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On principal types of stratified combinators

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Abstract

In this paper we study (in some cases) the relationship between the combinatory completeness of a set of stratified combinators for a system of λ-calculus and the axiomatic completeness, under substitution and modus ponens, of the respective set of principal types for the corresponding logical system. We show that combinatory completeness is a necessary, but not sufficient, condition for axiomatic completeness in the $\lambda\mathcal{K}$- and in the $\lambda\mathcal{I}$-calculus, while the two problems become equivalent for the **BCK**-λ- as well as for the **BCI**-λ-calculus. Furthermore, we present another proof for the fact that each **BCK**-λ-term in normal form is completely determined by its principal pairs.

1 Introduction

The connection between systems of λ-calculus and logical systems, already suggested in [4], was established in 1968 (see [9] and [5]) for the $\lambda\mathcal{K}$-calculus and intuitionistic logic[1]. In fact the types of stratified, i.e. typable, $\lambda\mathcal{K}$-combinators form the set of provable formulas in intuitionistic logic. The same correspondence exists between the $\lambda\mathcal{I}$-calculus and the system $R_{\rightarrow}$ of relevance logic (see [1]), as well as between the **BCK**-λ- and **BCI**-λ-calculus and **BCK**- and **BCI**-logic respectively, which have been introduced in [10]. Furthermore, it is well known that the principal types of some standard combinator bases for the four systems of λ-calculus form, together with the inference rules modus ponens and substitution, complete axiom sets for the respective logical systems. The aim of this paper is to study this pattern in the four cases, i.e. the relationship between the combinatory completeness of a set of stratified combinators and the axiomatic completeness of their principal types. We show that the two problems are equivalent in the **BCI**-λ- as well as in the **BCK**-λ-calculus, and that for the two remaining cases axiomatic completeness is a sufficient, but not necessary, condition for combinatory completeness. Finally, we give an alternative proof for the fact that each **BCK**-λ-term in normal form is completely determined by its principal pairs. This result, from which it follows that two **BCI**-λ-terms are β-convertible if and only if they have the same principal pairs, has already been proved by Hirokawa in [8], but his proof seems to be much more complex than the one presented here.

[1] Strictly speaking, we refer to Hilbert's positive implicational logic (cf. [11]).

2 Basic concepts

2.1 λ-calculus

We use standard notation from [2] and [3]. In order to distinguish the full λ-calculus from its different sub-systems, we usually refer to it as the $\lambda\mathcal{K}$-calculus. A $\lambda\mathcal{I}$-term is a λ-term such that each subterm $\lambda x.N$ has at least one free occurrence of x in N. If every variable occurs at most once free in any subterm of a λ-term M, then we call M a **BCK**-λ-term. If, in addition to that, for each subterm $\lambda x.N$ of M there is exactly one occurrence of x in N, then M is a **BCI**-λ-term.

Combinators are closed λ-terms. A set of combinators Γ is combinatorially complete or a basis for a system of λ-calculus if and only if for each combinator N in that system there is an applicative combination M of members of Γ such that $M =_\beta N$.

For type assignment we consider the simply typed λ-calculus à la Curry. Our notation differs from that in [3], since we denote type variables by $a, b, c, \ldots$ and arbitrary types by $\alpha, \beta, \gamma, \ldots$. The set of all types is denoted by $\mathcal{T}$. Besides standard results on typed terms and principal types, that can be found in [3], we will need the following result which has been proved by Hindley in [7].

Theorem 2.1 (Hindley) *Every* **BCK**-λ*-term is stratified.*

Note that, as a particular case, the same result is true for **BCI**-λ-terms. Furthermore, we will use the following property which follows directly from the definition of principal type.

Property 2.2 *Consider two closed λ-terms M and N such that the application MN is stratified. If α_1 and α_2 are respectively principal types for M and N, then there are substitutions $\star_1$ and $\star_2$ such that $\alpha_1^{\star_1} = \beta \to \gamma$ and $\alpha_2^{\star_2} = \beta$ and such that γ is a principal type for MN.*

2.2 Implicational logic

Propositional formulas are denoted by $\alpha, \beta, \gamma, \ldots$ and are built from propositional variables, denoted by $a, b, c, \ldots$, using the connective $\to$. Consider the following formulas.

$$\begin{array}{rcl}
\sigma & = & (a \to b \to c) \to (a \to b) \to a \to c \\
\kappa & = & a \to b \to a \\
\beta & = & (b \to c) \to (a \to b) \to a \to c \\
\gamma & = & (a \to b \to c) \to b \to a \to c \\
\iota & = & a \to a
\end{array}$$

The logical systems in this paper have two rules of inference, namely modus ponens and substitution, and are respectively based on the following sets of axioms.

Intuitionistic logic:	σ, κ
Relevance logic ($R_\to$):	$\sigma, \beta, \gamma, \iota$
BCK-logic:	β, γ, κ
BCI-logic:	β, γ, ι

Every formula that can be derived from the axioms of a logical systems using the inference rules modus ponens and substitution is called a theorem (of that system). If Σ is a set of formulas, such that the set of formulas that are derivable from Σ, by means of modus ponens and substitution, is exactly the set of theorems of one of the logical systems, then we say that Σ is axiomatically complete for that system.

As already mentioned in the introduction, the $\lambda\mathcal{K}$-, the $\lambda\mathcal{I}$-, the **BCK**-λ- and the **BCI**-λ-calculus are respectively related to intuitionistic, $R_{\rightarrow}$-, **BCK**- and **BCI**-logic. In particular, the set consisting of all types that can be assigned to the combinators of a system of λ-calculus, is exactly the set of theorems of the corresponding logical system. Now, consider the following combinator bases for the $\lambda\mathcal{I}$-calculus and the **BCK**-λ-calculus respectively:

$$\{\mathbf{B}, \mathbf{C}, \mathbf{I}, \mathbf{W}\} \qquad \text{and} \qquad \{\mathbf{B}, \mathbf{C}_*, \mathbf{K}\}$$

where

$$\begin{array}{lll} \mathbf{B} = \lambda xyz.x(yz) & \mathbf{W} = \lambda xy.xyy & \mathbf{I} = \lambda x.x \\ \mathbf{C} = \lambda xyz.xzy & \mathbf{C}_* = \lambda xy.yx & \mathbf{K} = \lambda xy.x \end{array}$$

One knows that the set of principal types of the combinators in the two sets form complete axiom sets, respectively for $R_{\rightarrow}$- and for **BCK**-logic. A great number of similar examples can be easily found for these as well as for the remaining cases. On the other hand, there are bases for $\lambda\mathcal{I}$, such as $\{\mathbf{B}, \mathbf{C}, \mathbf{I}, \mathbf{W}_*\}$, where $\mathbf{W}_* = \lambda x.xx$, which contain non-stratified combinators and the set of principal types of the remaining (typable) combinators is obviously not complete for $R_{\rightarrow}$. This suggests a pattern, that will be analysed in the next section, where we study the relationship between the combinatory completeness of a set of stratified combinators and the axiomatic completeness of the corresponding set of principal types.

3 Combinatory vs. axiomatic completeness

It is easy to show that in all four cases axiomatic completeness is a sufficient condition for combinatory completeness.

Proposition 3.1 *Let $\{\alpha_1, \ldots, \alpha_n\}$ be a complete axiom set for intuitionistic, $R_{\rightarrow}$-,* **BCK**- *or* **BCI**-*logic. If $M_1, \ldots, M_n$ are $\lambda\mathcal{K}$-, λI-,* **BCK**-λ- *or* **BCI**-λ-*terms, which admit $\alpha_1, \ldots, \alpha_n$ respectively as principal types, then $\{M_1, \ldots, M_n\}$ is a basis for the $\lambda\mathcal{K}$-, λI-,* **BCK**-λ- *or* **BCI**-λ-*calculus.*

Proof In each case it is enough to show that it is possible to define the elements of some basis of the respective system of λ-calculus by some applicative combination of $M_1, \ldots, M_n$. We show the result for the **BCK**-λ-calculus. The other cases are analogous.

If $\{\alpha_1, \ldots, \alpha_n\}$ is a complete axiom set for **BCK**-logic, then there are deductions of the formulas $\beta = (b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c$, $\gamma = (a \rightarrow b \rightarrow c) \rightarrow b \rightarrow a \rightarrow c$ and $\kappa = a \rightarrow b \rightarrow a$ from the hypotheses $\alpha_1, \ldots, \alpha_n$. Applying the corresponding terms for each application of the inference rule modus ponens in these deductions, one obtains by the substitution lemma (cf. [3], Proposition 3.1.10), derivations of $\vdash U : \beta$, $\vdash V : \gamma$ and $\vdash W : \kappa$, where U, V and W are combinations of $M_1, \ldots, M_n$. We conclude from the subject reduction property (cf. [3], Proposition 3.1.11) that the normal forms of U, V and W, whose existence follows from typability, admit equally the types β, γ and κ. Furthermore, it is easy to show that $\lambda xyz.x(yz)$, $\lambda xyz.xzy$ and $\lambda xy.x$ are the only terms in normal form for which one may infer β, γ and κ. Finally note that $\{\lambda xyz.x(yz), \lambda xyz.xzy, \lambda xy.x\}$ forms a basis for the **BCK**-λ-calculus. $\bullet$

The next example shows that the converse of this result does not hold for the $\lambda\mathcal{K}$-calculus. In fact there are bases of stratified combinators, whose principal types do not form a complete axiom set for intuitionistic logic.

Example 3.2 *Consider the $\lambda\mathcal{K}$-terms*

$$\mathbf{K} = \lambda xy.x \qquad \textit{and} \qquad \mathbf{S} = \lambda xyz.xz(yz)$$

which admit the following types

$$\tilde{\kappa} = a \rightarrow a \rightarrow a \qquad \textit{and} \qquad \tilde{\sigma} = (a \rightarrow a \rightarrow a) \rightarrow (a \rightarrow a) \rightarrow a \rightarrow a.$$

Hindley proved in [6] the existence of terms $\tilde{\mathbf{K}}$ *and* $\tilde{\mathbf{S}}$*, with principal types* $\tilde{\kappa}$ *and* $\tilde{\sigma}$*, and such that* $\tilde{\mathbf{K}} \rightarrow_\beta \mathbf{K}$ *and* $\tilde{\mathbf{S}} \rightarrow_\beta \mathbf{S}$*. Since* $\{\mathbf{S}, \mathbf{K}\}$ *is a basis for the* $\lambda\mathcal{K}$*-calculus, we conclude the same for* $\{\tilde{\mathbf{S}}, \tilde{\mathbf{K}}\}$*. On the other hand, it is easy to see that their principal types, i.e.* $\tilde{\kappa}$ *and* $\tilde{\sigma}$*, do not form a complete axiom set for intuitionistic logic.*

In the last example we made no restriction on the form of the combinators. The next example shows that in the $\lambda\mathcal{I}$-calculus, combinatory completeness of normal or even proper combinators does not imply axiomatic completeness of their principal types for $R_\rightarrow$-logic. It will become obvious, that the same holds in the $\lambda\mathcal{K}$-calculus.

Example 3.3 *Consider the proper combinators*

$$\begin{array}{rrcl} & \tilde{\mathbf{W}} & = & \lambda mxy.m(m(x(my)(my))) \\ & \tilde{\mathbf{B}} & = & \lambda mxyz.m(x(m(m(y(mz))))) \\ & \tilde{\mathbf{C}} & = & \lambda mxyz.m(m(x(mz)(my))) \\ \textit{and} & \mathbf{I} & = & \lambda x.x \end{array}$$

respectively with principal types

$$\begin{array}{rl} & (a \rightarrow a) \rightarrow (a \rightarrow a \rightarrow a) \rightarrow a \rightarrow a \\ & (a \rightarrow a) \rightarrow (a \rightarrow a) \rightarrow (a \rightarrow a) \rightarrow a \rightarrow a \\ & (a \rightarrow a) \rightarrow (a \rightarrow a \rightarrow a) \rightarrow a \rightarrow a \rightarrow a \\ \textit{and} & a \rightarrow a. \end{array}$$

Obviously the set of principal types is no complete axiom set for $R_\rightarrow$*-logic, since it is not possible to deduce the theorem* $(a \rightarrow a \rightarrow b) \rightarrow a \rightarrow b$*. On the other hand, we have*

$$\begin{array}{rrcrcl} & \tilde{\mathbf{W}}\mathbf{I} & \rightarrow_\beta & \mathbf{W} & = & \lambda xy.xyy \\ & \tilde{\mathbf{B}}\mathbf{I} & \rightarrow_\beta & \mathbf{B} & = & \lambda xyz.x(yz) \\ \textit{and} & \tilde{\mathbf{C}}\mathbf{I} & \rightarrow_\beta & \mathbf{C} & = & \lambda xyz.xzy \end{array}$$

thus $\{\tilde{\mathbf{W}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \mathbf{I}\}$ *is combinatorially complete for the* $\lambda\mathcal{I}$*-calculus, since* $\{\mathbf{W}, \mathbf{B}, \mathbf{C}, \mathbf{I}\}$ *is.*

The proof of proposition 3.1 relies on the fact that the typed λ-calculus has the subject-reduction property. We now show the converse result for systems with the subject-expansion property.

Definition 3.4 *A system of* λ*-calculus has the subject-expansion property if and only if for all terms* M *and* N *in that system, any basis* $\mathcal{B}$ *and any type* $\alpha \in \mathcal{T}$*, there is*

$$\mathcal{B} \vdash N : \alpha \quad \textit{and} \quad M \rightarrow_\beta N \quad \Rightarrow \quad \mathcal{B} \vdash M : \alpha.$$

Proposition 3.5 *Consider a system of* λ*-calculus* $\lambda_\mathcal{X}$ *with* $\Lambda_\mathcal{X} \subseteq \Lambda$ *where the subject-expansion property holds. Let* $\{M_1, \ldots, M_n\} \subseteq \Lambda^0_\mathcal{X}$ *be a set of stratified combinators, that is complete for* $\lambda_\mathcal{X}$*. If* $\alpha_1, \ldots, \alpha_n \in \mathcal{T}$ *are respectively the principal types of* $M_1, \ldots, M_n$*, then* $\{\alpha_1, \ldots, \alpha_n\}$ *is a complete axiom set for the propositional calculus* $P_\mathcal{X}(\rightarrow)$*, whose set of theorems is the set of admissible types for* $\Lambda^0_\mathcal{X}$*.*

Proof Let α be a theorem of $P_\mathcal{X}(\rightarrow)$. Then there is a term $N \in \Lambda^0_\mathcal{X}$ in normal form such that $\vdash N : \alpha$. From the completeness of $\{M_1, \ldots, M_n\}$ we conclude that there is some combination M of $M_1, \ldots, M_n$ that defines N. Consequently there is $M \rightarrow_\beta N$, hence $\vdash M : \alpha$. By property 2.2, it is possible to infer $\vdash M : \gamma$, where γ is a principal type of M, from $\vdash M_1 : \alpha_1, \ldots, \vdash M_n : \alpha_n$.

Hence there is some substitution $\star$ such that $\alpha = \gamma^\star$. Dropping the terms in this process, we obtain a deduction of γ from the hypotheses $\alpha_1, \ldots, \alpha_n$ from which we obtain a deduction of α by substitution. •

Hindley proved in [7] that two **BCI**-λ-terms M and N have the same principal pairs, whenever $M =_\beta N$. On the other hand, any **BCI**-λ-term is stratified. We conclude the following.

Corollary 3.6 *Consider a set of* **BCI**-*combinators* $\{M_1, \ldots, M_n\}$ *with principal types* $\{\alpha_1, \ldots, \alpha_n\}$. *Then,* $\{M_1, \ldots, M_n\}$ *is combinatorially complete for the* **BCI**-λ-*calculus if and only if* $\{\alpha_1, \ldots, \alpha_n\}$ *is a complete axiom-set for* **BCI**-*logic.*

Example 3.7 *Now consider the* **BCK**-λ-*terms*

$$M = (\lambda xyzw.xz(yw))(\lambda uv.u) \qquad \textit{and} \qquad N = \lambda yzw.z$$

for which one has

$$M \rightarrow_\beta N \qquad \textit{and} \qquad \vdash \lambda yzw.z : a \rightarrow b \rightarrow c \rightarrow b.$$

On the other hand, it is easy to see that

$$\nvdash (\lambda xyzw.xz(yw))(\lambda uv.u) : a \rightarrow b \rightarrow c \rightarrow b.$$

The example above shows that the subject-expansion property does not hold in the **BCK**-λ-calculus. Nevertheless, we will obtain a similar result to corollary 3.6 for this calculus, establishing first some properties (stated in lemma 3.9 and lemma 3.10) on principal types of **BCK**-λ-terms. Besides the stratification theorem for **BCK**-λ-terms (theorem 2.1) we will also need the next lemma, that has already been used and proved by Hindley in [7].

Lemma 3.8 (Hindley) *Let* M *and* M' *be* **BCK**-λ-*terms of the form*

$$M = (\lambda x.U)VP_1 \ldots P_t, \qquad M' = (U[V/x])P_1 \ldots P_t \qquad (t \geq 0)$$

such that $\mathcal{B} \vdash M' : \alpha$ *for some basis* $\mathcal{B}$ *and some type* $\alpha \in \mathcal{T}$.

1. *If* $x \in FV(U)$, *then* $\mathcal{B} \vdash M : \alpha$.
2. *If* $x \notin FV(U)$, *and there exists* $\beta \in \mathcal{T}$ *such that* $\mathcal{B} \vdash V : \beta$, *then* $\mathcal{B} \vdash M : \alpha$.

The following result is due to the absence of multiple occurrences of variables in **BCK**-λ-terms.

Lemma 3.9 *If* M *is a* **BCK**-λ-*term with* $FV(M) = \{x_1, \ldots, x_n\}$, *then there are types* $\alpha_1, \ldots, \alpha_n, \alpha$ *such that*

$$x_1 : \alpha_1, \ldots, x_n : \alpha_n \vdash M : \alpha$$

and such that each $\beta \in \{\alpha_1, \ldots, \alpha_n\}$ *is of the form* $\beta_1 \rightarrow \ldots \rightarrow \beta_k \rightarrow b$ *where the type variable* b *does not occur in the type schemes* $\beta_1, \ldots, \beta_k$.

Proof By induction on the length of M. The cases for $M = x$ and $M = \lambda x.M_1$ are easy.

Now consider $M = (\lambda x.U)VP_1 \ldots P_t$, such that $t \geq 0$ and $x \notin FV(U)$. From theorem 2.1 and by the induction hypothesis we conclude that there are bases $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfying the conditions of the lemma and such that $\mathcal{B}_1 \vdash UP_1 \ldots P_t : \alpha$ and $\mathcal{B}_2 \vdash V : \beta$. By the basis lemma (cf. [3], Proposition 3.1.7) and since no variable occurs more than once free in M we may suppose that $dom(\mathcal{B}_1) \cap dom(\mathcal{B}_2) = \emptyset$, as well as $x \notin dom(\mathcal{B}_1) \cup dom(\mathcal{B}_2)$. Consequently, $\mathcal{B}_1, \mathcal{B}_2 \vdash V : \beta$ follows equally from the basis lemma. On the other hand, we conclude from the generation lemma (cf. [3], Proposition 3.1.8) that there are types $\alpha_1, \ldots, \alpha_t$ such that $\mathcal{B}_1 \vdash U : \alpha_1 \rightarrow \ldots \rightarrow \alpha_t \rightarrow \alpha$ and $\mathcal{B}_1 \vdash P_i : \alpha_i$, $(1 \leq i \leq t)$. Hence, $\mathcal{B}_1, \mathcal{B}_2, x : \beta \vdash U : \alpha_1 \rightarrow \ldots \rightarrow \alpha_t \rightarrow \alpha$ and $\mathcal{B}_1, \mathcal{B}_2 \vdash P_i : \alpha_i$, for $(1 \leq i \leq t)$. Thus, $\mathcal{B}_1, \mathcal{B}_2 \vdash \lambda x.U : \beta \rightarrow \alpha_1 \rightarrow \ldots \rightarrow \alpha_t \rightarrow \alpha$ and consequently one has $\mathcal{B}_1, \mathcal{B}_2 \vdash (\lambda x.U)V : \alpha_1 \rightarrow \ldots \rightarrow \alpha_t \rightarrow \alpha$, and finally $\mathcal{B}_1, \mathcal{B}_2 \vdash (\lambda x.U)VP_1 \ldots P_t : \alpha$.

If $M = (\lambda x.U)VP_1 \ldots P_t$, with $t \geq 0$ and $x \in FV(U)$, we conclude from lemma 3.8 that the term $(\lambda x.U)VP_1 \ldots P_t$ admits exactly the same types as $U[V/x]P_1 \ldots P_t$. On the other hand, it is easy to see that $U[V/x]P_1 \ldots P_t$ has smaller length than $(\lambda x.U)VP_1 \ldots P_t$, since x occurs exactly once in U. Thus, the result follows from the induction hypothesis.

Finally let $M = xL_1 \ldots L_m$ with $m \geq 1$. By the induction hypothesis there are $\mathcal{B}_i$ and α_i such that $\mathcal{B}_i \vdash L_i : \alpha_i$, for $1 \leq i \leq m$. As before, we may suppose that the domains of $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are pairwise disjoint and do not contain the variable x. Choosing a new type variable a we have $\mathcal{B}_1, \ldots, \mathcal{B}_m, x : \alpha_1 \rightarrow \ldots \alpha_m \rightarrow a \vdash xL_1 \ldots L_m : a$. •

Lemma 3.10 *Consider a* **BCK**-λ*-term* M*, whose normal form is a proper combinator* $\lambda x_1 \ldots x_n.N$ *with principal type* $\alpha = \alpha_1 \rightarrow \ldots \rightarrow \alpha_n \rightarrow a$*. Then there are types* $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \in \mathcal{T}$ *such that*

- $\vdash M : \tilde{\alpha}_1 \rightarrow \ldots \rightarrow \tilde{\alpha}_n \rightarrow a$;
- *if* $x_i \in FV(N)$*, then* $\tilde{\alpha}_i = \alpha_i$*, for* $1 \leq i \leq n$;
- *if* $x_j \notin FV(N)$*, then* $\tilde{\alpha}_j = \beta_1 \rightarrow \ldots \rightarrow \beta_k \rightarrow b$*, with* $k \geq 0$ *and* $1 \leq j \leq n$*, and such that the type variable* b *does not occur in* $\beta_1, \ldots, \beta_k$ *and no type variable occurs both in* $\tilde{\alpha}_j$ *and in some* α_i *for which there is* $x_i \in FV(N)$ *and* $1 \leq i \leq n$.

Proof First note that α_i is a type variable, say a_i, with no further occurrences in α, whenever $x_i \notin FV(N)$ and $i \in \{1, \ldots, n\}$.

Now, we prove the result by induction on the length k of the normal reduction from M to $\lambda x_1 \ldots x_n.N$. Here we show that one may obtain a type for M from α, by substitution of some of the type variables a_i, for which there is $x_i \notin FV(N)$, but such that x_i occurs free in some subterm of M, by types of the form $\beta_1 \rightarrow \ldots \rightarrow \beta_k \rightarrow b$ containing only new type variables and such that b does not occur in $\beta_1, \ldots, \beta_k$. [2]

The result is trivially true for $k = 0$.

Consider a normal reduction from M to $\lambda x_1 \ldots x_n.N$ of length $k > 0$ and suppose that in the first step M reduces to some term M_1, which may be obtained from M by substitution of some subterm in M of the form $(\lambda x.P)Q$ by $P[Q/x]$. From lemma 3.8 we conclude that M admits exactly the same types as M_1, whenever $x \in FV(P)$. Hence the result follows from the induction hypothesis.

Now, suppose that there is $x \notin FV(P)$, i.e. $P[Q/x] = P$ and let $FV(Q) = \{y_1, \ldots, y_m\}$. Since $(\lambda x.P)Q$ is the leftmost redex in M and M reduces to the proper combinator $\lambda x_1 \ldots x_n.N$, it is easy to see that the term M is necessarily of the form

$$\lambda x_1 \ldots x_t.(X_1(X_2(\ldots(X_l((\lambda x.P)Q)\ldots)\ldots)\ldots)\ldots)$$

[2] Note that for such a substitution $\star = [a_1 := \beta_{11} \rightarrow \ldots \rightarrow \beta_{1k_1} \rightarrow b_1, a_2 := \beta_{21} \rightarrow \ldots \rightarrow \beta_{2k_2} \rightarrow b_2]$ the types $\beta_{11} \rightarrow \ldots \rightarrow \beta_{1k_1} \rightarrow b_1$ and $\beta_{21} \rightarrow \ldots \rightarrow \beta_{2k_2} \rightarrow b_2$ may have variables in common, while b_1 may occur in $\beta_{21} \rightarrow \ldots \rightarrow \beta_{2k_2} \rightarrow b_2$.

with $0 \leq t \leq n$, and $l = 0$ whenever $t < n$, and such that the terms $X_1, \ldots, X_l$ are applicative terms over $\{x_1, \ldots, x_n\} \setminus \{y_1, \ldots, y_m\}$. Furthermore, there is $\{y_1, \ldots, y_m\} \subseteq \{x_1, \ldots, x_t\}$ since M is a closed term. The term

$$M_1 = \lambda x_1 \ldots x_t.(X_1(X_2(\ldots(X_l P \ldots) \ldots) \ldots) \ldots)$$

admits, by the induction hypothesis a type $\tilde{\alpha}_1 \to \ldots \to \tilde{\alpha}_n \to a$ obtained from α by substitution of some of the type variables a_i by types containing only new type variables, where $1 \leq i \leq n$, and such that x_i occurs free in some subterm of M_1, while $x_i \notin FV(N)$. For $x_i \in \{y_1, \ldots, y_m\}$ follows $\tilde{\alpha}_i = \alpha_i = a_i$, with no further occurrence of the type variable a_i in $\tilde{\alpha}_1 \to \ldots \to \tilde{\alpha}_n \to a$. From lemma 3.9 we conclude that there are types

$$\vec{\beta}_1 = \beta_{11} \to \ldots \to \beta_{1k_1} \to b_1 \ , \ldots, \ \vec{\beta}_m = \beta_{m1} \to \ldots \to \beta_{mk_m} \to b_m \quad \text{and} \quad \gamma$$

such that b_i does not occur in $\beta_{i1}, \ldots, \beta_{ik_i}$ for $1 \leq i \leq m$ and such that

$$y_1 : \vec{\beta}_1, \ldots, y_m : \vec{\beta}_m \vdash Q : \gamma.$$

Without loss of generality we may suppose that $\vec{\beta}_1, \ldots, \vec{\beta}_m, \gamma$ and $\tilde{\alpha}_1 \to \ldots \to \tilde{\alpha}_n \to a$ have no variables in common. Now consider the inference process of the type $\tilde{\alpha}_1 \to \ldots \to \tilde{\alpha}_n \to a$ for M_1. As the variables $y_1, \ldots, y_m$ do not occur in $(X_1(X_2(\ldots(X_l P \ldots) \ldots) \ldots) \ldots)$ it follows that all steps in the inference of the type of this term are of the form

$$\mathcal{B}_U, y_1 : a_1, \ldots, y_m : a_m \vdash U : \sigma$$

where no use has been made of the type assignment $y_1 : a_1, \ldots, y_m : a_m$. Hence, one may substitute each a_i by $\vec{\beta}_i$, for $1 \leq i \leq m$. Furthermore, in the part which corresponds to the inference of a type, say δ, for P one may join the declaration $x : \gamma$ to the basis, thus obtaining

$$\mathcal{B}_P, y_1 : \vec{\beta}_1, \ldots, y_m : \vec{\beta}_m, x : \gamma \vdash P : \delta$$

and infering

$$\mathcal{B}_P, y_1 : \vec{\beta}_1, \ldots, y_m : \vec{\beta}_m \vdash \lambda x.P : \gamma \to \delta.$$

From the basis lemma we conclude

$$\mathcal{B}_P, y_1 : \vec{\beta}_1, \ldots, y_m : \vec{\beta}_m \vdash Q : \gamma$$

hence

$$\mathcal{B}_P, y_1 : \vec{\beta}_1, \ldots, y_m : \vec{\beta}_m \vdash (\lambda x.P)Q : \delta.$$

Consequently, it is possible to infer the type $\tilde{\tilde{\alpha}}_1 \to \ldots \to \tilde{\tilde{\alpha}}_n \to a$ for M, where $\tilde{\tilde{\alpha}}_i = \tilde{\alpha}_i$ whenever $x_i \notin \{y_1, \ldots, y_m\}$. •

Proposition 3.11 *Consider a set of* **BCK**-*combinators* $\{M_1, \ldots, M_n\}$ *with principal types* $\{\alpha_1, \ldots, \alpha_n\}$. *Then* $\{M_1, \ldots, M_n\}$ *combinatorially complete for the* **BCK**-λ-*calculus if and only if* $\{\alpha_1, \ldots, \alpha_n\}$ *forms a complete set of axioms for* **BCK**-*logic.*

Proof By proposition 3.1 it remains to show that the completeness of $\{\alpha_1, \ldots, \alpha_n\}$ follows from the completeness of $\{M_1, \ldots, M_n\}$. Since $\{M_1, \ldots, M_n\}$ is a basis, it is obvious that there are closed λ-terms $\tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{K}}$, combinations of $M_1, \ldots, M_n$, and such that

$$\begin{array}{lll} \tilde{\mathbf{B}}xyz & \to_\beta & x(yz) \\ \tilde{\mathbf{C}}xyz & \to_\beta & xzy \\ \tilde{\mathbf{K}}xy & \to_\beta & x \end{array}$$

It is easy to see that one has the same for the normal forms of $\tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$, whose existence results from theorem 2.1. On the other hand, note that $\lambda xyz.x(yz)$, $\lambda xyz.xzy$ and $\lambda xy.x$ are the only closed normal forms with this property and that their principal types

$$\begin{array}{ll} & (b \to c) \to (a \to b) \to a \to c \\ & (a \to b \to c) \to b \to a \to c \\ \text{and} & a \to b \to a \end{array}$$

form a complete set of axioms for **BCK**-logic. Since the terms $\tilde{\mathbf{B}}, \tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$ satisfy the conditions in lemma 3.10, there is

$$\begin{array}{l} \vdash \tilde{\mathbf{B}} : (b \to c) \to (a \to b) \to a \to c \\ \vdash \tilde{\mathbf{C}} : (a \to b \to c) \to b \to a \to c \\ \vdash \tilde{\mathbf{K}} : a \to \beta \to a \end{array}$$

where $\beta = \beta_1 \to \dots \beta_k \to b$ and $k \geq 0$, such that $a \neq b$ and neither a nor b occur in $\beta_1, \dots, \beta_k$.

We conclude the proof, showing that a term with type $a \to b \to a$ is definable from $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$. For this we construct a term that admits a type of the form

$$a \to (\star(\beta_2) \to \dots \to \star(\beta_k) \to b) \to a$$

where $\star$ is a substitution of some type variables by types containing only new type variables. Consequently, the type variable b does not occur in the types $\star(\beta_2), \dots, \star(\beta_k)$, while the number of "principal" implication symbols in $\star(\beta_2) \to \dots \to \star(\beta_k) \to b$ is strictly smaller than in the type $\beta_1 \to \dots \to \beta_k \to b$. Hence, it is sufficient to repeat this process until one obtains a term with a type of the form $a \to b \to a$.

In fact, there is

$$\begin{array}{l} \vdash \tilde{\mathbf{C}} : (((\beta \to a) \to c) \to (a \to (\beta \to a)) \to a \to c) \to \\ \qquad\qquad (a \to (\beta \to a)) \to ((\beta \to a) \to c) \to a \to c \\ \vdash \tilde{\mathbf{B}} : ((\beta \to a) \to c) \to (a \to (\beta \to a)) \to a \to c \\ \vdash \tilde{\mathbf{C}}\tilde{\mathbf{B}} : (a \to (\beta \to a)) \to ((\beta \to a) \to c) \to a \to c \\ \vdash \tilde{\mathbf{K}} : a \to \beta \to a \\ \vdash \tilde{\mathbf{C}}\tilde{\mathbf{B}}\tilde{\mathbf{K}} : ((\beta \to a) \to c) \to a \to c \\ \vdash \tilde{\mathbf{C}}\tilde{\mathbf{B}}\tilde{\mathbf{K}} : ((\beta' \to a') \to c') \to a' \to c' \\ \vdash \tilde{\mathbf{C}}\tilde{\mathbf{B}}\tilde{\mathbf{K}}(\tilde{\mathbf{C}}\tilde{\mathbf{B}}\tilde{\mathbf{K}}) : \star_1(a \to c) \end{array}$$

such that β' is β where each type variable d has been substituted by a new variable d' and such that $\star_1$ is the substitution following from the unification of

$$(\beta \to a) \to c \quad \text{with} \quad ((\beta' \to a') \to c') \to a' \to c'.$$

Note that the existence of $\star_1$ follows from theorem 2.1. Since $\star_1$ results from

$$\begin{array}{ll} & a = c', \\ & c = a' \to c', \\ & a' = \beta_2 \to \dots \to \beta_k \to b \\ \text{and} & \beta_1 = \beta_1' \to \dots \to \beta_k' \to b' \end{array}$$

we conclude

$$\star_1(a \to c) = c' \to \star(\beta_2 \to \dots \to \beta_k \to b) \to c'$$

where $\star$ results from the unification of β_1 with $\beta_1' \to \dots \to \beta_k' \to b'$. Furthermore, since b does not occur in β_1 there is

$$\vdash \tilde{\mathbf{C}}\tilde{\mathbf{B}}\tilde{\mathbf{K}}(\tilde{\mathbf{C}}\tilde{\mathbf{B}}\tilde{\mathbf{K}}) : c' \to (\beta_2^\star \to \dots \to \beta_k^\star \to b) \to c'. \quad \bullet$$

4 Principal pairs of BCK-λ-terms

In this section we reprove the fact, already presented by Hirokawa in [8], which states that each **BCK**-λ-term in normal form is completely (modulo α-conversion) determined by its principal pairs. This result, which has the equivalence of β-conversion and equality (modulo renaming of variables) of principal pairs for **BCI**-λ-terms as a consequence, is rather intuitive and stems from the fact that no variable occurs more than once free in any **BCK**-λ-term. In fact, without this restriction the result turns out to be false, even for $\lambda\mathcal{I}$-terms as shows the example below.

Example 4.1 *Both normal forms*

$$\lambda xyzu.x(yz)(yu) \;\; \neq_\beta \;\; \lambda xyzu.x(yu)(yz)$$

have principal type

$$(a \to a \to b) \to (c \to a) \to c \to c \to b.$$

At first sight, the proof of the result seems to be straightforward using induction on the term structure (or length). However, for the case $xM_1 \dots M_n$ the induction hypothesis is not applicable (see also comments in [8] on page 266). Hirokawa surrounded this difficulty defining a transformation of type assignment figures, which preserves principality. Furthermore, he introduced the notion of connection of types in type assignment figures in order to characterize principal type assignment for **BCK**-λ-terms. Finally the proof works by induction on the weight of terms. In the following, we describe an alternative way to overcome the encountered difficulty. It consists in taking a slightly stronger result, than the one we wish to prove, as the induction hypothesis in order to turn it applicable to the case $xM_1 \dots M_n$. Then, it is sufficient to use some direct consequences of the definition of principal pair as well as the following lemma.

Lemma 4.2 *Consider a* **BCK**-λ*-term* M *in normal form with principal pair* $(\mathcal{B}, \alpha)$*. Then each type variable has at most two occurrences in* $(\mathcal{B}, \alpha)$*.*

Proof By induction on the structure of M.

If $M = x$, then $(\mathcal{B}, \alpha)$ is of the form $(\{x : a\}, a)$.

Let $M = \lambda x.P$, where P is a **BCK**-λ-term in normal form. We conclude from the generation lemma that α is of the form $\alpha = \beta \to \gamma$ and $\mathcal{B}, x : \beta \vdash P : \gamma$. If $x \in FV(P)$, then $(\mathcal{B} \cup \{x : \beta\}, \gamma)$ is a principal pair of P and the result follows from the induction hypothesis. Otherwise, i.e. if $x \notin FV(P)$, then $(\mathcal{B}, \gamma)$ is a principal pair of P and $\beta = b$ where b is a type variable with no further occurrence in $\mathcal{B}$ and γ. By the induction hypothesis, each type variables occurs at most twice in $(\mathcal{B}, \gamma)$ and consequently in $(\mathcal{B}, b \to \gamma)$.

Finally suppose that $M = xN_1 \dots N_n$, with $n \geq 1$. Since M is a **BCK**-λ-term, it is easy to see that the variable x does not occur in $N_1, \dots, N_n$, that one has $\alpha = a$ and that

$$\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n \cup \{x : \alpha_1 \to \dots \alpha_n \to a\}$$

such that $(\mathcal{B}_i, \alpha_i)$ and $(\mathcal{B}_j, \alpha_j)$ have no common type variables, for $1 \leq i \neq j \leq n$, and that each pair $(\mathcal{B}_i, \alpha_i)$ is a principal pair of N_i. We conclude by the induction hypothesis that each type variable has at most two occurrences in each pair $(\mathcal{B}_i, \alpha_i)$ and consequently in $(\mathcal{B}, a)$. •

Proposition 4.3 *Let* M *and* N *be two* **BCK**-λ*-terms in normal form. If* M *and* N *have the same principal pairs, then* $M =_\alpha N$*.*

Proof We show by induction on the length of the terms M and N that whenever $\mathcal{B} = \{z_1 : \gamma_1, \dots, z_k : \gamma_k\}$ is a basis such that every type variable occurs at most twice in $(\mathcal{B}, \alpha)$ and such that

$(\mathcal{B} \restriction FV(M), \alpha)$ and $(\mathcal{B} \restriction FV(N), \alpha)$ are respectively principal pairs for M and N, then $M =_\alpha N$. Then it is sufficient to note that, by the basis lemma and by lemma 4.2, every principal pair of M and N satisfies the conditions above.

The result is trivial when M and N are variables. Otherwise, M and N are respectively of the form $M = \lambda u_1 \ldots u_{n_1}.y_1 M_1 \ldots M_{m_1}$ and $N = \lambda v_1 \ldots v_{n_2}.y_2 N_1 \ldots N_{m_2}$ with $n_1 > 0$, $n_2 > 0$, $m_1 > 0$ or $m_2 > 0$ and with principal pairs $(\mathcal{B}_1, \beta_1^1 \to \ldots \beta_{n_1}^1 \to b_1)$ and $(\mathcal{B}_2, \beta_1^2 \to \ldots \beta_{n_2}^2 \to b_2)$ which are respectively variants of $(\mathcal{B} \restriction FV(M), \alpha)$ and $(\mathcal{B} \restriction FV(N), \alpha)$. Thus, it follows that α is of the form $\alpha = \alpha_1 \to \ldots \to \alpha_n \to a$, where $n = n_1 = n_2 \geq 0$.

Next we substitute all occurrences of $u_1, \ldots, u_n$ in M and of $v_1, \ldots, v_n$ in N by new variables $x_1, \ldots, x_n$ obtaining $M' =_\alpha M$ and $N' =_\alpha N$. Hence, $(\mathcal{B} \restriction FV(M'), \alpha)$ and $(\mathcal{B} \restriction FV(N'), \alpha)$ are respectively principal pairs of $M' = \lambda x_1 \ldots x_n.y_1' M_1' \ldots M_{m_1}'$ and $N' = \lambda x_1 \ldots x_n.y_2' N_1' \ldots N_{m_2}'$ and consequently

$$(\mathcal{B} \cup \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\} \restriction FV(y_1' M_1' \ldots M_{m_1}'), a)$$

and

$$(\mathcal{B} \cup \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\} \restriction FV(y_2' N_1' \ldots N_{m_2}'), a)$$

are respectively principal pairs of $y_1' M_1' \ldots M_{m_1}'$ and of $y_2' N_1' \ldots N_{m_2}'$, where the type variable a occurs in the types assigned to both $y_1', y_2' \in dom(\mathcal{B}) \cup \{x_1, \ldots, x_n\}$. By lemma 4.2 each type variable occurs at most twice in $\{\gamma_1, \ldots, \gamma_k, \alpha\}$, hence $y_1' = y_2' = y$, with

$$y : \beta \in \mathcal{B} \cup \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\}.$$

Furthermore, note that the variable y does not occur in the terms $M_1', \ldots, M_{m_1}'$ or $N_1', \ldots, N_{m_2}'$. On the other hand, there is $FV(M_i') \cap FV(M_j') = \emptyset$, for $1 \leq i \neq j \leq m_1$, as well as $FV(N_l') \cap FV(N_k') = \emptyset$, for $1 \leq l \neq k \leq m_2$. Thus, $\beta = \beta_1 \to \ldots \to \beta_m \to a$ with $m = m_1 = m_2$ and for $i = 1, \ldots, m$ the pairs

$$(\mathcal{B} \cup \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\} \restriction FV(M_i'), \beta_i) \text{ and } (\mathcal{B} \cup \{x_1 : \alpha_1, \ldots, x_n : \alpha_n\} \restriction FV(N_i'), \beta_i)$$

are respectively principal pairs of M_i' and N_i'. Then $M_i' =_\alpha N_i'$ results from the induction hypothesis. •

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