The theorems of the formal system MIU

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The theorems of the formal system MIU

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Abstract

We show that, in the MIU formal system introduced by Hofstadter in [Hof79], every word $\mathtt{m}x$ where x is a sequence of u's and i's having a number of i′'s which is not a multiple of 3, is a theorem. We give an algorithm that outputs a standard proof of every suchn word.

1 The MIU formal system

The MIU formal system was introduced by Hofstadter ([Hof79]) as a simple example of an axiomatic formal system. It is defined as follows

1. Alphabet: $\Sigma = \{\mathtt{m}, \mathtt{i}, \mathtt{u}\}$
2. Axiom: mi
3. Inference rules
   (a) $x\mathrm{i} \to x\mathrm{iu}$
   (b) $\mathrm{m}x \to \mathrm{m}xx$
   (c) $x\mathrm{iii}y \to x\mathrm{u}y$

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(d) $x\mathrm{uu}y \to xy$

where x, $y \in \{\mathtt{m}, \mathtt{i}, \mathtt{u}\}^\star$.

Notation. "$\vdash x$" means that x is a theorem of MIU. "$x \to y$" means that y results from x by the application of an inference rule. Clearly $\vdash x$ iff $\mathtt{mi} \to^\star x$

2 The theorems of MIU

The following result is from [Hof79].

Theorem 1 *A theorem of MIU has the form* $\mathtt{m}x$ *where* $x \in \{\mathtt{i}, \mathtt{u}\}^\star$ *and the number of* $\mathtt{i}$*'s in* x *is not a multiple of 3.*

Proof. The only axiom, $\mathtt{mi}$, satisfies the property. If $x \to y$ and x has that property then so does y. This is obviously true for rules 3a, 3d (which do not change the number of $\mathtt{i}$'s) and 3c (which reduces the the number of $\mathtt{i}$'s by 3). Consider now rule 3b. If the number of $\mathtt{i}$'s in $\mathtt{m}x$ has the form $\dot{3}+1$ (respectively $\dot{3}+2$), the number of $\mathtt{i}$'s in $\mathtt{m}xx$ has the form $\dot{3}+2$ (respectively $\dot{3}+1$). $\diamond$

We now prove that the converse is also true. Let us begin with the following lemma.

Lemma 1 *Suppose that* $x \in \{\mathtt{i}, \mathtt{u}\}^\star$ *has* n_i $\mathtt{i}$*'s and* n_u $\mathtt{u}$*'s. Then, from the sequence of* $\mathtt{i}^l$*, where* $l = 3n_u + n_i + 3k$ *and* k *is a non-negative integer, we can derive* x.

Proof. We present an algorithm for the derivation.
Repeat one of the following steps until both x and i^l have been completed scanned (here "$x \leftarrow E$" represents the assignment of E to variable x).

1. If x has the form $\mathtt{i}x'$, advance one symbol ($\mathtt{i}$) in both x and i^m, that is, make $x \leftarrow x'$ and $m \leftarrow m-1$.
2. If x has the form $\mathtt{u}x'$, advance the symbol $\mathtt{u}$ in x and three $\mathtt{i}$'s in i^m (using rule 3c).
3. If x is the empty word, m must be a multiple of 3. Select the last sequence $\mathtt{iii}$ and use successively the rules 3a, 3c and 3d so the the number of $\mathtt{i}$'s is reduced by 3.

$\diamond$

Theorem 2 *If the number of* $\mathtt{i}$*'s in* $x \in \{\mathtt{i}, \mathtt{u}\}^\star$ *is not a multiple of 3, then* $\mathtt{m}x$ *is a theorem of MIU.*

Proof. Let $c = 3n_u + n_i$ where n_u and n_i denote respectively the number of $\mathtt{i}$'s and $\mathtt{u}$'s in x. By Lemma 1 it is enough to prove that

$$\vdash \mathtt{mi}^{c+3k}$$

where k is a non-negative integer. We show that, if c is not a multiple of 3, this is always possible.

The only theorem with length 2 is `mi` and its proof is just the line `mi`. We assume now that $c \geq 2$. From `mi`, using n times rule 3b, we get $\mathtt{mi}^{2^n}$. It is easy to see that

$$\begin{aligned} 2^n &\equiv 2 \pmod 3 \text{ for } n = 1, 3, 5, \ldots \\ 2^n &\equiv 1 \pmod 3 \text{ for } n = 2, 4, 6, \ldots \end{aligned}$$

If $c \equiv 2 \pmod 3$ (respectively $c \equiv 1 \pmod 3$), we select the minimum odd (respectively even) value of n such that $2^n = c + 3k$ with $k \geq 0$. $\diamond$

The proof of Theorem 2 shows that, whenever x has no `m`'s and a number of `i`'s which is not a multiple of 3, the algorithm in Figure 2 finds a MIU-proof of $\mathtt{m}x$ which we call the *standard proof.*

Example 1 *The standard proof of* `muiuuu` *is shown in Figure 2.*

In [Mat97] the length of minimum proofs are studied in some detail. In particularly, if n is the length of the theorem (in symbols), we have

- The number of lines of standard proofs is bounded by kn where k is a positive constant.
- For some families of theorems there are proofs with only about $\log n$ lines.
- In some cases there are non standard proofs shorter than standard ones.

References

[Hof79] Dougals R. Hofstadter. *Gödel, Escher, Bach, and Eternal Golden Braid.* Penguin, 1979.

[Mat97] Armando B. Matos. On the length of the theorems of the formal system MIU. Technical report, Laboratório de Inteligência Artificial e de Ciência de Computadores, Universidade do Porto, 1997.

Let **nu** be the number of `u`'s in x

Let **ni** be the number of `i`'s in x

`c` $\leftarrow 3 \times nu + ni$

if `c` $\equiv 2 \mod 3$ **then**

  let n be the minimum odd integer such that $2^n \geq c$

if `c` $\equiv 1 \mod 3$ **then**

  let n be the minimum even integer such that $2^n \geq c$

else print("mx **is not a theorem"); return**

Comment: `w` will contain successive proof lines

`w` $\leftarrow$ `mi`

Use n times rule 3b to set `w` $=$ `mi`$^{2^n}$

Use rule 3c to replace left to right, groups `iii` by `u` as necessary

Use rules 3a, 3c and 3d to eliminate the excess of `i`'s

  at the right (whose number is a multiple of 3

Comment: The last line is the theorem to prove, `w` $=$ `m`x

Figure 1: Algorithm which generates the proof in MIU of `m`x

`mi`	(axiom)
`mii`	(rule 3b)
`miiii`	(rule 3b)
`miiiiiiii`	(rule 3b)
`miiiiiiiiiiiiiiii`	(rule 3b)
`muiiiiiiiiiiiii`	(rule 3c)
`muiuiiiiiiiiii`	(rule 3c)
`muiuuiiiiii`	(rule 3c)
`muiuuuiii`	(rule 3c)
`muiuuuiiiu`	(rule 3a)
`muiuuuuu`	(rule 3c)
`muiuuu`	(rule 3d)

Figure 2: The standard proof of `muiuuu` in MIU; note that $3n_u + n_1 = 13$, $13 \equiv 1 \mod 3$, and $2^4 = 16 > 13$.