Why Some Elementary Functions Are Not Rational

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A classical exercise for college students is to ask them to prove that the sine function is not a polynomial or, more generally, a rational function. This follows from the fact that the sine function has an infinite number of zeros, but this cannot happen to a rational function unless it is identically zero. This, however does not rule out the possibility that the restriction of the sine function to an open interval is rational. One way to prove that a function defined on an open interval is not a polynomial function is to calculate successive derivatives of the function: Since the degree of the derivative of a non-constant polynomial function is smaller than the degree of the polynomial, after some time you will get a constant function. This approach, however, does not work for rational functions. Or does it?

In this note we prove, in an elementary way, that the restrictions to an open interval of certain elementary functions are not rational functions, and we do it by using the concept of degree of a rational function.

In what follows, the domain of all functions is a fixed nonempty open interval of the real line. When we speak about the exponential function, the sine function, and so on, what we actually mean is the restriction of these functions to this interval.

DEFINITION. *If R is a rational function and if $R \not\equiv 0$, then the degree of R (written $\deg(R)$) is the difference of the degrees of the numerator and denominators. Specifically, if P_1 and P_2 are polynomials such that $R = P_1/P_2$, then* $\deg(R) = \deg(P_1) - \deg(P_2)$.

It is easy to see that this definition makes sense, that is, that $\deg(R)$ does not depend upon the choice of P_1 and P_2 [**1**, §4.2].

THEOREM. *Suppose f and g are rational functions, neither of which is identically zero.*

1. *If $f + g \not\equiv 0$, then* $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$.
2. *The degree of a product satisfies* $\deg(f \cdot g) = \deg(f) + \deg(g)$.
3. *If $f' \not\equiv 0$, then* $\deg(f') < \deg(f)$. *More generally, if n is a natural number and if $f^{(n)} \not\equiv 0$, then* $\deg\left(f^{(n)}\right) \leq \deg(f) - n$.

Proof. As stated by Bourbaki [**1**, §4.2], the proofs of the first two statements are a direct consequence of the similar formulas for polynomials. Let P_1, P_2, Q_1, and Q_2 be polynomial functions such that $f = P_1/P_2$ and $g = Q_1/Q_2$. If $f+g \not\equiv 0$, then

$$\begin{aligned}
&\deg(f+g) = \\
&\quad = \deg(P_1Q_2 + P_2Q_1) - \deg(Q_1Q_2) \\
&\quad \le \max\{\deg(P_1)+\deg(Q_2), \deg(P_2)+\deg(Q_1)\} - \deg(Q_1) - \deg(Q_2) \\
&\quad = \max\{\deg(P_1)-\deg(Q_1), \deg(P_2)-\deg(Q_2)\} \\
&\quad = \max\{\deg(f), \deg(g)\}.
\end{aligned}$$

The second statement follows directly from the definition. Finally, if $f' \not\equiv 0$, then

$$\begin{aligned}
&\deg(f') = \\
&\quad = \deg\left(\frac{P_1'P_2 - P_2'P_1}{{P_2}^2}\right) \\
&\quad = \deg(P_1'P_2 - P_2'P_1) - 2\deg(P_2) \\
&\quad \le \max\{\deg(P_1')+\deg(P_2), \deg(P_2')+\deg(P_1)\} - 2\deg(P_2) \\
&\quad = \max\{\deg(P_1)-1+\deg(P_2), \deg(P_2)-1+\deg(P_1)\} - 2\deg(P_2) \\
&\quad = \deg(P_1) - \deg(P_2) - 1 \\
&\quad = \deg(f) - 1.
\end{aligned}$$

It follows by induction that, if $n \in \mathbb{N}$ and if $f^{(n)} \not\equiv 0$, then $\deg\bigl(f^{(n)}\bigr) \le \deg(f) - n$. ∎

This theorem already allows us to show that certain functions are not rational. Take, for instance, the function defined by $f(x) = \sqrt[3]{1+x^2}$. If it were rational then it would follow from the second statement of the theorem that $2 = \deg(f^3) = 3\deg(f)$, which is impossible since $\deg(f)$ is an integer. Since $(e^x)' = e^x$, the third statement tells us that the exponential function is not a rational function.

Note that although it is true that, for a non-constant polynomial function P, we always have $\deg(P') = \deg(P) - 1$, this is not true in general for rational functions: if $n \in \mathbb{N}$ and if $f(x) = (x^n-1)/(x^n+1)$, then $\deg(f) = 0$ but $\deg(f') = -n-1$.

As a consequence of the third statement of the theorem, we have the following

COROLLARY. *If f is a rational function (with $f \not\equiv 0$), $k \in \mathbb{R}$ and $n \in \mathbb{N}$, then $f^{(n)} \ne k \cdot f$, unless f is a polynomial function, $k = 0$ and $n > \deg(f)$.*

This corollary can be used (with $k = 1$ and $n = 1$) to prove again that the exponential function is not rational and also (with $k = -1$ and $n = 2$) to prove that neither the sine function nor the cosine function is rational. With a little extra effort it could also be deduced from the corollary that neither the tangent function nor the cotangent function is a rational function, but it is easier to observe that, since $\tan'(x) = \sec^2(x)$, if the tangent function was rational then $x \mapsto \cos^2(x)$ would also be rational as the reciprocal of a rational function. But

$$\cos^2(x) = \frac{\cos(2x)+1}{2}$$

and therefore the function $x \mapsto \cos(2x)$ would be rational. That this cannot be the case can be deduced from the corollary (with $k = -4$ and $n = 2$) or from the fact that cosine is not rational.

Reference

1. N. Bourbaki, *Algebra II*, Springer-Verlag, Berlin, 1980.