The fundamental theorem of algebra deduced from elementary calculus

José Carlos de Sousa Oliveira Santos

The Mathematical Gazette, July 2007

This article contains an elementary proof of the fundamental theorem of algebra which uses only well-known results from calculus. It also explains how the same basic idea can be used to provide a very short proof of that theorem based upon Cauchy's integral formula.

The fundamental theorem of Algebra states:

Every non-constant polynomial function from the complex field into itself has at least one zero.

In order to prove it, suppose that there is some polynomial function P from the complex field into itself which is not constant and which has no zeros. Since $P(z)$ is never 0, we can consider the function

$$\begin{array}{rccc} f\colon & [0,\infty) & \longrightarrow & \mathbb{C} \\ & r & \mapsto & \displaystyle\int_0^{2\pi} \frac{1}{P\left(re^{it}\right)}\,dt. \end{array}$$

Since $\lim_{r\to\infty}\left|P\left(re^{it}\right)\right| = \infty$, we have $\lim_{r\to\infty} f(r) = 0$. On the other hand, $f(0) = 2\pi/P(0) \neq 0$. Therefore f cannot be constant. We shall now prove that f is constant, thereby reaching a contradiction.

Since the domain of f is the interval $[0,\infty)$ and since f is clearly continuous, in order to prove that f is constant, it will be enough to prove that $f'(r) = 0$ when $r > 0$. To compute $f'(r)$ for $r > 0$, all that we have to do is to apply Leibniz's rule (see [2, ch. 9], for instance); in this particular case, Leibniz's rule says that

$$f'(r) = \int_0^{2\pi} \frac{\partial}{\partial r}\frac{1}{P\left(re^{ix}\right)}\,dx. \tag{1}$$

On the other hand, since

$$\frac{\partial}{\partial x}\frac{1}{P\left(re^{ix}\right)} = \frac{-ire^{ix}P'\left(re^{ix}\right)}{P^2\left(re^{ix}\right)} = ir\frac{-e^{ix}P'\left(re^{ix}\right)}{P^2\left(re^{ix}\right)} = ir\frac{\partial}{\partial r}\frac{1}{P\left(re^{ix}\right)},$$

it is a consequence of (1) that

$$f'(r) = \frac{1}{ir}\int_0^{2\pi} \frac{\partial}{\partial x}\frac{1}{P\left(re^{ix}\right)}\,dx = \frac{1}{ir}\left[\frac{1}{P\left(re^{ix}\right)}\right]_{x=0}^{x=2\pi} = 0.$$

This concludes the proof of the fundamental theorem of algebra, but let's see where the definition of f comes from; this will lead us to another proof of the theorem, which will be based upon Complex Analysis. According to Cauchy's integral formula (see [1, ch. III] or [3, ch. 7] for more details as well as for background), if A is an open subset of $\mathbb{C}$, g is an analytical function from A into $\mathbb{C}$, $a \in A$ and $r > 0$ is such that the closed disk $\overline{D(a,r)}$ is contained in A, then

$$g(a) = \frac{1}{2\pi i}\int_{\gamma(r,a)} \frac{g(z)}{z-a}\,dz,$$

where $\gamma(r,a)\colon [0,2\pi] \longrightarrow \mathbb{C}$ is the closed path defined by $t \mapsto a + re^{it}$.

Let us apply Cauchy's integral formula to the function $1/P$, taking $a = 0$ and an arbitrary $r > 0$; then we have

$$\frac{1}{P(0)} = \frac{1}{2\pi i}\int_{\gamma(r,0)} \frac{1}{zP(z)}\,dz. \tag{2}$$

But

$$\frac{1}{2\pi i}\int_{\gamma(r,0)} \frac{1}{zP(z)}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{P\left(re^{it}\right)}\,dt = \frac{f(r)}{2\pi}\cdot$$

Note that (2) then states that $f \equiv 2\pi/P(0)$; in particular, f is constant and non-null. Again, the fact that $\lim_{r\to\infty} f(r) = 0$ allows us to reach a contradiction. This approach through complex analysis provides a very short (although not elementary) proof of the fundamental theorem of algebra.

References

1. S. Lang, *Complex analysis*, Springer-Verlag (1999).
2. J. E. Marsden and M. J. Hoffman, *Elementary classical analysis*, W. H. Freeman (1993).
3. R. Remmert, *Theory of complex functions*, Springer-Verlag (1998).