HOMOLOGICALLY INDUCED $\mathfrak{sl}(1,2)$-MODULES

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Abstract. The $\mathfrak{sl}(1,2)$-modules that can be obtained from a parabolic subalgebra and a generalized Verma module by (co)homological induction are described. It is proved that, unlike the case of simple Lie algebras, the modules thus obtained starting from a Borel subalgebra depend upon the choice of this subalgebra. It is also proved that every indecomposable module such that the action of a Cartan subalgebra on that module is semisimple can be obtained by (co)homological induction.

Introduction

This article will use the notations and terminology introduced by Victor Kac in his seminal works [2] and [3]. We will start by defining the objects that we will be dealing with. The field we will be working with will be the complex number field. A *super vector space* is a vector space V endowed with a decomposition $V = V_0 \bigoplus V_1$ (where 0 and 1 should be seen as elements of the group $\mathbb{Z}_2$). A *superalgebra* is an algebra A endowed with super vector space structure $A = A_0 \bigoplus A_1$ such that, for each $i, j \in \{0,1\}$, $A_i \cdot A_j \subset A_{i+j}$. For each $i \in \{0,1\}$ and each $a \in A_i \setminus \{0\}$, let $|a|$ be equal to i. A *Lie superalgebra* is a superalgebra $(\mathfrak{g}, [\cdot,\cdot])$ such that

(1) $(\forall X, Y \in (\mathfrak{g}_0 \cup \mathfrak{g}_1) \setminus \{0\}) : [X,Y] = -(-1)^{|X|\cdot|Y|}[Y,X]$;
(2) $(\forall X, Y, Z \in (\mathfrak{g}_0 \cup \mathfrak{g}_1) \setminus \{0\}) : [X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X|\cdot|Y|}[Y,[X,Z]]$.

Note that $\mathfrak{g}_0$ is then a Lie algebra and that $\mathfrak{g}_1$ has a natural $\mathfrak{g}_0$-module structure. A *classical Lie superalgebra* is a finite-dimensional simple Lie superalgebra $\mathfrak{g}$ such that the natural action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is completely reducible. If $\mathfrak{g}$ is a classical Lie superalgebra, we will say that $\mathfrak{g}$ is *basic classical* if there is some non-degenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ which is *invariant*, that is, such that

(1) $(\forall X, Y \in (\mathfrak{g}_0 \cup \mathfrak{g}_1) \setminus \{0\}) : (X,Y) = (-1)^{|X|\cdot|Y|}(Y,X)$;
(2) $(\mathfrak{g}_0, \mathfrak{g}_1) = \{0\}$;
(3) $(\forall X, Y, Z \in \mathfrak{g}) : ([X,Y],Z) = (X,[Y,Z])$.

Of course, every simple finite-dimensional Lie algebra is a basic classical Lie superalgebra.

There are two Lie superalgebras that will be studied in this article. The first one is $\mathfrak{sl}(1,2)$, whose elements the matrices

$$\begin{pmatrix} a+d & x & y \\ z & a & b \\ t & c & d \end{pmatrix}$$

with $a, b, c, d, x, y, z, t \in \mathbb{C}$. Its superalgebra structure is the one such that

(1) $\mathfrak{sl}(1,2)_0$ (respectively $\mathfrak{sl}(1,2)_1$) is the set of those matrices in $\mathfrak{sl}(1,2)$ such that $x = y = z = t = 0$ (resp. $a = b = c = d = 0$);
(2) if $M, N \in (\mathfrak{sl}(1,2)_0 \cup \mathfrak{sl}(1,2)_1) \setminus \{0\}$, then $[M,N] = MN - (-1)^{|M|\cdot|N|}NM$.

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The second Lie superalgebra which will be studied in this article is

$$\mathfrak{gl}(1,1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{C} \right\}.$$

In this case, $\mathfrak{gl}(1,1)_0$ (respectively $\mathfrak{gl}(1,1)_1$) is the set of those matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $b=c=0$ (resp. $a=d=0$) and the the product is defined as in the case of $\mathfrak{sl}(1,2)$. As an example of how different the behaviour of these Lie superalgebras can be different from the behaviour of the reductive Lie algebras, it will be enough to observe that, whereas $\mathfrak{g}(2,\mathbb{C})$ has finite-dimensional irreducible representations of any dimension, every finite-dimensional irreducible representation of $\mathfrak{gl}(1,1)$ has dimension 1 or 2. See the theorem 4.2 for a more precise statement.

Let $\mathfrak{g}$ be a basic classical Lie superalgebra. A *Cartan subalgebra* of $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. When $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$, then "$\mathfrak{a}$-module" will mean "$\mathfrak{h}$-semisimple $\mathfrak{a}$-module", unless it is explicitly stated otherwise.

Let $(\cdot,\cdot)$ be a non-degenerate bilinear form on $\mathfrak{h}^*$ induced by an invariant non-degenerate bilinear form on $\mathfrak{g}$. A root α of the pair $(\mathfrak{g},\mathfrak{h})$ is called an *even* (respectively *odd*) root if $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$ (resp. $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$). Denote by Δ_0 and Δ_1 the set of all even roots and the set of all odd roots respectively and let $\overline{\Delta}_1$ be the set of all *isotropic* roots (that is, the roots α such that $(\alpha,\alpha)=0$, the subscript 1 being due to the fact that every isotropic root is odd). If $\lambda \in \mathfrak{h}^*$ is a weight, we will say that λ is *typical* when $(\lambda,\alpha) \neq 0$ whenever $\alpha \in \overline{\Delta}_1$. If $\mathfrak{b}$ is a *Borel subalgebra* of $\mathfrak{g}$ (that is, a maximal solvable subalgebra of $\mathfrak{g}$) such that $\mathfrak{h} \subset \mathfrak{g}$, set $\Delta_0(\mathfrak{b})$ (respectively $\Delta_1(\mathfrak{b})$) as the set of all even (resp. odd) roots α such that $\mathfrak{g}_\alpha \subset \mathfrak{b}$ and define

$$\rho_{\mathfrak{b}} = \frac{1}{2} \sum_{\alpha \in \Delta_0(\mathfrak{b})} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1(\mathfrak{b})} \alpha.$$

If $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ that contains a Borel subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$ and if, for every root α of the pair $(\mathfrak{g},\mathfrak{h})$, $\mathfrak{p}$ contains $\mathfrak{g}_\alpha$ or $\mathfrak{g}_{-\alpha}$, then it will be said that $\mathfrak{p}$ is a *parabolic subalgebra* of $\mathfrak{g}$. Then $\mathfrak{p} = \mathfrak{u} \bigoplus \mathfrak{s}$, where $\mathfrak{s}$ is a subalgebra of $\mathfrak{p}$ (it is the Levi factor of $\mathfrak{g}$ when $\mathfrak{g}$ is a semisimple Lie algebra) and $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$. If E is an $\mathfrak{s}$-module, then E becomes a $\mathfrak{p}$-module with $\mathfrak{u}$ acting trivially in E. Let

$$M(\mathfrak{p},E) = \mathcal{U}(\mathfrak{g}) \bigotimes_{\mathcal{U}(\mathfrak{p})} E.$$

Such a module is called a *generalized Verma module* (or simply a *Verma module*, when $\mathfrak{p}$ is a Borel subalgebra). Assume that E is irreducible. It was proved in [7, lemme 2.3] that it has then one and only one irreducible quotient. If $\mathfrak{p}$ is a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, then $\mathfrak{s}=\mathfrak{h}$ and $E=\mathbb{C}_\lambda$, for some $\lambda \in \mathfrak{h}^*$; in this case, $M(\mathfrak{b},E)$ will be denoted by $M(\mathfrak{b},\lambda)$ and its only irreducible quotient will be denoted by $L(\mathfrak{b},\lambda)$. Every finite-dimensional irreducible $\mathfrak{g}$-module M (distinct from $\{0\}$) is isomorphic to $L(\mathfrak{b},\lambda)$, for one and only one $\lambda \in \mathfrak{h}^*$. We will say that M is a *typical module* when $\lambda - \rho_{\mathfrak{b}}$ is typical.

If $\mathfrak{s}_0$ is a reductive subalgebra of $\mathfrak{g}_0$, then $\mathcal{HC}(\mathfrak{g},\mathfrak{s}_0)$ represents the category of $\mathfrak{g}$-modules which, as $\mathfrak{s}_0$-modules, are direct sum of finite-dimensional irreducible modules ($\mathcal{HC}$ stands for Harish-Chandra). A covariant right exact functor

$$\mathcal{L}_0^{\mathfrak{s}_0} : \mathcal{HC}(\mathfrak{g},\mathfrak{s}_0) \longrightarrow \mathcal{HC}(\mathfrak{g},\mathfrak{g}_0)$$

will be defined in section 1 with the following property: when V belongs to $\mathcal{HC}(\mathfrak{g},\mathfrak{s}_0)$ and is such that, up to isomorphism, there are only a finite number of finite-dimensional irreducible $\mathfrak{g}_0$-modules that are isomorphic to $\mathfrak{g}_0$-submodules of $\mathcal{L}_0^{\mathfrak{s}_0}(V)$, then $\mathcal{L}_0^{\mathfrak{s}_0}(V)$ is the greatest quotient of V in the category $\mathcal{HC}(\mathfrak{g},\mathfrak{g}_0)$; cf. proposition 1.1 for a more precise statement. In an analogous way (see [7, §4]), a left exact functor

$$\Gamma_{\mathfrak{s}_0}^0 : \mathcal{HC}(\mathfrak{g},\mathfrak{s}_0) \longrightarrow \mathcal{HC}(\mathfrak{g},\mathfrak{g}_0)$$

can be defined such that, when $V \in \mathcal{HC}(\mathfrak{g},\mathfrak{s}_0)$, $\Gamma_{\mathfrak{s}_0}^0(V)$ is the greatest submodule of V in the category $\mathcal{HC}(\mathfrak{g},\mathfrak{g}_0)$. The functors $\Gamma_{\mathfrak{s}_0}^0$ and $\mathcal{L}_0^{\mathfrak{s}_0}$ are similar to the Zuckerman functors and the dual Zuckerman functors; cf. [4, chap. II]. For each $i \geqslant 0$, let $\mathcal{L}_i^{\mathfrak{s}_0}$ (respectively $\Gamma_{\mathfrak{s}_0}^i$) denote the i^{th} derived functor of $\mathcal{L}_0^{\mathfrak{s}_0}$ (resp. $\Gamma_{\mathfrak{s}_0}^0$). It was proved in [7, §6] that, if $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, then every finite-dimensional typical $\mathfrak{g}$-module is isomorphic to $\mathcal{L}_i^{\mathfrak{h}}(M(\mathfrak{b},\lambda))$ for some $i \in \{0,1,\dots,\dim(\mathfrak{g}_0/\mathfrak{h})/2\}$ and some weight λ. It was also proved there that, if $\lambda \in \mathfrak{h}^*$ is a typical weight, then $\mathcal{L}_i^{\mathfrak{h}}(M(\mathfrak{b},\lambda-\rho_{\mathfrak{b}})) = \{0\}$ except for, at most, one single $i \geqslant 0$ and also that when $\mathcal{L}_i^{\mathfrak{h}}(M(\mathfrak{b},\lambda-\rho_{\mathfrak{b}}))$ is different from $\{0\}$, then it is irreducible.

As it will be seen (cf. theorem 2.1), even in such a simple case as when $\mathfrak{g} = \mathfrak{sl}(1,2)$, the situation changes drastically when one considers all modules of the form $\mathcal{L}_i^{\mathfrak{h}}(M(\mathfrak{b},\lambda))$. In fact:

(1) Not every finite-dimensional irreducible representation of $\mathfrak{sl}(1,2)$ is isomorphic to some $\mathcal{L}_i^{\mathfrak{h}}(M(\mathfrak{b},\lambda))$. Only the typical representations are.
(2) Sometimes $\mathcal{L}_i^{\mathfrak{h}}(M(\mathfrak{b},\lambda))$ is neither $\{0\}$ nor an irreducible module.
(3) It is not true that there is always, at most, one single $i \geqslant 0$ such that $\mathcal{L}_i^{\mathfrak{h}}(M(\mathfrak{b},\lambda)) \neq \{0\}$.

It will also be seen (cf. theorem 4.1) that every finite-dimensional irreducible representation of $\mathfrak{sl}(1,2)$ is isomorphic to $\mathcal{L}_i^{\mathfrak{s}_0}(M(\mathfrak{p},E))$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, some $i \geqslant 0$ and some irreducible $\mathfrak{s}$-module E.

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1. The functors

The aim of this section is to define the functors $\mathcal{L}_i^{\mathfrak{s}_0}$ and $\Gamma_{\mathfrak{s}_0}^i$ ($i \in \mathbb{Z}_+$). Since $\mathcal{L}_i^{\mathfrak{s}_0}$ (respectively $\Gamma_{\mathfrak{s}_0}^i$) will be defined as the i^{th} derived functor of the functor $\mathcal{L}_0^{\mathfrak{s}_0}$ (resp. $\Gamma_{\mathfrak{s}_0}^0$), this functor will be defined first. See [7, §4] for further details.

Note that $\mathfrak{sl}(1,2)_0 \simeq \mathfrak{gl}(2,\mathbb{C})$; we shall identify these Lie algebras. Let $G = SL(2,\mathbb{C})\times(\mathbb{C},+)$. Then G is a connected, simply connected complex Lie group such that its Lie algebra is isomorphic to $\mathfrak{gl}(2,\mathbb{C})$. Let $R(G)$ be the vector space generated by the matrix coefficients of the finite-dimensional semisimple representations of G. It turns out that $(\forall f,g \in R(G)) : f\cdot g \in R(G)$ and that, therefore, $R(G)$ has a natural structure of an algebra. The group G acts on $R(G)$ by the natural right action, denoted by r and defined as follows: if $g \in G$ and $f \in R(G)$, then

$$(\forall h \in G) : (r(g)(f))(h) = f(hg).$$

Note that, under this action, $R(G)$ can be decomposed as a direct sum of finite-dimensional G-modules and that, therefore, the right natural action of G on $R(G)$ induces an action of $\mathfrak{sl}(1,2)_0$. This action will also be denoted by r.

Let $\hat{\mathcal{M}}(G)$ be the dual space of $R(G)$ and let

$$\mathcal{M}(G) = \left\{\psi \in \hat{\mathcal{M}}(G) \middle| \#\left\{\gamma \in \mathfrak{sl}(1,2)_0^\wedge \,|\, \psi(R(G)_\gamma) \neq \{0\}\right\} < \infty\right\},$$

where $\mathfrak{sl}(1,2)_0^\wedge$ stands for the set of equivalent classes of finite-dimensional irreducible representations of $\mathfrak{sl}(1,2)_0$ and $R(G)_\gamma$ is the sum of the submodules of $R(G)$ that belong to the class γ. The multiplication in $R(G)$ induces an action of

$R(G)$ in $\mathcal{M}(G)$. Then, if V is an $\mathfrak{sl}(1,2)$-module that belongs to $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{s}_0)$ (where $\mathfrak{s}_0$ is a reductive subalgebra of $\mathfrak{sl}(1,2)_0$), we can define

$$\mathcal{L}_0^{\mathfrak{s}_0}(V) = \mathcal{M}(G) \bigotimes_{\mathcal{U}(\mathfrak{sl}(1,2)_0)} V.$$

An $\mathcal{U}(\mathfrak{sl}(1,2))$-module structure (and, therefore, an $\mathfrak{sl}(1,2)$-module structure) can be defined in $\mathcal{L}_0^{\mathfrak{s}_0}(V)$ in the following way: if $u \in \mathcal{U}(\mathfrak{sl}(1,2))$, choose elements $u_1, u_2, u_3, \ldots \in \mathcal{U}(\mathfrak{sl}(1,2))$ and elements $f_1, f_2, f_2, \ldots \in R(G)$ such that

$$(\forall g \in G) : \operatorname{Ad}\left(g^{-1}\right)(u) = \sum_j f_j(g) u_j.$$

Then, if $m \otimes v \in \mathcal{L}_0^{\mathfrak{s}_0}(V)$, the action of u on $m \otimes v$ is given by:

$$u \cdot (m \otimes v) = \sum_j (f_j m) \otimes (u_j v).$$

Let us now define the functor $\Gamma^0_{\mathfrak{s}_0}$. As a vector space,

$$\Gamma^0_{\mathfrak{s}_0}(V) = \left(R(G) \bigotimes V\right)^{\mathfrak{sl}(1,2)_0},$$

where the action of $\mathfrak{sl}(1,2)_0$ on $R(G) \bigotimes V$ is $r \otimes \theta$, θ being the action of $\mathfrak{sl}(1,2)$ on V. In order to define an action of $\mathfrak{sl}(1,2)$ on $\Gamma^0_{\mathfrak{s}_0}(V)$, note that $R(G) \bigotimes V$ (and, therefore, $\Gamma^0_{\mathfrak{s}_0}(V)$) can be seen as a space of functions from G into V. Let $X \in \mathfrak{sl}(1,2)$, $\psi \in \Gamma^0_{\mathfrak{s}_0}(V)$ and define $X\psi$ by

$$(\forall g \in G) : (X\psi)(g) = \operatorname{Ad}\left(g^{-1}\right)(X)(\psi(g)).$$

With these definitions, $\mathcal{L}_0^{\mathfrak{s}_0}$ and $\Gamma^0_{\mathfrak{s}_0}$ are, respectively, a right exact functor and a left exact functor.

Proposition 1.1. *Let $V \in \mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{s}_0)$ and suppose that, up to isomorphism, there are only a finite number of $\mathfrak{sl}(1,2)_0$-modules that are irreducible and isomorphic to $\mathfrak{sl}(1,2)_0$-submodules of $\mathcal{L}_0^{\mathfrak{s}_0}(V)$. Then there is a surjective homomorphism $\eta : V \twoheadrightarrow \mathcal{L}_0^{\mathfrak{s}_0}(V)$ and, furthermore, if W is a quotient of V that belongs to the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$ and $\pi : V \twoheadrightarrow W$ is the projection of V onto W, then there is a surjective homomorphism $\psi : \mathcal{L}_0^{\mathfrak{s}_0}(V) \twoheadrightarrow W$ such that $\psi \circ \eta = \pi$. Up to isomorphism, $\mathcal{L}_0^{\mathfrak{s}_0}(V)$ is the only $\mathfrak{sl}(1,2)$-module in the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$ with this property.*

See [7, proposition 4.3] for a proof of this proposition (in a more general context).

Note that in the previous proposition the hypothesis that says that, up to isomorphism, there are only a finite number of $\mathfrak{sl}(1,2)_0$-modules irreducible and isomorphic to $\mathfrak{sl}(1,2)_0$-submodules of $\mathcal{L}_0^{\mathfrak{s}_0}(V)$ is certainly valid when $\mathcal{L}_0^{\mathfrak{s}_0}(V)$ is finite-dimensional. It happens that, according to [7, proposition 4.10], this is always the case when $V = M(\mathfrak{p}, E)$ for some finite-dimensional $\mathfrak{s}$-module E. This observation, together with proposition 1.1, shows that when E is a finite-dimensional $\mathfrak{s}$-module, $\mathcal{L}_0^{\mathfrak{s}_0}(M(\mathfrak{p}, E))$ is the greatest quotient of $M(\mathfrak{p}, E)$ in $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$, that is, the greatest quotient of $M(\mathfrak{p}, E)$ that, as an $\mathfrak{sl}(1,2)_0$-module, can be written as a direct sum of finite-dimensional irreducible modules.

It was also proved in [7, §4] that if V belongs to the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{s}_0)$, then $\Gamma^0_{\mathfrak{s}_0}(V)$ is the greatest submodule of V in $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$.

2. Borel subalgebras

Let

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 + h_2 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_2 \end{pmatrix} \middle| \, h_1, h_2 \in \mathbb{C} \right\}.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(1,2)$. Let $\alpha, \beta \in \mathfrak{h}^*$ be defined by

$$\alpha \begin{pmatrix} h_1+h_2 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_2 \end{pmatrix} = h_1 \quad \text{and} \quad \beta \begin{pmatrix} h_1+h_2 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_2 \end{pmatrix} = h_2$$

and let

$$h_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The even roots of $(\mathfrak{sl}(1,2), \mathfrak{h})$ are $\pm(\alpha - \beta)$ and the odd roots are $\pm\alpha$ and $\pm\beta$. If $\lambda \in \mathfrak{h}^* \setminus \{0\}$, then we define

$$\mathfrak{sl}(1,2)_\lambda = \{X \in \mathfrak{sl}(1,2) | (\forall H \in \mathfrak{h}) : [H, X] = \lambda(H)X\}.$$

Note that this is done for $\lambda \neq 0$ only; the reason for this is that the notation $\mathfrak{sl}(1,2)_0$ is reserved for the even part of $\mathfrak{sl}(1,2)$. Let

$$\begin{aligned} \mathfrak{b}_+ &= \mathfrak{h} \bigoplus \mathfrak{sl}(1,2)_{\alpha-\beta} \bigoplus \mathfrak{sl}(1,2)_\alpha \bigoplus \mathfrak{sl}(1,2)_\beta, \\ \mathfrak{b}_- &= \mathfrak{h} \bigoplus \mathfrak{sl}(1,2)_{\alpha-\beta} \bigoplus \mathfrak{sl}(1,2)_{-\alpha} \bigoplus \mathfrak{sl}(1,2)_{-\beta} \end{aligned}$$

and

$$\mathfrak{b}_\pm = \mathfrak{h} \bigoplus \mathfrak{sl}(1,2)_{\alpha-\beta} \bigoplus \mathfrak{sl}(1,2)_\alpha \bigoplus \mathfrak{sl}(1,2)_{-\beta};$$

then $\mathfrak{b}_+$, $\mathfrak{b}_-$ and $\mathfrak{b}_\pm$ are Borel subalgebras of $\mathfrak{g}$ and, up to conjugacy, these are the only ones, as can be easily seen (or deduced from [2, §2.5.4]). They all have the same even part, namely $\mathfrak{h} \bigoplus \mathfrak{sl}(1,2)_{\alpha-\beta}$. If $\lambda \in \mathfrak{h}^*$, then λ is a weight if and only if $\lambda = n(\alpha-\beta)/2 + t(\alpha+\beta)$ with $n \in \mathbb{Z}$ and $t \in \mathbb{C}$. It will be said that λ is a dominant weight when $n \in \mathbb{Z}_+$ and that λ is a regular weight when $\lambda(h_\alpha - h_\beta) \neq 0 (\Longleftrightarrow (\lambda, \alpha-\beta) \neq 0)$.

In this article, when we speak of "dominant weight" or "highest weight" this is meant to correspond to the choice of $\{\alpha - \beta\}$ as a set of positive roots of $\mathfrak{sl}(1,2)_0$.

If $\lambda \in \mathfrak{h}^*$ is a dominant weight, then let $L_0(\lambda)$ denote the finite-dimensional irreducible $\mathfrak{sl}(1,2)_0$-module whose highest weight is λ. If

$$\mathfrak{sl}(1,2)_+ = \mathfrak{sl}(1,2)_\alpha \bigoplus \mathfrak{sl}(1,2)_\beta \text{ and } \mathfrak{sl}(1,2)_- = \mathfrak{sl}(1,2)_{-\alpha} \bigoplus \mathfrak{sl}(1,2)_{-\beta},$$

then $\mathfrak{sl}(1,2)_+$ and $\mathfrak{sl}(1,2)_-$ are supercommutative ideals of $\mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_+$ and $\mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_-$ respectively and therefore every $\mathfrak{sl}(1,2)_0$-module M becomes an $\mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_+$-module (resp. an $\mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_-$-module) with $\mathfrak{sl}(1,2)_+$ (resp. $\mathfrak{sl}(1,2)_-$) acting trivially in M. Let

$$K_+(\lambda) = \mathcal{U}(\mathfrak{sl}(1,2)) \bigotimes_{\mathcal{U}(\mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_+)} L_0(\lambda)$$

and

$$K_-(\lambda) = \mathcal{U}(\mathfrak{sl}(1,2)) \bigotimes_{\mathcal{U}(\mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_-)} L_0(\lambda);$$

these are the Kac modules in the specific case of $\mathfrak{sl}(1,2)$ (cf. [3, §2.2]).

Note that the functors $\mathcal{L}_i^{\mathfrak{s}_0}$ and $\Gamma_{\mathfrak{s}_0}^i$ can be defined for every basic classical Lie superalgebra. It will be useful for what will be done later to see now in more detail what is $\mathcal{L}_0^{\mathfrak{s}_0}(V)$ and $\mathcal{L}_1^{\mathfrak{s}_0}(V)$ in the specific case of the Lie algebra $\mathfrak{sl}(1,2)_0$ when $\mathfrak{s}(= \mathfrak{s}_0)$ is the standard Cartan subalgebra $\mathfrak{h}$ (that is, the subalgebra whose elements are the diagonal matrices), $\mathfrak{b}$ is the set of upper triangular matrices and V is a Verma module $M(\mathfrak{b}, \lambda)$ (where $\lambda \in \mathfrak{h}^*$ is a weight). In this situation, the functors $\mathcal{L}_0^{\mathfrak{s}_0}$ and $\mathcal{L}_1^{\mathfrak{s}_0}$ will be represented by $\mathcal{L}_0^0$ and $\mathcal{L}_1^0$ respectively. Notice that the Weyl group of $(\mathfrak{sl}(1,2)_0,\mathfrak{h})$ has only two elements: the identity and the automorphism $w:\mathfrak{h}\longrightarrow\mathfrak{h}$ that exchanges α and β. With this notation, the description is quite simple:

$$\mathcal{L}_0^0(M(\mathfrak{b},\lambda))\simeq\begin{cases}L(\mathfrak{b},\lambda) & \text{if } \lambda \text{ is dominant}\\ \{0\} & \text{otherwise}\end{cases}$$

and

$$\mathcal{L}_1^0(M(\mathfrak{b},\lambda))\simeq\begin{cases}L(\mathfrak{b},w(\lambda)-\alpha+\beta) & \text{if } w(\lambda)-\alpha+\beta \text{ is dominant}\\ \{0\} & \text{otherwise.}\end{cases}$$

This is just a particular case of the Borel-Weil-Bott theorem (see [4, §IV.11]).

In what follows, $\mathcal{L}_i^{\mathfrak{s}_0}$ and $\Gamma^i_{\mathfrak{s}_0}$ will be replaced by $\mathcal{L}_i$ and Γ^i whenever it is clear from the context what subalgebra of $\mathfrak{g}_0$ is being taken. It was proved in [7, §4] that, for any basic classical Lie superalgebra $\mathfrak{g}$, any parabolic subalgebra $\mathfrak{p}$ and any finite-dimensional $\mathfrak{s}$-module E, $\mathcal{L}_i(M(\mathfrak{p},E))=\{0\}$ when $i>\dim(\mathfrak{g}_0/\mathfrak{s}_0)/2$. Therefore, $(\forall\lambda\in\mathfrak{h}^*):\mathcal{L}_i(M(\mathfrak{b},\lambda))=\{0\}$ for any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}(1,2)$ and any $i>1$.

We are ready to state and prove the first of the three theorems which will describe $\mathcal{L}_0(M(\mathfrak{p},E))$ and $\mathcal{L}_1(M(\mathfrak{p},E))$ when E is an irreducible finite-dimensional $\mathfrak{s}$-module. However, when $\mathcal{L}_i(M(\mathfrak{p},E))$ $(i\in\{0,1\})$ is isomorphic to $\{0\}$ or to $\mathbb{C}$, the method that will be used to prove that assertion will always be the same and we will explain it now. The generalized Verma modules $M(\mathfrak{p},E)$ are always, as $\mathfrak{sl}(1,2)_0$-modules, direct sum of three modules, two of which are Verma modules whereas the third one admits a filtration by Verma modules. To be more precise, we shall denote the $\mathfrak{sl}(1,2)_0$-Verma module with highest weight λ by $M_0(\mathfrak{b}_0,\lambda)$. Then, for instance, in the case when $\mathfrak{p}=\mathfrak{b}_+$ and $E=\mathbb{C}_\lambda$, we have an $\mathfrak{sl}(1,2)_0$-module isomorphism

$$M(\mathfrak{b}_+,\lambda)\simeq M_0(\mathfrak{b}_0,\lambda)\bigoplus M_0(\mathfrak{b}_0,\lambda-\alpha-\beta)\bigoplus M, \tag{1}$$

where M is such that there is a short exact sequence

$$\{0\}\longrightarrow M_0(\mathfrak{b}_0,\lambda-\beta)\hookrightarrow M\twoheadrightarrow M_0(\mathfrak{b}_0,\lambda-\alpha)\longrightarrow\{0\}. \tag{2}$$

On the other hand, if V is an $\mathfrak{sl}(1,2)$-module from the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{h})$ then, as an $\mathfrak{sl}(1,2)_0$-module, $\mathcal{L}_i(V)\simeq\mathcal{L}_i^0(V)$ where $\mathcal{L}_i^0$ is the functor from the category $\mathcal{HC}(\mathfrak{sl}(1,2)_0,\mathfrak{h})$ to the category $\mathcal{HC}(\mathfrak{sl}(1,2)_0,\mathfrak{sl}(1,2)_0)$ built the same way as $\mathcal{L}_i$ (see [7, proposition 4.4]). Therefore, it follows from (1) that we have

$$\mathcal{L}_i^0(M_0(\mathfrak{b}_0,\lambda))\simeq\mathcal{L}_i^0(M_0(\mathfrak{b}_0,\lambda))\bigoplus\mathcal{L}_i^0(M_0(\mathfrak{b}_0,\lambda-\alpha-\beta))\bigoplus\mathcal{L}_i^0(M) \tag{3}$$

and it is a consequence of (2) that there is an exact sequence

$$\begin{aligned}&\{0\}\longrightarrow\mathcal{L}_1^0(M_0(\mathfrak{b}_0,\lambda-\beta))\longrightarrow\mathcal{L}_1^0(M)\longrightarrow\mathcal{L}_1^0(M_0(\mathfrak{b}_0,\lambda-\alpha))\longrightarrow\\&\quad\longrightarrow\mathcal{L}_0^0(M_0(\mathfrak{b}_0,\lambda-\beta))\longrightarrow\mathcal{L}_0^0(M)\longrightarrow\mathcal{L}_0^0(M_0(\mathfrak{b}_0,\lambda-\alpha))\longrightarrow\{0\}.\end{aligned} \tag{4}$$

So, to prove that some $\mathcal{L}_i(M(\mathfrak{p},E))$ is isomorphic to $\{0\}$ or to $\mathbb{C}$ it will be enough to use (3), (4) and the description made above of the $\mathfrak{sl}(1,2)_0$-modules of the type $\mathcal{L}_i^0(M_0(\mathfrak{b},\lambda))$ $(i\in\{0,1\})$.

Theorem 2.1. *Let $\lambda\in\mathfrak{h}^*$ be a weight.*

(1) *If λ is dominant, then*
   (a) $\mathcal{L}_0(M(\mathfrak{b}_+,\lambda))\simeq K_+(\lambda)$*;*
   (b) $\mathcal{L}_1(M(\mathfrak{b}_+,\lambda))\simeq\{0\}$*;*
   (c) $\mathcal{L}_0(M(\mathfrak{b}_-,\lambda))\simeq K_-(\lambda)$*;*
   (d) $\mathcal{L}_1(M(\mathfrak{b}_-,\lambda))\simeq\{0\}$*.*

(2) *If λ is not dominant and $\lambda+(\alpha-\beta)/2$ is regular, then,*

(a) $\mathcal{L}_0(M(\mathfrak{b}_+,\lambda)) \simeq \{0\}$;
(b) $\mathcal{L}_1(M(\mathfrak{b}_+,\lambda)) \simeq K_+(w(\lambda)-\alpha+\beta)$;
(c) $\mathcal{L}_0(M(\mathfrak{b}_-,\lambda)) \simeq \{0\}$;
(d) $\mathcal{L}_1(M(\mathfrak{b}_-,\lambda)) \simeq K_-(w(\lambda)-\alpha+\beta)$.

(3) *If λ is not dominant and $\lambda+(\alpha-\beta)/2$ is not regular, $\mathcal{L}_0(M(\mathfrak{b}_+,\lambda))$, $\mathcal{L}_1(M(\mathfrak{b}_+,\lambda))$, $\mathcal{L}_0(M(\mathfrak{b}_-,\lambda))$ and $\mathcal{L}_1(M(\mathfrak{b}_-,\lambda))$ are all isomorphic to $\{0\}$.*
(4) *One has:*
(a) $\lambda(h_\beta)\neq 0 \Longrightarrow M(\mathfrak{b}_\pm,\lambda)\simeq M(\mathfrak{b}_+,\lambda+\beta)$;
(b) $\lambda(h_\alpha)\neq 0 \Longrightarrow M(\mathfrak{b}_\pm,\lambda)\simeq M(\mathfrak{b}_-,\lambda-\alpha)$.
(5) $\mathcal{L}_0(M(\mathfrak{b}_\pm,0))\simeq\mathcal{L}_1(M(\mathfrak{b}_\pm,0)\simeq\mathbb{C}$.

Proof: Among these assertions, those which state that some $\mathcal{L}_i(M(\mathfrak{b},\lambda))$ is isomorphic to $\{0\}$ or to $\mathbb{C}$ (where $\mathfrak{b}$ can be either $\mathfrak{b}_+$ or $\mathfrak{b}_-$) can be proved using the method describe before the statement of the theorem. For instance, in order to prove 1b all that has to be proved is that $\mathcal{L}_1^0(M_0(\mathfrak{b},\eta))=\{0\}$, for each $\eta\in\{\lambda,\lambda-\alpha,\lambda-\beta,\lambda-\alpha-\beta\}$, but this is a consequence of the fact that, for each such η, $w(\eta)-\alpha+\beta$ is not dominant. Besides this, since there is an automorphism φ of $\mathfrak{sl}(1,2)$ such that $\varphi(\mathfrak{b}_+)=\mathfrak{b}_-$, it is clear that for any result concerning $\mathfrak{b}_+$ there is a similar result concerning $\mathfrak{b}_-$. Hence, the assertions that will be proved are 1a, 2b and 4a.

It follows from proposition 1.1 and from the fact that $\mathcal{L}_0(M(\mathfrak{b}_+,\lambda))$ is finite-dimensional (see [7, proposition 4.10]) that, to prove that $\mathcal{L}_0(M(\mathfrak{b}_+,\lambda))$ and $K_+(\lambda)$ are isomorphic, it will be enough to prove that when V is a quotient of $M(\mathfrak{b}_+,\lambda)$ that, as an $\mathfrak{sl}(1,2)_0$-module, can be written as a direct sum of finite-dimensional irreducible modules, then the projection $\pi: M(\mathfrak{b}_+,\lambda)\longrightarrow V$ factors through $K_+(\lambda)$. But since λ is a maximal weight of V (unless $V=\{0\}$, but in this case there is nothing to prove), $L_0(\lambda)$ is isomorphic to a submodule $V(\lambda)$ of V. On the other hand, every weight of $M(\mathfrak{b}_+,\lambda)$ has the form $\omega-n(\alpha-\beta)$ with $\omega\in\{\lambda,\lambda-\alpha,\lambda-\beta,\lambda-\alpha-\beta\}$ and n a non-positive integer; therefore, the weights of V have the same form. It follows that $\mathfrak{sl}(1,2)_\alpha\bigoplus\mathfrak{sl}(1,2)_\beta$ acts trivially in $V(\lambda)$ and this proves that the inclusion $V(\lambda)\hookrightarrow V$ induces an $\mathfrak{sl}(1,2)$-morphism

$$\eta: K_+(\lambda)\left(=\mathcal{U}(\mathfrak{sl}(1,2))\bigotimes_{\mathcal{U}(\mathfrak{sl}(1,2)_0\bigoplus\mathfrak{sl}(1,2)_+)}V(\lambda)\right)\longrightarrow V.$$

It is clear that, if Π is the projection of $M(\mathfrak{b}_+,\lambda)$ onto $K_+(\lambda)$, then $\eta\circ\Pi=\pi$. This proves the assertion 1a.

To prove the assertion 2b, take $\eta=w(\lambda)-\alpha+\beta$. Then η is a dominant weight and if π is a projection from $M(\mathfrak{b}_+,\eta)$ onto $K_+(\eta)$, then its kernel is isomorphic to $M(\mathfrak{b}_+,\lambda)$. In other words, one has a short exact sequence

$$\{0\}\longrightarrow M(\mathfrak{b}_+,\lambda)\overset{\iota}{\hookrightarrow} M(\mathfrak{b}_+,\eta)\overset{\pi}{\twoheadrightarrow} K_+(\eta)\longrightarrow\{0\}$$

which induces an exact sequence

$$\mathcal{L}_2(M(\mathfrak{b}_+,\eta))\longrightarrow\mathcal{L}_2(K_+(\eta))\longrightarrow\mathcal{L}_1(M(\mathfrak{b}_+,\lambda))\longrightarrow\mathcal{L}_1(M(\mathfrak{b}_+,\eta)).$$

But, since $M(\mathfrak{b}_+,\eta)$ is a Verma module, $\mathcal{L}_2(M(\mathfrak{b}_+,\eta))\simeq\{0\}$ and, since η is dominant, the assertion 1a shows that $\mathcal{L}_1(M(\mathfrak{b}_+,\eta))\simeq\{0\}$. Therefore,

$$\mathcal{L}_1(M(\mathfrak{b}_+,\lambda))\simeq\mathcal{L}_2(K_+(\eta)).$$

But, according to [7, proposition 4.8], $\mathcal{L}_2(K_+(\eta))$ is isomorphic to $\Gamma^0(K_+(\eta))$. Since $K_+(\eta)$ belongs to the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$, it follows that $\Gamma^0(K_+(\eta))\simeq K_+(\eta)$, because, as it was stated at the introduction, when V is an $\mathfrak{sl}(1,2)$-module that belongs to the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{s}_0)$, $\Gamma^0(V)$ is the greatest submodule of V in the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$.

Finally to prove the assertion 4a simply take $X_\beta \in \mathfrak{sl}(1,2)_\beta \setminus \{0\}$. It is easy to see that

$$\begin{array}{ccc} M(\mathfrak{b}_+,\lambda) & \longrightarrow & M(\mathfrak{b}_\pm,\lambda+\beta) \\ u\otimes c & \rightsquigarrow & uX_\beta\otimes c \end{array}$$

is an isomorphism.

Note that assertion 4 of the theorem is only a particular case of a much more general result; cf. [5, §0.1.5] or [6, p. 23].

3. Irreducible representations of $\mathfrak{gl}(1,1)$

Theorem 2.1 proves that an $\mathfrak{sl}(1,2)$-module of the form $\mathcal{L}_i(M(\mathfrak{b},\lambda))$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{sl}(1,2)$, is never an atypical finite-dimensional irreducible representation of $\mathfrak{sl}(1,2)$. It will be seen that such representations may be obtained by homological induction if one uses a parabolic subalgebra of $\mathfrak{sl}(1,2)$. To be more precise, it will be enough to use the parabolic subalgebras associated with one of the following sets of roots: $\{\pm\alpha,\beta\}$, $\{\pm\alpha,-\beta\}$, $\{\alpha,\pm\beta\}$ and $\{-\alpha,\pm\beta\}$. In each case, $\mathfrak{s}\simeq\mathfrak{gl}(1,1)$.

In order to study the finite-dimensional irreducible $\mathfrak{gl}(1,1)$-modules, we shall fix the notation. Put $X_\beta = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $X_{-\beta} = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ and $h_\beta = [X_\beta, X_{-\beta}] = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. If E is a finite-dimensional irreducible $\mathfrak{gl}(1,1)$-module, then the usual argument shows that E is $\mathfrak{h}$-semisimple and that, for some weight $\lambda\in\mathfrak{h}^*$, every weight of E has the form $\lambda - n\beta$, for some non negative integer n. It will be said then that λ is the highest weight of E.

If E is a super vector space, $\dim E = \dim E_0 + \epsilon\dim E_1$.

Proposition 3.1. *Let E be a finite-dimensional irreducible representation of the Lie superalgebra $\mathfrak{gl}(1,1)$ and let λ be its highest weight. Then the dimension of E is 1 or ϵ if $\lambda(h_\beta)=0$ and $1+\epsilon$ otherwise.*

Proof: Let $v_\lambda \in E_\lambda\setminus\{0\}$. Then $\mathbb{C}X_{-\beta}v_\lambda \bigoplus \mathbb{C}v_\lambda \bigoplus \mathbb{C}X_\beta v_\lambda$ is a submodule of E, since ${X_\beta}^2 v = {X_{-\beta}}^2 v = 0$ for every $v\in E$. In fact, since $\lambda+\beta$ is not a weight of E, $E = \mathbb{C}X_{-\beta}v_\lambda \bigoplus \mathbb{C}v_\lambda$. There are now two possibilities:

$\lambda(h_\beta)\neq 0$: In this case, $X_{-\beta}v_\lambda \neq 0$ since

$$X_\beta(X_{-\beta}v_\lambda) = X_\beta(X_{-\beta}v_\lambda) + X_{-\beta}(X_\beta v_\lambda) = h_\beta v_\lambda = \lambda(h_\beta)v_\lambda \neq 0.$$

$\lambda(h_\beta) = 0$: The same argument as above shows that $\mathbb{C}X_{-\beta}v_\lambda$ is a submodule of E; therefore, $E = \mathbb{C}X_{-\beta}v_\lambda$ or $X_{-\beta}v_\lambda = 0$. But, since $v_\lambda \in E$, $E \neq \mathbb{C}X_{-\beta}v_\lambda$ and this implies that $E = \mathbb{C}v_\lambda$.

4. Parabolic subalgebras

We will deal now with the case where $\mathfrak{p} = \mathfrak{h}\bigoplus\left(\bigoplus_{\eta\in\Psi}\mathfrak{sl}(1,2)_\eta\right)$, where Ψ is one among the following four sets: $\{\pm\alpha,\beta,-\alpha+\beta\}$, $\{\pm\alpha,-\beta,\alpha-\beta\}$, $\{\alpha,\pm\beta,\alpha-\beta\}$ and $\{-\alpha,\pm\beta,-\alpha+\beta\}$. Since all these cases are similar, it will be enough to do things in detail for one of these cases; this will be done with $\Psi = \{\alpha,\pm\beta,\alpha-\beta\}$. Therefore $\mathfrak{s} = \mathfrak{h}\bigoplus\mathfrak{sl}(1,2)_\beta\bigoplus\mathfrak{sl}(1,2)_{-\beta}(\simeq\mathfrak{gl}(1,1))$.

Theorem 4.1. *Let E be an irreducible finite-dimensional $\mathfrak{s}$-module and let λ be its highest weight.*

(1) *If λ is dominant, then*
   (a) *if $\lambda(h_\beta)\neq 0$, $\mathcal{L}_0(M(\mathfrak{p},E))\simeq K_+(\lambda)$;*
   (b) *if $\lambda(h_\beta) = 0$, $\mathcal{L}_0(M(\mathfrak{p},E))$ is irreducible with highest weight λ (in other words, $\mathcal{L}_0(M(\mathfrak{p},E))\simeq L(\mathfrak{b}_+,\lambda)$);*
   (c) *$\mathcal{L}_1(M(\mathfrak{p},E))\simeq\{0\}$.*
(2) *If λ is not dominant, then*

(a) $\mathcal{L}_0(M(\mathfrak{p},E)) \simeq \{0\}$;
(b) *if* $\lambda(h_\beta) \neq 0$, $\mathcal{L}_1(M(\mathfrak{p},E)) \simeq K_+(w(\lambda)-\alpha+\beta)$, *unless* $w(\lambda)-\alpha+\beta$ *is not dominant, in which case* $\mathcal{L}_1(M(\mathfrak{p},E)) \simeq \{0\}$;
(c) *if* $\lambda(h_\beta)=0$, $\mathcal{L}_1(M(\mathfrak{p},E))$ *is irreducible with highest weight* $w(\lambda)-\alpha+\beta$ *(that is, it is isomorphic with* $L(\mathfrak{b}_+, w(\lambda)-\alpha+\beta)$*), unless* $\lambda=\beta$, *in which case* $\mathcal{L}_1(M(\mathfrak{p},E)) \simeq \mathbb{C}$.

Proof: By the argument presented before the statement of theorem 2.1, the cases where it is stated that $\mathcal{L}_i(M(\mathfrak{p},E))$ is isomorphic either to $\{0\}$ or to $\mathbb{C}$ are easy to establish.

Take $X_{\pm\alpha} \in \mathfrak{sl}(1,2)_{\pm\alpha}$ and $X_{\pm\beta} \in \mathfrak{sl}(1,2)_{\pm\beta}$ such that $h_\alpha = [X_\alpha, X_{-\alpha}]$ and $h_\beta = [X_\beta, X_{-\beta}]$. Suppose that λ is dominant and that $\lambda(h_\beta) \neq 0$. If V is a quotient of $M(\mathfrak{p},E)$ that belongs to the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$ and if π is the projection of $M(\mathfrak{p},E)$ onto V, then, in order to be able to use proposition 1.1 (and the remark made after its statement), it must be proved that π factors through $K_+(\lambda)$. Let $v \in E_\lambda \setminus \{0\}$ and let $w = \pi(1 \otimes v)$; it can (and will) be assumed that $w \neq 0$. Since $w \in V_\lambda$ and since V belongs to the category $\mathcal{HC}(\mathfrak{sl}(1,2),\mathfrak{sl}(1,2)_0)$, there is an $\mathfrak{sl}(1,2)_0$-submodule of V isomorphic to $L_0(\lambda)$ and the inclusion of this module into V induces an $\mathfrak{sl}(1,2)$-morphism $F : K_+(\lambda) \longrightarrow V$ such that, for some $v^* \in L_0(\lambda)_\lambda \setminus \{0\}$, $F(1 \otimes v^*) = w$. Since $\lambda(h_\beta) \neq 0$, $\mathbb{C}(1 \otimes v^*) \bigoplus \mathbb{C}(X_{-\beta} \otimes v^*) \simeq E$ (as a $\mathfrak{gl}(1,1)$-module) and X_α acts trivially on $\mathbb{C}(1 \otimes v^*) \bigoplus \mathbb{C}(X_{-\beta} \otimes v^*)$, there is an $\mathfrak{sl}(1,2)$-morphism $\eta : M(\mathfrak{p},E) \longrightarrow K_+(\lambda)$ such that $\eta(v) = 1 \otimes v^*$. Therefore $(F \circ \eta)(v) = w = \pi(v)$; since v generates $M(\mathfrak{p},E)$, $F \circ \eta = \pi$. This proves the assertion 1a.

Suppose now that λ is dominant and that $\lambda(h_\beta) = 0$. In this case, and since proposition 3.1 tells us that $E = \mathbb{C}v$ for some $v \in E$ whose weight is λ, the $\mathfrak{sl}(1,2)_0$-module $M(\mathfrak{p},E)$ is the direct sum of two Verma modules whose highest weights are λ and $\lambda-\alpha$. Observe that, since $\lambda(h_\beta)=0$ and λ is dominant, $\lambda = n\alpha$ for some non-negative integer n. There are two possibilities:

$\lambda \neq 0$: Then λ and $\lambda-\alpha$ are both dominant weights and therefore, as an $\mathfrak{sl}(1,2)_0$-module, $\mathcal{L}_0(M(\mathfrak{p},E))$ is isomorphic to the direct sum

$$L_0(\lambda) \bigoplus L_0(\lambda-\alpha).$$

Let ω_1 be an element of $\mathcal{L}_0(M(\mathfrak{p},E))_\lambda$ different from 0 and define $\omega_2 = X_{-\alpha}\omega_1$. Then the weight of ω_2 is $\lambda-\alpha$ and $\omega_2 \neq 0$ since

$$X_\alpha \cdot \omega_2 = X_\alpha \cdot (X_{-\alpha} \cdot \omega_1) = -X_{-\alpha} \cdot (X_\alpha \cdot \omega_1) + h_\alpha \cdot \omega_1 = \lambda(h_\alpha)\omega_1$$

and both $\lambda(h_\alpha)$ and ω_1 are different from 0. Since, as an $\mathfrak{sl}(1,2)_0$-module, $\mathcal{L}_0(M(\mathfrak{p},E))$ is the direct sum of two irreducible modules, generated by ω_1 and ω_2, $\omega_2 = X_{-\alpha}\omega_1$, and $X_\alpha \cdot \omega_2 = \lambda(h_\alpha)\omega_1$, the $\mathfrak{sl}(1,2)$-module $\mathcal{L}_0(M(\mathfrak{p},E))$ is irreducible and generated by ω_1; it is therefore isomorphic to $L(\mathfrak{b}_+,\lambda)$.

$\lambda = 0$: Then $\lambda-\alpha = -\alpha$ and therefore it is not a dominant weight. In this case then, as a $\mathfrak{sl}(1,2)_0$-module, $\mathcal{L}_0(M(\mathfrak{p},E))$ is simply $L_0(\lambda) (= L_0(0))$, which is isomorphic to $\mathbb{C}$. Therefore, $\mathcal{L}_0(M(\mathfrak{p},E)) \simeq \mathbb{C} \simeq L(\mathfrak{b}_+,0) = L(\mathfrak{b}_+,\lambda)$.

Finally, the statements concerning $\mathcal{L}_1(M(\mathfrak{p},E))$ when λ is not dominant can be proved in the same way as in theorem 2.1.

There are only two parabolic subalgebras $\mathfrak{p}$ left (distinct from $\mathfrak{sl}(1,2)$), namely

$$\mathfrak{p} = \mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_+ \text{ and } \mathfrak{p} = \mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_-.$$

We shall simply describe the $\mathfrak{sl}(1,2)$-modules homologically induced that can be obtained from the first of these two parabolic subalgebras; the proof is similar (in fact, easier) to the proofs of theorems 2.1 and 4.1.

Theorem 4.2. *Let $\mathfrak{p} = \mathfrak{sl}(1,2)_0 \bigoplus \mathfrak{sl}(1,2)_+$ and let E be an irreducible finite-dimensional $\mathfrak{sl}(1,2)_0$-module with highest weight $\lambda \in \mathfrak{h}^*$. Then $\mathcal{L}_0(E) \simeq K_+(\lambda)$ and $\mathcal{L}_1(E) \simeq \{0\}$.*

5. Final remarks

The description made in sections 2 and 4 of the $\mathfrak{sl}(1,2)$-modules that are homologically induced show that

(1) no single parabolic subalgebra of $\mathfrak{sl}(1,2)$ is enough to obtain every irreducible finite-dimensional $\mathfrak{sl}(1,2)$-module by homological induction;
(2) every homologically induced $\mathfrak{sl}(1,2)$-module is indecomposable and, furthermore, every indecomposable $\mathfrak{sl}(1,2)$-module is homologically induced.

Remember that it was stated at the beginning of the article that all our modules are $\mathfrak{h}$-semisimple. In fact, there are $\mathfrak{sl}(1,2)$-modules which are indecomposable but that are not $\mathfrak{h}$-semisimple (see [1]).

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