Games and Economic Behavior 69 (2010) 42–71

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Price dynamics on a stock market with asymmetric information

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ARTICLE INFO

Article history:
Received 19 February 2007
Available online 6 February 2010

JEL classification:
G14
C72
C73
D44

Keywords:
Repeated games
Incomplete information
Price dynamics
Martingales of maximal variation

ABSTRACT

When two asymmetrically informed risk-neutral agents repeatedly exchange a risky asset for numéraire, they are essentially playing an n-times repeated zero-sum game of incomplete information. In this setting, the price L_q at period q can be defined as the expected liquidation value of the risky asset given players' past moves. This paper indicates that the asymptotics of this price process at equilibrium, as n goes to ∞, is completely independent of the "natural" trading mechanism used at each round: it converges, as n increases, to a Continuous Martingale of Maximal Variation. This martingale class thus provides natural dynamics that could be used in financial econometrics. It contains in particular Black and Scholes' dynamics. We also prove here a mathematical theorem on the asymptotics of martingales of maximal M-variation, extending Mertens and Zamir's paper on the maximal L^1-variation of a bounded martingale.

1. Introduction

One fundamental problem in financial analysis is to accurately identify the stock price dynamics: different dynamics for the underlying asset will lead to different pricing formulae for derivatives.

Financial econometrics does not completely solve this problem: Statistical methods can calibrate the parameters of a model, finding in a general class of possible dynamics, the one that best fits the historical data. But still, assumptions have to be made regarding the class of possible dynamics. Most of the classes used in practice (Bachelier's dynamics, Black and Scholes dynamics, diffusion models, stochastic volatility models, GARCH-models, etc.) are chosen by a kind of rule of thumb, with no real economic justification. The randomness of the prices is often conceived as completely exogenous. The first sentences in Bachelier's (1900) thesis illustrate quite well this kind of explanation: "The influences that determine the price variations on the stock market are uncountable. Past, present or even future expected events, having often nothing to do with the stock market, have repercussion on the prices."

In this paper however, we suggest that part of the randomness in the stock price dynamics is endogenous: it is introduced by the agents in order to maximize their profit. This idea was already present in De Meyer and Moussa-Saley (2003), where the Brownian term in the price dynamics was explained endogenously. Institutional investors clearly have better access to information on the market than the private ones: they are better skilled to analyze the flow of information and in some cases they are even part of the board of directors of the firms of which they are trading the shares. So, institutional investors are better informed and this informational advantage is known publicly. As a consequence, each of their moves on the markets is analyzed by the other agents to extract its informational content. If informed agents act naively, making

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[1] The main ideas of this paper were developed during my stay at the Cowles Foundation for research in Economics at Yale University. I am very thankfull to all the members of the Cowles foundation for their hospitality. I would like to thank John Geanakoplos and Pradeep Dubey for fruitful discussions.

0899-8256/$ – see front matter
doi:10.1016/j.geb.2010.01.011

moves that depend deterministically on their information, they will completely reveal this information to the other agents, and doing so, they will lose their strategic advantage for the future. The only way to benefit from the information without revealing it too fast is to introduce noise on their moves: this is tantamount to selecting random moves via lotteries that depend on their information. The main idea in De Meyer and Moussa-Saley (2003) is that the noise introduced by the informed agents in the day-to-day transactions will generate a Brownian motion.

The central result of this paper is that this kind of argument leads to a particular class of price dynamics, hereafter referred to as Continuous Martingales of Maximal Variation (CMMV), which is quite robust: CMMV will appear in any repeated exchange between two risk neutral asymmetrically-informed players, independently of the "natural" trading mechanism used in the exchanges.

The structure of the paper is as follows: in the next section, we define CMMV precisely. As suggested by our main result, this class of dynamics is a natural candidate that could be used in financial econometrics as the class of possible dynamics. As will be seen, this class of dynamics is a subclass of local volatility models that contains as particular cases Bachelier's dynamics as well as Black and Scholes' dynamics.

In Section 3, we introduce the game Γ_n to model the repeated exchanges between two risk neutral asymmetrically informed players in the most general way: Player 1 initially receives a private message concerning the liquidation value L of the risky asset traded. During n consecutive rounds, the players exchange the risky asset against a numéraire, using a general trading mechanism $\langle I, J, T\rangle$ which is simply a game with respective action spaces I and J for players 1 and 2, and whose outcome $T(i, j)$ is a transfer vector representing the quantities of risky asset and numéraire exchanged when actions are (i, j). At each round, actions are chosen simultaneously and are then publicly announced. The players aim to maximize the liquidation value of their final portfolio.

The game Γ_n is equivalent to a zero sum repeated game with one sided information à la Aumann–Maschler. In Section 4, we define the concepts of strategy, value and optimal strategy for Γ_n. Since players are risk neutral, the natural notion of price L_q of the risky asset at period q is defined as the conditional expected liquidation value given player 2's previous observations.

The trading mechanism introduced above should satisfy some properties in order to represent real exchanges on the stock market. In Section 5, we introduce 5 axioms that must be satisfied by a "natural" trading mechanism. Let us describe them very briefly here:

(H1) The game Γ_n has a value V_n whatever the distribution of the liquidation value is.
(H2) is a continuity assumption of V_1 as a function of the law of the liquidation value.
(H3) stipulates that the mechanism should be invariant with respect to the scale of numéraire: If two players use this mechanism to exchange a risky asset R against the dollar or against the cent, the same transactions in value will be observed in both cases. More specifically, the quantity of risky asset exchanged will be the same, but the counterpart in cents will be the counterpart in dollars multiplied by 100. (H3) is thus a 1-homogeneity property of the value.
(H4) is an invariance axiom with respect to the riskless part of the risky asset: If two players use the trading mechanism to exchange with the dollar as numéraire an asset R' consisting of one share of asset R and a \$100 bill, then the transaction observed will be the same in value as if they were exchanging R for the dollar. In other words, the quantity x of R- and R'-shares exchanged will be the same in both cases, but the x bills of \$100 exchanged within the R'-shares will be paid back in dollars, that is, if y and y' denotes the counterpart in numéraire when exchanging R and R' respectively, then $y' = y + 100x$. The value of the game must thus remain unchanged if one shifts the liquidation value by a constant amount.
(H5) There exists a situation in which player 1 can take a strictly positive profit from his private information: he is strictly better off with his message than without. This axiom is on the one hand completely natural to model the stock market: it seems indeed commonsense that private information has a strictly positive value on the market. Otherwise, there would clearly be no need for insider trading regulation, since no one would have incentive to make such trades.

On the other hand, however, this axiom is in a way unnatural. This game is zero sum and has a positive value. So why should the uninformed player participate in a game where he is loosing money? This is a particular case of Milgrom–Stokey's No Trade Theorem. Some agents on the market are in fact forced to trade: for instance, a market maker facing a more informed trader. Since the bid and the ask posted by a market maker is a commitment to buy or sell at these prices any quantity of shares up to a prefixed limit, the only way for the market maker to avoid trading would be to post a very large bid-ask spread. Most market regulations however impose explicit limit on market makers' bid-ask spread, thus steering past the No Trade paradox.

At the end of Section 5, we state the main result of the paper which is Theorem 1. It indicates that if the trading mechanism is natural in the above sense, if the price process $(L_q)_{q=0,\dots,n}$ at equilibrium in Γ_n is represented by the continuous time process $(\Pi^n_t)_{t\in[0,1]}$, with $\Pi^n_t := L_q$ on the time interval $[q/n, (q+1)/n[$, then Π^n converges in law to a particular CMMV Π^μ depending just on the law μ of the liquidation value of R. The limit is thus completely independent of the natural trading mechanism considered, showing in this way the robustness of the CMMV class of dynamics. This paper differs from the existing literature on trading with asymmetrical information (see e.g. Kyle, 1985) by the fact that the price randomness is essentially considered as endogenous. In Kyle's paper however, to get rid of the above mentioned No Trade paradox, noise traders have to be introduced, and the resulting dynamics will thus crucially depend on the hypotheses made on this exogenous source of randomness.

We will provide explicit examples of trading mechanism satisfying (H1) to (H5) is Section 6, and we will compare our results with De Meyer and Moussa-Saley (2003). In that paper a particular trading mechanism is analyzed for which optimal strategies can be explicitly computed and the convergence to the CMMV Π^{μ} can be proved directly. The result of this paper however is much more general and applies to games with abstract trading mechanisms. In the absence of closed-form formulae, a more subtle and abstract line of analysis has to be followed.

The proof of Theorem 1 is presented in Section 7. By linking more or less his moves to his initial message, player 1 can chose the rate of revelation of his private information and in this way, he can control the price process $(L_q)_{q=1,\dots,n}$, which is precisely the conditional expectation of L given the past moves. We then express the maximal amount player 1 can guarantee in Γ_n with a given revelation martingale $(L_q)_{q=1,\dots,n}$. We prove in particular that this amount is the V_1-variation of the martingale $(L_q)_{q=1,\dots,n}$: for a function M, such as V_1, that maps probability measures (the laws of the liquidation value for V_1) to $\mathbb{R}$, the M-variation of the martingale $(L_q)_{q=1,\dots,n}$ is defined as $E[\sum_{q=0}^{n-1} M([L_{q+1} - L_q|(L_s)_{s\leqslant q}])]$, where $[L_{q+1} - L_q|(L_s)_{s\leqslant q}]$ denotes the conditional law of the increment $L_{q+1} - L_q$ given the past at time q. The informed player is thus facing a martingale optimization problem: the price martingale (L_q) at equilibrium must maximize the V_1-variation.

Theorem 1 then follows at once from Theorem 5 which is a mathematical result proved in the second part of the paper. It is an interesting result that generalizes both Mertens and Zamir (1977) and De Meyer (1998) on the maximal L^1- and L^p-variation of a bounded martingale: It states that the continuous representation $(\Pi^n_t)_{t\in[0,1]}$ of a martingale $(L^n_q)_{q=0,\dots,n}$ (i.e. $\Pi^n_t = L^n_q$ if $t \in [q/n, (q+1)/n[$) with final distribution μ that maximizes the M-variation converges in law, as $n \to \infty$, to the CMMV Π^{μ}, provided M satisfies a homogeneity and a continuity property. This result justifies the terminology Continuous Martingale of Maximal Variation adopted in this paper. Since the limit Π^{μ} is independent of M, which is a completely general function that could even have no relation with a game, this result underlines even more the robustness of the CMMV class. The proof of this result is based on three ingredients: duality, the central limit theorem and Skorokhod's embedding techniques. We refer the reader to Section 8 for more details.

It is generally fairly difficult to prove that one particular mechanism satisfies to (H1): Games of any length must have a value. In Appendix A, we prove that it is often sufficient to prove that the one shot game has a value.

2. Continuous martingales of maximal variation

Let Δ be the set of probability distributions on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, where $B_{\mathbb{R}}$ is the Borel tribe on $\mathbb{R}$. In the following, the probability distribution of a random variable X will be denoted $[X]$. If $\mu \in \Delta$, we will use both notations $X \sim \mu$ or $[X] = \mu$ to indicate that the random variable X is μ-distributed. We also write $\|\mu\|_{L^p}$ for $\|X\|_{L^p}$ where $X \sim \mu$, and we set $\Delta^p := \{\mu \in \Delta\colon \|\mu\|_{L^p} < \infty\}$.

A continuous martingale of maximal variation is defined as a martingale Π on time interval $[0,1]$ that can be written as: $\forall t\colon \Pi_t = f(B_t, t)$ where B is a standard Brownian motion and $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ is increasing in the first variable.

Notice that we can recover the whole function $f(x,t)$ from its terminal values $g(x) := f(x,1)$: Indeed, using the Markov property of the Brownian motion, we get:

$$f(B_t, t) = \Pi_t = E\big[\Pi_1|(B_s)_{s\leqslant t}\big] = E\big[g(B_1)|B_t\big] = E\big[g\big(B_t + (B_1 - B_t)\big)|B_t\big].$$

Since, $B_1 - B_t \sim \mathcal{N}(0, (1-t))$ is independent of B_t, we get with h_t denoting the density function of $\mathcal{N}(0,(1-t))$:

$$f(x,t) = \int_{-\infty}^{\infty} g(x-y) h_t(y)\, dy = g * h_t(x),$$

$*$ representing the convolution product. Due to the smoothing property of the normal kernel, the function f is thus $\mathcal{C}^{\infty}$ on $\mathbb{R} \times [0,1[$. We may then also apply Itô's formula to the process Π_t and we get:

$$d\Pi_t = \partial_x f(B_t,t)\, dB_t + \left(\partial_t f(B_t,t) + \frac{1}{2}\partial^2_{x,x} f(B_t,t)\right) dt.$$

For Π to be a martingale, the drift term must vanish and f must thus satisfy the heat equation $\partial_t f(x,t) + \frac{1}{2}\partial^2_{x,x} f(x,t) = 0$.

One useful property of CMMV is that for a given $\mu \in \Delta^1$, there exists a unique CMMV denoted hereafter Π^{μ} such that $\Pi^{\mu}_1 \sim \mu$. Indeed, as is well known, there exists a unique (up to a null set) increasing function $f_{\mu} : \mathbb{R} \to \mathbb{R}$ that $f_{\mu}(Z) \sim \mu$ if $Z \sim \mathcal{N}(0,1)$. The function f_{μ} can be expressed in terms of the cumulative distribution functions F_{μ} and $F_{\mathcal{N}}$ of μ and $\mathcal{N}(0,1)$ by the following formula: $f_{\mu}(x) = F^{-1}_{\mu}(F_{\mathcal{N}}(x))$, where $F^{-1}_{\mu}(y) := \inf\{s\colon F_{\mu}(s) > y\}$.

Since $\Pi^{\mu}_1 = f(B_1, 1) \sim \mu$ and $f(x,1)$ is by hypothesis increasing in x, it must be the case that $f(x,1) = f_{\mu}(x)$, as $B_1 \sim \mathcal{N}(0,1)$. As mentioned above, we then get $f(x,t) = \int_{-\infty}^{\infty} f_{\mu}(x-y) h_t(y)\, dy$ which is clearly increasing in x, since so is f_{μ}: if $x > x'$: $f_{\mu}(x-y) \geqslant f_{\mu}(x'-y)$. Multiplying both sides of this inequality by $h_t(y) > 0$, we get indeed after integration $f(x,t) \geqslant f(x',t)$.

We conclude this section by showing that the CMMV class is a class of local volatility models for the price process that contains as particular cases Bachelier's Dynamics as well as Black and Scholes' one. Indeed, if μ is not a Dirac measure, f_{μ}

is not a constant function. Therefore, the strict inequality $f_\mu(x-y) > f_\mu(x'-y)$ will hold in the previous argument for a set of y of positive Lebesgue measure. As a result, for $t<1$, $f(x,t) > f(x',t)$, whenever $x > x'$. The strictly increasing function $x \to f(x,t)$ has thus an inverse ϕ_t and the relation $\Pi_t = f(B_t,t)$ yields $B_t = \phi_t(\Pi_t)$. Our above formula for $d\Pi_t$ thus becomes: $d\Pi_t = \partial_x f(B_t,t)\,dB_t = a(\Pi_t,t)\,dB_t$, where $a(y,t) := \partial_x f(\phi_t(y),t)$. We also have $d\Pi_t = a(\Pi_t,t)\,dB_t$ with $a(y,t)=0$ when μ is a Dirac measure. A model of actualized price dynamics under the risk-neutral probability measure satisfying this diffusion equation is referred to as a local volatility model in finance, a being the volatility function (e.g. Dupire, 1997). Since a must satisfy specific properties for the corresponding process to be a CMMV, the CMMV class is indeed a subclass of local volatility models.

Under the risk-neutral probability, Bachelier's dynamics and Black and Scholes' dynamics for the actualized price process are given by the formulae $d\Pi_t = \sigma\,dB_t$ and $d\Pi_t = \sigma\Pi_t\,dB_t$, where $\sigma > 0$ is the volatility parameter. As is well known, in both case we get $\Pi_t = f(B_t,t)$, with f defined respectively as $f(x,t) = \sigma x + c$ and $f(x,t) = c\,e^{\sigma x - \frac{\sigma^2}{2}t}$. These two functions are increasing in x.

3. The game $\Gamma_n(\mu)$

In the game Γ_n analyzed in this paper, two players are repeatedly trading a risky asset R against a numéraire N.

We first describe the *information asymmetry*: At the beginning of the game, player 1 (P1) receives a private message m concerning the risky asset. This message is randomly chosen by nature with a given probability distribution ν. P2 does not receive this message. He just knows that P1 has been informed and he also knows the probability distribution ν.

The message m will be publicly revealed at a future date T, say at the next shareholder meeting. The price L of R on the market at that date is called the liquidation value of R. It will depend on m and L is thus a function $L(.)$ of m. The liquidation value of N is independent of m and is fixed to be 1. We assume that both players know how to interpret the message m and they therefore know the function $L(.)$.

Let μ denote the probability distribution of $L(m)$ when $m \sim \nu$. We will assume in this paper that $\mu \in \Delta^1$. As $L(m)$ is the only relevant information in the message m, we may assume that P1 is only informed of $L(m)$.

So, the initial stage in $\Gamma_n(\mu)$ is simply the following: Nature picks L at random with probability μ. P1 is informed of L, while P2 is not. Both players know μ.

We next consider n rounds of exchange before the revelation date T. To model these repeated exchanges in the most general way, we introduce the notion of a *trading mechanism*. Such a mechanism is defined as a triple $\langle (I,\mathcal{I}),(J,\mathcal{J}),T\rangle$, where I and J are the respective action sets of P1 and P2, endowed with σ-algebras $\mathcal{I},\mathcal{J}$, and where $T: I \times J \to \mathbb{R}^2$ is the transfer function, assumed to be measurable from $(\mathcal{I}\otimes\mathcal{J})$ to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^2}$. If the players play (i,j), $T(i,j) = (A_{ij},B_{ij})$ represents the transfer from P2 to P1: A_{ij} and B_{ij} are the respective numbers of R and N shares that P1 receives from P2. (Typically one is positive and the other negative.)

So, in $\Gamma_n(\mu)$, at trading period q $(q=1,\dots,n)$, the players select an action pair (i_q,j_q) independently of each other, based on their prior observations and private information. Actions are then made public and the portfolios are then incremented. If $y_q = (y_q^R, y_q^N)$ and $z_q = (z_q^R, z_q^N)$ represent P1 and P2's portfolios at the end of round q, we have thus: $y_q = y_{q-1} + T(i_q,j_q)$ and $z_q = z_{q-1} - T(i_q,j_q)$. We assume that both players have sufficiently large portfolios, or have short-selling capacities, so as to honor the trade $T(i_q,j_q)$.

We will give examples of trading mechanisms in Section 6. The exchange at period q could result from a bargaining game or from an auction procedure, and actions in these setups are the strategies used by the players in these mechanisms. Another example of interest is a market game: Players' actions i and j are demand functions for asset R in a given class E. In this case, $I = J = E$ is a set of strictly decreasing functions i from $\mathbb{R}$ to $\mathbb{R}$ satisfying $\lim_{p\to-\infty} i(p) > 0$ and $\lim_{p\to\infty} i(p) < 0$. Once i and j are selected, the marked clearing price $p^*(i,j)$ is computed, whenever it exists: it is the value of p solving the equation $i(p) + j(p) = 0$. The transfer vector, representing the quantities of the risky asset and numéraire that player 1 receives during the exchange is then $T(i,j) := (i(p^*(i,j)), -p^*(i,j)i(p^*(i,j)))$, and $T(i,j) = (0,0)$, if $p^*(i,j)$ fails to exist. In other words, P1 receives the demanded quantity of asset R corresponding to the market clearing price. The counterpart in numéraire is that quantity multiplied by the clearing price.

Players' utility: The players are supposed to be risk-neutral and they aim to maximize the expected liquidation value of their final portfolio. So P1's utility is: $E[y_n^R L + y_n^N]$ and P2's is $E[z_n^R L + z_n^N]$. Since y_0 and z_0 are initially fixed, the liquidation values of the initial portfolios are constants that can be subtracted from player's utilities without affecting their behavior in the game. This turns out to be equivalent to assuming, as we will do in the remaining part of the paper, that $y_0 = z_0 = (0,0)$, allowing for negative entries in the portfolios.

With that hypothesis, we get clearly $y_n = -z_n$ and the game $\Gamma_n(\mu)$ is then a zero sum game.

Since $y_n = \sum_{q=1}^n T(i_q,j_q)$, P1's payoff in $\Gamma_n(\mu)$ becomes $\sum_{q=1}^n h(L,i_q,j_q)$, where $h(L,i,j) := LA_{ij} + B_{ij}$ is the stage payoff. $\Gamma_n(\mu)$ is thus a repeated zero-sum game with one sided information and full monitoring à la Aumann–Maschler. The stage payoff at stage q just depends on the current actions i_q, j_q and of the state of nature L initially chosen. The only difference with Aumann–Maschler's model is that there could be infinitely many states and actions.

4. Strategies, value, equilibria and price process in $\Gamma_n(\mu)$

Let us first define strategies in $\Gamma_n(\mu)$. A mixed strategy for P2 in $\Gamma_1(\mu)$ is a probability distribution τ on $(J, \mathcal{J})$. However, since A_{ij} and B_{ij} are a priori unbounded, we have to restrict a little bit this definition. Let $\Delta(J)$ be the set of probability distributions τ on $(J, \mathcal{J})$ such that,

$$\forall i \in I: \quad \int_J |A_{ij}|\, d\tau(j) < \infty \quad \text{and} \quad \int_J |B_{ij}|\, d\tau(j) < \infty.$$

For $\tau \in \Delta(J)$, we set: $A_{i\tau} := \int_J A_{ij}\, d\tau(j)$ and $B_{i\tau} := \int_J B_{ij}\, d\tau(j)$. In the same way, we define $\Delta(I)$ and, for $\sigma \in \Delta(I)$, $A_{\sigma,j}$ and $B_{\sigma,j}$.

A strategy τ for P2 in $\Gamma_n(\mu)$ is a sequence $(\tau_1, \dots, \tau_n)$ of $\Delta(J)$-valued transition probabilities $\tau_q : (H_{q-1}, \mathcal{H}_{q-1}) \to (J, \mathcal{J})$, where $(H_q, \mathcal{H}_q) := ((I \times J)^q, (\mathcal{I} \times \mathcal{J})^q)$. In other words: $\forall h_{q-1} \in H_{q-1}$, $\tau_q(h_{q-1}) \in \Delta(J)$ and $\forall A \in \mathcal{J}$: the map $h_{q-1} \to \tau_q(h_{q-1})[A]$ is $\mathcal{H}_{q-1}$-measurable. $\tau_q(h_{q-1})$ is thus the probability distribution used by P2 to select his action j_q. This probability depends on the past observation h_{q-1} of P2 at that stage.

In the same way, a strategy σ in $\Gamma_n(\mu)$ is a sequence $(\sigma_1, \dots, \sigma_n)$ of $\Delta(I)$-valued transition probabilities $\sigma_q : (\mathbb{R} \times H_{q-1}, \mathcal{B}_{\mathbb{R}} \times \mathcal{H}_{q-1}) \to (I, \mathcal{I})$. At stage q, P1 will pick his action i_q with the probability distribution σ_q that depends both on his private information L and on the past moves h_{q-1} of both players. $\mathcal{S}_n$ will denote hereafter the set of P1's strategies.

With Tulcea theorem, a triplet (μ, σ, τ) will induce a unique probability $\pi_{(\mu,\sigma,\tau)}$ on $(\mathbb{R} \times H_n)$.

Still the payoff function could be undefined in general for integrability reasons and we have to restrict our notion of strategy. The integrability problem can be illustrated as follows: suppose that τ_2 is just a function of j_1 so that $B_{i_2\tau_2(j_1)}$, is a finite function of j_1, but it could fail to be integrable with respect to τ_1.

This leads us to the definition of admissible strategy: A strategy τ is said admissible if for every history $h^1 \in I^n$, the probability $\pi^2_{(h^1,\tau)}$ induced on J^n by (h^1, τ) is such that for all q, the random variables $|A_{i_q,j_q}|$ and $|B_{i_q,j_q}|$ have finite expectation with respect to $\pi^2_{(h^1,\tau)}$. $\mathcal{T}_n^{\text{adm}}$ will denote the set of P2's admissible strategies. Observe that $\pi^2_{(h^1,\tau)}$ is just the conditional probability $\pi_{(\mu,\sigma,\tau)}$ on J^n given h^1.

So, $\mathcal{A}^n(h^1, \tau) := E_{\pi^2_{(h^1,\tau)}}[\sum_{q=1}^n A_{i_q,j_q}]$ and $\mathcal{B}^n(h^1, \tau) := E_{\pi^2_{(h^1,\tau)}}[\sum_{q=1}^n B_{i_q,j_q}]$ are the expected R and N quantities in y_n given that player 1 played h^1. These are finite measurable functions of h^1.

Let us now write formally the payoff in $\Gamma_n(\mu)$. Notice that y_n is independent of L conditionally to h^1, since P2's moves depend on h^1 but not on L. Therefore, with expectations taken with respect to $\pi_{(\mu,\sigma,\tau)}$, and assuming the integrability of $Ly_n^R + y_n^N$, we could write:

$$\begin{aligned} E\big[Ly_n^R + y_n^N\big] &= E\big[E\big[Ly_n^R|h^1\big]\big] + E\big[E\big[y_n^N|h^1\big]\big] \\ &= E\big[E\big[L|h^1\big] \cdot E\big[y_n^R|h^1\big]\big] + E\big[\mathcal{B}^n\big(h^1, \tau\big)\big] \\ &= E\big[L\mathcal{A}^n\big(h^1, \tau\big) + \mathcal{B}^n\big(h^1, \tau\big)\big]. \end{aligned} \tag{1}$$

Observing the last formula, the best player 1 can do to reply to τ is to play a history $h^1(L)$ depending on L, that solves the problem:

$$\phi_\tau^n(L) := \sup_{h^1} L\mathcal{A}^n\big(h^1, \tau\big) + \mathcal{B}^n\big(h^1, \tau\big). \tag{2}$$

Optimal solutions could fail to exist, but measurable ϵ-solutions exist. Therefore a strategy τ guarantees $E_\mu[\phi_\tau^n(L)]$ to P2. As supremum of a family of affine functions of L, $\phi_\tau^n(L)$ is a convex l.s.c. function from $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$ and is therefore measurable. Since $\mu \in \Delta^1$, we also have: $E_\mu[\phi_\tau^n(L)] > -\infty$.

Notice that there could be integrability problems in Eq. (1) in general and it could be the case that $E[Ly_n^R + y_n^N]$ is undetermined (meaning that both the positive and the negative part of $Ly_n^R + y_n^N$ have infinite expectation) although $E[L\mathcal{A}^n(h^1,\tau) + \mathcal{B}^n(h^1,\tau)]$ is finite. This remark leads us to define the payoff function in $\Gamma_n(\mu)$ as $g_n(\mu,\sigma,\tau) := E_{\pi_{(\mu,\sigma,\tau)}}[L\mathcal{A}^n(h^1,\tau) + \mathcal{B}^n(h^1,\tau)]$.

This definition of the payoff could still be undetermined for some pairs of strategies. However, if $E_\mu[\phi_\tau^n(L)] < \infty$, then the payoff function $g_n(\mu,\sigma,\tau)$ is possibly equal to $-\infty$, but there is never indeterminacy, whatever the strategy σ is.

The minimal amount player 2 can guarantee is $\overline{V}_n(\mu) := \inf_{\tau \in \mathcal{T}_n} E_\mu[\phi_\tau^n(L)]$.

A strategy τ of player 2 is optimal in $\Gamma_n(\mu)$ if $\overline{V}_n(\mu) = E_\mu[\phi_\tau^n(L)]$.

A strategy σ is admissible for player 1, if, for all admissible strategy τ:

$$E\big[\min\big(L\mathcal{A}^n\big(h^1, \tau\big) + \mathcal{B}^n\big(h^1, \tau\big), 0\big)\big] > -\infty,$$

which implies that $g_n(\mu,\sigma,\tau)$ is well defined in $\mathbb{R} \cup \{\infty\}$. Let $\mathcal{S}_n^{\text{adm}}$ be the set of admissible strategies for P1.

A strategy $\sigma \in \mathcal{S}_n^{\text{adm}}$ guarantees α to P1 if, $\forall \tau \in \mathcal{T}_n^{\text{adm}}$: $g_n(\mu,\sigma,\tau) \geqslant \alpha$.

The maximum amount P1 can guarantee in $\Gamma_n(\mu)$ is

$$\underline{V}_n(\mu) := \sup_{\sigma \in \mathcal{S}_n^{\mathrm{adm}}} \inf_{\tau \in \mathcal{T}_n^{\mathrm{adm}}} g_n(\mu, \sigma, \tau).$$

A strategy σ is optimal if it guarantees $\underline{V}_n(\mu)$.

It is always true that $\underline{V}_n(\mu) \leqslant \overline{V}_n(\mu)$. When equality holds, the game $\Gamma_n(\mu)$ is said to have a value $V_n(\mu) := \underline{V}_n(\mu) = \overline{V}_n(\mu)$.

If the game has a value, and if σ^* and τ^* are optimal strategies, then (σ^*, τ^*) is a Nash equilibrium of the game. Conversely, if (σ^*, τ^*) is a Nash equilibrium of the game, then the game has a value and σ^* and τ^* are optimal strategies.

In the "market game" example of trading mechanism described in the previous section, where both players submit a demand function, a natural notion of price at period q would be the market clearing price. However, with general trading mechanisms, actions by the players are not necessarily price related and we have to define what we mean by the price L_q of the risky asset R at period q. One possible definition of L_q could be based on the quotient $-\frac{B_{i_q,j_q}}{A_{i_q,j_q}}$, that is the counterpart in numéraire paid per R-shares acquired. But this definition is not completely satisfying: on one hand, it would lead to technical problems in case $A_{i_q,j_q} = 0$. On the other hand, both B_{i_q,j_q} and A_{i_q,j_q} are random quantities. At stage 1 for instance, the counterpart to the lottery yielding A_{i_1,j_1} units of R is a lottery yielding $-B_{i_1,j_1}$ of N. So, should the price prevailing at the first exchange be defined as the quotient $-\frac{E[B_{i_1,j_1}]}{E[A_{i_1,j_1}]}$ or as $E[-\frac{B_{i_1,j_1}}{A_{i_1,j_1}}]$? These two notions do not coincide in general.

To avoid these difficulties, we prefer in this paper to define the price L_q as the price at which the risk-neutral player P2 would agree to trade with another uninformed player: $L_q = E[L|i_1, \ldots, i_q, j_1, \ldots, j_q]$. This definition implies in particular that the price process is a martingale. It will also be convenient in this paper to represent this discrete time price process $(L_q)_{q=0,\ldots,n}$ by a continuous time process $(\Pi_t^n)_{t\in[0,1]}$, where the time t represents the proportion of already elapsed rounds: $\Pi_t^n := L_{[\![nt]\!]}$, where $[\![x]\!]$ denotes the largest integer below x.

5. Natural exchange mechanism

The notion of a trading mechanism involved in the definition of $\Gamma_n(\mu)$ is completely general and some mechanisms could be unrealistic to represent actual exchanges on the stock market, for instance a dictatorial mechanism where P1 selects the transfer vectors, regardless to P2's action.

Rather than focusing on an explicit trading mechanism representing one particular market, we isolate in this section five axioms (H1) to (H5) that should be met by any natural modelization of an exchange. These are expressed in terms of the value of the one shot game. We also give some additional conditions (H1′) to (H4′) that imply the corresponding (H) axiom and that are easier to check.

We are concerned in this paper with qualitative properties of the price process at equilibrium in $\Gamma_n(\mu)$ when such equilibria exist. Even if equilibria could fail to exist in $\Gamma_n(\nu)$, for $\nu \neq \mu$, we however have to assume that the value exists. This is the content of the two versions of axiom (H1) for $k = 2$ or ∞:

(H1-Δ^k) Existence of the value: *The game $\Gamma_n(\mu)$ has a value $V_n(\mu)$ for all distribution $\mu \in \Delta^k$ and all $n \in \mathbb{N}$.*

As we will see, axioms (H3) and (H4) hereafter will imply in particular that action spaces I and J contain infinitely many actions. Even when I and J are subsets of $\mathbb{R}^m$, the transfer function T will typically have some discontinuities in order to satisfy (H5). So proving that a particular mechanism satisfies (H1) is in general fairly difficult. The problem is essentially to prove that:

(H1′): *$\forall \mu \in \Delta^\infty$, the game $\Gamma_1(\mu)$ has a value.*

Indeed, we prove in Appendix A of this paper that (H1′) joint to our following continuity assumption (H2) implies (H1-Δ^∞). We also prove that (H1′) and (H2′) imply (H1-Δ^2).

The second axiom is a continuity assumption of $V_1(\mu)$ as a function of μ: if two assets R and R' have nearly the same the liquidation values L and L', i.e. there is a join distribution of (L, L') where, for some $p \in [1, 2[$, $\|L - L'\|_p$ is small, then it is natural to expect that exchanging R or R' will lead to similar profits for P1: $V_1([L])$ and $V_1([L'])$ should be close to each other. Axiom (H2) is in fact a Lipschitz continuity assumption:

(H2) Continuity: $\exists p \in [1, 2[, \exists A \in \mathbb{R}$: $\forall L, L' \in L^2$:

$$\left|V_1\big([L]\big) - V_1\big([L']\big)\right| \leqslant A \cdot \left\|L - L'\right\|_{L^p}.$$

The Wasserstein distance $W_p(\mu, \mu')$ of order p between μ and $\mu' \in \Delta^2$ is defined as $W_p(\mu, \mu') := \inf\{\|X - X'\|_{L^p}$: $X \sim \mu, X' \sim \mu'\}$, and therefore (H2) is simply a Lipschitz continuity assumption in terms of $W_p(\mu, \mu')$: $\exists p \in [1, 2[$, $\exists A \in \mathbb{R}$: $\forall \mu, \mu' \in \Delta^2$: $|V_1(\mu) - V_1(\mu')| \leqslant A \cdot W_p(\mu, \mu')$.

Notice that a mechanism T satisfying (H2′) will clearly satisfy (H2) with $p = 1$:

(H2′) Bounded exchanges: $\forall i, j$: $|A_{i,j}| \leqslant A$.

Indeed, if nature initially selects jointly the liquidation values (L, L') of two asset R and R' and informs P1 of its joint choice, the additional information contained in L' is useless to play in $\Gamma_1([L])$, and similarly, the additional information contained in L is useless to play in $\Gamma_1([L'])$. Viewing thus a strategy of P1 as a map from (L, L') to $\Delta(I)$, we may consider that strategy spaces are identical in $\Gamma_1([L])$ and $\Gamma_1([L'])$. If both players use the same strategies (σ, τ) in these games, the payoffs are respectively $E[A_{i,j}L + B_{i,j}]$ and $E[A_{i,j}L' + B_{i,j}]$. The difference of P1's payoffs in both games $E[A_{i,j}(L - L')]$ is then bounded by $A\|L - L'\|_1$ and, as announced, (H2) is thus satisfied.

For the next two axioms, we consider a market as a system with which agents agree on trades of some asset in counterpart for some numéraire. The same system and the trading mechanism representing it could be used to trade different assets and numéraires.

In axiom (H3), we consider the effect of a change of numéraire: we compare two situations where the same risky asset is exchanged for numéraire N_1 and N_2 respectively, with one unit of N_1 being worth $\alpha > 0$ units of N_2. If L is the liquidation value of R in terms of N_1, the liquidation value of R in terms of N_2 will then be αL and we are thus comparing $\Gamma_1([L])$ with $\Gamma_1([\alpha L])$. Quite intuitively, we expect to observe same value trades in both situations and the value of both game will just differ by the scaling factor α. This is our hypothesis (H3):

(H3): Invariance with respect with the numéraire scale:

$$\forall \alpha > 0\colon\ \forall L \in L^2\colon\quad V_1\big([\alpha L]\big) = \alpha V_1\big([L]\big).$$

We could see this invariance property of the value V_1 as a consequence of an invariance property of the trading mechanism itself: when comparing $\Gamma_1([L])$ with $\Gamma_1([\alpha L])$, we expect that the number of exchanged R-shares will be the same in both games, but the number of N_2-shares given in counterpart in $\Gamma_1([\alpha L])$ will just be the product of α and the N_1-counterpart in $\Gamma_1([L])$.

Clearly, the players will not use the same actions when trading with N_1 or N_2, but to each action in the N_1-trading game corresponds an action in the N_2-game with similar effect. These correspondences between actions are represented by the translation rules in the following axiom:

(H3′): *For all $\alpha > 0$, there are one-to-one translation rules $\psi_1 : I \to I$ and $\psi_2 : J \to J$ with the property that $\forall i, j$: $A_{\psi_1(i),\psi_2(j)} = A_{i,j}$ and $B_{\psi_1(i),\psi_2(j)} = \alpha \cdot B_{i,j}$.*

If P1 plays $\psi_1(i)$ in $\Gamma_1([\alpha L])$ whenever he would play i in $\Gamma_1([L])$, then his payoff in $\Gamma_1([\alpha L])$ against an action $j = \psi_2(j')$ of P2 will be:

$$E[A_{\psi_1(i),\psi_2(j')}\alpha L + B_{\psi_1(i),\psi_2(j')}] = \alpha E[A_{i,j'}L + B_{i,j'}].$$

Therefore P1 can guarantee the same amount, up to a factor α in $\Gamma_1([L])$ and $\Gamma_1([\alpha L])$. A similar argument holds for P2, and therefore our hypothesis (H3′) implies (H3).

With the next axiom, we analyze the effect of shifting the liquidation value by a constant quantity β: we compare the game $\Gamma_1([L])$, where a risky asset R with liquidation value L is traded for numéraire N, with the game $\Gamma_1([L + \beta])$ where a risky asset R' is traded for N, the asset R' consisting of one unit of R and β units of N. We expect to observe similar trades in both games: the same number x of R shares will be exchanged in both cases, but the $x.\beta$ units of N included in the x units of R' in the second game are immediately paid back in N. In both games, the players will not use the same actions, however, their actions in the first game can be translated into actions in the second one. This leads to the following condition, in terms of the trading mechanism:

(H4′): *For all $\beta \in \mathbb{R}$, there exist one-to-one translation rules $\phi_1 : I \to I$ and $\phi_2 : J \to J$ that map Pi's actions in $\Gamma_1([L])$ to Pi's actions in $\Gamma_1([L + \beta])$ with the property that $\forall i, j$: $A_{\phi_1(i),\phi_2(j)} = A_{i,j}$ and $B_{\phi_1(i),\phi_2(j)} = B_{i,j} - \beta \cdot A_{i,j}$.*

As we will see, (H3′) and (H4′) are quite natural when "price related" mechanisms are concerned, that is mechanisms where actions are prices or demand curves as the two first examples presented in the next section. For these mechanisms, the translation rules ψ_i and ϕ_i appearing in (H3′) and (H4′) are respectively a rescaling by a factor α or a shift by a constant β of the corresponding price or demand curve.

Since the trades in $\Gamma_1([L])$ and $\Gamma_1([L+\beta])$ will have the same value whenever the players use their translated strategies, it follows from (H4′) that:

(H4): Invariance with respect to the risk-less part of the risky asset:

$$\forall \beta \in \mathbb{R}\colon\quad V_1\big([L + \beta]\big) = V_1\big([L]\big).$$

Using successively (H4), (H3) and (H2) we infer that

$$\forall \beta \in \mathbb{R}\colon\quad V_1\big([\beta]\big) = V_1\big([0 + \beta]\big) = V_1\big([0]\big) = \lim_{\alpha \to 0} V_1\big([\alpha]\big) = \lim_{\alpha \to 0} \alpha V_1(1) = 0.$$

In other words, no benefit can be made of exchanging a riskless asset for a numéraire. Notice that this last property would be automatically satisfied by a symmetrical trading mechanism ($I = J$ and $T(i, j) = -T(j, i)$). Indeed, in this case, the only asymmetry in $\Gamma_1(\mu)$ is the initial message sent to P1. When the liquidation value of R is a constant β, the game is thus a completely symmetric zero sum game (with anti-symmetric payoff matrix) and its value $V_1([\beta])$ must then be 0.

In the game $\Gamma_1([L])$, P1 could decide to ignore his private information L. Since the players are risk-neutral, the risky asset R could then be replaced by a risk-less asset with constant liquidation value $\beta := E[L]$: without his information, P1 could then guarantee $V_1([\beta]) = 0$. So, using his information in $\Gamma_1([L])$, P1 can guarantee a positive amount: $V_1([L]) \geqslant 0$. Our next axiom is that there exists a situation in which P1 can make a strict profit out of his information:

(H5): Positive value of information: $\exists L \in L^2\colon\ V_1([L]) > 0$.

As already explained in the introduction, this last axiom is paradoxical: on one hand, the fact that private information has a strictly positive value on the market is so obvious that a model without (H5) would be completely unrealistic. Rating firms can sell information at a positive price because it has a positive value. Insider trading regulation would not exist if private information had no value. On the other hand, (H5) is only possible if the uninformed player is forced to play the game: he would be better off if he could avoid trading. As previously mentioned, we argue that some agents, for instance market makers, are in fact forced to trade. Notice also that most market models with asymmetric information avoid this no trade paradox by introducing noise traders, that is traders that are forced to trade for exogenous liquidity reasons.

A trading mechanism will be referred to as k**-natural** if it satisfies the previous five axioms (H1-Δ^k), (H2), ..., (H5). We are now ready to state the main result of the paper:

Theorem 1. *For $k \in \{2, \infty\}$ and μ in Δ^k, consider a sequence $\{(\sigma_n, \tau_n)\}_{n\in\mathbb{N}}$ of equilibria in the repeated exchange games $\Gamma_n(\mu)$ indexed by the length n of the game. Let L^n be the price process at equilibrium in $\Gamma_n(\mu)$:*

$$L^n_q := E_{\pi(\mu,\sigma_n,\tau_n)}[L|i_1,\ldots,i_q,j_1,\ldots,j_q].$$

Then, if the trading mechanism is k-natural, the continuous time representation Π^n of L^n defined as $\Pi^n_t := L^n_{[\![nt]\!]}$ converges in finite-dimensional distribution to the continuous martingale of maximal variation Π^μ.

The asymptotics of the price process is thus completely independent of the natural trading mechanism. Before proving this theorem in Section 7, we provide in the next section some examples of natural trading mechanism.

6. Examples of natural trading mechanism

6.1. Market maker game

As a first example, let us consider the game between two asymmetrically informed market makers. At each period, they post simultaneously a price i and $j \in \mathbb{R}$ which are a commitment to sell or buy one unit of the risky asset at this price. If $i \neq j$, an arbitrageur will see a possibility of arbitrage: he will buy one share of R at the lowest price and sell it immediately at the highest one. This game, referred to hereafter as the market maker game with arbitrageur is not an exchange between two individuals and does not belong to the class of zero-sum games analyzed in this paper. It can however be approximated by the following trading mechanism: when $i > j$, one share of R goes from P2 to P1 as in the market maker with arbitrageur game, but we now assume that both players are exchanging R at a common price $f(i,j)$ in numéraire, where f is a function satisfying $\min(i,j) \leqslant f(i,j) \leqslant \max(i,j)$, and $\forall\alpha > 0$, $\forall\beta$: $f(\alpha i+\beta, \alpha j+\beta) = \alpha f(i,j)+\beta$. Symmetric transfers happen when $i < j$, and there is no transfer if $i = j$. So, formally, the trading mechanism is represented by the transfer function

$$T(i,j) := \begin{cases} (1, -f(i,j)) & \text{if } i > j, \\ (0,0) & \text{if } i = j, \\ (-1, f(i,j)) & \text{if } i < j. \end{cases}$$

In De Meyer and Moussa-Saley (2003), the particular case $f(i,j) = \max(i,j)$ was considered. Similar results would hold for other function f as $f(i,j) = \lambda\max(i,j) + (1-\lambda)\min(i,j)$, with $\lambda \in [0,1]$. In that paper, the game $\Gamma_n(\mu)$ was analyzed for distribution μ concentrated on the two points 0, 1. The analysis was extended to general distributions $\mu \in \Delta^2$ in De Meyer and Moussa-Saley (2002). We prove in these papers that $\Gamma_n(\mu)$ have a value and both players have optimal strategies. We further have a closed form formula for the value: $V_n(\mu) = \max E[LS_n]$, where the maximum is taken over the joint distributions of (L, S_n) satisfying $L \sim \mu$ and $S_n = \sum_{q=1}^n U_q$ is a sum of n independent random variables U_q that are uniformly distributed on $[-1,1]$. This is a Monge–Kantorovich mass transportation problem and its optimal solution (L, S_n) is characterized by the fact that L can be expressed as the unique increasing function $g^n_\mu(S_n)$ satisfying $g^n_\mu(S_n) \sim \mu$. It follows from this formula that (H1-Δ^2)–(H5) are satisfied by the mechanism. Based on these explicit formulae, we can also prove that the price process at equilibrium is given by $L_q = E[g^n_\mu(S_n)|U_1,\ldots,U_q]$. Using the central limit theorem, the asymptotics of this price process can then be derived, and we prove in De Meyer and Moussa-Saley (2003, 2002) that the above Theorem 1 holds for this precise trading mechanism. The proof of Theorem 1 presented in the present paper is however much more general as it applies to arbitrary natural trading mechanism where no closed form formulae are known for $V_n(\mu)$.

The particular mechanism dealt with in these papers is “price related” in the sense that the players' actions are the proposed prices for the exchange. In this setting, a natural notion of the price process for R could be the sequence of P1's posted prices. We prove in De Meyer and Moussa-Saley (2003) that the price i_q posted at period q will be very close to L_q, and the posted price process will have the same asymptotic as $(L_q)_{q=1,\dots,n}$.

De Meyer and Marino (2005a, 2005b), Domansky (2007) analyze the same trading mechanism as in De Meyer and Moussa-Saley (2003), but market makers have to post prices within a discrete grid. In this case, however, the price process fails to converge to a CMMV. Theorem 1 does not apply in this setting since neither (H3) nor (H4) are satisfied by these discretized mechanism: the grid should be both shift- and scale-invariant, which is impossible.

6.2. Market game

The second class of examples of natural mechanisms we will present are the following market games: Players' actions i and j are demand functions for asset R in terms of numéraire N. In this case, I and J are sets of strictly decreasing functions i from $\mathbb{R}$ to $\mathbb{R}$ satisfying $\lim_{p\to-\infty} i(p) > 0$ and $\lim_{p\to\infty} i(p) < 0$. Once i and j are selected, the market clearing price $p^*(i,j)$ is computed, whenever it exists: it is the value of p solving the equation $i(p)+j(p)=0$. The transfer vector, representing the quantities of the risky asset and numéraire that player 1 receives during the exchange is then $T(i,j) := (i(p^*(i,j)), -p^*(i,j)i(p^*(i,j)))$, and $T(i,j)=(0,0)$, if $p^*(i,j)$ fails to exist.

The set I and J must be sufficiently rich to model the market: a natural condition on action spaces is to assume that I and J are closed by dilatation and shift: for all $\alpha > 0$ and all $\beta \in \mathbb{R}$, if i belongs to I then so does the function $i_{\alpha,\beta}: p \to i_{\alpha,\beta}(p) := i(\alpha p + \beta)$. In the same way, $j \in J$ will imply $j_{\alpha,\beta} \in J$, for all $a > 0$ and $\beta \in \mathbb{R}$.

Axioms (H3) and (H4) will then be satisfied by this mechanism: we just have to prove that (H3′) and (H4′) hold. For $\alpha > 0$, define the translation maps $\psi_1(i) := i_{\alpha^{-1},0}$ and $\psi_2(j) := j_{\alpha^{-1},0}$. Since $p^*(i_{\alpha^{-1},0}, j_{\alpha^{-1},0}) = \alpha p^*(i,j)$, we get indeed $A_{\psi_1(i),\psi_2(j)} = i_{\alpha^{-1},0}(p^*(i_{\alpha^{-1},0}, j_{\alpha^{-1},0})) = i(p^*(i,j)) = A_{i,j}$ and $B_{\psi_1(i),\psi_2(j)} = -p^*(i_{\alpha^{-1},0}, j_{\alpha^{-1},0}) i_{\alpha^{-1},0}(p^*(i_{\alpha^{-1},0}, j_{\alpha^{-1},0})) = \alpha B_{i,j}$. (H3′) is thus proved.

To prove (H4′), define, for $\beta \in \mathbb{R}$, the translation maps $\phi_1(i) := i_{1,-\beta}$ and $\phi_2(j) := j_{1,-\beta}$. Then $p^*(i_{1,-\beta}, j_{1,-\beta}) = p^*(i,j) + \beta$ and (H4′) follows:

$$A_{\phi_1(i),\phi_2(j)} = i_{1,-\beta}\big(p^*(i_{1,-\beta}, j_{1,-\beta})\big) = i\big(p^*(i,j)\big) = A_{i,j}$$

and

$$\begin{aligned} B_{\phi_1(i),\phi_2(j)} &= -i_{1,-\beta}\big(p^*(i_{1,-\beta}, j_{1,-\beta})\big)p^*(i_{1,-\beta}, j_{1,-\beta}) \\ &= -i\big(p^*(i,j)\big)\big(p^*(i,j)+\beta\big) \\ &= B_{i,j} - A_{i,j}\beta. \end{aligned}$$

We next discuss axiom (H5) in the market game model. This axiom precludes P2 from no trading and the zero demand function could therefore not belong to J nor be approximated by an element of J in order to (H5) to hold. In fact, this condition will imply that any function $j \in J$ is discontinuous at the point $\gamma := \sup\{p\colon\ j(p) > 0\}$. At this threshold price γ, P2 passes from a strictly buying position ($\lim_{p \overset{<}{\to} \gamma} j(p) > 0$) to a strictly selling one ($\lim_{p \overset{>}{\to} \gamma} j(p) < 0$). Would indeed j be continuous at γ, the function $j_{\alpha,-\alpha\gamma}$, which belongs to J by our previous hypothesis, would tend to the zero demand function as $\alpha \to 0$, and P2 could guarantee the zero trade.

To illustrate the above discussion, let us consider the following simple example: Let I be the set of functions $i^{\alpha,q}$ for $\alpha \geqslant 0, q \in \mathbb{R}$, where $i^{\alpha,q}(p) := \frac{q-p}{\alpha}$ if $\alpha > 0$ and $i^{0,q}$ is the inelastic demand function at price q. An action i of P1 can thus be identified with the corresponding pair (q,α). Let J be the set of functions j^r for some $r \in \mathbb{R}$, where $j^r(p) := 1$ if $p < r$, $j^r(p) := -1$ if $p > r$ and $j^r(p) := 0$ if $p = r$. The market clearing price in this setup will be $p^*(\alpha,q,r) = q - \alpha$ if $q - \alpha > r$, $p^*(\alpha,q,r) = q + \alpha$ if $q + \alpha < r$, and there will be no clearing price if $q - \alpha \leqslant r \leqslant q + \alpha$. It is convenient to represent a pair (α,q) by a pair (x,y), with $x := q - \alpha \leqslant y =: q + \alpha$. With these notations, the trading mechanism is thus

$$T(x,y,r) = \begin{cases} (1,x) & \text{if } r < x, \\ (-1,y) & \text{if } r > y, \\ (0,0) & \text{if } x \leqslant r \leqslant y. \end{cases}$$

This game could then also be viewed as an exchange between an informed market maker (P1) posting a bid x and an ask y and an investor who is forced to trade: he can just decide the price r at which he switches from a buying position to a selling one.

We now prove that this mechanism satisfies (H1′). Let $\mu \in \Delta^\infty$ and let K be a compact interval of positive length such that $\mu(K) = 1$. A pure strategy for P1 in $\Gamma_1(\mu)$ is a pair of functions (X,Y) that maps the liquidation value L to the action $X(L), Y(L)$. If P2 plays a pure strategy r, the payoff is $E_\mu[g(L, X(L), Y(L), r)]$, where, with $\chi(x) := \mathbb{1}_{\{x<0\}}$:

$$g(L,x,y,r) := (L-x)\chi(r-x) + (y-L)\chi(y-r). \tag{3}$$

Notice that a strategy (X,Y) is always dominated by (X',Y') where $X'(L) := \min(L, X(L))$ and $Y'(L) := \max(L, Y(L))$. We may therefore restrict P1's pure strategy space to the set of pairs (X,Y) satisfying $\forall L\colon\ X(L) \leqslant L \leqslant Y(L)$. Next observe that if one player plays in K with probability 1, the other player has K-valued ϵ-best reply: $\Gamma_1(\mu)$ will have the same value as the restricted game where both players have to play in K. This last game has indeed a value $V_1(\mu)$ in mixed strategies and

P2 has an optimal strategy τ^*, as it results from Proposition 1.17 in Mertens et al. (1994): K is compact, and due to our assumption $\forall L$: $X(L) \leqslant L \leqslant Y(L)$, the coefficients of χ in (3) are positive and $g(X, Y, r)$ is thus lower semicontinuous in r.

With this mechanism, there is at most one share of R exchanged, and thus (H2′) holds. With Theorem 24, we thus conclude that (H1-Δ^2) holds and axiom (H2) is implied by (H2′). Since I and J are closed for shift and dilatation, (H3) and (H4) will hold. Finally, to prove that (H5) holds, consider the measure μ that puts a weight 1/2 on both 0 and 1. The following strategy (X, Y) guarantees 1/8 to P1 in $\Gamma(\mu)$

$$X(L) := \begin{cases} 3/4 & \text{if } L = 1, \\ 0 & \text{if } L = 0 \end{cases} \quad \text{and} \quad Y(L) := \begin{cases} 1/4 & \text{if } L = 0, \\ 1 & \text{if } L = 1, \end{cases}$$

and $V_1(\mu) \geqslant 1/8 > 0$. The mechanism is thus 2-natural.

6.3. Canonical games

We conclude this section with a technical remark to show the class of games we are analyzing in this paper is far from empty: We provide a general way to generate natural trading mechanisms. If a function $V : \Delta^2 \to \mathbb{R}$ is the value of a game with one sided information where the state of nature space is $\mathbb{R}$, it is well known that V must be both concave and Blackwell increasing. This last property means that if (Y_1, Y_2) is a random vector such that $E[Y_2|Y_1] = Y_1$, then $V([Y_2]) \geqslant V([Y_1])$. We argue here that any function V with these properties and satisfying further (H2) to (H5) can be implemented as the value of a game with a natural trading mechanism. $V(\mu) := (E_\mu[|L - E_\mu[L]|^p])^{1/p}$, $1 \leqslant p < 2$, is an example of such a function. A concave function V is the infimum of the linear functionals by which it is dominated. In this setting, the continuous linear functionals of measures are maps of the form $\mu \to E_\mu[\phi(L)]$, where ϕ is a continuous function. Therefore if Φ is the set of continuous functions ϕ such that $\forall \mu$: $E_\mu[\phi(L)] \geqslant V(\mu)$, we will have $\forall \mu$: $V(\mu) = \inf_{\phi \in \Phi} E_\mu[\phi(L)]$. The Blackwell monotonicity indicates that only convex functions ϕ in Φ are to be considered: in other words, if Φ^{vex} is the set of convex ϕ in Φ then $\forall \mu$: $V(\mu) = \inf_{\phi \in \Phi^{\text{vex}}} E_\mu[\phi(L)]$. A proof of this property can be found in De Meyer et al. (2009).

By a density argument, the set $\Phi^{1,\text{vex}}$ of continuously differentiable functions ϕ in Φ^{vex} will have the same property:

$$\forall \mu: \quad V(\mu) = \inf_{\phi \in \Phi^{1,\text{vex}}} E_\mu\big[\phi(L)\big]. \tag{4}$$

Consider then the following trading mechanism: $I = \mathbb{R}$, $J = \Phi^{1,\text{vex}}$ and $\forall i \in I, \phi \in \Phi^{1,\text{vex}}$, $T(i, \phi)$ is defined as: $T(i, \phi) := (\phi'(i), \phi(i) - i\phi'(i))$.

With this trading mechanism, both players can guarantee $V(\mu)$ in $\Gamma_1(\mu)$. Indeed if P2 plays ϕ in $\Gamma_1(\mu)$, for all pure strategy $i(L)$, the payoff will be $E_\mu[\phi'(i(L))(L - i(L)) + \phi(i(L))]$ which is less than $E_\mu[\phi(L)]$, as a consequence of the convexity of ϕ. Minimizing this in ϕ, P2 can the guarantee $V(\mu)$ in $\Gamma_1(\mu)$, as it follows from (4).

P1 can guarantee the same amount with the pure strategy $i^*(L) := L$. Indeed, if P2 plays ϕ the payoff is then $E_\mu[\phi(L)] \geqslant V(\mu)$.

Therefore $V(\mu)$ is the value of $\Gamma_1(\mu)$ and thus $V_1 = V$ will satisfy (H1′) and thus (H1-Δ^∞), as it follows from Theorem 24. According to our assumption on V the mechanism will also satisfy (H2) to (H5): It thus is an ∞-natural mechanism.

In the above game, the strategy space J could be reduced to any subset F of $\phi^{1,\text{vex}}$ such that $\forall \mu$: $V(\mu) = \inf_{\phi \in F} E_\mu[\phi(L)]$. As an application of this remark, let f be a strictly positive continuously differentiable convex function such that $f(x)/(1 + |x|^p)$ is bounded on $\mathbb{R}$ for some $p \in [1, 2[$. If F denotes then the set of ϕ of the form as $\phi(L) = \alpha f(\frac{L-\beta}{\alpha})$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the function $V(\mu) := \inf_{\alpha > 0, \beta} E_\mu[\alpha f(\frac{L-\beta}{\alpha})]$ is the value of the game $\Gamma_1(\mu)$ with the following trading mechanism: In this case a function $\phi \in E$ can be identified with the corresponding pair (α, β), and thus $J = (\mathbb{R}^+ \times \mathbb{R})$, $I = \mathbb{R}$ and $T(i, \alpha, \beta) = (f'(\frac{i-\beta}{\alpha}), \alpha f(\frac{i-\beta}{\alpha}) - i f'(\frac{i-\beta}{\alpha}))$. The resulting mechanism will be ∞-natural.

7. P1's martingale optimization problem

In this section we show that the choice of a strategy for player 1 turns out to be a choice of the optimal rate of revelation of his private information. If, at period q, P1 uses a strategy that depends on his information L, P2 will make a Bayesian reevaluation of his beliefs about L. Therefore, the price L_q, which is the expectation of L with respect to P2's believe after stage q, will be different of L_{q-1}. The amount of information revealed can thus be represented by the price process. The optimal rate of revelation for P1 will be that for which the price process L is optimal in the maximization problem (6) here below.

If Y is a random variable on a probability space $(\Omega, \mathcal{A}, P)$ and if $\mathcal{H} \subset \mathcal{A}$ is a σ-algebra, the probability distribution of Y will be denoted $[Y]$, and the conditional distribution of Y given $\mathcal{H}$ will be $[Y|\mathcal{H}]$. We will also write $\Gamma_n[Y]$ and $V_n[Y]$

instead of $\Gamma_n([Y])$ and $V_n([Y])$. In particular $V_1[Y|\mathcal{H}]$ is an $\mathcal{H}$-measurable random variable.[2] Let $\mathcal{W}_n(\mu)$ be the set of pairs $(\mathcal{F}, X)$ where $\mathcal{F} := (\mathcal{F}_q)_{q=0,\dots,n+1}$ is a filtration on a probability space, and $X = (X_q)_{q=0,\dots,n+1}$ is an $\mathcal{F}$-martingale whose $n+1$-th value X_{n+1} is μ-distributed. For $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, we define $\mathcal{V}_n(\mathcal{F}, X)$ as:

$$\mathcal{V}_n(\mathcal{F}, X) := E\left[\sum_{q=0}^{n-1} V_1[X_{q+1}|\mathcal{F}_q]\right]. \tag{5}$$

Let us also define $\overline{\mathcal{V}}_n(\mu)$ as

$$\overline{\mathcal{V}}_n(\mu) := \sup\bigl\{\mathcal{V}_n(\mathcal{F}, X)\colon (\mathcal{F}, X) \in \mathcal{W}_n(\mu)\bigr\}. \tag{6}$$

Lemma 2. *For all $\mu \in \Delta^2$: $V_n(\mu) \geqslant \overline{\mathcal{V}}_n(\mu)$.*

Proof. Given $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$ on a probability space $(\Omega, \mathcal{A}, P)$, we have to prove that P1 can guarantee $\mathcal{V}_n(\mathcal{F}, X)$ in $\Gamma_n(\mu)$. At the initial stage of $\Gamma_n(\mu)$, nature selects a μ distributed random variable L and informs P1 of its choice. We can clearly assume that nature uses the probability space $(\Omega, \mathcal{A}, P)$ as lottery and sets $L = X_{n+1}$, since $X_{n+1} \sim \mu$. We can also assume that P1 observes the whole space $(\Omega, \mathcal{A}, P)$. He can therefore adopt the following strategy in $\Gamma_n[X_{n+1}]$: at stage q he plays an ϵ-optimal strategy σ_q^* in $\Gamma_1[X_{q+1}|\mathcal{F}_q]$. The probability distribution σ_q^* used by P1 to select i_q is thus measurable with respect to $\sigma(\mathcal{F}_q, X_{q+1}) \subset \mathcal{F}_{q+1}$ and i_q is chosen independently of X_{n+1} conditionally to $\sigma(\mathcal{F}_q, X_{q+1})$. Therefore $E[X_{n+1}|\sigma(\mathcal{F}_q, X_{q+1}, i_q)] = E[X_{n+1}|\sigma(\mathcal{F}_q, X_{q+1})] = X_{q+1}$, since X is an $\mathcal{F}$-martingale. The payoff at stage q is thus:

$$E[A_{i_q,\tau_q} X_{n+1} + B_{i_q,\tau_q}] = E[A_{i_q,\tau_q} X_{q+1} + B_{i_q,\tau_q}] = E\bigl[E[A_{i_q,\tau_q} X_{q+1} + B_{i_q,\tau_q}|\mathcal{F}_q]\bigr].$$

Since the move of P1 is ϵ-optimal in $\Gamma_1[X_{q+1}|\mathcal{F}_q]$, he gets at least $V_1[X_{q+1}|\mathcal{F}_q] - \epsilon$ conditionally to $\mathcal{F}_q$. The result follows then easily since $\epsilon > 0$ is arbitrary. □

Lemma 3. *For all $\mu \in \Delta^2$: $V_n(\mu) \leqslant \overline{\mathcal{V}}_n(\mu)$.*

Proof. We will prove that P1 will not be able to guarantee a higher payoff than $\overline{\mathcal{V}}_n(\mu)$. Indeed, let σ be a strategy of P1 in $\Gamma_n(\mu)$. To reply to σ, P2 may adopt the following strategy: since he knows σ_1, he may compute the distribution of $L_1 := E[L|i_1]$. He plays then an ϵ-optimal strategy τ_1 in $\Gamma_1[L_1]$. At period q, he computes $[L_q|\mathcal{H}_{q-1}]$, with $\mathcal{H}_q := \sigma(i_1, j_1 \dots i_q, j_q)$, where $L_q := E[L|i_q, \mathcal{H}_{q-1}]$, and plays an ϵ-optimal strategy τ_q in $\Gamma_1[L_q|\mathcal{H}_{q-1}]$. Clearly, we also have $L_q = E[L|\mathcal{H}_q]$, since conditionally to $\mathcal{H}_{q-1}$, the move j_q is independent of L. Therefore, with $\mathcal{H}_{n+1} := \sigma(L, \mathcal{H}_n)$, $L_{n+1} := L$, $\mathcal{H}_0 := \{\emptyset, (I \times J)^n \times \mathbb{R}\}$, $L_0 := E[L]$, $\bar{L} := (L_q)_{q=0,\dots,n+1}$ and $\mathcal{H} := (\mathcal{H}_q)_{q=0,\dots,n+1}$, the pair $(\mathcal{H}, \bar{L})$ belongs to $\mathcal{W}_n(\mu)$.

With that reply P1's conditional payoff at period q, given $\mathcal{H}_{q-1}$ is the at most $V_1[L_q|\mathcal{H}_{q-1}] + \epsilon$ and the overall payoff in $\Gamma_n(\mu)$ is then less than $\mathcal{V}_n(\mathcal{H}, \bar{L}) + n\epsilon \leqslant \overline{\mathcal{V}}_n(\mu) + n\epsilon$. □

Theorem 4. *If the mechanism satisfies* (H1-Δ^k) ($k \in \{2, \infty\}$)*, then for all $\mu \in \Delta^k$: $V_n(\mu) = \overline{\mathcal{V}}_n(\mu)$. Furthermore, if σ, τ are optimal strategies in $\Gamma_n(\mu)$, if $L_q := E_{\pi(\mu,\sigma,\tau)}[L|\mathcal{H}_q]$, where $\mathcal{H}_q := \sigma(i_1, j_1 \dots i_q, j_q)$, and $\bar{L} := (L_q)_{q=0,\dots,n+1}$, then $(\mathcal{H}, \bar{L})$ solves the maximization problem* (6)*.*

Proof. The first claim follows the two previous lemmas and the fact that the value exists as stated in (H1-Δ^k).

Let us next assume that the players are playing a pair (σ, τ) of optimal strategies in $\Gamma_n(\mu)$. Let X_q denote the expectation, conditional to $\mathcal{H}_q$, of the sum of the $n-q$ stage payoffs following stage q.

Let us then first observe that, for all q, X_q must be at least $V_{n-q}[L|\mathcal{H}_q]$. Otherwise P1 could deviate from stage $q+1$ on to an ϵ-optimal strategy in $\Gamma_{n-q}[L|\mathcal{H}_q]$, obtaining thus a higher payoff against τ than with σ, which is impossible since (σ, τ) is an equilibrium of the game.

At period q, P2 may compute $v_{q-1} := V_1[L_q|\mathcal{H}_{q-1}]$ and $u_{q-1} := E[A_{i_q,j_q} L_q + B_{i_q,j_q}|\mathcal{H}_{q-1}]$. On the event $\{v_{q-1} < u_{q-1}\}$, P2 could then deviate at stage q with an ϵ-optimal strategy in $\Gamma_1[L_q|\mathcal{H}_{q-1}]$, bringing the expected payoff of that stage to less than $v_{q-1} + \epsilon$. Provided ϵ is small enough, this is strictly less than the payoff u_{q-1} P1 would obtain with τ. This deviation will not change the behavior of P1 at period q, and the distribution $[L|\mathcal{H}_q]$ remains thus unchanged.

[2] The set Δ of probability measures on $\mathbb{R}$ may be endowed with the weak*-topology: the weakest topology such that $\phi_g : \mu \to E_\mu[g]$ is continuous, for all continuous bounded function $g : \mathbb{R} \to \mathbb{R}$. If Y is a random variable on a probability space $(\Omega, \mathcal{A}, P)$ and if $\mathcal{H} \subset \mathcal{A}$ is a σ-algebra, the conditional distribution $[Y|\mathcal{H}]$ can then be seen as a measurable map from $(\Omega, \mathcal{H})$ to $(\Delta, \mathcal{B}_\Delta)$ where $\mathcal{B}_\Delta$ denotes the Borel σ-algebra on Δ corresponding to the weak*-topology. Let Δ_r^2 be the set of $\mu \in \Delta$ such that $\|\mu\|_{L^2} \leqslant r$. Δ_r^2 is a closed subset of Δ. (Indeed $\Delta_r^2 = \bigcap_n \phi_{g_n}^{-1}([0, r^2])$, where $g_n(x) := \min(x^2, n)$.) We next prove that the restriction of V_1 to Δ_r^2 is continuous: If $\{\mu_n\}_{n\in\mathbb{N}} \subset \Delta_r^2$ converges weakly to μ, then, according to Skorokhod representation theorem, there exists a sequence X_n of random variables with $X_n \sim \mu_n$ that converges a.s. to $X \sim \mu$. Since $\|X_n - X\|_{L^2} \leqslant 2r$, we conclude that $|X_n - X|^p$ is a uniformly integrable sequence ($p < 2$), and thus $\|X_n - X\|_{L^p} \to 0$, implying with (H2) that $V_1[X_n] \to V_1[X]$. So V_1 is indeed continuous on Δ_r^2. Therefore $V_1 : \Delta^2 \to \mathbb{R}$ is measurable on the trace $\mathcal{B}_{\Delta^2}$ of $\mathcal{B}_\Delta$ on Δ^2. Let next $[Y]$ be in Δ^2. As the composition of the measurable maps $[Y|\mathcal{H}] : (\Omega, \mathcal{H}) \to (\Delta^2, \mathcal{B}_{\Delta^2})$ and $V_1 : (\Delta^2, \mathcal{B}_{\Delta^2}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$, $V_1[Y|\mathcal{H}]$, is thus $\mathcal{H}$-measurable, as announced.

If, after stage q, P2 follows with an ϵ-optimal strategy in $\Gamma_{n-q}[L|\mathcal{H}_q]$, the payoff Y_q of P1 in the $n-q$ last stages will be less than or equal to $V_{n-q}[L|\mathcal{H}_q] + \epsilon \leqslant X_q + \epsilon$. Therefore, $\pi_{(\mu,\sigma,\tau)}[v_{q-1} < u_{q-1}] = 0$, since otherwise, P2 would have a profitable deviation: $v_{q-1} + \epsilon + Y_q \leqslant v_{q-1} + X_q + 2\epsilon \leqslant u_{q-1} + X_q$, if ϵ is small enough.

So for all q, $E[v_{q-1}] \geqslant E[u_{q-1}]$. Summing up all these inequalities, we get $\mathcal{V}_n(\mathcal{H}, \bar{L}) \geqslant g_n(\mu,\sigma,\tau) = V_n(\mu) = \overline{\mathcal{V}}_n(\mu)$, and the second assertion is proved. □

8. The asymptotic behavior of martingales of maximal variation

In this section, we define the M-variation of a martingale and we analyze in Theorem 5 the limit behavior of the martingales maximizing this M-variation. As we will see at the end of this section, Theorem 1 is a simple corollary of Theorem 4 and Theorem 5.

Let us start with some notations. L^2_0 will denote hereafter the set of random variables $X \in L^2$ such that $E[X] = 0$, and Δ^2_0 be the set of measure μ on $\mathbb{R}$ such that $X \sim \mu$ implies $X \in L^2_0$. For a function $M : \Delta^2_0 \to \mathbb{R}$, and a random variable X in L^2_0, we will write $M[X]$ instead of $M([X])$. $\mathcal{W}_n(\mu)$ was defined in Section 7 as the set of pairs $(\mathcal{F}, X)$ where $\mathcal{F} := (\mathcal{F}_q)_{q=0,\dots,n+1}$ is a filtration on a probability space, and $X = (X_q)_{q=0,\dots,n+1}$ is an $\mathcal{F}$-martingale X whose $n+1$-th value X_{n+1} is μ-distributed. Observe that if $\mu \in \Delta^2$ and $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, then $[X_{q+1} - X_q|\mathcal{F}_q] \in \Delta^2_0$, and we may therefore define the M-variation $\mathcal{V}^M_n(\mathcal{F}, X)$ as

$$\mathcal{V}^M_n(\mathcal{F}, X) := E\left[\sum_{q=0}^{n-1} M[X_{q+1} - X_q|\mathcal{F}_q]\right]. \tag{7}$$

Since we only will deal in this paper with functions M that are Lipschitz in L^p-norm for $p < 2$, we refer to footnote 2 on page 52 for a proof of the measurability of $M[X_{q+1} - X_q|\mathcal{F}_q]$. Let us next define $\overline{\mathcal{V}}^M_n(\mu)$ as

$$\overline{\mathcal{V}}^M_n(\mu) := \sup\{\mathcal{V}^M_n(\mathcal{F}, X)\colon\ (\mathcal{F}, X) \in \mathcal{W}_n(\mu)\}. \tag{8}$$

For $\mu \in \Delta^2$, the function f_μ was defined in Section 2 as the unique increasing function such that $f_\mu(Z) \sim \mu$ when $Z \sim \mathcal{N}(0,1)$. The CMMV Π^μ was also defined there as $\Pi^\mu_t := E[f_\mu(B_1)|(B_s)_{s\leqslant t}]$, where B is a standard Brownian motion. With these definitions, we are ready to state the main result of this section:

Theorem 5. *If M satisfies*:

(i) *For all random variable $X \in L^2_0$, $\forall \alpha \geqslant 0$: $M[\alpha X] = \alpha M[X]$.*
(ii) *There exist $p \in [1, 2[$ and $A \in \mathbb{R}$ such that for all $X, Y \in L^2_0$:*

$$\left|M[X] - M[Y]\right| \leqslant A\|X - Y\|_{L^p}.$$

Then for all $\mu \in \Delta^2$,

$$\lim_{n\to\infty} \frac{\overline{\mathcal{V}}^M_n(\mu)}{\sqrt{n}} = \rho \cdot E\left[f_\mu(Z)Z\right],$$

where $Z \sim \mathcal{N}(0,1)$ and $\rho := \sup\{M(\nu)\colon\ \nu \in \Delta^2_0,\ \|\nu\|_{L^2} \leqslant 1\}$.

Furthermore, if $\rho > 0$ and if, for all n, $(\mathcal{F}^n, X^n) \in \mathcal{W}_n(\mu)$ satisfies $\mathcal{V}^M_n(\mathcal{F}^n, X^n) = \overline{\mathcal{V}}^M_n(\mu)$, then the continuous time representation Π^n of X^n defined as $\Pi^n_t := X^n_{[\![n\cdot t]\!]}$ converges in finite-dimensional distribution to the CMMV Π^μ.

This theorem justifies our terminology when referring to Π^μ as the continuous martingale of maximal variation corresponding to μ.

With $M[X] := \|X\|_{L^1}$, $\mathcal{V}^M_n(\mathcal{F}, X)$ is just the L^1-variation of the martingale X and we recover here Mertens and Zamir's (1977) result on the maximal variation of a bounded martingale, taking μ such that $\mu(\{1\}) = s$, $\mu(\{0\}) = 1 - s$. With $M[X] := \|X\|_{L^p}$, with $p \in [1, 2[$ we also recover the results of De Meyer (1998). The proof presented here is in fact inspired by this last paper.

The previous theorem implies also Theorem 1.

Proof of Theorem 1. Indeed, (H4) indicates that for all variable X: $V_1[X] = V_1[(X - E[X])]$ and thus also, if $\mathcal{F}$ is a σ-algebra: $V_1[X|\mathcal{F}] = V_1[(X - E[X|\mathcal{F}])|\mathcal{F}]$. Therefore, if $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, the function $\mathcal{V}_n(\mathcal{F}, X)$ defined in (5) is just equal to:

$$\mathcal{V}_n(\mathcal{F}, X) = \mathcal{V}^{V_1}_n(\mathcal{F}, X).$$

Since (H3) indicates that V_1 satisfies the first condition of Theorem 5, (H2) indicates that V_1 fulfills the second one and (H5) indicates that $\rho > 0$, Theorem 1 follows then from Theorem 4. As a byproduct, we also get that $\lim_{n\to\infty} \frac{V_n(\mu)}{\sqrt{n}} = \rho \cdot E[f_\mu(Z)Z]$. □

Notice that the techniques presented in this paper could also be applied to analyze general repeated zero-sum game, and Mertens and Zamir's (1976) result is in fact an easy consequence of Theorem 5 and of an adapted version of Theorem 4.

The remaining sections of this paper are devoted to the proof of Theorem 5 that relies on duality techniques, a central limit theorem for martingales and a Skorokhod embedding of martingales in the Brownian filtration. With these techniques, the problem of maximizing the M-variation of martingales will become a problem of maximizing a covariation as introduced in the next section.

We provide an upper bound for M in Section 10 that leads to an upper bound for $\mathcal{V}_n^M$ in the next one. We will then conclude in Section 13 that $\rho \cdot E[f_\mu(Z)Z]$ dominates the lim sup of $\frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}}$, using a central limit theorem for martingales presented in Section 12.

In Section 14, we prove that $\rho \cdot E[f_\mu(Z)Z]$ is the lim inf of $\frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}}$, and in Section 15, we prove the convergence of the Π^n to Π^μ.

Notice that the case $\rho = 0$ is trivial in Theorem 5, since then $M[\mu] \leqslant 0$ for all μ in Δ_0^2, and thus the constant martingale $X_q = E[X_{n+1}]$, for all $q \leqslant n$ and $X_{n+1} \sim \mu$ will be optimal in the maximization problem (8), and so $\overline{\mathcal{V}}_n^M(\mu) = 0$. In the sequel, we therefore assume $\rho > 0$.

As a remark, observe that the hypothesis $p < 2$ in (ii) of Theorem 5 could not be weakened in $p \leqslant 2$. A counterexample of this is given at the end of Section 13.

9. The maximal covariation

In this section, we solve an auxiliary optimization problem that will be central in our argument: it explains where the expression $E[f_\mu(Z)Z]$ appearing in Theorem 5 comes from. It is a classical Monge–Kantorovich mass transportation problem and claim (1) in the next theorem is well known. However, claim (2) is original and a proof is needed.

In this section all random variables will be on a probability space $(\Omega, \mathcal{A}, P)$, and Z will denote an $\mathcal{N}(0,1)$ random variable.

Theorem 6. *For $\mu \in \Delta^{1^+}$, let us define $\alpha(\mu) := \sup\{E[XZ]\colon\ X \sim \mu\}$ then*

(1) $\alpha(\mu) = E[f_\mu(Z)Z]$.
(2) *If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of μ-distributed random variables such that $E[X_n Z]$ converges to $\alpha(\mu)$ then X_n converges in L^1-norm to $f_\mu(Z)$.*

Proof. Let X be a μ-distributed random variable. For a real number x, we set $x^+ := \max\{x, 0\}$ and $x^- := (-x)^+$. Since $\mu \in \Delta^{1^+}$, X^+Z and X^-Z are in L^1 and with Fubini–Tonelli theorem

$$E\big[X^+Z\big] = E\left[\int_0^\infty \mathbb{1}_{\{c\leqslant X\}} \cdot Z\,dc\right] = \int_0^\infty E[\mathbb{1}_{\{c\leqslant X\}}Z]\,dc.$$

Similarly, $E[X^- \cdot Z] = \int_0^\infty E[\mathbb{1}_{\{c<-X\}}Z]\,dc$. Therefore

$$E\big[\big(f_\mu(Z)\big)^+ \cdot Z\big] - E\big[X^+ \cdot Z\big] = \int_0^\infty E\big[(\mathbb{1}_{\{c\leqslant f_\mu(Z)\}} - \mathbb{1}_{\{c\leqslant X\}})Z\big]\,dc,$$

$$E\big[\big(f_\mu(Z)\big)^- \cdot Z\big] - E\big[X^- \cdot Z\big] = \int_0^\infty E\big[(\mathbb{1}_{\{c<-f_\mu(Z)\}} - \mathbb{1}_{\{c<-X\}})Z\big]\,dc.$$

Now observe that X and $f_\mu(Z)$ have the same distribution. Therefore

$$\forall c\colon\quad E\big[(\mathbb{1}_{\{c\leqslant f_\mu(Z)\}} - \mathbb{1}_{\{c\leqslant X\}})\big] = 0 = E\big[(\mathbb{1}_{\{c<-f_\mu(Z)\}} - \mathbb{1}_{\{c<-X\}})\big]$$

and we infer that

$$E\big[\big(f_\mu(Z)\big)^+ \cdot Z\big] - E\big[X^+ \cdot Z\big] = \int_0^\infty E\big[(\mathbb{1}_{\{c\leqslant f_\mu(Z)\}} - \mathbb{1}_{\{c\leqslant X\}})\big(Z - f_\mu^{-1}(c)\big)\big]\,dc,$$

$$E\big[\big(f_\mu(Z)\big)^- \cdot Z\big] - E\big[X^- \cdot Z\big] = \int_0^\infty E\big[(\mathbb{1}_{\{c<-f_\mu(Z)\}} - \mathbb{1}_{\{c<-X\}})\big(Z - f_\mu^{-1}(-c)\big)\big]\,dc$$

where f_μ^{-1} is the left continuous inverse of f_μ: $f_\mu^{-1}(c) := \inf\{s\colon\ f_\mu(s) \geqslant c\}$.

An easy computation shows that

$$\mathbb{1}_{\{c\leqslant f_\mu(Z)\}} - \mathbb{1}_{\{c\leqslant X\}} = \mathbb{1}_{\{X<c\leqslant f_\mu(Z)\}} - \mathbb{1}_{\{f_\mu(Z)<c\leqslant X\}},$$
$$\mathbb{1}_{\{c<-f_\mu(Z)\}} - \mathbb{1}_{\{c<-X\}} = \mathbb{1}_{\{f_\mu(Z)<-c\leqslant X\}} - \mathbb{1}_{\{X<-c\leqslant f_\mu(Z)\}}.$$

Since $c \leqslant f_\mu(Z)$ if and only if $f_\mu^{-1}(c) \leqslant Z$, we conclude that

$$E\big[\big(f_\mu(Z)\big)^+ \cdot Z\big] - E\big[X^+ \cdot Z\big] = \int_0^\infty E\big[h(X,Z,c)\big]\,dc,$$
$$E\big[\big(f_\mu(Z)\big)^- \cdot Z\big] - E\big[X^- \cdot Z\big] = \int_0^\infty E\big[-h(X,Z,-c)\big]\,dc$$

where

$$h(X,Z,c) := (\mathbb{1}_{\{X<c\leqslant f_\mu(Z)\}} + \mathbb{1}_{\{f_\mu(Z)<c\leqslant X\}}) \cdot \big|Z - f_\mu^{-1}(c)\big|.$$

Since $x = x^+ - x^-$, we get thus $E[f_\mu(Z) \cdot Z] - E[X \cdot Z] = \int_{-\infty}^{\infty} E[h(X,Z,c)]\,dc$. Observing that $h(X,Z,c) \geqslant 0$, we get $E[f_\mu(Z) \cdot Z] \geqslant E[X \cdot Z]$, and assertion (1) follows.

We next prove claim (2). Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of μ-distributed random variables such that $E[X_n Z]$ converges to $\alpha(\mu)$. Then $\int_{-\infty}^{\infty} E[h(X_n,Z,c)]\,dc$ converges to 0, and, since $h(X_n,Z,c) \geqslant 0$, we conclude that $h(X_n(\omega), Z(\omega), c)$ converges to 0 in $P \otimes \lambda$-measure on the measure space $(\Omega \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}, P \otimes \lambda)$, where λ is the Lebesgue measure and $\mathcal{B}_{\mathbb{R}}$ is the Borelean tribe on $\mathbb{R}$.

As a consequence, there exists a subsequence $\{X'_n\}_{n\in\mathbb{N}}$ of $\{X_n\}_{n\in\mathbb{N}}$, such that $h(X'_n(\omega), Z(\omega), c)$ converges $P \otimes \lambda$-a.e. to 0. Next

$$h\big(X'_n, Z, c\big) = l\big(X'_n, Z, c\big) \cdot \big|Z - f_\mu^{-1}(c)\big|$$

with $l(X'_n, Z, c) := (\mathbb{1}_{\{X'_n<c\leqslant f_\mu(Z)\}} + \mathbb{1}_{\{f_\mu(Z)<c\leqslant X'_n\}})$, so that

$$l\big(X'_n(\omega), Z(\omega), c\big).\mathbb{1}_{\{Z(\omega)\neq f_\mu^{-1}(c)\}}$$

converges $P \otimes \lambda$-a.e. to 0.

Since $\forall c$: $P(Z(\omega) = f_\mu^{-1}(c)) = 0$, $l(X'_n(\omega), Z(\omega), c)$ converges also $P \otimes \lambda$-a.e. to 0, and since l is bounded by 2, we conclude with Lebesgue dominated convergence theorem that for all $K < \infty$: $\lim_{n\to\infty} \int_{-K}^{K} E[l(X'_n, Z, c)]\,dc = 0$. Now, observe that with Tonelli's theorem:

$$\int_{-K}^{K} E\big[l\big(X'_n, Z, c\big)\big]\,dc = E\Bigg[\int_{-K}^{K} (\mathbb{1}_{\{X'_n<c\leqslant f_\mu(Z)\}} + \mathbb{1}_{\{f_\mu(Z)<c\leqslant X'_n\}})\,dc\Bigg]$$
$$= E\big[\big|T_K\big(X'_n\big) - T_K\big(f_\mu(Z)\big)\big|\big],$$

where $T_K(x) := \max(\min(x, K), -K)$. Now

$$\big\|X'_n - f_\mu(Z)\big\|_{L^1} \leqslant \big\|X'_n - T_K\big(X'_n\big)\big\|_{L^1} + \big\|T_K\big(X'_n\big) - T_K\big(f_\mu(Z)\big)\big\|_{L^1} + \big\|T_K\big(f_\mu(Z)\big) - f_\mu(Z)\big\|_{L^1}.$$

Since X'_n and $f_\mu(Z)$ are μ-distributed, the first and the third terms are equal and are just a function $g(K)$. So, $\forall K$, $\limsup_{n\to\infty} \|X'_n - f_\mu(Z)\|_{L^1} \leqslant 2g(K)$. Next, since $X'_n \in L^1$, we get $\lim_{K\to\infty} g(K) = 0$, and we conclude therefore that X'_n converges to $f_\mu(Z)$ in L^1.

We thus have proved that any maximizing sequence $\{X_n\}_{n\in\mathbb{N}}$ (i.e. such that $E[X_n Z] \to \alpha(\mu)$) contains a subsequence $\{X'_n\}_{n\in\mathbb{N}}$ that converges in L^1 to $f_\mu(Z)$. This implies clearly that any maximizing sequence converges to $f_\mu(Z)$ in L^1 and claim (2) is thus proved. $\square$

10. An upper bound for M

On a probability space $(\Omega, \mathcal{A}, P)$, for $q \geqslant 1$ and $r > 0$, let $B^q_r(\Omega, \mathcal{A}, P)$ be the set $B^q_r(\Omega, \mathcal{A}, P) := \{X \in L^2(\Omega, \mathcal{A}, P) | \|X\|_{L^q} \leqslant r\}$. Let next $\mathcal{B}^*(\Omega, \mathcal{A}, P)$ be defined as:

$$\mathcal{B}^*(\Omega, \mathcal{A}, P) := B^2_\rho(\Omega, \mathcal{A}, P) \cap B^{p'}_{2A}(\Omega, \mathcal{A}, P),$$

with A, ρ and p as in Theorem 5 and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let us finally define, for $X \in L^2(\Omega, \mathcal{A}, P)$:

$$B(X) := \sup\big\{E[XY]\colon\ Y \in \mathcal{B}^*(\Omega, \mathcal{A}, P)\big\}. \tag{9}$$

Due to Jensen's inequality, for all $r \geqslant 1$, $\|E[Y|X]\|_{L^r} \leqslant \|Y\|_{L^r}$. Therefore, if $Y \in \mathcal{B}^*(\Omega, \mathcal{A}, P)$ then $E[Y|X] \in \mathcal{B}^*(\Omega, \mathcal{A}, P)$. So, we infer that $B(X) = \sup\{E[Xf(X)]\colon\ f(X) \in \mathcal{B}^*(\Omega, \mathcal{A}, P)\}$, and therefore $B(X)$ just depends on the distribution $[X]$. In other words, if $[X] = [X']$, then $B(X) = B(X')$, even if X and X' are defined on different probability spaces. We will therefore abuse the notations and write $B[X]$ instead of $B(X)$.

Lemma 7.

(1) *For all* $X \in L^2_0(\Omega, \mathcal{A}, P)$: $M[X] \leqslant \rho \cdot \|X\|_{L^2}$.
(2) *If* $\mathcal{B}(\Omega, \mathcal{A}, P) := \{X \in L^2(\Omega, \mathcal{A}, P)\colon\ B[X] \leqslant 1\}$, *then*

$$\mathcal{B}(\Omega, \mathcal{A}, P) \subset \operatorname{conv}\big(B^2_{\frac{1}{\rho}}(\Omega, \mathcal{A}, P) \cup B^p_{\frac{1}{2A}}(\Omega, \mathcal{A}, P)\big).$$

(3) *For all* $X \in L^2_0(\Omega, \mathcal{A}, P)$: $B[X] \geqslant M[X]$.

Proof. Claim (1) is an obvious consequence of the definition of ρ as $\sup\{M[X]\colon\ X \in L^2_0, \|X\|_{L^2} \leqslant 1\}$ and of the 1-homogeneity of M.

We next prove claim (2): Let $\mathcal{C}$ denote $\operatorname{conv}(B^2_{\frac{1}{\rho}}(\Omega, \mathcal{A}, P) \cup B^p_{\frac{1}{2A}}(\Omega, \mathcal{A}, P))$. Since $B^2_{\frac{1}{\rho}}(\Omega, \mathcal{A}, P)$ and $B^p_{\frac{1}{2A}}(\Omega, \mathcal{A}, P)$ are bounded convex closed sets in L^2-norm ($p < 2$), they are weakly compact. $\mathcal{C}$ is therefore closed. Therefore, if $Z \in L^2(\Omega, \mathcal{A}, P)$ does not belong to $\mathcal{C}$, we can separate $\{Z\}$ and $\mathcal{C}$ in $L^2(\Omega, \mathcal{A}, P)$ by a separating vector Y: $E[YZ] > \alpha := \sup\{E[YX]\colon\ X \in \mathcal{C}\}$. In particular $\alpha \geqslant \sup\{E[YX]\colon\ X \in B^2_{\frac{1}{\rho}}\} = \frac{1}{\rho} \cdot \|Y\|_{L^2}$, and $\alpha \geqslant \sup\{E[YX]\colon\ X \in B^p_{\frac{1}{2A}}\} = \frac{1}{2A} \cdot \|Y\|_{L^{p'}}$. This indicates that $Y' := \frac{Y}{\alpha} \in \mathcal{B}^*(\Omega, \mathcal{A}, P)$. Therefore $B[Z] \geqslant E[Y'Z] > 1$ and so $Z \notin \mathcal{B}(\Omega, \mathcal{A}, P)$. So the complementary of $\mathcal{C}$ is included in the complementary of $\mathcal{B}(\Omega, \mathcal{A}, P)$, or equivalently: $\mathcal{B}(\Omega, \mathcal{A}, P) \subset \mathcal{C}$.

To prove claim (3) observe that both M and B are 1-homogeneous on $L^2_0(\Omega, \mathcal{A}, P)$. Therefore, we just have to prove that, for all $X \in L^2_0(\Omega, \mathcal{A}, P)$, $M[X] \leqslant 1$ whenever $B[X] \leqslant 1$. But if $B[X] \leqslant 1$, then $X \in \mathcal{B}(\Omega, \mathcal{A}, P)$. So, by the previous claim $X \in \mathcal{C}$. Since $\mathcal{C}$ is the convex hull of two convex sets, we get that $X = \lambda Y + \lambda' Y'$, with $\lambda, \lambda' \geqslant 0$, $\lambda + \lambda' = 1$, $Y \in B^2_{\frac{1}{\rho}}$ and $Y' \in B^p_{\frac{1}{2A}}$. Since $E[X] = 0$, we also have $X = \lambda(Y - E[Y]) + \lambda'(Y' - E[Y'])$. Due to property (ii) in Theorem 5, we get:

$$M[X] \leqslant M\big[\lambda\big(Y - E[Y]\big)\big] + A\big\|\lambda'\big(Y' - E\big[Y'\big]\big)\big\|_{L^p}.$$

Since $\|(Y - E[Y])\|_{L^2} \leqslant \|Y\|_{L^2} \leqslant \frac{1}{\rho}$, it follows from claim (1) that the first term is bounded by λ. The second one is bounded by λ' since $\|Y' - E[Y']\|_{L^p} \leqslant \|Y'\|_{L^p} + \|E[Y']\|_{L^p} \leqslant 2\|Y'\|_{L^p} \leqslant \frac{1}{A}$. Thus $M[X] \leqslant 1$ and the lemma is proved. □

11. An upper bound for $\overline{\mathcal{V}}^M_n(\mu)$

For $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, $\mathcal{V}^M_n(\mathcal{F}, X)$ was defined in (7). The term $M[X_{q+1} - X_q|\mathcal{F}_q]$ involved there is then dominated by $B[X_{q+1} - X_q|\mathcal{F}_q]$, and we will next focus on the B variation $\mathcal{V}^B_n(\mathcal{F}, X)$ that dominates the M-variation.

The next lemma presents $E[B[X_{q+1} - X_q|\mathcal{F}_q]]$ as the result of an optimization problem.

Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two σ-algebras on a probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ denote the set of $Y \in L^2(\mathcal{F}_2)$ such that $E[Y^2|\mathcal{F}_1] \leqslant \rho^2$ and $E[|Y|^{p'}|\mathcal{F}_1] \leqslant (2A)^{p'}$. Let $\mathcal{B}^*_{(\rho,C)}(\mathcal{F}_2|\mathcal{F}_1)$ denote the set of $Y \in L^2(\mathcal{F}_2)$ such that

(1) $E[Y|\mathcal{F}_1] = 0$;
(2) $E[Y^2|\mathcal{F}_1] = \rho^2$;
(3) $E[|Y|^{p'}|\mathcal{F}_1] \leqslant C^{p'}$.

Lemma 8.

(1) *For all* $X \in L^2_0(\mathcal{F}_2)$:

$$E\big[B[X|\mathcal{F}_1]\big] = \sup\big\{E[XY]|Y \in \mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)\big\}.$$

(2) *If there exists in $L^2(\Omega, \mathcal{F}_2, P)$ a random variable U that is independent of $\sigma(\mathcal{F}_1, X)$ and taking the values 1 and -1 with probability $1/2$ then*

$$E\big[B[X|\mathcal{F}_1]\big] \leqslant \sup\big\{E[XY] | Y \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}_2|\mathcal{F}_1)\big\}.$$

Proof. If $X \in L^2(\mathcal{F}_2)$ and $Y \in \mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ then $E[XY] = E[E[XY|\mathcal{F}_1]]$. Since conditionally to $\mathcal{F}_1$, Y belongs to $\mathcal{B}^*$, we get $E[XY|\mathcal{F}_1] \leqslant B[X|\mathcal{F}_1]$, and thus $E[B[X|\mathcal{F}_1]] \geqslant \sup\{E[XY] | Y \in \mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)\}$.

To prove the reverse inequality, we just have to prove that, $\forall \epsilon > 0$, there exists a measurable map $\phi : \Delta^2 \times \mathbb{R} \to \mathbb{R}$ such that, $\forall \mu \in \Delta^2$, if $X \sim \mu$, then $E[\phi(\mu, X)^2] \leqslant \rho^2$, $E[|\phi(\mu, X)|^{p'}] \leqslant (2A)^{p'}$ and $B(\mu) - \epsilon \leqslant E[X\phi(\mu, X)]$. Indeed, the random variable $Y := \phi([X|\mathcal{F}_1], X)$ belongs then to $\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ and $E[XY] \geqslant E[B[X|\mathcal{F}_1]] - \epsilon$.

We now prove the existence of such a measurable map. As argued just after Eq. (9), $B(\mu) = \sup\{E_\mu[Xf(X)]: E_\mu[f(X)^2] \leqslant \rho^2; E_\mu[|f(X)|^p] \leqslant (2A)^p\}$. Since there exists a countable set $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions, with $f_0 \equiv 0$, that is dense in $L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)$, for all $\mu \in \Delta^2$, we also have $B(\mu) = \sup_n g_n(\mu)$, where $g_n(\mu) := E_\mu[Xf_n(X)]$ if $E_\mu[f_n(X)^2] \leqslant \rho^2$ and $E_\mu[|f_n(X)|^p] \leqslant (2A)^p$, and $g_n(\mu) := 0$ otherwise. The maps $\mu \to g_n(\mu)$ are measurable with respect to the Borel σ-algebra $\mathcal{B}_{\Delta^2}$ associated with the weak topology on Δ^2. Indeed, the maps $\mu \to E_\mu[Xf_n(X)]$, $\mu \to E_\mu[f_n(X)^2]$ and $\mu \to E_\mu[|f_n(X)|^p]$ are weakly continuous and thus measurable with respect to $\mathcal{B}_{\Delta^2}$. We infer therefore that $\mu \to B(\mu)$ and thus $\mu \to B(\mu) - g_n(\mu)$ are also $\mathcal{B}_{\Delta^2}$-measurable. For $\epsilon > 0$, we may then define $N(\mu)$ as the smallest integer n such that $B(\mu) - g_n(\mu) \leqslant \epsilon$. The map $\phi : (\mu, x) \to f_{N(\mu)}(x)$ will therefore be measurable.

It has also the required properties: $\forall \mu \in \Delta^2$, if $X \sim \mu$, then $E[\phi(\mu, X)^2] \leqslant \rho^2$, $E[|\phi(\mu, X)|^{p'}] \leqslant (2A)^{p'}$ and $B(\mu) - \epsilon \leqslant E[X\phi(\mu, X)]$.

Indeed, for all $\mu \in \Delta^2$, either $B(\mu) \leqslant \epsilon$, which implies $N(\mu) = 0$ and thus $E_\mu[f_{N(\mu)}(X)X] = 0 \geqslant B(\mu) - \epsilon$, since $f_0 \equiv 0$. We also have in this case $E_\mu[f^2_{N(\mu)}(X)] = 0 \leqslant \rho^2$ and $E_\mu[|f_{N(\mu)}(X)|^p] = 0 \leqslant (2A)^p$.

Or, $B(\mu) > \epsilon$, and thus $g_{N(\mu)}(\mu) > 0$, which indicates that $E_\mu[f^2_{N(\mu)}(X)] \leqslant \rho^2$, $E_\mu[|f_{N(\mu)}(X)|^p] \leqslant (2A)^p$ and also $E_\mu[f_{N(\mu)}(X)X] = g_{N(\mu)}(\mu) \geqslant B(\mu) - \epsilon$, as it results from the definition of g_n.

We next turn to claim (2): Just observe that if $Y \in \mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$, then $Y' := E[Y|X]$ also belongs to $\mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$ and has thus the property that $E[XY] = E[XY']$. Now, consider $Y'' := Y' - E[Y'|\mathcal{F}_1]$. Since $X \in L^2_0(\mathcal{F}_2|\mathcal{F}_1)$, we have $E[XE[Y'|\mathcal{F}_1]] = 0$. Therefore $E[XY] = E[XY'']$. Now, observe that

$$\theta^2 := E\big[\big(Y''\big)^2|\mathcal{F}_1\big] = E\big[\big(Y'\big)^2|\mathcal{F}_1\big] - \big(E\big[Y'|\mathcal{F}_1\big]\big)^2 \leqslant E\big[Y^2|\mathcal{F}_1\big] \leqslant \rho^2.$$

Finally, let Y''' be defined as $Y''' := Y'' + \sqrt{\rho^2 - \theta^2}U$, since U is independent of $\sigma(\mathcal{F}_1, X)$, and since Y'' is measurable on this σ-algebra, we get obviously

$$E[XY] = E\big[XY'''\big], \qquad E\big[Y'''|\mathcal{F}_1\big] = 0 \quad \text{and} \quad E\big[\big(Y'''\big)^2|\mathcal{F}_1\big] = \rho^2.$$

Observing that $(E[(Y''')^{p'}|\mathcal{F}_1])^{\frac{1}{p'}}$ is just a conditional $L^{p'}$-norm, we get

$$\begin{aligned}
\big(E\big[\big|Y'''\big|^{p'}|\mathcal{F}_1\big]\big)^{\frac{1}{p'}} &\leqslant \big(E\big[\big|Y''\big|^{p'}|\mathcal{F}_1\big]\big)^{\frac{1}{p'}} + \rho \\
&\leqslant \big(E\big[\big|Y'\big|^{p'}|\mathcal{F}_1\big]\big)^{\frac{1}{p'}} + \big(E\big[\big|E\big[Y'|\mathcal{F}_1\big]\big|^{p'}|\mathcal{F}_1\big]\big)^{\frac{1}{p'}} + \rho \\
&\leqslant 2\big(E\big[\big|Y'\big|^{p'}|\mathcal{F}_1\big]\big)^{\frac{1}{p'}} + \rho \\
&\leqslant 2\big(E\big[|Y|^{p'}|\mathcal{F}_1\big]\big)^{\frac{1}{p'}} + \rho \\
&\leqslant 4A + \rho.
\end{aligned}$$

Therefore, for all $Y \in \mathcal{B}^*(\mathcal{F}_2|\mathcal{F}_1)$, there is a $Y''' \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}_2|\mathcal{F}_1)$ such that $E[XY] = E[XY''']$, and claim (2) then follows from claim (1). □

Let us now use this lemma to compute $\mathcal{V}^M_n(\mathcal{F}, X)$ for a pair $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$ defined on a probability space $(\Omega, \mathcal{A}, P)$. Let us first enlarge this space obtaining a new one $(\Omega', \mathcal{A}', P')$ where $\mathcal{A}$ may be seen as a sub-σ-algebra of $\mathcal{A}'$, P' and P coincide on $\mathcal{A}$ and where there is a system of n independent random variables $(U_q)_{q=1,\dots,n}$, independent of $\mathcal{A}$, with $P'(U_q = 1) = P'(U_q = -1) = 1/2$. Consider next the filtration $\mathcal{F}'$ defined by $\mathcal{F}'_q := \sigma(\mathcal{F}_q, U_k, k \leqslant q)$. X is then also a martingale on $\mathcal{F}'$ and $[X_{q+1} - X_q|\mathcal{F}_q] = [X_{q+1} - X_q|\mathcal{F}'_q]$. Therefore

$$\mathcal{V}^M_n(\mathcal{F}, X) \leqslant \mathcal{V}^B_n(\mathcal{F}, X) = \mathcal{V}^B_n\big(\mathcal{F}', X\big).$$

We will denote $\mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')$ the set of $\mathcal{F}'$- adapted processes Y such that for all $q = 1, \dots, n$: $Y_q \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}'_q|\mathcal{F}'_{q-1})$. Then, since $X_q - X_{q-1} \in L^2_0(\mathcal{F}'_q|\mathcal{F}'_{q-1})$, we may apply claim (2) of last lemma to get

$$\mathcal{V}_n^B(\mathcal{F}', X) = \sum_{q=1}^n E\big[B\big[X_q - X_{q-1}|\mathcal{F}'_{q-1}\big]\big]$$
$$\leqslant \sup_{Y \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')} \sum_{q=1}^n E\big[(X_q - X_{q-1}) \cdot Y_q\big].$$

Since X is an $\mathcal{F}'$-martingale and $E[Y_q|\mathcal{F}'_{q-1}] = 0$, we get

$$E\big[(X_q - X_{q-1}) \cdot Y_q\big] = E[X_q \cdot Y_q] = E\big[E\big[X_{n+1}|\mathcal{F}'_q\big] \cdot Y_q\big] = E[X_{n+1} \cdot Y_q],$$

and therefore

$$\mathcal{V}_n^M(\mathcal{F}', X) \leqslant \sup_{Y \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')} E\left[X_{n+1} \cdot \sum_{q=1}^n Y_q\right]. \tag{10}$$

For a given $Y \in \mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, let S_q be defined as $S_0 := 0$, $S_q := S_{q-1} + Y_q$. Observe then that S is an $\mathcal{F}'$-martingale. We will denote $\mathcal{S}_{(\rho,4A+\rho)}(\mathcal{F}')$ the set of $\mathcal{F}'$-martingale S whose increments $S_{q+1} - S_q$ belong to $\mathcal{B}^*_{(\rho,4A+\rho)}(\mathcal{F}'_q|\mathcal{F}'_{q-1})$, for all q, and such that $S_0 = 0$. The last formula becomes then

$$\mathcal{V}_n^M(\mathcal{F}', X) \leqslant \sup_{S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')} E[X_{n+1} \cdot S_n]. \tag{11}$$

Let us make two comments on the last formula: the quantity $\mathcal{V}_n^M(\mathcal{F}', X)$ depends on the laws $[X_{q+1} - X_q|\mathcal{F}_q]$ which are intimately related to the filtration $\mathcal{F}$. The bound we found in the last formula just depends on the laws $[X_{q+1} - X_q|X_1, \dots, X_q]$. Therefore, if we create a martingale $\tilde{X}$ on another filtration $\mathcal{G}$ with the same law as X — we call this procedure the embedding of X in the filtration $\mathcal{G}$ —, the right-hand side of last inequality can equivalently be evaluated on $\tilde{X}$.

The second comment is that we will have to deal with $\frac{\mathcal{V}_n^M(\mathcal{F}', X)}{\sqrt{n}}$, and will then have to evaluate $E[X_{n+1} \cdot \frac{S_n}{\sqrt{n}}]$, for $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$. Since the increments of S have a conditional variance equal to ρ^2, $\frac{S_n}{\sqrt{n}}$ will be approximatively normally distributed, due to a central limit theorem for martingales. We need however precise bounds for this approximation. These bounds are provided in the next section which is in fact the crux point of the argument. We embed there both martingales $\frac{S}{\sqrt{n}}$ and X in a Brownian filtration.

12. The embedding in the Brownian filtration

Let B be a Brownian motion on a probability space $(\Omega_0, \mathcal{A}_0, P_0)$ and let $\mathcal{G}$ be the natural filtration of B. Skorokhod posed the following question: Given a probability distribution $\mu \in \Delta^{p'}$, is there a $\mathcal{G}$-stopping time θ such that $B_\theta \sim \mu$? To avoid trivial uninteresting solutions to this problem, one further requires that $E[\theta^{\frac{p'}{2}}] < \infty$. It is well known that Skorokhod's problem has a solution for all $\mu \in \Delta_0^{p'}$ (see for instance Azéma and Yor, 1979) and we will denote θ_μ one of these solutions.

We also will need the following fact: For all $p' > 1$, there exist two non-negative constants $c_{p'}$ and $C_{p'}$, called the Burkholder–Davis–Gundy constants (see Burkholder, 1973), such that, for all $\mathcal{G}$-stopping times $\tau \geqslant \tau'$:

$$E\big[\tau^{\frac{p'}{2}}\big] < \infty \quad \Longrightarrow \quad c_{p'} \cdot E\big[(\tau - \tau')^{\frac{p'}{2}}|\mathcal{G}_{\tau'}\big] \leqslant E\big[|B_\tau - B_{\tau'}|^{p'}|\mathcal{G}_{\tau'}\big] \leqslant C_{p'} \cdot E\big[(\tau - \tau')^{\frac{p'}{2}}|\mathcal{G}_{\tau'}\big].$$

In the particular case $p' = 2$, we have $c_2 = C_2 = 1$.

Lemma 9. *Let $R = (R_q)_{q:=0,\dots,n}$ be a martingale with $R_n \in L_0^{p'}$, then there is an increasing sequence of stopping times $\{\tau_q\}_{q:=0,\dots,n}$ such that $E[\tau_n^{\frac{p'}{2}}] < \infty$ and such that both processes R and $\hat{R}$ have the same distribution where $\hat{R}_q := B_{\tau_q}$.*

Proof. Just take $\tau_0 := \theta_{[R_0]}$ so that $[\hat{R}_0] = [B_{\tau_0}] = [R_0]$. Once τ_q is defined, define τ_{q+1} as follows: $B'_t := B_{\tau_q + t} - B_{\tau_q}$ is another Brownian motion on its natural filtration $\mathcal{G}'$. For all $(r_0, \dots, r_q) \in \mathbb{R}^{q+1}$ define $\tilde{\theta}(r_0, \dots, r_q) := \theta'_{[R_{q+1} - R_q|R_0 = r_0; \dots; R_q = r_q]}$, where θ'_μ is a solution of μ-Skorokhod's problem for the Brownian motion B'. This mapping can be chosen measurable from $\mathbb{R}^{q+1}$ to $(\Omega_0, \mathcal{A}_0)$, and define $\tau_{q+1} := \tau_q + \tilde{\theta}(\hat{R}_0, \dots, \hat{R}_q)$. Then $\hat{R}_{q+1} - \hat{R}_q = B_{\tau_{q+1}} - B_{\tau_q} = B'_{\tilde{\theta}(\hat{R}_0, \dots, \hat{R}_q)}$. Therefore $[\hat{R}_{q+1} - \hat{R}_q|\hat{R}_0, \dots, \hat{R}_q] = [R_{q+1} - R_q|R_0, \dots, R_q]$. We then conclude that R and $\hat{R}$ have the same laws. Next, since $c_p \cdot E[\tilde{\theta}(\hat{R}_0, \dots, \hat{R}_q)^{\frac{p'}{2}}|\mathcal{G}_{\tau_q}] \leqslant E[(\hat{R}_{q+1} - \hat{R}_q)^{p'}|\mathcal{G}_{\tau_q}]$, we conclude by induction that $E[\tau_n^{\frac{p'}{2}}] < \infty$. □

We are now ready to start the embedding procedure. Let us consider $(\mathcal{F}', X) \in \mathcal{W}_n(\mu)$ and $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, as in the last section.

Let R denote $R := \frac{S}{\rho\sqrt{n}}$. To embed both R and X, we will have to slightly perturb the above procedure: For $\epsilon > 0$ we define $\tau_0 = 0$, $\tau_{\frac{1}{2}} := \epsilon$ and $\hat{R}_0 := 0$. There exists a function f_0 such that $[f_0(\sqrt{\epsilon}Z)] = [X_0]$ if $Z \sim \mathcal{N}(0,1)$ and we set $\hat{X}_0 := f_0(B_{\tau_{\frac{1}{2}}} - B_{\tau_0})$. The random vectors (R_0, X_0) and $(\hat{R}_0, \hat{X}_0)$ will thus have the same distribution.

For $q = 0, \dots, n-1$, we then define τ_{q+1}, $\tau_{q+\frac{3}{2}}$, $\hat{R}_{q+1}$ recursively as follows: Let $\mathcal{G}'$ be the natural σ-algebra of $B'_t := B_{t+\tau_{q+\frac{1}{2}}} - B_{\tau_{q+\frac{1}{2}}}$. Define as above

$$\tilde{\theta}(r_0, \dots, r_q, x_0, \dots, x_q) := \theta'_{[R_{q+1}-R_q | R_0=r_0,\dots,R_q=r_q, X_0=x_0,\dots,X_q=x_q]},$$

and then set $\tau_{q+1} := \tau_{q+\frac{1}{2}} + \tilde{\theta}(\hat{R}_0, \dots, \hat{R}_q, \hat{X}_0, \dots, \hat{X}_q)$, $\tau_{q+\frac{3}{2}} := \tau_{q+1} + \epsilon$ and finally $\hat{R}_{q+1} := \hat{R}_q + B_{\tau_{q+1}} - B_{\tau_{q+\frac{1}{2}}}$. It follows from this definition that $\hat{R}_{q+1} - \hat{R}_q$ has the same law conditionally to $(\hat{R}_s, \hat{X}_s)_{s\leqslant q}$ as $R_{q+1} - R_q$ conditionally to $(R_s, X_s)_{s\leqslant q}$, and thus $((\hat{R}_s)_{s\leqslant q+1}, (\hat{X}_s)_{s\leqslant q})$ and $((R_s)_{s\leqslant q+1}, (X_s)_{s\leqslant q})$ have the same law. There exists a function $f_{q+1} : \mathbb{R}^{q+1} \times \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}$ such that, $\forall ((r_s)_{s\leqslant q+1}, (x_s)_{s\leqslant q}) \in \mathbb{R}^{q+1} \times \mathbb{R}^q$, with $Z \sim \mathcal{N}(0,1)$:

$$\bigl[f_{q+1}\bigl((r_s)_{s\leqslant q+1}, (x_s)_{s\leqslant q}, \sqrt{\epsilon}Z\bigr)\bigr] = \bigl[X_{q+1} | (R_s = r_s)_{s\leqslant q+1}, (X_s = x_s)_{s\leqslant q}\bigr].$$

We then set $\hat{X}_{q+1} := f_{q+1}((\hat{R}_s)_{s\leqslant q+1}, (\hat{X}_s)_{s\leqslant q}, B_{\tau_{q+\frac{3}{2}}} - B_{\tau_{q+1}})$, and it follows that $(R_s, X_s)_{s\leqslant q+1}$ and $(\hat{R}_s, \hat{X}_s)_{s\leqslant q+1}$ are equally distributed.

It is convenient to define also τ_{n+1} as $\tau_{n+1} = \tau_{n+\frac{1}{2}} := \tau_n + \epsilon$ and $\hat{X}_{n+1} := f_{n+1}((\hat{R}_s)_{s\leqslant n}, (\hat{X}_s)_{s\leqslant n}, B_{\tau_{n+\frac{1}{2}}} - B_{\tau_n})$, where f_{n+1} is such that, for all $(r_s), (x_s)$:

$$\bigl[f_{q+1}\bigl((r_s)_{s\leqslant n}, (x_s)_{s\leqslant n}, \sqrt{\epsilon}Z\bigr)\bigr] = \bigl[X_{n+1} | (R_s = r_s)_{s\leqslant n}, (X_s = x_s)_{s\leqslant n}\bigr],$$

whenever $Z \sim \mathcal{N}(0,1)$.

The resulting process $(\hat{R}, \hat{X})$ has the same distribution as (R, X). It follows from the above definition that $(\hat{R}_q, \hat{X}_q)$ is $\mathcal{G}_{\tau_q}$-measurable and the law $(\hat{R}_q, \hat{X}_q)$ conditionally to $\mathcal{G}_{\tau_{q-1}}$ is just the law of $(\hat{R}_q, \hat{X}_q)$ conditionally to $(\hat{R}_s, \hat{X}_s)_{s<q}$. Therefore $(\hat{R}, \hat{X})$ is a $\mathcal{G}_{\tau_q}$-martingale. Observe next that

$$B_{\tau_n} = \sum_{q=0}^{n-1} B_{\tau_{q+1}} - B_{\tau_{q+\frac{1}{2}}} + \sum_{q=0}^{n-1} B_{\tau_{q+\frac{1}{2}}} - B_{\tau_q}.$$

Since $B_{\tau_{q+1}} - B_{\tau_{q+\frac{1}{2}}} = \hat{R}_{q+1} - \hat{R}_q$ and $\hat{R}_0 = 0$ we get $B_{\tau_n} - \hat{R}_n$ as a sum of iid $\mathcal{N}(0,\epsilon)$ random variables: $B_{\tau_{q+\frac{1}{2}}} - B_{\tau_q} = B_{\tau_{q+\epsilon}} - B_{\tau_q}$. Therefore $\hat{R}_n - B_{\tau_n} \sim \mathcal{N}(0, n\epsilon)$ and, in particular

$$\|\hat{R}_n - B_{\tau_n}\|_{L^2} = \sqrt{\epsilon \cdot n}. \tag{12}$$

In order to obtain our central limit result for $\hat{R}_n$, we will prove hereafter that τ_n is close to be a constant stopping time, which indicates that B_{τ_n} follows approximately a normal distribution.

Lemma 10.

(1) *For all* $t \in [0,1]$: $E[\tau_{[\![nt]\!]}] = [\![nt]\!](\epsilon + \frac{1}{n})$, *where* $[\![a]\!]$ *is the greatest integer less than or equal to* a.

(2) $E[|\tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}]|] \leqslant \kappa^2 \cdot n^{\frac{2}{p'\wedge 4}-1}$, *where* $\kappa := 2^{\frac{2}{p'\wedge 4}} \frac{4A+\rho}{(c_{p'})^{\frac{1}{p'}}\rho}$.

(3) $\|B_t - B_{\tau_{[\![nt]\!]}}\|_{L^2} \leqslant \kappa \cdot n^{\frac{1}{p'\wedge 4}-\frac{1}{2}} + \sqrt{t - \frac{[\![nt]\!]}{n} + \epsilon[\![nt]\!]}$.

Proof. To prove claim (1), observe that $\tau_{[\![nt]\!]} = \tau_0 + \sum_{q=1}^{[\![nt]\!]}(\tau_q - \tau_{q-1})$. Then, since $S \in S^*_{(\rho,4A+\rho)}(\mathcal{F}')$, we get:

$$\begin{aligned}
E[\tau_q - \tau_{q-1} | \mathcal{G}_{\tau_{q-1}}] &= \epsilon + E[\tau_q - \tau_{q-\frac{1}{2}} | \mathcal{G}_{\tau_{q-1}}] \\
&= \epsilon + E\bigl[(B_{\tau_q} - B_{\tau_{q-\frac{1}{2}}})^2 | \mathcal{G}_{\tau_{q-1}}\bigr] \\
&= \epsilon + E\bigl[(\hat{R}_q - \hat{R}_{q-1})^2 | \hat{R}_k, \hat{X}_k,\ k \leqslant q-1\bigr] \\
&= \epsilon + \frac{1}{\rho^2 n} E\bigl[(S_q - S_{q-1})^2 | S_k, X_k,\ k \leqslant q-1\bigr] \\
&= \epsilon + \frac{1}{n}.
\end{aligned}$$

Therefore, $E[\tau_{[\![nt]\!]}] = [\![nt]\!](\epsilon + \frac{1}{n})$, as announced.

We next prove claim (2). Since $E[\tau_q - \tau_{q-1}|\mathcal{G}_{\tau_{q-1}}] = E[\tau_q - \tau_{q-1}]$, we get

$$\tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}] = \sum_{q=1}^{[\![nt]\!]} \big((\tau_q - \tau_{q-1}) - E[\tau_q - \tau_{q-1}]\big) = Q_{[\![nt]\!]},$$

where

$$Q_s := \sum_{q=1}^{s} \big((\tau_q - \tau_{q-1}) - E[\tau_q - \tau_{q-1}|\mathcal{G}_{\tau_{q-1}}]\big) = \sum_{q=1}^{s} \left(\tau_q - \tau_{q-\frac{1}{2}} - \frac{1}{n}\right).$$

The process $Q = (Q_s)_{s=0,\dots,n}$ is clearly a $\mathcal{G}_{\tau_s}$-martingale starting at 0. Since $p \in [1,2[$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we get $p' > 2$, and so, $\tilde{p} := \frac{\min(p',4)}{2} \in]1,2]$. Therefore

$$\big\| \tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}] \big\|_{L^1} = \| Q_{[\![nt]\!]} \|_{L^1} \leqslant \| Q_n \|_{L^1} \leqslant \| Q_n \|_{L^{\tilde{p}}}. \tag{13}$$

We claim next that

$$E\big[|Q_n|^{\tilde{p}}\big] \leqslant 2^{2-\tilde{p}} \sum_{k=0}^{n-1} E\big[|Q_{k+1} - Q_k|^{\tilde{p}}\big]. \tag{14}$$

This follows at once from a recursive use of the relation:

$$E\big[|x+Y|^{\tilde{p}}\big] \leqslant |x|^{\tilde{p}} + 2^{2-\tilde{p}} E\big[|Y|^{\tilde{p}}\big],$$

that holds for all x in $\mathbb{R}$, whenever Y is a centered random variable: Indeed,

$$|x+Y|^{\tilde{p}} - |x|^{\tilde{p}} = Y \int_0^1 \tilde{p}|x+sY|^{\tilde{p}-1} \operatorname{sgn}(x+sY)\, ds.$$

Thus, since $E[Y] = 0$, we get

$$E\big[|x+Y|^{\tilde{p}}\big] - |x|^{\tilde{p}} = E\left[Y \int_0^1 \tilde{p}\big(|x+sY|^{\tilde{p}-1} \operatorname{sgn}(x+sY) - |x|^{\tilde{p}-1} \operatorname{sgn}(x)\big)\, ds \right].$$

Since $\tilde{p} \leqslant 2$, straightforward computation indicates that, for fixed a, the function $g(x) := ||x+a|^{\tilde{p}-1} \operatorname{sgn}(x+a) - |x|^{\tilde{p}-1} \operatorname{sgn}(x)|$ reaches its maximum at $x = -a/2$, implying $g(x) \leqslant 2^{2-\tilde{p}}|a|^{\tilde{p}-1}$.

So, $E[|x+Y|^{\tilde{p}}] - |x|^{\tilde{p}} \leqslant E[|Y| \int_0^1 2^{2-\tilde{p}} \tilde{p}|sY|^{\tilde{p}-1}\, ds] = 2^{2-\tilde{p}} E[|Y|^{\tilde{p}}]$, as announced and inequality (14) follows.

Next $\| Q_{k+1} - Q_k \|_{L^{\tilde{p}}} = \| \tau_{k+1} - \tau_{k+\frac{1}{2}} - \frac{1}{n} \|_{L^{\tilde{p}}} \leqslant \| \tau_{k+1} - \tau_{k+\frac{1}{2}} \|_{L^{\tilde{p}}} + \frac{1}{n}$. Since $\frac{1}{n} = E[\tau_{k+1} - \tau_{k+\frac{1}{2}}]$, we also have $\frac{1}{n} \leqslant \| \tau_{k+1} - \tau_{k+\frac{1}{2}} \|_{L^{\tilde{p}}}$, and thus

$$\| Q_{k+1} - Q_k \|_{L^{\tilde{p}}} \leqslant 2\| \tau_{k+1} - \tau_{k+\frac{1}{2}} \|_{L^{\tilde{p}}} \leqslant 2\| \tau_{k+1} - \tau_{k+\frac{1}{2}} \|_{L^{\frac{p'}{2}}}.$$

Finally, $\hat{R}_{k+1} - \hat{R}_k = B_{\tau_{k+1}} - B_{\tau_{k+\frac{1}{2}}}$. Therefore, we get with Burkholder–Davis–Gundy inequality, and since $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$:

$$E\big[(\tau_{k+1} - \tau_{k+\frac{1}{2}})^{\frac{p'}{2}}\big] \leqslant \frac{1}{c_{p'}} E\big[|\hat{R}_{k+1} - \hat{R}_k|^{p'}\big] = \frac{1}{c_{p'}\rho^{p'} n^{\frac{p'}{2}}} E\big[|S_{k+1} - S_k|^{p'}\big] \leqslant \frac{(4A+\rho)^{p'}}{c_{p'}\rho^{p'} n^{\frac{p'}{2}}}.$$

So: $E[|Q_{k+1} - Q_k|^{\tilde{p}}] \leqslant 2^{\tilde{p}}(4A+\rho)^{2\tilde{p}}(c_{p'})^{-\frac{2\tilde{p}}{p'}}\rho^{-2\tilde{p}} n^{-\tilde{p}}$, and, with (14), we conclude

$$E\big[|Q_n|^{\tilde{p}}\big] \leqslant 2^2(4A+\rho)^{2\tilde{p}}(c_{p'})^{-\frac{2\tilde{p}}{p'}}\rho^{-2\tilde{p}} n^{1-\tilde{p}}.$$

Therefore, with (13), we get:

$$\big\| \tau_{[\![nt]\!]} - E[\tau_{[\![nt]\!]}] \big\|_{L^1} \leqslant 2^{\frac{2}{\tilde{p}}} \left(\frac{4A+\rho}{(c_{p'})^{\frac{1}{p'}}\rho} \right)^2 n^{\frac{1}{\tilde{p}}-1}$$

and claim (2) is proved.

We next prove claim (3): let $\overline{\tau}$ denote $E[\tau_{[\![nt]\!]}]$. Then

$$\| B_{\tau_{[\![nt]\!]}} - B_t \|_{L^2} \leqslant \| B_{\tau_{[\![nt]\!]}} - B_{\overline{\tau}} \|_{L^2} + \| B_{\overline{\tau}} - B_t \|_{L^2} = \sqrt{\| \tau_{[\![nt]\!]} - \overline{\tau} \|_{L^1}} + \sqrt{|\overline{\tau} - t|}.$$

The first term is bounded by claim (2), and the second one by claim (1). □

13. An upper bound for $\limsup \overline{\mathcal{V}}_n^M(\mu)/\sqrt{n}$

Let us consider $(\mathcal{F}', X) \in \mathcal{W}_n(\mu)$. According to (11), we have:

$$\frac{\mathcal{V}_n^M(\mathcal{F}', X)}{\sqrt{n}} \leqslant \sup_{S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')} E\left[X_{n+1} \cdot \frac{S_n}{\sqrt{n}}\right].$$

For $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, let us define $R_n := \frac{S_n}{\rho\sqrt{n}}$ and let us embed (X, R) in the Brownian filtration, as done in the last section, for $\epsilon > 0$.

Then $E[X_{n+1} \cdot \frac{S_n}{\sqrt{n}}] = \rho \cdot E[X_{n+1} \cdot R_n] = \rho \cdot E[\hat{X}_{n+1} \cdot \hat{R}_n]$.

Claim (3) for $t = 1$ in Lemma 10 yields $\|B_1 - B_{\tau_n}\|_{L^2} \leqslant \kappa \cdot n^{\frac{1}{p' \wedge 4} - \frac{1}{2}} + \sqrt{\epsilon n}$. With (12), we get then $\|\hat{R}_n - B_1\|_{L^2} \leqslant \kappa \cdot n^{\frac{1}{p'\wedge 4} - \frac{1}{2}} + 2\sqrt{\epsilon \cdot n}$.

Therefore, since $\hat{X}_{n+1} \sim \mu$ and $B_1 \sim \mathcal{N}(0, 1)$,

$$\begin{aligned} E[\hat{X}_{n+1} \cdot \hat{R}_n] &\leqslant E[\hat{X}_{n+1} \cdot B_1] + E\big[\hat{X}_{n+1} \cdot (\hat{R}_n - B_1)\big] \\ &\leqslant E[\hat{X}_{n+1} \cdot B_1] + \|\hat{X}_{n+1}\|_{L^2} \cdot \|\hat{R}_n - B_1\|_{L^2} \\ &\leqslant \alpha(\mu) + \|\mu\|_{L^2} \cdot \big(\kappa \cdot n^{\frac{1}{p'\wedge 4} - \frac{1}{2}} + 2\sqrt{\epsilon \cdot n}\big), \end{aligned}$$

where $\alpha(\mu) = E[f_\mu(Z)Z]$ was defined in Theorem 6. Since this holds for all $\epsilon > 0$ and all $S \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}')$, we conclude that for all $(\mathcal{F}', X) \in \mathcal{W}_n(\mu)$:

$$\frac{\mathcal{V}_n^M(\mathcal{F}', X)}{\sqrt{n}} \leqslant \rho \cdot \alpha(\mu) + \rho \cdot \|\mu\|_{L^2} \cdot \kappa \cdot n^{\frac{1}{p'\wedge 4} - \frac{1}{2}}.$$

Since $p' > 2$, and since the constant κ in Lemma 10 is independent of n, we thus have proved:

Theorem 11. *Under the hypotheses of Theorem* 5,

$$\limsup_{n\to\infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} \leqslant \rho \cdot E\big[f_\mu(Z)Z\big].$$

We will prove in the next section that $\rho \cdot \alpha(\mu)$ is the limit of $\frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}}$ as n increases to ∞. To conclude this section, we give here an example to illustrate that p must be strictly less than 2 in hypothesis (ii) of Theorem 5 in order to get the result: Clearly, the function $M[\mu] := \|\mu\|_{L^2}$ satisfies hypothesis (i) of Theorem 5, and would also satisfy hypothesis (ii) with $A = 1$ if $p = 2$ was allowed. For this M, $\rho = 1$. Let then μ be the probability that assigns a weight $1/2$ to $+1$ and -1. Let X^n be the unique martingale of length $n+1$ such that $\forall q = 0, \ldots, n$, $|X^n_q| = \sqrt{\frac{q}{n}}$ and such that $X^n_{n+1} := X^n_n$. In other words, if, for $q < n$, $X^n_q = \sqrt{\frac{q}{n}}$, then X^n_{q+1} jumps to $\sqrt{\frac{q+1}{n}}$ with probability γ or $-\sqrt{\frac{q+1}{n}}$ with probability $1 - \gamma$, where $\gamma := \frac{1}{2}(1 + \sqrt{\frac{q}{q+1}})$, and symmetric jumps are made if $X^n_q = -\sqrt{\frac{q}{n}}$. An easy computation shows that $E[(X^n_{q+1} - X^n_q)^2 | X^n_1, \ldots, X^n_q] = \frac{1}{n}$, and thus, if $\mathcal{F}^n$ denotes the natural filtration of X^n, we get $\mathcal{V}_n^M(\mathcal{F}^n, X^n) = \sqrt{n}$. Since for all pair $(\mathcal{F}', X) \in \mathcal{W}_n(\mu)$, we can write as in (11): $\mathcal{V}_n^M(\mathcal{F}', X) = E[X_{n+1}S_n]$, where $S_n = \sum_{k=1}^n Y_k$, with $E[Y_k^2] = 1$ we get $E[S_n^2] = n$, and due to Cauchy–Schwarz inequality, it comes $\mathcal{V}_n^M(\mathcal{F}', X) \leqslant \|X_{n+1}\|_{L^2}\sqrt{n} = \|\mu\|_{L^2}\sqrt{n} = \sqrt{n}$. Therefore:

$$\sqrt{n} \geqslant \overline{\mathcal{V}}_n^M(\mu) \geqslant \mathcal{V}_n^M\big(\mathcal{F}^n, X^n\big) = \sqrt{n},$$

and thus

$$\lim_{n\to\infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} = 1 > \rho \cdot E\big[f_\mu(Z)Z\big] = E\big[|Z|\big] = \sqrt{\frac{2}{\pi}},$$

since $f_\mu(Z) = \mathbb{1}_{\{Z \geqslant 0\}} - \mathbb{1}_{\{Z<0\}}$.

14. A lower bound for $\liminf \overline{\mathcal{V}}_n^M(\mu)/\sqrt{n}$

Let Y be a random variable in L^4 with $E[Y] = 0$, $E[Y^2] = 1$. We will provide in this section a sequence $(\mathcal{F}^n, X^n) \in \mathcal{W}_n(\mu)$ such that

$$\liminf_{n\to\infty} \frac{\mathcal{V}_n^M(\mathcal{F}^n, X^n)}{\sqrt{n}} \geqslant M[Y] \cdot \alpha(\mu).$$

Using Lemma 9, we can construct, for each n, an increasing sequence $(\tau_q^n)_{q=0,\dots,n}$ of stopping times on the Brownian filtration $\mathcal{G}$ such that $Y_q^n := \sqrt{n}\cdot(B_{\tau_q^n} - B_{\tau_{q-1}^n})$ is an i.i.d. sequence with $[Y_q^n] = [Y]$. Observe in particular that $\tau_q^n - \tau_{q-1}^n$ is also an i.i.d. sequence. We also set $\tau_{n+1}^n := \tau_n^n \vee 1$.

The argument of Lemma 10 can be applied to this sequence of stopping times, replacing ρ by 1, p' by 4, ϵ by 0 and $4A + \rho$ by $\|Y\|_{L^4}$. We obtain in this way:

Lemma 12.

(1) *For all* $t \in [0,1]$: $E[\tau_{[\![nt]\!]}^n] = \frac{[\![nt]\!]}{n}$.
(2) $\|\tau_{[\![nt]\!]}^n - E[\tau_{[\![nt]\!]}^n]\|_{L^2} \leqslant \gamma^2 \cdot n^{-\frac{1}{2}}$, *where* $\gamma := \frac{\|Y\|_{L^4}}{(c_4)^{\frac{1}{4}}}$.
(3) $\forall u \in [0,1]$, $\forall q \geqslant [\![nu]\!]$: $p(\tau_q^n < u) \leqslant \frac{\gamma^4}{n(\frac{q}{n} - u)^2}$.
(4) $\|B_1 - B_{\tau_n^n}\|_{L^2} \leqslant \gamma \cdot n^{-\frac{1}{4}}$.

Proof. By construction of the sequence τ_q^n, $\theta_q^n := \tau_q^n - \tau_{q-1}^n$ is an i.i.d. sequence of random variables and $1 = E[(Y_q^n)^2] = n \cdot E[(B_{\tau_q^n} - B_{\tau_{q-1}^n})^2] = n \cdot E[\theta_q^n]$. Burkholder–Davis–Gundy inequality indicates that

$$c_4 \cdot \mathrm{var}\big[\theta_q^n\big] \leqslant c_4 \cdot E\big[\big(\theta_q^n\big)^2\big] \leqslant E\big[(B_{\tau_q^n} - B_{\tau_{q-1}^n})^4\big] = E\big[Y^4\big]/n^2.$$

Therefore, since $\tau_{[\![nt]\!]}^n = \sum_{q=1}^{[\![nt]\!]} \theta_q^n$, we get $E[\tau_{[\![nt]\!]}^n] = [\![nt]\!]/n$ and

$$\big\|\tau_{[\![nt]\!]}^n - E\big[\tau_{[\![nt]\!]}^n\big]\big\|_{L^2}^2 = \mathrm{var}\big(\tau_{[\![nt]\!]}^n\big) \leqslant \frac{E[Y^4]\cdot[\![nt]\!]}{c_4 \cdot n^2}. \leqslant \frac{E[Y^4]}{c_4 \cdot n}.$$

Claim (3) is just a one-sided Chebichev inequality: Indeed, with $t := q/n$, claims (1) and (2) indicate that $E[\tau_q^n] = q/n$ and $var(\tau_q^n) \leqslant \gamma^4/n$. Therefore: $p(\tau_q^n < u) = p(\tau_q^n - E[\tau_q^n] < u - q/n) \leqslant \frac{var(\tau_q^n)}{(q/n - u)^2}$.

We finally conclude $\|B_{\tau_n^n} - B_1\|_{L^2}^2 = E[|\tau_n^n - 1|] \leqslant \frac{\|Y\|_{L^4}^2}{\sqrt{c_4 \cdot n}}$, and the lemma is proved. □

We define next $\mathcal{F}_q^n := \mathcal{G}_{\tau_q^n}$ and $X_q^n := E[f_\mu(B_1)|\mathcal{F}_q^n]$, for $q = 0,\dots,n+1$. Since $\tau_{n+1}^n \geqslant 1$, we have $X_{n+1}^n = f_\mu(B_1)$, and due to the definition of f_μ, we have $X_{n+1}^n \sim \mu$. Therefore $(\mathcal{F}^n, X^n) \in \mathcal{W}_n(\mu)$.

We will have to compute $\mathcal{V}_n^M(\mathcal{F}^n, X^n)$. To do so, it is convenient to introduce an approximation $\tilde{X}^n$ of X^n. As explained in Section 2, due to the Markov property of the Brownian motion, $\Pi_t^\mu := E[f_\mu(B_1)|\mathcal{G}_t] = f(B_t, t)$ where $f(x,t) := E[f_\mu(x + \sqrt{1-t}\cdot Z)]$ with $Z \sim \mathcal{N}(0,1)$. As a convolution with a normal density, f is twice continuously differentiable on $\mathbb{R} \times [0,1[$, and it further satisfies the heat equation, so that $\Pi_t^\mu = f(0,0) + \int_0^t r_s\,dB_s$, with $r_s = 0$ for $s \geqslant 1$ and $r_s = \frac{\partial}{\partial x} f(B_s, s)$ for $s < 1$.

Let us observe here that $f(x,t)$ is increasing in x at fixed t since so is $f_\mu(x)$, and thus $r_s \geqslant 0$ for all s. Observe also that r_s is continuous on $[0,1[$ and that $X_q^n = f(0,0) + \int_0^{\tau_q^n} r_s\,dB_s$. We will then define $\tilde{X}^n$ by:

$$\tilde{X}_q^n = f(0,0) + \int_0^{\tau_q^n} r_s^n\,dB_s, \tag{15}$$

where $r^n := T_n(r)$ is the image of the process r by the map T_n we now define. Let $\mathcal{H}^2$ be the linear space of $\mathcal{G}$-progressively measurable processes a such that

$$\|a\|_{\mathcal{H}^2}^2 := E\left[\int_0^\infty a_s^2\,ds\right] < \infty.$$

Let also $\mathcal{H}_{[0,1]}^2$ denote the set of $a \in \mathcal{H}^2$ such that $a_s = 0$, for all $s \geqslant 1$. For $a \in \mathcal{H}_{[0,1]}^2$, we define $T_n(a)$ as the simple process

$$T_n(a)_t := \sum_{q=0}^{n-1} n \cdot E\left[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s\,ds \Big| \mathcal{G}_{\tau_q^n}\right] \cdot \mathbb{1}_{[\tau_q^n, \tau_{q+1}^n[}(t).$$

Lemma 13. *T_n is a linear mapping from $\mathcal{H}_{[0,1]}^2$ to $\mathcal{H}^2$, and*

$$\forall a \in \mathcal{H}_{[0,1]}^2: \quad \big\|T_n(a)\big\|_{\mathcal{H}^2} \leqslant \|a\|_{\mathcal{H}^2}.$$

Proof. As a simple process, $T_n(a)$ is progressively measurable and

$$\|T_n(a)\|^2_{\mathcal{H}^2} = E\left[\sum_{q=0}^{n-1}\left(n\cdot E\left[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s\,ds|\mathcal{G}_{\tau^n_q}\right]\right)^2\cdot\left(\tau^n_{q+1}-\tau^n_q\right)\right].$$

Since $Y^n_{q+1} := \sqrt{n}\cdot(B_{\tau^n_{q+1}} - B_{\tau^n_q})$ satisfies $[Y^n_{q+1}|\mathcal{G}_{\tau^n_q}] = [Y]$, we get

$$E\left[\tau^n_{q+1}-\tau^n_q|\mathcal{G}_{\tau^n_q}\right] = E\left[(B_{\tau^n_{q+1}} - B_{\tau^n_q})^2|\mathcal{G}_{\tau^n_q}\right] = \frac{E[Y^2]}{n} = \frac{1}{n}.$$

Furthermore, with Jensens inequality:

$$\left(E\left[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s\,ds|\mathcal{G}_{\tau^n_q}\right]\right)^2 \leqslant E\left[\left(\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s\,ds\right)^2|\mathcal{G}_{\tau^n_q}\right],$$

and by Cauchy–Schwarz inequality: $(\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s\,ds)^2 \leqslant \int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s^2\,ds\cdot\frac{1}{n}$. Therefore

$$\|T_n(a)\|^2_{\mathcal{H}^2} \leqslant E\left[\sum_{q=0}^{n-1} E\left[\int_{\frac{q}{n}}^{\frac{q+1}{n}} a_s^2\,ds|\mathcal{G}_{\tau^n_q}\right]\right] = \|a\|^2_{\mathcal{H}^2},$$

and the lemma is proved. □

Lemma 14. $\forall a\in\mathcal{H}^2_{[0,1]}$: $\lim_{n\to\infty}\|T_n(a)-a\|_{\mathcal{H}^2} = 0$.

Proof. As it follows from the last lemma, the linear maps W_n defined by $W_n(a) := T_n(a) - a$ form an equi-continuous sequence of linear mappings. Therefore, we just have to prove the result for elementary processes a of the form: $a_s := \psi_u\cdot\mathbb{1}_{[u,v[}$, where $u<v<1$ and $\psi_u\in L^\infty(\mathcal{G}_u)$. Indeed, these elementary processes engender a dense subspace of $\mathcal{H}^2_{[0,1]}$. If $\psi_t := E[\psi_u|\mathcal{G}_t]$, the process ψ is a martingale on the Brownian filtration and, as such, has continuous sample paths. It is further uniformly integrable since $\psi_u\in L^\infty(\mathcal{G}_u)$, and with the stopping theorem, we conclude that $E[\psi_u|\mathcal{G}_{\tau^n_q}] = \psi_{\tau^n_q}$. Next:

$$T_n(a) = \psi_{\tau^n_{[\![nu]\!]}}\cdot\left([\![nu]\!]-nu\right)\cdot\mathbb{1}_{[\tau^n_{[\![nu]\!]},\tau^n_{[\![nu]\!]+1}[} + \psi_{\tau^n_{[\![nv]\!]}}\cdot\left(nv-[\![nv]\!]\right)\cdot\mathbb{1}_{[\tau^n_{[\![nv]\!]},\tau^n_{[\![nv]\!]+1}[} + \sum_{q=[\![nu]\!]}^{[\![nv]\!]-1}\psi_{\tau^n_q}\cdot\mathbb{1}_{[\tau^n_q,\tau^n_{q+1}[}.$$

The $\mathcal{H}^2$ norm of the two first terms goes to 0 with n since $\|\psi\|_{L^\infty}<\infty$ and $E[\tau^n_{q+1}-\tau^n_q] = 1/n$.

For all positive numbers x, x', y, y' such that $x\leqslant y$ and $x'\leqslant y'$, it is easy to check that $\int_0^\infty(\mathbb{1}_{[x,y[}-\mathbb{1}_{[x',y'[})^2\,dt = |x-x'|+|y-y'|$ if $[x,y[\cap[x',y'[\neq\emptyset$. Otherwise, $\int_0^\infty(\mathbb{1}_{[x,y[}-\mathbb{1}_{[x',y'[})^2\,dt = y-x+y'-x'$. So, with $D_n := \{\tau^n_{[\![nu]\!]}\geqslant v\}\cup\{\tau^n_{[\![nv]\!]}\leqslant u\}$, we get: $\|a-\psi_u\cdot\mathbb{1}_{[\tau^n_{[\![nu]\!]},\tau^n_{[\![nv]\!]}[}\|^2_{\mathcal{H}^2} = \|\psi_u\cdot\mathbb{1}_{[u,v[}-\psi_u\cdot\mathbb{1}_{[\tau^n_{[\![nu]\!]},\tau^n_{[\![nv]\!]}[}\|^2_{\mathcal{H}^2}$ which is also equal to $E[\mathbb{1}_{D_n^c}\psi_u^2(|u-\tau^n_{[\![nu]\!]}|+|v-\tau^n_{[\![nv]\!]}|)] + E[\mathbb{1}_{D_n}\psi_u^2(\tau^n_{[\![nv]\!]}-\tau^n_{[\![nu]\!]}+v-u)]$. The first expectation converges to 0 as n increases, since $\|\psi_u\|_{L^\infty}<\infty$ and both $\|v-\tau^n_{[\![nv]\!]}\|_{L^1}$ and $\|u-\tau^n_{[\![nu]\!]}\|_{L^1}$ converge to 0, according to Lemma 12. The same lemma indicates that $\|\tau^n_{[\![nt]\!]}\|_{L^2}\leqslant 1+\gamma^2$ and $\psi_u^2(\tau^n_{[\![nv]\!]}-\tau^n_{[\![nu]\!]}+v-u)$ is thus bounded in L^2. With Cauchy–Schwarz inequality, we conclude then that $E[\mathbb{1}_{D_n}\psi_u^2(\tau^n_{[\![nv]\!]}-\tau^n_{[\![nu]\!]}+v-u)]$ converges to 0 as $n\to\infty$, since $p(D_n) = p(\tau^n_{[\![nu]\!]}\geqslant v)+p(\tau^n_{[\![nv]\!]}\leqslant u)$ also converges to 0: $\tau^n_{[\![nu]\!]}$ converges in L^1 to $u<v$, and $\tau^n_{[\![nv]\!]}$ to $v>u$. Therefore $\|a-\psi_u\cdot\mathbb{1}_{[\tau^n_{[\![nu]\!]},\tau^n_{[\![nv]\!]}[}\|^2_{\mathcal{H}^2}$ converges to 0 and, to prove the lemma, it just remains to prove that η_n converges to 0, where

$$\eta_n := \left\|\psi_u\cdot\mathbb{1}_{[\tau^n_{[\![nu]\!]},\tau^n_{[\![nv]\!]}[} - \sum_{q=[\![nu]\!]}^{[\![nv]\!]-1}\psi_{\tau^n_q}\cdot\mathbb{1}_{[\tau^n_q,\tau^n_{q+1}[}\right\|^2_{\mathcal{H}^2}.$$

Now

$$\eta_n = \left\|\sum_{q=[\![nu]\!]}^{[\![nv]\!]-1}(\psi_u-\psi_{\tau^n_q})\cdot\mathbb{1}_{[\tau^n_q,\tau^n_{q+1}[}\right\|^2_{\mathcal{H}^2} = \sum_{q=[\![nu]\!]}^{[\![nv]\!]-1}E\left[(\psi_u-\psi_{\tau^n_q})^2\cdot\left(\tau^n_{q+1}-\tau^n_q\right)\right].$$

It results from the definition of ψ_t that $\psi_t = \psi_u$ if $t \geqslant u$. Therefore, we infer that: $(\psi_u - \psi_{\tau_q^n})^2 \leqslant 4\|\psi_u\|_\infty^2 \mathbb{1}_{\tau_q^n < u}$ and thus

$$\eta_n \leqslant 4\|\psi_u\|_\infty^2 E\left[\sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} \mathbb{1}_{\tau_q^n<u} \cdot \left(\tau_{q+1}^n - \tau_q^n\right)\right] = 4\|\psi_u\|_\infty^2 \sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} p\left(\tau_q^n < u\right)/n,$$

since $\{\tau_q^n < u\} \in \mathcal{G}_{\tau_q^n}$ and $E[(\tau_{q+1}^n - \tau_q^n)|\mathcal{G}_{\tau_q^n}] = 1/n$. Due to claim (3) in Lemma 12 we have: $\sum_{q=[\![nu]\!]}^{[\![nv]\!]-1} p(\tau_q^n < u)/n \leqslant \sum_{q=[\![nu]\!]}^{n} \min(\frac{\gamma^4}{n(\frac{q}{n}-u)^2}, 1)/n$. For q between $[\![nu]\!]$ and $nu + \gamma^2\sqrt{n}$, the *min* appearing in the corresponding term is equal to 1. The sum of these first terms is thus bounded by $(1+\gamma^2\sqrt{n})/n$ which goes to 0 as $n \to \infty$. The sum S_n of the remaining terms is thus $S_n := \frac{\gamma^4}{n}(\sum_{q=[\![nu+\gamma^2\sqrt{n}]\!]}^{n} \frac{1}{(\frac{q}{n}-u)^2}\frac{1}{n})$. The expression in between the parentheses can be viewed as a Riemann sum of $\int_{\frac{[\![nu+\gamma^2\sqrt{n}]\!]-1}{n}}^{1+\frac{1}{n}} \frac{1}{(x-u)^2}\,dx = n(\frac{1}{[\![nu+\gamma^2\sqrt{n}]\!]-1-nu} - \frac{1}{n(1-u)+1}) \leqslant \frac{n}{\gamma^2\sqrt{n}-2}$. The function $\frac{1}{(x-u)^2}$ being decreasing in x on the integration range, the Riemann sum is below the corresponding integral. Multiplying this by a factor $\frac{\gamma^4}{n}$, we get a bound for S_n and we conclude that S_n goes to 0 as $n \to \infty$. The lemma is thus proved. □

We defined $\tilde{X}^n$ in Eq. (15) with $r^n := T_n(r)$. We next take benefit of last lemma to prove that $\tilde{X}^n$ is a good approximation of X^n.

Lemma 15.

(1) $\lim_{n\to\infty} \|\tilde{X}_n^n - X_n^n\|_{L^2} = 0$.
(2) $\lim_{n\to\infty} \frac{|\mathcal{V}_n^M(\mathcal{F}^n, X^n) - \mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)|}{\sqrt{n}} = 0$.

Proof. Since Itô's integral is isometric from $\mathcal{H}^2$ to L^2, we get:

$$\left\|\tilde{X}_n^n - X_n^n\right\|_{L^2} \leqslant \left\|\tilde{X}_{n+1}^n - X_{n+1}^n\right\|_{L^2} = \left\|r - r^n\right\|_{\mathcal{H}^2},$$

and claim (1) then follows from last lemma.

We prove now claim (2). With $\Delta X_{q+1}^n := X_{q+1}^n - X_q^n$ and $\Delta\tilde{X}_{q+1}^n := \tilde{X}_{q+1}^n - \tilde{X}_q^n$, we have, with assumption (ii) in Theorem 5:

$$\begin{aligned}
\left|\mathcal{V}_n^M(\mathcal{F}^n, X^n) - \mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)\right| &= \left|E\left[\sum_{q=0}^{n-1} M\left[\Delta X_{q+1}^n|\mathcal{F}_q^n\right] - M\left[\Delta\tilde{X}_{q+1}^n|\mathcal{F}_q^n\right]\right]\right| \\
&\leqslant E\left[\sum_{q=0}^{n-1}\left|M\left[\Delta X_{q+1}^n|\mathcal{F}_q^n\right] - M\left[\Delta\tilde{X}_{q+1}^n|\mathcal{F}_q^n\right]\right|\right] \\
&\leqslant A \cdot E\left[\sum_{q=0}^{n-1} E\left[\left|\Delta X_{q+1}^n - \Delta\tilde{X}_{q+1}^n\right|^p|\mathcal{F}_q^n\right]^{\frac{1}{p}}\right] \\
&\leqslant A \cdot E\left[\sum_{q=0}^{n-1} E\left[\left|\Delta X_{q+1}^n - \Delta\tilde{X}_{q+1}^n\right|^2|\mathcal{F}_q^n\right]^{\frac{1}{2}}\right].
\end{aligned}$$

Due to Cauchy–Schwarz inequality, we have for all real numbers $x_0, \ldots, x_{n-1}$:

$$\sum_{q=0}^{n-1} x_q \leqslant \sqrt{n} \cdot \sqrt{\sum_{q=0}^{n-1} x_q^2}.$$

Therefore and since $\sqrt{x}$ is concave in x, we get with Jensens inequality:

$$\begin{aligned}
\left|\mathcal{V}_n^M(\mathcal{F}^n, X^n) - \mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)\right| &\leqslant \sqrt{n} \cdot A \cdot E\left[\sqrt{\sum_{q=0}^{n-1} E\left[\left|\Delta X_{q+1}^n - \Delta\tilde{X}_{q+1}^n\right|^2|\mathcal{F}_q^n\right]}\right] \\
&\leqslant \sqrt{n} \cdot A \cdot \sqrt{\sum_{q=0}^{n-1} E\left[\left|\Delta X_{q+1}^n - \Delta\tilde{X}_{q+1}^n\right|^2\right]}
\end{aligned}$$

$$= \sqrt{n} \cdot A \cdot \sqrt{E\big[\big|X_n^n - \tilde{X}_n^n\big|^2\big]}.$$

Claim (2) follows then from claim (1). □

We will next compute $\mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)$. Defining λ_q^n as: $\lambda_q^n := n \cdot E[\int_{\frac{q}{n}}^{\frac{q+1}{n}} r_s\, ds | \mathcal{G}_{\tau_q^n}]$, we have $r_t^n := \sum_{q=0}^{n-1} \lambda_q^n \cdot \mathbb{1}_{[\tau_q^n, \tau_{q+1}^n[}(t)$. Since r is a positive process, we clearly have $\lambda_q^n \geqslant 0$. Next, $\tilde{X}_{q+1}^n - \tilde{X}_q^n = \lambda_q^n \cdot (B_{\tau_{q+1}^n} - B_{\tau_q^n}) = a_q^n \cdot Y_{q+1}^n$, where $a_q^n := \frac{\lambda_q^n}{\sqrt{n}}$. a_q^n is thus positive and $\mathcal{F}_q^n$-measurable. Therefore, since $M[X]$ is 1-homogeneous in X according to assumption (i) in Theorem 5, since $[Y_{q+1}^n | \mathcal{F}_q] = [Y]$, and since $E[Y^2] = 1$, $E[Y] = 0$, we get:

$$\begin{aligned}
\mathcal{V}_n^M\big(\mathcal{F}^n, \tilde{X}^n\big) &= E\left[\sum_{q=0}^{n-1} M\big[\tilde{X}_{q+1}^n - \tilde{X}_q^n | \mathcal{F}_q^n\big]\right] \\
&= E\left[\sum_{q=0}^{n-1} M\big[a_q^n \cdot Y_{q+1}^n | \mathcal{F}_q^n\big]\right] \\
&= E\left[\sum_{q=0}^{n-1} a_q^n \cdot M[Y]\right] \\
&= M[Y] \cdot E\left[\sum_{q=0}^{n-1} a_q^n \cdot \big(Y_{q+1}^n\big)^2\right] \\
&= M[Y] \cdot E\left[\left(\sum_{q=0}^{n-1} a_q^n \cdot Y_{q+1}^n\right) \cdot \left(\sum_{q=0}^{n-1} Y_{q+1}^n\right)\right] \\
&= \sqrt{n} \cdot M[Y] \cdot E\big[\tilde{X}_n^n \cdot B_{\tau_n^n}\big].
\end{aligned}$$

Since $E[B_{\tau_n^n}^2] = 1$, we also have

$$\begin{aligned}
\frac{\mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)}{M[Y] \cdot \sqrt{n}} &\geqslant E\big[X_n^n \cdot B_{\tau_n^n}\big] - \big\|\tilde{X}_n^n - X_n^n\big\|_{L^2} \\
&= E\big[X_{n+1}^n \cdot B_{\tau_n^n}\big] - \big\|\tilde{X}_n^n - X_n^n\big\|_{L^2} \\
&\geqslant E\big[X_{n+1}^n \cdot B_1\big] - \big\|X_{n+1}^n\big\|_{L^2} \cdot \|B_{\tau_n^n} - B_1\|_{L^2} - \big\|\tilde{X}_n^n - X_n^n\big\|_{L^2} \\
&= E\big[f_\mu(B_1) \cdot B_1\big] - \|\mu\|_{L^2} \cdot \|B_{\tau_n^n} - B_1\|_{L^2} - \big\|\tilde{X}_n^n - X_n^n\big\|_{L^2} \\
&= \alpha(\mu) - \|\mu\|_{L^2} \cdot \|B_{\tau_n^n} - B_1\|_{L^2} - \big\|\tilde{X}_n^n - X_n^n\big\|_{L^2}.
\end{aligned}$$

With claim (4) in Lemma 12 and claim (1) in Lemma 15, we conclude then that:

$$\liminf_{n\to\infty} \frac{\mathcal{V}_n^M(\mathcal{F}^n, X^n)}{\sqrt{n}} = \liminf_{n\to\infty} \frac{\mathcal{V}_n^M(\mathcal{F}^n, \tilde{X}^n)}{\sqrt{n}} \geqslant M[Y] \cdot \alpha(\mu).$$

Since $\overline{\mathcal{V}}_n^M(\mu) \geqslant \mathcal{V}_n^M(\mathcal{F}^n, X^n)$, we thus have proved that for all $Y \in L^4$ with $E[Y] = 0$ and $E[Y^2] = 1$:

$$\liminf_{n\to\infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} \geqslant M[Y] \cdot \alpha(\mu).$$

Since $\tilde{\mathcal{D}} := \{\tilde{Y} \in L^4\colon\ E[\tilde{Y}] = 0 \text{ and } E[\tilde{Y}^2] \leqslant 1\}$ is dense for the L^2-norm in $\mathcal{D} := \{\tilde{Y} \in L^2\colon\ E[\tilde{Y}] = 0 \text{ and } E[\tilde{Y}^2] \leqslant 1\}$, and since M is continuous for the L^p-norm and thus for the L^2-norm, we infer that there exists a sequence $\{\tilde{Y}_n\}_{n\in\mathbb{N}} \subset \tilde{\mathcal{D}}$ such that

$$\lim_{n\to\infty} M[\tilde{Y}_n] = \rho := \sup\big\{M[\tilde{Y}]\colon\ \tilde{Y} \in \mathcal{D}\big\} > 0.$$

We may further assume that $M[\tilde{Y}_n] > 0$, so that, since M is 1-homogeneous, we have that $M[Y_n] \geqslant M[\tilde{Y}_n]$, where $Y_n = \frac{\tilde{Y}_n}{\|\tilde{Y}_n\|_{L^2}}$. Since $Y_n \in L^4$ satisfies $E[Y_n] = 0$ and $E[Y_n^2] = 1$, we thus have proved that

$$\liminf_{n\to\infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} \geqslant \lim_{n\to\infty} M[Y] \cdot \alpha(\mu) = \rho \cdot \alpha(\mu).$$

With Theorem 11, we get then

Theorem 16. *Under the hypotheses of Theorem* 5*,*

$$\lim_{n\to\infty} \frac{\overline{\mathcal{V}}_n^M(\mu)}{\sqrt{n}} = \rho\cdot\alpha(\mu).$$

The first part of Theorem 5 is thus proved. The second part will be proved in the next section.

15. Convergence to the continuous martingale of maximal variation

Let B be a standard one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{A}, P, (\mathcal{G}_t)_{t\geqslant 0})$. If $\mu \in \Delta^{1^+}$, the martingale $\Pi_t^\mu := E[f_\mu(B_1)|\mathcal{G}_t]$ is referred to in this paper as the continuous martingales of maximal variation of final distribution μ. This terminology is justified by the next result that clearly implies the second part of Theorem 5.

If $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, we define the continuous time representation $\tilde{X}$ of X as the process $(\tilde{X}_t)_{t\in[0,1]}$ with $\tilde{X}_t := X_{[\![nt]\!]}$, where $[\![a]\!]$ is the greatest integer less or equal to a.

Theorem 17. *Assume that M satisfies the hypotheses* (i) *and* (ii) *of Theorem* 5*, that* $\rho > 0$*, that* $\mu \in \Delta^2$ *and that* $\{(\mathcal{F}^n, X^n)\}_{n\in\mathbb{N}}$ *is a sequence of martingales with for all n* $(\mathcal{F}^n, X^n) \in \mathcal{M}_n(\mu)$*, that asymptotically maximizes the M-variation, i.e.*:

$$\lim_{n\to\infty} \frac{\mathcal{V}_n^M(\mathcal{F}^n, X^n)}{\sqrt{n}} = \rho\cdot\alpha(\mu).$$

Then $\tilde{X}^n$ converges in finite-dimensional distribution to Π^μ: For all finite set $J \subset [0,1]$, $(\tilde{X}_t^n)_{t\in J}$ converges in law to $(\Pi_t^\mu)_{t\in J}$.

Proof. Let $\{(\mathcal{F}^n, X^n)\}_{n\in\mathbb{N}}$ be an asymptotically maximizing sequence. Without loss of generality, we may assume that $\mathcal{F}^n$ contains an adapted system $(U_q)_{q=0,\dots,n}$ of independent uniform random variables, independent of X^n (otherwise $\mathcal{F}^n$ could be widened). Therefore, with (11), there exists $S^n \in \mathcal{S}^*_{(\rho,4A+\rho)}(\mathcal{F}^n)$ such that $\mathcal{V}_n^M(\mathcal{F}^n, X^n) - 1 \leqslant E[X_{n+1}^n \cdot S_n^n]$, and thus

$$\lim_{n\to\infty} \frac{E[X_{n+1}^n \cdot S_n^n]}{\rho\cdot\sqrt{n}} = \alpha(\mu).$$

As in Section 12, for $\epsilon_n > 0$ to be determined later, we may embed (X^n, R^n) in the Brownian filtration $\mathcal{G}$, where $R^n := \frac{S^n}{\rho\cdot\sqrt{n}}$, obtaining thus an increasing sequence $(\tau_q^n)_{q=0,\dots,n+1}$ and a pair $(\hat{X}^n, \hat{R}^n)$ of $\hat{\mathcal{F}}^n$ martingales, where $\hat{\mathcal{F}}_q^n := \mathcal{G}_{\tau_q^n}$ such that (X^n, R^n) and $(\hat{X}^n, \hat{R}^n)$ are equally distributed. We then have

$$E\big[\hat{X}_{n+1}^n B_1\big] \geqslant E\big[\hat{X}_{n+1}^n \hat{R}_n^n\big] - \|\mu\|_{L^2}\cdot\big\|B_1 - \hat{R}_n^n\big\|_{L^2}.$$

Since $E[\hat{X}_{n+1}^n \hat{R}_n^n] = E[X_{n+1}^n R_n^n]$, the first term in the right-hand side of the last inequality converges to $\alpha(\mu)$. Next, according to (12) and claim (3) in Lemma 10 with $t = 1$:

$$\begin{aligned}\big\|B_1 - \hat{R}_n^n\big\|_{L^2} &\leqslant \|B_1 - B_{\tau_n^n}\|_{L^2} + \big\|B_{\tau_n^n} - \hat{R}_n^n\big\|_{L^2}\\ &\leqslant \kappa\cdot n^{\frac{1}{p'\wedge 4}-\frac{1}{2}} + 2\sqrt{\epsilon_n n}.\end{aligned}$$

So if ϵ_n is chosen so as to ensure $n\epsilon_n \to 0$ as $n\to\infty$, we get

$$\lim_{n\to\infty} \frac{E[\hat{X}_{n+1}^n \cdot B_1]}{\rho\cdot\sqrt{n}} = \alpha(\mu).$$

Since $B_1 \sim \mathcal{N}(0,1)$ and $\hat{X}_{n+1}^n \sim \mu$, we may then apply claim (2) in Theorem 6 to infer that $\hat{X}_{n+1}^n$ converges in L^1-norm to $f_\mu(B_1) = \Pi_1^\mu$.

Next observe that $\|\hat{X}_{[\![nt]\!]}^n - \Pi_t^\mu\|_{L^1} \leqslant \|\hat{X}_{[\![nt]\!]}^n - \Pi_{\tau_{[\![nt]\!]}^n}^\mu\|_{L^1} + \|\Pi_{\tau_{[\![nt]\!]}^n}^\mu - \Pi_t^\mu\|_{L^1}$. But

$$\big\|\hat{X}_{[\![nt]\!]}^n - \Pi_{\tau_{[\![nt]\!]}^n}^\mu\big\|_{L^1} = \big\|E\big[\hat{X}_{n+1}^n - \Pi_1^\mu|\mathcal{G}_{\tau_{[\![nt]\!]}^n}\big]\big\|_{L^1} \leqslant \big\|\hat{X}_{n+1}^n - \Pi_1^\mu\big\|_{L^1}.$$

On the other hand, with claims (1) and (2) in Lemma 10 and our choice of ϵ_n we infer that $\tau_{[\![nt]\!]}^n \to t$ in L^1. Since Π^μ is uniformly integrable and, as a martingale on the Brownian filtration, it has continuous sample paths, we then conclude that $\|\Pi_{\tau_{[\![nt]\!]}^n}^\mu - \Pi_t^\mu\|_{L^1} \to 0$ as n increases. Therefore $\hat{X}_{[\![nt]\!]}^n$ converges to Π_t^μ in L^1.

This implies in particular the convergence in finite distribution of the process $(\hat{X}_{[\![nt]\!]}^n)_{t\geqslant 0}$ to Π^μ, and this process has same distribution as $\tilde{X}^n$. □

16. Conclusion

We conclude this paper with a list of open problems and some possible extensions of the model.

(1) It is assumed in the description of Γ_n that actions are observed at each round. This is however an unrealistic hypothesis in the case of a market game, where the complete individual demand functions remain typically private. Only a sample of this demand function will be revealed during the tâtonnement process leading to the market clearing price. What is certainly observed by P2 in an exchange game is the trade. This raises the question whether similar dynamics will appear in a game where only transfers are observed at each round. This however is a game with imperfect monitoring and is thus more difficult to analyze.

(2) It is quite reductive to represent the market by a two player game. It is interesting to note at this respect that, in Γ_n, P1 has no need to observe P2's actions to play optimally, and thus, at each stage, P2 is essentially maximizing his stage payoff, since his action will not influence P1's future behavior. Therefore, we could replace P2 by a succession of players, one per stage, playing against a single informed P1, and we would obtain the same equilibria.
Our result relies crucially on the min max approach and the notion of value that characterizes zero-sum games. Dealing with more general model where N-players interact at each round is more difficult as the notion of optimal strategies has to be replaced by that of Nash equilibrium which has much less structure. There could in particular exist multiple equilibria with different payoffs.

(3) One criticism of the model concerns hypothesis (H5) and the fact that it implies that P2 is forced to trade (see the discussion of (H5) is Section 5). The only way to avoid this no trade paradox is to consider risk proclivity for P2. This however turns also to be a non-zero sum game. Instead of considering one single risk-seeking player 2, we are currently analyzing a game where P1 faces a succession of risk-seeking P2s and we expect the same price dynamics to appear. Indeed, in this setting, the one shot game has a single equilibrium for all μ and P1's payoff at equilibrium is also a function M of the law of $L_{q+1} - L_q$. P1 thus maximizes the M variation. However, due to the risk proclivity, this function M is not 1-homogeneous any more, but it is locally 1-homogeneous around 0. Apparently this is enough to get our asymptotic results: Martingales with maximal variation converge to continuous processes: as n increases, the size of the increments goes to 0 and only the local behavior of M around zero seems to matter.

(4) As mentioned above, analyzing non-zero sum games is much more difficult. As a first step in that direction, we are currently analyzing the market maker game with arbitrageur model mentioned is Section 6.1. This game is non-zero sum and we find in this model a sequence of equilibria in which the price process converges to continuous martingales close to the CMMV class.

(5) Since CMMV is a quite robust class of dynamics, it seems natural to use it in financial econometrics. As a particular local volatility model, it could be used to price derivatives, taking into account the volatility smile, as Dupire's (1997) method. We are currently working on a pricing method with volatility smile using CMMV. As compared to Dupire's one, less information is needed on the volatility manifold to calibrate the model accurately.

Appendix A

In this appendix we aim to prove that (H1′) joint (H2) implies (H1-Δ^∞) as stated in Theorem 24. The trading mechanism considered in this section is thus assumed to satisfy both (H1′) and (H2). We are concerned in this section with measures in Δ^∞ having thus a compact support. Let $K = [\underline{K}, \overline{K}]$ be a compact interval. All measures μ considered in this section are in $\Delta(K)$, the set of probability distribution over K. In Lemma 2, we proved that P1 can guarantee $\overline{\mathcal{V}}_n(\mu)$ in $\Gamma_n(\mu)$. We will prove that player 2 can guarantee the same amount, proving thus that $\overline{\mathcal{V}}_n(\mu)$ is the value of $\Gamma_n(\mu)$, without assuming that this value exists. With the next lemma, we analyze the continuity of $\overline{\mathcal{V}}_n(\mu)$.

Lemma 18.

(1) *If V_1 satisfies* (H2) *in L^p-norm with the Lipschitz constant A, then for all random variables L_1, L_2, for all n:*

$$\left|\overline{\mathcal{V}}_n([L_1]) - \overline{\mathcal{V}}_n([L_2])\right| \leqslant nA\|L_1 - L_2\|_{L^p}.$$

(2) *In particular $\overline{\mathcal{V}}_n$ is continuous for the weak topology on $\Delta(K)$.*

(3) *$\overline{\mathcal{V}}_n(\mu)$ is further concave in μ.*

(4) *Let Φ_n denote the set of continuous functions ϕ on K satisfying for all $\nu \in \Delta(K)$, $E_\nu[\phi(L)] \geqslant \overline{\mathcal{V}}_n(\nu)$, then*

$$\forall \mu \in \Delta(K): \quad \overline{\mathcal{V}}_n(\mu) = \inf_{\phi \in \Phi_n} E_\mu\left[\phi(L)\right].$$

(5) $\forall \epsilon > 0$, *there exists a finite subset $\Phi' \in \Phi_n$ such that*

$$\forall \mu \in \Delta(K): \quad \min_{\phi \in \Phi'} E_\mu\left[\phi(L)\right] \leqslant \overline{\mathcal{V}}_n(\mu) + \epsilon.$$

Proof. (1) Let (L_1, L_2) be a random vector with marginals μ_1, μ_2 in $\Delta(K)$. Let $(\mathcal{F}, X) \in \mathcal{W}_n(\mu_1)$ be such that $\mathcal{V}_n(\mathcal{F}, X) \geqslant \overline{\mathcal{V}}_n(\mu_1) - \epsilon$. Let then Y_{n+1} be a random variable on the same probability space as X and such that the random vectors (X_{n+1}, Y_{n+1}) and (L_1, L_2) are equally distributed. Set then $Y_q := E[Y_{n+1}|\mathcal{F}_q]$. Then $(\mathcal{F}, Y) \in \mathcal{W}_n(\mu_2)$. It follows from Jensen's inequality:

$$\begin{aligned} E\big[\big|V_1[X_{q+1}|\mathcal{F}_q] - V_1[Y_{q+1}|\mathcal{F}_q]\big|\big] &\leqslant A \cdot E\big[\big(E\big[\big|X_{q+1} - Y_{q+1}\big|^p|\mathcal{F}_q\big]\big)^{\frac{1}{p}}\big] \\ &\leqslant A \cdot \big(E\big[E\big[|X_{q+1} - Y_{q+1}|^p|\mathcal{F}_q\big]\big]\big)^{\frac{1}{p}} \\ &= A \cdot \|X_{q+1} - Y_{q+1}\|_{L^p} \\ &\leqslant A \cdot \|X_{n+1} - Y_{n+1}\|_{L^p} \\ &= A \cdot \|L_1 - L_2\|_{L^p}. \end{aligned}$$

Therefore $|\mathcal{V}_n(\mathcal{F}, X) - \mathcal{V}_n(\mathcal{F}, Y)| \leqslant nA\|Y_{n+1} - X_{n+1}\|_{L^p}$, and thus $\overline{\mathcal{V}}_n(\mu_1) - \epsilon \leqslant \mathcal{V}_n(\mathcal{F}, X) \leqslant \mathcal{V}_n(\mathcal{F}, Y) + nA\|L_1 - L_2\|_{L^p} \leqslant \overline{\mathcal{V}}_n(\mu_2) + nA\|L_1 - L_2\|_{L^p}$. Since $\epsilon > 0$ is arbitrary, we get $\overline{\mathcal{V}}_n([L_1]) - \overline{\mathcal{V}}_n([L_2]) \leqslant nA\|L_1 - L_2\|_{L^p}$. Interchanging L_1 and L_2, we get claim (1).

(2) Next, let μ_m be weakly convergent in $\Delta(K)$ to μ. According to Skorokhod's representation theorem, there exists a sequence X_m of μ_m-distributed random variables that converges a.s. to a μ-distributed limit X. Since all variables are K-valued, we conclude with Lebesgue dominated convergence theorem that X_m converges to X in L^p-norm. Claim (1) implies then that $\overline{\mathcal{V}}_n(\mu_m) = \overline{\mathcal{V}}_n([X_m])$ converges to $\overline{\mathcal{V}}_n([X]) = \overline{\mathcal{V}}_n(\mu)$, and $\overline{\mathcal{V}}_n$ is thus weakly continuous as announced.

(3) If $\mu = \lambda_1\mu_1 + \lambda_2\mu_2$, with $\lambda_i \geqslant 0$, $\lambda_1 + \lambda_2 = 1$, let, for $\epsilon > 0$, $(\mathcal{F}^i, X^i) \in \mathcal{W}_n(\mu_i)$ be such that $\mathcal{V}_n(\mathcal{F}^i, X^i) \geqslant \overline{\mathcal{V}}_n(\mu_i) - \epsilon$. Assume that X^1 and X^2 are on two independent probability spaces. On the product space, we can then consider $X := \mathbb{1}_A X^1 + (1 - \mathbb{1}_A)X^2$, where A is an event of probability λ_1 independent of X^1, X^2. Then set $\mathcal{F}_q := \sigma(A, \mathcal{F}^1_q, \mathcal{F}^2_q)$. It follows that $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$. Therefore $\overline{\mathcal{V}}_n(\mu) \geqslant \mathcal{V}_n(\mathcal{F}, X) = \sum_i \lambda_i \mathcal{V}_n(\mathcal{F}^i, X^i) \geqslant \sum_i \lambda_i \overline{\mathcal{V}}_n(\mu_i) - \epsilon$. Letting ϵ go to 0, we get the announced concavity.

(4) This claim follows at once from the fact that a continuous concave function f on a Banach space is the infimum of the set of continuous linear functionals that dominate f. In this case, $\overline{\mathcal{V}}_n$ is a function on the closed subset $\Delta(K)$ of the Banach space of bounded measures on K and $\mathcal{C}(K)$ is the dual of this space.

(5) For $\epsilon > 0$ and $\phi \in \Phi_n$, define C_ϕ as $\{\mu \in \Delta(K) | E_\mu[\phi(L)] - \overline{\mathcal{V}}_n(\mu) < \epsilon\}$. Since $\overline{\mathcal{V}}_n$ is weakly continuous, C_ϕ is an open set for the weak topology of $\Delta(K)$. Claim (4) indicates that $\{C_\phi\}_{\phi \in \Phi_n}$ forms an open covering of the weakly compact set $\Delta(K)$. There exists thus a finite subset Φ' of Φ_n such that $\{C_\phi\}_{\phi \in \Phi'}$ is subcovering $\Delta(K)$. This clearly implies our claim. □

Lemma 19.

(1) *Let $\mathcal{F}$ be a σ-algebra on a probability space $(\Omega, \mathcal{A}, P)$ and X a random variable. Then $E[\overline{\mathcal{V}}_n[X|\mathcal{F}]] = \inf_{\phi_. \in F} E[\phi_\omega(X(\omega))]$, where F is the set of $\mathcal{F}$-measurable maps $\phi_. : \Omega \to \Phi_n$ taking finitely many values.*
(2) *If $\mathcal{F}_1 \subset \mathcal{F}_2$, then $E[\overline{\mathcal{V}}_n[X|\mathcal{F}_2]] \leqslant E[\overline{\mathcal{V}}_n[X|\mathcal{F}_1]]$.*

Proof. (1) Let $\phi_.$ be in F and let $\phi^1, \dots, \phi^R$ be the list of the values taken by $\phi_.$. Then, if D_r denotes the event $\phi_\omega = \phi^r$, we get $\phi_\omega = \sum_r \mathbb{1}_{D_r}(\omega)\phi^r$. Therefore, since the sets D_r form a partition of Ω, and since $\phi^r \in \Phi_n$, we have: $E[\phi_\omega(X(\omega))|\mathcal{F}] = \sum_r \mathbb{1}_{D_r}(\omega)E[\phi^r(X(\omega))|\mathcal{F}] \geqslant \overline{\mathcal{V}}_n([X|\mathcal{F}])$. On the other hand, for $\epsilon > 0$, let Φ' as in claim (5) of Lemma 18. Let $\phi^1, \dots, \phi^R$ be an enumeration of Φ'. Then, if $r^*(\omega)$ denotes the smallest r that minimizes $E[\phi^r(X)|\mathcal{F}](\omega)$, $r^*(\omega)$ is $\mathcal{F}$ measurable and we clearly get with $\phi_\omega := \phi^{r^*(\omega)}$: $E[\phi_\omega(X(\omega))] = E[\min_r\{E[\phi^r(X)|\mathcal{F}]\}] \leqslant E[\overline{\mathcal{V}}_n([X|\mathcal{F}])] + \epsilon$. Letting ϵ go to 0, we get claim (1).

(2) Let F_i the set corresponding to $\mathcal{F}_i$ in claim (1). Then $F_1 \subset F_2$ and thus $\inf_{\phi_. \in F_2} E[\phi_\omega(X(\omega))] \leqslant \inf_{\phi_. \in F_1} E[\phi_\omega(X(\omega))]$, and the result follows. □

Throughout this paper, $\overline{\mathcal{V}}_n(\mu)$, as defined in (6), has been considered as a problem of maximization of $\mathcal{V}_n(\mathcal{F}, X)$ over a martingale space $\mathcal{W}_n(\mu)$. The next lemma will allow us to view $\overline{\mathcal{V}}_n(\mu)$ as a maximization problem over a measure space. Let Δ_n^{mart} be the subset of $\rho \in \Delta(K^{n+1})$ such that if $(L_1, \dots, L_{n+1})$ is ρ-distributed then $\forall q$: $E_\rho[L_{q+1}|L_{\leqslant q}] = L_q$, where $L_{\leqslant q}$ is a notation for $(L_1, \dots, L_q)$. The set of $\rho \in \Delta(K^{n+1})$ that further satisfy $L_{n+1} \sim \mu$ will be denoted $\Delta_n^{\text{mart}}(\mu)$. This last set is thus the set of laws of martingales in $\mathcal{W}_n(\mu)$. Clearly, $\rho \in \Delta(K^{n+1})$ belongs to Δ_n^{mart} if and only if for all continuous function f: $E_\rho[f(L_{\leqslant q})(L_{q+1} - L_q)] = 0$, and it belongs to $\Delta_n^{\text{mart}}(\mu)$ if furthermore $E_\rho[f(L_{n+1})] = E_\mu[f(L)]$. Since all these conditions are linear continuous in ρ, Δ_n^{mart} and $\Delta_n^{\text{mart}}(\mu)$ are closed convex subsets of the weakly compact space $\Delta(K^{n+1})$.

Lemma 20.

(1) $\forall(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$: $\mathcal{V}_n(X, \mathcal{F}) \leqslant \sum_{q=1}^n E[V_1[X_q|X_{<q}]]$.
(2) $\forall\mu \in \Delta(K)$: $\overline{\mathcal{V}}_n(\mu) = \sup_{\rho \in \Delta_n^{\text{mart}}(\mu)} \sum_{q=1}^n E_\rho[V_1([L_q|L_{<q}]_\rho)]$, *where $[L_q|L_{\leqslant q}]_\rho$ is the conditional law of L_q given $L_{<q}$ induced by ρ.*

(3) $\forall\mu \in \Delta(K)$: $\overline{\mathcal{V}}_n(\mu) = \sup_{\rho\in\Delta_1^{\mathrm{mart}}(\mu)} V_1([L_1]_\rho) + E_\rho[\overline{\mathcal{V}}_{n-1}([L_2|L_1]_\rho)]$.

(4) *The map $\rho \to E_\rho[\overline{\mathcal{V}}_n([L_2|L_1]_\rho)]$ is concave in ρ and weakly upper semicontinuous on Δ_1^{mart}.*

Proof. (1) For $(\mathcal{F}, X) \in \mathcal{W}_n(\mu)$, set $\mathcal{F}'_q := \sigma(X_{\leqslant q})$. Since X_q is $\mathcal{F}_q$-measurable, we get $\mathcal{F}'_q \subset \mathcal{F}_q$. Therefore, with claim (2) Lemma 19,

$$\mathcal{V}_n(X,\mathcal{F}) = \sum_{q=1}^{n} E\big[V_1[X_q|\mathcal{F}_{q-1}]\big] \leqslant \sum_{q=1}^{n} E\big[V_1\big[X_q|\mathcal{F}'_{q-1}\big]\big],$$

and claim (1) is proved.

(2) If ρ denotes the law of X, then $\rho \in \Delta_n^{\mathrm{mart}}(\mu)$ and the coordinate process $(L_{\leqslant n+1})$ on the probability space $(K^{n+1}, \mathcal{B}_{K^{n+1}}, \rho)$ has the same law as X. If $\mathcal{G}_q := \sigma(L_{\leqslant q})$, we get thus $\sum_{q=1}^n E[V_1[X_q|\mathcal{F}'_{q-1}]] = \sum_{q=1}^n E_\rho[V_1[L_q|\mathcal{G}_{q-1}]] = \mathcal{V}_n(\mathcal{G}, L) \leqslant \overline{\mathcal{V}}_n(\mu)$, and thus $\overline{\mathcal{V}}_n(\mu) = \sup_{\rho\in\Delta_n^{\mathrm{mart}}(\mu)} \sum_{q=1}^n E_\rho[V_1[L_q|L_{<q}]]$.

(3) Observe that, conditionally to L_1, $L_{>1}$ is a martingale of length n, with final distribution $[L_{n+1}|L_1]$. Therefore, for all ρ : $E_\rho[\sum_{q=2}^n V_1[L_q|L_{\leqslant q}]|L_1] \leqslant \overline{\mathcal{V}}_{n-1}([L_{n+1}|L_1])$. Thus, $\overline{\mathcal{V}}_n(\mu) \leqslant \sup_{\rho\in\Delta_n^{\mathrm{mart}}(\mu)} V_1([L_1]_\rho) + E_\rho[\overline{\mathcal{V}}_{n-1}([L_{n+1}|L_1])]$. Conversely, let ρ be ϵ-optimal in the right-hand side of this formula. Let then ρ_{L_1} denote the law of an ϵ-optimal martingale in this in $\overline{\mathcal{V}}_{n-1}([L_{n+1}|L_1])$, then selecting L_1 with ρ and $L_{>1}$ with ρ_{L_1} gives a martingale that satisfies $\sum_{q=1}^n E[V_1[L_q|L_{<q}]] \geqslant \sup_{\rho\in\Delta_n^{\mathrm{mart}}(\mu)} V_1([L_1]_\rho) + E_\rho[\overline{\mathcal{V}}_{n-1}([L_{n+1}|L_1])] - 2\epsilon$.

(4) According to claim (1) in Lemma 19: $E[\overline{\mathcal{V}}_{n-1}[L_2|L_1]] = \inf_{\phi_.\in F} E[\phi_{L_1}(L_2)]$, where F is the set of $\mathcal{F}$-measurable maps $\phi_. : K \to \Phi_{n-1}$ taking finitely many values. If $\phi^1, \dots, \phi^R \in \Phi_{n-1}$ are the possible values taken by such a map $\phi_. \in F$, then $\phi_{L_1} = \sum_{r=1}^R \mathbb{1}_{D_r}(L_1)\phi^r$, D_r is the measurable set of $L \in K$ where $\phi_L = \phi^r$. The map $L \to (\mathbb{1}_{D_r}(L))_{r=1}^R$ is a measurable map from K to the R-dimensional simplex Δ_R and is thus the limit in L^1 of a sequence θ^m in $\mathcal{C}$, where $\mathcal{C}$ is the set of continuous functions $\theta : K \to \Delta_R$. For such a function θ, we will denote $\phi_{\theta(L)} := \sum_{r=1}^R \theta_r(L_1)\phi^r$. We get thus $E[\phi_{L_1}(L_2)] = \lim_{m\to\infty} E[\phi_{\theta^m(L_1)}(L_2)]$. Since Φ_{n-1} is a convex set, $\forall L$: $\phi_{\theta(L)} \in \Phi_{n-1}$, and thus $E[\phi_{\theta(L_1)}(L_2)|L_1] \geqslant \overline{\mathcal{V}}_{n-1}[L_2|L_1]$, implying $E[\phi_{\theta(L_1)}(L_2)] \geqslant E[\overline{\mathcal{V}}_{n-1}[L_2|L_1]]$. It follows that

$$E\big[\overline{\mathcal{V}}_{n-1}[L_2|L_1]\big] = \inf_{\phi_.\in F} \inf_{\theta\in\mathcal{C}} E\big[\phi_{\theta(L_1)}(L_2)\big].$$

Since $\phi_{\theta(L_1)}(L_2)$ is continuous in the pair (L_1, L_2), the map $\rho \to E_\rho[\phi_{\theta(L_1)}(L_2)]$ is linear weakly continuous and $\rho \to E_\rho[\overline{\mathcal{V}}_{n-1}[L_2|L_1]]$, as an infimum of continuous linear maps, is concave weakly u.s.c. □

The three previous lemma were dealing with functional properties of the function $\overline{\mathcal{V}}_n$. We will now focus on the game $\Gamma_n(\mu)$ and prove that P2 can guarantee $\overline{\mathcal{V}}_n$ in this game. We now will use hypothesis (H1′) that for all $\mu \in \Delta(K)$, the game $\Gamma_1(\mu)$ has a value.

For an admissible strategy τ in $\Gamma_1(\mu)$, the function $\phi_\tau(L) := \sup_i A_{i\tau}L + B_{i\tau}$ involved in formula (2) is finite on the support of μ, but could take infinite values outside of this support. The next lemma indicates that we may restrict our analysis to the set $\mathcal{T}_K$ of strategies τ such that ϕ_τ is finite and thus continuous on K.

Lemma 21. *For all $\mu \in \Delta(K)$: $V_1(\mu) = \inf_{\tau\in\mathcal{T}_K} E_\mu[\phi_\tau(L)]$.*

Proof. Let L_1 be a μ distributed random variable with values in the compact interval $K := [\underline{K}, \overline{K}]$. Consider a random variable L_2^n having the following distribution conditionally to L_1: $L_2^n = L_1$ with probability $1 - 1/n$, and otherwise, with probability $1/n$, L_2^n jumps to either $\overline{K}$ or $\underline{K}$, with weights selected so as $E[L_2^n|L1] = L_1$. If μ is not a Dirac measure on $\bar{L}$ or $\underline{L}$, then for all n, the law μ^n of L_2^n gives a strictly positive probability to both $\bar{L}$ and $\underline{L}$. Therefore any admissible strategy τ in $\Gamma_1(\mu^n)$ must be such that $\phi_\tau(\bar{L}) < \infty$ and $\phi_\tau(\underline{L}) < \infty$. Due to convexity of ϕ_τ, we infer that ϕ_τ is finite on K and τ belongs thus to $\mathcal{T}_K$. Since $\|L_2^n - L_1\|_{L^p}$ goes to 0 with n, we conclude with (H2) that for n high enough $|V_1(\mu) - V_1(\mu_n)|$ will be smaller than an arbitrarily fixed $\epsilon > 0$. Let τ be an ϵ-optimal strategy in $\Gamma_1(\mu_n)$. Then $V_1(\mu) \leqslant E[\phi_\tau(L_1)] = E[\phi_\tau(E[L_2^n|L1])] \leqslant E[\phi_\tau(L_2^n)] \leqslant V_1(\mu_n) + \epsilon \leqslant V_1(\mu) + 2\epsilon$. Since $\tau \in \mathcal{T}_K$ and ϵ is arbitrarily small, the lemma follows for μ. Note that the result can be proved for Dirac measures on $\bar{L}$ or $\underline{K}$ by extending the interval K. □

Lemma 22. *There exists a countable subset $\mathcal{T}_1$ of $\mathcal{T}_K$ such that, if $\phi \in \Phi_1$ then, for all $\epsilon > 0$, there exists $\tau \in \mathcal{T}_1$ such that $\forall L \in K$: $\phi(L) + \epsilon \geqslant \phi_\tau(L)$.*

Proof. Our hypothesis on ϕ indicates that

$$0 \geqslant \sup_{\mu\in\Delta(K)} V_1(\mu) - E_\mu\big[\phi(L)\big] = \sup_{\mu\in\Delta(K)} \inf_{\tau\in\mathcal{T}_K} E_\mu\big[\phi_\tau(L) - \phi(L)\big].$$

Since $E_\mu[\phi_\tau(L)] = \sup_{i(.)} E_\mu[A_{i(L),\tau}L + B_{i(L),\tau}L]$, we conclude that $\tau \to E_\mu[\phi_\tau(L)]$ is convex, as supremum of linear functionals. Since ϕ_τ and ϕ are continuous, the map $\mu \to E_\mu[\phi_\tau(L) - \phi(L)]$ is linear weakly continuous. Since $\Delta(K)$ is weakly compact, we conclude with Proposition 1.8 in Mertens et al. (1994) that inf and sup comute in the previous formula and thus:

$$0 \geqslant \inf_{\tau\in\mathcal{T}_K}\max_{\mu\in\Delta(K)} E_\mu\big[\phi_\tau(L)-\phi(L)\big].$$

If τ in and ϵ-optimal strategy in this inf sup, we get $\forall\mu$: $\epsilon \geqslant E_\mu[\phi_\tau(L)-\phi(L)]$, and in particular $\forall L$: $\epsilon \geqslant \phi_\tau(L)-\phi(L)$.

We thus have proved that $\forall\phi\in\Phi_1, \forall\epsilon>0$, there exists $\tau\in\mathcal{T}_K$ such that $\phi+\epsilon\geqslant\phi_\tau$.

Let E be a countable dense subset in the separable space $(\mathcal{C}(K), \|.\|_\infty)$. For all $f\in E\cap\Phi_1, \forall n\in\mathbb{N}$, let $\tau_{f,n}\in\mathcal{T}_K$ be such that $f+1/n\geqslant\phi_{\tau_{f,n}}$. The countable set $\mathcal{T}_1$ of $\{\tau_{f,n}|f\in E\cap\Phi_1, n\in\mathbb{N}\}$ will have the required property. Indeed, if $\phi\in\Phi_1$, for all $\epsilon>0$, there exists $f\in E$ such that $\phi+\epsilon/2\geqslant f\geqslant\phi$. In particular $f\in\Phi_1$. If $n\geqslant 2/\epsilon$, then $\phi_{\tau_{f,n}}\leqslant f+1/n\leqslant\phi+\epsilon$. □

We next prove recursively a similar property for Γ_n:

Lemma 23. *There exists a countable set $\mathcal{T}_n$ of* P2*'s strategies in Γ_n such that, if $\phi\in\Phi_n$ then, for all $\epsilon>0$, there exists a strategy τ in $\mathcal{T}_n$ such that $\forall L\in K$: $\phi(L)+\epsilon\geqslant\phi^n_\tau(L)$, where ϕ^n_τ was defined in* (2)*.*

Proof. According to the previous lemma, the result holds for $n=1$. Assume next it holds for $n-1$. We argue that it will also hold for n. Indeed, $\phi\in\Phi_n$ implies that $\forall\mu\in\Delta(K)$: $E_\mu[\phi(L)]\geqslant\overline{\mathcal{V}}_n(\mu)$. Therefore $0\geqslant\sup_\mu\overline{\mathcal{V}}_n(\mu)-E_\mu[\phi(L)]$. According to claim (3) in Lemma 20, we get thus

$$0\geqslant\sup_{\mu}\sup_{\rho\in\Delta_1^{\text{mart}}(\mu)} V_1\big([L_1]_\rho\big)+E_\rho\big[\overline{\mathcal{V}}_{n-1}\big([L_2|L_1]_\rho\big)\big]-E_\rho\big[\phi(L_2)\big].$$

Since $\Delta_1^{\text{mart}}=\bigcup_\mu\Delta_1^{\text{mart}}(\mu)$, we get with Lemma 21:

$$0\geqslant\sup_{\rho\in\Delta_1^{\text{mart}}}\inf_{\tau\in\mathcal{T}_K} E_\rho\big[\phi_\tau(L_1)\big]+E_\rho\big[\overline{\mathcal{V}}_{n-1}\big([L_2|L_1]_\rho\big)\big]-E_\rho\big[\phi(L_2)\big].$$

The payoff in this sup inf is finite since all the ϕ_τ and ϕ are continuous on K and thus bounded. It is further concave weakly u.s.c. in ρ as it results from claim (4) in Lemma 20. Δ_1^{mart} is weakly compact as a closed subset of $\Delta(K^2)$. On the other hand the map $\tau\to E_\rho[\phi_\tau(L_1)]$ is convex. We conclude with Proposition 1.8 in Mertens et al. (1994) that inf and sup comute in the previous formula and thus: $0\geqslant\inf_{\tau\in\mathcal{T}_K}\sup_{\rho\in\Delta_1^{\text{mart}}}E_\rho[\phi_\tau(L_1)]+E_\rho[\overline{\mathcal{V}}_{n-1}([L_2|L_1]_\rho)]-E_\rho[\phi(L_2)]$.

Let then $\tau^*\in\mathcal{T}_K$ be an $\epsilon/2$-optimal strategy in this inf sup. We get thus for all $\rho\in\Delta_1^{\text{mart}}$: $\epsilon/2\geqslant E_\rho[\phi_{\tau^*}(L_1)]+E_\rho[\overline{\mathcal{V}}_{n-1}([L_2|L_1]_\rho)]-E_\rho[\phi(L_2)]$. For all $\mu\in\Delta(K)$, if ρ is the law of a vector (L_1,L_2) with $L_2\sim\mu$ and $L_1=E_\mu[L_2]$, then $\rho\in\Delta_1^{\text{mart}}$ and the last formula yields: $\epsilon/2\geqslant\phi_{\tau^*}(E_\mu[L_2])+\overline{\mathcal{V}}_{n-1}(\mu)-\phi(E_\mu[L_2])$ and in particular, as it results from the definition of ϕ_{τ^*}, for all action i of P1: $\epsilon/2\geqslant A_{i\tau^*}E_\mu[L_2]+B_{i\tau^*}+\overline{\mathcal{V}}_{n-1}(\mu)-\phi(E_\mu[L_2])$. In other words, the function $\phi_i(L):=\epsilon/2+\phi(L)-A_{i,\tau^*}L-B_{i\tau^*}$ satisfies $\forall\mu\in\Delta(K)$: $E_\mu[\phi_i(L)]\geqslant\overline{\mathcal{V}}_{n-1}(\mu)$ and thus belongs to Φ_{n-1}. According to our hypothesis that the lemma holds for $n-1$, there must be a strategy τ in $\mathcal{T}_{n-1}$ such that $\epsilon/2+\phi_i\geqslant\phi^{n-1}_\tau$. Let $\tau(i)$ denote the first such τ in a given enumeration of $\mathcal{T}_{n-1}$. The function $\tau(i)$ will then be measurable in i.

Consider then the following strategy $\overline{\tau}$ of P2 in Γ_n: at the first stage he plays $\tau_1:=\tau^*$ and starting from the second stage on, he plays according to $\tau(i_1)$ if P1's first move was i_1.

For this strategy,

$$\begin{aligned}
\phi^n_{\overline{\tau}}(L) &= \sup_{i_1,\dots i_n} A_{i_1\tau^*}L+B_{i_1\tau^*}+L\left(\sum_{q=2}^n A_{i_q,\tau_{q-1}(i_1)}\right)+\left(\sum_{q=2}^n B_{i_q,\tau_{q-1}(i_1)}\right)\\
&= \sup_{i_1} A_{i_1\tau^*}L+B_{i_1\tau^*}+\phi^{n-1}_{\tau(i_1)}(L)\\
&\leqslant \sup_{i_1} A_{i_1\tau^*}L+B_{i_1\tau^*}+\phi_{i_1}(L)+\epsilon/2\\
&= \phi_{\tau^*}(L)+\epsilon.
\end{aligned}$$

We thus have proved that for all $\phi\in\Phi_n$, $\forall\epsilon>0$, there exists a strategy $\overline{\tau}$ such that $\epsilon+\phi\geqslant\phi^n_{\overline{\tau}}$. The construction of $\mathcal{T}_n$ is then similar to that of $\mathcal{T}_1$ in the previous lemma. □

Theorem 24.

(1) *If the trading mechanism satisfies* (H1′) *and* (H2) *then for all $\mu\in\Delta(\infty)$, $\overline{\mathcal{V}}_n(\mu)$ is the value of $\Gamma_n(\mu)$. The mechanism satisfies thus to* (H1-Δ^∞)*.*
(2) *If the mechanism further satisfies to* (H2′)*, then the above assertion holds for all $\mu\in\Delta^2$, and* (H1-Δ^2) *is satisfied.*

Proof. (1) Any $\mu \in \Delta^\infty$ has a compact support, and the previous results apply. In particular, with claim (4) in Lemma 18, for all $\epsilon > 0$, there exists $\phi \in \Phi_n$ such that $\epsilon + \overline{\mathcal{V}}_n(\mu) \geqslant E_\mu[\phi]$. According to Lemma 23, P2 has a strategy τ in $\Gamma_n(\mu)$ such that $\phi + \epsilon \geqslant \phi^n_\tau$. The maximal amount P1 can get if P2 uses this strategy is $E_\mu[\phi^n_\tau(L)] \leqslant E_\mu[\phi(L)] + \epsilon \leqslant \overline{\mathcal{V}}_n(\mu) + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that P2 can guarantee $\overline{\mathcal{V}}_n(\mu)$ in $\Gamma_n(\mu)$. The fact that P1 can guarantee the same amount was proved in Lemma 2.

(2) If $\forall i, j$: $|A_{i,j}| \leqslant A$, as stated in (H2′), then for all admissible strategy τ in Γ_n, the function $\phi^n_\tau(L)$ defined in (2) will be Lipschitz in L with constant nA. Let then $\mu \in \Delta^2$ and let L be a μ-distributed random variable. There exists a sequence L_m of random variables in L^∞ that converges to L in L^1. Let then τ_m be a $1/m$-optimal strategy of P2 in $\Gamma_n([Y_m])$. Then, since $E[\phi^n_{\tau_m}(L)] \leqslant E[\phi^n_{\tau_m}(L_m)] + nA\|L - L_m\|_{L^1} \leqslant \overline{\mathcal{V}}_n([L_m]) + 1/m + nA\|L - L_m\|_{L^1} \leqslant \overline{\mathcal{V}}_n([L]) + 2nA\|Y - Y_m\|_{L^1} + 1/m$. Since the right-hand side converges to $\overline{\mathcal{V}}_n([Y])$ as $m \to \infty$, we infer that P2 can guarantee $\overline{\mathcal{V}}_n(\mu)$ in $\Gamma(\mu)$, for all $\mu \in \Delta^2$. □

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