{Hom}_R\,({}_RS, {}_RS/R)_R$ where the action of S on M is defined as $sf : x \mapsto (sx)f$ and the action of R on M is defined as $fr : x \mapsto (xr)f$ for all $s \in S, f \in M, r \in R$ and $x \in S$. Consider the (S,R)-submodule $N := \{f \in M | (R)f = 0\}$. Clearly the canonical projection $\pi_R : {}_RS \to {}_RS/R$ is in N. For all $s \in S$ we have $sN \subseteq N \Leftrightarrow s \in R$ since whenever $sN \subseteq N$ then $(s)\pi_R = (1)s\pi_R = 0$ implies $s \in R$. On the other hand if $s \in R$, then $Rs \subseteq R$ and so $sf \in N$ for all $f \in N$. Clearly $f \in N \Leftrightarrow fR \subseteq N$ holds. Let

$$T := \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \text{ and } I := \begin{pmatrix} 0 & N \\ 0 & R \end{pmatrix}.$$

Then I is a right ideal in T. The idealizer I' of the right ideal I is defined as $I' := \{t \in T : tI \subseteq I\}$. Hence the idealizer of I is

$$I' = \begin{pmatrix} R & N \\ 0 & R \end{pmatrix}$$

because an element $\begin{pmatrix} s & f \\ 0 & r \end{pmatrix}$ is in I' if and only if $sN \subseteq N$ and $fR \subseteq N$. Hence $s \in R$ and $f \in N$ by the foregoing. Applying [51, Proposition 1] we get $End(T/I) = I'/I = R$. Every proper right ideal of T containing I is of the form $\begin{pmatrix} J & K \\ 0 & R \end{pmatrix}$, where J is a proper right ideal of S, and K is a submodule of M containing N such that $JM \subseteq K$. Now assume that S is local. Then $J \subseteq \mathrm{Jac}\,(S)$ holds for every right ideal J of S. Hence every right T-submodule of T/I is contained in $\begin{pmatrix} \mathrm{Jac}\,(S) & M \\ 0 & R \end{pmatrix}$. Thus T/I is a local right T-module with endomorphism ring R.
(2) Assume R is semilocal and S is not semilocal (e.g. $R := K$ a field and $S := K[X]$), thus S allows an infinite coindependent family $\{A_i\}_{\mathbb{N}}$ of right ideals. The right ideals of T, $\begin{pmatrix} A_i & M \\ 0 & R \end{pmatrix}_{i \in \mathbb{N}}$ will give an infinite family of coindependent submodules of T/I. Thus T/I has infinite hollow dimension, but its endomorphism ring is the semilocal ring R. $\Box$

Using the fact, that epimorphisms in modules with finite hollow dimensions are small (see 3.1.10) we can dualize [10, 5.16].

3.4.18. Endomorphism rings and artinian projective covers.
Let M be an indecomposable R-module with an artinian projective cover P in $\sigma[M]$. Then $S := End(M)$ is local and Jac (S) *is nil.*

Proof: An artinian module is amply supplemented, thus P is semiperfect in $\sigma[M]$ (see [67, 42.3(1)]). M is a factor module of P and so artinian and semiperfect in $\sigma[M]$ as well. Let $f \in S$. The descending chain

$$\mathrm{Im}\,(f) \supset \mathrm{Im}\,(f^2) \supset \mathrm{Im}\,(f^3) \supset \cdots$$

of submodules of M becomes stationary and hence $\mathrm{Im}\,(f^n) = \mathrm{Im}\,(f^{n+1})$ for some $n \in \mathbb{N}$. For $K = \mathrm{Ker}\,(f^n)$ we have $M = K + \mathrm{Im}\,(f^n)$. Since $(K)f \subseteq K$, f induces an epimorphism $\bar{f} : M/K \to M/K, (m + K) \mapsto (m)f + K$. Since M/K has finite hollow dimension, we get by 3.1.10 that $\bar{f}$ is small. As P is semiperfect every factor module of P has a projective cover. Let P_0 be a projective cover of M/K with small epimorphism $\pi : P_0 \to M/K$. Since (P_0, π) and $(P_0, \pi\bar{f})$ are projective covers of M/K we get an automorphism $g : P_0 \to P_0$ such that $g\pi = \pi\bar{f}$ (see [67, 19.5]).
We show, that $\bar{f}$ is an automorphism. Let $L := \mathrm{Ker}\,(\pi\bar{f}) \subset P_0$. For every $x \in L$ we have $(x)g\pi\bar{f} = (x)\pi\bar{f}\bar{f} = (0)\bar{f} = 0$. Thus $Lg \subseteq L$ holds. Since L is artinian the chain $Lg \supset Lg^2 \supset Lg^3 \supset \cdots$ has to stop. So there is a number k such that $Lg^k = Lg^{k+1}$. But since g is a monomorphism, we get $L = Lg$. This yields $\mathrm{Ker}\,(\pi\bar{f}) = L = Lg = (\mathrm{Ker}\,(g\pi))g \subseteq \mathrm{Ker}\,(\pi)$. Thus $\bar{f}$ is a monomorphism and hence an automorphism.
Consider an arbitrary element $m \in \mathrm{Ker}\,(f^{2n})$, then $(m)f^n \in K$ and hence $(m + K)\bar{f}^n = 0$. Since $\bar{f}$ is monomorph, $m \in K$ holds showing $K = \mathrm{Ker}\,(f^n) = \mathrm{Ker}\,(f^{2n})$. Since $\mathrm{Im}\,(f^n) = \mathrm{Im}\,(f^{2n})$ holds we get $M = \mathrm{Im}\,(f^n) \oplus \mathrm{Ker}\,(f^n)$.
But as M is indecomposable $\mathrm{Im}\,(f^n) = 0$ or $\mathrm{Ker}\,(f^n) = 0$ must hold. Thus f is nilpotent or an isomorphism. □

Remarks: A similar proof of the above theorem can be found in Takeuchi [58].

3.5 Chain conditions and hollow dimension

In this section we will state some results about the relationship between chain conditions and hollow dimension. We will need the first two lemmas to prove our first theorem of this section.

Lemma 3.5.1. *Let M be an R-module and $N_1 \subseteq N_2 \subseteq M$ submodules of M such that N_1 and N_2 have the same supplement in M. Then N_2 lies above N_1.*

Proof: Let L be a supplement of N_1 and N_2. Then $M = N_1 + L = N_2 + L$ implies $N_2 = N_1 + (N_2 \cap L)$. Assume that there is a submodule X of M with $M = N_2 + X$.

Then $M = N_1 + (N_2 \cap L) + X = N_1 + X$ as $N_2 \cap L \ll L$. By 1.1.2 N_2 lies above N_1 in M. $\square$

Lemma 3.5.2. *Let M be an R-module and $\{N_\lambda\}_\Lambda$ a coindependent family of proper submodules of M. Let $\mu \in \Lambda$ and assume that N_μ has a weak supplement L in M. Then $\{(L + (N_\lambda \cap N_\mu))/L\}_{\Lambda \setminus \{\mu\}}$ is a coindependent family of proper submodules in M/L.*

Proof: Let $\Lambda' := \Lambda \setminus \{\mu\}$ and $\lambda \in \Lambda'$. If $M = L + (N_\lambda \cap N_\mu)$ then $N_\mu = (N_\mu \cap L) + (N_\lambda \cap N_\mu)$ with $N_\mu \cap L \ll M$ since L is a weak supplement of N_μ in M. Hence

$$M = N_\lambda + N_\mu = N_\lambda + (N_\mu \cap L) = N_\lambda$$

holds. This is a contradiction to N_λ being a proper submodule. Moreover for $\lambda \in \Lambda'$ and a finite subset $F \subseteq \Lambda' \setminus \{\lambda\}$ we have:

$$\begin{aligned}(L + (N_\lambda \cap N_\mu)) + \bigcap_{i \in F}(L + (N_i \cap N_\mu)) &\supseteq L + (N_\lambda \cap N_\mu) + \left(\bigcap_{i \in F} N_i \cap N_\mu\right)\\ &= L + N_\mu \cap \left(N_\lambda + \bigcap_{i \in F \cup \{\mu\}} N_i\right)\\ &= L + N_\mu = M.\end{aligned}$$

$\square$

Let us recall that a coclosed submodule N of a module M has no proper submodule K such that N lies above K (i.e. $N/K \ll M/K$).

3.5.3. Chain conditions on coclosed submodules. *([59, 4.5, 4.6, 4.11])*
Let M be an R-module.

1. *If M has finite hollow dimension then M satisfies DCC and ACC on coclosed submodules.*
2. *If M is amply supplemented then the following are equivalent:*
   (a) *M has finite hollow dimension;*
   (b) *M has DCC on coclosed submodules;*
   (c) *M has ACC on coclosed submodules.*

Proof: (1) If M has finite hollow dimension, then for every descending chain $N_1 \supset N_2 \supset \cdots$ of submodules of M, there is an integer n, such that N_n lies above N_k for every $k \geq n$ (see 3.1.2(d)). If the N_i are coclosed, then this yields $N_n = N_k$ for all $k \geq n$. Let $0 =: N_0 \subset N_1 \subset N_2 \subset \cdots$ be an ascending chain of coclosed submodules of M. Since N_k does not lie above N_{k-1} for all $k > 0$ we get by 3.1.3 that M contains an infinite coindpendent family of submodules.
(2) Recall that coclosed submodules of a weakly supplemented module are supplements (see 1.2.1). $(a) \Rightarrow (b), (c)$ clear by (1).
$(b) \Rightarrow (c)$ If there is an ascending chain $N_1 \subset N_2 \subset \cdots$ of coclosed submodules of M, then, by hypothesis, for every integer i, there are supplements L_i of N_i, such that $L_1 \supset L_2 \supset \cdots$. Supplements are coclosed, so there is an integer n, such that $L_n = L_k$ for every $k \geq n$. By Lemma 3.5.1, N_k lies above N_n for every $k \geq n$ and thus $N_n = N_k$, because N_k is coclosed.
$(c) \Rightarrow (a)$ Assume that M contains an infinite coindependent family $\{N_\lambda\}_\Lambda$ of proper submodules. We show by induction that there exists a strictly ascending chain of supplements

$$L_1 \subset L_2 \subset L_3 \subset \cdots$$

in M such that M/L_k contains an infinite coindependent family of proper submodules for all $k \in \mathbb{N}$. Let $\mu \in \Lambda$, $\Lambda' := \Lambda \setminus \{\mu\}$ and L_1 a supplement of N_μ in M. By Lemma 3.5.2 we know, that $\{(L_1 + (N_\lambda \cap N_\mu))/L_1\}_{\Lambda'}$ is an infinite coindependent family of proper submodules of M/L_1. Now assume $k \geq 1$ and there exists an ascending chain $L_1 \subset L_2 \subset \cdots \subset L_k$ such that each L_i is a supplement in M and M/L_i contains an infinite coindependent family for all $1 \leq i \leq k$. Let $\{N_\lambda/L_k\}_\Lambda$ be an infinite coindependent family of proper submodules of M/L_k. Let $\mu \in \Lambda$ and choose a supplement L' of N_μ in M. Let $L_{k+1} := L_k + L'$. Then $L_{k+1}/L_k + N_\mu/L_k = M/L_k$ holds. As $N_\mu \cap L' \ll L'$ we get

$$(L_{k+1} \cap N_\mu)/L_k = (L_k + (L' \cap N_\mu))/L_k \ll (L_k + L')/L_k = L_{k+1}/L_k.$$

Thus L_{k+1}/L_k is a supplement of N_μ/L_k in M/L_k. Applying Lemma 3.5.2 M/L_{k+1} contains an infinite coindependent family. Hence if M contains an infinite coindependent family of proper submodules we can construct an ascending chain of supplements in M. $\square$

Remarks:

1. M need only to be supplemented for $(c) \Rightarrow (a)$.

2. Takeuchi defined in [59, pp 18] the notion of a *supplement composition series*:

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_n = M$$

such that for all $1 \leq i < n$ L_i is a supplement in M and there exists no supplement between L_{i+1} and L_i. If M is supplemented this is equivalent to L_i is a supplement in M and L_{i+1}/L_i is hollow for all $1 \leq i < n$. Let

$s.lg(M) := sup\{k :$ there exists a supplement composition series of length k in$M\}$.

Takeuchi proved in [62, 4.13] that for a supplemented module M $hdim(M) = s.lg(M)$ holds. Moreover Varadarajan proved a similar result in [62, 2.28].

In [6] Camillo gave a characterization of modules whose factor modules have finite uniform dimension. We will examine a dual version of Camillo's result in terms of hollow dimension.

Our first observation is easy, but useful.

Lemma 3.5.4. *Let M be an R-module. Then* $\mathrm{Soc}\,(M)$ *is finitely generated if and only if there exists a submodule K of M such that M/K is finitely cogenerated and* $\mathrm{Soc}\,(K) = 0$.

Proof: ($\Rightarrow$) Let K be a complement of $\mathrm{Soc}\,(M)$ in M. Note that $K \cap \mathrm{Soc}\,(M) = 0$, so $\mathrm{Soc}\,(K) = 0$, and $K \oplus \mathrm{Soc}\,(M) \trianglelefteq M$. Since K is closed in M, $(\mathrm{Soc}\,(M) \oplus K)/K$ is a finitely generated semisimple essential submodule of M/K. Hence M/K is finitely cogenerated as $\mathrm{Soc}\,(M/K) = (\mathrm{Soc}\,(M) \oplus K)/K$.
($\Leftarrow$) Since $K \cap \mathrm{Soc}\,(M) = 0$, we have $\mathrm{Soc}\,(M) \simeq (\mathrm{Soc}\,(M) \oplus K)/K \subseteq \mathrm{Soc}\,(M/K)$. Hence $\mathrm{Soc}\,(M)$ is finitely generated. □

Let us state Camillo's result (see [6]) and extend it a little bit (property (d)).

3.5.5. Modules whose factor modules have finite uniform dimension.
The following statements are equivalent for an R-module M:

(a) *Every factor module of M has finite uniform dimension;*

(b) *every factor module of M has finitely generated socle;*

(c) *every submodule N of M contains a finitely generated submodule K such that N/K has no maximal submodules;*

(d) every non-zero factor module M/N of M, has a finitely cogenerated factor module M/K such that K/N has no simple submodules.

Proof: For (a),(b),(c) see [10, Theorem 5.11]. For $(b) \Leftrightarrow (d)$ apply Lemma 3.5.4. □

Remarks:

1. Modules whose factor modules have finite uniform dimension are also called *q.f.d.* (quotients are finite dimensional).

2. Properties (c) and (d) in Theorem 3.5.5 can be seen as dual to each other.

3. A module is called a *Maxmodule* if every non-zero factor module contains a maximal submodule. It can be shown that M is a Maxmodule if and only if every submodule has small radical (see [57]). Moreover every submodule of a Maxmodule is a Maxmodule. Thus we see by property (c) from the above theorem that a Maxmodule whose factor modules have finite uniform dimension is noetherian.

Trying to state a similar theorem for hollow dimension we get the following:

3.5.6. Modules whose submodules have finite hollow dimension.
Let M be an R-module. Consider the following statements.

(i) Every submodule of M has finite hollow dimension.

(ii) For every submodule N of M, $N/Rad(N)$ is finitely cogenerated (and hence finitely generated, semisimple).

(iii) Every non-zero factor module M/N of M has a finitely cogenerated factor module M/K such that K/N has no simple submodule.

(iv) Every factor module of M has finite uniform dimension.

Then the following holds: $(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv)$.
Moreover, if $N/Rad(N)$ has essential socle for every $N \subseteq M$, then $(iii) \Rightarrow (ii)$ holds. Also if $Rad(N) \ll N$ for every $N \subseteq M$ (e.g. M is a Maxmodule), then $(ii) \Rightarrow (i)$ holds.

Proof: (i) $\Rightarrow$ (ii) For a module N with finite hollow dimension $N/Rad(N)$ is finitely generated and semisimple by 3.3.3.
(ii) $\Rightarrow$ (iii) Let N be a proper submodule of M. Then $\mathrm{Soc}\,(M/N) = H/N$ for some

$H \subseteq M$. Since H/N is semisimple it follows that $Rad(H) \subseteq N$ and hence H/N is finitely generated since it is a factor module of the finitely generated semisimple module $H/Rad(H)$. By Lemma 3.5.4, there exists a submodule K/N such that K/N has no simple submodules and M/K is finitely cogenerated.
(iii) $\Leftrightarrow$ (iv) By Theorem 3.5.5 above.
If $N/Rad(N)$ has essential socle for every $N \subseteq M$ then:
(iii) $\Rightarrow$ (ii). Let N be a submodule of M. Then, by assumption, for every $N \subseteq M$, $M/Rad(N)$ has a finitely cogenerated factor module M/K such that $K/Rad(N)$ has no simple submodules. So $(N \cap K)/Rad(N)$ has zero socle and is a submodule of $N/Rad(N)$ having essential socle. Hence $N \cap K = Rad(N)$ yielding $N/Rad(N) = N/(N \cap K) \simeq (N + K)/K$ is finitely cogenerated as M/K is finitely cogenerated.
If $Rad(N) \ll N$ for every $N \subseteq M$, then:
(ii) $\Rightarrow$(i) $hdim(N) = hdim(N/K)$ holds for $K \ll N$ (see 3.1.10). Thus $hdim(N) = hdim(N/Rad(N)) < \infty$ for every submodule N of M. $\square$

Remarks:

1. It is not true that a module M with finite hollow dimension has finite uniform dimension. For example consider $\begin{pmatrix} K & V \\ 0 & K \end{pmatrix}$ where K is a field and ${}_KV$ a vector space. Then $hdim(R) = 1$ as R is local but $udim({}_RR)$ is finite if and only if $dim_K(V)$ is finite.

2. In general, the converse of $(iv) \Rightarrow (ii)$ is false. For example consider $\mathbb{Z}$: ${}_{\mathbb{Z}}\mathbb{Z}$ is noetherian, hence ${}_{\mathbb{Z}}\mathbb{Z}$ has property (iv), but not property (ii) since ${}_{\mathbb{Z}}\mathbb{Z}/Rad({}_{\mathbb{Z}}\mathbb{Z})$ is not semisimple.

3. If a module M has property (i) of the theorem above, then every subfactor of M has finite uniform and finite hollow dimension and hence a semilocal endomorphism ring by the previous section.

Recall that a module is called *uniserial* if its lattice of submodules is linearly ordered.

Proposition 3.5.7. *Let M be an R-module. Then the following statements are equivalent:*

(a) M is uniserial;

(b) every non-zero submodule of M is hollow;

(c) every non-zero factor module of M is uniform.

Proof: $(a) \Rightarrow (b)$ Clear, since for two proper submodules K, L of M, $K+L = L \neq M$ or $K + L = K \neq M$ holds.
$(b) \Rightarrow (c)$ Let $0 \neq N \subset M$ and assume $L \cap N = 0$ for a submodule $L \subseteq M$. Then $N \oplus L \subseteq M$. But since $N \oplus L$ is hollow we have $L = 0$. Hence M is uniform. Since factor modules of hollow modules are hollow the same argument can be applied to any factor module of M.
$(c) \Rightarrow (a)$ Let $K \neq L$ be non-zero proper submodules of M. By hypothesis $M/(K \cap L)$ is uniform and $K/(K \cap L) \cap L/(K \cap L) = 0$ implies $K = K \cap L \subseteq L$ or $L = K \cap L \subseteq K$. □

Recall the definitions of modules with $AB5^*$ and completely coindependent families from Chapter 1.

Lemma 3.5.8. *Let M be an R-module and $\{N_\lambda\}_\Lambda$ a completely coindependent family of proper submodules in M. Let $N := \bigcap_\Lambda N_\lambda$. Then $\{N_\lambda/N\}_\Lambda$ is a completely coindependent family of proper submodules of M/N and if $|\Lambda| = \infty$, then M/N contains an infinite direct sum of submodules.*

Proof: Clearly $\{N_\lambda/N\}_\Lambda$ is a completely coindependent family in M/N. Thus by induction one can easily see, that for every finite subset $J \subseteq \Lambda$ there exists a decomposition

$$M/N \simeq \left(\bigoplus_{j \in J} (M/N_j) \right) \bigoplus \left(M/ \bigcap_{\mu \in \Lambda \setminus J} N_\mu \right).$$

If $|\Lambda| = \infty$, then M/N cannot have finite uniform dimension and must contain an infinite direct sum of submodules. □

Under certain conditions we can state a converse of Theorem 3.5.6.

3.5.9. $AB5^*$ modules whose factor module have finite uniform dimension. *Assume M satisfies $AB5^*$ such that every factor module of M has finite uniform dimension. Then every submodule of M has finite hollow dimension.*

Proof: (see Lemma 6 in [29]) If M has infinite hollow dimension, then there exists an infinite coindependent family of proper submodules $\{N_\lambda\}_\Lambda$. Since M has $AB5^*$; $\{N_\lambda\}_\Lambda$ is completely coindependent by Lemma 1.4.4. By Lemma 3.5.8 $M/\bigcap_\Lambda N_\lambda$ contains an infinite direct sum. Thus it does not have finite uniform dimension. The same argument applies for every submodule of M. □

Remarks: The above observations about hollow and uniform dimensions can also be found in [5] and [63, Proposition 13].

It is well-known that a linearly compact module M has property $AB5^*$ and has finite uniform dimension (see [67, 29.8]). Since every factor module of a linearly compact module is linearly compact (see [67, 29.8]) every factor module has finite uniform dimension. Thus we get as a corollary of the above theorem:

Corollary 3.5.10. *([69, Proposition 6],[59, 4.10])*
Every submodule of a linearly compact R-module M has finite hollow dimension.

Applying 3.4.1(3) this yields:

Corollary 3.5.11. *A linearly compact module has semilocal endomorphism ring.*

Al-Khazzi and Smith characterized modules with noetherian (artinian) radical in [2]. This dualizes [10, 5.15] and will be useful for the following observations.

3.5.12. Chain conditions on small submodules.
Let M be an R-module.

1. *M has ACC on small submodules if and only if* $\mathrm{Rad}\,(M)$ *is noetherian;*

2. *M has DCC on small submodules if and only if* $\mathrm{Rad}\,(M)$ *is artinian;*

Proof: (see [2, Proposition 2 and Theorem 5]) $\square$

Definition. An R-module M is called *semiartinian* if every non-zero factor module of M has a simple submodule.

Semiartinian modules are also called *Loewy modules* or *Min modules* (see [57]). They can be characterized by the following lemma:

Lemma 3.5.13. *([57, Proposition 2.1])*
A non-zero R-module M is semiartinian if and only if every factor module has essential socle.

Proof: If M is semiartinian then every factor module of M is semiartinian so it remains to show, that a semiartinian module has essential socle. Let N be a non-zero submodule of M and K a complement of N in M. Since K is closed, $N \simeq (N \oplus K)/K$ is essential in M/K. This implies $\mathrm{Soc}\,(M/K) = \mathrm{Soc}\,((N \oplus K)/K) \simeq \mathrm{Soc}\,(N)$. By hypothesis $0 \neq \mathrm{Soc}\,(M/K)$. Thus for every submodule N of M, $0 \neq \mathrm{Soc}\,(N) = N \cap \mathrm{Soc}\,(M)$ holds. Hence M has an essential socle. The converse is clear. $\square$

Remarks: It is easy to see, that for a semiartinian module M every subfactor (to be more precise every module in $\sigma[M]$) is semiartinian and combining this property with condition (iii) of 3.5.6, we see, that every factor module of M is finitely cogenerated, i.e. M is artinian.

Now we are able to state a comprehensive characterization of artinian modules in terms of hollow dimension.

3.5.14. Artinian modules.
The following statements are equivalent for an R-module M.

(a) M is artinian;

(b) every submodule of M is semiartinian with finite hollow dimension;

(c) M is semiartinian and one of the following properties hold:

  (i) M is linearly compact or

  (ii) every submodule of M has finite hollow dimension or

  (iii) every factor module of M has finite uniform dimension;

(d) M has finite hollow dimension and one of the following properties hold:

  (i) $\mathrm{Rad}\,(M)$ *is artinian or*

  (ii) M/N is finitely cogenerated for every small submodule N of M or

  (iii) every small submodule is artinian.

Proof: (a) $\Leftrightarrow$ (b) by 3.5.6 and above remarks;
(a) $\Leftrightarrow$ (c)(i) by applying [67, 41.10(2)];
(c)(i) $\Rightarrow$ (c)(ii) by 3.5.10; (c)(ii) $\Rightarrow$ (c)(iii) by 3.5.6; (c)(iii) $\Rightarrow$ (a) Assume every factor module has finite uniform dimension, then by 3.5.5 every factor module has finitely generated socle. Because M is semiartinian, the socle of every factor module is essential and hence every factor module of M is finitely cogenerated (see [67, 21.3]). Thus M is artinian.
(a) implies all properties in (d). Further (d)(ii) $\Rightarrow$ (d)(iii) and (d)(iii) $\Leftrightarrow$ (d)(i) by the Al-Khazzi Smith Theorem 3.5.12 so it remains to prove (d)(i) $\Rightarrow$ (a). But since M has finite hollow dimension $M/\mathrm{Rad}\,(M)$ is artinian. Hence M is artinian as $\mathrm{Rad}\,(M)$ is artinian. $\Box$

Remarks: (a) $\Leftrightarrow$ (c)(iii) was also proved by Shock in [57, Proposition 3.1], but with a different proof. Moreover Hanna & Shamsuddin proved (a) $\Leftrightarrow$ (d) in [24] without using Al-Khazzi and Smith's Theorem. $(d)(ii) \Rightarrow (a)$ and $(d)(iii) \Rightarrow (a)$ was proven in [49, 4.2,4.3].

For torsionfree abelian groups A the uniform dimension coincides with the ordinary finite rank of A; $udim(A) = dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} A)$ (see [19, 4L]). Moreover the only uniform $\mathbb{Z}$-modules are the ones that are isomorphic to $\mathbb{Z}_{p^k}$ for a prime p and $1 \leq k \leq \infty$ (see [33, Theorem 10]) or that are isomorphic to a torsionfree $\mathbb{Z}$-module with $dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$.

Let us now examine the situation for hollow dimension of *abelian groups*. Let $t(A)$ denote the torsion submodule of a $\mathbb{Z}$-module A. For basic group-theoretical notions we refer to [33].

3.5.15. Abelian groups with finite hollow dimension.
Let A be a $\mathbb{Z}$-module.

1. *If A is non-zero and torsionfree then $hdim(A) = \infty$.*
2. *A is hollow if and only if $A \simeq \mathbb{Z}_{p^k}$ for a prime number p and $1 \leq k \leq \infty$.*
3. *A has finite hollow dimension if and only if it is a finite direct sum of hollow modules.*

Proof: (1) If A is not reduced, then it contains a direct summand isomorphic to $\mathbb{Q}$ (see [33, Theorem 4]). Since $\mathbb{Q}/\mathbb{Z}$ is an infinite direct sum of non-zero modules, we conclude that $\mathbb{Q}$ has infinite hollow dimension.

Suppose A is reduced, and let P be the set of prime numbers p for which $pA \neq A$. Then $\{pA\}_{p \in P}$ forms a coindependent family of submodules of A since $pA + qA = A$ and $pA \cap qA = (pq)A$ holds for all relatively prime numbers p and q. If P is infinite, then A has infinite hollow dimension.

Assume that P is finite. Then $pA = A$ holds for infinitely many prime numbers p. Let these be $p_1, p_2, \ldots$ and let $0 \neq a \in A$. Since A is torsionfree and $p_i A = A$ holds division in A by each p_i is unique for all $i = 1, 2, \ldots$. Hence the elements $p_i^{-1}a + \mathbb{Z}a, p_i^{-2}a + \mathbb{Z}a, \ldots$ generate a direct summand of $A/\mathbb{Z}a$ isomorphic to $\mathbb{Z}_{p_i^\infty}$ for every $i = 1, 2, \ldots$. Hence $A/\mathbb{Z}a$ cannot have finite hollow dimension and so A cannot have finite hollow dimension.

(2) Let A be hollow. Since $A/t(A)$ is hollow and torsionfree, we get by (1) that $A = t(A)$. By [33, Theorem 10] we get that an abelian indecomposable torsion group is isomorphic to $\mathbb{Z}_{p^k}$ for a prime p and $1 \leq k \leq \infty$. Conversely $\mathbb{Z}_{p^k}$ is uniserial for all primes p and $1 \leq k \leq \infty$ and therefore hollow.

(3) Suppose that A has finite hollow dimension. Let $t(A)$ be the torsion submodule of A. By (1) $hdim(A/t(A)) = \infty$ and hence $hdim(A) = \infty$. Hence $A = t(A)$ is torsion. By induction on $hdim(A)$ we show that A is a finite direct sum of hollow $\mathbb{Z}$-modules. If A is hollow, we are done. Assume that all $\mathbb{Z}$-modules with $1 \leq hdim(A) \leq n$ are a finite direct sum of hollow modules. Let A be an abelian torsion group with $hdim(A) = n + 1$ and $n \geq 1$. Then A cannot be indecomposable. Thus there exists a decomposition $A = A_1 \oplus A_2$ with $hdim(A) = hdim(A_1) + hdim(A_2)$ and $A_1, A_2 \neq 0$. Hence $hdim(A_1) \leq n$ and by assumption A_1 is a finite direct sum of hollow modules. The same argument holds for A_2. So A is a finite direct sum of hollow modules. □

Remarks:

1. (1) and (3) of the above theorem were obtained from [24, Theorem 2.8]. See also [62, Proposition 1.13].

2. (2) was obtained from [62, Proposition 1.14] which arises from a characterization of hollow modules over Dedekind domains by Rangaswamy in [47].

Corollary 3.5.16. *A $\mathbb{Z}$-module has finite hollow dimension if and only if it is artinian.*

Proof: By 3.5.15(3) every $\mathbb{Z}$-module with finite hollow dimension is a finite direct sum of hollow modules. By 3.5.15(2) every hollow $\mathbb{Z}$-module is isomorphic to an artinian module of the form $\mathbb{Z}_{p^k}$ with p a prime and $1 \leq k \leq \infty$. Hence every $\mathbb{Z}$-module with finite hollow dimension is artinian. The converse is always true. □

Remarks: More general Zöschinger proved that a module over a commutative noetherian domain with infinitely many maximal ideals has finite hollow dimension if and only if it is artinian (see [74, Beispiel 3.9]).

A well-known theorem by Goodearl (see [18, Proposition 3.6] or [2, Proposition 4]) asserts that $M/\mathrm{Soc}\,(M)$ is noetherian if and only if every factor module M/N with N essential in M is noetherian. This can easily be extended to show that $M/\mathrm{Soc}\,(M)$ has Krull dimension if and only if M/N has Krull dimension for every essential submodule N of M (see [46, Proposition 2]). Dual to Goodearl's result Al-Khazzi and Smith proved that $Rad(M)$ is artinian if and only if every small submodule of M is artinian (see 3.5.12). Puczyłowski asked if Al-Khazzi and Smith's

Theorem can be extended for arbitrary Krull dimension and answered this question in the negative by showing that there exists a $\mathbb{Z}$-module M such that every small submodule is noetherian and hence has Krull dimension but $Rad(M)$ does not have Krull dimension (see [46, Example]).

We will show that the Al-Khazzi-Smith Theorem can be extended for arbitrary Krull dimension to modules which satisfy property $AB5^*$.

Let us first prove a useful lemma.

Lemma 3.5.17. *Let M be an R-module and $\{N_\lambda\}_\Lambda$ a completely coindependent family of proper submodules. Assume that for every $\lambda \in \Lambda$ there exists a submodule L_λ such that $N_\lambda \subset L_\lambda$. Let $L := \bigcap_{\lambda\in\Lambda} L_\lambda$. Then $\{N_\lambda \cap L\}_\Lambda$ forms a completely coindependent family of proper submodules in L.*

Proof: Let $\lambda \in \Lambda$, $L' := \bigcap_{\mu\neq\lambda} L_\mu$ and $N' := \bigcap_{\mu\neq\lambda} N_\mu$. Then

$$N_\lambda + L = N_\lambda + (L_\lambda \cap L') = L_\lambda \cap (N_\lambda + L') = L_\lambda \cap M = L_\lambda$$

(because $N_\lambda + L' \supseteq N_\lambda + N' = M$). Thus $N_\lambda \cap L$ is a proper submodule of L (otherwise $L \subseteq N_\lambda$ would imply $N_\lambda = N_\lambda + L = L_\lambda$, a contradiction).
Moreover:

$$\begin{aligned}(N_\lambda \cap L) + \bigcap_{\mu\neq\lambda}(N_\mu \cap L) &= L \cap \left(N_\lambda + \left(\bigcap_{\mu\neq\lambda} N_\mu \cap L\right)\right)\\ &= L \cap (N_\lambda + (N' \cap L' \cap L_\lambda))\\ &= L \cap (N_\lambda + (N' \cap L_\lambda))\\ &= L \cap (L_\lambda \cap (N_\lambda + N'))\\ &= L \cap L_\lambda \cap M = L.\end{aligned}$$

Thus $\{N_\lambda \cap L\}_\Lambda$ forms a completely coindependent family of proper submodules of L. □

3.5.18. Small submodules with finite hollow dimension.
Let M be an R-module having $AB5^*$ such that every small submodule of M has finite hollow dimension. Then every submodule of $Rad(M)$ has finite hollow dimension.

Proof: Consider first the following fact:
Let L, N be submodules of M such that L lies above N in M. We will show, that L/N has finite hollow dimension. First note that M is amply supplemented as it

has $AB5^*$ (see [67, 47.9]). If L is small then by hypothesis L and so L/N has finite hollow dimension.
Assume L is not small in M and let K be a (weak) supplement of L in M. Then $M = L + K = N + K$ implies $L = N + (L \cap K)$. Hence $L/N \simeq (L \cap K)/(N \cap K)$. Since $L \cap K \ll M$ we get by hypothesis, that $L \cap K$, and so L/N has finite hollow dimension.
Let G be a submodule of $Rad(M)$ with $G \not\ll M$ and assume H is a (weak) supplement for G in M. Then $H \cap G \ll M$ and the following sequence is exact:

$$0 \longrightarrow H \cap G \longrightarrow G \longrightarrow M/H \longrightarrow 0.$$

Thus $hdim(G) \leq hdim(H \cap G) + hdim(M/H)$ (see 3.1.10(6)).
Since $H \cap G$ is small in M, $hdim(H \cap G) < \infty$, by assumption. It is enough to show that M/H has finite hollow dimension:
Assume that M/H contains an infinite coindependent family $\{N_\lambda/H\}_\Lambda$ of proper submodules of M/H. For any $\lambda \in \Lambda$ we have $N_\lambda + G = M$. Since $G \subseteq Rad(M)$ and N_λ is a proper submodule of M, there exists an element $x \in Rad(M) \setminus N_\lambda$ such that $Rx \ll M$ and

$$L_\lambda := N_\lambda + Rx \neq N_\lambda.$$

Let $N := \bigcap_\Lambda N_\lambda$ and $L := \bigcap_\Lambda L_\lambda$. Applying Lemma 1.4.4, every coindependent family is completely coindependent and, applying Lemma 3.5.17, we get that $\{N_\lambda \cap L\}_\Lambda$ is a completely coindependent family of L. Since $N \subseteq N_\lambda \cap L \neq L$ holds for all $\lambda \in \Lambda$ we get that $N \subsetneq L$. By Lemma 3.5.8 L/N does not have finite hollow dimension. But since L_λ lies above N_λ for all $\lambda \in \Lambda$, we get by applying Lemma 1.4.5 that L lies above N in M, and thus, by the above argument, L/N has finite hollow dimension. This contradiction shows that M/H must have finite hollow dimension. Hence every submodule $G \subseteq Rad(M)$ has finite hollow dimension. □

We refer to [10, Chapter 6] for the definition of *Krull dimension*. Note the following result by Lemonnier. This will help us to prove a corollary to the above theorem.

Proposition 3.5.19. *Let M be an R-module such that every non-zero factor module of M has finite uniform dimension and contains a non-zero submodule having Krull dimension. Then M has Krull dimension.*

Proof: See [35, Proposition 1.3]. □

Corollary 3.5.20. *Let M be an R-module having $AB5^*$ such that every small submodule of M has Krull dimension. Then $Rad(M)$ has Krull dimension.*

Proof: It is well-known that a module having Krull dimension has finite uniform dimension (see [10, 6.2]). Hence every factor module of a small submodule N of M has finite uniform dimension. Since N has $AB5^*$ every submodule of N has finite hollow dimension by 3.5.9. Hence by 3.5.18 every submodule of $Rad(M)$ has finite hollow dimension. By 3.5.6 every factor module of $Rad(M)$ has finite uniform dimension. In order to apply Lemonnier's proposition, we need to show, that every non-zero factor module of $Rad(M)$ contains a non-zero submodule having Krull dimension. Let $L \subset Rad(M)$ and $x \in Rad(M) \setminus L$; then $Rx \ll M$ so that Rx has Krull dimension and hence $(Rx+L)/L \subseteq Rad(M)/L$ has Krull dimension. Applying Proposition 3.5.19, $Rad(M)$ has Krull dimension. □

Corollary 3.5.21. *Let M be an R-module such that $Rad(M)$ has $AB5^*$ and every small submodule of M has Krull dimension. Then every submodule of $Rad(M)$ that has a weak supplement in M has Krull dimension.*

Proof: By Corollary 3.5.20, the radical of every submodule contained in $Rad(M)$ has Krull dimension. Since $Rad(N) = N \cap Rad(M)$ holds for every supplement N in M (see 1.2.1), every supplement in M that is a submodule of $Rad(M)$ has Krull dimension. Let $L \subseteq Rad(M)$ such that there exists a $K \subseteq M$ with $L + K = M$ and $L \cap K \ll M$. Then $Rad(M) = L + (Rad(M) \cap K)$. Since $Rad(M)$ has $AB5^*$ it is amply supplemented. Thus there exists a supplement $N \subseteq L$ in $Rad(M)$ such that $Rad(M) = N + (Rad(M) \cap K)$ and $N \cap Rad(M) \cap K = N \cap K \ll N$ holds. Moreover $L = N + (L \cap K)$ and $M = N + K$ holds. Thus N is a supplement of K in M, implying that N has Krull dimension. Because $L/N \simeq (L \cap K)/(N \cap K)$ with $L \cap K \ll M$, L/N has Krull dimension and hence so has L. □

The following result is an attempt to dualize [2, Proposition 3].

3.5.22. Essential submodules with finite hollow dimension.
Consider the following statements for an R-module M.

(i) *$M/\mathrm{Soc}\,(M)$ has finite hollow dimension.*

(ii) *There exists an integer $n \in \mathbb{N}$ such that for every essential submodule N of M, $hdim(M/N) \leq n$.*

(iii) There exists an integer $n \in \mathbb{N}$ such that every coindependent family of essential submodules of M has at most n elements.
Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) holds.

Proof: (i) $\Rightarrow$ (ii) If $M/\mathrm{Soc}(M)$ has finite hollow dimension, then so has every factor module of $M/\mathrm{Soc}\,(M)$. Set $n := hdim(M/\mathrm{Soc}\,(M))$.
(ii) $\Rightarrow$ (iii) Note that the intersection of a finite number of essential modules is essential again. Let $\{N_1, \ldots, N_k\}$ be a coindependent family of essential submodules in M and $N := N_1 \cap \ldots \cap N_k$. By 1.4.1 $M/N \simeq M/N_1 \oplus \cdots \oplus M/N_k$ holds and thus $n \geq hdim(M/N) \geq k$ implies (iii).
(iii) $\Rightarrow$ (ii) Let N be an essential submodule of M and $\{N_1/N, \ldots, N_k/N\}$ a coindependent family of M/N. Then $\{N_1, \ldots, N_k\}$ is a coindependent family of M too. Hence $n \geq k$ implies (ii).
□

3.6 $AB5^*$ and hollow dimension

In this chapter we will establish equivalent conditions for a module to be lattice anti-isomorphic to a linearly compact module. First note the following lemma:

Lemma 3.6.1. *Let M be an R-module with property $AB5^*$ such that the socle of every factor module of M contains only a finite number of non-isomorphic simple modules. Then every factor module of M has finite uniform dimension.*

Proof: Every submodule of a factor module has $AB5^*$ (see [67, 47.9(i)]) and whenever N is a module with property $AB5^*$ such that $N \simeq E^{(\Lambda)}$ holds then Λ must be finite (see [67, 47.9(iii)]). Hence the socle of a module having $AB5^*$ cannot contain a summand that is isomorphic to an infinite direct sum of copies of a simple module. By hypothesis every factor module has only a finite number of non-isomorphic simple modules. Hence we conclude that the socle of every factor module has to be a finite direct sum of simple modules. By 3.5.5 every factor module has finite uniform dimension. □

Remarks:

1. Note that every simple module in $\sigma[M]$ is a factor module of a submodule of M. This can easily be verified: let E be a simple submodule of a M-generated module X. Let $f : M^{(\Lambda)} \to X$ be an epimorphism for an index set

Λ. Since $E = Rx$ with $x \in X$ we get that there is an element $(m_\lambda)_\Lambda$ such that $((m_\lambda)_\Lambda)f = x$. Only finitely many m_λ's are not zero; say $m_1, \ldots, m_k$. Thus f induces an epimorphism from $\sum_{i=1}^k Rm_i \subseteq M$ to E.

2. An R-module is called a *self-generator* if it generates all its submodules. Let M be a self-generator such that $M/\mathrm{Rad}\,(M)$ is semisimple and finitely generated. Then every simple module in $\sigma[M]$ is isomorphic to a simple module of $M/\mathrm{Rad}\,(M)$. Thus $\sigma[M]$ contains only a finite number of non-isomorphic simple modules. Thus every module $N \in \sigma[M]$ with $AB5^*$ satisfies the hypothesis of Lemma 3.6.1 and hence every submodule of N has finite hollow dimension by 3.5.9. In case $M = R$ we get: If R is semilocal and M an R-module with $AB5^*$ then every submodule of M has finite hollow dimension.

Definition. Let R and T be rings, ${}_RM$ a left R-module and N_T a right T-module. A mapping $\alpha : \mathcal{L}({}_RM) \to \mathcal{L}(N_T)$ is called a *lattice anti-isomorphism* if it is an order reversing lattice isomorphism.

Lemma 3.6.2. *Let R and T be rings, $M \in R-\mathrm{Mod}$ and $N \in \mathrm{Mod}-T$. Assume that $\alpha : \mathcal{L}({}_RM) \to \mathcal{L}(N_T)$ is a lattice anti-isomorphism. Then ${}_RM$ and N_T have property $AB5^*$.*

Proof: This lemma is quite obvious, but for the sake of completeness we will state a proof here. Let $\{K_\lambda\}_\Lambda$ be a family of submodules of M. For every $\lambda \in \Lambda$ we have $\alpha(K_\lambda) \subseteq \alpha(\bigcap_\Lambda K_\lambda)$. Thus $\sum_\Lambda \alpha(K_\lambda) \subseteq \alpha(\bigcap_\Lambda K_\lambda)$. On the other hand let $\{L_\lambda\}_\Lambda$ be a family of submodules of N. Then for every $\lambda \in \Lambda$ we have $\alpha^{-1}(L_\lambda) \supseteq \alpha^{-1}(\sum_\Lambda L_\lambda)$. Thus $\bigcap_\Lambda \alpha^{-1}(L_\lambda) \supseteq \alpha^{-1}(\sum_\Lambda L_\lambda)$. Hence $\alpha(\bigcap_\Lambda \alpha^{-1}(L_\lambda)) \subseteq \sum_\Lambda L_\lambda$ holds. Letting $L_\lambda := \alpha(K_\lambda)$ for every $\lambda \in \Lambda$ we get $\alpha(\bigcap_\Lambda K_\lambda) = \sum_\Lambda \alpha(K_\lambda)$. Let L be a submodule of M and $\{K_\lambda\}$ be an inverse family of submodules of M. It is easy to see that α carries inverse families of M to direct families of N. Together with the foregoing we get:

$$\begin{aligned}
\alpha(L + \bigcap_{\lambda\in\Lambda} K_\lambda) &= \alpha(L) \cap \alpha(\bigcap_{\lambda\in\Lambda} K_\lambda) = \alpha(L) \cap \sum_{\lambda\in\Lambda} \alpha(K_\lambda) \\
&= \sum_{\lambda\in\Lambda} (\alpha(L) \cap \alpha(K_\lambda)) = \sum_{\lambda\in\Lambda} (\alpha(L + K_\lambda)) \\
&= \alpha(\bigcap_{\lambda\in\Lambda} (L + K_\lambda)).
\end{aligned}$$

Hence $L + \bigcap_\Lambda K_\lambda = \bigcap_\Lambda (L + K_\lambda)$ implies that M has property $AB5^*$. The same argument holds for N. $\square$

Remarks: Let M be an R-module and let $\{E_\lambda\}_\Lambda$ be a minimal representing set of the isomorphism classes of simple modules in $\sigma[M]$. Then the M-injective hull of $\bigoplus_\Lambda E_\lambda$ always forms a 'minimal' injective cogenerator in $\sigma[M]$ with essential socle (see [67, 16.5, 17.12]). Hence there always exists an injective cogenerator with essential socle in $\sigma[M]$.

Let $_RQ$ be an injective cogenerator in $\sigma[M]$. Let $T = \text{End}\,(Q)$, $N \in \sigma[M]$ and $N^* := \text{Hom}\,(N, Q)$. Recall the definitions from Chapter 3.1 for submodules $K \subseteq N$ and $X \subseteq N^*$: $An(K) := \{f \in N^* | (K)f = 0\}$ and $Ke(X) := \bigcap_{g \in X} \text{Ker}\,(g)$ and the properties $(AC1) - (AC3)$. Note that the mappings $An(-)$ and $Ke(-)$ are order reversing. By definition we have for all $X, Y \subseteq N^*$:

$$Ke(X) \cap Ke(Y) \supseteq Ke(X + Y) \text{ and } Ke(X) + Ke(Y) \subseteq Ke(X \cap Y).$$

Lemma 3.6.3. *Let M be an R-module, $_RQ$ an injective cogenerator in $\sigma[M]$, $T :=$ End (Q) and $N \in \sigma[M]$. Then the mappings $An : \mathcal{L}(N) \to \mathcal{L}(N^*)$ and $Ke : \mathcal{L}(N^*) \to \mathcal{L}(N)$ carry inverse families to direct families and direct families to inverse families.*

Proof: This follows easily from the following four observations:

Let K_λ, K_μ, K_ν be submodules of N.

(1) If $K_\lambda + K_\mu \subseteq K_\nu$ then $An(K_\lambda) \cap An(K_\mu) = An(K_\lambda + K_\mu) \supseteq An(K_\nu)$.

(2) If $K_\lambda \cap K_\mu \supseteq K_\nu$ then $An(K_\lambda) + An(K_\mu) = An(K_\lambda \cap K_\mu) \subseteq An(K_\nu)$.

Let X_λ, X_μ, X_ν be submodules of N^*.

(3) If $X_\lambda + X_\mu \subseteq X_\nu$ then $Ke(X_\lambda) \cap Ke(X_\mu) \supseteq Ke(X_\lambda + X_\mu) \supseteq Ke(X_\nu)$.

(4) If $X_\lambda \cap X_\mu \supseteq X_\nu$ then $Ke(X_\lambda) + Ke(X_\mu) \subseteq Ke(X_\lambda \cap X_\mu) \subseteq Ke(X_\nu)$. $\square$

Remarks: (1) Let $\{X_\lambda\}_\Lambda$ be a direct family of submodules of N^* then $Ke(\sum_\Lambda X_\lambda) \subseteq \bigcap_\Lambda Ke(X_\lambda)$ holds. On the other hand let $x \in \bigcap_\Lambda Ke(X_\lambda)$ and $g \in \sum_\Lambda X_\lambda$. Then $g \in X_{\lambda_1} + \cdots + X_{\lambda_k}$. Since $\{X_\lambda\}_\Lambda$ is direct we get $g \in X_\mu$ for an index $\mu \in \Lambda$. Thus $(x)g = 0$ and hence

$$Ke(\sum_\Lambda X_\lambda) = \bigcap_\Lambda Ke(X_\lambda).$$

(2) Let $\{K_\lambda\}_\Lambda$ be a direct family of submodules of N then a similar argument as in (1) shows that

$$An(\sum_\Lambda K_\lambda) = \bigcap_\Lambda An(K_\lambda)$$

holds.

The next theorem was obtained from Ánh, Herbera and Menini in [3].

3.6.4. Modules anti-isomorphic to a linearly compact module*([3, 1.2].*
Let M be an R-module, ${}_RQ$ be an injective cogenerator in $\sigma[M]$ and $T := \mathrm{End}\,({}_RQ)$. For every module $N \in \sigma[M]$ the following statements are equivalent:

(a) For any inverse family $\{K_\lambda\}_\Lambda$ of N,

$$An\left(\bigcap_{\lambda\in\Lambda} K_\lambda\right) = \sum_{\lambda\in\Lambda} An(K_\lambda).$$

(b) The mapping $K \mapsto An(K)$ is a lattice anti-isomorphism of $\mathcal{L}({}_RN)$ into $\mathcal{L}(N_T^*)$ whose inverse is given by the mapping $X \mapsto Ke(X)$.

In this case N_T^* is linearly compact, ${}_RN$ has property $AB5^*$ and every submodule of ${}_RN$ has finite hollow dimension. If ${}_RQ$ has essential socle then (a) and (b) are also equivalent to:

(c) ${}_RN$ has property $AB5^*$ and every factor module of ${}_RN$ does not contain an infinite number of non-isomorphic simple modules.

(d) ${}_RN$ is anti-isomorphic to a linearly compact right S-module with S a ring.

Proof: (a) $\Rightarrow$ (b) Let $X \subseteq N^*$ and denote by $\mathcal{F}$ the set of all finitely generated submodules of X. Since $F_1 + F_2$ is again a finitely generated submodule of X, $\{F\}_{F\in\mathcal{F}}$ forms a direct family of submodules of N^*. By 3.6.3 $\{Ke(F)\}_{F\in\mathcal{F}}$ forms an inverse family of submodules of M. Thus by (a) and (AC2) we get:

$$X = \sum_{F\in\mathcal{F}} F = \sum_{F\in\mathcal{F}} An(Ke(F)) = An\left(\bigcap_{F\in\mathcal{F}} Ke(F)\right) = An(Ke(X)),$$

since $Ke(X) = \bigcap_{f\in X} \mathrm{Ker}\,(f) = \bigcap_{f\in X} Ke(fT) = \bigcap_{F\in\mathcal{F}} Ke(F)$. Thus $X = An(Ke(X))$ for all $X \subseteq N^*$. By (AC1) we have $K = Ke(An(K))$ for all $K \subseteq N$. For submodules $X, Y \subseteq N^*$ we have

$$X + Y = An(Ke(X)) + An(Ke(Y)) = An(Ke(X) \cap Ke(Y)).$$

Hence $Ke(X+Y) = Ke(X) \cap Ke(Y)$ holds. $Ke(X \cap Y) = Ke(X) + Ke(Y)$ can be shown similarly. Thus $An(-)$ and $Ke(-)$ are lattice anti-isomorphisms and each others inverses.

(b) $\Rightarrow$ (a) By the above remarks, we have for a direct family $\{X_\lambda\}_\Lambda$ of submodules of N^*

$$Ke(\sum_{\lambda\in\Lambda} X_\lambda) = \bigcap_{\lambda\in\Lambda} Ke(X_\lambda).$$

Hence for any inverse family $\{N_\lambda\}_\Lambda$ of submodules of N^*:

$$An\left(\bigcap_{\lambda\in\Lambda} N_\lambda\right) = An\left(\bigcap_{\lambda\in\Lambda} Ke(An(N_\lambda))\right) = An(Ke(\sum_{\lambda\in\Lambda} An(N_\lambda))) = \sum_{\lambda\in\Lambda} An(N_\lambda).$$

N and N^* have property $AB5^*$ by 3.6.2. Let us check that N^* is linearly compact. Let $\{X_\lambda\}_\Lambda$ be an inverse family of submodules of N^* and $(f_\lambda + X_\lambda)_\Lambda \in \varprojlim N^*/X_\lambda$. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \sum_{\lambda\in\Lambda} Ke(X_\lambda) & \xrightarrow{i} & N \\ & & {\scriptstyle\alpha}\big\downarrow & & \\ & & Q & & \end{array}$$

with canonical inclusion i and α defined as follows: $(\sum_\Lambda k_\lambda)\,\alpha := \sum_\Lambda (k_\lambda) f_\lambda$ for all elements $k_\lambda \in Ke(X_\lambda)$ and $\lambda \in \Lambda$. Clearly α is an R-module homomorphism. Since ${}_RQ$ is injective in $\sigma[M]$ we get a homomorphism $f \in N^*$ such that $\alpha = if$. Thus for every $\lambda \in \Lambda$ we have $0 = (Ke(X_\lambda))(f-\alpha) = (Ke(X_\lambda))(f - f_\lambda)$. Hence $f - f_\lambda \in An(Ke(X_\lambda)) = X_\lambda$ implies that $f \equiv f_\lambda \mod X_\lambda$ holds for every $\lambda \in \Lambda$. Hence the following sequence

$$0 \longrightarrow \bigcap_\Lambda X_\lambda \longrightarrow N^* \longrightarrow \varprojlim N^*/X_\lambda \longrightarrow 0$$

is exact. Thus N^* is linearly compact.

Since N^* is linearly compact every factor module has finite uniform dimension. By 3.1.12 we get for every submodule $K \subseteq N$: $hdim(K) = udim(\mathrm{Hom}\,(K,Q))$. Since $\mathrm{Hom}\,(K,Q)$ is a factor module of $N^* = \mathrm{Hom}\,(N,Q)$ it has finite uniform dimension. Hence every submodule of ${}_RN$ has finite hollow dimension.

$(a)+(b) \Rightarrow (c)$ Assuming (a) or (b) yields that every submodule of N has finite hollow dimension and by 3.5.6, 3.5.5 that every factor module of N has finitely generated socle. Thus every factor module of N contains only a finite number of simple modules.

Assume that ${}_RQ$ has essential socle. We show (c) $\Rightarrow$(a). By Lemma 3.6.1 every factor module of N has finitely generated socle. Hence for every $f \in N^*$ we have $\mathrm{Soc}\,(N/\mathrm{Ker}\,(f)) = \mathrm{Soc}\,(\mathrm{Im}\,(f)) \subseteq Q$ is finitely generated. Since ${}_RQ$ has essential socle, $\mathrm{Soc}\,(\mathrm{Im}\,(f))$ is essential in $\mathrm{Im}\,(f)$. Thus $\mathrm{Im}\,(f) \simeq N/\mathrm{Ker}\,(f)$ is finitely cogenerated.

Let $\{K_\lambda\}_\Lambda$ be an inverse family of submodules of N. Since $An(K_\lambda) \subseteq An(\bigcap_\Lambda K_\lambda)$ for all $\lambda \in \Lambda$ we get $\sum_\Lambda An(K_\lambda) \subseteq An(\bigcap_\Lambda K_\lambda)$. We will show that $An(\bigcap_\Lambda K_\lambda) \subseteq \sum_\Lambda An(K_\lambda)$ holds. Let $f \in An(\bigcap_\Lambda K_\lambda)$. Then $\mathrm{Ker}\,(f) \supseteq \bigcap_\Lambda K_\lambda$. Since N has $AB5^*$

we have

$$\bigcap_{\lambda\in\Lambda}((K_\lambda+\mathrm{Ker}\,(f))/\mathrm{Ker}\,(f)) = \left((\bigcap_{\lambda\in\Lambda}K_\lambda)+\mathrm{Ker}\,(f)\right)/\mathrm{Ker}\,(f)=0.$$

By the above remarks, $N/\mathrm{Ker}\,(f)$ is finitely cogenerated. Hence by [67, 14.7] there exists a finite subset $F\subseteq\Lambda$ such that $\bigcap_{i\in F}(K_i+\mathrm{Ker}\,(f)) = \mathrm{Ker}\,(f)$. Hence $\bigcap_{i\in F}K_i\subseteq\mathrm{Ker}\,(f)$. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & N/\bigcap_{i\in F}K_i & \xrightarrow{\epsilon} & \bigoplus_{i\in F}N/K_i \\ & & \alpha\big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}\,(f) & \longrightarrow & Q \end{array}$$

with $\alpha: n+\bigcap_F K_i\mapsto n+\mathrm{Ker}\,(f)$ and ϵ the inclusion map. Since ${}_RQ$ is injective in $\sigma[M]$ there exists a homomorphism $\phi:\bigoplus_{i\in F}N/K_i\to Q$ which makes the diagram commute. Hence for every $n\in N$:

$$(n)f=((n+K_i)_{i\in F})\phi$$

holds. Define for every $k\in F$ the following composed map

$$f_k: N\xrightarrow{\pi} N/K_k\xrightarrow{i}\bigoplus_{i\in F}N/K_i\xrightarrow{\phi} Q,$$

with π the canonical projection and i the inclusion map. Note that $(n)f_k := ((\delta_{ik}n+K_i)_{i\in F})\phi$ holds for all $n\in N$ where $\delta_{ik}\in R$ denotes the Kronecker symbol. Then clearly $(K_k)f_k=0$ and hence $f_k\in An(K_k)$ holds for every $k\in F$. Since $(n)f = ((n+K_i)_{i\in F})\phi=\sum_{i\in F}(n)f_i$ holds, we get $f\in\sum_{i\in F}An(K_i)\subseteq\sum_\Lambda An(K_\lambda)$. Hence we have proved that (a) holds.

(b) $\Rightarrow$ (d) is obvious. We show $(d)\Rightarrow(c)$: By 3.6.2 we see that N has property $AB5^*$. Let us assume that Y_S is a linearly compact module over an appropriate ring S. Note that for any submodule K of N, $\mathcal{L}(N/K)$ can be seen as the sublattice $[K,N]\in\mathcal{L}(N)$ and $\alpha([K,N])$ can be seen as a lattice of submodules of a submodule of Y. It is easy to check, that independent families of submodules of Y are carried over by α to coindependent families of submodules of N. Since Y is linearly compact, every submodule has finite hollow dimension, i.e. contains no infinite coindependent family of submodules. Thus every factor module of N contains no infinite independent family of submodules, i.e. it has finite uniform dimension. □

Corollary 3.6.5. *([3, 1.3]) Let R be a ring, ${}_RQ$ an injective cogenerator in $R-$Mod and $T:=\mathrm{End}\,(Q)$. Then the following statements are equivalent:*

(a) For any inverse family $\{L_\lambda\}_\Lambda$ of left ideals of R

$$An(\bigcap_\Lambda L_\lambda) = \sum_\Lambda An(L_\lambda).$$

(b) $An : \mathcal{L}({}_RR) \to \mathcal{L}(Q_T)$ is a lattice anti-isomorphism with inverse $Ke(-)$.

In this case Q_T is linearly compact and ${}_RR$ has $AB5^*$. Moreover every submodule of a finitely generated R-module has finite uniform dimension, finite hollow dimension and a semilocal endomorphism ring. If ${}_RQ$ has an essential socle then (a) and (b) are also equivalent to

(c) ${}_RR$ has $AB5^*$.

Proof: Recall that $hdim(M) = udim(\text{Hom}\,(M, Q))$ holds for all $M \in R-$Mod.

(c) $\Rightarrow$ (a) ${}_RR$ is semiperfect whenever it has $AB5^*$ (see [67, 47.9]). Hence there is only a finite number of non-isomorphic simple R-modules. □

Corollary 3.6.6. *Let M be an R-module such that there is only a finite number of non-isomorphic simple modules in $\sigma[M]$. Let ${}_RQ$ be an injective cogenerator in $\sigma[M]$ with essential socle. Then for every $N \in \sigma[M]$ the following statements are equivalent:*

(a) ${}_RN$ has $AB5^*$;

(b) $An : \mathcal{L}({}_RN) \to \mathcal{L}(N_T^*)$ is a lattice anti-isomorphism;

(c) ${}_RN$ is lattice anti-isomorphic to a linearly compact module.

In this case every module in $\sigma_f[N]$ has finite uniform dimension, finite hollow dimension and a semilocal endomorphism ring.

Proof: Since there are only finitely many non-isomorphic simple modules in $\sigma[M]$ every factor module of $N \in \sigma[M]$ has only finitely many non-isomorphic simple modules. Then $(a) \Leftrightarrow (b)$ and $(a) \Leftrightarrow (c)$ follow by 3.6.4.

Since every submodule of N has finite hollow dimension by 3.6.4 every submodule of a finitely N-generated module has finite hollow dimension. Hence every $L \in \sigma_f[N]$ has finite hollow dimension, finite uniform dimension by 3.5.6 and a semilocal endomorphism ring by 3.4.1. □

Remarks: A semilocal ring R has the property that there are only finitely many non-isomorphic simple modules in $R-$Mod. Since we can always choose an injective cogenerator with essential socle we get by the last corollary that an R-module M has $AB5^*$ if and only if it is anti-isomorphic to a linearly compact module.

Let us summarize the relationship between uniform and hollow dimension under the hypothesis of $AB5^*$.

Corollary 3.6.7. *Let M be an R-module. Then the following statements are equivalent:*

(a) M has $AB5^*$ and one of the following properties hold:

(i) every submodule of M has finite hollow dimension, or

(ii) every factor module $N/\mathrm{Rad}\,(N)$ with $N \subseteq M$ is finitely generated, or

(iii) every factor module of M has finite uniform dimension.

(b) ${}_RM$ is lattice anti-isomorphic to a linearly compact module.

In this case every module $N \in \sigma_f[M]$ has property $AB5^*$, finite uniform dimension, finite hollow dimension and a semilocal endomorphism ring.

Proof: (a)(i) $\Rightarrow$ (a)(ii) by 3.5.6. By [67, 47.9(1)] every module N with $AB5^*$ is amply supplemented. Hence $N/\mathrm{Rad}\,(N)$ is semisimple. Thus (a)(ii) $\Rightarrow$ (a)(iii) by 3.5.6. (a)(iii) $\Rightarrow$ (a)(i) follows from 3.5.9.

(a) $\Leftrightarrow$ (b) Choose an injective cogenerator ${}_RQ$ in $\sigma[M]$ with essential socle and apply 3.6.4. $\square$

Chapter 4

The lifting property

Consider the following list of properties for an R-module M (see [39, pp 18]):

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is also a direct summand.

(C_3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

An R-module M is called *continuous* if it has (C_1) and (C_2); M is called *quasi-continuous* or *π-injective* if it has (C_1) and (C_3) and M is called an *extending* or *CS-module* if it has property (C_1). For more information about these notions we refer to [10] and [39].

Extending modules can be seen as a generalization of injective modules and the development of this notion can be tracked down to von Neumann's work on continuous geometry (see [40]). The following hierarchy of properties holds:

$$\text{injective} \Rightarrow \text{self-injective} \Rightarrow \text{continuous} \Rightarrow \pi\text{-injective} \Rightarrow \text{extending}.$$

Let us dualize each property $(C_1), (C_2), (C_3)$ (see [39, pp 57]):

(D_1) Every submodule of M lies above a direct summand of M.

(D_2) If $N \subseteq M$ such that M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

(D_3) If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a direct summand of M.

A module M is called *discrete* if it has (D_1) and (D_2); M is called *quasi-discrete* if it has (D_1) and (D_3) and M is called *lifting* if it has property (D_1). In [59] Takeuchi called lifting modules *codirect*. The properties $(D2)$ and $(D3)$ are called *Condition (I)* and *Condition (II)*. In [41] Oshiro called (quasi-)discrete modules *(quasi-)semiperfect*.

A module M is called *π-projective* or (*co-continuous*) if for every two submodules N, L of M with $N + L = M$ there exists an endomorphism $f \in \mathrm{End}\,(M)$ with

$$\mathrm{Im}\,(f) \subset N \text{ and } \mathrm{Im}\,(1-f) \subset L.$$

Theorem 4.1.9 shows that a module is quasi-discrete if and only if it is supplemented and π-projective .

A lot of use was made of the existence of complements in the study of extending modules. Under the assumption that there are supplements in a module we get the following dualized hierarchy for **supplemented** modules.

$$\text{projective} \Rightarrow \text{self-projective} \Rightarrow \text{discrete} \Rightarrow \pi\text{-projective} \Rightarrow \text{lifting}.$$

Clearly 'projective' $\Rightarrow$ 'self-projective' and 'discrete' $\Rightarrow$ 'quasi-discrete' $\Rightarrow$ 'lifting'. As mentioned we will see that a module is quasi-discrete if and only if it is supplemented and π-projective . A self-projective supplemented module is π-projective supplemented and hence lifting. It is easy to check that a self-projective module has property (D_2). A projective supplemented module is nothing but a semiperfect module. Therefore discrete, quasi-discrete and lifting modules can be seen as a generalization of semiperfect modules.

4.1 Lifting modules

Recall that we say for submodules $L \subseteq N \subseteq M$, N *lies above* L (in M) if $N/L \ll M/L$ and we say that a submodule N of M is *coclosed* (in M) if N does not lie above any submodule of N.

A submodule $N \subseteq M$ is a *supplement* in M if and only if it is a coclosed, weak supplement in M (cf. 1.2.1). Hence in a weakly supplemented module M every submodule that is coclosed in M is a supplement in M.

Note that an R-module is hollow if and only if it is indecomposable lifting.

This is clear since in an indecomposable module M the only proper direct summand is 0. Hence every submodule of M lies above 0 (i.e. every submodule of M is small in M).

¿From [39, Proposition 4.8] we get the following characterization of lifting modules:

4.1.1. Lifting modules.
Let M be an R-module. Then the following statements are equivalent:

(a) M is lifting;

(b) for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll M$;

(c) every submodule N of M can be written as $N = N_1 \oplus N_2$ with N_1 a direct summand of M and $N_2 \ll M$;

(d) M is amply supplemented and every coclosed submodule of M is a direct summand of M.

Proof: $(a) \Rightarrow (b)$ Every submodule N of M lies above a direct summand M_1 of M. Thus there is a decomposition $M = M_1 \oplus M_2$ with $N/M_1 \ll M/M_1$. Since $M/M_1 \simeq M_2$ and $N/M_1 \simeq (N \cap M_2)$ we get $N \cap M_2$ is small in M_2 and hence in M.
$(b) \Rightarrow (c)$ For every submodule N there is a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $N \cap M_2 \ll M$. Hence $N = M_1 \oplus (N \cap M_2)$.
$(c) \Rightarrow (d)$ Let $M = L + K$ for submodules $K, L \subset M$. We will show, that K contains a supplement of L. By hypothesis: $K = N \oplus H$ with $H \ll M$ and N a direct summand of M. Hence $M = L + N$. By hypothesis $L \cap N = N_1 \oplus S$ with $S \ll M$ and N_1 a direct summand of M. Hence N_1 is a direct summand of N and $S \ll N$. Let $N = N_1 \oplus N_2$ for some submodule N_2 of N. N_2 is a supplement of N_1 in N. We claim that N_2 is a supplement of $N_1 + S$ in N. To see this consider a submodule $X \subseteq N_2$ such that $N = X + N_1 + S$. Then $N = X + N_1$ holds as $S \ll N$ and $X = N_2$ as N_2 is a supplement of N_1 in N. Hence N_2 is a supplement of $N_1 + S = L \cap N$ in N. So $M = L + N = L + (L \cap N) + N_2 = L + N_2$ and $L \cap N_2 = (L \cap N) \cap N_2 \ll N_2$ holds. Thus N_2 is a supplement of L in M.
Let N be a coclosed submodule in M, then $N = M_1 \oplus S$ with S small in M. Clearly N lies above M_1 in M. Hence $N = M_1$ as N is coclosed.
$(d) \Rightarrow (a)$ By 1.2.2 every submodule of M that is not small in M lies above a coclosed submodule and hence above a direct summand. □

Remarks:

1. For a characterization of "lying above direct summands" we refer to [67, 41.11].

2. Lifting modules are exactly the amply supplemented modules whose supplements are direct summands.

In general, direct sums of lifting modules are not lifting. Dual to [10, 7.4] we state an example from [43]:

Lemma 4.1.2. *Assume M is an uniserial module with composition series $0 \neq V \subset U \subset M$. Then the module $M \oplus (U/V)$ is not lifting.*

Proof: See [43, Lemma 2.3]. □

A module is called *uniform-extending* if every uniform submodule is essentially contained in a direct summand. As an attempt to dualize the notion of *uniform-extending* we will consider the following definition.

Definition. An R-module M is called *hollow-lifting* if M is amply supplemented and every hollow submodule of M lies above a direct summand of M.

Equivalently M is hollow-lifting if and only if M is amply supplemented and every hollow, coclosed submodule of M is a direct summand of M. Of course, every lifting module is hollow-lifting.

Lemma 4.1.3. *Any coclosed submodule (and so every direct summand) of a (hollow-)lifting module is (hollow-)lifting.*

Proof: Let M be a (hollow-)lifting R-module and N a coclosed submodule of M. Then N is a supplement in M. By [67, 41.7(1)] N is amply supplemented. Let K be a (hollow) submodule of N, that is coclosed in N. Since N is a supplement in M we get K is coclosed in M by 1.2.2 (3). Hence K is a direct summand of M and hence of N. □

The next lemma is dual to [10, 7.7].

Lemma 4.1.4. *Let M be a hollow-lifting module and $K \subset M$ a coclosed submodule with finite hollow dimension. Then K is a direct summand of M.*

Proof: Since K has finite hollow dimension, there is a submodule L of K such that K/L is hollow. By the previous lemma, K is hollow-lifting. Let N be a supplement of L in K; then N is hollow since $N/(N \cap L) \simeq K/L$ and $N \cap L \ll N$. Furthermore N is coclosed in K and K is a supplement in M, so N is coclosed in M (cf. 1.2.2). Hence N is a direct summand of M. Let $M = N \oplus N'$. Then $K = N \oplus (K \cap N')$

and $hdim(K) = hdim(N) + hdim(K \cap N')$ hold. By induction $K \cap N'$ is a direct summand of M. Let $M = (K \cap N') \oplus N''$ then $N' = (K \cap N') \oplus (N' \cap N'')$ such that $M = K \oplus (N' \cap N'')$. $\square$

Corollary 4.1.5. *Let M be an R-module with finite hollow dimension. Then M is hollow-lifting if and only if M is lifting.*

In the following proposition we show that a lifting module with a finiteness condition can be decomposed into a finite direct sum of hollow modules.

Proposition 4.1.6. *Let M be a non-zero R-module with finite uniform dimension or finite hollow dimension. Then the following holds:*

1. *If M is lifting, then $M = \bigoplus_{i=1}^{n} H_i$ with $0 \neq H_i$ hollow and $n = hdim(M)$;*
2. *If M is extending, then $M = \bigoplus_{i=1}^{n} U_i$ with $0 \neq U_i$ uniform and $n = udim(M)$.*

Proof: (1) Assume M to have finite uniform dimension or finite hollow dimension. Because the additive dimension formula for direct summands holds for both dimension notions, the result can be proved by induction on *udim* or *hdim*. In the following *dim* will denote either *udim* or *hdim*. If M is indecomposable or $dim(M) = 1$ then M is hollow since an indecomposable lifting module is hollow. Let $n \geq 1$ be a number and assume that for all R-modules with $dim(M) < n$ our hypothesis holds. Assume $dim(M) = n + 1$ and that is decomposable $M = M_1 \oplus M_2$ with M_1 and M_2 non-zero submodules of M. Then $dim(M) = dim(M_1) + dim(M_2) = n + 1$ and $dim(M_1)$ and $dim(M_2)$ are at most equal to n. By hypothesis M_1 and M_2 are finite direct sums of hollow modules. Thus the result follows.
The proof of (2) is similar to (1), since an indecomposable extending module is uniform. $\square$

The *Osofsky-Smith* Theorem (cf. [10, 7.13]) states, that a cyclic module whose cyclic subfactors are extending can be expressed as a finite direct sum of uniform submodules. The next corollary can be regarded as an attempt to dualize this theorem.

Corollary 4.1.7. *If M is a lifting R-module that is either finitely generated or finitely cogenerated, then M is a finite direct sum of hollow submodules.*

Proof: By 3.3.4 a finitely generated, weakly supplemented module has finite hollow dimension. A finitely cogenerated module has finitely generated essential socle and hence finite uniform dimension. Thus the result follows by applying 4.1.6. $\square$

Remarks: Recall the definiton of a π-projective module. It can easily be seen, that the condition π-projective is equivalent to the splitting of the epimorphism

$$N \oplus L @ >>> M$$

$$(n, l) \mapsto n + l.$$

In [71] Zöschinger calls these modules *ko-stetig* (i.e. *co-continuous*) as a dualization of Utumi's definition of continuous modules in [61].

Proposition 4.1.8. *([67]) For a π-projective R-module M the following statements hold:*

1. *Each direct summand of M is π-projective .*
2. *If N and L are mutual supplements in M, then $N \cap L = 0$.*

Proof: (1) Let N be a direct summand of M with an idempotent $e \in \mathrm{End}\,(M)$ and $Me = N$. Then $M = Me \oplus M(1-e)$ holds. If $Me = K + L$, then $M = K + L + M(1-e)$, and there exists $f \in \mathrm{End}\,(M)$ with $\mathrm{Im}\,(f) \subseteq K$ and $\mathrm{Im}\,(1-f) \subseteq L + M(1-e)$. Now fe and $e - fe = (1-f)e$ can be seen as endomorphisms of $\mathrm{End}\,(Me)$, satisfying $\mathrm{Im}\,(fe) \subset K$ and $\mathrm{Im}\,((1-f)e) \subseteq L$.
(2) If N, L are mutual supplements, then we have $N \cap L \ll N$ and $N \cap L \ll L$. Let ϕ denote the epimorphism $N \oplus L \to M$, $(n, l) \mapsto n + l$. Then the kernel of ϕ

$$\mathrm{Ker}\,(\phi) = \{(n, -n) : n \in N \cap L\} \subseteq (N \cap L, 0) \oplus (0, N \cap L) \ll N \oplus L$$

is small and splits by assumption. Thus $\mathrm{Ker}\,(\phi) = 0$ and so $N \cap L = 0$. $\square$

Remarks: More properties and characterizations of π-projective modules can be found in [71] or [67, 41.14 - 41.17].

As mentioned in the begining of this section π-projective supplemented modules are exactely the quasi-discrete module. We state a characterization of such modules from [67, 41.15]:

4.1.9. Quasi-discrete modules.

For an R-module M the following assertions are equivalent:

(a) *M is supplemented and π-projective ;*

(b) (i) *M is amply supplemented, and* *(ii) the intersection of mutual supplements is zero;*

(c) (i) M is lifting, and

(ii) if U, V are direct summands of M with $M = U + V$, then $U \cap V$ is a direct summand of M.

Proof: (see [67, 41.15]) □

Remarks:

1. Recall that property (c) of above theorem is the definiton of quasi-discrete.
2. There exists a decomposition theorem for quasi-discrete module (see [67, 41.17] or [39, Theorem 4.15]) that states that any quasi-discrete module can be expressed as a (not necessarily finite) direct sum of hollow modules.

The next proposition dualizes [10, 7.5] and was obtained from [39, Lemma 4.47] and [67, 41.14].

Lemma 4.1.10. *Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. Then M_1 is $M_2 - projective$ if and only if for every $N \subset M$ with $M = N + M_2$ there is a submodule $L \subseteq N$ with $M = L \oplus M_2$.*

Proof: ($\Rightarrow$) Let $p : M \to M/N$ be the canonical projection and $p_i = p|_{M_i}$ for $i = 1, 2$. Then p_2 is epimorph since $M/N \simeq M_2/(M_2 \cap N)$ and by hypothesis the commutative diagram:

$$\begin{array}{ccccc} & & M_1 & & \\ & & \downarrow{\scriptstyle p_1} & & \\ M_2 & \xrightarrow{p_2} & M/N & \longrightarrow & 0 \end{array}$$

can be extended by a homomorphism $f : M_1 \to M_2$. Let

$$L := \{x - (x)f | x \in M_1\} \subset M,$$

then $L \cap M_2 = 0$, since $M_1 \cap M_2 = 0$ and $L \oplus M_2 = M$ as $x = (x - (x)f) + ((x)f)$ for all $x \in M_1$ implies $M_1 \subseteq L + \mathrm{Im}\,(f)$. Also $L \subseteq N$ holds, since $(L)p = 0$.
($\Leftarrow$) Consider any factor module F of M_2 with projection p and a homomorphism $f : M_1 \to F$.

$$\begin{array}{ccccc} & & M_1 & & \\ & & \downarrow{\scriptstyle f} & & \\ M_2 & \xrightarrow{p} & F & \longrightarrow & 0 \end{array}$$

Set

$$N := \{m_2 - m_1 \in M | m_1 \in M_1, m_2 \in M_2 \text{ and } (m_2)p = (m_1)f\}.$$

Every element $m \in M$ can be expressed as $m = m_1 + m_2$ with $m_1 \in M_1$ and $m_2 \in M_2$. Since p is epimorph we will find for any $m_1 \in M_1$ an element $x \in M_2$ with $(x)p = (m_1)f$. Thus $m = (m_1 - x) + (x + m_2)$ implies $M = N + M_2$. By hypothesis there is a submodule $L \subseteq N$ such that $M = L \oplus M_2$. Let $e : M \to M_2$ be the projection with respect to this decomposition. This yields a homomorphism from M_1 to M_2:

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] & M_1 \oplus M_2 \arrow[r] \arrow[d] & M_2 \arrow[r] & 0 \\ 0 \arrow[r] & L \arrow[r] & L \oplus M_2 \arrow[r, "e"] & M_2 \arrow[r] & 0 \end{tikzcd}$$

Since $M_1(1-e) \subseteq L \subseteq N$ we get $m_1 - (m_1)e \in N$ for all $m_1 \in M_1$ and hence $(m_1)f = (m_1 e)p$. Thus $f = ep$. Therefore M_1 is M_2-projective. □

Definition. R-modules M_i $(i \in I\}$ are called *relative projective* if M_i is M_j-projective for all distinct $i, j \in I$.

The next corollary is dual to [10, 7.6]

Corollary 4.1.11. *An R-module M is quasi-discrete if and only if M is a lifting module such that whenever $M = M_1 \oplus M_2$ is a direct sum of submodules, then M_1 and M_2 are relatively projective.*

Proof: ($\Rightarrow$) By 4.1.9 M is lifting. Let $M = M_1 \oplus M_2$. Assume $M = N + M_2$ for a submodule $N \subset M$, then N lies above a direct summand L. Hence $M = L + M_2$. By 4.1.9 $L \cap M_2$ is a direct summand of M and so it is a direct summand of L, say $L = K \oplus (L \cap M_2)$, which yields $M = K \oplus M_2$. By 4.1.10 M_1 is M_2-projective. A similar argument shows that M_2 is M_1-projective.
($\Leftarrow$) Assume $U, V \subseteq M$ with $M = U + V$. Since M is lifting, U lies above a direct summand M_1. Let $M = M_1 \oplus M_2$. Clearly $M_1 + V = M$. Since M_2 is M_1-projective we get by 4.1.10 a submodule $W \subseteq V$ such that $M = M_1 \oplus W$. Consider the canonical projection $\pi : M \to M_1$ with kernel W and the inclusion map $\epsilon : M_1 \to M$ with respect to the decomposition $M = M_1 \oplus W$. Then $f := \pi\epsilon$ is an endomorphism of M such that $(M)f \subseteq U$ and $(M)(1-f) \subseteq V$. Thus M is π-projective and since M is lifting it is supplemented. □

Remarks: Baba and Harada studied in [4] when a finite direct sum of hollow modules with local endomorphism rings is lifting. They showed that this is closely related to a generalized projectivity condition between the direct summands. Let M and N be two R-modules. M is called *almost N-projective* if every diagram

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow{\scriptstyle g} & & \\ M & \xrightarrow{f} & F & \longrightarrow & 0 \end{array}$$

can be either extended commutatively by a homomorphism $h : N \to M$ or there exists a direct summand M_1 of M and $h : M_1 \to N$ such that $hg = e_1 f$ where $e_1 : M_1 \to M$ is the canonical inclusion map. They proved in [4, Theorem 1] that a finite direct sum $M = \bigoplus_{i=1}^{n} H_i$ of hollow modules whose endomorphism rings are local is lifting if and only if H_i is almost H_j-projective for all $i \neq j$. For more information about direct sums of lifting modules and almost projectivity we refer to [4], [26], [27] and [28].

4.2 Lifting modules with chain conditions

The following results are dual to [10, 18.5-18.7]. Let us first observe an easy lemma.

Lemma 4.2.1. *Let M be an R-module with essential radical. For every direct summands $D_1 \subseteq D_2$ of M we have* $\mathrm{Rad}\,(D_1) = \mathrm{Rad}\,(D_2)$ *if and only if $D_1 = D_2$.*

Proof: Let $M = D_1 \oplus D_1'$. Then $D_2 = D_1 \oplus (D_2 \cap D_1')$ and $\mathrm{Rad}\,(D_2) = \mathrm{Rad}\,(D_1) \oplus \mathrm{Rad}(D_2 \cap D_1')$. If $\mathrm{Rad}(D_1) = \mathrm{Rad}(D_2)$ then $0 = \mathrm{Rad}(D_2 \cap D_1') = \mathrm{Rad}(M) \cap D_2 \cap D_1'$. This implies $D_2 \cap D_1' = 0$ since $\mathrm{Rad}\,(M) \trianglelefteq M$ and hence $D_1 = D_2$. $\square$

Remarks: Let M be an R-module. It follows from this lemma that if $Rad(M) \trianglelefteq M$ and $\mathrm{Rad}\,(M)$ has ACC (DCC) on direct summands, then M has ACC (DCC) on direct summands.

4.2.2. Lifting modules with radical chain condition.
Let M be a lifting module such that $\mathrm{Rad}\,(M)$ *has ACC on direct summands. Then M is a direct sum of a semisimple module and a finite direct sum of hollow modules.*

Proof: By 1.3.2, every weakly supplemented module M can be decomposed as $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 has essential radical. Since $\mathrm{Rad}(M) = \mathrm{Rad}\,(M_2) \trianglelefteq M_2$ has ACC on direct summands M_2 has ACC on direct summands by 4.2.1. Since M_2 is lifting, it is amply supplemented and every coclosed submodule is a direct summand by 4.1.1. By 3.5.3 M_2 has finite hollow dimension and by 4.1.6 M_2 is a finite direct sum of hollow modules. $\square$

Corollary 4.2.3. *Let M be a lifting module.*

1. *If M has ACC on small submodules, then $M = S \oplus N$, where S is semisimple and N is noetherian.*

2. *If M has DCC on small submodules, then $M = S \oplus A$, where S is semisimple and A is artinian.*

Proof: (1) By 3.5.12 $\mathrm{Rad}\,(M)$ is noetherian and hence it has ACC on direct summands. By 4.2.2 $M = S \oplus N$, where S is semisimple and N is a finite direct sum of hollow modules. Let $N = \bigoplus_{i=1}^{n} H_i$ then $\mathrm{Rad}\,(H_i)$ is noetherian for all i. Since $H_i/\mathrm{Rad}\,(H_i)$ is simple (or zero) we get that H_i is noetherian. Thus N is noetherian. (2) By 3.5.12 $\mathrm{Rad}\,(M)$ is artinian and by 1.3.2 $M = S \oplus A$ with $\mathrm{Rad}\,(M) \trianglelefteq A$. By 4.2.1 A has DCC on direct summands and by 3.5.3 A has finite hollow dimension. Applying 3.5.14 A is artinian. $\square$

Corollary 4.2.4. *Let M be a lifting module with finite hollow dimension or finite uniform dimension.*

1. *If M has ACC on small submodules, then M is noetherian.*

2. *If M has DCC on small submodules, then M is artinian.*

Chapter 5

Dual polyform modules with finite hollow dimension

In this chapter we will give an attempt to dualize the notions of singular and non-M-singular modules, rational submodules and polyform modules. The notion of polyform modules was defined by Zelmanowitz in [70], where he generalizes Goldie's Theorem (see [10, 5.19]).

5.1 Non-M-small modules

A module N in $\sigma[M]$ is called *M-singular* (or *singular in* $\sigma[M]$) if $N \simeq L/K$ with K essential in $L \in \sigma[M]$. In case $M = R$ we just say *singular* (or *cosmall* in [48]) instead of *R-singular*. The M-singular modules

$$\mathcal{S}_M = \{N \in \sigma[M] | N \text{ is } M\text{-singular}\}$$

are closed under submodules, homomorphic images and direct sums. Any $N \in \sigma[M]$ contains a largest *M-singular submodule*

$$\mathcal{S}_M(N) := Tr(\mathcal{S}_M, N) = \sum\{\mathrm{Im}\,(f) | f \in \mathrm{Hom}\,(L, N), L \in \mathcal{S}_M\}.$$

Then $\mathcal{S}_M(N) = \sum\{L \subseteq N | L \in \mathcal{S}_M\}$ holds. A module N in $\sigma[M]$ is called *non-M-singular* if $\mathcal{S}_M(N) = 0$, i.e. N has no M-singular submodule. For basic facts about these modules we refer to [10, Chapter 2]. Let us now dualize these notions.

Definition. Let M, N be R-modules. N is called *M-small* (or *small in* $\sigma[M]$) if $N \simeq K \ll L$ for $K, L \in \sigma[M]$. In case $M = R$ we just say *small* instead of *R-small*.

Remarks: Let N be M-small with K and L as above. Denote by $\widehat{K}$ the M-injective hull of K. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{e} & L \\ & & {\scriptstyle i}\big\downarrow & & \\ & & \widehat{K} & & \end{array}$$

with i the inclusion map from K into its M-injective hull $\widehat{K}$ and e the inclusion map from K to L. Since $\widehat{K}$ is injective in $\sigma[M]$, the diagram can be extended commutatively by a homomorphism $f : L \to \widehat{K}$. Then $ef = i$ holds. Since $\mathrm{Im}\,(e) \ll L$ we get $K = \mathrm{Im}\,(ef) \ll \widehat{K}$. Since $K \simeq N$ it follows, that N is small in its M-injective hull as well. Thus a module is M-small if and only if it is small in its M-injective hull (see [36, Theorem 1]). Dual to this fact Rayar proved in [48, Proposition 1] that a module M is singular (or cosmall) if and only if the kernel of every epimorphism from a projective module P to M is essential in P.

Definition. Denote the class of all M-small modules in $\sigma[M]$ by

$$\mathcal{T}_M^* := \{N \in \sigma[M] | N \text{ is } M\text{-small}\}.$$

Then $\mathcal{T}_M^*$ is closed under submodules, homomorphic images and finite direct sums. For any $N \in \sigma[M]$ define

$$\mathcal{T}_M^*(N) := \mathrm{Re}\,(N, \mathcal{T}_M^*) = \bigcap\{\mathrm{Ker}\,(g) | g \in \mathrm{Hom}_R\,(N, L), L \in \mathcal{T}_M^*\}$$

Then $\mathcal{T}_M^*(N) = \bigcap\{L \subseteq N | N/L \in \mathcal{T}_M^*\}$ holds. A module $N \in \sigma[M]$ is called *non-M-small* if $\mathcal{T}_M^*(N) = N$, i.e. N has no non-zero M-small factor module. In case $M = R$ we just say *non-small* instead of *non-R-small*. Clearly N is not M-small if it is non-M-small. Moreover the class $\mathcal{F}_M^*$ of non-M-small submodules can be described as

$$\mathcal{F}_M^* := \{L \in \sigma[M] | \text{for all } N \in \mathcal{T}_M^* : \mathrm{Hom}\,(L, N) = 0\}.$$

Remarks:

1. In [36] Leonard defined a module N to be *small* in $R-$Mod if it is a small submodule of some R-module. He showed that N is small if and only if N is small in its injective hull. M.Rayar in [48] and in her thesis calls a module N *non-small* if it is not small in any module. In our sense, N is *non-M-small* if N has no non-zero M-small factor module, dual to the definition of a *non-M-singular* module N which has no non-zero *M-singular* submodules.

2. Oshiro called a ring R a left *H-ring* if every injective left R-module is lifting (see [42]). He showed that a ring R is a left *H-ring* if and only if R is left artinian and every left R-module that is not small contains a non-zero injective left R-module. Moreover he showed that a ring is a left *H-ring* if and only if every left R-module is a direct sum of an injective module and a small module. Oshiro and Wisbauer studied this situation in $\sigma[M]$ and showed that every injective module in $\sigma[M]$ is lifting if and only if every module in $\sigma[M]$ is a direct sum of an M-injective module and an M-small module (see [43]).

3. While every module over a left *H-ring* is a direct sum of an injective module and a small module, Rayar showed in [48, Theorem 7] that every left R-module is a direct sum of a projective module and a small module if and only if the ring R is QF.

The next statement dualizes [10, 4.1].

5.1.1. Non-M-small modules.
Let M be an R-module.

1. *The following are equivalent:*

   (a) *N is non-M-small;*

   (b) *for any $0 \neq K \in \sigma[M]$ and $0 \neq f : N \to K$, $\mathrm{Im}\,(f)$ is coclosed in K;*

   (c) *for any $0 \neq K \in \sigma[M]$ and $0 \neq f : N \to K$, $\mathrm{Im}\,(f)$ is not small in K.*

2. *Assume that M has a projective cover P in $\sigma[M]$. Then any module $N \in \sigma[M]$ with $\mathrm{Hom}\,(P, N) = 0$ is M-small.*

3. *Assume M is non-M-small and has a projective cover P in $\sigma[M]$. Then*

   (i) *$\mathcal{T}_M^* = \{N \in \sigma[M] | \mathrm{Hom}\,(P, N) = 0\}$.*

   (ii) *$\mathcal{T}_M^*$ is closed under extensions, direct sums and products (in $\sigma[M]$).*

   (iii) *Let $N \in \sigma[M]$ and consider the following exact sequence*

$$0 \longrightarrow \mathcal{T}_M^*(N) \longrightarrow N \longrightarrow N/\mathcal{T}_M^*(N) \longrightarrow 0.$$

   Then $\mathcal{T}_M^*(N)$ is non-M-small and $N/\mathcal{T}_M^*(N)$ is M-small.

Proof: (1)$(a) \Rightarrow (b)$ Let $f : N \to K$ be a non-zero homomorphism and assume $L \subseteq \mathrm{Im}\,(f) \subseteq K$ such that $\mathrm{Im}\,(f)/L \ll K/L$. Then $\mathrm{Im}\,(f)/L \in \mathcal{T}^*_M$. Let $\pi : K \to K/L$ denote the canonical projection; then $f\pi : N \to \mathrm{Im}\,(f)/L$ is a homomorphism. Since N is non-M-small, $\mathrm{Ker}\,(f\pi) = N$ implies $\mathrm{Im}\,(f) = L$. Hence $\mathrm{Im}\,(f)$ is coclosed in K.
$(b) \Rightarrow (c)$ Clear;
$(c) \Rightarrow (a)$ If there is a $g : N \to L$ with $L \in \mathcal{T}^*_M$, then $L \ll \widehat{L}$. Let $i : L \to \widehat{L}$ be the inclusion map. Then $gi : N \to \widehat{L}$ is a non-zero homomorphism with $\mathrm{Im}\,(gi) \ll \widehat{L}$.

(2) Let $\widehat{N}$ denote the M-injective hull of N. By [67, 17.9] $\widehat{N} = Tr(M, E(N))$ is M-generated (where $E(N)$ denotes the injective hull of N in $R-$Mod). Since M is P-generated, $\widehat{N}$ is P-generated. If N is not M-small, then it is not small in its M-injective hull $\widehat{N}$. Assume there exists a submodule $K \subset \widehat{N}$ such that $N + K = \widehat{N}$. Then $N/(N \cap K) \simeq \widehat{N}/K$ is a non-zero P-generated R-module. Thus there exists an index set Λ and a non-zero epimorphism f such that the following diagram

$$\begin{array}{ccccc} & & P^{(\Lambda)} & & \\ & & {\scriptstyle f}\downarrow & & \\ N & \xrightarrow{\pi} & N/(N \cap K) & \longrightarrow & 0 \end{array}$$

can be extended commutatively by a non-zero homomorphism $g : P^{(\Lambda)} \to N$ such that $g\pi = f$ holds. Hence there exists a non-zero homomorphism in $\mathrm{Hom}\,(P, N)$.

(3) Let us first note that P is non-M-small. Denote $M \simeq P/K$ with $K \ll P$. Assume there exist a submodule $L \subseteq P$ such that P/L is M-small. Then $P/(L+K)$ is M-small as well. But since M is non-M-small, we have $L + K = P$ and hence $L = P$. Thus $\mathcal{S}_M(P) = P$.

(i) Since P is non-M-small, $P \in \mathcal{F}^*_M$. Hence for every M-small $N \in \sigma[M]$ we get $\mathrm{Hom}\,(P, N) = 0$. The converse follows from (2).

(ii) Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an exact sequence with N_1 and N_3 M-small. Applying (i) and the exact functor $\mathrm{Hom}(P, -)$ we get that $\mathrm{Hom}(P, N_2) = 0$. Thus N_2 is M-small. Let $\prod_\Lambda N_\lambda$ be a product of M-small modules; then $\mathrm{Hom}\,(P, \prod_\Lambda N_\lambda) \simeq \prod_\Lambda \mathrm{Hom}\,(P, N_\lambda) = 0$ (see [67, 11.10]). Let $\bigoplus_\Lambda N_\lambda$ be a direct sum of M-small modules. As $\bigoplus_\Lambda N_\lambda$ is a submodule of $\prod_\Lambda N_\lambda$ we get that $\bigoplus_\Lambda N_\lambda$ is M-small. Moreover direct products of M-small modules are M-small if the product is in $\sigma[M]$. In general $\sigma[M]$ is not closed under taking direct products, but it has a product (in the categorical sense) defined as $\prod^M_\Lambda N_\lambda := Tr(U_e, \prod_\Lambda N_\lambda)$ with $U_e := \bigoplus\{U \subseteq M^{(\mathbb{N})} | U$ finitely generated $\}$ (see [67, 14.1]). By definition $\prod^M_\Lambda N_\lambda \subseteq \prod_\Lambda N_\lambda$. Thus the product in $\sigma[M]$ of M-small modules is M-small.

(iii) As a consequence of (i) and (ii) we get Hom $(P, N/\mathcal{T}_M^*(N)) = 0$ since $N/\mathcal{T}_M^*(N)$ is isomorphic to a submodule of a product of M-small modules. Assume there exists a submodule $L \subseteq \mathcal{T}_M^*(N)$ such that $\mathcal{T}_M^*(N)/L$ is M-small. Then

$$0 \to \mathcal{T}_M^*(N)/L \to N/L \to N/\mathcal{T}_M^*(N) \to 0$$

is an exact sequence and by (ii) we get N/L is M-small. But then $\mathcal{T}_M^*(N) \subseteq L$ and hence $\mathcal{T}_M^*(N) = L$. Thus $\mathcal{T}_M^*(N)$ is non-M-small. $\square$

Remarks:

1. In [37] McMaster defines the notion of a *cotorsion theory* induced by a projective R-module P. He defines the class of *cotorsion modules* to be all R-modules N such that Hom $(P, N) = 0$ and the class of *cotorsionfree modules* to be all R-modules L such that Hom $(L, N) = 0$ for all cotorsion R-modules N. Under the hypothesis of 5.1.1(3) we see that $(\mathcal{F}_M^*, \mathcal{T}_M^*)$ is the cotorsion theory (in $\sigma[M]$) that is induced by P.

2. A class of modules is called a *TTF class* or *Jansian class* (see [37]) if it is closed under submodules, direct products, homomorphic images, extensions and isomorphic images. Hence we see that under the assumptions made in 5.1.1(3) $\mathcal{T}_M^*$ forms a Jansian class.

Cotorsion theories can be described by trace ideals (see [37, 1.2, 1.3]).

5.1.2. Projective non-small modules.
Let P be a projective, non-small R-module. Let $T := Tr(P, R)$ be the trace ideal of P. Then the following holds for a module $N \in R{-}\mathrm{Mod}$:

(1) If N is small then $TN = 0$.

(2) The following statements are equivalent:

(a) $N = TN$;

(b) $R/T \otimes_R N = 0$;

(c) N is P-generated.

In this case N is non-small.

Proof: (1) Let N be small. If there exists an element $n \in N$ such that $Tn \neq 0$ then there exists a non-zero homomorphism $P \to T \to Tn \subseteq N$. Hence $Tn = 0$ for all $n \in N$. Thus $TN = 0$.

(2) (a) $\Leftrightarrow$ (b) By [67, 12.11] $R/T \otimes_R N \simeq N/TN$ holds.
(a) $\Leftrightarrow$ (c) Since $P = TP$ holds for a projective R-module, we have $Tr(P, N) = (P)\mathrm{Hom}\,(P, N) = T(P)\mathrm{Hom}\,(P, N) = T(Tr(P, N)) \subseteq TN$. On the other hand let $p \in P$, $f : P \to R$ and $n \in N$. Then $(p)fn \in TN$ and every element in TN is a finite sum of elements of this form. Clearly f and n induces a homomorphism $\bar{f}_n : P \to N$ such that $p \mapsto (p)fn$. Hence $(p)fn \in \mathrm{Im}\,(\bar{f}_n) \subseteq Tr(P, N)$ implies $TN \subseteq Tr(P, N)$. Thus $N = TN$ if and only if $Tr(P, N) = N$.
Assume N satisfies (a). For every small R-module X we have $Tr(N, X) = (N)\mathrm{Hom}\,(N, X) = T(N)\mathrm{Hom}\,(N, X) \subseteq TX = 0$. Hence N is non-small.
□

Remarks: Let $Gen(P)$ denote the set of all P-generated modules. Then under the assumptions of 5.1.2 we have $\mathcal{T}^*_M = R/T-\mathrm{Mod}$ and $\mathcal{F}^*_M = Gen(P)$. (Note that the trace ideal of a projective R-module is a two-sided ideal; see [67, pp. 154]). Moreover if R is commutative and P a finitely generated projective R-module then $R = T \oplus Ann_R(P)$ holds (see [67, 18.10]). Thus if P is non-small we have $\mathcal{T}^*_M = Ann_R(P)-\mathrm{Mod}$ and $\mathcal{F}^*_M = Gen(T)$.

Definition. An R-module M is called *co-semisimple* if every simple module in $\sigma[M]$ is M-injective (see [67, 23.1]). A ring R that is co-semisimple as a left R-module is called a left *V-ring*.

Corollary 5.1.3. *Assume M to be projective in $\sigma[M]$. Then the following statements are equivalent:*

(a) M is non-M-small and a generator in $\sigma[M]$;

(b) $\mathcal{T}^*_M = 0$;

(c) $\mathrm{Rad}\,(N) = 0$*, for every $N \in \sigma[M]$;*

(d) M is co-semisimple.

Proof: (a) $\Rightarrow$ (b) clear by 5.1.1(2); (b) $\Leftrightarrow$ (c) clear by definition; (c) $\Leftrightarrow$ (d) by [67, 23.1]; (d) $\Rightarrow$ (a) by [67, 23..8(1)] M is a generator in $\sigma[M]$ and by (b) $\mathcal{T}^*_M = 0$ hence M is non-M-small. □

Remarks: The last corollary shows, that a ring R is (left) non-small if and only if it is a (left) V-ring (see also [25, Proposition 2.2]).

Recall that every simple module in $\sigma[M]$ is a factor module of a submodule of M (see Chapter 3.6). The next statement dualizes [10, Proposition 4.2].

Proposition 5.1.4. *Let M be an R-module.*

1. *Every simple R-module is M-small or M-injective.*
2. *If $\mathcal{T}_M^*(M) + \mathrm{Rad}\,(M) = M$, then M/L is injective in $\sigma[M]$ for every maximal submodule L of M.*
3. *If $\mathcal{T}_M^*(M) + \mathrm{Soc}\,(M) = M$, then every maximal submodule L such that M/L is M-small is a direct summand of M.*
4. *If $\mathrm{Rad}\,(M) = M$ then every simple module in $\sigma[M]$ is M-small.*

Proof: (1) A simple module which does not belong to $\sigma[M]$ is trivially M-injective. Assume the simple module $E \in \sigma[M]$ is not M-small; then it is not small in its M-injective hull $\widehat{E}$. Therefore there exists a proper submodule $K \subset \widehat{E}$ such that $\widehat{E} = E + K$. $E \cap K = 0$ holds since K is proper and hence E is a direct summand of $\widehat{E}$ and hence M-injective.

(2) Let L be a maximal submodule of M. Assume M/L is M-small, then $\mathcal{T}_M^*(M) \subseteq L$ implies $M = \mathcal{T}_M^*(M) + \mathrm{Rad}\,(N) \subseteq L$ a contradiction to L being a maximal (proper) submodule. Hence by (1) for all maximal submodule $L \subset M$ we have M/L is M-injective (and hence injective in $\sigma[M]$; see [67, 16.3]).

(3) Let L be a maximal submodule of M with M/L being M-small. Then $\mathcal{T}_M^*(M) \subseteq L$ and hence there must be a simple module E in $\mathrm{Soc}(M)$ with $L \oplus E = M$.

(4) Every simple module E in $\sigma[M]$ is a factor module of a submodule of M. Hence there exists a submodule $L \subseteq M$ such that the following holds:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \longrightarrow & M \\ & & {\scriptstyle f}\downarrow & & \\ & & E & & \end{array}$$

with f an epimorphism. If E is M-injective, then the diagram can be commutatively extended by an epimorphism from M to E. By hypothesis M has no simple factor module. Thus there are no M-injective simple modules in $\sigma[M]$ and by (1) every simple module in $\sigma[M]$ is M-small. □

The next statement dualizes [10, Proposition 4.5].

Proposition 5.1.5. *Let M be an R-module and N an M-small module. If M is self-injective then for any $f \in \mathrm{Hom}\,(N, M)$, $\mathrm{Im}\,(f) \ll M$.*

Proof: Let $\widehat{N}$ denote the M-injective hull of N.

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & \widehat{N} \\ & & {\scriptstyle f}\big\downarrow & & \\ & & M & & \end{array}$$

Since M is injective in $\sigma[M]$, the diagram can be extended commutatively by an homomorphism $g : \widehat{N} \to M$. Since $N \ll \widehat{N}$ we get $\mathrm{Im}\,(f) = \mathrm{Im}\,(g) \ll M$. $\square$

5.2 Co-rational submodules

In this section we will define dual notions for rational submodules and polyform modules. A submodule U of a module M is called *rational* if $\mathrm{Hom}\,(M/U, \widehat{M}) = 0$ where $\widehat{M}$ denotes the M-injective hull of M. Equivalently U is rational in M if and only if for all submodules $U \subseteq V \subseteq M$, $\mathrm{Hom}\,(V/U, M) = 0$. Moreover every rational submodule is an essential submodule of M. Zelmanowitz called a module *polyform* if every essential submodule is rational. These notions were used to generalize Goldie's Theorem (see [10, 5.19]).

Definition. Let M be an R-module. A submodule N of M is called *co-rational* in M if for every $L \subseteq N$, $\mathrm{Hom}\,(M, N/L) = 0$.

This is a slightly different definition of co-rationality than the one by Courter (see [9]).

Proposition 5.2.1. *Let M be an R-module having a projective cover P in $\sigma[M]$.*

1. *Let $N \subset M$ then the following are equivalent:*

   (a) N is co-rational in M;

   (b) $\mathrm{Hom}_R\,(P, N) = 0$.

2. *Every co-rational submodule of M is small in M.*

Proof: Denote $M \simeq P/K$ with $K \ll P$.

(1) (a) $\Rightarrow$ (b) Let $g \in \mathrm{Hom}\,(P, N)$ and $L := (K)g$ and h be the induced homomorphism $h : M \to N/L$ with $p + K \mapsto (p)g + L$. But then $h = 0$ since N is

co-rational. Hence $(P)g = (K)g$ and for all $p \in P$ there exist $k \in K$ and $l \in \mathrm{Ker}\,(g)$ such that $p = k + l$. Thus $P = K + \mathrm{Ker}\,(g) = \mathrm{Ker}\,(g)$, implying $g = 0$. (b) $\Rightarrow$ (a) For $0 \neq f \in \mathrm{Hom}\,(M, N/L)$, the diagram

$$\begin{array}{ccccc} P & \longrightarrow & M & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \\ N & \longrightarrow & N/L & \longrightarrow & 0 \end{array}$$

can be extended commutatively by a non-zero homomorphism from P to N.

(2) Let N be a submodule of M such that $N + L = M$ for $L \subset M$. Write $M = P/K$, $N = X/K$, $L = Y/K$ for some submodules X and Y of P such that $X + Y = P$, $Y \neq P$ and $K \subseteq X \cap Y$. Then

$$0 \to Y \to P \to X/(Y \cap X) \to 0$$

holds. Since P is projective we get a non-zero homomorphism $P \to X/K = N$. Thus $\mathrm{Hom}_R\,(P, N) \neq 0$ and N is not co-rational in M. $\square$

Definition. A module M is called *co-polyform* if every small submodule of M is co-rational.

Proposition 5.2.2. *Let M be an R-module such that M has a projective cover P in $\sigma[M]$. Then the following are equivalent:*

(a) M is co-polyform;

(b) $\mathrm{Jac}\,(\mathrm{End}\,(P)) = 0$.

Moreover in this case every $f \in \mathrm{End}\,(M)$ lifts to an $\bar{f} \in \mathrm{End}\,(P)$ and every small epimorphism in $\mathrm{End}\,(M)$ *is invertible in* $\mathrm{End}\,(P)$.

Proof: Recall, that the Jacobson radical of the endomorphism ring of a self-projective module P can be expressed as $\mathrm{Jac}\,(\mathrm{End}\,(P)) = \{f \in \mathrm{End}\,(P) : \mathrm{Im}\,(f) \ll P\}$ [67, 22.2].

(b) $\Rightarrow$ (a) Let $f : P \to K$ be a non-zero homomorphism with K a small submodule of M. Consider the following diagram

$$\begin{array}{ccccc} P & \xrightarrow[f]{} & K & & \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i} & & \\ P & \xrightarrow[p]{} & M & \longrightarrow & 0 \end{array}$$

Where p denotes the projection from P to M, and there is a non-zero $g \in \text{End}\,(P)$ with $f = gp$ and $\text{Im}\,(f) = \text{Im}\,(gp) \ll M$. Hence $\text{Im}\,(g) \ll P$ because p has small kernel, implying $g = 0$ and so $f = 0$, a contradiction. This shows $\text{Hom}\,(P, K) = 0$ for every small submodule K of M and by 5.2.1 every small submodule of M is co-rational.

(a) $\Rightarrow$ (b) Consider $f \in \text{End}\,(P)$ with $\text{Im}\,(f) \ll P$. Then for all $g \in \text{Hom}\,(P, M)$, $U := \text{Im}\,(fg) \ll M$, but then $fg \in \text{Hom}\,(P, U)$ and hence is zero. This implies $\text{Im}\,(f) \subseteq \text{Ker}\,(g)$ and so

$$\text{Im}\,(f) \subseteq \bigcap_{g \in \text{Hom}\,(P,M)} \text{Ker}\,(g) = \text{Re}\,(P, M).$$

But P is cogenerated by M (see [67, 18.4]). Hence $\text{Re}\,(P, M) = 0$ implies $f = 0$. Thus $\text{Jac}\,(\text{End}\,(P)) = 0$.

For every $f \in \text{End}\,(M)$, the following diagram can be extended by an $\bar{f} \in \text{End}\,(P)$.

$$\begin{array}{ccccc} P & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow f & & \\ P & \underset{p}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

If f is a small epimorphism, then $\bar{f}$ is a small epimorphism and hence an isomorphism. $\square$

Remarks: The above theorem shows that a ring is co-polyform as a left R-module (right R-module) if and only if $\text{Jac}\,(R) = 0$. Moreover this shows us that these notions differ in their behaviour from their duals, since an R-module M is non-M-singular if and only if it is polyform. In contrast to that: ${}_{\mathbb{Z}}\mathbb{Z}$ has zero radical (i.e. it is co-polyform) but it is not a V-ring (e.g. $\mathbb{Z}/p^k\mathbb{Z}$ with p prime and $k \geq 2$ has non-zero radical).

5.2.3. Co-polyform and non-small modules

Let M be an R-module.

1. *If N is non-M-small, then N is co-polyform.*
2. *If M is co-polyform and self-injective then* $\text{Hom}\,(M/N, M) = 0$ *for all* $N \subseteq M$ *with* $M/N \in \mathcal{T}_M^*$.
3. *Assume M is self-injective and* $\text{Hom}\,(M/N, M) \neq 0$ *for all non-zero* $N \subseteq M$. *Then M is co-polyform if and only if M is non-M-small.*

Proof: (1) Let $L \ll N$, then $L/K \ll N/K$ for every $K \subseteq L$. Let $f \in \mathrm{Hom}\,(N, L/K)$, then $f = 0$ since $\mathcal{T}^*_M(N) = N$
(2) Let $f \in \mathrm{Hom}\,(M/N, M)$. Then $\mathrm{Im}\,(f) \ll M$ holds by 5.1.5. Since M is co-polyform we have $\mathrm{Hom}\,(\mathrm{Im}\,(f), M) = 0$. Thus $f = 0$.
(3) This is a consequence of (1) and (2). □

Combining 5.2.2 and 3.4.6 we get for co-polyform modules:

5.2.4. Semisimple artinian endomorphism ring.
For an R-module M having a projective cover P in $\sigma[M]$ the following are equivalent:

(a) M has finite hollow dimension and is co-polyform;

(b) $\mathrm{End}\,(P)$ *is semisimple artinian.*

If M has this property, then every epimorphism $f \in \mathrm{End}(M)$ is invertible in $\mathrm{End}(P)$.

Proof: By 3.1.10 M having finite hollow dimension is equivalent to P having finite hollow dimension. By 3.4.6 and 3.3.5 this is equivalent to $S := \mathrm{End}\,(P)$ being semilocal. By 5.2.2 M co-polyform is equivalent to $\mathrm{Jac}\,(S) = 0$. So M having finite hollow dimension and being co-polyform is equivalent to S being semisimple artinian. □

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