Left Relations

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Abstract

Recently, Yamamoto presented a new method for the conversion from regular expressions (REs) to non-deterministic finite automata (NFA) based on the Thompson ε-NFA ($\mathcal{A}_T$). The $\mathcal{A}_T$ automaton has two quotients considered: the suffix automaton $\mathcal{A}_{suf}$ and the prefix automaton, $\mathcal{A}_{pre}$. Eliminating ε-transitions in $\mathcal{A}_T$, the Glushkov automaton ($\mathcal{A}_{pos}$) is obtained. Thus, it is easy to see that $\mathcal{A}_{suf}$ and the partial derivative automaton ($\mathcal{A}_{pd}$) are the same. In this paper, we characterise the $\mathcal{A}_{pre}$ automaton as a solution of a system of left RE equations and express it as a quotient of $\mathcal{A}_{pos}$ by a specific left-invariant equivalence relation. We define and characterise the right-partial derivative automaton ($\overleftarrow{\mathcal{A}}_{pd}$). Finally, we study the average size of all these constructions both experimentally and from an analytic combinatorics point of view.

1 Introduction

Regular expressions (REs), because of their succinctness and clear syntax, are the common choice to represent regular languages. The minimal deterministic finite automaton (DFA) equivalent to a RE can be exponentially larger than the RE. However, nondeterministic finite automata (NFAs) equivalent to REs can have the number of states linear with respect to (w.r.t) the size of the REs.

Brzozowski [4] proposed a method to convert REs in equivalent DFAs, based on the derivatives of regular expressions. Conversion methods from REs to equivalent NFAs can produce NFAs without or with transitions labelled with the empty word (ε-NFA). The standard conversion without ε-transitions is the position automaton ($\mathcal{A}_{pos}$) [11, 16]. Other conversions such as partial derivative automata ($\mathcal{A}_{pd}$) [1, 17], which is a nondeterministic version of the Brzozowski automata, follow automata ($\mathcal{A}_f$) [13], or the construction by Garcia et al. ($\mathcal{A}_u$) [10] were proved to be quotients of the position automata, by specific right-equivalence relations. The Thompson automaton $\mathcal{A}_T$ is the typical conversion method with ε-transitions. Yamamoto [20] present a new conversion method based on the left languages of the states of $\mathcal{A}_T$, instead of the right languages commonly used. This method has as result the $\mathcal{A}_{pre}$ automaton.

In this paper we define the left derivative and the left partial derivative automaton and show its relation with Brzozowski automaton and partial derivative automaton, respectively. We also construct the $\mathcal{A}_{pre}$ automaton directly from the regular expression without use the $\mathcal{A}_T$ automaton, and show that it also is a quotient of the $\mathcal{A}_{pos}$ automaton.

2 Regular Expressions and Automata

Given an alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ of size k, the set RE of *regular expressions* α over Σ is defined by the following grammar:

$$\alpha := \emptyset \mid \varepsilon \mid \sigma_1 \mid \cdots \mid \sigma_k \mid (\alpha + \alpha) \mid (\alpha \cdot \alpha) \mid (\alpha)^\star, \tag{1}$$

where the symbol $\cdot$ is often omitted. If two regular expressions α and β are syntactically equal, we write $\alpha \equiv \beta$. The *size* of a regular expression α, $|\alpha|$, is its number of symbols, disregarding parenthesis; its *alphabetic size*, $|\alpha|_\Sigma$, is the number of occurrences of letters from Σ; and $|\alpha|_\varepsilon$ denotes the number of occurrences of ε in α. A regular expression α is *linear* if all its letters are distinct.

The language represented by a RE α is denoted by $\mathcal{L}(\alpha)$. Two REs α and β are *equivalent* if $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$, and one writes $\alpha = \beta$. We define $\varepsilon(\alpha) = \varepsilon$ if $\varepsilon \in \mathcal{L}(\alpha)$ and $\varepsilon(\alpha) = \emptyset$, otherwise. Given a set $S \subseteq \mathsf{RE}$, we define $\varepsilon(S) = \bigcup_{\alpha \in S} \varepsilon(\alpha)$. We can inductively define $\varepsilon(\alpha)$ as follows:

$$\begin{array}{lcl} \varepsilon(\sigma) = \varepsilon(\emptyset) & = & \emptyset \\ \varepsilon(\varepsilon) & = & \varepsilon \\ \varepsilon(\alpha^*) & = & \varepsilon \end{array} \qquad \begin{array}{lcl} \varepsilon(\alpha + \beta) & = & \begin{cases} \varepsilon & \text{if } (\varepsilon(\alpha) = \varepsilon) \text{ or } (\varepsilon(\beta) = \varepsilon) \\ \emptyset & \text{otherwise} \end{cases} \\ \varepsilon(\alpha\beta) & = & \begin{cases} \varepsilon & \text{if } (\varepsilon(\alpha) = \varepsilon) \text{ and } (\varepsilon(\beta) = \varepsilon) \\ \emptyset & \text{otherwise} \end{cases} \end{array}$$

The algebraic structure $(\mathsf{RE}, +, ., \emptyset, \varepsilon)$ constitutes an idempotent semiring, and with the Kleene star operator $\star$, a Kleene algebra. There are several well-known complete axiomatizations of Kleene algebras [14]. In the following we consider regular expressions reduced by the following rules: $\varepsilon \cdot \alpha = \alpha = \alpha \cdot \varepsilon$, $\emptyset + \alpha = \alpha = \alpha + \emptyset$, and $\emptyset \cdot \alpha = \emptyset = \alpha \cdot \emptyset$. Let ACI denote the associativity, commutativity and idempotence of $+$. Given a language $L \subseteq \Sigma^\star$ and a word $w \in \Sigma^\star$, the *left-quotient* of L w.r.t. w is the language $w^{-1}L = \{x \mid wx \in L\}$, and the *rigth-quotient* of L w.r.t. w is the language $Lw^{-1} = \{x \mid xw \in L\}$. It is not difficult to verify that $Lw^{-1} = (w^R)^{-1}L^R$. We define $\mathsf{Prefix}(w) = \{v \in \Sigma^+ \mid \exists u \in \Sigma^\star : vu = w\}$. The *reversal of a word* $w = \sigma_1\sigma_2 \cdots \sigma_n$ is the word written backwards, i.e. $w^R = \sigma_n \cdots \sigma_2\sigma_1$. The reversal of a language L, denoted by L^R, is the set of words whose reversal is on L. The reversal of a regular expression $\alpha \in \mathsf{RE}$ can be inductively define by the following rules [12]:

$$\begin{array}{rcl} \sigma^R & = & \sigma \\ \emptyset^R & = & \emptyset \\ \varepsilon^R & = & \varepsilon \end{array} \qquad \begin{array}{rcl} (\alpha + \beta)^R & = & \alpha^R + \beta^R \\ (\alpha\beta)^R & = & \beta^R\alpha^R \\ (\alpha^*)^R & = & (\alpha^R)^\star \end{array} \tag{2}$$

A *nondeterministic finite automaton* (NFA) is a five-tuple $A = (Q, \Sigma, \delta, I, F)$ where Q is a finite set of states, Σ is a finite alphabet, $I \subseteq Q$ is the set of initial states, $F \subseteq Q$ is the set of final states, and $\delta : Q \times \Sigma \to 2^Q$ is the transition function. This transition function can be extended to words in the natural way. To simplify the notation, when $|I| = 1$, instead of use $I = \{q_0\}$, $q_0 \in Q$, we use $I = q_0$. Given a state $q \in Q$, the right language of q is $\mathcal{L}(A, q) = \{w \in \Sigma^\star \mid \delta(q, w) \cap F \neq \emptyset\}$, and the left language is $\overleftarrow{\mathcal{L}}(A, q) = \{w \in \Sigma^\star \mid \delta(q_0, w) = q\}$. The language accepted by A is $\mathcal{L}(A) = \mathcal{L}(A, q_0)$. Two NFAs are *equivalent* if they accept the same language. If two NFAs A and B are isomorphic, we write $A \simeq B$. An NFA is deterministic if for all $(q, \sigma) \in Q \times \Sigma$, $|\delta(q, \sigma)| \leq 1$. A DFA is minimal if there is no equivalent DFA with fewer states. Minimal DFAs are unique up to isomorphism. The *size* of an automaton A, $|A|$, is its number of states. The reversal of an automaton A is the automaton A^R, where the initial and final states are interchanged and all transitions are reversed.

If R is a equivalence relation on Q the *quotient automaton* ${}^{A}\!/\!_{R}$ can be constructed by ${}^{A}\!/\!_{R} = ({}^{Q}\!/\!_{R}, \Sigma, {}^{\delta}\!/\!_{R}, [q_0], {}^{F}\!/\!_{R})$, where $[q]$ is the equivalence class that contains $q \in Q$; ${}^{S}\!/\!_{R} = \{[q] \mid q \in S\}$, with $S \subseteq Q$; and ${}^{\delta}\!/\!_{R} = \{([p], \sigma, [q]) \mid (p, \sigma, q) \in \delta\}$. It is easy to see that $\mathcal{L}({}^{A}\!/\!_{R}) = \mathcal{L}(A)$. R is right invariant w.r.t. A if and only if: $R \subseteq (Q-F)^2 \cup F^2$ and $\forall p, q \in Q, \sigma \in \Sigma, \text{if } pRq, \text{ then } {}^{\delta(p,\sigma)}\!/\!_{R} = {}^{\delta(q,\sigma)}\!/\!_{R}$. R is a left invariant relation w.r.t. A if and only if it is a right invariant relation w.r.t. A^R.

The right languages of the states of an automaton A with $Q = [0, n]$ define a system of right equations, $\mathcal{L}_i = \bigcup_{j=1}^{k} \sigma_j \left(\bigcup_{m \in I_{ij}} \mathcal{L}_m \right) \cup \varepsilon(\mathcal{L}_i)$, where $I_{ij} \subseteq [0, n]$ and $m \in I_{ij} \Leftrightarrow \mathcal{L}_m \in \delta(\mathcal{L}_i, \sigma_j)$. In the same manner, the left languages of the states of A define a system of left equations $\overleftarrow{\mathcal{L}}_i = \bigcup_{j=1}^{k} \left(\bigcup_{m \in I_{ij}} \overleftarrow{\mathcal{L}}_m \right) \sigma_j \cup \varepsilon(\overleftarrow{\mathcal{L}}_i)$, where $I_{ij} \subseteq [0, n]$ and $m \in I_{ij} \Leftrightarrow \overleftarrow{\mathcal{L}}_i \in \delta(\overleftarrow{\mathcal{L}}_m, \sigma_j)$; and $\mathcal{L}(A) = \bigcup_{i \in F} \overleftarrow{\mathcal{L}}_i$.

3 Left Derivatives

The left derivative [4] of a regular expression α with respect to a symbol $\sigma \in \Sigma$, denoted $\sigma^{-1}\alpha$, is defined recursively on the structure of α as follows:

$$
\begin{aligned}
\sigma^{-1}\emptyset &= \sigma^{-1}\varepsilon = \emptyset \\
\sigma^{-1}\sigma' &= \begin{cases} \{\varepsilon\}, & \text{if } \sigma' = \sigma \\ \emptyset, & \text{otherwise} \end{cases}
\end{aligned}
\qquad
\begin{aligned}
\sigma^{-1}(\alpha + \beta) &= \sigma^{-1}(\alpha) + \sigma^{-1}(\beta) \\
\sigma^{-1}(\alpha\beta) &= \begin{cases} (\sigma^{-1}\alpha)\beta & \text{if } \varepsilon(\alpha) \neq \varepsilon \\ (\sigma^{-1}\alpha)\beta + \sigma^{-1}\beta & \text{otherwise} \end{cases} \\
\sigma^{-1}(\alpha^\star) &= (\sigma^{-1}\alpha)\alpha^\star
\end{aligned}
\tag{3}
$$

This notion can be extended to derivatives w.r.t. a word $w = w_1 \cdots w_n,\ w_i \in \Sigma^\star$ in the following way:

$$
\begin{aligned}
\varepsilon^{-1}\alpha &= \alpha \\
(w_1 \cdots w_n)^{-1}\alpha &= (w_2 \cdots w_n)^{-1}(w_1^{-1}\alpha)
\end{aligned}
$$

or, more generally,

$$(ps)^{-1}\alpha = s^{-1}(p^{-1}\alpha) \tag{4}$$

for every factorization $w = ps,\ p, s \in \Sigma^\star$.

Brzozowski [4] proved that $L(w^{-1}\alpha) = w^{-1}L(\alpha)$. Let $\mathcal{D}(\alpha)$ be the quotient of the set of all derivatives of a regular expression α modulo the ACI-equivalence relation. Brzozowski also proved that the set $\mathcal{D}(\alpha)$ is finite. Using this result it is possible to define the *Brzozowski's automaton*: $A_{\mathcal{B}}(\alpha) = (\mathcal{D}(\alpha), \Sigma, \delta, [\alpha], F)$, where $F = \{[d] \in \mathcal{D}(\alpha) \mid \varepsilon(d) = \varepsilon\}$, and $\delta([q], \sigma) = [\sigma^{-1}q]$, for all $[q] \in \mathcal{D}(\alpha), \sigma \in \Sigma$. It was proved that this automaton recognizes $L(\alpha)$.

4 Right derivatives

Similar to what happens in the previous section, the right derivative of a regular expression α with respect to a symbol $\sigma \in \Sigma$, denoted $\alpha\sigma^{-1}$, is defined recursively on the structure of

α as follows:

$$\begin{array}{rcl} \emptyset\sigma^{-1} & = & (\varepsilon)\sigma^{-1} = \emptyset \\ \sigma'\sigma^{-1} & = & \begin{cases} \{\varepsilon\}, & \text{if } \sigma' = \sigma \\ \emptyset, & \text{otherwise} \end{cases} \end{array} \qquad \begin{array}{rcl} (\alpha+\beta)\sigma^{-1} & = & (\alpha)\sigma^{-1} + (\beta)\sigma^{-1} \\ (\alpha\beta)\sigma^{-1} & = & \begin{cases} \alpha(\beta\sigma^{-1}) & \text{if } \varepsilon(\beta) \neq \varepsilon \\ \alpha(\beta\sigma^{-1}) + \alpha\sigma^{-1} & \text{otherwise} \end{cases} \\ (\alpha^\star)\sigma^{-1} & = & \alpha^\star(\alpha\sigma^{-1}) \end{array} \tag{5}$$

This definition can be extended to derivatives w.r.t. a word $w = w_1 \cdots w_n,\ w_i \in \Sigma^\star$ in the following way:

$$\begin{array}{rcl} \alpha\varepsilon^{-1} & = & \alpha \\ \alpha(w_1 \cdots w_n)^{-1} & = & (\alpha w_n^{-1})(w_1 \cdots w_{n-1})^{-1} \end{array}$$

More generally we can use:

$$\alpha(ps)^{-1} = (\alpha s^{-1})p^{-1}$$

for every factorization $w = ps,\ p, s \in \Sigma^\star$.

Let $\overleftarrow{\mathcal{D}}(\alpha)$ be the quotient of the set of all right derivatives of a regular expression α modulo the ACI-equivalence relation.

The two following results establish a relationship between the right and the left derivatives.

Proposition 1. *For any regular expression $\alpha \in \mathsf{RE}$ and any $\sigma \in \Sigma$, the following result holds:*

$$\alpha\sigma^{-1} = (\sigma^{-1}\alpha^R)^R$$

Proof. Let us prove the result by induction on α. For the base cases the result is obviously true. Assume that the equality holds for $\alpha_1, \alpha_1 \in \mathsf{RE}$. Let $\alpha = \alpha_1 + \alpha_2$, then:

$$\begin{array}{rcl} (\alpha_1+\alpha_2)\sigma^{-1} & = & \alpha_1\sigma^{-1} + \alpha_2\sigma^{-1} \text{ by (5)} \\ & = & (\sigma^{-1}\alpha_1^R)^R + (\sigma^{-1}\alpha_2^R)^R \text{ by inductive hypothesis} \\ & = & (\sigma^{-1}\alpha_1^R + \sigma^{-1}\alpha_2^R)^R \text{ by (2)} \\ & = & (\sigma^{-1}(\alpha_1^R + \alpha_2^R))^R \text{ by (3)} \\ & = & (\sigma^{-1}(\alpha_1+\alpha_2)^R)^R \text{ by (2)} \end{array}$$

If $\alpha = \alpha_1\alpha_2$, then we have:

$$\begin{array}{rcl} (\alpha_1\alpha_2)\sigma^{-1} & = & \begin{cases} \alpha_1(\alpha_2\sigma^{-1}) & \text{If } \varepsilon(\alpha_2) \neq \varepsilon \\ \alpha_1(\alpha_2\sigma^{-1}) + \alpha_1\sigma^{-1} & \text{otherwise} \end{cases} \text{ by (5)} \\ & = & \begin{cases} \alpha_1(\sigma^{-1}\alpha_2^R)^R & \text{If } \varepsilon(\alpha_2) \neq \varepsilon \\ \alpha_1(\sigma^{-1}\alpha_2^R)^R + (\sigma^{-1}\alpha_1^R)^R & \text{otherwise} \end{cases} \text{ by inductive hypothesis} \\ & = & \begin{cases} ((\sigma^{-1}\alpha_2^R)\alpha_1^R)^R & \text{If } \varepsilon(\alpha_2) \neq \varepsilon \\ ((\sigma^{-1}\alpha_2^R)\alpha_1^R + \sigma^{-1}\alpha_1^R)^R & \text{otherwise} \end{cases} \text{ by (2)} \\ & = & (\sigma^{-1}(\alpha_2^R\alpha_1^R))^R = (\sigma^{-1}(\alpha_1\alpha_2)^R)^R \text{ by (3) and (2), respectively.} \end{array}$$

Finally, if $\alpha = \alpha_1^\star$, then:

$$\begin{array}{rcl} \alpha_1^\star\sigma^{-1} & = & \alpha_1^\star(\alpha_1\sigma^{-1}) \;\; \text{by (5)} \\ & = & \alpha_1^\star(\sigma^{-1}\alpha_1^R)^R \text{ by inductive hypothesis} \\ & = & (\sigma^{-1}(\alpha_1^R)(\alpha_1^R)^\star)^R \;\; \text{by (2)} \\ & = & (\sigma^{-1}(\alpha_1^\star)^R)^R \;\; \text{by (3)} \\ & = & (\sigma^{-1}(\alpha_1^\star)^R)^R \;\; \text{by (2)} \end{array}$$

□

Proposition 2. *For any regular expression $\alpha \in \mathsf{RE}$ and any $w \in \Sigma^+$, the following reult holds:*

$$\alpha w^{-1} = ((w^R)^{-1}\alpha^R)^R.$$

Proof. Let us prove the result by induction on w. If $|w| = 1$, then $w = \sigma$. Thus, in this case, the equality is true by Proposition 1. Assuming that the equality holds for some $w \in \Sigma^+$, let us prove it for $w' = \sigma w$:

$$\begin{array}{rcl} \alpha(\sigma w)^{-1} & = & (\alpha w^{-1})\sigma^{-1} \\ & = & ((w^R)^{-1}\alpha^R)^R\sigma^{-1} \text{ by inductive hypothesis} \\ & = & (\sigma^{-1}(((w^R)^{-1}\alpha^R)^R)^R)^R \;\; \text{by Proposition 1} \\ & = & (\sigma^{-1}((w^R)^{-1}\alpha^R))^R \\ & = & ((w^R\sigma)^{-1}\alpha^R)^R \;\; \text{by (4)} \\ & = & (((\sigma w)^R)^{-1}\alpha^R)^R \;\; \text{by (2)} \end{array}$$

□

Using these relations is not difficult to prove that the following result holds.

Proposition 3. *For any regular expression $\alpha \in \mathsf{RE}$ and any word $w \in \Sigma^\star$ the following holds:*

$$L(\alpha w^{-1}) = L(\alpha)w^{-1}.$$

Proof. It is known that $L(w^{-1}\alpha) = w^{-1}L(\alpha)$. Thus, we have:

$$\begin{array}{rcl} L(\alpha w^{-1}) & = & L(((w^R)^{-1}\alpha^R)^R), \text{ because } \alpha w^{-1} = ((w^R)^{-1}\alpha^R)^R \\ & = & (L((w^R)^{-1}\alpha^R))^R \\ & = & ((w^R)^{-1}L(\alpha^R))^R, \text{ because } L(w^{-1}\alpha) = w^{-1}L(\alpha) \\ & = & L(\alpha)w^{-1}, \text{ because } Lw^{-1} = (w^R)^{-1}L^R \end{array}$$

□

Using the Proposition 1 and Proposition 2 we can conclude that:

Corollary 1. *For any regular expression $\alpha \in \mathsf{RE}$, $\overleftarrow{\mathcal{D}}(\alpha) = (\mathcal{D}(\alpha^R))^R$*

As we know that the set $\mathcal{D}(\alpha)$ of derivatives modulo ACI-equivalence is finite, by the Corollary 1 we can conclude that the set $\overleftarrow{\mathcal{D}}(\alpha)$ is also finite. Thus, we can define the NFA $\overleftarrow{A}_{\mathcal{B}}(\alpha)$, whose states are the right derivatives of α modulo ACI-equivalence.

The *right Brzozowski's automaton* of a regular expression α is defined by $\overleftarrow{A}_{\mathcal{B}}(\alpha) = (\overleftarrow{\mathcal{D}}(\alpha), \Sigma, \delta, I, \{[\alpha]\})$, where $i = \{[d] \in \overleftarrow{\mathcal{D}}(\alpha) \mid \varepsilon(d) = \varepsilon\}$, and $\delta([q], \sigma) = \{[q'] \in \overleftarrow{\mathcal{D}}(\alpha) \mid [q'\sigma^{-1}] = [q]\}$.

Corollary 2. *For any regular expression* $\alpha \in \mathsf{RE}$, $\overleftarrow{A}_{\mathcal{B}}(\alpha) \simeq (A_{\mathcal{B}}(\alpha^R))^R$.

Corollary 3. *For any regular expression* $\alpha \in \mathsf{RE}$, $L(\overleftarrow{A}_{\mathcal{B}}(\alpha)) = L(\alpha)$.

As $(\overleftarrow{A}_{\mathcal{B}}(\alpha))^R = A_{\mathcal{B}}(\alpha^R)$ and $A_{\mathcal{B}}(\alpha^R)$ is deterministic, for any regular expression $\alpha \in \mathsf{RE}$, $\overleftarrow{A}_{\mathcal{B}}$ is a disjoint NFA [19] or a partial átomaton [5].

5 Left Partial Derivates Automaton

The partial derivative automaton of a regular expression was introduced independently by Mirkin [17] and Antimirov [1]. Champarnaud and Ziadi [7] proved that the two formulations are equivalent. For a regular expression $\alpha \in \mathsf{RE}$ and a symbol $\sigma \in \Sigma$, the set of left-partial derivatives of α w.r.t. σ is defined inductively as follows:

$$
\begin{array}{rclrcl}
\partial_\sigma(\emptyset) & = & \partial_\sigma(\varepsilon) = \emptyset & \partial_\sigma(\alpha+\beta) & = & \partial_\sigma(\alpha) \cup \partial_\sigma(\beta) \\
\partial_\sigma(\sigma') & = & \begin{cases} \{\varepsilon\}, & \text{if } \sigma' = \sigma \\ \emptyset, & \text{otherwise} \end{cases} & \begin{array}{r}\partial_\sigma(\alpha\beta) \\ \partial_\sigma(\alpha^\star)\end{array} & \begin{array}{c} = \\ = \end{array} & \begin{array}{l}\partial_\sigma(\alpha)\beta \cup \varepsilon(\alpha)\partial_\sigma(\beta) \\ \partial_\sigma(\alpha)\alpha^\star\end{array}
\end{array}
\tag{6}
$$

where for any $S \subseteq \mathsf{RE}$, $S\emptyset = \emptyset S = \emptyset$, $S\varepsilon = \varepsilon S = S$, and $S\beta = \{\alpha\beta | \alpha \in S\}$ if $\beta \neq \emptyset$, and $\beta \neq \varepsilon$.

The definition of left-partial derivative can be extended to sets of regular expressions, words, and languages. Given a regulare expression $\alpha \in \mathsf{RE}$, a symbol $\sigma \in \Sigma$, a word $w \in \Sigma^\star$ and a set $S \subseteq \mathsf{RE}$, we define $\partial_\sigma(S) = \bigcup_{\alpha \in S} \partial_\sigma(\alpha)$, $\partial_\varepsilon(\alpha) = \alpha$, $\partial_{w\sigma}(\alpha) = \partial_\sigma(\partial_w(\alpha))$, and $\partial_L(\alpha) = \bigcup_{w \in L} \partial_w(\alpha)$ for $L \subseteq \Sigma^\star$. It is known that $\bigcup_{\tau \in \partial_w(\alpha)} \mathcal{L}(\tau) = w^{-1}\mathcal{L}(\alpha)$. The set of all partial derivatives of α w.r.t. words is denoted by $\mathsf{PD}(\alpha) = \bigcup_{w \in \Sigma^\star} \partial_w(\alpha)$. Note that the set $\mathsf{PD}(\alpha)$ is always finite [1], as opposed to what happens for the Brzozowski derivatives set which is only finite modulo ACI.

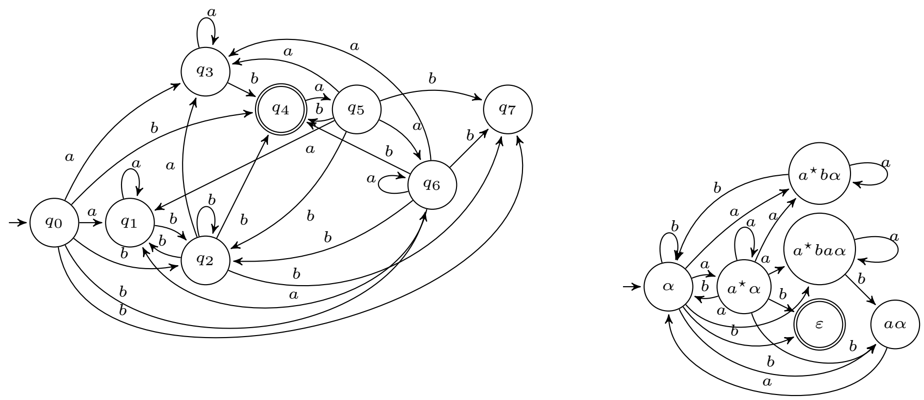
The partial derivative automaton is defined by $\mathcal{A}_{pd}(\alpha) = (\mathsf{PD}(\alpha), \Sigma, \delta_{pd}, \alpha, F_{pd})$, where $\delta_{pd} = \{(\tau, \sigma, \tau') \mid \tau \in \mathsf{PD}(\alpha),\ \sigma \in \Sigma,\ \tau' \in \partial_\sigma(\tau)\}$ and $F_{pd} = \{\tau \in \mathsf{PD}(\alpha) \mid \varepsilon(\tau) = \varepsilon\}$.

As noted by Broda et al. [3] and Maia et al. [15], following Mirkin's construction, the partial derivative automaton of α can be inductively constructed. A *(right) support* for α is a set of regular expressions $\{\alpha_1, \ldots, \alpha_n\}$ such that $\alpha_i = \sigma_1\alpha_{i1} + \cdots + \sigma_k\alpha_{ik} + \varepsilon(\alpha_i)$, $i \in [0, n]$, $\alpha_0 \equiv \alpha$ and α_{ij} is a linear combination of α_l, $l \in [1, n]$ and $j \in [1, k]$. The set $\pi(\alpha)$ inductively defined as follows is a right support of α:

$$
\begin{array}{rclrcl}
\pi(\emptyset) & = & \emptyset & \pi(\alpha+\beta) & = & \pi(\alpha) \cup \pi(\beta) \\
\pi(\varepsilon) & = & \emptyset & \pi(\alpha\beta) & = & \pi(\alpha)\beta \cup \pi(\beta) \\
\pi(\sigma) & = & \{\varepsilon\} & \pi(\alpha^\star) & = & \pi(\alpha)\alpha^\star.
\end{array}
\tag{7}
$$

Champarnaud and Ziadi [7] proved that $\mathsf{PD}(\alpha) = \pi(\alpha) \cup \{\alpha\}$.

We also need an inductive definition of the set of transitions of $\mathcal{A}_{pd}(\alpha)$. Let $\varphi(\alpha) = \{(\sigma, \gamma) \mid \gamma \in \partial_\sigma(\alpha), \sigma \in \Sigma\}$ and $\lambda(\alpha) = \{\alpha' \mid \alpha' \in \pi(\alpha),\ \varepsilon(\alpha') = \varepsilon\}$, where both sets can be inductively defined using (6) and (7). We have, $\delta_{pd} = \{\alpha\} \times \varphi(\alpha) \cup F(\alpha)$ where the result of



(a) $\mathcal{A}_{pos}(\alpha)$ (b) $\mathcal{A}_{pd}(\alpha)$

Figure 1: $\alpha = (a^\star b + a^\star ba + a^\star)^\star b$.

the $\times$ operation is seen as a set of triples and the set F is defined inductively by:

$$\begin{array}{rcl} F(\emptyset) & = & F(\varepsilon) = F(\sigma) = \emptyset,\ \sigma \in \Sigma \\ F(\alpha+\beta) & = & F(\alpha) \cup F(\beta) \\ F(\alpha\beta) & = & F(\alpha)\beta \cup F(\beta) \cup \lambda(\alpha)\beta \times \varphi(\beta) \\ F(\alpha^\star) & = & F(\alpha)\alpha^\star \cup (\lambda(\alpha) \times \varphi(\alpha))\alpha^\star. \end{array} \tag{8}$$

Note that the concatenation of a transition (α, σ, β) with a regular expression is defined by $(\alpha, \sigma, \beta)\gamma = (\alpha\gamma, \sigma, \beta\gamma)$, if $\gamma \notin \{\emptyset, \varepsilon\}$, $(\alpha, \sigma, \beta)\emptyset = \emptyset$ and $(\alpha, \sigma, \beta)\varepsilon = (\alpha, \sigma, \beta)$. Then,

$$\mathcal{A}_{pd}(\alpha) = (\pi(\alpha) \cup \{\alpha\}, \Sigma, \{\alpha\} \times \varphi(\alpha) \cup F(\alpha), \alpha, \lambda(\alpha) \cup \varepsilon(\alpha)\{\alpha\}).$$

In Fig. 1 are represented $\mathcal{A}_{pos}(\alpha)$ and $\mathcal{A}_{pd}(\alpha)$, where $\alpha = (a^\star b + a^\star ba + a^\star)^\star b$.

Champarnaud and Ziadi [8] showed that the partial derivative automaton is a quotient of the Glushkov automaton by the right-invariant equivalence relation $\equiv_c$, such that $i \equiv_c j$ if $\partial_{w\sigma_i}(\overline{\alpha}) = \partial_{w\sigma_j}(\overline{\alpha})$, where it is known that $\partial_{w\sigma_i}(\overline{\alpha})$ is either empty or an unique singleton for all $w \in \Sigma^\star$.

6 Right Partial Derivates Automaton

In the previous section we define the partial derivative automaton using the left-partial derivatives. We can also define a partial derivative automaton using the right-partial derivatives.

The concept of right-partial derivatives was introduced by Champarnaud *et. al* [6]. For a regular expression $\alpha \in \mathsf{RE}$ and a symbol $\sigma \in \Sigma$, the set of right-partial derivatives of α w.r.t. σ, $\overleftarrow{\partial}_\sigma(\alpha)$, is defined inductively as follows:

$$\begin{array}{rcl} \overleftarrow{\partial}_\sigma(\emptyset) & = & \overleftarrow{\partial}_\sigma(\varepsilon) = \emptyset \\ \overleftarrow{\partial}_\sigma(\sigma') & = & \begin{cases} \{\varepsilon\}, & \text{if } \sigma' = \sigma \\ \emptyset, & \text{otherwise} \end{cases} \end{array} \qquad \begin{array}{rcl} \overleftarrow{\partial}_\sigma(\alpha+\beta) & = & \overleftarrow{\partial}_\sigma(\alpha) \cup \overleftarrow{\partial}_\sigma(\beta) \\ \overleftarrow{\partial}_\sigma(\alpha\beta) & = & \alpha\overleftarrow{\partial}_\sigma(\beta) \cup \varepsilon(\beta)\overleftarrow{\partial}_\sigma(\alpha) \\ \overleftarrow{\partial}_\sigma(\alpha^\star) & = & \alpha^\star\overleftarrow{\partial}_\sigma(\alpha). \end{array} \tag{9}$$

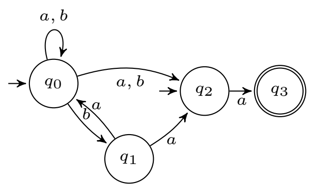


Figure 2: $\overleftarrow{\mathcal{A}}_{pd}(\alpha) : q_0 = (a^\star b + a^\star ba + a^\star)^\star a^\star, q_1 = (a^\star b + a^\star ba + a^\star)^\star a^\star b, q_2 = (a^\star b + a^\star ba + a^\star)^\star, q_3 = (a^\star b + a^\star ba + a^\star)^\star b$

The definition of right-partial derivative can be extended to sets of regular expressions, words, and languages. Given a regulare expression $\alpha \in \mathsf{RE}$, a symbol $\sigma \in \Sigma$, a word $w \in \Sigma^\star$ and a set $S \subseteq \mathsf{RE}$, we define $\overleftarrow{\partial}_\sigma(S) = \bigcup_{\alpha \in S} \overleftarrow{\partial}_\sigma(\alpha)$, $\partial_\varepsilon(\alpha) = \alpha$, $\overleftarrow{\partial}_{\sigma w}(\alpha) = \overleftarrow{\partial}_\sigma(\overleftarrow{\partial}_w(\alpha))$, and $\overleftarrow{\partial}_L(\alpha) = \bigcup_{w \in L} \overleftarrow{\partial}_w(\alpha)$ for $L \subseteq \Sigma^\star$. The set of all right-partial derivatives of α w.r.t. words is denoted by $\overleftarrow{\mathsf{PD}}(\alpha) = \bigcup_{w \in \Sigma^\star} \overleftarrow{\partial}_w(\alpha)$. In Fig. 6 is represented the $\overleftarrow{\mathcal{A}}_{pd}(a^\star b + a^\star ba + a^\star)^\star b)$.

The next results establish a parallel between the left and the right partial derivatives.

Proposition 4. *For a regular expression α and a symbol $\sigma \in \Sigma$, $(\partial_\sigma(\alpha^R))^R = \overleftarrow{\partial}_\sigma(\alpha)$.*

Proof. Let us prove the result by induction on the structure of α. For the base cases the equality is obvious. Let $\alpha = \alpha_1 + \alpha_2$, then

$$\begin{array}{rcl} (\partial_\sigma((\alpha_1 + \alpha_2)^R))^R & = & (\partial_\sigma(\alpha_1^R) \cup \partial_\sigma(\alpha_2^R))^R \\ & = & (\partial_\sigma(\alpha_1^R))^R \cup (\partial_\sigma(\alpha_2^R))^R \\ & = & \overleftarrow{\partial}_\sigma(\alpha_1) \cup \overleftarrow{\partial}_\sigma(\alpha_2) \\ & = & \overleftarrow{\partial}_\sigma(\alpha) \end{array}$$

Let $\alpha = \alpha_1\alpha_2$, then

$$\begin{array}{rcl} (\partial_\sigma((\alpha_1\alpha_2)^R))^R & = & (\partial_\sigma(\alpha_2^R\alpha_1^R))^R \\ & = & (\partial_\sigma(\alpha_2^R)\alpha_1^R \cup \varepsilon(\alpha_2^R)\partial_\sigma(\alpha_1^R))^R \\ & = & (\partial_\sigma(\alpha_2^R)\alpha_1^R)^R \cup (\varepsilon(\alpha_2^R)\partial_\sigma(\alpha_1^R))^R \\ & = & \alpha_1 \overleftarrow{\partial}_\sigma(\alpha_2) \cup \varepsilon(\alpha_2) \overleftarrow{\partial}_\sigma(\alpha_1) \\ & = & \overleftarrow{\partial}_\sigma(\alpha) \end{array}$$

Let $\alpha = \alpha_1^\star$

$$\begin{array}{rcl} (\partial_\sigma((\alpha_1^\star)^R))^R & = & (\partial_\sigma((\alpha_1^R)^\star))^R \\ & = & (\partial_\sigma(\alpha_1^R)(\alpha_1^R)^\star)^R \\ & = & \alpha_1^\star(\partial_\sigma(\alpha_1^R))^R \\ & = & \alpha_1^\star \overleftarrow{\partial}_\sigma(\alpha_1) \\ & = & \overleftarrow{\partial}_\sigma(\alpha) \end{array}$$

Thus the equality in the Proposition holds. □

Proposition 5. *For a regular expression α and a word $w \in \Sigma^\star$, $(\partial_{w^R}(\alpha^R))^R = \overleftarrow{\partial}_w(\alpha)$.*

Proof. Let us proceed by induction on the size of w. If $w = \varepsilon$ the result is obviously true. If $w = \sigma$ the result is true by the Proposition 4. Assuming that the result is true for w, let us prove it for $w' = \sigma w$:

$$\begin{aligned}
\overleftarrow{\partial}_{\sigma w}(\alpha) &= \overleftarrow{\partial}_{\sigma}(\overleftarrow{\partial}_{w}(\alpha)) = \overleftarrow{\partial}_{\sigma}((\partial_{w^R}(\alpha^R))^R) \\
&= (\partial_{\sigma}(((\partial_{w^R}(\alpha^R))^R)^R))^R \\
&= (\partial_{\sigma}(\partial_{w^R}(\alpha^R)))^R \\
&= (\partial_{w^R\sigma}(\alpha^R))^R = (\partial_{(\sigma w)^R}(\alpha^R))^R
\end{aligned}$$

□

Proposition 6. *For a regular expression* α, $\overleftarrow{\mathsf{PD}}(\alpha) = (\mathsf{PD}(\alpha^R))^R$.

Proof.

$$\begin{aligned}
\overleftarrow{\mathsf{PD}}(\alpha) &= (\mathsf{PD}(\alpha^R))^R \\
(\overleftarrow{\mathsf{PD}}(\alpha))^R &= \mathsf{PD}(\alpha^R)) \\
\left(\bigcup_{w\in\Sigma^\star} \overleftarrow{\partial}_{w}(\alpha)\right)^R &= \bigcup_{w\in\Sigma^\star} \partial_w(\alpha^R) \\
\left(\bigcup_{w\in\Sigma^\star} (\partial_{w^R}(\alpha^R))^R\right)^R &= \bigcup_{w\in\Sigma^\star} \partial_w(\alpha^R) \\
\bigcup_{w\in\Sigma^\star} \partial_{w^R}(\alpha^R) &= \bigcup_{w\in\Sigma^\star} \partial_w(\alpha^R)
\end{aligned}$$

As $\bigcup_{w\in\Sigma^\star} w = \bigcup_{w\in\Sigma^\star} w^R$ the equality holds. □

Using the last result is not difficult to conclude that $\overleftarrow{\mathsf{PD}}$ is finite. Thus, the right partial derivative automaton for α is

$$\overleftarrow{\mathcal{A}}_{pd}(\alpha) = (\overleftarrow{\mathsf{PD}}(\alpha) \cup \{\alpha\}, \Sigma, \overleftarrow{\delta}_{pd}, \overleftarrow{F}_{pd}(\alpha), \alpha),$$

where $\overleftarrow{\delta}_{pd} = \{(q', a, q) \mid q \in \overleftarrow{\mathsf{PD}}(\alpha),\ q' \in \overleftarrow{\partial}_a(q),\ \text{and } a \in \Sigma\}$, $\overleftarrow{F}_{pd} = \{q \in \overleftarrow{\mathsf{PD}}(\alpha) \mid \varepsilon(q) = \varepsilon\}$. Note that $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$ has always just one final state and it can has more than one initial state.

As what happens for $\mathcal{A}_{pd}$, the $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$ can also be defined inductively as a solution of a left system of expression equations, $\alpha_i = \alpha_{i1}\sigma_1 + \cdots + \alpha_{ik}\sigma_k + \varepsilon(\alpha_i)$, $i \in [0, n]$, $\alpha_0 \equiv \alpha$, α_{ij} is a linear combination of α_l, $l \in [1, n]$ and $j \in [1, k]$.

Proposition 7. *The set of regular expressions* $\overleftarrow{\pi}(\alpha)$ *is a solution of a left system of expression equations,*

$$\begin{aligned}
\overleftarrow{\pi}(\emptyset) &= \emptyset & \qquad \overleftarrow{\pi}(\alpha+\beta) &= \overleftarrow{\pi}(\alpha) \cup \overleftarrow{\pi}(\beta) \\
\overleftarrow{\pi}(\varepsilon) &= \emptyset & \overleftarrow{\pi}(\alpha\beta) &= \alpha\overleftarrow{\pi}(\beta) \cup \overleftarrow{\pi}(\alpha) \\
\overleftarrow{\pi}(\sigma) &= \{\varepsilon\} & \overleftarrow{\pi}(\alpha^\star) &= \alpha^\star\overleftarrow{\pi}(\alpha).
\end{aligned} \tag{10}$$

Proof. For $\alpha = \emptyset$ and $\alpha = \varepsilon$ is obvious that the solution is $\emptyset$. For $\alpha = \sigma$, $\alpha = \varepsilon\sigma$ thus the solution set is $\{\varepsilon\}$. Let us suppose that

$$\begin{aligned} \beta &= \beta_0 \\ \beta_i &= \beta_{i1}\sigma_1 + \cdots + \beta_{ik}\sigma_k + \varepsilon(\beta_i), \end{aligned}$$

with $\overleftarrow{\pi}(\beta) = \{\beta_1, \cdots, \beta_n\}$ and

$$\begin{aligned} \gamma &= \gamma_0 \\ \gamma_i &= \gamma_{i1}\sigma_1 + \cdots + \gamma_{ik}\sigma_k + \varepsilon(\gamma_i), \end{aligned}$$

with $\overleftarrow{\pi}(\gamma) = \{\gamma_1, \cdots, \gamma_m\}$. Consider $\alpha = \beta + \gamma$, then

$$\beta + \gamma = \beta_0 + \gamma_0$$

As we need all β_i, $i \in [1, n]$ to define β, and all γ_i, $i \in [1, m]$ to define γ, $\overleftarrow{\pi}(\alpha\beta) = \{\beta_1, \cdots, \beta_n\} \cup \{\gamma_1, \cdots, \gamma_m\}$. Consider $\alpha = \beta\gamma$ then

$$\begin{aligned} \beta\gamma &= \beta\gamma_0 \\ &= \beta(\gamma_{01}\sigma_1 + \cdots + \gamma_{0k}\sigma_k + \varepsilon(\gamma_0)) \\ &= \beta\gamma_{01}a_1 + \cdots + \beta\gamma_{0k}a_k + \varepsilon(\gamma_0)\beta \\ &= (\beta\gamma_{01} + \varepsilon(\gamma_0)\beta_{01})\sigma_1 + \cdots + (\beta\gamma_{0k} + \varepsilon(\gamma_0)\beta_{0k}) + \varepsilon(\gamma_0)\varepsilon(\beta_0) \end{aligned}$$

And,

$$\beta\gamma_i = (\beta\gamma_{0i} + \varepsilon(\gamma_0)\beta_{01})\sigma_1 + \cdots + (\beta\gamma_{ik} + \varepsilon(\gamma_i)\beta_{0k})\sigma_k + \varepsilon(\gamma_i)\varepsilon(\beta_0)$$

As we know that exists $i \in [0, m]$ such that $\varepsilon(\gamma_i) = \varepsilon$, then $\overleftarrow{\pi}(\beta\gamma) = \{\beta\gamma_1, \cdots, \beta\gamma_m\} \cup \{\beta_1, \cdots, \beta_n\}$.

Consider $\alpha = \beta^\star$ then

$$\begin{aligned} \beta^\star &= \beta^\star\beta + \varepsilon \\ &= \beta^\star(\beta_{01}\sigma_1 + \cdots + \beta_{0k}\sigma_k + \varepsilon(\beta_i)) + \varepsilon \\ &= \beta^\star\beta_{01}\sigma_1 + \cdots + \beta^\star\beta_{0k}\sigma_k + \varepsilon(\beta_i)\beta^\star + \varepsilon \end{aligned}$$

And,

$$\beta^\star\beta_i = \beta^\star\beta_{0i}\sigma_1 + \cdots + \beta^\star\beta_{0i}\sigma_k + \varepsilon(\beta_i)\beta^\star$$

Thus, $\overleftarrow{\pi}(\beta^\star) = \{\beta^\star\beta_1, \cdots, \beta^\star\beta_n\}$.

Therefore the set $\overleftarrow{\pi}(\alpha)$ is a solution of the system of equations. □

Let $\overleftarrow{\varphi}(\alpha) = \{(\gamma, \sigma) \mid \gamma \in \overleftarrow{\partial}_\sigma(\alpha), \sigma \in \Sigma\}$ and $\overleftarrow{\lambda}(\alpha) = \{\alpha' \mid \alpha' \in \overleftarrow{\pi}(\alpha),\ \varepsilon(\alpha') = \varepsilon\}$, where both sets can be inductively defined as follows:

$$
\begin{array}{rclrcl}
\overleftarrow{\varphi}(\emptyset) & = & \emptyset & \overleftarrow{\varphi}(\alpha+\beta) & = & \overleftarrow{\varphi}(\alpha)\cup\overleftarrow{\varphi}(\alpha)\\
\overleftarrow{\varphi}(\varepsilon) & = & \emptyset & \overleftarrow{\varphi}(\alpha\beta) & = & \alpha\overleftarrow{\varphi}(\beta)\cup\varepsilon(\beta)\overleftarrow{\varphi}(\alpha)\\
\overleftarrow{\varphi}(\sigma) & = & \{(\varepsilon,\sigma)\},\ \sigma\in\Sigma & \overleftarrow{\varphi}(\alpha^\star) & = & \alpha^\star\overleftarrow{\varphi}(\alpha)\\
\overleftarrow{\lambda}(\emptyset) & = & \emptyset & \overleftarrow{\lambda}(\alpha+\beta) & = & \overleftarrow{\lambda}(\alpha)\cup\overleftarrow{\lambda}(\beta)\\
\overleftarrow{\lambda}(\varepsilon) & = & \emptyset & \overleftarrow{\lambda}(\alpha\beta) & = & \varepsilon(\alpha)\alpha\overleftarrow{\lambda}(\beta)\cup\overleftarrow{\lambda}(\alpha)\\
\overleftarrow{\lambda}(\sigma) & = & \{\varepsilon\},\ \sigma\in\Sigma & \overleftarrow{\lambda}(\alpha^\star) & = & \alpha^\star\overleftarrow{\lambda}(\alpha).
\end{array}
\tag{11}
$$

The set of transitions of $\overleftarrow{\mathcal{A}}_{pd}$ is $\overleftarrow{\varphi}(\alpha)\times\{\alpha\}\cup\overleftarrow{F}(\alpha)$ where the set $\overleftarrow{F}$ is defined inductively by:

$$
\begin{array}{rcl}
\overleftarrow{F}(\emptyset) & = & \overleftarrow{F}(\varepsilon)=\overleftarrow{F}(\sigma)=\emptyset,\ \sigma\in\Sigma\\
\overleftarrow{F}(\alpha+\beta) & = & \overleftarrow{F}(\alpha)\cup\overleftarrow{F}(\beta)\\
\overleftarrow{F}(\alpha\beta) & = & \alpha\overleftarrow{F}(\beta)\cup\overleftarrow{F}(\alpha)\cup\varphi(\alpha)\times(\alpha\overleftarrow{\lambda}(\beta))\\
\overleftarrow{F}(\alpha^\star) & = & \alpha^\star\overleftarrow{F}(\alpha)\cup\alpha^\star(\overleftarrow{\varphi}(\alpha)\times\overleftarrow{\lambda}(\alpha)).
\end{array}
\tag{12}
$$

Note that the concatenation of a regular expression with a transition (α,σ,β) is defined by $\gamma(\alpha,\sigma,\beta)=(\gamma\alpha,\sigma,\gamma\beta)$, if $\gamma\notin\{\emptyset,\varepsilon\}$, $\emptyset(\alpha,\sigma,\beta)=\emptyset$ and $\varepsilon(\alpha,\sigma,\beta)=(\alpha,\sigma,\beta)$.

Proposition 8. *The inductive definitions given in (11) and (12) follows from the resolution of the above system of equations.*

Proof. For the base cases it is obvious. Let us suppose that

$$
\begin{array}{rcl}
\beta & = & \beta_0\\
\beta_i & = & \beta_{i1}\sigma_1+\cdots+\beta_{ik}\sigma_k+\varepsilon(\beta_i),
\end{array}
$$

with $\overleftarrow{\lambda}(\beta)=\{\beta_i\mid\varepsilon(\beta_i)=\varepsilon\}$, $\overleftarrow{\varphi}(\beta)=\{(\beta_i,\sigma_1)\mid\beta_i\in\beta_{i1},\ \beta=\beta_{i1}\sigma_1\}$, and $\overleftarrow{F}(\beta)=\{(\beta_i,\sigma_1,\beta_j)\mid\beta_i\in\beta_{i1},\ \beta_j=\beta_{i1}\sigma_1\}$.

and

$$
\begin{array}{rcl}
\gamma & = & \gamma_0\\
\gamma_i & = & \gamma_{i1}\sigma_1+\cdots+\gamma_{ik}\sigma_k+\varepsilon(\gamma_i),
\end{array}
$$

with $\overleftarrow{\lambda}(\gamma)=\{\beta_i\mid\varepsilon(\gamma_i)=\varepsilon\}$, $\overleftarrow{\varphi}(\gamma)=\{(\gamma_i,\sigma_1)\mid\gamma_i\in\gamma_{i1},\ \gamma=\gamma_{i1}\sigma_1\}$, and $\overleftarrow{F}(\gamma)=\{(\gamma_i,\sigma_1,\gamma_j)\mid\gamma_i\in\gamma_{i1},\ \gamma_j=\gamma_{i1}\sigma_1\}$ For the case $\alpha=\beta+\gamma$, it is obvious. Consider $\alpha=\beta\gamma$ then

$$
\begin{array}{rcl}
\beta\gamma & = & \beta\gamma_0\\
 & = & \beta(\gamma_{01}\sigma_1+\cdots+\gamma_{0k}\sigma_k+\varepsilon(\gamma_0))\\
 & = & \beta\gamma_{01}a_1+\cdots+\beta\gamma_{0k}a_k+\varepsilon(\gamma_0)\beta
\end{array}
$$

From the last row of the equation is not difficult to conclude that $\overleftarrow{\varphi}(\beta\gamma)=\beta\overleftarrow{\varphi}(\gamma)\cup\varepsilon(\gamma)\overleftarrow{\varphi}(\beta)$. We know that $\overleftarrow{\lambda}(\alpha)=\{\alpha_i\mid\varepsilon(\alpha_i)=\varepsilon\}$, then in this case $\overleftarrow{\lambda}(\alpha)=\{\beta_i\mid\varepsilon(\beta_i)=$

$\varepsilon\}\cup\{\beta\gamma_i \mid \varepsilon(\gamma_i)=\varepsilon,\ \varepsilon(\beta)=\varepsilon\}$. Thus, $\overleftarrow{\lambda}(\alpha)=\overleftarrow{\lambda}(\beta)\cup\varepsilon(\beta)\beta\overleftarrow{\lambda}(\gamma)$. Considering the solutions β_i we can conclude that $\overleftarrow{F}(\beta)\subseteq\overleftarrow{F}(\alpha)$, and considerinf the solutions $\beta\gamma_i$ we conclude that $\beta\overleftarrow{F}(\gamma)\cup\overleftarrow{\varphi}(\beta)\times(\beta\overleftarrow{\lambda}(\gamma))\subseteq Fr(\alpha)$. Thus, $\overleftarrow{F}(\alpha)=\overleftarrow{F}(\beta)\cup\beta\overleftarrow{F}(\gamma)\cup\overleftarrow{\varphi}(\beta)\times(\beta\overleftarrow{\lambda}(\gamma))$.

Consider $\alpha=\beta^\star$ then

$$\begin{aligned}\beta^\star &= \beta^\star\beta+\varepsilon\\ &= \beta^\star(\beta_{01}\sigma_1+\cdots+\beta_{0k}\sigma_k+\varepsilon(\beta_i))+\varepsilon\end{aligned}$$

From this it is not difficult to conclude that $\overleftarrow{\varphi}(\beta^\star)=\beta^\star\overleftarrow{\varphi}(\beta)$. Looking at the definition of β_i, it is also not difficult to see that $\overleftarrow{\lambda}(\beta^\star)=\beta^\star\overleftarrow{\lambda}(\beta)$.

We know that

$$\begin{aligned}\beta^\star\beta_i &= \beta^\star\beta_{0i}\sigma_1+\cdots+\beta^\star\beta_{0i}\sigma_k+\varepsilon(\beta_i)\beta^\star\\ &= \beta^\star\beta_{0i}\sigma_1+\cdots+\beta^\star\beta_{0i}\sigma_k+\varepsilon(\beta_i)(\beta^\star\beta+\varepsilon)\end{aligned}$$

Thus, $\overleftarrow{F}(\beta^\star)=\beta^\star\overleftarrow{F}(\beta)\cup\beta^\star(\overleftarrow{\varphi}(\beta)\times\overleftarrow{\lambda}(\beta))$.

□

Then, the right-partial derivative automaton of α is

$$\mathcal{A}_{pd}(\alpha)=(\overleftarrow{\pi}(\alpha)\cup\{\alpha\},\Sigma,\overleftarrow{\varphi}(\alpha)\times\{\alpha\}\cup\overleftarrow{F}(\alpha),\overleftarrow{\lambda}(\alpha)\cup\varepsilon(\alpha)\{\alpha\},\{\alpha\}).$$

We can relate the set π with the set $\overleftarrow{\pi}$:

Proposition 9. *Let α be a regular expression. Then $(\pi(\alpha^R))^R=\overleftarrow{\pi}(\alpha)$.*

Proof. Let us prove by induction on the structure of α. For $\alpha=\varepsilon$, $\alpha=\emptyset$ and $\alpha=\sigma\in\Sigma$ it is obvious. Suppose that the equality is true for any subexpression of α, and let us prove that it is also true for α. If $\alpha=\alpha_1+\alpha_2$, then

$$\begin{aligned}(\pi(\alpha^R))^R &= (\pi(\alpha_1^R)\cup\pi(\alpha_2^R))^R\\ &= (\pi(\alpha_1^R))^R\cup(\pi(\alpha_2^R))^R\\ &= \overleftarrow{\pi}(\alpha_1)\cup\overleftarrow{\pi}(\alpha_2)\\ &= \overleftarrow{\pi}(\alpha_1+\alpha_2).\end{aligned}$$

If $\alpha=\alpha_1\alpha_2$, then

$$\begin{aligned}(\pi((\alpha_1\alpha_2)^R))^R &= (\pi(\alpha_2^R\alpha_1^R))^R\\ &= (\pi(\alpha_2^R)\alpha_1^R\cup\pi(\alpha_1^R))^R\\ &= (\pi(\alpha_2^R)\alpha_1^R)^R\cup(\pi(\alpha_1^R))^R\\ &= (\alpha_1^R)^R(\pi(\alpha_2^R))^R\cup\overleftarrow{\pi}(\alpha_1)\\ &= \alpha_1\overleftarrow{\pi}(\alpha_2)\cup\overleftarrow{\pi}(\alpha_1)\\ &= \overleftarrow{\pi}(\alpha_1\alpha_2)\end{aligned}$$

If $\alpha=\alpha_1^\star$, then

$$\begin{aligned}(\pi((\alpha_1^\star)^R))^R &= (\pi^((\alpha_1^R)^\star))^R\\ &= (\pi(\alpha_1^R)(\alpha_1^R)^\star)^R\\ &= ((\alpha_1^\star)^R)^R(\pi(\alpha_1^R))^R\\ &= \alpha_1^\star\overleftarrow{\pi}(\alpha_1)\\ &= \overleftarrow{\pi}(\alpha_1^\star)\end{aligned}$$

$\square$

Note that the sizes of $\pi(\alpha)$ and $\overleftarrow{\pi}(\alpha)$ are not comparable in general. For example, if $\alpha = (a^\star b + a^\star ba + a^\star)^\star b$ then $|\pi(\alpha)| > |\overleftarrow{\pi}(\alpha)|$, but if we consider $\beta = b(ba^\star + aba^\star + a^\star)^\star$ then $|\pi(\beta)| < |\overleftarrow{\pi}(\beta)|$. As $\pi(\alpha)$ and $\overleftarrow{\pi}(\alpha)$ are subsets of the set of states of $\mathcal{A}_{pd}$ and $\overleftarrow{\mathcal{A}}_{pd}$ respectively, the number of states of $\mathcal{A}_{pd}$ and of $\overleftarrow{\mathcal{A}}_{pd}$ are also not related.

Then we can conclude that the set of states of the right-partial derivative automaton is partially given by the set $\overleftarrow{\pi}(\alpha)$, moreover

Corollary 4. *For a regular expression α, $\overleftarrow{\mathsf{PD}}(\alpha) = \overleftarrow{\pi}(\alpha) \cup \{\alpha\}$.*

Proof. For any regular expression $\alpha \in RE$ we know that

$$\begin{array}{rll} & \mathsf{PD}(\alpha) & = \pi(\alpha) \cup \{\alpha\} \\ \Leftrightarrow & \mathsf{PD}(\alpha^R) & = \pi(\alpha^R) \cup \{\alpha^R\} \\ \Leftrightarrow & (\overleftarrow{\mathsf{PD}}(\alpha))^R & = (\overleftarrow{\pi}(\alpha))^R \cup \{\alpha^R\} \\ \Leftrightarrow & \overleftarrow{\mathsf{PD}}(\alpha) & = \overleftarrow{\pi}(\alpha) \cup \{\alpha\} \end{array}$$

$\square$

Let us define that $\forall \alpha \in RE, \sigma \in \Sigma, \{(\sigma, \alpha)\}^R = \{(\alpha^R, \sigma)\}$. The following facts show the relationship between the functions λ and $\overleftarrow{\lambda}$, φ and $\overleftarrow{\varphi}$, and F and $\overleftarrow{F}$.

Corollary 5. *Let α be a regular expression, $(\lambda(\alpha^R))^R = \overleftarrow{\lambda}(\alpha)$.*

Proof. Let us prove the result by induction on α. For the base cases the result is obviously true. If $\alpha = \alpha_1 + \alpha_2$, then $(\lambda((\alpha_1 + \alpha_2)^R))^R = (\lambda(\alpha_1^R + \alpha_2^R))^R = (\lambda(\alpha_1^R))^R \cup (\lambda(\alpha_2^R))^R = \overleftarrow{\lambda}(\alpha_1) \cup \overleftarrow{\lambda}(\alpha_2) = \overleftarrow{\lambda}(\alpha_1 + \alpha_2)$. If $\alpha = \alpha_1\alpha_2$ then

$$\begin{array}{rcl} (\lambda((\alpha_1\alpha_2)^R))^R & = & (\lambda(\alpha_2^R\alpha_1^R))^R \\ & = & (\lambda(\alpha_1^R) \cup \varepsilon(\alpha_1^R)\lambda(\alpha_2^R)\alpha_1^R)^R \\ & = & (\lambda(\alpha_1^R))^R \cup \varepsilon(\alpha_1)\alpha_1(\lambda(\alpha_2^R))^R \\ & = & \overleftarrow{\lambda}(\alpha_1) \cup \varepsilon(\alpha_1)\alpha_1\overleftarrow{\lambda}(\alpha_2) \\ & = & \overleftarrow{\lambda}(\alpha_1\alpha_2) \end{array}$$

If $\alpha = \alpha_1^\star$ then $(\lambda((\alpha_1^\star)^R))^R = (\lambda((\alpha_1^R)^\star))^R = (\lambda(\alpha_1^R)(\alpha_1^R)^\star)^R = \alpha_1^\star\overleftarrow{\lambda}(\alpha_1) = \overleftarrow{\lambda}(\alpha_1^\star)$.

$\square$

Note that while $\lambda(\alpha)$ defines the final states of $\mathcal{A}_{pd}(\alpha)$, $\overleftarrow{\lambda}(\alpha)$ defines the initial states of $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$.

Corollary 6. *Let α be a regular expression, $(\varphi(\alpha^R))^R = \overleftarrow{\varphi}(\alpha)$.*

Proof. Let us prove the result by structural induction. For $\alpha = \emptyset$, and $\alpha = \varepsilon$ the result is obviously true. If $\alpha = \sigma \in \Sigma$ then $(\varphi(\sigma))^R = \{(\sigma, \varepsilon)\}^R$. Thus, $(\varphi(\sigma))^R = \overleftarrow{\varphi}(\sigma)$.

If $\alpha = \alpha_1 + \alpha_2$ then $(\varphi((\alpha_1 + \alpha_2)^R))^R = (\varphi(\alpha_1^R + \alpha_2^R))^R = (\varphi(\alpha_1^R))^R \cup (\varphi(\alpha_2^R))^R = \overleftarrow{\varphi}(\alpha_1) \cup \overleftarrow{\varphi}(\alpha_2) = \overleftarrow{\varphi}(\alpha_1 + \alpha_2)$.

If $\alpha = \alpha_1\alpha_2$ then

$$\begin{aligned}(\varphi((\alpha_1\alpha_2)^R))^R &= (\varphi(\alpha_2^R\alpha_1^R))^R\\ &= (\varphi(\alpha_2^R)\alpha_1^R \cup \varepsilon(\alpha_2^R)\varphi(\alpha_1^R))^R\\ &= \alpha_1(\varphi(\alpha_2^R))^R \cup \varepsilon(\alpha_2)(\varphi(\alpha_1^R))^R\\ &= \alpha_1\overleftarrow{\varphi}(\alpha_2) \cup \varepsilon(\alpha_2)\overleftarrow{\varphi}(\alpha_1)\\ &= \overleftarrow{\varphi}(\alpha_1\alpha_2).\end{aligned}$$

If $\alpha = \alpha_1^\star$ then $(\varphi((\alpha_1^\star)^R))^R = (\varphi(\alpha_1^R)(\alpha_1^R)^\star)^R = \alpha_1^\star(\varphi(\alpha_1^R))^R = \alpha^\star\varphi(\alpha_1)$. Thus the equality of the Proposition holds.

□

Corollary 7. *Let α be a regular expression, $(F(\alpha^R))^R = \overleftarrow{F}(\alpha)$.*

Proof. Let us prove the result by structural induction. For $\alpha = \emptyset$, and $\alpha = \varepsilon$ and $\alpha = \sigma$ the result is obviously true.

If $\alpha = \alpha_1 + \alpha_2$ then $(F((\alpha_1+\alpha_2)^R))^R = (F(\alpha_1^R + \alpha_2^R))^R = (F(\alpha_1^R))^R \cup (F(\alpha_2^R))^R = \overleftarrow{F}(\alpha_1) \cup \overleftarrow{F}(\alpha_2) = \overleftarrow{F}(\alpha_1+\alpha_2)$.

If $\alpha = \alpha_1\alpha_2$ then

$$\begin{aligned}F((\alpha_1\alpha_2)^R))^R &= (F(\alpha_2^R\alpha_1^R))^R\\ &= (F(\alpha_2^R)\alpha_1^R \cup F(\alpha_1^R) \cup \lambda(\alpha_2^R)\alpha_1^R \times \varphi(\alpha_1^R))^R\\ &= (F(\alpha_2^R)\alpha_1^R)^R \cup (F(\alpha_1^R))^R \cup (\lambda(\alpha_2^R)\alpha_1^R \times \varphi(\alpha_1^R))^R\\ &= \alpha_1(F(\alpha_2^R))^R \cup \overleftarrow{F}(\alpha_1) \cup (\varphi(\alpha_1^R))^R \times (\alpha_1(\lambda(\alpha_2^R))^R)\\ &= \alpha_1\overleftarrow{F}(\alpha_2) \cup \overleftarrow{F}(\alpha_1) \cup \overleftarrow{\varphi}(\alpha_1) \times (\alpha_1\overleftarrow{\lambda}(\alpha_2)).\end{aligned}$$

If $\alpha = \alpha_1^\star$ then

$$\begin{aligned}(F((\alpha_1^\star)^R))^R &= (F(\alpha_1^R)(\alpha_1^R)^\star \cup (\lambda(\alpha_1^R) \times \varphi(\alpha_1^R))(\alpha_1^R)^\star)^R\\ &= (F(\alpha_1^R)(\alpha_1^R)^\star)^R \cup ((\lambda(\alpha_1^R) \times \varphi(\alpha_1^R))(\alpha_1^R)^\star)^R\\ &= \alpha_1^\star\overleftarrow{F}(\alpha_1) \cup \alpha_1^\star(\overleftarrow{\varphi}(\alpha_1) \times \overleftarrow{\lambda}(\alpha_1))\end{aligned}$$

Thus the equality of the Proposition holds.

□

Proposition 10. *For any $\alpha \in \mathsf{RE}$ and $w \in \Sigma^\star$, the following holds: $L(\overleftarrow{\partial}_w(\alpha)) = L(\alpha)w^{-1}$.*

Proof. We know that $L(\partial_w(\alpha)) = w^{-1}L(\alpha)$. Thus,

$$\begin{aligned}L(\overleftarrow{\partial}_w(\alpha)) &= L((\partial_{w^R}(\alpha^R))^R) = (L(\partial_{w^R}(\alpha^R)))^R\\ &= ((w^R)^{-1}L(\alpha^R))^R = L(\alpha)w^{-1}\end{aligned}$$

□

Proposition 11. *For any $\alpha \in \mathsf{RE}$ and $w \in \Sigma^\star$, the following holds: $\alpha w^{-1} = \sum \overleftarrow{\partial}_w(\alpha)$.*

Proof. It known that $w^{-1}\alpha = \sum \partial_w(\alpha)$. Thus,

$$\begin{aligned}\sum \overleftarrow{\partial}_w(\alpha) &= \sum \big((\partial_{w^R}(\alpha^R))^R\big) = (\sum \partial_{w^R}(\alpha^R))^R \\ &= ((w^R)^{-1}\alpha^R)^R = \alpha w^{-1}\end{aligned}$$

□

Using the previous results we can associate $\mathcal{A}_{pd}$ with $\overleftarrow{\mathcal{A}}_{pd}$.

Proposition 12. *Let α be a regular expression. Then $(\mathcal{A}_{pd}(\alpha^R))^R \simeq \overleftarrow{\mathcal{A}}_{pd}(\alpha)$.*

Proof. Follows from the Proposition 9, Corollary 5, Corollary 6, and Corollary 7. □

Proposition 13. *Let α be a regular expression. Then $L(\overleftarrow{\mathcal{A}}_{pd}(\alpha)) = L(\alpha)$.*

Proof. We know that $L(\alpha) = L(\mathcal{A}_{pd}(\alpha))$. Thus,

$$\begin{aligned}L(\alpha) = L(\mathcal{A}_{pd}(\alpha)) &\Leftrightarrow L(\alpha^R) = L(\mathcal{A}_{pd}(\alpha^R)) \\ &\Leftrightarrow L((\alpha^R)^R) = L((\mathcal{A}_{pd}(\alpha^R))^R) \\ &\Leftrightarrow L(\alpha) = L(\overleftarrow{\mathcal{A}}_{pd}(\alpha))\end{aligned}$$

□

As we know that $\mathcal{A}_{pd}(\alpha) \simeq {}^{\mathcal{A}_{pos}(\alpha)}\!\big/_{\equiv_c}$ we can conclude that:

Corollary 8. *For any $\alpha \in \mathsf{RE}$, $\overleftarrow{\mathcal{A}}_{pd}(\alpha) \simeq {}^{(\mathcal{A}_{pos}(\alpha^R))^R}\!\big/_{\equiv_c}$.*

It is not difficult to see that:

Proposition 14. *For any $\alpha \in \mathsf{RE}$, $|(\mathcal{A}_{pos}(\alpha^R))^R| = |\mathcal{A}_{pos}(\alpha)|$.*

Proof. The reversal of an automaton does not change its number of states and transitions. Thus, to prove this equality is sufficient prove that $|\mathcal{A}_{pos}(\alpha^R)| = |\mathcal{A}_{pos}(\alpha)|$. For any $\beta \in \mathsf{RE}$, we know that $|\mathcal{A}_{pos}(\beta)| = |\beta| + 1$, and $|\beta| = |\beta^R|$. Thus, the equality $|\mathcal{A}_{pos}(\alpha^R)| = |\mathcal{A}_{pos}(\alpha)|$ holds. □

6.1 Properties of Right Partial Derivates

Proposition 15. *For any regular expressions α, β and word $w \in \Sigma^+$ the following holds:*

$$\overleftarrow{\partial}_w(\alpha+\beta) \subseteq \overleftarrow{\partial}_w(\alpha) \cup \overleftarrow{\partial}_w(\beta) \tag{13}$$

$$\overleftarrow{\partial}_w(\alpha\beta) \subseteq \alpha\overleftarrow{\partial}_w(\beta) \cup \bigcup_{v \in \mathsf{Prefix}(w)} \overleftarrow{\partial}_v(\alpha) \tag{14}$$

$$\overleftarrow{\partial}_w(\alpha^\star) \subseteq \alpha^\star \bigcup_{v \in \mathsf{Prefix}(w)} \overleftarrow{\partial}_v(\alpha) \tag{15}$$

Proof. Let us prove all the inclusions by induction on the size of w. Note that if $|w| = 1$, then $w = \sigma$ and the inclusions correspond to the rules presented in (6). Assuming that all the inclusions hold for some $w \in \Sigma^+$, we will prove each one for $w' = xw$. Considering the inclusion (13) we have that:

$$\begin{aligned}
\overleftarrow{\partial}_{xw}(\alpha+\beta) = \overleftarrow{\partial}_x(\overleftarrow{\partial}_w(\alpha+\beta)) &= \overleftarrow{\partial}(\overleftarrow{\partial}_w(\alpha) \cup \overleftarrow{\partial}_w(\beta)) \\
&= \overleftarrow{\partial}_{xw}(\alpha) \cup \overleftarrow{\partial}_{xw}(\beta)
\end{aligned}$$

Let us prove the inclusion (14):

$$\begin{aligned}
\overleftarrow{\partial}_{xw}(\alpha\beta) = \overleftarrow{\partial}_x(\overleftarrow{\partial}_w(\alpha\beta)) &\subseteq \overleftarrow{\partial}_x\left(\alpha\overleftarrow{\partial}_w(\beta) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \overleftarrow{\partial}_v(\alpha)\right) \\
&\subseteq \overleftarrow{\partial}_x(\alpha\overleftarrow{\partial}_w(\beta)) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \overleftarrow{\partial}_x(\overleftarrow{\partial}_v(\alpha)) \\
&\subseteq \alpha\overleftarrow{\partial}_x(\overleftarrow{\partial}_w(\beta)) \cup \overleftarrow{\partial}_x(\alpha) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \overleftarrow{\partial}_{xv}(\alpha) \\
&\subseteq \alpha\overleftarrow{\partial}_{xw}(\beta) \cup \bigcup_{v\in\mathsf{Prefix}(xw)} \overleftarrow{\partial}_v(\alpha)
\end{aligned}$$

Finally, we prove the inclusion (15):

$$\begin{aligned}
\overleftarrow{\partial}_{xw}(\alpha^\star) = \partial_x(\overleftarrow{\partial}_w(\alpha^\star)) &\subseteq \overleftarrow{\partial}_x\left(\alpha^\star \bigcup_{v\in\mathsf{Prefix}(w)} \overleftarrow{\partial}_v(\alpha)\right) \\
&\subseteq \bigcup_{v\in\mathsf{Prefix}(w)} \overleftarrow{\partial}_x(\alpha^\star\overleftarrow{\partial}_v(\alpha)) \\
&\subseteq \left(\bigcup_{v\in\mathsf{Prefix}(w)} \alpha^\star\overleftarrow{\partial}_x(\overleftarrow{\partial}_v(\alpha))\right) \cup \overleftarrow{\partial}_x(\alpha^\star) \\
&\subseteq \left(\bigcup_{v\in\mathsf{Prefix}(w)} \alpha^\star\partial_{xv}(\alpha)\right) \cup \alpha^\star\overleftarrow{\partial}_x(\alpha) \\
&\subseteq \bigcup_{v\in\mathsf{Prefix}(xw)} \alpha^\star\overleftarrow{\partial}_v(\alpha)
\end{aligned}$$

□

Proposition 16. *For any $\alpha \in \mathsf{RE}$ the following hold:*

$$|\overleftarrow{\pi}(\alpha)| \leq |\alpha| \tag{16}$$

$$|\overleftarrow{\mathsf{PD}}(\alpha)| \leq |\alpha| + 1. \tag{17}$$

Proof. Since $\overleftarrow{\mathsf{PD}}(\alpha) = \overleftarrow{\pi}(\alpha) \cup \{\alpha\}$, the first inequality implies the second one, thus we only need to prove (16). We proceed by induction on α. The base cases are obvious. Let us suppose that the inequality (16) holds for some $\alpha_1,\ \alpha_2 \in \mathsf{RE}$ and consider three subcases. First, consider $\alpha = \alpha_1 + \alpha_2$. Then we have:

$$|\overleftarrow{\pi}(\alpha_1+\alpha_2)| = |\overleftarrow{\pi}(\alpha_1) \cup \overleftarrow{\pi}(\alpha_2)| = |\overleftarrow{\pi}(\alpha_1)| + |\overleftarrow{\pi}(\alpha_2)| \leq |\alpha_1 + \alpha_2|$$

For the second case, consider $\alpha = \alpha_1\alpha_2$, then:

$$\begin{aligned}|\overleftarrow{\pi}(\alpha_1\alpha_2)| = |\alpha_1\overleftarrow{\pi}(\alpha_2) \cup \overleftarrow{\pi}(\alpha_1)| &= |\alpha_1\overleftarrow{\pi}(\alpha_2)| + |\overleftarrow{\pi}(\alpha_1)| \\ &\leq |\alpha_2| + |\alpha_1| = |\alpha_1\alpha_2|\end{aligned}$$

Finally, consider $\alpha = \alpha_1^\star$, thus we have that:

$$|\overleftarrow{\pi}(\alpha_1^\star)| = |\alpha_1^\star\overleftarrow{\pi}(\alpha_1)| \leq |\alpha_1| = |\alpha_1^\star|$$

□

Observing the Proposition 15 is not difficult to conclude the following result, which is similar to what happens for left-partial derivatives:

Corollary 9. *Given a regular expression $\alpha \in \mathsf{RE}$, any right partial derivative of α is either ε, or a subterm of α, or a concatenation $\alpha_1 \cdot \alpha_2 \cdots \alpha_n$ of several such subterms, where n is not greater than the number of ocurrences of concatenation and Kleene star in α.*

6.2 Average Case

In [3] the authors proved that the average number of states of partial derivative automaton is, on average, half the number of states of the Glushkov automaton. As $\overleftarrow{\mathcal{A}}_{pd}(\alpha) = (\mathcal{A}_{pd}(\alpha^R))^R$, we can conclude that:

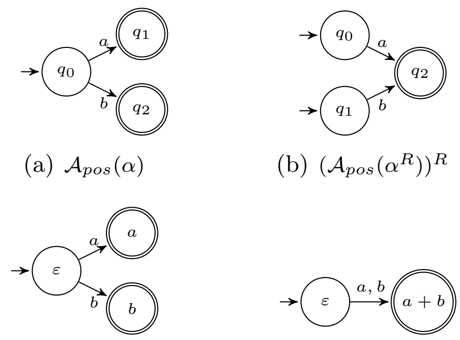
Corollary 10. *The number of states of $\overleftarrow{\mathcal{A}}_{pd}$ are, on average, half of the number of states of $\mathcal{A}_{pos}$.*

7 Prefix Automaton

Yamamoto [20] presented a new algorithm for converting a regular expression into an equivalent NFA. First, a labeled version of the usual Thompson NFA $(Q, \Sigma, \delta, I, F)$ is obtained, where each state q is labeled with two regular expressions, one that corresponds to its left language, $LP(q)$, and the other to its right language, $LS(q)$. States that in-transitions are labeled with a letter are called sym-states. Then the equivalence relations $\equiv_{pre}$ and $\equiv_{suf}$ are defined on the set of *sym-states*: for two states $p, q \in Q$, $p \equiv_{pre} q$ if and only if $LP(p) = LP(q)$; and $p \equiv_{suf} q$ if and only if $LS(p) = LS(q)$. The *prefix automaton* $\mathcal{A}_{pre}$ and the *suffix automaton* $\mathcal{A}_{suf}$ are the quotient automata by these relations. The final automaton is a combination of these two. The author also shows that $\mathcal{A}_{suf}$ automaton coincides with $\mathcal{A}_{pd}$. This relation between $\mathcal{A}_{pd}$ and $\mathcal{A}_{suf}$ could lead us to think that $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$ coincide with $\mathcal{A}_{pre}(\alpha)$, which is not true. For instance, considering $\alpha = a + b$, the $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$ has 2 states and the $\mathcal{A}_{pre}(\alpha)$ has 3 states (see Fig. 3). Note that both automata are obtained from another automaton by merging the states with the same left language: while the $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$ is obtained from $(\mathcal{A}_{pos}(\alpha^R))^R$, we will see that the $\mathcal{A}_{pre}(\alpha)$ is obtained from $\mathcal{A}_{pos}(\alpha)$.

We can define the set of states of the automaton $\mathcal{A}_{pre}$ inductively on the structure of α by the following rules, which are based in the left languages labelling scheme defined by the author:

$$\begin{aligned}\mathsf{Pre}(\emptyset) &= \emptyset & \mathsf{Pre}(\alpha+\beta) &= \mathsf{Pre}(\alpha) \cup \mathsf{Pre}(\beta)\\ \mathsf{Pre}(\varepsilon) &= \emptyset & \mathsf{Pre}(\alpha\beta) &= \alpha\mathsf{Pre}(\beta) \cup \mathsf{Pre}(\alpha)\\ \mathsf{Pre}(\sigma) &= \{\sigma\} & \mathsf{Pre}(\alpha^\star) &= \alpha^\star\mathsf{Pre}(\alpha).\end{aligned} \tag{18}$$



(a) $\mathcal{A}_{pos}(\alpha)$ (b) $(\mathcal{A}_{pos}(\alpha^R))^R$

(c) $\mathcal{A}_{pre}(\alpha)$ (d) $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$

Figure 3: $\alpha = a + b$.

The definition of Pre can be extended to sets of regular expressions: $\mathsf{Pre}(S) = \bigcup_{\alpha \in S} \mathsf{Pre}(\alpha)$ for $S \subseteq \mathsf{RE}$.

Remark 1. *For any $\alpha \in \mathsf{RE}$, the elements of $\mathsf{Pre}(\alpha)$ are always of the form $\emptyset$ or $\alpha'\sigma$, where α' is a subexpression of α or ε, and $\sigma \in \Sigma$.*

In order to obtain the final states of $\mathcal{A}_{pre}$ automaton we need to calculate the left languages of the $\mathcal{A}_{pos}$ final states. For that we define the function Pr':

$$\begin{aligned} \mathsf{Pr}'(\emptyset) &= \emptyset & \mathsf{Pr}'(\alpha\beta) &= \alpha\mathsf{Pr}'(\beta) \cup \varepsilon(\beta)\mathsf{Pr}'(\alpha) \\ \mathsf{Pr}'(\varepsilon) &= \varepsilon & \mathsf{Pr}'(\alpha+\beta) &= \mathsf{Pr}'(\alpha) \cup \mathsf{Pr}'(\beta) \\ \mathsf{Pr}'(\sigma) &= \{\sigma\} & \mathsf{Pr}'(\alpha^\star) &= \alpha^\star\mathsf{Pr}'(\alpha). \end{aligned} \tag{19}$$

Similarly to what happens for $\mathcal{A}_{pre}$, the definition of Pr' can be extended to sets of regular expressions: $\mathsf{Pr}'(S) = \bigcup_{\alpha \in S} \mathsf{Pr}'(\alpha)$ for $S \subseteq \mathsf{RE}$.

Lemma 17. *For any $\alpha \in \mathsf{RE}$ the following holds: $\mathsf{Pr}'(\alpha) \subseteq \varepsilon \cup \mathsf{Pre}(\alpha)$.*

Proof. Let us prove the inclusion by induction on α. Is not difficult to see that the results is true for the base cases. Let $\alpha = \alpha_1 + \alpha_2$, then $\mathsf{Pr}'(\alpha_1 + \alpha_2) = \mathsf{Pr}'(\alpha_1) \cup \mathsf{Pr}'(\alpha_1) \subseteq \mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_2)$. If $\alpha = \alpha_1\alpha_2$, then $\mathsf{Pr}'(\alpha_1\alpha_2) = \alpha_1\mathsf{Pr}'(\alpha_2) \cup \mathsf{Pr}'(\alpha_1) \subseteq \alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pre}(\alpha_1)$. If $\alpha = \alpha_1^\star$, the $\mathsf{Pr}'(\alpha_1^\star) = \alpha_1^\star\mathsf{Pr}'(\alpha_1) \subseteq \alpha_1^\star\mathsf{Pre}(\alpha_1)$. □

Remark 2. *For any $\alpha \in \mathsf{RE}$, the elements of $\mathsf{Pr}'(\alpha)$ are always of the form ε, $\emptyset$ or $\alpha'\sigma$, where α' is a subexpression of α or ε, and $\sigma \in \Sigma$.*

Corollary 11. *For any $j \in \mathsf{Pr}'(\alpha)$, $\alpha \in \mathsf{RE}$, if $\varepsilon(j) = \varepsilon$ then $\varepsilon(\alpha) = \varepsilon$.*

Proof. By Remark 2 we know that $j = \varepsilon$, $j = \emptyset$ or $j = \alpha'\sigma$. In the last two cases is obvious that $\varepsilon(j) \neq \varepsilon$. By (19) we can conclude that if $j = \varepsilon$ then $\alpha = \varepsilon$, $\alpha = \varepsilon + \alpha'$ or $\alpha = \alpha' + \varepsilon$. Thus the proposition holds. □

Now, we need to define the $\mathcal{A}_{pre}(\alpha)$ transition function, i.e. for each state γ of $\mathcal{A}_{pre}$ and each $\sigma \in \Sigma$ we need to determine from which states we reach γ. Thus, we have to find which β satisfies $\beta\sigma = \gamma$, and then calculate the left languages of the states of $\mathcal{A}_{pos}(\alpha)$ that correspond to β. The following function permit us to define such transitions:

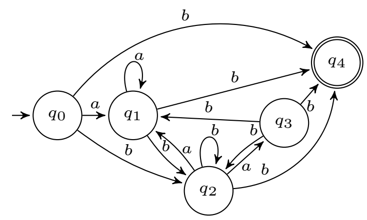


Figure 4: $\mathcal{A}_{pre}((a^\star b + a^\star ba + a^\star)^\star b) : q_0 = \varepsilon, q_1 = (a^\star b + a^\star ba + a^\star)^\star(a^\star a), q_2 = (a^\star b + a^\star ba + a^\star)^\star(a^\star b), q_3 = (a^\star b + a^\star ba + a^\star)^\star((a^\star b)a), q_4 = (a^\star b + a^\star ba + a^\star)^\star b$.

$$
\begin{aligned}
\mathsf{Pr}_\sigma(\emptyset) &= \mathsf{Pr}_\sigma(\varepsilon) = \emptyset \\
\mathsf{Pr}_\sigma(\sigma') &= \begin{cases} \{\varepsilon\}, & \text{if } \sigma' = \sigma \\ \emptyset, & \text{otherwise} \end{cases} \\
\mathsf{Pr}_\sigma(\alpha + \beta) &= \mathsf{Pr}_\sigma(\alpha) \cup \mathsf{Pr}_\sigma(\beta) \\
&\cup\ \varepsilon(\mathsf{Pr}_\sigma(\alpha) \cup \mathsf{Pr}_\sigma(\beta))
\end{aligned}
\qquad
\begin{aligned}
\mathsf{Pr}_\sigma(\alpha\beta) &= \mathsf{Pr}'(\alpha \mathsf{Pr}_\sigma(\beta)) \\
&\cup\ \varepsilon(\alpha \mathsf{Pr}_\sigma(\beta)) \\
&\cup\ \varepsilon(\beta)(\mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha)) \\
&\cup\ \varepsilon(\mathsf{Pr}_\sigma(\alpha))) \\
\mathsf{Pr}_\sigma(\alpha^\star) &= \mathsf{Pr}'(\alpha^\star \mathsf{Pr}_\sigma(\alpha)) \\
&\cup\ \varepsilon(\mathsf{Pr}_\sigma(\alpha))
\end{aligned}
\tag{20}
$$

The definition of Pr_σ can be extended to sets of regular expressions, words, and languages. Given $\alpha \in \mathsf{RE}$ and $\sigma \in \Sigma$, $\mathsf{Pr}_\sigma(S) = \bigcup_{\alpha \in S} \mathsf{Pr}_\sigma(\alpha)$ for $S \subseteq \mathsf{RE}$, $\mathsf{Pr}_\varepsilon(\alpha) = \mathsf{Pr}'(\alpha)$ and $\mathsf{Pr}_{\sigma w}(\alpha) = \mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha))$, for any $w \in \Sigma^\star, \sigma \in \Sigma$.

Therefore, the automaton $\mathcal{A}_{pre}$ can be inductively defined by

$$\mathcal{A}_{pre}(\alpha) = (\{\varepsilon\} \cup \mathsf{Pre}(\alpha), \Sigma, \delta_{pre}, \varepsilon, \mathsf{Pr}'(\alpha) \cup \varepsilon(\alpha)\{\varepsilon\}),$$

where $\delta_{pre} = \{(s', \sigma, s) \mid s \in \mathsf{Pre}(\alpha),\ s' \in \mathsf{Pr}_\sigma(s),\ \sigma \in \Sigma\}$.

As we know that we only apply the function Pr_σ to an s in $\mathsf{Pre}(\alpha)$, by Remark 1 we can redefine this function in a simplest way:

$$
\begin{aligned}
\mathsf{Pr}_\sigma(\emptyset) &= \mathsf{Pr}_\sigma(\varepsilon) = \emptyset \\
\mathsf{Pr}_\sigma(\sigma') &= \begin{cases} \{\varepsilon\}, & \text{if } \sigma' = \sigma \\ \emptyset, & \text{otherwise} \end{cases}
\end{aligned}
\qquad
\mathsf{Pr}_\sigma(\alpha'\sigma') = \begin{cases} \mathsf{Pr}'(\alpha') \cup \varepsilon(\alpha'), & \text{if } \sigma' = \sigma \\ \emptyset, & \text{otherwise} \end{cases}
\tag{21}
$$

In Fig.4 we can see the $\mathcal{A}_{pre}((a^\star b + a^\star ba + a^\star)^\star b)$.

The following facts describe some properties of the functions Pr and Pr'.

Lemma 18. *For any $\alpha \in \mathsf{RE}$, and any $\sigma \in \Sigma$, the following inclusion hold $\mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha)) \subseteq \mathsf{Pr}_\sigma(\alpha)$.*

Proof. Let us prove the result by structural induction on α. For $\alpha = \emptyset$,$\alpha = \varepsilon$ and $\alpha = \sigma$ the inclusion is obviously true. If $\alpha = \alpha_1 + \alpha_2$, then

$$
\begin{aligned}
\mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1 + \alpha_2)) &= \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1)) \cup \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_2)) \\
&\subseteq \mathsf{Pr}_\sigma(\alpha_1) \cup \mathsf{Pr}_\sigma(\alpha_2)
\end{aligned}
$$

If $\alpha = \alpha_1\alpha_2$, then

$$
\begin{aligned}
\mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1\alpha_2)) &= \begin{cases} \mathsf{Pr}_\sigma(\alpha_1 \mathsf{Pr}'(\alpha_2) \cup \varepsilon(\alpha_2)\mathsf{Pr}'(\alpha_1)) & \text{if } \alpha_2 \neq \varepsilon \\ \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1)) & \text{if } \alpha_2 = \varepsilon \end{cases} \\
&= \begin{cases} \begin{array}{l} \mathsf{Pr}'(\alpha_1 \mathsf{Pr}_\sigma(\mathsf{Pr}(\alpha_2))) \cup \varepsilon(\alpha_1 \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_2))) \cup \\ \varepsilon(\mathsf{Pr}'(\alpha_2))(\mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha_1)) \cup \varepsilon(\mathsf{Pr}_\sigma(\alpha_1))) \cup \varepsilon(\alpha_2)\mathsf{Pr}'(\alpha_1)) \end{array} & \text{if } \alpha_2 \neq \varepsilon \\ \mathsf{Pr}_\sigma(\alpha_1) & \text{if } \alpha_2 = \varepsilon \end{cases} \\
&\subseteq \begin{cases} \begin{array}{l} \mathsf{Pr}'(\alpha_1 \mathsf{Pr}_\sigma(\alpha_2)) \cup \varepsilon(\alpha_1 \mathsf{Pr}_\sigma(\alpha_2)) \cup \\ \varepsilon(\mathsf{Pr}'(\alpha_2))(\mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha_1)) \cup \varepsilon(\mathsf{Pr}_\sigma(\alpha_1))) \cup \varepsilon(\alpha_2)\mathsf{Pr}'(\alpha_1)) \end{array} & \text{if } \alpha_2 \neq \varepsilon \\ \mathsf{Pr}_\sigma(\alpha_1) & \text{if } \alpha_2 = \varepsilon \end{cases} \\
&\text{by Corollary 11} \\
&\subseteq \mathsf{Pr}_\sigma(\alpha_1\alpha_2)
\end{aligned}
$$

If $\alpha = \alpha_1^\star$, then

$$
\begin{aligned}
\mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1^\star)) &= \mathsf{Pr}_\sigma(\alpha_1^\star \mathsf{Pr}'(\alpha_1)) \\
&= \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1))) \cup \varepsilon(\mathsf{Pr}(\mathsf{Pr}'(\alpha_1)))\{\varepsilon\} \\
&= \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_\sigma(\alpha_1)) \cup \varepsilon(\mathsf{Pr}_\sigma(\alpha_1)) \\
&= \mathsf{Pr}_\sigma(\alpha_1^\star)
\end{aligned}
$$

$\square$

Lemma 19. *For any $\alpha \in \mathsf{RE}$, the following inclusion hold $\mathsf{Pr}'(\mathsf{Pr}'(\alpha)) = \mathsf{Pr}'(\alpha)$.*

Proof. Let us prove the equality by structural induction on w. For the base cases it is obvious. Let $\alpha = \alpha_1 + \alpha_2$, then

$$
\begin{aligned}
\mathsf{Pr}'(\mathsf{Pr}'(\alpha_1 + \alpha_2)) &= \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1) \cup \mathsf{Pr}'(\alpha_2)) \\
&= \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1)) \cup \mathsf{Pr}'(\mathsf{Pr}'(\alpha_2)) \\
&= \mathsf{Pr}'(\alpha_1) \cup \mathsf{Pr}'(\alpha_2)
\end{aligned}
$$

If $\alpha = \alpha_1\alpha_2$, then

$$
\begin{aligned}
\mathsf{Pr}'(\mathsf{Pr}'(\alpha_1\alpha_2)) &= \begin{cases} \mathsf{Pr}'(\alpha_1 \mathsf{Pr}'(\alpha_2) \cup \varepsilon(\alpha_2)\mathsf{Pr}'(\alpha_1)) & \text{if } \alpha_2 \neq \varepsilon \\ \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1)) & \text{if } \alpha_2 = \varepsilon \end{cases} \\
&= \begin{cases} \alpha_1 \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1)) \cup \varepsilon(\mathsf{Pr}'(\mathsf{Pr}'(\alpha_2)))\mathsf{Pr}'(\alpha_1) \cup \varepsilon(\alpha_2)\mathsf{Pr}'(\alpha_1) & \text{if } \mathsf{Pr}'(\alpha_1) \neq \varepsilon \wedge \alpha_2 \neq \varepsilon \\ \mathsf{Pr}'(\alpha_1) & \text{if } \mathsf{Pr}(\alpha_2) = \varepsilon \\ \mathsf{Pr}'(\alpha_1) & \text{if } \alpha_2 = \varepsilon \end{cases} \\
&= \begin{cases} \alpha_1 \mathsf{Pr}'(\alpha_1) \cup \varepsilon(\mathsf{Pr}'(\alpha_2))\mathsf{Pr}'(\alpha_1) \cup \varepsilon(\alpha_2)\mathsf{Pr}'(\alpha_1) & \text{if } \mathsf{Pr}'(\alpha_1) \neq \varepsilon \wedge \alpha_2 \neq \varepsilon \\ \mathsf{Pr}'(\alpha_1) & \text{if } \mathsf{Pr}(\alpha_2) = \varepsilon \\ \mathsf{Pr}'(\alpha_1) & \text{if } \alpha_2 = \varepsilon \end{cases} \\
&\text{by Corollary 11} \\
&= \mathsf{Pr}'(\alpha_1\alpha_2)
\end{aligned}
$$

If $\alpha = \alpha_1^\star$, then

$$\begin{aligned}
\mathsf{Pr}'(\mathsf{Pr}'(\alpha_1^\star)) &= \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}'(\alpha_1)) \\
&= \begin{cases} \alpha_1^\star \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1)) \cup \varepsilon(\mathsf{Pr}'(\alpha_1))\mathsf{Pr}'(\alpha_1^\star) & \text{if } \mathsf{Pr}'(\alpha_1) \neq \varepsilon \\ \mathsf{Pr}'(\alpha_1^\star) & \text{otherwise} \end{cases} \\
&\text{as } \mathsf{Pr}'(\alpha_1^\star) = \alpha_1^\star \mathsf{Pr}'(\alpha_1) \\
&= \alpha_1^\star \mathsf{Pr}'(\alpha_1)
\end{aligned}$$

□

Lemma 20. *For any $\alpha \in \mathsf{RE}$ and $\sigma \in \Sigma$, the following inclusion hold $\mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha)) = \mathsf{Pr}_\sigma(\alpha)$.*

Proof. Follows directly from the definition and from the previous lemma. □

Lemma 21. *For any regular expressions α_1, α_2 and word $w \in \Sigma^\star$ the following inclusions hold:*

$$\mathsf{Pr}_w(\alpha_1 + \alpha_2) \subseteq \mathsf{Pr}_w(\alpha_1) \cup \mathsf{Pr}_w(\alpha_2) \cup \{\varepsilon\} \tag{22}$$

$$\mathsf{Pr}_w(\alpha_1\alpha_2) \subseteq \mathsf{Pr}'(\alpha\mathsf{Pr}_w(\alpha_2)) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}_v(\alpha)) \cup \{\varepsilon\} \tag{23}$$

$$\mathsf{Pr}_w(\alpha_1^\star) \subseteq \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\alpha^\star\mathsf{Pr}_v(\alpha)) \cup \{\varepsilon\} \tag{24}$$

Proof. We will prove the three results by induction on the size of w. The base case, $|w| = 1$, for all the results it is obvious and coincide with the definition of $\mathsf{Pr}_\sigma(\alpha)$. Assuming that the inclusions holds for some $w \in \Sigma^+$, we prove it for $w' = \sigma w$. Consider the result (22). Then,

$$\begin{aligned}
\mathsf{Pr}_{\sigma w}(\alpha_1 + \alpha_2) &= \mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha_1 + \alpha_2)) \\
&\subseteq \mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha_1) \cup \mathsf{Pr}_w(\alpha_2) \cup \{\varepsilon\}) \\
&\subseteq \mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha_1)) \cup \mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha_2)) \\
&\subseteq \mathsf{Pr}_{\sigma w}(\alpha_1) \cup \mathsf{Pr}_{\sigma w}(\alpha_2) \cup \{\varepsilon\}
\end{aligned}$$

Let us consider the inclusion (23). Then,

$$\begin{aligned}
\mathsf{Pr}_{\sigma w}(\alpha_1\alpha_2) &= \mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha_1\alpha_2)) \\
&\subseteq \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1\mathsf{Pr}_w(\alpha_2)) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\}) \\
&\subseteq \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1\mathsf{Pr}_w(\alpha_2))) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}_\sigma(\mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1))) \\
&\quad \text{as } \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha)) \subseteq \mathsf{Pr}_\sigma(\alpha) \\
&\subseteq \mathsf{Pr}_\sigma(\alpha_1\mathsf{Pr}_w(\alpha_2)) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}_\sigma(\mathsf{Pr}_v(\alpha_1)) \\
&\subseteq \mathsf{Pr}'(\alpha_1\mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha_2))) \cup \{\varepsilon\} \cup \mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha_1)) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}_\sigma(\mathsf{Pr}_v(\alpha_1)) \\
&\quad \text{as } \mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha)) \subseteq \mathsf{Pr}_\sigma(\alpha) \text{ and } \mathsf{Pr}_\sigma(\alpha_1) \subseteq \bigcup_{v\in\mathsf{Prefix}(\sigma w)} \mathsf{Pr}_v(\alpha_1) \\
&\subseteq \mathsf{Pr}'(\alpha_1\mathsf{Pr}_{\sigma w}(\alpha_2))) \cup \{\varepsilon\} \cup \bigcup_{v\in\mathsf{Prefix}(\sigma w)} \mathsf{Pr}_v(\alpha_1)
\end{aligned}$$

Finally consider the inclusion (24). Then

$$\begin{array}{rcl}
\mathsf{Pr}_{\sigma w}(\alpha_1^\star) & = & \mathsf{Pr}_\sigma(\mathsf{Pr}_w(\alpha_1^\star)) \\
 & \subseteq & \mathsf{Pr}_\sigma(\bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\}) \\
 & \subseteq & \bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_v(\alpha_1))) \\
 & & \text{as } \mathsf{Pr}_\sigma(\mathsf{Pr}'(\alpha)) \subseteq \mathsf{Pr}_\sigma(\alpha) \\
 & \subseteq & \bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}_\sigma(\alpha_1^\star \mathsf{Pr}_v(\alpha_1)) \\
 & \subseteq & \bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_\sigma(\mathsf{Pr}_v(\alpha_1))) \cup \{\varepsilon\} \cup \mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha_1^\star)) \\
 & & \text{as } \mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha)) \subseteq \mathsf{Pr}_\sigma(\alpha) \\
 & \subseteq & \bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_{\sigma v}(\alpha_1))) \cup \{\varepsilon\} \cup \mathsf{Pr}_\sigma(\alpha_1^\star) \\
 & & \text{as } \mathsf{Pr}_\sigma(\alpha_1^\star) \subseteq \bigcup\limits_{v\in\mathsf{Prefix}(\sigma w)} \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_v(\alpha_1)) \\
 & \subseteq & \bigcup\limits_{v\in\mathsf{Prefix}(\sigma w)} \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_v(\alpha_1))) \cup \{\varepsilon\}
\end{array}$$

□

Lemma 22. *For any* $\alpha \in \mathsf{RE}$ *and any* $w \in \Sigma^+$, *the following holds:* $\mathsf{Pr}'(\mathsf{Pr}_w(\alpha)) \subseteq \mathsf{Pr}_w(\alpha)$

Proof. Let us prove the result by structural inductions on α. For the base cases the result is obviously true. Consider $\alpha = \alpha_1 + \alpha_2$, then

$$\begin{array}{rcl}
\mathsf{Pr}'(\mathsf{Pr}_w(\alpha_1+\alpha_2)) & \subseteq & \mathsf{Pr}'(\mathsf{Pr}_w(\alpha_1) \cup \mathsf{Pr}_w(\alpha_2) \cup \{\varepsilon\}) \\
 & \subseteq & \mathsf{Pr}'(\mathsf{Pr}_w(\alpha_1)) \cup \mathsf{Pr}'(\mathsf{Pr}_w(\alpha_2)) \cup \mathsf{Pr}'(\{\varepsilon\}) \\
 & \subseteq & \mathsf{Pr}_w(\alpha_1) \cup \mathsf{Pr}_w(\alpha_2) \cup \{\varepsilon\} \\
 & \subseteq & \mathsf{Pr}_w(\alpha_1+\alpha_2)
\end{array}$$

If $\alpha = \alpha_1\alpha_2$ then

$$\begin{array}{rcl}
\mathsf{Pr}'(\mathsf{Pr}_w(\alpha_1\alpha_2)) & \subseteq & \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1 \mathsf{Pr}_w(\alpha_2)) \cup \bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\}) \\
 & \subseteq & \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1 \mathsf{Pr}_w(\alpha_2))) \cup \bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1))) \cup \mathsf{Pr}'(\{\varepsilon\}) \\
 & \subseteq & \mathsf{Pr}'(\alpha_1 \mathsf{Pr}_w(\alpha_2)) \cup \bigcup\limits_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\} \\
 & \subseteq & \mathsf{Pr}_w(\alpha_1\alpha_2)
\end{array}$$

Finally if $\alpha = \alpha_1^\star$ the

$$\begin{array}{rcl}\mathsf{Pr}'(\mathsf{Pr}_w(\alpha_1^\star)) & \subseteq & \mathsf{Pr}'(\bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\}) \\ & \subseteq & \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_v(\alpha_1))) \cup \mathsf{Pr}'(\{\varepsilon\}) \\ & \subseteq & \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\alpha_1^\star \mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\} \\ & \subseteq & \mathsf{Pr}_w(\alpha_1^\star)\end{array}$$

□

The next results ensure that the transitions of $\mathcal{A}_{pre}(\alpha)$ only goes to states in the set $\{\varepsilon\} \cup \mathsf{Pre}(\alpha)$.

Lemma 23. *For any $\alpha \in \mathsf{RE}$ and any $\sigma \in \Sigma$, the following holds: $\mathsf{Pr}_\sigma(\alpha) \subseteq \mathsf{Pre}(\alpha) \cup \{\varepsilon\}$.*

Proof. Let us proceed by induction on α. For the base cases the inclusion is obvious. Let $\alpha = \alpha_1 + \alpha_2$, the $\mathsf{Pr}_\sigma(\alpha_1 + \alpha_2) = \mathsf{Pr}_\sigma(\alpha_1) \cup \mathsf{Pr}_\sigma(\alpha_2) \subseteq \mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_2)$.

If $\alpha = \alpha_1\alpha_2$, then

$$\begin{array}{rcl}\mathsf{Pr}_\sigma(\alpha_1\alpha_2) & \subseteq & \mathsf{Pr}'(\alpha_1\mathsf{Pr}_\sigma(\alpha_2)) \cup \{\varepsilon\} \cup \mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha_1)) \\ & \subseteq & \mathsf{Pr}'(\alpha_1\mathsf{Pr}_\sigma(\alpha_2)) \cup \{\varepsilon\} \cup \mathsf{Pr}_\sigma(\alpha_1) \\ & \subseteq & \alpha_1\mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha_2)) \cup \mathsf{Pr}'(\alpha_1) \cup \{\varepsilon\} \cup \mathsf{Pr}_\sigma(\alpha_1) \\ & & \text{as } \mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha)) \subseteq \mathsf{Pr}_\sigma(\alpha) \\ & \subseteq & \alpha_1\mathsf{Pr}_\sigma(\alpha_2) \cup \mathsf{Pr}'(\alpha_1) \cup \{\varepsilon\} \cup \mathsf{Pr}_\sigma(\alpha_1) \\ & & \text{as } \mathsf{Pr}'(\alpha)) \subseteq \mathsf{Pre}(\alpha) \cup \{\varepsilon\} \text{ and by inductive hypothesis} \\ & \subseteq & \alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pre}(\alpha_1) \cup \{\varepsilon\}\end{array}$$

If $\alpha = \alpha_1^\star$ then

$$\begin{array}{rcl}\mathsf{Pr}_\sigma(\alpha_1^\star) & = & \mathsf{Pr}'(\alpha_1^\star\mathsf{Pr}_\sigma(\alpha_1)) \\ & \subseteq & \alpha_1^\star\mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha_1)) \cup \mathsf{Pr}'(\alpha_1^\star) \\ & & \text{as } \mathsf{Pr}'(\mathsf{Pr}_\sigma(\alpha)) \subseteq \mathsf{Pr}_\sigma(\alpha) \\ & \subseteq & \alpha_1^\star\mathsf{Pr}_\sigma(\alpha_1) \cup \alpha_1^\star\mathsf{Pr}'(\alpha_1) \\ & & \text{as } \mathsf{Pr}'(\alpha)) \subseteq \mathsf{Pre}(\alpha) \cup \{\varepsilon\} \text{ and by inductive hypothesis} \\ & \subseteq & \alpha_1^\star\mathsf{Pre}(\alpha_1) \cup \{\varepsilon\}\end{array}$$

□

Lemma 24. *For any $R \subseteq \mathsf{RE}$ and any $\sigma \in \Sigma$, the following holds: $\mathsf{Pr}_\sigma(R) \subseteq \mathsf{Pre}(R) \cup \{\varepsilon\}$.*

Proof. Let us prove the result by induction on the size of the set R. If $|R| = 1$ the inclusion is true by Lemma 23. Assuming that the inclusion holds for some $R \subseteq \mathsf{RE}$, let us prove it for $R' = R \cup \{\alpha\}$:

$$\begin{array}{rcl}\mathsf{Pr}_x(R \cup \{\alpha\}) & = & \mathsf{Pr}_x(R) \cup \mathsf{Pr}_x(\alpha) \\ & \subseteq & \mathsf{Pre}(R) \cup \mathsf{Pre}(\alpha) \cup \{\varepsilon\} \subseteq \mathsf{Pre}(R \cup \{\alpha\}) \cup \{\varepsilon\}.\end{array}$$

□

Lemma 25. *For any* $\alpha \in \mathsf{RE}$ *the following holds:* $\mathsf{Pre}(\mathsf{Pre}(\alpha)) = \mathsf{Pre}(\alpha)$.

Proof. Let us prove the result by induction on α. For the base cases the result is obviously true. Let $\alpha = \alpha_1 + \alpha_2$ then $\mathsf{Pre}(\mathsf{Pre}(\alpha_1 + \alpha_2)) = \mathsf{Pre}(\mathsf{Pre}(\alpha_1)) \cup \mathsf{Pre}(\mathsf{Pre}(\alpha_2)) = \mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_2)$. If $\alpha = \alpha_1\alpha_2$, then

$$\begin{aligned}
\mathsf{Pre}(\mathsf{Pre}(\alpha_1\alpha_2)) &= \mathsf{Pre}(\alpha_1\mathsf{Pre}(\alpha_2)) \cup \mathsf{Pre}(\mathsf{Pre}(\alpha_1)) \\
&= \alpha_1\mathsf{Pre}(\mathsf{Pre}(\alpha_2)) \cup \mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_1) \\
&= \alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pre}(\alpha_1)
\end{aligned}$$

If $\alpha = \alpha_1^\star$ then $\mathsf{Pre}(\mathsf{Pre}(\alpha_1^\star)) = \mathsf{Pre}(\alpha_1^\star\mathsf{Pre}(\alpha_1)) = \alpha_1^\star\mathsf{Pre}(\mathsf{Pre}(\alpha_1)) \cup \mathsf{Pre}(\alpha_1^\star) = \alpha_1^\star\mathsf{Pre}(\alpha_1)$. $\square$

Lemma 26. *For any* $\alpha \in \mathsf{RE}$ *and any* $w \in \Sigma^+$*, the following holds:*

$$\bigcup_{v \in \mathsf{Prefix}(w)} \mathsf{Pr}_v(\alpha) \subseteq \mathsf{Pre}(\alpha) \cup \{\varepsilon\}$$

.

Proof. Let us prove the result by induction on the size of w. If $|w| = 1$, then $w = \sigma$ and $\mathsf{Prefix}(\sigma) = \{\sigma\}$. Thus, $\bigcup_{v \in \mathsf{Prefix}(w)} \mathsf{Pr}_v(\alpha) = \mathsf{Pr}_\sigma(\alpha)$, and by Lemma 23 we can conclude that $\mathsf{Pr}_\sigma(\alpha) \subseteq \mathsf{Pre}(\alpha) \cup \{\varepsilon\}$. Assuming that the inclusion holds for some $w \in \Sigma^+$, and let us prove it for $w' = xw$, where $x \in \Sigma$:

$$\begin{aligned}
\bigcup_{v \in \mathsf{Prefix}(xw)} \mathsf{Pr}_v(\alpha) &\subseteq \bigcup_{v \in \mathsf{Prefix}(w)} \mathsf{Pr}_{xv}(\alpha) \cup \mathsf{Pr}_x(\alpha) \\
&\subseteq \bigcup_{v \in \mathsf{Prefix}(w)} \mathsf{Pr}_x(\mathsf{Pr}_v(\alpha)) \cup \mathsf{Pr}_x(\alpha) \\
&\subseteq \mathsf{Pr}_x(\bigcup_{v \in \mathsf{Prefix}(w)} \mathsf{Pr}_v(\alpha)) \cup \mathsf{Pr}_x(\alpha) \\
&\subseteq \mathsf{Pr}_x(\mathsf{Pre}(\alpha)) \cup \mathsf{Pr}_x(\{\varepsilon\}) \cup \mathsf{Pr}_x(\alpha) \\
&\quad \text{as } \mathsf{Pr}_x(\alpha) \cup \{\varepsilon\} \subseteq \mathsf{Pre}(\alpha) \text{ and } \mathsf{Pr}_x(R) \subseteq \mathsf{Pre}(R) \cup \{\varepsilon\}, \text{ where } R \subseteq \mathsf{RE} \\
&\subseteq \mathsf{Pre}(\mathsf{Pre}(\alpha)) \cup \mathsf{Pre}(\alpha) \cup \{\varepsilon\} \\
&\subseteq \mathsf{Pre}(\alpha) \cup \{\varepsilon\}
\end{aligned}$$

$\square$

Proposition 27. *For any regular expressions* α_1, α_2 *and word* $w \in \Sigma^\star$ *the following inclusion holds:* $\mathsf{Pr}_w(\alpha) \subseteq \mathsf{Pre}(\alpha) \cup \{\varepsilon\}$.

Proof. Let us proceed by induction on α. It is not difficult to see that the inclusion is true for the base cases. If $\alpha = \alpha_1 + \alpha_2$ then $\mathsf{Pr}_w(\alpha_1 + \alpha_2) \subseteq \mathsf{Pr}_w(\alpha_1) \cup \mathsf{Pr}_w(\alpha_2) \cup \{\varepsilon\} \subseteq \mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_2) \cup \{\varepsilon\}$.

Let $\alpha = \alpha_1\alpha_2$ then

$$\begin{aligned}
\mathsf{Pr}_w(\alpha_1\alpha_2) &\subseteq \mathsf{Pr}'(\alpha_1\mathsf{Pr}_w(\alpha_2)) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\} \\
&\subseteq \alpha_1\mathsf{Pr}'(\mathsf{Pr}_w(\alpha_2)) \cup \mathsf{Pr}'(\alpha_1) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\} \\
&\quad \text{as } \mathsf{Pr}'(\mathsf{Pr}_w(\alpha)) \subseteq \mathsf{Pr}_w(\alpha) \\
&\subseteq \alpha_1\mathsf{Pr}_w(\alpha_2) \cup \mathsf{Pr}'(\alpha_1) \cup \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}_v(\alpha_1) \cup \{\varepsilon\} \\
&\quad \text{by inductive hypothesis and Lemma 26} \\
&\subseteq \alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pr}'(\alpha_1) \cup \mathsf{Pre}(\alpha_1) \cup \{\varepsilon\} \\
&\quad \text{as } \mathsf{Pr}'(\alpha) \subseteq \mathsf{Pre}(\alpha) \\
&\subseteq \alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pre}(\alpha_1) \cup \{\varepsilon\} \\
&\subseteq \mathsf{Pre}(\alpha_1\alpha_2) \cup \{\varepsilon\}
\end{aligned}$$

If $\alpha = \alpha_1^\star$, then

$$\begin{aligned}
\mathsf{Pr}_w(\alpha_1^\star) &\subseteq \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\alpha_1^\star\mathsf{Pr}_v(\alpha_1)) \cup \{\varepsilon\} \\
&\subseteq \bigcup_{v\in\mathsf{Prefix}(w)} (\alpha_1^\star\mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1)) \cup \mathsf{Pr}'(\alpha_1^\star)) \cup \{\varepsilon\} \\
&\subseteq \alpha_1^\star \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}'(\mathsf{Pr}_v(\alpha_1)) \cup \mathsf{Pr}'(\alpha_1^\star) \cup \{\varepsilon\} \\
&\quad \text{as } \mathsf{Pr}'(\mathsf{Pr}_w(\alpha)) \subseteq \mathsf{Pr}_w(\alpha) \\
&\subseteq \alpha_1^\star \bigcup_{v\in\mathsf{Prefix}(w)} \mathsf{Pr}_v(\alpha_1) \cup \mathsf{Pr}'(\alpha_1^\star) \cup \{\varepsilon\} \\
&\quad \text{by inductive hypothesis and Lemma 17} \\
&\subseteq \alpha_1^\star\mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_1^\star) \cup \{\varepsilon\} \\
&\subseteq \mathsf{Pre}(\alpha_1^\star) \cup \{\varepsilon\}
\end{aligned}$$

$\square$

Inductive construction. If $\alpha = \emptyset$ then $\mathcal{A}_{pre}(\emptyset) = (\{\varepsilon, \emptyset\}, \Sigma, \emptyset, \varepsilon, \{\emptyset\})$. If $\alpha = \varepsilon$ then $\mathcal{A}_{pre}(\varepsilon) = (\{\varepsilon\}, \Sigma, \emptyset, \varepsilon, \{\varepsilon\})$. If $\alpha = \sigma$ then $\mathcal{A}_{pre}(\varepsilon) = (\{\varepsilon, \sigma\}, \Sigma, \{(\varepsilon, \sigma, \sigma)\}, \varepsilon, \{\sigma\})$.

Suppose $\mathcal{A}_{pre}(\alpha_i) = (\{\varepsilon\} \cup \mathsf{Pre}(\alpha_i), \Sigma, \delta_{pre}(\alpha_i), \varepsilon, \mathsf{Pr}'(\alpha_i) \cup \varepsilon(\alpha_i))$, for $i \in \{1, 2\}$.

If $\alpha = \alpha_1 + \alpha_2$ then $\mathcal{A}_{pre}(\alpha) = (\{\varepsilon\} \cup \mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_2), \Sigma, \delta_{pre}(\alpha_1) \cup \delta_{pre}(\alpha_2), \varepsilon, \mathsf{Pr}'(\alpha_1) \cup \mathsf{Pr}'(\alpha_2) \cup \varepsilon(\alpha))$.

If $\alpha = \alpha_1\alpha_2$ then $\mathcal{A}_{pre}(\alpha) = (\{\varepsilon\} \cup \alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pre}(\alpha_1), \Sigma, \delta_{pre}(\alpha_1) \cup D, \varepsilon, \alpha_1\mathsf{Pr}'(\alpha_2) \cup \varepsilon(\alpha_2)\mathsf{Pr}'(\alpha_1) \cup \varepsilon(\alpha))$, where $D = \delta' \cup \delta''$, $\delta' = \{(t, \sigma, \alpha_1 s_2) \mid t \in (\mathsf{Pr}'(\alpha_1) \cup \varepsilon(\alpha_1)), (\varepsilon, \sigma, s_2) \in \delta_{pre}(\alpha_2)\}$ and $\delta'' = \{(t, \sigma, \alpha_1 s_2) \mid t \in (\alpha_1\mathsf{Pr}'(s_2') \cup \varepsilon(s_2')\mathsf{Pr}'(\alpha_1) \cup \varepsilon(\alpha_1 s_2')), s_2' \neq \varepsilon, (s_2', \sigma, s_2) \in \delta_{pre}(\alpha_2)\}$.

If $\alpha = \alpha_1^\star$ then $\mathcal{A}_{pre}(\alpha) = (\{\varepsilon\}, \cup\alpha_1^\star\mathsf{Pre}(\alpha_1), \Sigma, \delta, \varepsilon, \alpha_1^\star\mathsf{Pr}'(\alpha_1) \cup \{\varepsilon\})$. where $\delta = \{(t, \sigma, \alpha_1^\star s_1) \mid t \in (\alpha_1^\star\mathsf{Pr}'(s_1') \cup \varepsilon(s_1')\mathsf{Pr}'(\alpha_1^\star) \cup \varepsilon(s_1')), s_1' \neq \varepsilon, (s_1', \sigma, s_1) \in \delta_{pre}(\alpha_1)\} \cup \{(t, \sigma, \alpha_1^\star s_1) \mid t \in (\mathsf{Pr}'(\alpha_1^\star) \cup \{\varepsilon\}), (\varepsilon, \sigma, s_1) \in \delta_{pre}(\alpha_1)\}$.

Proposition 28. *Let α be a regular expression. Then $L(\mathcal{A}_{pre}(\alpha)) = L(\alpha)$.*

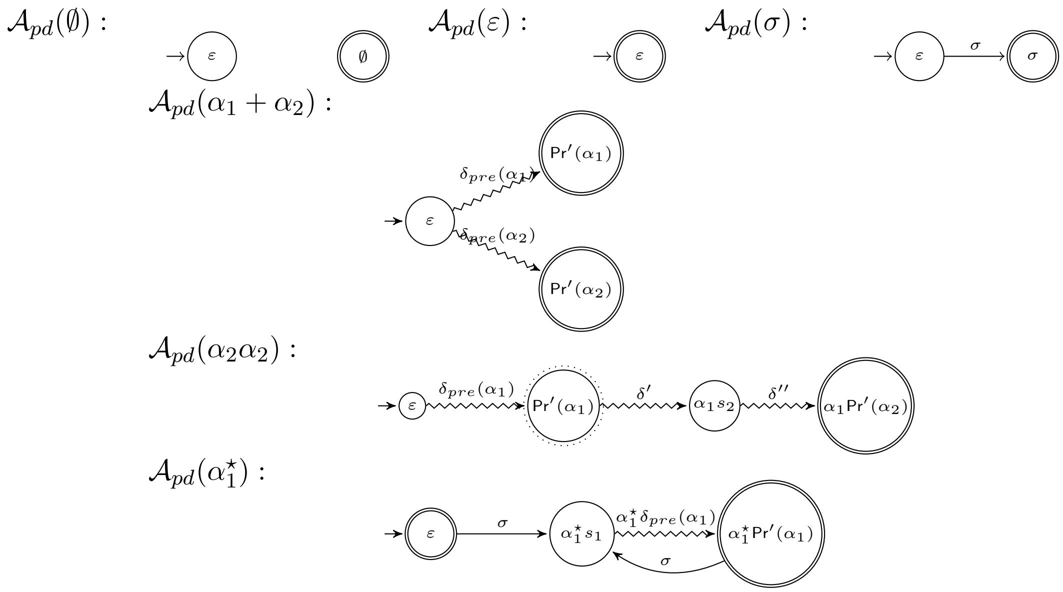


Figure 5: Inductive construction of $\mathcal{A}_{pd}$. The initial states are final if ε belongs to α language. Note that only if $\varepsilon(\alpha_2) = \varepsilon$ the dotted arrow in $\mathcal{A}_{pd}(\alpha_1\alpha_2)$ exists.

Proof. For the base cases it is obvious (see Fig. 5). If $\alpha = \alpha_1 + \alpha_2$, then the initial state of $\mathcal{A}_{pre}(\alpha)$ coincides with the initial state of $\mathcal{A}_{pre}(\alpha_1)$ and $\mathcal{A}_{pre}(\alpha_2)$. The transitions of $\mathcal{A}_{pre}(\alpha_1)$ and $\mathcal{A}_{pre}(\alpha_2)$ are maintained in $\mathcal{A}_{pre}(\alpha)$. Thus, $L(\mathcal{A}_{pre}(\alpha)) = L(\mathcal{A}_{pre}(\alpha_1)) \cup L(\mathcal{A}_{pre}(\alpha_2)) = L(\alpha_1) \cup L(\alpha_2) = L(\alpha)$. Let $\alpha = \alpha_1\alpha_2$. The words accepted by $\mathcal{A}_{pre}(\alpha)$ are the words xy such that x goes from the initial state of $\mathcal{A}_{pre}(\alpha)$, which corresponds to the initial state of $\mathcal{A}_{pre}(\alpha_1)$, to a state $s \in \mathsf{Pr}'(\alpha_1)$, which corresponds to a final state of $\mathcal{A}_{pre}(\alpha_1)$; and y goes from a state $s \in \mathsf{Pr}'(\alpha_1)$, which corresponds to the initial state of $\mathcal{A}_{pre}(\alpha_2)$ to a state $s' \in \alpha_1\mathsf{Pr}'(\alpha_2)$, which corresponds to a final state of $\mathcal{A}_{pre}(\alpha_2)$. Then $L(\mathcal{A}_{pre}(\alpha)) = \{xy \mid x \in L(\mathcal{A}_{pre}(\alpha_1)) \wedge y \in L(\mathcal{A}_{pre}(\alpha_2))\}$. Thus, $L(\mathcal{A}_{pre}(\alpha)) = L(\mathcal{A}_{pre}(\alpha_1))L(\mathcal{A}_{pre}(\alpha_2)) = L(\alpha_1)L(\alpha_2) = L(\alpha)$. Consider $\alpha = \alpha_1^\star$. The words accepted by $\mathcal{A}_{pre}(\alpha)$ are the words of the form $x = x_1x_2x_3\cdots x_n$ such that $x_i \in L(\alpha_1), \forall i \in [1, n]$. Any word x_i goes from the initial state of $\mathcal{A}_{pre}(\alpha)$, which corresponds to the initial state of $\mathcal{A}_{pre}(\alpha_1)$, to a final state $f \subseteq \alpha_1^\star\mathsf{Pr}'(\alpha_1)$, which corresponds to a final state of $\mathcal{A}_{pre}(\alpha_1)$. After recognise a word x_i we can recognize a word x_{i+1}, because when the recognising process reaches a final state, it can proceed as if it has reached the initial state – $\delta_{pre}(\varepsilon) \subseteq \delta_{pre}(f)$. Note that if $x = \varepsilon$, x is also recognised, because the initial state is also a final state. Thus, $L(\mathcal{A}_{pre}(\alpha)) = L(\mathcal{A}_{pre}(\alpha_i)^\star) = L(\alpha_1^\star)$. □

7.1 $\mathcal{A}_{pre}$ inductive definition

The LP labelling scheme proposed by Yamamoto can be obtained as a solution of a system of expression equations for a RE α, as done both for $\mathcal{A}_{pd}$ and $\overleftarrow{\mathcal{A}}_{pd}$. Consider a system of left equations $\alpha_i = \alpha_{i1}\sigma_1 + \cdots + \alpha_{ik}\sigma_k$, $i \in [1, n]$, where $\alpha = \sum_{i \in I \subseteq [0,n]} \alpha_i$, $\alpha_{ij} = \sum_{l \in I_{ij} \subseteq [0,n]} \alpha_l$ and $\alpha_0 \equiv \varepsilon$.

Proposition 29. *The set* $\mathsf{Pre}(\alpha)$ *inductively defined as follows:*

$$\begin{array}{rclcrcl} \mathsf{Pre}(\emptyset) & = & \emptyset & \quad & \mathsf{Pre}(\alpha+\beta) & = & \mathsf{Pre}(\alpha)\cup\mathsf{Pre}(\beta) \\ \mathsf{Pre}(\varepsilon) & = & \emptyset & & \mathsf{Pre}(\alpha\beta) & = & \alpha\mathsf{Pre}(\beta)\cup\mathsf{Pre}(\alpha) \\ \mathsf{Pre}(\sigma) & = & \{\sigma\} & & \mathsf{Pre}(\alpha^{\star}) & = & \alpha^{\star}\mathsf{Pre}(\alpha). \end{array} \tag{25}$$

is a solution (left support) of the system of left equations defined above.

Proof. For $\alpha=\emptyset$ and $\alpha=\varepsilon$ is obvious that the solution is $\emptyset$. For $\alpha=\sigma$,

$$\begin{array}{rcl} \alpha & = & \alpha_1 \\ \alpha_1 & = & \alpha_0\sigma \\ \alpha_0 & = & \varepsilon \end{array}$$

Thus $\mathsf{Pre}(\alpha)=\{\sigma\}$. Let us suppose that

$$\begin{array}{rcl} \beta & = & \sum_{i\in I\subseteq[0,n]}\beta_i \\ \beta_i & = & \beta_{i1}a_1+\cdots+\beta_{ik}a_k, \end{array}$$

with $\mathsf{Pre}(\beta)=\{\beta_1,\cdots,\beta_n\}$ and

$$\begin{array}{rcl} \gamma & = & \sum_{i\in I\subseteq[0,m]}\gamma_i \\ \gamma_i & = & \gamma_{i1}a_1+\cdots+\gamma_{ik}a_k, \end{array}$$

with $\mathsf{Pre}(\gamma)=\{\gamma_1,\cdots,\gamma_m\}$. Consider $\alpha=\beta+\gamma$, then

$$\beta+\gamma \quad = \quad \sum_{i\in I\subseteq[1,n]}\beta_i+\sum_{i\in I'\subseteq[1,n']}\gamma_i$$

As we need all β_i, $i\in[1,n]$ to define β, and all γ_i, $i\in[1,m]$ to define γ, $\mathsf{Pre}(\alpha)=\{\beta_1,\cdots,\beta_n\}\cup\{\gamma_1,\cdots,\gamma_m\}$. Consider $\alpha=\beta\gamma$ then

$$\begin{array}{rcl} \beta\gamma & = & \beta(\sum_{i\in I\subseteq[0,m]}\gamma_i) \\ & & \text{Note that if i=0, then } \varepsilon(\gamma)=\varepsilon \\ & = & \beta(\sum_{i\in I\subseteq[0,m]}\gamma_i)+\varepsilon(\gamma)\sum_{i\in I\subseteq[0,n]}\beta_i+\varepsilon(\gamma)\varepsilon(\beta) \end{array}$$

And

$$\beta\gamma_i \quad = \quad \beta(\gamma_{i1}\sigma_1+\cdots+\gamma_{ik}\sigma_k)$$

As we know that $\gamma_0\subseteq\gamma_{i,k}$ for some $i\in[0,m]$, the solution set is $\mathsf{Pre}(\alpha)=\{\beta\gamma_1,\cdots,\beta\gamma_m\}\cup\{\beta_1,\cdots,\beta_n\}$.

Consider $\alpha = \beta^\star$ then

$$\begin{array}{rcl}\beta^\star & = & \beta^\star\beta + \varepsilon \\ & = & \beta^\star(\sum_{i\in I\subseteq[1,n]} \beta_i) + \varepsilon\end{array}$$

Thus, $\mathsf{Pre}(\alpha) = \{\beta^\star\beta_1, \cdots, \beta^\star\beta_n\}$. □

The set $\mathsf{Pre}_0(\alpha) = \mathsf{Pre}(\alpha) \cup \{\varepsilon\}$ constitutes the set of states of the prefix automaton $\mathcal{A}_{pre}(\alpha)$. It also follows from the resolution of the above system of equations, that the set of transitions of $\mathcal{A}_{pre}(\alpha)$ can be inductively defined. Let $\mathsf{Pr}'(\alpha)$, $\psi(\alpha)$ and $\mathsf{T}(\alpha)$ be defined, respectively, as follows

$$\begin{array}{rclrcl}\mathsf{Pr}'(\emptyset) & = & \emptyset & \mathsf{Pr}'(\alpha+\beta) & = & \mathsf{Pr}'(\alpha) \cup \mathsf{Pr}'(\beta) \\ \mathsf{Pr}'(\varepsilon) & = & \{\varepsilon\} & \mathsf{Pr}'(\alpha\beta) & = & \alpha\mathsf{Pr}'(\beta) \cup \varepsilon(\beta)\mathsf{Pr}'(\alpha) \\ \mathsf{Pr}'(\sigma) & = & \{\sigma\} & \mathsf{Pr}'(\alpha^\star) & = & \alpha^\star\mathsf{Pr}'(\alpha).\end{array} \tag{26}$$

$$\begin{array}{rclrcl}\psi(\emptyset) & = & \emptyset & \psi(\alpha+\beta) & = & \psi(\alpha) \cup \psi(\alpha) \\ \psi(\varepsilon) & = & \emptyset & \psi(\alpha\beta) & = & \psi(\alpha) \cup \varepsilon(\alpha)\ \alpha\ \psi(\beta) \\ \psi(\sigma) & = & \{(\sigma,\sigma)\} & \psi(\alpha^\star) & = & \alpha^\star\psi(\alpha)\end{array} \tag{27}$$

$$\begin{array}{rcl}\mathsf{T}(\emptyset) & = & \mathsf{T}(\varepsilon) = \mathsf{T}(\sigma) = \emptyset,\ \sigma \in \Sigma \\ \mathsf{T}(\alpha+\beta) & = & \mathsf{T}(\alpha) \cup \mathsf{T}(\beta) \\ \mathsf{T}(\alpha\beta) & = & \mathsf{T}(\alpha) \cup \alpha\mathsf{T}(\beta) \cup \mathsf{Pr}'(\alpha) \times (\alpha\psi(\beta)) \\ \mathsf{T}(\alpha^\star) & = & \alpha^\star\mathsf{T}(\alpha) \cup \alpha^\star(\mathsf{Pr}'(\alpha) \times \psi(\alpha)).\end{array} \tag{28}$$

Proposition 30. *The inductive definitions given in (26), (27) and (28) follows from the resolution of the above system of equations.*

Proof. For the base cases it is obvious. Let us suppose that

$$\begin{array}{rcl}\beta & = & \sum_{i\in I\subseteq[0,n]} \beta_i \\ \beta_i & = & \beta_{i1}a_1 + \cdots + \beta_{ik}a_k,\end{array}$$

with $\psi(\beta) = \{(\sigma_1, \beta_i) \mid \beta_i = \beta_{i1}\sigma_1,\ \beta_0 \in \beta_{i1}\}$, $\mathsf{Pr}'(\beta) = \{\beta_i \mid i \in I \subseteq [0,n]\}$, and $\mathsf{T}(\beta) = \{(\beta_i, \sigma_1, \beta_j) \mid \beta_j = \beta_{j1}\sigma_1,\ \beta_i \in \beta_{j1}\}$, and

$$\begin{array}{rcl}\gamma & = & \sum_{i\in I\subseteq[0,m]} \gamma_i \\ \gamma_i & = & \gamma_{i1}a_1 + \cdots + \gamma_{ik}a_k,\end{array}$$

with $\psi(\gamma) = \{(\sigma_1, \gamma_i) \mid \gamma_i = \gamma_{i1}\sigma_1,\ \gamma_0 \in \gamma_{i1}\}$, $\mathsf{Pr}'(\gamma) = \{\gamma_i \mid i \in I \subseteq [0,n]\}$, and $\mathsf{T}(\gamma) = \{(\gamma_i, \sigma_1, \gamma_j) \mid \gamma_j = \gamma_{j1}\sigma_1,\ \gamma_i \in \gamma_{j1}\}$.

In the case $\alpha = \beta + \gamma$ the definitions are obvious. Consider $\alpha = \beta\gamma$, then from the equation

$$\beta\gamma\beta = (\sum_{i\in I\subseteq[0,m]} \gamma_i) + \varepsilon(\gamma) \sum_{i\in I\subseteq[0,n]} \beta_i + \varepsilon(\gamma)\varepsilon(\beta)$$

it is obvious that $\mathsf{Pr}' = \beta\mathsf{Pr}'(\gamma) \cup \varepsilon(\gamma)\mathsf{Pr}'(\beta)$. Considering the equations $\beta\gamma_i$ and β_i is not difficult to conclude that $psi(\beta\gamma) = \psi(\beta) \cup \varepsilon(\beta)\beta\psi(\gamma)$. From the same equations we can also conclude that $\mathsf{T}(\beta\gamma) = \mathsf{T}(\beta) \cup \beta\mathsf{T}(\gamma) \cup \mathsf{Pr}'(\beta) \times \beta\psi(\gamma)$. The definition for the functions when $\alpha = \beta^\star$ are obtained in a similar way. $\square$

Therefore,

$$\mathcal{A}_{pre}(\alpha) = (\mathsf{Pre}_0(\alpha), \Sigma, \{\varepsilon\} \times \psi(\alpha) \cup \mathsf{T}(\alpha), \varepsilon, \mathsf{Pr}'(\alpha) \cup \varepsilon(\alpha)).$$

7.2 $\mathcal{A}_{pre}$ as $\mathcal{A}_{pos}$ Quotient

We now show that the $\mathcal{A}_{pre}(\alpha)$ is a quotient of $\mathcal{A}_{pos}(\alpha)$. If α is a linear regular expression, $\mathcal{A}_{pos}(\alpha)$ is deterministic and thus all its states have distinct left languages. Therefore, in this case, $\mathcal{A}_{pre}(\alpha)$ coincides with $\mathcal{A}_{pos}(\alpha)$.

Proposition 31. *For any linear regular expression α, $|\mathsf{Pre}(\alpha)| = |\alpha|_\Sigma$.*

Proof. Let us prove the result by induction on α. For the base cases the result is obviously true. Assuming that the result holds for $\alpha_1, \alpha_2 \in \mathsf{RE}$, we prove it for the operations. If $\alpha = \alpha_1 + \alpha_2$, then $|\mathsf{Pre}(\alpha_1 + \alpha_2)| = |\mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_2)|$. As $\Sigma_{\alpha_1} \cap \Sigma_{\alpha_2} = \emptyset$, $|\mathsf{Pre}(\alpha_1) \cup \mathsf{Pre}(\alpha_2)| = |\alpha_1| + |\alpha_2| = |\alpha_1 + \alpha_2|$. Considering $\alpha = \alpha_1\alpha_2$ we have that $|\mathsf{Pre}(\alpha_1\alpha_2)| = |\alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pre}(\alpha_1)|$. By the same reason of the previous case $|\alpha_1\mathsf{Pre}(\alpha_2) \cup \mathsf{Pre}(\alpha_1)| = |\alpha_2| + |\alpha_1| = |\alpha_1\alpha_2|$. Finally, if $\alpha = \alpha_1^\star$, then $|\mathsf{Pre}(\alpha_1^\star)| = |\alpha_1^\star\mathsf{Pre}(\alpha_1)| = |\alpha_1| = |\alpha_1^\star|$. $\square$

In particular, for an arbitrary RE α, $\mathcal{A}_{pre}(\overline{\alpha}) \simeq \mathcal{A}_{pos}(\overline{\alpha})$.

Proposition 32. *Let α be a regular expression. Then $\overline{\mathcal{A}_{pre}(\overline{\alpha})} \simeq \mathcal{A}_{pos}(\overline{\alpha})$.*

Proof. By the Proposition 31 we know that the both automata have the same number of states. However we need to prove that the automata are isomorphic. We proceed by induction on α. For $\alpha = \emptyset$ or $\alpha = \varepsilon$ the result is obvious. If $\alpha = \sigma$, $\overline{\mathcal{A}_{pre}(\overline{\sigma})} = (\{\sigma_1, \varepsilon\}, \{\sigma\}, (\varepsilon, \sigma, \sigma_1), \varepsilon, \{\sigma_1\})$ and $\mathcal{A}_{pos}(\sigma) = (\{0, (\sigma, 1)\}, \{\sigma\}, (0, \sigma, (\sigma, 1)), 0, \{(\sigma, 1)\})$. Thus the automata are isomorphic. Let us assume that the result holds for $\alpha_1, \alpha_2 \in \mathsf{RE}$, and consider that

$\overline{\mathcal{A}_{pre}(\overline{\alpha_i})} = (Q_{\overline{\alpha}_i}, \Sigma, \delta_{\alpha_i}, \varepsilon, \mathsf{Pr}'(\overline{\alpha}_i) \cup \varepsilon(\overline{\alpha}_i))$ and

$\mathcal{A}_{pos}(\alpha_i) = (\mathsf{Pos}(\alpha_i) \cup \{0\}, \Sigma, \delta_{p_{\alpha_i}}, 0, Last(\alpha_i)\varepsilon(\alpha_i)\{\varepsilon\}),$ for $i \in \{1, 2\}$.

If $\alpha = \alpha_1 + \alpha_2$, then $\overline{\mathcal{A}_{pre}(\overline{\alpha_1 + \alpha_2})} = (\mathsf{Pre}(\overline{\alpha}_1) \cup \mathsf{Pre}(\overline{\alpha}_2) \cup \{\varepsilon\}, \Sigma, \delta_{\alpha_1} \cup \delta_{\alpha_2}, \varepsilon, \mathsf{Pr}'(\overline{\alpha}_1) \setminus \{\varepsilon\} \cup \mathsf{Pr}'(\overline{\alpha}_2) \setminus \{\varepsilon\} \cup \varepsilon(\overline{\alpha_1} + \overline{\alpha_2}))$; and $\mathcal{A}_{pos}(\alpha_1 + \alpha_2) = (\mathsf{Pos}(\alpha_1 + \alpha_2) \cup \{\varepsilon\}, \Sigma, \delta_{p_{\alpha_1}} \cup \delta_{p_{\alpha_2}}, 0, Last(\alpha_1) \setminus \{0\} \cup Last(\alpha_2) \setminus \{0\} \cup \varepsilon(\alpha_1 + \alpha_2)\{0\})$. Therefore, is not difficult to see that the automata are isomorphic.

If $\alpha = \alpha_1\alpha_2$ then $\overline{\mathcal{A}_{pre}(\overline{\alpha_1\alpha_2})} = (\overline{\alpha}_1\mathsf{Pre}(\overline{\alpha}_2) \cup \overline{\alpha}_1 \cup \{\varepsilon\}, \Sigma, \delta_{\alpha_1\alpha_2}, \varepsilon, \mathsf{Pr}'(\overline{\alpha}_1\overline{\alpha}_2) \setminus \{\varepsilon\} \cup \varepsilon(\overline{\alpha}_1\overline{\alpha}_2))$, where

$\delta_{\alpha_1\alpha_2} = A \cup B$, whith $A = \{(s', \sigma, s) \mid s \in \overline{\alpha}_1 q \wedge q \in \mathsf{Pre}(\overline{\alpha}_2) \wedge \mathsf{Pr}_\sigma(q) \neq \varepsilon \wedge s' \in (\overline{\alpha}_1\mathsf{Pr}'(\mathsf{Pr}_\sigma(q)) \cup \varepsilon(\mathsf{Pr}_\sigma(q))\mathsf{Pr}'(\overline{\alpha}_1) \cup \varepsilon(\overline{\alpha}_1\mathsf{Pr}_\sigma(q))\{\varepsilon\} \cup \varepsilon(q)(\mathsf{Pr}'(\mathsf{Pr}_\sigma(\overline{\alpha}_1)) \cup \varepsilon(\mathsf{Pr}_\sigma(\overline{\alpha}_1))))\}$,

$B = \{(s', \sigma, s) \mid s \in \overline{\alpha}_1 q \wedge q \in \mathsf{Pre}(\overline{\alpha}_2) \wedge \mathsf{Pr}_\sigma(q) = \varepsilon \wedge s' \in (\mathsf{Pr}'(\overline{\alpha}_1) \cup \varepsilon(\overline{\alpha}_1))\} \cup \delta_{\alpha_1}$, and

$$\mathsf{Pr}'(\overline{\alpha}_1\overline{\alpha}_2) = \begin{cases} \overline{\alpha}_1\mathsf{Pr}'(\overline{\alpha}_2) \cup \varepsilon(\overline{\alpha}_2)\mathsf{Pr}'(\overline{\alpha}_1) & \text{if } \overline{\alpha}_2 \neq \varepsilon \\ \mathsf{Pr}'(\overline{\alpha}_1) & \text{otherwise} \end{cases}.$$

Still considering the same α,

$\mathcal{A}_{pos}(\alpha_1\alpha_2) = (\mathsf{Pos}(\alpha_1\alpha_2) \cup \{0\}, \sigma, \delta_p, 0, (Last(\alpha_2) \cup \varepsilon(\alpha_2)Last(\alpha_1)) \setminus \{0\} \cup \varepsilon(\alpha_1\alpha_2)\{0\})$, where

$\delta_p(0,\sigma) = \delta_{p_{\alpha_1}}(0,\sigma) \cup \varepsilon(\alpha_1)\delta_{p_{\alpha_2}}(0,\sigma)$, and

$\delta_p(x,\sigma) = A' \cup B'$, with $A' = \{y \mid y \in Follow(\alpha_1, x) \wedge \overline{y} = \sigma \wedge x \in (\mathsf{Pos}(\alpha_1) \setminus Last(\alpha_1))\} \cup \{y \mid y \in Follow(\alpha_1, x) \cup \mathsf{First}(\alpha_2) \wedge \overline{y} = \sigma \wedge x \in Last(\alpha_1)\}$ and $B' = \{y \mid y \in Follow(\alpha_2, x) \wedge \overline{y} = \sigma \wedge x \in \mathsf{Pos}(\alpha_2)\}$.

It is not difficult to conclude that exists an isomorphism between the sets A and B'. The same happens with the remaining transitions of δ_p and $\delta_{\alpha_1\alpha_2}$. Therefore we can conclude that, also in this case, the automata are isomorphic.

Finally, consider $\alpha = \alpha_1^\star$. Then, we have that $\overline{\mathcal{A}_{pre}(\overline{\alpha_1^\star})} = (\alpha_1^\star \mathsf{Pre}(\alpha_1) \cup \{\varepsilon\}, \Sigma, \delta_{\alpha_1^\star}, \varepsilon, \alpha_1^\star \mathsf{Pr}'(\alpha_1) \cup \{\varepsilon\})$, where

$\delta_{\alpha_1^\star} = \{(s', \sigma, s) \mid s \in \alpha_1^\star t \wedge t \in \mathsf{Pre}(\alpha_1) \wedge \mathsf{Pr}_\sigma(t) \neq \varepsilon \wedge s' \in (\alpha_1^\star \mathsf{Pr}'(\mathsf{Pr}_\sigma(t)) \cup \varepsilon(\mathsf{Pr}_\sigma(t))(\{\varepsilon\} \cup \mathsf{Pr}'(\alpha_1^\star)))\} \cup \{(s', \sigma, s) \mid s \in \alpha_1^\star t \wedge t \in \mathsf{Pre}(\alpha_1) \wedge \mathsf{Pr}_\sigma(t) = \varepsilon \wedge s' \in (\mathsf{Pr}'(\alpha_1^\star) \cup \{\varepsilon\})\}$.

$\mathcal{A}_{pos}(\alpha_1^\star) = (\mathsf{Pos}(\alpha_1), \Sigma, \delta_p, Last(\alpha_1) \cup \{0\})$, where $\delta_p(0,\sigma) = \delta_{p_{\alpha_1}}(0,\sigma)$, and $\delta_p(x,\sigma) = \{y \mid y \in Follow(\alpha_1, \sigma) \wedge \overline{y} = \sigma \wedge x \in (\mathsf{Pos}(\alpha_1 \setminus Last(\alpha_1)))\} \cup \{y \mid y \in (Follow(\alpha_1, x) \cup First(\alpha_1)) \wedge \overline{y} = \sigma \wedge x \in Last(\alpha_1)\}$.

We can establish an isomorphism between δ_p and $\delta_{\alpha_1^\star}$. The same happens with the other components of the automata. Thus, the automata are isomorphic. $\square$

Let us define the equivalence relation $\equiv_l$ such that for any regular expression α, $\forall s, s' \in \mathsf{Pre}(\overline{\alpha})$, $s \equiv_l s' \Leftrightarrow \overline{s} = \overline{s'}$. In the following we show that $\equiv_l$ is a left-invariant relation.

Corollary 12. *Let $\alpha \in \mathsf{RE}$. For any $s \in \mathsf{Pre}(\overline{\alpha})$ the following holds: $\forall w \in \mathsf{Pr}_\sigma(s) : \overline{w} \in \mathsf{Pr}_{\overline{\sigma}}(\overline{s})$.*

Proposition 33. *The relation $\equiv_l$ is a left-invariant relation.*

Proof. We want to prove that $\equiv_l$ is a left-invariant relation. In this case it is obvious that the initial state ε is not $\equiv_l$ equivalent to any other state. We also need to prove that $s \equiv_l s' \Rightarrow w \in \mathsf{Pr}_\sigma(s) \equiv_l z \in \mathsf{Pr}_\sigma(s')$. Let us suppose that $w \in \mathsf{Pr}_\sigma(s)$, then

$$\begin{aligned}
w \in \mathsf{Pr}_\sigma(s) \quad &\Rightarrow \quad \overline{w} \in \mathsf{Pr}_{\overline{\sigma}}(\overline{s}) \\
&\Rightarrow \quad \overline{w} \in \mathsf{Pr}_{\overline{\sigma}}(\overline{s}') \\
&\qquad \text{Thus } \exists z \in \mathsf{Pr}_\sigma(s') : w \equiv_l z \\
&\qquad \text{because } \forall w \in \mathsf{Pr}_\sigma(s) : \overline{w} \in \mathsf{Pr}_{\overline{\sigma}}(\overline{s})
\end{aligned}$$

Let us suppose that $z \in \mathsf{Pr}_\sigma(s')$, then

$$\begin{aligned}
z \in \mathsf{Pr}_\sigma(s') \quad &\Rightarrow \quad \overline{z} \in \mathsf{Pr}_{\overline{\sigma}}(\overline{s'}) \\
&\Rightarrow \quad \overline{z} \in \mathsf{Pr}_{\overline{\sigma}}(\overline{s}) \\
&\qquad \text{Thus } \exists w \in \mathsf{Pr}_\sigma(s) : w \equiv_l z
\end{aligned}$$

Thus $\equiv_l$ is a left-invariant relation. $\square$

After all these results is not difficult to conclude that $\mathcal{A}_{pre}$ automaton is a quotient of $\mathcal{A}_{pos}$.

Corollary 13. *Let α be a regular expression. Then $\mathcal{A}_{pre}(\alpha) \simeq \overline{\mathcal{A}_{pre}(\overline{\alpha})}/_{\equiv_l}$.*

By construction, the Glushkov automaton is homogeneous, i.e. the in- transitions of each state are all labelled by the same letter. It follows from Corollary 13 that this property also holds for $\mathcal{A}_{pre}$.

k	$\lvert\alpha\rvert$	$\lvert\mathsf{Pos}_0\rvert$	$\lvert\delta_{pos}\rvert$	$\lvert\mathsf{PD}\rvert$	$\lvert\delta_\pi\rvert$	$\frac{\lvert\pi\rvert}{\lvert\mathsf{Pos}\rvert}$	$\lvert\overleftarrow{\mathsf{PD}}\rvert$	$\lvert\delta_{\overleftarrow{\pi}}\rvert$	$\frac{\lvert\overleftarrow{\pi}\rvert}{\lvert\mathsf{Pos}\rvert}$	$\lvert\mathsf{Pre}_0\rvert$	$\lvert\delta_{pre}\rvert$	$\frac{\lvert\mathsf{Pre}\rvert}{\lvert\mathsf{Pos}\rvert}$	$1-\eta_k$
2	100	28.9	167.5	15.7	56.0	0.55	15.9	56.4	0.55	20.1	73.7	0.71	0.90
	500	139.9	1486.5	71.6	389.8	0.51	71.5	393.1	0.51	91.9	530.8	0.66	
10	100	42.5	159.4	23.8	73.7	0.56	23.8	72.9	0.56	38.5	130.4	0.91	0.99
	500	207.1	1019.1	113.2	423.8	0.55	112.4	425.6	0.54	186	807.1	0.90	
	1000	412.1	2182.1	223.7	884.1	0.54	223.1	884.5	0.54	369.5	1717.6	0.90	

Table 1: Experimental results for uniform random generated regular expressions.

8 Average-Case Complexity

We conducted some experimental tests in order to compare the sizes of $\mathcal{A}_{pos}$, $\mathcal{A}_{pd}$, $\overleftarrow{\mathcal{A}}_{pd}$ and $\mathcal{A}_{pre}$ automata. We used the FAdo library[1] that includes implementations of the NFA conversions and also several tools for uniformly random generate regular expressions. In order to obtain regular expressions uniformly generated in the size of the syntactic tree, we use a prefix notation version of the grammar. For each alphabet size, k, and $|\alpha|$, samples of 10 000 REs were generated, which is sufficient to ensure a 95% confidence level within a 1% error margin. Table 1 presents the average values obtained for $|\alpha| \in \{100, 500, 1000\}$ and $k \in \{2, 10\}$. These experiments suggest that the $\overleftarrow{\mathcal{A}}_{pd}$ and the $\mathcal{A}_{pd}$ have the same size and the $\mathcal{A}_{pre}$ is not significantly smaller then the $\mathcal{A}_{pos}$.

By Proposition **??**, $|\alpha^R|_\Sigma = |\alpha|_\Sigma$ and by the fact that $\varepsilon \in \pi(\alpha)$ if and only if $\varepsilon \in \overleftarrow{\pi}(\alpha)$, the analysis of the average size of $\mathcal{A}_{pd}(\alpha)$ presented in Broda *et al* [2] carries on to $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$. Thus the average sizes of $\mathcal{A}_{pd}$ and $\overleftarrow{\mathcal{A}}_{pd}$ are asymptotically the same. However, $\overleftarrow{\mathcal{A}}_{pd}(\alpha)$ has only one final state and its number of initial states is the number of final states of $\mathcal{A}_{pd}(\alpha^R)$. As studied by Nicaud [18], the size of $Last(\alpha)$ tends asymptotically to a constant depending on k and $|\lambda(\alpha)|$ is half that size [3]. Following, again, the ideas in Broda *et al.*, we estimate the number of mergings of states that arise when computing $\mathcal{A}_{pre}$ from $\mathcal{A}_{pos}$. The $\mathcal{A}_{pre}$ has at most $|\alpha|_\Sigma + 1$ states and this only occurs when all unions in $\mathsf{Pre}(\alpha)$ are disjoint. However there are cases in which this does not happen. For instance, when $\sigma \in \mathsf{Pre}(\beta) \cap \mathsf{Pre}(\gamma)$, then $|\mathsf{Pre}(\beta+\gamma)| = |\mathsf{Pre}(\beta) \cup \mathsf{Pre}(\gamma)| \le |\mathsf{Pre}(\beta)| + |\mathsf{Pre}(\gamma)| - 1$ and $|\mathsf{Pre}(\beta^\star\gamma)| = |\beta^\star\mathsf{Pre}(\gamma) \cup \beta^\star\mathsf{Pre}(\beta)| \le |\mathsf{Pre}(\beta)| + |\mathsf{Pre}(\gamma)| - 1$. In what follows we estimate the number of these non-disjoint unions, which correspond to a lower bound for the number of states merged in the $\mathcal{A}_{pos}$ automaton. This is done by the use of the methods of analytic combinatorics as expounded by Flajolet and Sedgewick [9]. These apply to generating functions $A(z) = \sum_n a_n z^n$ for a combinatorial class $\mathcal{A}$ with a_n objects of size n, denoted by $[z^n]A(z)$, and also bivariate functions $C(u,z) = \sum_\alpha u^{c(\alpha)} z^{|\alpha|}$, where $c(\alpha)$ is some measure of the object $\alpha \in \mathcal{A}$.

The regular expressions α_σ for which $\sigma \in \mathsf{Pre}(\alpha_\sigma)$, $\sigma \in \Sigma$ are generated by following grammar

$$\alpha_\sigma \quad := \quad \sigma \mid \alpha_\sigma + \alpha \mid \alpha_{\overline{\sigma}} + \alpha_\sigma \mid \alpha_\sigma \cdot \alpha \mid \varepsilon \cdot \alpha_\sigma \tag{29}$$

The regular expressions that are not generated by α_σ are denoted by $\alpha_{\overline{\sigma}}$. The generating function for α_σ, $R_{\sigma,k}(z)$ satisfies

[1] http://fado.dcc.fc.up.pt

$$R_{\sigma,k}(z) = z + zR_{\sigma,k}(z)R_k(z) + z(R_k(z) - R_{\sigma,k}(z))R_{\sigma,k}(z) + zR_{\sigma,k}(z)R_k(z) + z^2R_{\sigma,k}(z)$$

that is equivalent to

$$zR_{\sigma,k}(z)^2 - (3zR_k(z) + z^2 - 1)R_{\sigma,k}(z) - z = 0. \tag{30}$$

From this one gets

$$R_{\sigma,k}(z) = \frac{(z^2 + 3zR_k(z) - 1) + \sqrt{(z^2 + 3zR_k(z) - 1)^2 + 4z^2}}{2z}. \tag{31}$$

As we know that $R_k(z) = \frac{1-z-\sqrt{\Delta_k(z)}}{4z}$, which is the generating function for REs given by grammar (1) but omitting the $\emptyset$, one has

$$8zR_\sigma(z) = -b(z) - 3\sqrt{\Delta_k(z)} + \sqrt{a(z) + 6b(z)\sqrt{\Delta_k(z)} + 9\Delta_k(z)} \tag{32}$$

where $a(z) = 16z^4 - 24z^3 + 65z^2 + 6z + 1$, $b(z) = -4z^2 + 3z + 1$, and $\Delta_k(z) = 1 - 2z - (7 + 8k)z^2$. Using the binomial theorem, we know that

$$\sqrt{a(z) + 6b(z)\sqrt{\Delta_k(z)} + 9\Delta_k(z)} = \sqrt{a(z)} + 3\frac{b(z)}{\sqrt{a(z)}}\sqrt{\Delta_k(z)} + o(\Delta_k(z)^{\frac{1}{2}}).$$

Thus,

$$8zR_{\sigma,k}(z) = -b(z) + \sqrt{a(z)} + 3\frac{b(z)}{\sqrt{a(z)}}\sqrt{\Delta_k(z)} + o(\Delta_k(z)^{\frac{1}{2}}). \tag{33}$$

As we know that the following equalities are true:

$$\sqrt{\Delta_k(z)} = \sqrt{(7 + 8k)\rho_k(z - \overline{\rho}_k)}\sqrt{1 - z/\rho_k},$$

$$\sqrt{(7 + 8k)\rho_k(\rho_k - \overline{\rho}_k)} = \sqrt{2 - 2\rho_k}$$

and using the techniques in Broda *et. al* and namely Proposition 3

$$[z^n]R_\sigma(z) \sim \frac{3}{16\sqrt{\pi}}\left(1 - \frac{b(\rho_k)}{\sqrt{a(\rho_k)}}\right)\sqrt{2(1 - \rho_k)}\rho_k^{-(n+1)}n^{-\frac{3}{2}}. \tag{34}$$

Thus the asymptotic ratio of regular expressions with $\sigma \in \mathsf{Pre}(\alpha)$ is:

$$\frac{[z^n]R_{\sigma,k}(z)}{[z^n]R_k(z)} \sim \frac{3}{2}\left(1 - \frac{b(\rho_k)}{\sqrt{a(\rho_k)}}\right). \tag{35}$$

As $\lim\limits_{k\to\infty} \rho_k = 0$, $\lim\limits_{k\to\infty} a(\rho_k) = 1$, and $\lim\limits_{k\to\infty} b(\rho_k) = 1$, this asymptotic ratio approaches 0 when $k \to \infty$.

Let $i(\alpha)$ be the number of non-disjoint unions appearing during the computation of $\mathsf{Pre}(\alpha)$, $\alpha \in \mathsf{RE}$ originated by the previous two cases. Then $i(\alpha)$ verifies

$$\begin{array}{rclcrcl}
i(\varepsilon) & = & i(\sigma) \ = \ 0 & \quad & i(\alpha_\sigma^\star\alpha_\sigma) & = & i(\alpha_\sigma^\star) + i(\alpha_\sigma) + 1 \\
i(\alpha_\sigma + \alpha_\sigma) & = & i(\alpha_\sigma) + i(\alpha_\sigma) + 1 & & i(\overline{\alpha_\sigma^\star}\alpha_\sigma) & = & i(\overline{\alpha_\sigma^\star}) + i(\alpha_\sigma) \\
i(\alpha_\sigma + \alpha_{\overline{\sigma}}) & = & i(\alpha_\sigma) + i(\alpha_{\overline{\sigma}}) & & i(\alpha\alpha_{\overline{\sigma}}) & = & i(\alpha) + i(\alpha_{\overline{\sigma}}) \\
i(\alpha_{\overline{\sigma}} + \alpha) & = & i(\alpha_{\overline{\sigma}}) + i(\alpha) & & i(\alpha^\star) & = & i(\alpha).
\end{array}$$

From these equations we can obtain the cost generating function of the mergings, $I_a(z)$:

$$I_a(z) = \frac{(z+z^2)R_a(z)^2}{\sqrt{\Delta_k(z)}}. \tag{36}$$

Using again the same Proposition 3 from Broda *et al.*, we conclude that:

$$[z^n]I_a(z) \sim \frac{1+\rho_k}{64} \frac{\left(a(\rho_k) + b(\rho_k)^2 - 2b(\rho_k)\sqrt{a(\rho_k)}\right)}{\sqrt{\pi}\sqrt{2-2\rho_k}} \rho_k^{-(n+1)} n^{-\frac{1}{2}}. \tag{37}$$

The asymptotic estimate for the average number of mergings is given by:

$$\frac{[z^n]I_\sigma(z)}{[z^n]L_k(z)} \sim \frac{1-\rho_k}{4\rho_k^2}\lambda_k = \eta_k, \tag{38}$$

where $\lambda_k = \frac{(1+\rho_k)}{16(1-\rho_k)}\left(a(\rho_k) + b(\rho_k)^2 - 2b(\rho_k)\sqrt{a(\rho_k)}\right)$. It is not difficult to conclude that $\lim\limits_{k\to\infty} \lambda_k = 0$, therefore $\lim\limits_{k\to\infty} \eta_k = 0$. As it is evident from the last two columns of Table 1, for small values of k, the lower bound η_k does not capture all the mergings that occur in $\mathcal{A}_{pre}$. Although we must study other contributions for those mergings, it seems that for larger values of k, the average number of states of the $\mathcal{A}_{pre}$ automaton approaches the number of states of the $\mathcal{A}_{pos}$ automaton.

8.0.1 A more general case

Instead of consider the cases previous described to calculate the number of non-disjoint unions, we can consider the following ones. Whenever $\sigma \in \mathsf{Pre}(\beta) \cap \mathsf{Pre}(\gamma)$,

$$|\mathsf{Pre}(\beta+\gamma)| = |\mathsf{Pre}(\beta) \cup \mathsf{Pre}(\gamma)| \leq |\mathsf{Pre}(\beta)| + |\mathsf{Pre}(\gamma)| - 1;$$

and also whenever $\sigma \in \mathsf{Pre}(\gamma)$ and $\beta\sigma \in \mathsf{Pre}(\beta)$

$$|\mathsf{Pre}(\beta\gamma)| = |\beta\mathsf{Pre}(\gamma) \cup \mathsf{Pre}(\beta)| \leq |\mathsf{Pre}(\beta)| + |\mathsf{Pre}(\gamma)| - 1.$$

Note that the previous cases are a subset of these ones. Now we need to consider one more grammar, R_r, that generate the regular expressions α_r such that $\alpha_r\sigma \in \mathsf{Pre}(\alpha_r)$:

$$\alpha_r \quad := \quad \alpha \cdot \alpha_r \mid \alpha_\sigma^\star$$

The generating function for R_r, $R_r(z)$ is:

$$R_r(z) = \frac{zR_a(z)}{1-zR_k(z)}. \tag{39}$$

In this case the $i(\alpha)$ is given by:

$$\begin{array}{rcl} i(\varepsilon) & = & i(\sigma) = 0 \\ i(\alpha_\sigma + \alpha_\sigma) & = & i(\alpha_\sigma) + i(\alpha_\sigma) + 1 \\ i(\alpha_\sigma + \alpha_{\overline{\sigma}}) & = & i(\alpha_\sigma) + i(\alpha_{\overline{\sigma}}) \\ i(\alpha_{\overline{\sigma}} + \alpha) & = & i(\alpha_{\overline{\sigma}}) + i(\alpha) \end{array} \qquad \begin{array}{rcl} i(\alpha_r\alpha_\sigma) & = & i(\alpha_r) + i(\alpha_\sigma) + 1 \\ i(\alpha_{\overline{r}}\alpha_\sigma) & = & i(\alpha_{\overline{r}}) + i(\alpha_\sigma) \\ i(\alpha\alpha_{\overline{\sigma}}) & = & i(\alpha) + i(\alpha_{\overline{\sigma}}) \\ i(\alpha^\star) & = & i(\alpha) \end{array}$$

Thus, the cumulative generating function of the mergings is:

$$Ir_\sigma(z) = \frac{zR_a(z)\left(R_a(z) + R_r(z)\right)}{1-4zR_k(z)-z}. \tag{40}$$

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