Statistical Stability for Chaotic Dynamical Systems

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Resumo

Neste trabalho estudamos a estabilidade estatística, no sentido da variação contínua de medidas físicas, de certos sistemas dinâmicos caóticos. A nossa atenção centra-se em dois tipos de sistemas.

O primeiro tipo diz respeito à família quadrática definida no intervalo $I = [-1, 1]$, dada pela expressão $f_a(x) = 1 - ax^2$, para os parâmetros Benedicks-Carleson. Neste conjunto de medida de Lebesgue positiva de parâmetros situado perto de $a = 2$, f_a exibe crescimento exponencial da derivada ao longo da órbita crítica e possui uma única medida invariante absolutamente contínua relativamente a Lebesgue, comummente designada por medida de Sinai-Ruelle-Bowen (SRB). Mostramos que o volume dos pontos de I que, até um dado instante, ainda não apresentam crescimento exponencial da derivada ao longo da sua órbita decai exponencialmente com a passagem do tempo. Provamos ainda que o mesmo é válido para o volume dos pontos que resistem a apresentar recorrência lenta ao conjunto crítico até um dado instante. Como consequência obtemos a variação contínua das medidas SRB na norma L^1, e das suas entropias métricas, com o parâmetro no conjunto de Benedicks-Carleson.

O segundo tipo de sistemas que consideramos é a família de aplicações de Hénon no plano dada por $f_{a,b}(x, y) = (1 - ax^2 + y, bx)$. Para um conjunto de medida de Lebesgue positiva de parâmetros – parâmetros de Benedicks-Carleson – estas aplicações de Hénon apresentam um atractor não-hiperbólico que suporta uma única medida SRB que se desintegra em medidas condicionais absolutamente contínuas com respeito à medida de Lebesgue 1-dimensional em cada folha instável. Provamos que a medida SRB varia continuamente na topologia fraca com o parâmetro dentro do conjunto de Benedicks-Carleson.

Abstract

In this work we address the problem of proving statistical stability, in the sense of continuous variation of physical measures, for certain chaotic dynamical systems. We consider two types of systems.

The first one is the quadratic family given by $f_a(x) = 1 - ax^2$ on $I = [-1, 1]$, for the Benedicks-Carleson parameters. On this positive Lebesgue measure set of parameters, close to $a = 2$, f_a presents exponential growth of the derivative along the orbit of the critical point and has an absolutely continuous Sinai-Ruelle-Bowen (SRB) invariant measure. We show that the volume of the set of points of I that at a given time fail to present exponential growth of the derivative decays exponentially as time passes. We also prove that the same holds for the volume of the set of points of I that are not slowly recurrent to the critical set. As a consequence we obtain continuous variation of the SRB measures, in the L^1 norm, and associated metric entropies with the parameter on the referred set. For this purpose we elaborate on the Benedicks-Carleson techniques in the phase space setting.

The second type is the family of Hénon maps in the plane given by $f_{a,b}(x, y) = (1 - ax^2 + y, bx)$. For a positive Lebesgue measure set of parameters – the Benedicks-Carleson parameter set – these Hénon maps exhibit a non-hyperbolic strange attractor supporting a unique SRB measure that disintegrates into absolutely continuous conditional measures with respect to the 1-dimensional Lebesgue measure on each unstable leaf. We prove that the SRB measures vary continuously in the weak* topology within the set of Benedicks-Carleson parameters.

Résumé

Dans ce travail nous étudions la stabilité statistique, en termes de la continuité de la variation des mesures physiques, pour certains systèmes dynamiques chaotiques. Nous considérons deux types de systèmes.

Le premier c'est la famille des applications $f_a(x) = 1 - ax^2$ avec $x \in [-1, 1]$ pour les paramètres de Benedicks-Carleson. Pour chaqun de ces paramètres il y a une unique mesure de Sinai-Ruelle-Bowen (SRB) qui est absolument continue par rapport à la mesure de Lebesgue. Nous montrons que ces mesures SRB et ses entropies métriques varient continûment avec le paramètre dans l'ensemble de Benedicks-Carleson.

Le deuxième c'est la famille des applications de Hénon sur $\mathbb{R}^2$, c'est-à-dire $f_{a,b}(x, y) = (1 - ax^2 + y, bx)$. Pour un ensemble de paramètres avec mesure de Lebesgue positive - les paramètres de Benedicks-Carleson – ces applications ont un attracteur étrange qui supporte une unique mesure SRB. Nous montrons que ces mesures SRB varient continûment avec le paramètre dans l'ensemble de Benedicks-Carleson.

Contents

Introduction

In general terms, the study of dynamics focuses on the long term behavior of evolving systems. As a theory it has grown to touch several branches of Mathematics and other sciences which interact, motivate and provide examples and applications. Among a wide list we mention the connections with Physics, Chemistry, Ecology, Economics, Computer Science, Communications, Meteorology. The main ingredients of a dynamical system are the *phase space*, the *time* and the *evolution law* governing the time progress of the system. The *phase space* is a set, say M, usually with some additional structure like topological, measurable or differentiable. The elements of M represent the possible states of the system which are commonly described by observable quantities, like position, velocity, acceleration, temperature, pressure, population density, concentration and many others. The *time* may be discrete or continuous. For discrete systems, *time* is parametrized by the group $\mathbb{Z}$ or the semigroup $\mathbb{N}$, depending on whether it is reversible or not; while for continuous systems the *time* is usually parametrized by $\mathbb{R}$ or $\mathbb{R}_0^+$. The *evolution law* is the rule that represents the action of *time* in the *phase space*. In the case of discrete systems the action is given by a map $f : M \to M$ and the time progress corresponds to successive iterations by f of each initial state $x \in M$. For continuous time, the action is given by a flow that usually appears as a solution of a differential equation determining the infinitesimal evolution of the system at each state $x \in M$. The *evolution law* in most applications preserves the additional structure of the *phase space*. There are natural constructions to pass from a flow to a map or vice versa; most of the main dynamical phenomena is already present in the discrete case. In the present exposition we restrict ourselves to discrete time dynamical systems.

At the end of the 19th century, Poincaré addressed the problem of evolution and stability of the solar system, which arose many surprising questions and his techniques gave birth to the Modern Theory of Dynamical Systems as a qualitative study of the asymptotic evolution of systems. By analogy to Celestial Mechanics, the time evolution of a particular state $x \in M$ is called the *orbit* of x. Hence, the main goal of this Theory is to study the *typical behavior* of orbits for a given dynamical system. The next natural aim is to understand how this behavior changes when we perturb the system and the extent to which it is robust. In the present work we are specially concerned with this problem of *stability* of the systems.

The first fundamental concept of robustness, *structural stability*, was formulated in the late 1930's by Andronov and Pontryagin. It requires the persistence of the orbit topological structure under small perturbations, expressed in terms of a homeomorphism sending orbits of the initial system onto orbits of the perturbed one. This concept is tied with the notion of *uniform hyperbolicity* introduced by Smale in the mid 1960's. In fact, structural stability had been proved for Anosov and Morse-Smale systems. A complete connection was conjectured by Palis and Smale in 1970: a diffeomorphism is structurally stable if and only if it is uniformly hyperbolic and satisfies the so-called transversality condition. During the 1970's the "if" part of the conjecture was solved due to the contributions of Robbin, de Melo and Robinson. It was only at the mid 1980's that Mañé settled the C^1-stability conjecture (perturbations are taken to be small in the C^1 topology). The flow case was solved by Aoki and Hayashi, independently, in the early 1990's (also in the C^1-category).

In spite of these astonishing successes, structural stability proved to be somewhat restrictive. Several important models, such as Lorenz flows, Hénon maps and other non-uniformly hyperbolic systems fail to present structural stability, although some key aspects of a statistical nature persist after small perturbations. After the 1960's, the contributions of Kolmogorov, Sinai, Ruelle, Bowen, Oseledets, Pesin, Katok, Mañé and many others turned the attention of the study of dynamical systems from a topological perspective to a more statistical essence and Ergodic Theory experienced an unprecedent development. A notion of stability with a statistical flavor was introduced by Kolmogorov and Sinai in the 1970's. It is known as *stochastic stability* and in broad terms asserts that time-averages of continuous functions are only slightly affected when iteration by f is perturbed by a small random noise. In trying to capture this statistical persistence of phenomena, Alves and Viana [**AV02**] proposed another notion, called *statistical stability*, which expresses the persistence of statistical properties in terms of continuous variation of physical measures as a function of the evolution law governing the systems. More precisely, consider a manifold M and a smooth map $f : M \to M$. A *physical measure* is a Borel probability measure μ on M for which there is a positive Lebesgue measure set of points $x \in M$ called typical such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(f_a^j(x)\right) = \int \varphi \, d\mu, \tag{1}$$

for any continuous function $\varphi : M \to \mathbb{R}$. The set of typical points forms the *basin* of μ. Physical measures provide a fairly description of the statistical behavior of orbits in the sense that, for the large set of points that constitute their basins, computing the time average of any specific function along their orbits is accomplished simply by integrating that function with respect to μ (spatial average). Now, suppose that f admits a forward invariant region $U \subset M$, meaning that $f(U) \subset U$, and there exists a (unique) physical measure μ_f supported on U such that (1) holds for Lebesgue almost every point $x \in U$. Following Alves and Viana [**AV02**], we say that f is *statistically stable* (restricted to U) if similar facts are true for any C^k nearby map g, for some $k \geq 1$, and the map $g \mapsto \mu_g$, associating to each g its physical measure μ_g, is continuous at f in the weak* topology (two measures are close to each other if they assign close-by integrals to each continuous function). Observe that with this definition we are guaranteeing that if a system is *statistically stable* then time averages of continuous functions are only slightly affected when the system is perturbed.

Physical measures are intimately connected with *Sinai-Ruelle-Bowen measures* (SRB for short). An f-invariant Borel probability measure μ is said to be SRB if it has a positive Lyapunov exponent and the conditional measures of μ on unstable leaves are absolutely continuous with respect to the Riemannian measure induced on these leaves (see Chapter 2, Section 3.8 for a more precise definition). The existence of SRB measures for general dynamical systems is usually a difficult problem. Starting with the work of Sinai, Ruelle and Bowen, in the case of Axiom A attractors, we know that SRB measures exist and qualify as physical measures. Moreover, Axiom A diffeomorphisms are statistically stable.

In the case of uniformly expanding or non-uniformly expanding systems, proving the existence of SRB measures can be reduced, by Birkhoff's Ergodic Theorem, to the problem of finding ergodic, absolutely continuous invariant measures. Clearly, these SRB measures are also physical measures. In 1969, Krzyzewski and Szlenk [**KS69**] have proved the existence of such measures for uniformly expanding maps which are also well known to be statistically stable. The existence of SRB measures for a considerably large set of one-dimensional quadratic maps exhibiting non-uniformly expanding behavior has been established in the pioneer work of Jakobson [**Ja81**]; see also [**CE80a, BC85**] for different approaches of the same result. Viana introduced in [**Vi97**] an open class of maps in higher dimensions with non-uniformly expanding behavior. Alves in [**Al00**] proved the existence of SRB measures for the Viana maps. Motivated by these results, Alves, Bonatti and Viana [**ABV00**] obtained general conclusions on the existence of SRB measures for non-uniformly expanding dynamical systems.

With the Viana maps in mind, Alves and Viana [**AV02**] built an abstract model to derive strong statistical stability for these transformations. By strong statistical stability we mean convergence of the SRB measures in the L^1-norm (recall that these SRB measures are absolutely continuous with respect to Lebesgue measure). Soon after, Alves in [**Al03**] showed that, under some general conditions, non-uniformly expanding maps with slow recurrence to the critical region fit the abstract model in [**AV02**]. Hence, they are statistically stable in the strong sense. The conditions at stake have to do with the volume decay of the *tail set*, which is the set of points that resist satisfying either the non-uniformly expanding requirement or the slow recurrence, up to a given time.

Recently, Freitas [**Fr05**] has proved that the Benedicks-Carleson quadratic maps are non-uniformly expanding, slowly recurrent to the critical set and the volume of their tail sets decays sufficiently fast so that the results in [**Al03**] apply. Thus, these maps are statistically stable in the strong sense. Similar results had already been obtained by Rychlik and Sorets [**RS97**] for Misiurewicz quadratic maps; and by Tsujii [**Ts96**] for convergence in the weak* topology. Chapter 1 of this dissertation is essentially the content of [**Fr05**] with the following improvement: while in [**Fr05**] it was obtained that the volume of the tail set of Benedicks-Carleson quadratic maps decays sub-exponentially, in this work we conclude the exponential decay.

In the remarkable paper [**BC91**], Benedicks and Carleson showed that for a positive Lebesgue measure set of parameters the Hénon map exhibits a non-hyperbolic attractor. Afterwards, Benedicks and Young in [**BY93**] proved that each of these non-hyperbolic attractors supports a unique SRB measure which is also a physical measure. Thus, a natural question is: are the Hénon maps of the Benedicks-Carleson type statistically stable? The main result of Chapter 2 is the positive answer to this question.

Let us add that stochastic stability may imply statistical stability if we are allowed to have a deterministic noise. Although we have stochastic stability for Benedicks-Carleson quadratic maps (see for example [**BY92, BV96**]), it does not follow from this that these maps are statistically stable. The same is true for the Hénon maps of Benedicks-Carleson type. In fact, Benedicks and Viana [**BV06**] showed that these maps are stochastically stable but statistical stability is not a direct consequence of it.

Before ending, let us mention that one can study the stability of the statistical behavior of a system in a broader perspective, namely, also investigate the variation of entropy. Entropy is related to the unpredictability of the system. Topological entropy measures the complexity of a dynamical system in terms of the exponential growth rate of the number of orbits distinguishable over long time intervals, within a fixed small precision. Metric entropy with respect to a physical measure quantifies the average level of uncertainty every time we iterate, in terms of exponential growth rate of the number of statistically significant paths an orbit can follow. In [**Fr05**] it was also shown that the metric entropy with respect to the SRB measure varies continuously with the parameter within the Benedicks-Carleson quadratic maps. This was achieved through the use of the results in [**AOT**] for non-uniformly expanding maps with slow recurrence to the critical set. The same problem within Hénon maps is, up to our knowledge, still open.

CHAPTER 1

Statistical stability for Benedicks-Carleson quadratic maps

1. Motivation and statement of results

Our object of study is the logistic family. As regards the asymptotic behavior of orbits of points $x \in I = [-1, 1]$ we know that:

(1) The set of parameters H for which f_a has an attracting periodic orbit, is open and dense in $[0, 2]$.
(2) There is a positive Lebesgue measure set of parameters, close to the parameter value 2, for which f_a has no attracting periodic orbit and exhibits a chaotic behavior, in the sense of existence of an ergodic, f_a-invariant measure absolutely continuous with respect to the Lebesgue measure on $I = [-1, 1]$.
(3) There is also a well studied set of parameters where f_a is infinitely renormalizable.

The first result is a conjecture with long history, which was finally proved by Graczyk, Swiatek [**GS97**] and Lyubich [**Ly97, Ly00**]. The second one was studied in Jakobson's pioneer work [**Ja81**], in the work of P. Collet and J.P. Eckmann [**CE80a, CE80b**] and latter by Benedicks and Carleson in their celebrated papers [**BC85, BC91**], just to mention a few. For the third type of parameters we refer to [**MS93**] where an extensive treatment of the subject can be found.

The crucial role played by the orbit of the unique critical point $\xi_0 = 0$ on the determination of the dynamical behavior of f_a is remarkable. It is well known that if f_a has an attracting periodic orbit then $\xi_0 = 0$ belongs to its *basin of attraction*, which is the set of points $x \in I$ whose ω-limit set is the attracting periodic orbit. Also, the *basin of attraction* of the periodic orbit is an open and dense full Lebesgue measure subset of I. See [**MS93**], for instance. Benedicks and Carleson [**BC85, BC91**] show the existence of a positive Lebesgue measure set of parameters $\mathcal{B}C_1$ for which there is exponential growth of the derivative of the orbit of the critical point ξ_0. This implies the non-existence of attracting periodic orbits and leads to a new proof of Jakobson's theorem.

In this work, we study the regularity in the variation of invariant measures and their metric entropy for small perturbations in the parameters. We are interested in investigating *statistical stability* of the system, that is, the persistence of its statistical properties for small modifications of the parameters. Alves and Viana [**AV02**] formalized the concept of *statistical stability* in terms of continuous variation of *physical measures* as a function of the governing law of the dynamical system. It is not difficult to conclude that if $a \in H$, and $\{p, f_a(p), \ldots, f_a^{k-1}(p)\}$ is the attracting periodic orbit then

$$\eta_a = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{f_a^i(p)},$$

where δ_x is the Dirac probability measure at $x \in I$, is a physical measure whose *basin* coincides with the *basin of attraction* of the periodic orbit. Moreover, the quadratic family is *statistically stable* for $a \in H$, i.e. the physical measure η_a varies continuously with $a \in H$, in a weak sense (convergence of measures in the weak* topology).

The infinitely renormalizable quadratic maps also admit a physical measure with the whole interval I for *basin*. In fact, any absolutely continuous f_a-invariant measure is SRB and describes (statistically speaking) the asymptotic behavior of almost all points, which is to say that its *basin* is I(see pp 348-352 [**MS93**]).

Benedicks and Young [**BY92**] proved that for each Benedicks-Carleson parameter $a \in \mathcal{BC}_1$, there is a unique, ergodic, f_a-invariant, absolutely continuous measure (with respect to Lebesgue measure on I) μ_a. These SRB measures qualify as physical measures by Birkhoff's ergodic theorem and their *basin* is the whole interval I. Hence, it is a natural question to wonder if the Benedicks-Carleson quadratic maps are *statistically stable*.

In the subsequent sections it will be shown that the answer is in the affirmative. In fact, we will prove that the quadratic family is *statistically stable*, in strong sense, for $a \in \mathcal{BC}_1$. To be more precise, we will show that the densities of the SRB measures vary continuously, in L^1-norm, with the parameter $a \in \mathcal{BC}_1$. This result relates to those of Tsujii, Rychlik and Sorets. In [**Ts96**], Tsujii showed the continuity of SRB measures, in weak topology, on a positive Lebesgue measure set of parameters. Rychlik and Sorets [**RS97**], on the other hand, obtained the continuous variation of the SRB measures, in terms of convergence in L^1- norm, for Misiurewicz parameters, which form a subset of zero Lebesgue measure. We also would like to refer the work of Thunberg [**Th01**] who proved that on any full Lebesgue measure set of parameters there is no continuous variation of the physical measures with the parameter.

With a view to studying the stability of the statistical behavior of the system in a broader perspective, we are also specially interested in the variation of entropy. It is known that topological entropy varies continuously with $a \in [0, 2]$ (see [**MS93**]). This is not the case with the metric entropy of physical measures. We note that the metric entropy associated to η_a, with $a \in H$, is zero. H is an open and dense set which means we can find a sequence of parameters $(a_n)_{n\in\mathbb{N}}$, such that $a_n \in H$ and thus with zero metric entropy with respect to the physical measure η_{a_n}, accumulating on $a \in \mathcal{BC}_1$, whose metric entropy associated with the SRB measure, μ_a, is strictly positive.

However, we will show that the metric entropy of the SRB measure μ_a varies continuously on the Benedicks-Carleson parameters, $a \in \mathcal{BC}_1$. We would like to stress that the continuous variation of the metric entropy is not a direct consequence of the continuous variation of the SRB measures and the entropy formula, because $\log(f_a')$ is not continuous on the interval I.

1.1. Brief description of the strategy. The work developed by Alves and Viana on [**AV02**] led Alves [**Al03**] to obtain sufficient conditions for the strong statistical stability of certain classes of *non-uniformly expanding* maps with *slow recurrence to the critical set*. By *non-uniformly expanding*, we mean that for Lebesgue almost all points we have exponential growth of the derivative along their orbits. *Slow recurrence to the critical set* means, roughly speaking, that almost none of the points can have its orbit making frequent visits to very small vicinities of the critical set.

Alves, Oliveira and Tahzibi [**AOT**] determined abstract conditions for continuous variation of metric entropy with respect to SRB measures. They also obtained conditions for *non-uniformly expanding* maps with *slow recurrence to the critical set* to satisfy their initial abstract conditions.

In both cases, the conditions obtained for continuous variation of SRB measures and their metric entropy are tied with the volume decay of the *tail set*, which is the set of points that resist to satisfy either the *non-uniformly expanding* or the *slow recurrence to the critical set* conditions, up to a given time.

Consequently, our main objective is to show that on the Benedicks-Carleson set of parameter values, where we have exponential growth of the derivative along the orbit of the critical point $\xi_0 = 0$, the maps f_a are *non-uniformly expanding*, have *slow recurrence to the critical set*, and the volume of the *tail set* decays sufficiently fast. In fact, we will show that the volume of the tail set decays exponentially fast. Finally we apply the results on [**Al03, AOT**] to obtain the continuous variation of the SRB measures and their metric entropy inside the set of Benedicks-Carleson parameters $\mathcal{BC}_1$.

We also refer to the recent work [**ACP06**] from which we conclude, by the *non-uniformly expanding* character of these maps, that for almost every $x \in I$ and any y on a pre-orbit of x, one has an exponential growth of the derivative of y.

1.2. Statement of results. In what follows, we will only consider parameter values $a \in \mathcal{BC}_1$ that are Benedicks-Carleson parameters, in the sense that for those $a \in \mathcal{BC}_1$ we have exponential growth of the derivative of $f_a(\xi_0)$,

$$\left|\left(f_a^j\right)'\left(f_a(\xi_0)\right)\right| \geq \mathrm{e}^{cj}, \ \forall j \in \mathbb{N}, \tag{EG}$$

where $c \in \left[\frac{2}{3}, \log 2\right)$ is fixed, and the basic assumption is valid, namely

$$\left|f_a^j\left(\xi_0\right)\right| \geq \mathrm{e}^{-\alpha j}, \ \forall j \in \mathbb{N}, \tag{BA}$$

where α is a small constant. Note that $\mathcal{BC}_1$ is a set of parameter values of positive Lebesgue measure, very close to $a = 2$. (See Theorem 1 of [**BC91**] or [**Mo92**] for a detailed version of its proof).

We say that f_a is *non-uniformly expanding* if there is a $d > 0$ such that for Lebesgue almost every point in $I = [-1, 1]$

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\left|f_a'\left(f_a^i(x)\right)\right| > d, \tag{1.1}$$

while having *slow recurrence to the critical set* means that for every $\epsilon > 0$, there exists $\gamma > 0$ such that for Lebesgue almost every $x \in I$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\gamma\left(f_a^j(x), 0\right) < \epsilon, \tag{1.2}$$

where

$$\operatorname{dist}_\gamma(x,y) = \begin{cases} |x-y| & \text{if} \ \ |x-y| \leq \gamma \\ 0 & \text{if} \ \ |x-y| > \gamma \end{cases} . \tag{1.3}$$

Observe that by (EG) it is obvious that ξ_0 satisfies (1.1) for all $a \in \mathcal{B}C_1$. However, with reference to condition (1.2) the matter is far more complicated and one has that ξ_0 satisfies it for Lebesgue almost all parameters $a \in \mathcal{B}C_1$. We provide a heuristic argument for the validity of the last statement on remark 8.2.

It is well known that the validity of (1.1) Lebesgue almost everywhere (a.e.) derives from the existence of an ergodic absolutely continuous invariant measure. Nevertheless we are also interested in knowing how fast does the volume of the points that resist to satisfy (1.1) up to n, decays to 0 as n goes to ∞. With this in mind, we define the *expansion time function*, first introduced on [**ALP05**]

$$\mathcal{E}^a(x) = \min\left\{N \geq 1 : \frac{1}{n}\sum_{i=0}^{n-1} \log\left|f_a'\left(f_a^i(x)\right)\right| > d, \ \forall n \geq N\right\}, \tag{1.4}$$

which is defined and finite almost everywhere on I if (1.1) holds a.e.

Similarly, we define the *recurrence time function*, also introduced on [**ALP05**]

$$\mathcal{R}^a(x) = \min\left\{N \geq 1 : \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\gamma\left(f_a^j(x), 0\right) < \epsilon, \ \forall n \geq N\right\}, \tag{1.5}$$

which is defined and finite almost everywhere in I, as long as (1.2) holds a.e.

We are now able to define the *tail set*, at time $n \in \mathbb{N}$,

$$\Gamma_n^a = \left\{x \in I : \ \mathcal{E}^a(x) > n \text{ or } \mathcal{R}^a(x) > n\right\}, \tag{1.6}$$

which can be seen as the set of points that at time n have not reached a satisfactory exponential growth of the derivative or could not be sufficiently kept away from $\xi_0 = 0$.

First we study the volume contribution to the *tail set*, Γ_n^a, of the points where f_a fails to present *non-uniformly expanding* behavior. We claim that in fact, (1.1) a.e. holds to be true and the volume of the set of points whose derivative has not achieved a satisfactory exponential growth at time n, decays exponentially as n goes to ∞. In what follows λ denotes Lebesgue measure on $\mathbb{R}$.

Theorem A. *Assume that $a \in \mathcal{B}C_1$. Then f_a is non-uniformly expanding, which is to say that* (1.1) *holds for Lebesgue almost all points $x \in I$. Moreover, there are positive real numbers C_1 and τ_1 such that for all $n \in \mathbb{N}$:*

$$\lambda\left\{x \in I : \ \mathcal{E}^a(x) > n\right\} \leq C_1 e^{-\tau_1 n}.$$

Second, we study the volume contribution to Γ_n^a, of the points that fail to be *slowly recurrent* to ξ_0. We claim that (1.2) a.e. also holds true and the volume of the set of points that at time n, have been too close to the critical point, in mean, decays exponentially with n.

Theorem B. *Assume that $a \in \mathcal{B}C_1$. Then f_a has slow recurrence to the critical set, or in other words,* (1.2) *holds for Lebesgue almost all points $x \in I$. Moreover, there are positive real numbers C_2 and τ_2 such that for all $n \in \mathbb{N}$:*

$$\lambda\left\{x \in I : \ \mathcal{R}^a(x) > n\right\} \leq C_2 e^{-\tau_2 n}.$$

REMARK 1.1. The constants d in (1.1), ϵ, γ in (1.2), c, α from (EG) and (BA) can be chosen uniformly on $\mathcal{B}C_1$. Moreover, the constants C_1, τ_1 given by theorem A and the constants C_2, τ_2 given by theorem B depend on the previous ones but are independent of the parameter $a \in \mathcal{B}C_1$. Thus, we may say that $\{f_a\}_{a\in\mathcal{B}C_1}$ is a *uniform family* in the sense considered in [**Al03**]. For a further discussion on this subject see section 9.

REMARK 1.2. Both theorems easily imply that the volume of the *tail set* decays to 0 at least exponentially as n goes to ∞, i.e. for all $n \in \mathbb{N}$, $\lambda\left(\Gamma_n^a\right) \leq \text{const}\, \mathrm{e}^{-\tau n}$, for some $\tau > 0$ and const > 0.

The exponential volume decay of the *tail set* allows us to apply theorem A from [**Al03**] to obtain, in a strong sense, continuous variation of the ergodic invariant measures under small perturbations on the set of parameters. By strong sense we mean convergence of the densities of the ergodic invariant measures in the L^1 norm.

COROLLARY C. *Let μ_a be the SRB measure invariant for f_a. Then $\mathcal{B}C_1 \ni a \mapsto \frac{d\mu_a}{d\lambda}$ is continuous.*

Theorems A and B also make it possible to apply corollary C from [**AOT**] to get the continuous variation of metric entropy with the parameter.

COROLLARY D. *The entropy of the SRB measure invariant of f_a varies continuously with $a \in \mathcal{B}C_1$.*

Theorem A alone, also allows us to apply corollary 1.2 from [**ACP06**] to obtain backward contraction on every pre-orbit of Lebesgue almost every point.

COROLLARY E. *For Lebesgue almost every $x \in I$, there exists $C_x > 0$ and $b > 0$ such that $\left|(f_a^n)'(y)\right| > C_x e^{bn}$, for every $y \in f^{-n}(x)$ and for all $n \in \mathbb{N}$.*

2. Benedicks-Carleson techniques on phase space and notation

The first thing we need to establish is the meaning of "close to the critical set" and "distant from the critical set", for which we introduce the following neighborhoods of $\xi_0 = 0$:

$$U_m = \left(-\mathrm{e}^{-m}, \mathrm{e}^{-m}\right), \quad U_m^+ = U_{m-1}, \quad \text{for } m \in \mathbb{N}$$

and consider a large positive integer Δ that will indicate when closeness to the critical region is relevant. In fact, here and henceforth, we define $\delta = \mathrm{e}^{-\Delta}$.

We will use λ to refer to Lebesgue measure on $\mathbb{R}$, although, sometimes we will write $|\omega|$ as an abbreviation of $\lambda(\omega)$, for $\omega \subset \mathbb{R}$.

We follow [**BC85, BC91**] and proceed for each point $x \in I$ as was done in ξ_0, by splitting the orbit of x into *free periods*, *returns*, *bound periods*, which occur in this order. Before we explain these concepts we introduce the following notation for the orbit of the critical point, $\xi_n = f_a^n(0)$, for all $n \in \mathbb{N}_0$.

The *free periods* correspond to periods of time in which we are certain that the orbit never visits the vicinity $U_\Delta = (-\delta, \delta)$ of ξ_0. During these periods the orbit of x experiences an exponential growth of its derivative $|(f_a^n)'(x)|$, provided we are close enough to the parameter value 2. In fact, the following lemma gives a first approach to the set $\mathcal{B}C_1$ by stating that we may have an exponential growth rate $0 < c_0 < \log 2$ of the derivative of the orbit of x during free periods, for all $a \in [a_0, 2]$, where a_0 is chosen sufficiently close to 2.

LEMMA 2.1. *For every* $0 < c_0 < \log 2$ *and* Δ *sufficiently large there exists* $1 < a_0(c_0, \Delta) < 2$ *such that for every* $x \in I$ *and* $a \in [a_0, 2]$ *one has:*

(1) *If* $x, f_a(x), \dots, f_a^{k-1}(x) \notin U_{\Delta+1}$ *then* $\left|(f_a^k)'(x)\right| \geq e^{-(\Delta+1)} e^{c_0 k}$;
(2) *If* $x, f_a(x), \dots, f_a^{k-1}(x) \notin U_{\Delta+1}$ *and* $f_a^k(x) \in U_\Delta^+$, *then* $\left|(f_a^k)'(x)\right| \geq e^{c_0 k}$;
(3) *If* $x, f_a(x), \dots, f_a^{k-1}(x) \notin U_{\Delta+1}$ *and* $f_a^k(x) \in U_1$, *then* $\left|(f_a^k)'(x)\right| \geq \frac{4}{5} e^{c_0 k}$.

The proof relies on the fact that $f_2(x) = 1 - 2x^2$ is conjugate to $1 - 2|x|$. So it is only a question of choosing a sufficiently close to 2 for f_a to inherit the expansive behavior of f_2. See [**BC85**] or [**Al92, Mo92**] for detailed versions. In what follows, we assume that a_0 is sufficiently close to 2 so that $c_0 \geq \frac{2}{3}$.

Due to this exponential expansion outside the critical region one can prove that, for almost every point $x \in I$, it is impossible to keep its orbit away from U_Δ. We have a *return* of the orbit of a point to the neighborhood of $\xi_0 = 0$ if for some $j \in \mathbb{N}$, $f_a^j(x) \in U_\Delta = (-\delta, \delta)$. So a free period ends with what we call a *free return.* There are two types of free returns: the *essential* and *inessential* ones. In order to distinguish each type we need a sequence $\mathcal{P}_0, \mathcal{P}_1, \dots$ of partitions of I into intervals. We begin by partitioning U_Δ in the following way:

$$I_m = \left[\mathrm{e}^{-(m+1)}, \mathrm{e}^{-m}\right), \quad I_m^+ = \left[\mathrm{e}^{-(m+1)}, \mathrm{e}^{-(m-1)}\right) \quad , \text{ for } m \geq \Delta,$$
$$I_m = \left(-\mathrm{e}^{-m}, -\mathrm{e}^{-(m+1)}\right], I_m^+ = \left(-\mathrm{e}^{-(m-1)}, -\mathrm{e}^{-(m+1)}\right], \text{ for } m \leq -\Delta.$$

We say that the return had a *depth* of $\mu \in \mathbb{N}$ if $\mu = [-\log \operatorname{dist}_\delta(f_a^j(x), 0)]$, which is equivalent to saying that $f_a^j(x) \in I_{\pm\mu}$.

Next we subdivide each I_m, $m \geq \Delta$ into m^2 pieces of the same length in order to obtain bounded distortion on each member of the partition. For each $m \geq \Delta - 1$ and $k = 1, \dots, m^2$, we introduce the following notation

$$I_{m,k} = \left[\mathrm{e}^{-m} - k\frac{\lambda(I_m)}{m^2}, \mathrm{e}^{-m} - (k-1)\frac{\lambda(I_m)}{m^2}\right)$$
$$I_{-m,k} = -I_{m,k}\,, \qquad I_{m,k}^+ = I_{m_1,k_1} \cup I_{m,k} \cup I_{m_2,k_2},$$

where I_{m_1,k_1} and I_{m_2,k_2} are the adjacent intervals of $I_{m,k}$.

The sequence of partitions will be built in full detail on section 4 but we note the following:

For Lebesgue almost every $x \in I$, $\{x\} = \cap_{n\geq 0}\omega_n(x)$, where $\omega_n(x)$ is the element of $\mathcal{P}_n$ containing x. For such x there is a sequence $t_1, t_2, \dots$ corresponding to the instants when the orbit of x experiences an essential return, which means $I_{m,k} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset I_{m,k}^+$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$. In contrast we say that v is a free return time for x of inessential type if $f_a^v(\omega_v(x)) \subset I_{m,k}^+$, for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$, but $f_a^v(\omega_v(x))$ is not large enough to contain an interval $I_{m,k}$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$.

Now let us see some consequences of the returns. Since

$$|(f_a^n)'(x)| = \prod_{j=1}^{n} \left|2af_a^j(x)\right|,$$

the returns introduce some small factors in the derivative of the orbit of x. Also if we define for a point $x \in I$ and $n \in \mathbb{N}$,

$$T_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} - \log \operatorname{dist}_\gamma \left(f_a^j(x), 0\right), \tag{2.1}$$

where $\gamma = \mathrm{e}^{-\Theta}$ is the same of condition (1.2) and $\operatorname{dist}_\gamma$ is given by (1.3). We note that the only points of the orbit of x that contribute to the sum in (2.1) are those considered to be *deep* returns with depth above the threshold $\Theta \geq \Delta$ which is to be determined later. To compensate for the loss in the expansion of the derivative, we will show that a property very similar to (BA) holds for the orbit of $x \in I$ which can be seen as follows: we allow the orbit of x to get close to ξ_0 but we put some restraints on the velocity of possible accumulation on ξ_0. This will be the basis of the proof of theorem A. As for the proof of theorem B the strategy will be of different kind, it will be based on a statistical analysis of the depth of the returns, specially of the essential returns, which, fortunately, are very unlikely to reach large depths.

Finally, we are lead to the notion of *bound period* that follows a return during which the orbit of x is bounded to the orbit of ξ_0, or in other words: if at a return the orbit of x falls in a tight vicinity of the critical point we expect it to shadow the early iterates of ξ_0 at least for some period of time.

Let $\beta > 0$ be a small number such that $\beta > \alpha$; for example, take $10^{-2} > \beta = 2\alpha$.

Definition 2.2. Suppose $x \in U_m^+$. Let $p(x)$ be the largest p such that the following binding condition holds:

$$\left|f_a^j(x) - \xi_j(a)\right| \leq \mathrm{e}^{-\beta j}, \quad \text{for all } i = 1, \ldots, p-1 \tag{BC}$$

The time interval $1, \ldots, p(x) - 1$ is called the *bound period* for x.

If $p(m)$ is the largest p such that (BC) holds for all $x \in I_m^+$, which is the same to define

$$p(m) = \min_{x \in I_m^+} p(m, x),$$

then the time interval $1, \ldots, p(m) - 1$ is called the *bound period* for I_m^+.

One expects that the deeper is the return, the longer is its associated bound period. Next lemma confirms this, in particular.

Lemma 2.3. *If Δ is sufficiently large, then for each $|m| \geq \Delta$, $p(m)$ has the following properties:*

(1) *There is a constant $B_1 = B_1(\beta - \alpha)$ such that $\forall y \in f_a\left(U_{|m|-1}\right)$*

$$\frac{1}{B_1} \leq \left|\frac{(f_a^j)'(y)}{(f_a^j)'(\xi_1)}\right| \leq B_1, \quad \textit{for } j = 0, 1 \ldots, p(m) - 1;$$

(2) $\frac{2}{3}|m| < p(m) < 3|m|$*;*

(3) $\left|(f_a^p)'(x)\right| \geq e^{(1-4\beta)|m|}$*, for $x \in I_m^+$ and $p = p(m)$.*

The proof of this lemma depends heavily on the conditions (EG) and (BA). It can be found in [**Al92, Mo92**]. (See [**BC85**] for a similar version of the lemma but with sub-exponential estimates).

We call the attention to the fact that after the bound period not only have we recovered from the loss on the growth of the derivative caused by the return that originated the bound period, but we even have some exponential gain.

Also note that nothing prevents the orbit of a point x from entering in U_Δ during a bound period. These instants are called the *bound return* times.

Hence, we may speak of three types of returns: *essential*, *inessential* and *bound*. The essential returns are the ones that will play a prominent role in the reasoning. Let, as before, the sequence $t_1, t_2, \ldots$ denote the instants corresponding to essential returns of the orbit of x. When $n \in \mathbb{N}$ is given, we can define s_n to be the number of essential returns of the orbit of x, occurring up to n. We denote by $sd_n(x)$ the number of those essential returns occurring up to n that correspond to deep essential returns of the orbit of x with return depths above the threshold $\Theta \geq \Delta$. Let η_i denote the depth of the i-th essential return. Each t_i may be followed by bounded returns at times $u_{i,j}$, $j = 1, \ldots, u$ and these can be followed by inessential returns at times $v_{i,j}$, $j = 1, \ldots, v$. We will write $\eta_{i,j}$ to denote the depth of the inessential return correspondent to $v_{i,j}$. Note that each $v_{i,j}$ has a bound evolution where new bound returns may occur and, although we refer to these returns later, it is not necessary to introduce here a notation for them. Sometimes, for the sake of simplicity, it is convenient not to distinguish between essential and inessential returns, so we introduce the notation $z_1 < z_2 < \ldots$ for the instants of occurrence of free returns of the orbit of x.

We call attention to the fact that t_i, for example, depends on the point $x \in I$ considered- $t_i(x)$ corresponds to the i-th instant of essential return of the orbit of x. So, t_i, s_n, η_i, $u_{i,j}$, $v_{i,j}$, $\eta_{i,j}$ and z_i, should be regarded as functions of the point $x \in I$.

The sequence of partitions $\mathcal{P}_n$ of the set I will be such that all $x \in \omega \in \mathcal{P}_n$ have the same return times and return depths up to n. In fact, if, for example, $t_i(x) \leq n$ for some $x \in \omega \in \mathcal{P}_n$, then t_i and η_i are constant on ω. The same applies to the other above mentioned functions of x. The construction of the partition will also guarantee that f_a has bounded distortion on each component which will be shown to be of extreme importance.

3. Insight into the reasoning

We are now in condition to sketch the proofs of theorems A and B. The following two basic ideas are determinant for both the proofs.

(I) Not only the depth of the inessential and bound returns is smaller than the depth of the essential return preceding them (as we will show in lemmas 5.1 and 5.2) but also the total sum of the depths of bounded and inessential returns is less than a quantity proportional to the depth of the essential return preceding them, as we will show in propositions 5.4 and 5.5.

(II) The chances of occurring a very deep essential return are very small, in fact, they are less than $\mathrm{e}^{-\tau\rho}$, where $\tau > 0$ is constant and ρ is the depth in question. See proposition 6.2 and corollary 6.3.

The first one derives from (BA), (EG) and other properties of the critical orbit, while the main ingredient of the proof of the second is the bounded distortion on each element of the partition.

In order to prove theorem A, we define the following sets for a sufficiently large n.

$$E_1(n) = \left\{x \in I:\ \exists i \in \{1,\ldots,n\},\ |f_a^i(x)| < \mathrm{e}^{-\alpha n}\right\}. \tag{3.1}$$

Next, we will see that if $x \in I - E_1(n)$ then $\left|(f_a^n)'(x)\right| > \mathrm{e}^{dn}$, for some $d = d(\alpha,\beta) > 0$.

Let us fix a large n. Assume that z_i, $i = 1,\ldots,\gamma$ are the instants of return of the orbit of x, either essential or inessential. Let p_i denote the length of the bound period associated with the return z_i. We set $z_0 = 0$, whether $x \in U_\Delta$ or not; $p_0 = 0$ if $x \notin U_\Delta$ and as usual if not. We define $q_i = z_{i+1} - (z_i + p_i)$, for $i = 0,1,\ldots,\gamma-1$ and

$$q_\gamma = \begin{cases} 0 & \text{if } \ n < z_\gamma + p_\gamma \\ n - (z_\gamma + p_\gamma) & \text{if } \ n \geq z_\gamma + p_\gamma \end{cases}.$$

Finally, let

$$d = \min\left\{c, \frac{1-4\beta}{3}\right\} - 2\alpha = \frac{1-4\beta}{3} - 2\alpha. \tag{3.2}$$

If $n \geq z_\gamma + p_\gamma$ then

$$|(f_a^n)'(x)| = \prod_{i=0}^{\gamma} \left|(f_a^{q_i})'(f_a^{z_i+p_i}(x))\right| |(f_a^{p_i})'(f_a^{z_i}(x))|.$$

Using lemmas 2.1 and 2.3, we have

$$|(f_a^n)'(x)| \geq \mathrm{e}^{-\Delta+1}\mathrm{e}^{c_0\sum_{i=0}^{\gamma} q_i}\mathrm{e}^{\frac{1-4\beta}{3}\sum_{i=0}^{\gamma} p_i} \geq \mathrm{e}^{-\Delta+1}\mathrm{e}^{dn}\mathrm{e}^{2\alpha n} \geq \mathrm{e}^{dn}, \tag{3.3}$$

for n large enough.

If $n < z_\gamma + p_\gamma$ then

$$|(f_a^n)'(x)| = |f_a'(f_a^{z_\gamma}(x))| \left|(f_a^{n-(z_\gamma+1)})'(f_a^{z_\gamma+1}(x))\right| \prod_{i=0}^{\gamma-1} \left|(f_a^{q_i})'(f_a^{z_i+p_i}(x))\right| |(f_a^{p_i})'(f_a^{z_i}(x))|.$$

Now, by lemmas 2.1 and 2.3 together with the assumption that $x \in I - E_1(n)$, for n large enough we have

$$\begin{aligned} |(f_a^n)'(x)| &\geq |f_a'(f_a^{z_\gamma}(x))| \tfrac{1}{B_1} \left|(f_a^{n-(z_\gamma+1)})'(1)\right| \mathrm{e}^{c_0\sum_{i=0}^{\gamma-1} q_i}\mathrm{e}^{\frac{1-4\beta}{3}\sum_{i=0}^{\gamma-1} p_i} \\ &\geq \mathrm{e}^{-\alpha n}\tfrac{1}{B_1}\mathrm{e}^{c_0(n-(z_\gamma+1))}\mathrm{e}^{c_0\sum_{i=0}^{\gamma-1} q_i}\mathrm{e}^{\frac{1-4\beta}{3}\sum_{i=0}^{\gamma-1} p_i} \\ &\geq \mathrm{e}^{-\alpha n-\log B_1}\mathrm{e}^{(d+2\alpha)(n-1)} \\ &\geq \mathrm{e}^{-2\alpha n}\mathrm{e}^{dn}\mathrm{e}^{2\alpha n} \\ &\geq \mathrm{e}^{dn}. \end{aligned} \tag{3.4}$$

Using (I) and (II) we will show that

$$\lambda\left(E_1(n)\right) \leq \mathrm{e}^{-\tau_1 n}, \tag{3.5}$$

for a constant $\tau_1(\alpha,\beta) > 0$ and for all $n \geq N_1^*(\Delta,\tau_1)$. We consider $N_1(\Delta,\alpha,B_1,d,N_1^*)$ such that for all $n \geq N_1$ estimates (3.3), (3.4) and (3.5) hold. Hence for every $n \geq N_1$ we have that $|(f_a^n)'(x)| \geq \mathrm{e}^{dn}$, except for a set $E_1(n)$ of points $x \in I$ satisfying (3.5).

We take $E_1 = \bigcap_{k\geq N_1}\bigcup_{n\geq k} E_1(n)$. Since $\forall k \geq N_1$

$$\sum_{n\geq k} \lambda\left(E_1(n)\right) \leq \mathrm{const}\,\mathrm{e}^{-\tau_1 k},$$

we have by the Borel Cantelli lemma that $\lambda(E_1) = 0$. Thus on the full Lebesgue measure set $I - E_1$ we have that (1.1) holds. We note that $\{x \in I:\ \mathcal{E}^a(x) > k\} \subset \bigcup_{n\geq k} E_1(n)$, where $\mathcal{E}^a$ is defined in (1.4). So for $k \geq N_1$

$$\lambda\left(\{x \in I:\ \mathcal{E}^a(x) > k\}\right) \leq \mathrm{const}\,\mathrm{e}^{-\tau_1 k}.$$

At this point we just have to compute an adequate $C_1 = C_1(N_1) > 0$ such that

$$\lambda\left(\{x \in I:\ \mathcal{E}^a(x) > n\}\right) \leq C_1\mathrm{e}^{-\tau_1 n}, \tag{3.6}$$

for all $n \in N$.

For the proof of theorem B, we define for $n \in \mathbb{N}$ the sets:

$$E_2(n) = \{x \in I:\ T_n(x) > \epsilon\}. \tag{3.7}$$

Note that it is the depth of the deep returns that counts for the sum on $T_n(x)$. Taking note of the basic idea (I), in order to obtain a bound for T_n one only needs to take into consideration the deep essential returns.

Thus if we define

$$F_n(x) = \sum_{i=1}^{sd_n} \eta_i, \tag{3.8}$$

where sd_n is the number of essential returns with depths above Θ that occur up to n and η_i their respective depths, we have $T_n(x) \leq \frac{C_5}{n}F_n(x)$, from which we conclude that

$$\lambda(E_2(n)) \leq \lambda\left\{x : F_n(x) > \frac{\epsilon n}{C_5}\right\}.$$

Fact (II) and a large deviation argument allow us to obtain for $n \geq N_2(\Theta)$

$$\lambda\left\{x : F_n(x) > \frac{\epsilon n}{C_5}\right\} \leq \mathrm{const}\,\mathrm{e}^{-\tau_2 n}$$

where $\tau_2 = \tau_2(\epsilon, \Theta) > 0$ is constant, which implies for $k \geq N_2$

$$\sum_{n\geq k} \lambda\left(E_2(n)\right) \leq \mathrm{const}\,\mathrm{e}^{-\tau_2 k}.$$

Consequently, applying Borel Cantelli's lemma, we get $\lambda(E_2) = 0$, where $E_2 = \cap_{k\geq 1}\cup_{n\geq k} E_2(n)$ and finally conclude that (1.2) holds on the full Lebesgue measure set $I - E_2$. Observe that $\{x \in I:\ \mathcal{R}^a(x) > k\} \subset \bigcup_{n\geq k} E_2(n)$, and thus, for all $n \in \mathbb{N}$,

$$\lambda\left(\{x \in I:\ \mathcal{R}^a(x) > n\}\right) \leq C_2\mathrm{e}^{-\tau_2 n},$$

where $C_2 = C_2(N_2, \tau_2) > 0$ is constant. Recall that $\mathcal{R}^a$ is defined in (1.5).

At this point we would like to bring the reader's attention to the fact that most proofs and lemmas that follow are standard, in the sense that they are very resemblant to the ones on [**Al92, BC85, BC91, BY92, Mo92**] (just to cite a few), that deal with the same subject. Nevertheless, we could not find the right version for our needs, either because in some cases they refer to sub-exponential estimates when we want exponential estimates or because the partition is built on the space of parameters instead of the set I, as we wish. Hence, we decided for the sake of completeness to include them in this work.

4. Construction of the partition and bounded distortion

We are going to build inductively a sequence of partitions $\mathcal{P}_0, \mathcal{P}_1, \dots$ of I (modulo a zero Lebesgue measure set) into intervals. We will also define inductively the sets $R_n(\omega) = \{z_1, \dots, z_{\gamma(n)}\}$ which is the set of the return times of $\omega \in \mathcal{P}_n$ up to n and a set $Q_n(\omega) = \{(m_1, k_1), \dots, (m_{\gamma(n)}, k_{\gamma(n)})\}$, which records the indices of the intervals such that $f_a^{z_i}(\omega) \subset I^+_{m_i,k_i}$, $i = 1, \dots, z_{\gamma(n)}$.

Along with the construction of the partition, we will show, inductively, that for all $n \in \mathbb{N}_0$

$$\forall \omega \in \mathcal{P}_n \quad f_a^{n+1}|_\omega \text{ is a diffeomorphism,} \tag{4.1}$$

which is vital for the construction itself.

For $n = 0$ we define

$$\mathcal{P}_0 = \{[-1, -\delta], [\delta, 1]\} \cup \left\{I_{m,k} : |m| \geq \Delta,\ 1 \leq k \leq m^2\right\}.$$

It is obvious that $\mathcal{P}_0$ satisfies (4.1). We set $R_0([-1, -\delta]) = R_0([\delta, 1]) = \emptyset$ and $R_0(I_{m,k}) = \{0\}$.

Assume that $\mathcal{P}_{n-1}$ is defined, satisfies (4.1), and R_{n-1}, Q_{n-1} are also defined on each element of $\mathcal{P}_{n-1}$. We fix an interval $\omega \in \mathcal{P}_{n-1}$. We have three possible situations:

(1) If $R_{n-1}(\omega) \neq \emptyset$ and $n < z_{\gamma(n-1)} + p(m_{\gamma(n-1)})$ then we say that n is a *bound time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega)$, $Q_n(\omega) = Q_{n-1}(\omega)$.

(2) If $R_{n-1}(\omega) = \emptyset$ or $n \geq z_{\gamma(n-1)} + p(m_{\gamma(n-1)})$, and $f_a^n(\omega) \cap U_\Delta \subset I_{\Delta,1} \cup I_{-\Delta,1}$, then we say that n is a *free time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega)$, $Q_n(\omega) = Q_{n-1}(\omega)$.

(3) If the above two conditions do not hold we say that ω has a *free return situation* at time n. We have to consider two cases:
   (a) $f_a^n(\omega)$ does not cover completely an interval $I_{m,k}$, with $|m| \geq \Delta$ and $k = 1, \dots, m^2$. Because f_a^n is continuous and ω is an interval, $f_a^n(\omega)$ is also an interval and thus is contained in some $I^+_{m,k}$, for a certain $|m| \geq \Delta$ and $k = 1, \dots, m^2$, which is called the *host interval* of the return. We say that n is an *inessential return time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega) \cup \{n\}$, $Q_n(\omega) = Q_{n-1}(\omega) \cup \{(m, k)\}$.
   (b) $f_a^n(\omega)$ contains at least an interval $I_{m,k}$, with $|m| \geq \Delta$ and $k = 1, \dots, m^2$, in which case we say that ω has an *essential return situation* at time n. Then we consider the sets

$$\begin{aligned} \omega'_{m,k} &= f_a^{-n}(I_{m,k}) \cap \omega \quad \text{ for } |m| \geq \Delta \\ \omega'_+ &= f_a^{-n}([\delta, 1]) \cap \omega \\ \omega'_- &= f_a^{-n}([-1, -\delta]) \cap \omega \end{aligned}$$

   and if we denote by $\mathcal{A}$ the set of indices (m, k) such that $\omega'_{m,k} \neq \emptyset$ we have

$$\omega - \left\{f_a^{-n}(0)\right\} = \bigcup_{(m,k) \in \mathcal{A}} \omega'_{m,k}. \tag{4.2}$$

By the induction hypothesis $f_a^n|_\omega$ is a diffeomorphism and then each $\omega'_{m,k}$ is an interval. Moreover $f_a^n(\omega'_{m,k})$ covers the whole $I_{m,k}$ except eventually for the two end intervals. When $f_a^n(\omega'_{m,k})$ does not cover $I_{m,k}$ entirely, we join it with its adjacent interval in (4.2). We also proceed likewise when $f_a^n(\omega'_+)$ does not cover $I_{\Delta-1,(\Delta-1)^2}$ or $f_a^n(\omega'_-)$ does not contain the whole interval $I_{1-\Delta,(\Delta-1)^2}$. In this way we get a new decomposition of $\omega - \{f_a^{-n}(0)\}$ into intervals $\omega_{m,k}$ such that

$$I_{m,k} \subset f_a^n(\omega_{m,k}) \subset I_{m,k}^+,$$

when $|m| \geq \Delta$.
We define $\mathcal{P}_n$, by putting $\omega_{m,k} \in \mathcal{P}_n$ for all indices (m,k) such that $\omega_{m,k} \neq \emptyset$, with $|m| \geq \Delta$, which results in a refinement of $\mathcal{P}_{n-1}$ at ω. We set $R_n(\omega_{m,k}) = R_{n-1}(\omega) \cup \{n\}$ and n is called an *essential return time* for $\omega_{m,k}$. The interval $I_{m,k}^+$ is called the *host interval* of $\omega_{m,k}$ and $Q_n(\omega_{m,k}) = Q_n(\omega) \cup \{(m,k)\}$.
In the case when the set ω_+ is not empty we say that n is an *escape time* or *escape situation* for ω_+ and $R_n(\omega_+) = R_{n-1}(\omega)$, $Q_n(\omega_+) = Q_{n-1}(\omega)$. We proceed likewise for ω_-. We also refer to ω_+ or ω_- as *escaping components*. Note that the points in escaping components are in free period.

To end the construction we need to verify that (4.1) holds for $\mathcal{P}_n$. Since for any interval $J \subset I$

$$\left.\begin{array}{l} f_a^n|_J \text{ is a diffeomorphism} \\ 0 \notin f_a^n(J) \end{array}\right\} \Rightarrow f_a^{n+1}|_J \text{ is a diffeomorphism},$$

all we are left to prove is that $0 \notin f_a^n(\omega)$ for all $\omega \in \mathcal{P}_n$. So take $\omega \in \mathcal{P}_n$. If n is a free time for ω then we have nothing to prove. If n is a return for ω, either essential or inessential, we have by construction that $f_a^n(\omega) \subset I_{m,k}^+$ for some $|m| \geq \Delta$, $k = 1, \ldots, m^2$ and thus $0 \notin f_a^n(\omega)$. If n is a bound time for ω then by definition of bound period and (BA) we have for all $x \in \omega$

$$\begin{aligned}
|f_a^n(x)| &\geq \left|f_a^{n-z_{\gamma(n-1)}}(0)\right| - \left|f_a^n(x) - f_a^{n-z_{\gamma(n-1)}}(0)\right| \\
&\geq \mathrm{e}^{-\alpha(n-z_{\gamma(n-1)})} - \mathrm{e}^{-\beta(n-z_{\gamma(n-1)})} \\
&\geq \mathrm{e}^{-\alpha(n-z_{\gamma(n-1)})}\left(1 - \mathrm{e}^{-(\beta-\alpha)(n-z_{\gamma(n-1)})}\right) \\
&> 0 \quad \text{since } \beta - \alpha > 0.
\end{aligned}$$

Now we will obtain estimates of the length of $|f_a^n(\omega)|$.

LEMMA 4.1. *Suppose that z is a return time for $\omega \in \mathcal{P}_{n-1}$, with host interval $I_{m,k}^+$. Let $p = p(m)$ denote the length of its bound period. Then*

(1) *Assuming that $z^* \leq n-1$ is the next return time for ω (either essential or inessential) and defining $q = z^* - (z+p)$ we have, for a sufficiently large Δ, $\left|f_a^{z^*}(\omega)\right| \geq e^{c_0 q} e^{(1-4\beta)|m|} \left|f_a^z(\omega)\right| \geq 2\left|f_a^z(\omega)\right|$.*

(2) *If z is the last return time of ω up to $n-1$ and n is either a free time for ω or a return situation for ω, then putting $q = n - (z+p)$ we have, for a sufficiently large Δ,*
   (a) $|f_a^n(\omega)| \geq e^{c_0 q-(\Delta+1)} e^{(1-4\beta)|m|} \left|f_a^z(\omega)\right|$
   (b) $|f_a^n(\omega)| \geq e^{c_0 q-(\Delta+1)} e^{-5\beta|m|}$ *if z is an essential return.*

(3) *If z is the last return time of ω up to $n-1$, n is a return situation for ω and $f_a^n(\omega) \subset U_1$, then putting $q = n-(z+p)$ we have, for a sufficiently large Δ,*
  (a) $|f_a^n(\omega)| \geq e^{c_0 q} e^{(1-5\beta)|m|} |f_a^z(\omega)| \geq 2|f_a^z(\omega)|$;
  (b) $|f_a^n(\omega)| \geq e^{c_0 q} e^{-5\beta|m|}$ *if z is an essential return.*

Proof. By the mean value theorem, for some $\zeta \in \omega$,

$$|f_a^n(\omega)| \geq \left|\left(f_a^{n-z}\right)'\left(f_a^z(\zeta)\right)\right| |f_a^z(\omega)|\,.$$

Using lemma 2.1 part 2 and lemma 2.3 part 3 we get

$$\begin{aligned}|f_a^n(\omega)| &\geq \left|(f_a^q)'\left(f_a^{z+p}(\zeta)\right)\right| \left|(f_a^p)'\left(f_a^z(\zeta)\right)\right| |f_a^z(\omega)| \\ &\geq \tfrac{4}{5}\mathrm{e}^{c_0 q}\mathrm{e}^{(1-4\beta)|m|} |f_a^z(\omega)| \\ &\geq \tfrac{4}{5}\mathrm{e}^{\beta|m|}\mathrm{e}^{c_0 q}\mathrm{e}^{(1-5\beta)|m|} |f_a^z(\omega)| \\ &\geq 2\mathrm{e}^{c_0 q}\mathrm{e}^{(1-5\beta)|m|} |f_a^z(\omega)|\,,\end{aligned}$$

if Δ is sufficiently large in order to have $\frac{4}{5}\mathrm{e}^{\beta|m|} \geq 2$.

Note that part 3a is proved. To demonstrate part 1 it is only a matter of using lemma 2.1 part 2 instead of 3, while for proving part 2a one has to use lemma 2.1 part 1 instead.

To obtain 3b observe that because z is an essential return time $I_{m,k} \subset f_a^z(\omega)$ which implies $\lambda(f_a^z(\omega)) \geq \frac{\mathrm{e}^{-|m|}}{2m^2}$ and so

$$\begin{aligned}|f_a^n(\omega)| &\geq \tfrac{4}{5}\mathrm{e}^{\beta|m|}\mathrm{e}^{c_0 q}\mathrm{e}^{(1-5\beta)|m|} |f_a^z(\omega)| \\ &\geq \mathrm{e}^{c_0 q}\mathrm{e}^{(1-5\beta)|m|}\mathrm{e}^{-|m|}\frac{2\mathrm{e}^{\beta|m|}}{5m^2} \\ &\geq \mathrm{e}^{c_0 q}\mathrm{e}^{-5\beta|m|},\end{aligned}$$

if Δ is large enough.

The same argument can easily be applied to obtain part 2b. ☐

The next lemma asserts that an escaping component returns considerably large in the return situation immediately after the escaping time, which means in particular that it will be an essential return situation.

Lemma 4.2. *Suppose that $\omega \in \mathcal{P}_t$ is an escape component. Then in the next return situation t_1 for ω we have that*

$$\left|f_a^{t_1}(\omega)\right| \geq e^{-\beta\Delta}.$$

Proof. Since ω is an escaping component at time t it follows that

$$f_a^t(\omega) \supset I_{m,m^2}, \text{ with } |m| = \Delta - 1$$

and so there exists $x_* \in \omega$ such that $|f_a^t(x_*)| = \mathrm{e}^{-\Delta}$. Therefore $f_a^{t+1}(x^*) = 1 - a\mathrm{e}^{-2\Delta} \geq 1 - 2\mathrm{e}^{-2\Delta}$. Thus, if $t_1 = t+1$ the result would follow easily.

Now suppose that $t_1 \geq t+2$. Writing

$$f_a^{t+2}(x_*) = f_2\left(f_a^{t+1}(x_*)\right) + f_a\left(f_a^{t+1}(x_*)\right) - f_2\left(f_a^{t+1}(x_*)\right)$$

and taking into account that $f_2\left(f_a^{t+1}(x_*)\right) \leq f_2(1-2\mathrm{e}^{-2\Delta})$, $f_a(y) - f_2(y) \leq 2 - a$, $\forall y \in I$, it follows that

$$f_a^{t+2}(x_*) \leq -1 + 4.2\mathrm{e}^{-2\Delta},$$

if we choose a_0 sufficiently close to 2 such that

$$(2-a_0) \leq 8\mathrm{e}^{-4\Delta}. \tag{4.3}$$

By induction, using the same argument we can state that for $k \geq 2$, providing that $-1+4^{k-2}2\mathrm{e}^{-2\Delta} \leq 1$ (this is to ensure that we are inside the domain I), we have

$$f_a^{t+k}(x_*) \leq -1+4^{k-1}2\mathrm{e}^{-2\Delta}.$$

Therefore if $-1+4^{t_1-t-1}2\mathrm{e}^{-2\Delta} \leq -\frac{1}{2}$ then $f_a^{t_1}(x_*) \leq -\frac{1}{2}$ and so $|f_a^{t_1}(\omega)| \geq \mathrm{e}^{-\beta\Delta}$, providing Δ is large enough.

In order to complete the proof it remains to consider the case when $-1+4^{t_1-t-1}2\mathrm{e}^{-2\Delta} > -\frac{1}{2}$. Under this condition we have that

$$2^{t_1-t} \geq \mathrm{e}^{\Delta}. \tag{4.4}$$

First we note that we can assume $f_a^t(\omega) \subset U_1$ otherwise we have the conclusion immediately.

Now, we know that there is $x \in \omega$ such that

$$\begin{aligned}
\left|f_a^{t_1}(\omega)\right| &\geq \left|\left(f_a^{t_1-t}\right)'\left(f_a^t(x)\right)\right|\left|f_a^t(\omega)\right| \\
&\geq \frac{\left|\left(h^{-1}\right)'\left(f_a^t(x)\right)\right|}{\left|\left(h^{-1}\right)'\left(f_a^{t_1}(x)\right)\right|}\left|\left(g_a^{t_1-t}\right)'\left(h^{-1}(f_a^t(x))\right)\right|\frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2},
\end{aligned}$$

where $h:[-1,1]\to[-1,1]$ is the homeomorphism that conjugates $f_2(x)$ to the tent map $1-2|x|$ and $g_a = h^{-1}\circ f_a \circ h$.

Using lemma 3.1 from [**Mo92**] it follows that

$$\left|f_a^{t_1}(\omega)\right| \geq L\left[2-\tfrac{3\pi}{\delta^3}(2-a)\right]^{t_1-t}\frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2},$$

with

$$L=\sqrt{\frac{1-\left(f_a^{t_1}(x)\right)^2}{1-\left(f_a^t(x)\right)^2}}.$$

Since $f_a^{t_1}(\omega) \subset U_1$,

$$\begin{aligned}
\left|f_a^{t_1}(\omega)\right| &\geq \sqrt{1-\mathrm{e}^{-2}}\left[2-\tfrac{3\pi}{\delta^3}(2-a)\right]^{t_1-t}\frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2} \\
&\geq \tfrac{4}{5}\left[2-\tfrac{3\pi}{\delta^3}(2-a)\right]^{t_1-t}\frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2}
\end{aligned}$$

Now, we remark that our choice of a_0 can provide that

$$\left[2-\tfrac{3\pi}{\delta^3}(2-a)\right] \geq \mathrm{e}^{c_0} \tag{4.5}$$

and then since $|f_a^{t_1}(\omega)| \leq 2$, it follows that

$$\mathrm{e}^{c_0(t_1-t)} \leq \tfrac{5}{2}\mathrm{e}^{\Delta}(\Delta-1)^2,$$

which implies, for Δ large that $t_1-t \leq 2\Delta$.

Again restraining a_0 in such a way that

$$\left[2-\tfrac{3\pi}{\delta^3}(2-a)\right]^{2\Delta} \geq 2^{2\Delta-1} \tag{4.6}$$

we have

$$\left|f_a^{t_1}(\omega)\right| \geq \tfrac{2}{5}2^{t_1-t}\frac{\mathrm{e}^{-\Delta}}{(\Delta-1)^2}.$$

Taking into account 4.4 we have

$$\left|f_a^{t_1}(\omega)\right| \geq \frac{2}{5(\Delta-1)^2} \geq \mathrm{e}^{-\beta\Delta},$$

for Δ large enough. $\square$

LEMMA 4.3 (Bounded Distortion). *For some $n \in \mathbb{N}$ let $\omega \in \mathcal{P}_{n-1}$ be such that $f_a^n(\omega) \subset U_1$. Then there is a constant $C(\beta-\alpha)$ such that for every $x, y \in \omega$*

$$\frac{\left|(f_a^n)'(x)\right|}{\left|(f_a^n)'(y)\right|} \leq C$$

PROOF. Let $R_{n-1}(\omega) = \{z_1, \ldots, z_\gamma\}$ and $Q_{n-1}(\omega) = \{(m_1, k_1), \ldots, (m_\gamma, k_\gamma)\}$, be, respectively, the sets of return times and host indices of ω, defined on the construction of the partition. Note that for $i = 1, \ldots, \gamma$, $f_a^{z_i}(\omega) \subset I^+_{m_i,k_i}$. Let $\sigma_i = f_a^{z_i}(\omega)$, $p_i = p(m_i)$, $x_i = f_a^i(x)$ and $y_i = f_a^i(y)$.

Observe that

$$\left|\frac{(f_a^n)'(x)}{(f_a^n)'(y)}\right| = \prod_{j=0}^{n-1}\left|\frac{f_a'(x_j)}{f_a'(y_j)}\right| = \prod_{j=0}^{n-1}\left|\frac{x_j}{y_j}\right| \leq \prod_{j=0}^{n-1}\left(1+\left|\frac{x_j-y_j}{y_j}\right|\right)$$

Hence the result is proved if we manage to bound uniformly

$$S = \sum_{j=0}^{n-1}\left|\frac{x_j-y_j}{y_j}\right|.$$

For the moment assume that $n \leq z_\gamma + p_\gamma - 1$.

We first estimate the contribution of the free period between z_{q-1} and z_q for the sum S

$$F_q = \sum_{j=z_{q-1}+p_{k-1}}^{z_q-1}\left|\frac{x_j-y_j}{y_j}\right| \leq \sum_{j=z_{q-1}+p_{k-1}}^{z_q-1}\left|\frac{x_j-y_j}{\delta}\right|$$

For $j = z_{q-1}+p_{k-1}, \cdots, z_q - 1$ we have

$$\begin{aligned}
\lambda(\sigma_q) &\geq |f_a^{z_q-j}(x_j) - f_a^{z_q-j}(y_j)| \\
&= \left|(f_a^{z_q-j})'(\zeta)\right| \cdot |x_j-y_j|, \text{for some } \zeta \text{ between } x_j \text{ and } y_j \\
&\geq \mathrm{e}^{c_0(z_q-j)}|x_j-y_j|, \text{ by Lemma 2.1}
\end{aligned}$$

and so

$$\begin{aligned}
F_q &\leq \sum_{j=z_{q-1}+p_{k-1}}^{z_q-1} \mathrm{e}^{-c_0(z_q-j)} \cdot \frac{\lambda(\sigma_q)}{\delta} \\
&\leq \sum_{j=1}^{\infty} \mathrm{e}^{-cj} \cdot \frac{\lambda(I_{m_q})}{\delta} \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \\
&\leq a_1 \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \text{ for some constant } a_1 = a_1(c).
\end{aligned}$$

The contribution of the return z_q is

$$\left|\frac{x_{z_q} - y_{z_q}}{y_{z_q}}\right| \leq \frac{\lambda(\sigma_q)}{\mathrm{e}^{-|m_q|-2}} \leq a_2 \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \text{ where } a_2 \text{ is a constant.}$$

Finally, let us compute the contribution of bound periods

$$B_q = \sum_{j=1}^{p_q-1} \left|\frac{x_{z_q+j} - y_{z_q+j}}{y_{z_q+j}}\right|$$

We have that

$$\begin{aligned}
|x_{z_q+j} - y_{z_q+j}| &= |(f_a^j)'(\zeta)| \cdot |x_{z_q} - y_{z_q}|, \text{for some } \zeta \text{ between } x_{z_q} \text{ and } y_{z_q} \\
&= \left|(f_a^{j-1})'\left(f_a(\zeta)\right)\right| \cdot |f_a'(\zeta)| \cdot |x_{z_q} - y_{z_q}| \\
&= \left|(f_a^{j-1})'\left(f_a(\zeta)\right)\right| \cdot 2a|\zeta| \cdot |x_{z_q} - y_{z_q}| \\
&\leq B_1 |(f_a^{j-1})'(\xi_1)| \cdot 2a\mathrm{e}^{-|m_q|+1} \cdot \lambda(\sigma_q).
\end{aligned}$$

On the other hand, we have

$$|y_{z_q+j} - \xi_j| = |(f_a^{j-1})'(\theta)| \cdot |y_{z_q+1} - \xi_1|$$

for some $\theta \in [y_{z_q+1}, \xi_1]$. Noting that $[y_{z_q+1}, \xi_1] \subset f_a\left(U^+_{|m_q|}\right)$, we apply Lemma 2.3 and get

$$\begin{aligned}
|y_{z_q+j} - \xi_j| &\geq \frac{1}{B_1} |(f_a^{j-1})'(\xi_1)| \cdot |y_{z_q+1} - \xi_1| \\
&= \frac{1}{B_1} |(f_a^{j-1})'(\xi_1)| \cdot 2a y_{z_q}^2 \\
&\geq \frac{1}{B_1} |(f_a^{j-1})'(\xi_1)| \cdot 2a\mathrm{e}^{-2|m_q|-4}.
\end{aligned}$$

Combining what we know about $|x_{z_q+j} - y_{z_q+j}|$ and $|y_{z_q+j} - \xi_j|$ we obtain

$$\begin{aligned}
\frac{|x_{z_q+j} - y_{z_q+j}|}{|y_{z_q+j}|} &= \frac{|x_{z_q+j} - y_{z_q+j}|}{|y_{z_q+j} - \xi_j|} \cdot \frac{|y_{z_q+j} - \xi_j|}{|y_{z_q+j}|} \\
&\leq B_1^2 \frac{\mathrm{e}^5}{\mathrm{e}^{-|m_q|}} \cdot \lambda(\sigma_q) \cdot \frac{|y_{z_q+j} - \xi_j|}{|y_{z_q+j}|} \\
&\leq B_1^2 \cdot \mathrm{e}^5 \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \cdot \frac{\mathrm{e}^{-\beta j}}{\mathrm{e}^{-\alpha j} - \mathrm{e}^{-\beta j}}
\end{aligned}$$

since

$$|y_{z_q+j}| \geq |\xi_j| - |y_{z_q+j} - \xi_j| \geq \mathrm{e}^{-\alpha j} - \mathrm{e}^{-\beta j}.$$

Clearly,

$$\sum_{j=1}^{\infty} \frac{\mathrm{e}^{-\beta j}}{\mathrm{e}^{-\alpha j} - \mathrm{e}^{-\beta j}} < \infty$$

and, therefore,

$$B_q \leq a_3 \cdot \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})}$$

for some constant $a_3 = a_3(\alpha - \beta)$.

From the estimates obtained above, we get

$$S \leq a_4 \cdot \sum_{q=0}^{\gamma} \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})}, \text{ where } a_4 = a_1 + a_2 + a_3.$$

Defining $q(m) = \max\{q : m_q = m\}$ and using the fact that $\lambda(\sigma_{q+1}) \geq 2\lambda(\sigma_q)$ (lemma 4.1 part 1), we can easily see that

$$\sum_{\{q:m_q=m\}} \lambda(\sigma_q) \leq 2\lambda(\sigma_{q(m)}),$$

and so

$$\sum_{q=0}^{\gamma} \frac{\lambda(\sigma_q)}{\lambda(I_{m_q})} \leq \sum_{m\geq\Delta} \frac{1}{\lambda(I_m)} \sum_{\{q:m_q=m\}} \lambda(\sigma_q) \leq \sum_{m\geq\Delta} \frac{2\lambda(\sigma_{q(m)})}{\lambda(I_m)}.$$

Since

$$\frac{\lambda(\sigma_{q(m)})}{\lambda(I_m)} \leq \frac{10}{m^2},$$

it follows that

$$\sum_{m\geq\Delta} \frac{2\lambda(\sigma_{q(m)})}{\lambda(I_m)} \leq 20 \sum_{m\geq\Delta} \frac{1}{m^2},$$

which proves that S is uniformly bounded .

Now, if $n \geq z_\gamma + p_\gamma$ we are left with a last piece of free period to study:

$$F_{\gamma+1} = \sum_{j=z_\gamma+p_\gamma}^{n} \left| \frac{x_j - y_j}{y_j} \right|$$

We consider two cases. In the first one we suppose that $|f_a^n(\omega)| \leq \mathrm{e}^{-2\Delta}$. Proceeding as before we have for $j = z_\gamma + p_\gamma, \ldots, n-1$,

$$\begin{aligned}
\lambda(\sigma_n) &\geq |f_a^{n-j}(x_j) - f_a^{n-j}(y_j)| \\
&= \left|(f^{n-j})'(\zeta)\right| \cdot |x_j - y_j|, \text{for some } \zeta \text{ between } x_j \text{ and } y_j \\
&\geq \mathrm{e}^{-(\Delta+1)}\mathrm{e}^{c_0(n-j)}|x_j - y_j|, \text{ by Lemma 2.1 part 1.}
\end{aligned}$$

So,

$$\begin{aligned} F_{\gamma+1} &\le \sum_{j=z_\gamma+p_\gamma}^{n} \frac{\mathrm{e}^{\Delta+1}\mathrm{e}^{-c_0(n-j)}\lambda(\sigma_n)}{\delta} \\ &\le \sum_{j=z_\gamma+p_\gamma}^{n} \mathrm{e}^{2\Delta+1}\mathrm{e}^{-c_0(n-j)}\mathrm{e}^{-2\Delta} \\ &\le e\sum_{j=1}^{\infty} \mathrm{e}^{-cj} \le a_5, \end{aligned}$$

where a_5 is constant.

In the second case we assume that $|f_a^n(\omega)| > \mathrm{e}^{-2\Delta}$. Let q_1 be the first integer such that $q_1 \ge z_\gamma + p_\gamma$, $|f_a^{q_1}(\omega)| > \mathrm{e}^{-2\Delta}$. From the previous argumentation we have that

$$\left|\frac{(f_a^{q_1})'(x)}{(f_a^{q_1})'(y)}\right| \le C.$$

At this point we consider the time-interval $[q_1, q_2 - 1]$ (eventually empty) defined to be the largest interval such that $i \in [q_1, q_2 - 1] \Rightarrow y_i \notin U_1$. Then, using lemma 2.1 part 3 (here we use for the first time the hypothesis $f_a^n(\omega) \subset U_1$),

$$\begin{aligned} \sum_{i=q_1}^{q_2-1} \frac{|x_i - y_i|}{|y_i|} &\le e\sum_{i=q_1}^{q_2-1} |x_i - y_i| \le 3\sum_{i=q_1}^{q_2-1} \tfrac{5}{4}\mathrm{e}^{-c_0(n-1)}|f_a^n(\omega)| \\ &\le \tfrac{15}{2}\sum_{i=1}^{\infty} \mathrm{e}^{-ci} \le a_6, \end{aligned}$$

where a_6 is a constant.

If $q_2 = n$ the lemma is proved. Otherwise writing:

$$\left|\frac{(f_a^n)'(x)}{(f_a^n)'(y)}\right| = \left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| \left|\frac{(f_a^{q_2})'(x)}{(f_a^{q_2})'(y)}\right|,$$

we observe that in order to obtain the result we need only to bound the first factor. We do this considering, again, two cases:

1. $x_{q_2} \ge \frac{1}{2}$. Then since $|y_{q_2}| \le \mathrm{e}^{-1}$ (by definition of q_2), we have $|x_{q_2} - y_{q_2}| \ge \frac{1}{10}$. Therefore, by lemma 2.1 part 3

$$\tfrac{4}{5}\mathrm{e}^{c_0(n-q_2)}\tfrac{1}{10} \le |f_a^n(\omega)| \le 1,$$

which implies that $n - q_2 \le \frac{3}{2}\log\left(\frac{25}{2}\right)$ (remember that by hypothesis $c_0 \ge \frac{2}{3}$).

Taking into account the facts: $|(f_a^{n-q_2})'(x_{q_2})| \le 4^{n-q_2}$ and $|(f_a^{n-q_2})'(y_{q_2})| \ge \frac{4}{5}\mathrm{e}^{c_0(n-q_2)}$, we have

$$\left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| \le a_7,$$

for some constant a_7.

2. $x_{q_2} < \frac{1}{2}$. We can write (see Lemma 2.2 of [**Al92**] or Lemma 3.3 of [**Mo92**] for details)

$$\left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| = L\left|\frac{(g_a^{n-q_2})'(h^{-1}(x_{q_2}))}{(f_a^{n-q_2})'(h^{-1}(y_{q_2}))}\right|,$$

where

$$L = \sqrt{\frac{1-(f_a^{n-q_2}(x_{q_2}))^2}{1-x_{q_2}^2}}\sqrt{\frac{1-y_{q_2}^2}{1-(f_a^{n-q_2}(y_{q_2}))^2}} \leq \sqrt{\frac{1}{1-\frac{1}{4}}}\sqrt{\frac{1}{1-\mathrm{e}^{-2}}} \leq \frac{3}{4},$$

$h : [-1,1] \to [-1,1]$ is the homeomorphism that conjugates $f_2(x)$ to the tent map $1-2|x|$ and $g_a = h^{-1} \circ f_a \circ h$.

For the second factor, we have (see Lemma 3.1 of [**Mo92**] for details)

$$\left|\frac{(g_a^{n-q_2})'(h^{-1}(x_{q_2}))}{(f_a^{n-q_2})'(h^{-1}(y_{q_2}))}\right| \leq \left(\frac{2+\frac{3\pi}{\delta^3}(2-a)}{2-\frac{3\pi}{\delta^3}(2-a)}\right)^{n-q_2}.$$

Note that $|f_a^{q_1}(\omega)| > \mathrm{e}^{-2\Delta}$ and $\frac{4}{5}\mathrm{e}^{c_0(n-q_1)}|f_a^{q_1}(\omega)| \leq |f_a^n(\omega)| \leq 1$, from which we conclude that $n-q_2 \leq n-q_1 \leq 4\Delta$. So if a is sufficiently close to 2 in order to have

$$\left(\frac{2+\frac{3\pi}{\delta^3}(2-a)}{2-\frac{3\pi}{\delta^3}(2-a)}\right)^{4\Delta} \leq 2, \tag{4.7}$$

then

$$\left|\frac{(f_a^{n-q_2})'(x_{q_2})}{(f_a^{n-q_2})'(y_{q_2})}\right| \leq \frac{8}{3}.$$

□

5. Return depths and time between consecutive returns

In this section we justify the preponderance of the depths of essential returns over the depths of bound and inessential returns, stated in basic idea (I). We also get an upper bound for the elapsed time between two consecutive essential returns.

As we have already mentioned, there are three types of returns: essential, bounded and inessential, which we denote by t, u and v respectively. Remember that up to time n, the essential return that occurs at time t_i has depth η_i, for $i = 1, \ldots, s_n$; each t_i might be followed by bounded returns $u_{i,j}$, $j = 1, \ldots, u$ and these can be followed by inessential returns $v_{i,j}$, $j = 1, \ldots, v$.

The following lemma states that the depth of an inessential return is not greater than the depth of the essential return that precedes it.

LEMMA 5.1. *Suppose that t_i is an essential return for $\omega \in \mathcal{P}_{t_i}$, with $I_{\eta_i,k_i} \subset f_a^{t_i}(\omega) \subset I^+_{\eta_i,k_i}$. Then the depth of each inessential return occurring on $v_{i,j}$, $j = 1, \ldots, v$ is not greater than η_i.*

PROOF. By lemma 4.1 part 1 we have

$$\lambda\left\{f_a^{v_{i,j}}(\omega)\right\} \geq 2^j \lambda\left\{f_a^{t_i}(\omega)\right\} \geq 2^j \lambda\left(I_{\eta_i,k_i}\right)$$

Thus,

$$\lambda\left\{f_a^{v_{i,j}}(\omega)\right\} \geq \lambda\left\{I_{\eta_i,k_i}\right\} = \frac{\mathrm{e}^{-\eta_i}\left(1-\mathrm{e}^{-1}\right)}{\eta_i^2}.$$

But, since $v_{i,j}$ is an inessential return time we must have $f_a^{v_{i,j}}(\omega) \subset I_{m,k}$ for some $m \geq \Delta$; then, out of necessity, $m \leq \eta_i$, because $f_a^{v_{i,j}}(\omega)$ is too large to fit on some $I_{m,k}$ with $m > \eta_i$. □

In the next lemma, we prove a similar result for bounded returns.

LEMMA 5.2. *Let t be a return time (either essential or inessential) for $\omega \in \mathcal{P}_t$, with $f_a^t(\omega) \subset I_{\eta,k}^+$. Let $p = p(\eta)$ be the bound period length associated to this return. Then, for all $x \in \omega$, if the orbit of x returns to U_Δ between t and $t+p$, then the depth of this bound return will not be grater than η, if Δ is sufficiently large.*

PROOF. Consider a point $x \in \omega$. We will show that if Δ is large enough then $|f_a^{t+j}(x)| \geq \mathrm{e}^{-\eta}$, $\forall j \in \{1, \ldots, p-1\}$.

$$\left|f_a^j(1)\right| - \left|f_a^{t+j}(x)\right| \leq \left|f_a^{t+j}(x) - f_a^j(1)\right| \leq \mathrm{e}^{-\beta j}$$

which implies that

$$\begin{aligned}
\left|f_a^{t+j}(x)\right| &\geq \left|f_a^j(1)\right| - \mathrm{e}^{-\beta j} \overset{\text{(BA)}}{\geq} \mathrm{e}^{-\alpha j} - \mathrm{e}^{-\beta j} \geq \mathrm{e}^{-\alpha j}\left(1-\mathrm{e}^{(\alpha-\beta)j}\right)\\
&\geq \mathrm{e}^{-\alpha j}\left(1-\mathrm{e}^{(\alpha-\beta)}\right), \text{ since } \alpha-\beta<0\\
&\geq \mathrm{e}^{-\alpha p}\left(1-\mathrm{e}^{(\alpha-\beta)}\right), \text{ since } j<p\\
&\geq \mathrm{e}^{-3\alpha\eta}\left(1-\mathrm{e}^{(\alpha-\beta)}\right), \text{ since } p \leq 3\eta \text{ by lemma 2.3}\\
&\geq \mathrm{e}^{-4\alpha\eta}, \text{ if we choose a large } \Delta \text{ so that } 1-\mathrm{e}^{\alpha-\beta} \geq \mathrm{e}^{-\alpha\eta}\\
&\geq \mathrm{e}^{-\eta}, \text{ since } \alpha < \frac{1}{4}
\end{aligned}$$

□

The next lemma gives an upper bound for the time we have to wait between two essential return situations.

LEMMA 5.3. *Suppose t_i is an essential return for $\omega \in \mathcal{P}_{t_i}$, with $I_{\eta_i,k_i} \subset f_a^{t_i}(\omega) \subset I_{\eta_i,k_i}^+$. Then the next essential return situation t_{i+1} satisfies:*

$$t_{i+1} - t_i < 5\left|\eta_i\right|.$$

PROOF. Let $v_{i,1} < \ldots < v_{i,v}$ denote the inessential returns between t_i and t_{i+1}, with host intervals $I_{\eta_{i,1},k_{i,1}}, \ldots, I_{\eta_{i,v},k_{i,v}}$, respectively. We also consider $v_{i,0} = t_i$; $v_{i,v+1} = t_{i+1}$; for $j = 0, \ldots, v+1$, $\sigma_j = f_a^{v_{i,j}}(\omega)$ and for $j = 0, \ldots, v$, $q_j = v_{i,j+1} - (v_{i,j} + p_j)$, where p_j is the length of the bound period associated to the return $v_{i,j}$.

We consider two different cases: $v = 0$ and $v > 0$.

(1) $v = 0$

In this situation $t_{i+1} - t_i = p_0 + q_0$. Applying lemma 4.1 part 2b we get that

$$|\sigma_1| \geq \mathrm{e}^{-5\beta|\eta_i|}\mathrm{e}^{c_0 q_0 - (\Delta+1)}.$$

Taking into account the fact that $|\sigma_1| \leq 2$, we have

$$\begin{aligned} c_0 q_0 &\leq 1 + 5\beta|\eta_i| + \Delta + 1 \\ q_0 &\leq 8\beta|\eta_i| + \frac{3}{2}\Delta + 3, \text{ since } c_0 \geq \frac{2}{3} \\ q_0 &\leq 9\beta|\eta_i| + \frac{3}{2}\Delta, \text{for } \Delta \text{ large enough so that } \beta|\eta_i| > 3. \end{aligned}$$

Therefore,

$$\begin{aligned} t_{i+1} - t_i &= p_0 + q_0 \\ &\leq 3|\eta_i| + 9\beta|\eta_i| + \frac{3}{2}\Delta \\ &\leq 4|\eta_i| + \Delta, \text{since } 9\beta < \frac{1}{2} \\ &\leq 5|\eta_i|. \end{aligned}$$

(2) $v > 0$

In this case, $t_{i+1} - t_i = \sum_{j=0}^{v}(p_j + q_j)$. We separate this sum into three parts and control each separately:

$$t_{i+1} - t_i = p_0 + \left(\sum_{j=1}^{v-1} p_j + \sum_{j=0}^{v-1} q_j\right) + (p_v + q_v)$$

(i) For p_0 we have by lemma 2.3 that $p_0 \leq 3|\eta_i|$.
(ii) By lemma 4.1 we get

$$|\sigma_1| \geq \mathrm{e}^{c_0 q_0}\mathrm{e}^{-5\beta|\eta_i|} \text{ and } \frac{|\sigma_{j+1}|}{|\sigma_j|} \geq \mathrm{e}^{c_0 q_j}\mathrm{e}^{(1-5\beta)|\eta_{i,j}|},$$

for $j = 1, \ldots, v-1$. Now, we observe that $p_j \leq 3|\eta_{i,j}| \leq 4(1-5\beta)|\eta_{i,j}|$ and $q_j \leq 4c_0 q_j$, for all $j = 0, \ldots, v$. This means that controlling the second parcel resumes to bound

$$\sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}| + \sum_{j=0}^{v-1} c_0 q_j. \tag{5.1}$$

We achieve our goal by noting that (5.1) corresponds to the growth rate of the size of the σ_j, which cannot be very large, since every σ_j, $j = 1, \ldots, v$ is contained in some $I_{m,k} \subset U_\Delta$. Writing

$$|\sigma_v| = |\sigma_1| \prod_{j=1}^{v-1} \frac{|\sigma_{j+1}|}{|\sigma_j|},$$

and taking into account that $\sigma_v \in I_{\eta_{i,v},k_{i,v}}$, with $|\eta_{i,v}| \geq \Delta$ and thus $|\sigma_v| \leq \mathrm{e}^{-(\Delta+1)}$, it follows that

$$\exp\left\{-5\beta|\eta_i| + \sum_{j=0}^{v-1} c_0 q_j + \sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}|\right\} \leq \exp\{-(\Delta+1)\}$$

and consequently

$$\sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}| + \sum_{j=0}^{v-1} c_0 q_j \leq 5\beta|\eta_i| - (\Delta+1)$$

(iii) For the last term $p_v + q_v$ we proceed in a very similar manner to what we did in the case $v = 0$. By lemma 4.1we have

$$\frac{|\sigma_{v+1}|}{|\sigma_v|} \geq \mathrm{e}^{c_0 q_v - (\Delta+1)} \mathrm{e}^{(1-4\beta)|\eta_{i,v}|} \geq \mathrm{e}^{c_0 q_v - (\Delta+1)} \mathrm{e}^{(1-5\beta)|\eta_{i,v}|}.$$

From part 1 of lemma 4.1 we have $|\sigma_v| \geq 2^{v-1}|\sigma_1| \geq |\sigma_1|$, from which we get

$$2 \geq |\sigma_{v+1}| \geq |\sigma_1| \frac{|\sigma_{v+1}|}{|\sigma_v|}$$

and consequently

$$\exp\{-5\beta|\eta_i| + c_0 q_v - (\Delta+1) + (1-5\beta)|\eta_{i,v}|\} \leq \mathrm{e}^{\log 2}$$

implying

$$c_0 q_v + (1-5\beta)|\eta_{i,v}| \leq \Delta + 2 + 5\beta|\eta_i|.$$

Putting together the three parts we get

$$\begin{aligned}
t_{i+1} - t_i &= p_0 + \left(\sum_{j=1}^{v-1} p_j + \sum_{j=0}^{v-1} q_j\right) + (p_v + q_v) \\
&\leq p_0 + 4\left\{\sum_{j=1}^{v-1}(1-5\beta)|\eta_{i,j}| + \sum_{j=0}^{v-1} c_0 q_j + c_0 q_v + (1-5\beta)|\eta_{i,v}|\right\} \\
&\leq 3|\eta_i| + 4\{5\beta|\eta_i| - (\Delta+1) + (\Delta+1) + 1 + 5\beta|\eta_i|\} \\
&\leq 3|\eta_i| + 40\beta|\eta_i| + 4 \\
&\leq 4|\eta_i|.
\end{aligned}$$

□

The next two propositions allow us to obtain a bound for $T_n(x)$ (see (2.1) for definition) by a quantity proportional to $\frac{1}{n}F_n(x)$ (defined in (3.8)).

In the proof of the following proposition we will use directly and for the first time the condition known as the *free assumption* for the critical orbit. This condition essentially asserts that the set of Benedicks-Carleson parameters is built in such a way that the amount of time spent by the critical orbit in bound periods totally makes up a small fraction of the whole time (see [**BC91**, Section 2] or [**Mo92**, condition FA(n)]).

PROPOSITION 5.4. *Let t be a free return time (either essential or inessential) for $\omega \in \mathcal{P}_t$ with $f_a^t(\omega) \subset I_{\eta,k}^+$. Let $p = p(\eta)$ be the bound period associated with this return. Let S denote the sum of the depths of all the bound returns plus the depth of the return that originated the bound period. Then $S \leq C_3\eta$, with constant $C_3 = C_3(\alpha)$.*

PROOF. Recall that by lemma 2.3 we know that $\frac{2}{3}\eta \leq p \leq 3\eta$. Let $x \in \omega$. We say that a bound return is of *level i* if, at the moment of this bound return, x has already initiated exactly i bindings to the critical point ξ_0 and all of them are still active. By active we mean that the respective bound periods have not finished yet. To illustrate, suppose that u_1 is the first time between t and $t+p$ that the orbit of x enters U_Δ. Obviously, at this moment, the only active binding to ξ_0 is the one initiated at time t. Thus, u_1 is a bound return of level 1. Now, at time u_1, the orbit of x establishes a new binding to the critical point which ends at the end of the corresponding bound period that we denote by p_1 which depends on the depth η_1 of the bound return in question. During the period from u_1 to u_1+p_1 new returns may happen and their level is at least 2 since there are at least 2 active bindings: the one initiated at t and the one initiated at u_1. If $u_1 + p_1 < t + p$ then new bound returns of level 1 may occur after $u_1 + p_1$.

We may redefine the notion of bound period so that the bound periods are nested (see [**BC91**], section 6.2). This means that we may suppose that no binding of level i extends beyond the bound period of level $i-1$ during which it was initiated.

Taking into account the *free assumption* condition for the critical orbit we may assume that in a period of length $n \in \mathbb{N}$, the time spent by the critical orbit in bound periods is at most αn (see [**Mo92**, condition FA(n)]).

Since, when a point initiates a binding with ξ_0, it shadows the early iterates of the critical point, the same applies to any of these points $x \in \omega$ bounded to ξ_0. Thus in the period of time from t to $t+p$, the orbit of x can spend at most the fraction of time αp in bound periods. So if l denotes the number of bound returns of level 1, $u_1, \dots, u_l$ their instants of occurrence, $\eta_1, \dots \eta_l$ their respective depths and $p_1, \dots, p_l$ their respective bound periods, then we have by lemma 2.3 and the above observation that:

$$\frac{2}{3}\sum_{i=1}^{l}\eta_i \leq \sum_{i=1}^{l} p_i \leq \alpha p \leq 3\alpha\eta$$

from where we easily obtain $\sum_{i=1}^{l}\eta_i \leq 5\alpha\eta$. The same argument applies to the bound returns of level 2 within the i-th bound period of level 1. So if l_i denotes the number of bound returns of level 2 within the i-th bound period of level 1, $u_{i1}, \dots, u_{il_i}$ their instants of occurrence, $\eta_{i1}, \dots \eta_{il}$ their respective depths and $p_{i1}, \dots, p_{il}$ their respective bound periods, then we have

$$\frac{2}{3}\sum_{j=1}^{l_i}\eta_{ij} \leq \sum_{i=1}^{l_i} p_{ij} \leq \alpha p_i \leq 3\alpha\eta_i$$

from where we easily obtain $\sum_{i=1}^{l}\sum_{j=1}^{l_i}\eta_{ij} \leq (5\alpha)^2\eta$.

Thus a simple induction argument gives that

$$S \leq \sum_{i=0}^{\infty}(5\alpha)^i\eta \leq C_3\eta,$$

where

$$C_3 = \frac{1}{1-5\alpha}, \tag{5.2}$$

remember that by choice $5\alpha < 1$. □

PROPOSITION 5.5. *Let t be an essential return time for $\omega \in \mathcal{P}_t$ with $I_{\eta,k} \subset f_a^t(\omega) \subset I_{\eta,k}^+$. Let p_0 denote the associated bound period. Let S denote the sum of the depths of all the free inessential returns before the next essential return situation. Then $S \leq C_4\eta$, with constant $C_4 = C_4(\beta)$.*

PROOF. Suppose that v is the number of inessential returns before the next essential return situation of ω, which occur at times $v_1, \dots, v_v$, with respective depths $\eta_1, \dots, \eta_v$ and respective bound periods $p_1, \dots, p_v$. Also denote by v_{v+1} the next essential return situation of ω. Let $\sigma_i = f_a^{v_i}(\omega)$.

By lemma 4.1 we get

$$|\sigma_1| \geq \mathrm{e}^{c_0 q_0}\mathrm{e}^{-5\beta|\eta|} \text{ and } \frac{|\sigma_{j+1}|}{|\sigma_j|} \geq \mathrm{e}^{c_0 q_i}\mathrm{e}^{(1-5\beta)|\eta_i|},$$

where $q_i = v_{i+1} - (v_i + p_i)$, for $i = 0, \dots, v$. We also know that $|\sigma_{v+1}| \leq 2$.

Since

$$|\sigma_{v+1}| = |\sigma_1| \prod_{i=1}^{v} \frac{|\sigma_{i+1}|}{|\sigma_i|},$$

we have that

$$\exp\left\{c_0 q_0 - 5\beta\eta + \sum_{i=1}^{v}(c_0 q_i + (1-5\beta)\eta_i)\right\} \leq \mathrm{e},$$

from where we obtain that

$$\sum_{i=1}^{v}(c_0 q_i + (1-5\beta)\eta_i) \leq 5\beta\eta + 1,$$

which easily implies that $S \leq C_4\eta$, where

$$C_4 = \frac{6\beta}{1-5\beta} \tag{5.3}$$

. □

From these propositions we easily conclude that

$$T_n(x) \leq \frac{C_5}{n} F_n(x)$$

with

$$C_5 = C_5(\alpha, \beta) = (C_3 + C_3 C_4). \tag{5.4}$$

6. Probability of an essential return reaching a certain depth

Now, that we know that only the essential returns matter, we prove that the chances of very deep essential returns occurring are very small. In fact, the probability of an essential return hitting the depth of ρ will be shown to be less than $\mathrm{e}^{-\tau\rho}$, with $\tau > 0$.

We must make our statements more precise and we begin by defining a probability space. We define the probability measure λ^* on I by renormalizing the Lebesgue measure so that $\lambda^*(I) = 1$. We may now speak of expectations $E(\cdot)$, events and their probability of occurrence.

For each $x \in I$, let $u_n(x)$ denote the number of essential return situations of x between 1 and n, $s_n(x)$ be the number those which are actual essential return times and sd_n the number of the latter that correspond to deep essential returns of the orbit of x with return depths above a threshold $\Theta \geq \Delta$. Observe that $u_n(x) - s_n(x)$ is the exact number of escaping situations of the orbit of x, up to n.

Given the integers $0 \leq s \leq \frac{3n}{2\Theta}$, $s \leq u \leq n$ and s integers $\rho_1, \ldots, \rho_s$, each greater than or equal to Θ, we define the event:

$$A^{u,s}_{\rho_1,\ldots,\rho_s}(n) = \left\{ x \in I \;:\; u_n(x) = u,\, sd_n(x) = s, \;\; \begin{array}{r} \text{and the depth of the i-th deep essen-} \\ \text{tial return is } \rho_i \; \forall i \in \{1, \ldots, s\} \end{array} \right\}.$$

Remark 6.1. Observe that the upper bound $\frac{3n}{2\Theta}$ for the number of deep essential returns up to time n derives from the fact that each deep essential return originates a bound period of length at least $\frac{2}{3}\Theta$ (see lemma 2.3). Since during the bound periods there cannot be any essential return, the number of deep essential returns occurring in a period of length n is at most $\frac{n}{\frac{2}{3}\Theta}$.

Proposition 6.2. *Given the integers $0 \leq s \leq \frac{3n}{2\Theta}$ and $s \leq u \leq n$, consider s integers $\rho_1, \ldots, \rho_s$, each greater than or equal to Θ. If Θ is large enough, then*

$$\lambda^* \left(A^{u,s}_{\rho_1,\ldots,\rho_s}(n) \right) \leq \binom{u}{s} Exp \left\{ -(1 - 6\beta) \sum_{i=1}^{s} \rho_i \right\}$$

Proof. Fix $n \in \mathbb{N}$ and take $\omega_0 \in \mathcal{P}_0$. Note that the functions u_n, s_n and sd_n are constant in each $\omega \in \mathcal{P}_n$. Let $\omega \in \omega_0 \cap \mathcal{P}_n$ be such that $u_n(\omega) = u$. Then, there is a sequence $1 \leq t_1 \leq \ldots \leq t_u \leq n$ of essential return situations. Let ω_i denote the element of the partition $\mathcal{P}_{t_i}$ that contains ω. We have $\omega_0 \supset \omega_1 \supset \ldots \supset \omega_u = \omega$. Consider that $\omega_j = \emptyset$ whenever $j > u$. For each $j \in \{0, \ldots, n\}$ we define the set:

$$Q_j = \bigcup_{\omega \in \mathcal{P}_n \cap \omega_0} \omega_j,$$

and its partition

$$\mathcal{Q}_j = \{\omega_j : \omega \in \mathcal{P}_n \cap \omega_0\}.$$

Let $\omega \in \mathcal{P}_n$ be such that $sd_n(\omega) = s$. Then, we may consider $1 \leq r_1 \leq \ldots \leq r_s \leq u$ with r_i indicating that the i-th deep essential return occurs in the r_i-th essential return situation. Now, set $V(0) = Q_0 = \omega_0$. Fix s integers $1 \leq r_1 \leq \ldots \leq r_s \leq u$. Next, for each $j \leq u$ we define recursively the sets $V(j)$. Although the set $V(u)$ will depend on the fixed integers $1 \leq r_1 \leq \ldots \leq r_s \leq u$, we do not indicate this so that the notation is not overloaded. Suppose that $V(j-1)$ is already defined and $r_{i-1} < j < r_i$. Then, we set

$$V(j) = \bigcup_{\omega \in \mathcal{Q}_j} \omega \bigcap f_a^{-t_j}(I - U_\Theta) \bigcap V(j-1).$$

If $j = r_i$ then we define

$$V(j) = \bigcup_{\omega \in \mathcal{Q}_j} \omega \bigcap f_a^{-t_j}(I_{\rho_i} \cup I_{-\rho_i}) \bigcap V(j-1)$$

Observe that for every $j \in \{1, \ldots, u\}$ we have $\frac{|V(j)|}{|V(j-1)|} \leq 1$. Therefore, we concentrate in finding a better estimate for $\frac{|V(r_i)|}{|V(r_i-1)|}$. Consider that $\omega_{r_i} \in \mathcal{Q}_{r_i} \cap V(r_i - 1)$ and let $\omega_{r_i-1} \in \mathcal{Q}_{r_i-1} \cap V(r_i-1)$ contain ω_{r_i}. We have to consider two situations depending on whether t_{r_i-1} is an escaping situation or an essential return.

Let us suppose first that t_{r_i-1} was an essential return with return depth η. Then,

$$\begin{aligned}
\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} &\leq \frac{|\omega_{r_i}|}{|\widehat{\omega}_{r_i-1}|}, \quad \text{where } \widehat{\omega}_{r_i-1} = \omega_{r_i-1} \cap f_a^{-t_{r_i}}(U_1) \\
&\leq C \frac{\left| f_a^{t_{r_i}}(\omega_{r_i}) \right|}{\left| f_a^{t_{r_i}}(\widehat{\omega}_{r_i-1}) \right|}, \text{ by the mean value theorem and lemma 4.3} \\
&\leq C \frac{2\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-5\beta\eta}}, \text{ by lemma 4.1 part 3b and definition of } \omega_{r_i}
\end{aligned}$$

Note that when $r_{i-1} = r_i - 1$ then $\eta = \rho_{i-1} \geq \Theta$. If, on the other hand, $r_{i-1} > r_i - 1$ then t_{r_i-1} is an essential return with depth $\eta < \Theta \leq \rho_{i-1}$. Then in both situations we have

$$\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} \leq 2C \frac{\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-5\beta\rho_{i-1}}}.$$

When t_{r_i-1} is an escape situation instead of using lemma 4.1 we can use lemma 4.2 and obtain

$$\frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} \leq 2C \frac{\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-\beta\Delta}} \leq 2C \frac{\mathrm{e}^{-\rho_i}}{\mathrm{e}^{-5\beta\rho_{i-1}}}.$$

Observe also that if $\widehat{\omega}_{r_i-1} \neq \omega_{r_i-1}$ then, because we are assuming that $\omega_{r_i} \neq \emptyset$, we have $\lambda\left(f_a^{t_{r_i}}(\widehat{\omega}_{r_i-1})\right) \geq \mathrm{e}^{-1} - \mathrm{e}^{-\Theta} \geq \mathrm{e}^{-5\beta\rho_{i-1}}$, for large Θ.

At this point we have

$$\begin{aligned}
|V(r_i)| &= \sum_{\omega_{r_i} \in \mathcal{Q}_{r_i} \cap V(r_i-1)} \frac{|\omega_{r_i}|}{|\omega_{r_i-1}|} |\omega_{r_i-1}| \\
&\leq 2C\mathrm{e}^{-\rho_i}\mathrm{e}^{5\beta\rho_{i-1}} \sum_{\omega_{r_i} \in \mathcal{Q}_{r_i} \cap V(r_i-1)} |\omega_{r_i-1}| \\
&\leq 2C\mathrm{e}^{-\rho_i}\mathrm{e}^{5\beta\rho_{i-1}} |V(r_i-1)|.
\end{aligned}$$

We are now in conditions to obtain that

$$|V(u)| \leq (2C)^s \mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\} \mathrm{e}^{5\beta\rho_0}|V(0)|$$

where ρ_0 is given by the interval $\omega_0 \in \mathcal{P}_0$. If $\omega_0 = I_{(\eta_0,k_0)}$ with $|\eta_0| \geq \Delta$ and $1 \leq k_0 \leq \eta_0^2$, then $\rho_0 = |\eta_0|$. If $\omega_0 = (\delta, 1]$ or $\omega_0 = [-1, -\delta)$, then we can take $\rho_0 = 0$.

Now, we have to take into account the number of possibilities of having the occurrence of the event $V(u)$ implying the occurrence of the event $A^{u,s}_{\rho_1,\ldots,\rho_s}(n)$. The number of possible configurations related with the different values that the integers $r_1, \ldots r_s$ can take is $\binom{u}{s}$. Hence, it follows that

$$\begin{aligned}
\lambda^*\left(A^{u,s}_{\rho_1,\ldots,\rho_s}(n)\right) &\leq (2C)^s \binom{u}{s} \mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\} \sum_{\omega_o\in\mathcal{P}_0} \mathrm{e}^{5\beta|\rho_0|}|\omega_0| \\
&\leq (2C)^s \binom{u}{s} \mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\} \left(2(1-\delta) + \sum_{|\eta_0|\geq\Delta} \mathrm{e}^{5\beta\eta_0}\mathrm{e}^{-|\eta_0|}\right) \\
&\leq 3(2C)^s \binom{u}{s} \mathrm{Exp}\left\{-(1-5\beta)\sum_{i=1}^{s}\rho_i\right\}, \quad \text{for } \Delta \text{ large enough} \\
&\leq \binom{u}{s} \mathrm{Exp}\left\{-(1-6\beta)\sum_{i=1}^{s}\rho_i\right\}.
\end{aligned}$$

The last inequality results from the fact that $s\Theta \leq \sum_{i=1}^{s}\rho_i$ and the freedom to choose a sufficiently large Θ. □

Fix $n \in \mathbb{N}$, the integers $1 \leq s \leq \frac{n}{\frac{2}{3}\Theta}$, $s \leq u \leq n$ and integer $j \leq s$. Given an integer $\rho \geq \Theta$, consider the event

$$A^{u,s}_{\rho,j}(n) = \left\{ x \in I \ : \ u_n(x) = u,\ sd_n(x) = s, \ \begin{array}{r}\text{and the depth of the j-th deep} \\ \text{essential return is } \rho\end{array} \right\}.$$

COROLLARY 6.3. *If Δ is large enough, then*

$$\lambda^*\left(A^{u,s}_{\rho,j}(n)\right) \leq \binom{u}{s} e^{-(1-6\beta)\rho}$$

PROOF. Since $A^{u,s}_{\rho,j}(n) = \bigcup_{\substack{\rho_i\geq\Theta \\ i\neq j}} A^{u,s}_{\rho_1,\ldots,\rho_{j-1},\rho,\rho_{j+1},\ldots,\rho_s}(n)$, then by proposition 6.2 we have

$$\lambda^*\left(A^{u,s}_{\rho,j}(n)\right) \leq \binom{u}{s} \mathrm{e}^{-(1-6\beta)\rho} \left(\sum_{\eta=\Theta}^{\infty} \mathrm{e}^{-(1-6\beta)\eta}\right)^{s-1} \leq \binom{u}{s} \mathrm{e}^{-(1-6\beta)\rho},$$

as long as Θ is sufficiently large so that $\sum_{\eta=\Theta}^{\infty} \mathrm{e}^{-(1-6\beta)\eta} \leq 1$. □

REMARK 6.4. Observe that the bound for the probability of the event $A^{u,s}_{\rho,j}(n)$ does not depend on the $j \leq s$ chosen.

REMARK 6.5. Observe that proposition 6.2 and corollary 6.3 also apply when $\Theta = \Delta$ in which case we have $sd_n = s_n$.

7. Non-uniform expansion

According to section 3 to finish the proof we only need to show that

$$\lambda\left(E_1(n)\right) \leq \mathrm{e}^{-\tau_1 n}, \quad \forall n \geq N_1^*$$

for some constant $\tau_1(\alpha,\beta) > 0$ and an integer $N_1^* = N_1^*(\Delta, \tau_1)$.

For each $x \in I$, recall that $u_n(x)$ denotes the number of essential return situations of x between 1 and n, and $s_n(x)$ the number of those which correspond to essential returns of the orbit of x. In this section we consider that the threshold $\Theta = \Delta$. Also remember that $u_n(x) - s_n(x)$ is the exact number of escaping situations the orbit of x goes through until the time n.

We define the following events:

$$A_\rho^{u,s}(n) = \left\{ \begin{array}{r} x \in I: \; u_n(x) = u, \; s_n(x) = s \text{ and there is one essential return} \\ \text{reaching the depth } \rho \end{array} \right\},$$

for fixed $n \in \mathbb{N}$, $s \leq n$ and $\rho \geq \Delta$;

$$A_\rho(n) = \left\{ x \in I : \exists t \leq n : t \text{ is essential return time and } |f_a^t(x)| \in I_\rho \right\},$$

for fixed n and $\rho \geq \Delta$.

Now, because $A_\rho^{u,s}(n) = \bigcup_{j=1}^{s} A_{\rho,j}^{u,s}(n)$, by corollary 6.3, we have

$$\lambda^*\left(A_\rho^{u,s}(n)\right) \leq \sum_{j=1}^{s} \lambda^*\left(A_{\rho,j}^{u,s}(n)\right) \leq s\binom{u}{s}\mathrm{e}^{-(1-6\beta)\rho}. \tag{7.1}$$

Observing that $A_\rho(n) = \bigcup_{s=1}^{\frac{3n}{2\Delta}} \bigcup_{u=s}^{n} A_\rho^s(n)$, then by (7.1) we get

$$\begin{aligned} \lambda^*\left(A_\rho(n)\right) &\leq \sum_{s=1}^{\frac{3n}{2\Delta}} \sum_{u=s}^{n} \lambda^*\left(A_\rho^{u,s}(n)\right) \leq \sum_{s=1}^{\frac{3n}{2\Delta}} \sum_{u=s}^{n} s\binom{u}{s}\mathrm{e}^{-(1-6\beta)\rho} \\ &\leq \mathrm{e}^{-(1-6\beta)\rho} \sum_{s=1}^{\frac{3n}{2\Delta}} s \sum_{u=s}^{n} \binom{n}{s} \leq n\mathrm{e}^{-(1-6\beta)\rho} \sum_{s=1}^{\frac{3n}{2\Delta}} s\binom{n}{s} \\ &\leq n\binom{n}{\frac{3n}{2\Delta}}\mathrm{e}^{-(1-6\beta)\rho} \sum_{s=1}^{\frac{3n}{2\Delta}} s \leq \frac{4n^3}{\Delta}\binom{n}{\frac{3n}{2\Delta}}\mathrm{e}^{-(1-6\beta)\rho}. \end{aligned}$$

By the Stirling formula, we have

$$\sqrt{2\pi m}m^m\mathrm{e}^{-m} \leq m! \leq \sqrt{2\pi m}m^m\mathrm{e}^{-m}\left(1 + \frac{1}{4m}\right),$$

which implies that

$$\binom{n}{\frac{3n}{2\Delta}} \leq \text{const}\frac{(n)^n}{(n - \frac{3n}{2\Delta})^{n-\frac{3n}{2\Delta}}(\frac{3n}{2\Delta})^{\frac{3n}{2\Delta}}}.$$

So, if we choose Δ large enough we have

$$\binom{n}{\frac{3n}{2\Delta}} \leq \text{const}\left(\left(1+\frac{\frac{3}{2\Delta}}{1-\frac{3}{2\Delta}}\right)\left(1+\frac{1-\frac{3}{2\Delta}}{\frac{3}{2\Delta}}\right)^{\frac{\frac{3}{2\Delta}}{1-\frac{3}{2\Delta}}}\right)^{(n-\frac{3n}{2\Delta})} \leq \text{const}\mathrm{e}^{h(\Delta)n},$$

where $h(\Delta) \to 0$, as $\Delta \to \infty$. The last inequality derives from the fact that each factor in the middle expression can be made arbitrarily close to 1 by taking Δ sufficiently large.

Since we know, by lemmas 5.1 and 5.2, that the depths of inessential and bound returns are not greater than the depth of the essential return preceding them we have, for all $n \geq N_1'$, where N_1' is such that $\alpha N_1' \geq \Delta$,

$$E_1(n) = \left\{x \in I:\ \exists i \in \{1,\dots,n\},\ |f_a^i(x)| < \mathrm{e}^{-\alpha n}\right\} \subset \bigcup_{\rho=\alpha n}^{\infty} A_\rho(n).$$

Consequently, taking $\tau_1 = \frac{(1-6\beta)\alpha}{4}$ and Δ large enough such that $h(\Delta) \leq \frac{(1-6\beta)\alpha}{2}$

$$\begin{aligned}\lambda^*(E_1(n)) &\leq \text{const}\tfrac{4n^3}{\Delta}\mathrm{e}^{h(\Delta)n}\sum_{\rho=\alpha n}^{\infty}\mathrm{e}^{-(1-6\beta)\rho}\\ &\leq \text{const}'\tfrac{4n^3}{\Delta}\mathrm{e}^{h(\Delta)n}\mathrm{e}^{-(1-6\beta)\alpha n}\\ &\leq \text{const}'\tfrac{4n^3}{\Delta}\mathrm{e}^{-2\tau_1 n}\\ &\leq \mathrm{e}^{-\tau_1 n},\end{aligned}$$

when $n \geq N_1^*$, where N_1^* is such that $N_1^* \geq N_1'$ and for all $n \geq N_1^*$ we have

$$\text{const}'\,\tfrac{4n^3}{\Delta}\mathrm{e}^{-\tau_1 n} \leq 1. \tag{7.2}$$

8. Slow recurrence to the critical set

As referred in section 3, we are left with the burden of having to show that for all $n \in \mathbb{N}$, and for a given ϵ, we may choose a small $\gamma = \mathrm{e}^{-\Theta}$ such that

$$\lambda^*\{E_2(n)\} \leq \lambda^*\left\{x : F_n(x) > \frac{\epsilon n}{C_5}\right\} \leq \mathrm{e}^{-\tau_2 n},$$

in order to complete the proof.

We achieve this goal, by means of a large deviation argument. Essentially we show that the moment generating function of F_n is bounded above by $\mathrm{e}^{h(\Theta)n}$, where $h(\Theta) \xrightarrow{\Theta\to\infty} 0$; then we use the Tchebychev inequality to obtain the desired result.

Lemma 8.1. *Take $0 < t \leq \frac{1-6\beta}{3}$. If Θ is sufficiently large, then there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have $E\left(e^{tF_n}\right) \leq e^{h(\Theta)n}$. Moreover $h(\Theta) \to 0$, as $\Theta \to \infty$.*

PROOF.

$$
\begin{aligned}
E\left(\mathrm{e}^{tF_n}\right) = E\left(\mathrm{e}^{t\sum_{i=1}^{s}\eta_i}\right) &= \sum_{u,s,(\rho_1,\ldots,\rho_s)} \mathrm{e}^{t\sum_{i=1}^{s}\rho_i}\lambda^*\left(A^{u,s}_{\rho_1,\ldots,\rho_s}(n)\right)\\
&\leq \sum_{u,s,(\rho_1,\ldots,\rho_s)} \mathrm{e}^{t\sum_{i=1}^{s}\rho_i}\binom{u}{s}\mathrm{e}^{-3t\sum_{i=1}^{s}\rho_i}, \text{ by proposition 6.2}\\
&\leq \sum_{u,s,R}\binom{u}{s}\zeta(s,R)\mathrm{e}^{-2tR},
\end{aligned}
$$

where $\zeta(s,R)$ is the number of integer solutions of the equation $x_1+\ldots+x_s=R$ satisfying $x_i\geq\Theta$ for all i. We have

$$
\zeta(s,R)\leq\#\{\text{solutions of } x_1+\ldots+x_s=R, x_i\in\mathbb{N}_0\}=\binom{R+s-1}{s-1}.
$$

By the Stirling formula, we may write

$$
\sqrt{2\pi m}m^m\mathrm{e}^{-m}\leq m!\leq\sqrt{2\pi m}m^m\mathrm{e}^{-m}\left(1+\frac{1}{4m}\right),
$$

which implies that

$$
\binom{R+s-1}{s-1}\leq\text{const}\frac{(R+s-1)^{R+s-1}}{R^R(s-1)^{s-1}}.
$$

So, if we choose Θ large enough we have

$$
\zeta(s,R)\leq\left(\text{const}^{\frac{1}{R}}\left(1+\tfrac{s-1}{R}\right)\left(1+\tfrac{R}{s-1}\right)^{\frac{s-1}{R}}\right)^R\leq\mathrm{e}^{tR}.
$$

The last inequality derives from the fact that $s\Theta\leq R$, and so each factor in the middle expression can be made arbitrarily close to 1 by taking Θ sufficiently large.

Continuing from where we stopped,

$$
\begin{aligned}
E\left(\mathrm{e}^{tF_n}\right) &\leq \sum_{u,s,R}\binom{u}{s}\mathrm{e}^{tR}\mathrm{e}^{-2tR}\\
&\leq \sum_{u,s,R}\binom{u}{s}\mathrm{e}^{-tR}\\
&\leq \sum_{u,s}\binom{u}{s}, \quad \text{for } \Theta \text{ sufficiently large.}
\end{aligned}
$$

Now, we have

$$
\sum_{u,s}\binom{u}{s}\leq\sum_{s=1}^{\frac{3n}{2\Theta}}\sum_{u=s}^{n}\binom{u}{s}\leq n\sum_{s=1}^{\frac{3n}{2\Theta}}\binom{n}{s}\leq n\sum_{s=1}^{\frac{3n}{2\Theta}}\binom{n}{\frac{3n}{2\Theta}}\leq\frac{3n^2}{2\Theta}\binom{n}{\frac{3n}{2\Theta}}.
$$

Using the Stirling formula again, and arguing like in section 7 it follows that we may take $N_2 = N_2(\Theta) \in \mathbb{N}$ sufficiently large so that for all $n \geq N_2$ we obtain

$$E\left(\mathrm{e}^{tF_n}\right) \leq \mathrm{e}^{h(\Theta)n},$$

where $h(\Theta) \to 0$, as $\Theta \to \infty$. □

If we take $t = \frac{1-6\beta}{3}$ and Θ large enough so that $\tau_2 = \frac{t\epsilon}{C_5} - h(\Theta) > 0$, then we have

$$\begin{aligned}\lambda^*\left(F_n > \frac{\epsilon n}{C_5}\right) &\leq \mathrm{e}^{-t\frac{\epsilon n}{C_5}} E\left(\mathrm{e}^{tF_n}\right), \text{ by Tchebychev's inequality}\\ &\leq \mathrm{e}^{-\frac{t\epsilon n}{C_5}} \mathrm{e}^{h(\Theta)n}, \text{ by lemma 8.1}\\ &\leq \mathrm{e}^{-\tau_2 n}.\end{aligned}$$

Consequently, $\lambda^*\{E_2(n)\} \leq \mathrm{e}^{-\tau_2 n}$, for all $n \geq N_2$.

Remark 8.2. Since the growth properties of the space and parameter derivatives along orbits are equivalent (see lemma 4 of [**BC85**] or lemma 3.4 of [**Mo92**]), it is possible to build a similar partition on the parameters as Benedicks and Carleson ([**BC85, BC91**]) did when they built $\mathcal{BC}_1$. Then, using the same kind of arguments as in sections 6 and 8, it is not difficult to bound, on a full Lebesgue measure subset of $\mathcal{BC}_1$, the value of $\frac{C_5}{n}F_n(\xi_0) = \frac{C_5}{n}\sum_{i=1}^{sd_n}\eta_i$, where η_i stands for the depth of the i-th deep essential return of the orbit of ξ_0. This way one obtains the validity of condition (1.2) for the critical point ξ_0, on a full Lebesgue measure subset of $\mathcal{BC}_1$.

9. Uniformness on the choice of the constants

As referred in remark 1.1 all constants involved must not depend on the parameter $a \in \mathcal{BC}_1$. Because there are many constants in question and because they depend on each other in an intricate manner we dedicate this section to clarifying their interdependencies.

We begin by considering the constants appearing in (EG) and (BA) that determine the space $\mathcal{BC}_1$ of parameters. So we fix $c \in [\frac{2}{3}, \log 2]$ and $0 < \alpha < 10^{-3}$.

Then, we consider $\beta > 0$ of definition 2.2 concerning the bound period, to be a small constant such that $\alpha < \beta < 10^{-2}$. A good choice for β would be $\beta = 2\alpha$.

We next fix a sufficiently large Δ such that we have validity on all estimates throughout the text. Most of the times the choice of a large Δ depends on the values of α and β. Note that at no time does the choice of a large Δ depend on the parameter value considered.

After fixing Δ we choose $\frac{2}{3} \leq c_0 \leq \log 2$ (take, for example, $c_0 = c$), and compute a_0 given by lemma 2.1, and such that (4.3), (4.5), (4.6) and (4.7) hold. Note that this might bring about a reduction in the set of parameters since we will only have to consider parameter values on $\mathcal{BC}_1 \cap [a_0, 2]$ which is still a positive Lebesgue measure set. If necessary we redefine $\mathcal{BC}_1$ to be $\mathcal{BC}_1 \cap [a_0, 2]$.

Finally, we fix any small $\epsilon > 0$ referring to (1.2), and explicit the dependence of the rest of the constants in the table 1

In conclusion, all the constants involved depend ultimately on α, β, Δ and ϵ, which were chosen uniformly on $\mathcal{BC}_1$, thus we may say that $(f_a)_{a \in \mathcal{BC}_1}$ is a uniform family in the sense referred to in [**Al03**].

Constant	Dependencies	Main References
B_1	α, β	lemma 2.3
C	α, β	lemma 4.3
d	α, β	(1.1) and (3.2)
τ_1	α	theorem A and section 7
N_1^*	α, Δ, τ_1	(7.2)
N_1	Δ, α, B_1, d, N_1^*	section 3
C_1	N_1, τ_1	theorem A and (3.6)
C_3	α	(5.2)
C_4	β	(5.3)
C_5	α, β	(5.4)
Θ	ϵ, C_5, Δ	sections 5 and 8
γ	Θ	section 2
N_2	Θ	section 8
τ_2	ϵ, C_5, Θ	theorem B and section 8
C_2	N_2, τ_2	theorem B and section 3

TABLE 1. Constants interdependency

CHAPTER 2

Statistical stability for Hénon maps of Benedicks-Carleson type

1. Motivation and statement of the result

Hénon [**He76**] proposed the two-parameter family of maps

$$\begin{array}{rccc} f_{a,b}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (1-ax^2+y, bx) \end{array}$$

as a model for non-linear two-dimensional dynamical systems. Numerical experiments for the parameters a = 1.4 and b = 0.3 indicated that $f_{a,b}$ has a global attractor. Hénon conjectured that this dynamical system should have a "strange attractor" and that it should be more favorable to analysis than the Lorenz system.

In principle most initial points could be attracted to a long periodic cycle, so it was not at all a priori clear that the attractor seen by Hénon in his computer pictures was not a long stable periodic orbit. However, in an outstanding paper, Benedicks and Carleson [**BC91**] managed to prove that what Hénon conjectured was true - not for the parameters $(a,b) = (1.4, 0.3)$ that Hénon studied - but for small $b > 0$. In fact, for such small $b > 0$ values $f_{a,b}$ is strongly dissipative, and may be seen as an "unfolded" version of a quadratic map in the interval. Simple arguments show that for these values of b there is a forward invariant region which by successive iterations of $f_{a,b}$ accumulates in a topological attractor that coincides with the closure of the unstable manifold W of a fixed point z^* of $f_{a,b}$. The *Benedicks-Carleson Theorem* states that as long as $b > 0$ is kept sufficiently small there is a positive Lebesgue measure set of parameters $a \in [1,2]$ (very close to $a = 2$) for which there is a dense orbit in $\overline{W}$ along which the derivative grows exponentially fast, which makes it a non-trivial transitive non-hyperbolic attractor. We denote by $\mathcal{BC}$ this positive Lebesgue measure set of parameters to which we refer as the Benedicks-Carleson parameter set.

This remarkable breakthrough and the techniques developed in [**BC91**] promoted the emergence of several results not specific to the context of Hénon maps. In the basis of this enlargement of the perspective is the work of Mora and Viana [**MV93**]. They proposed a renormalization scheme that when applied to a generic unfolding of a homoclinic tangency associated to a dissipative saddle reveals the presence of Hénon-like families, which they proved to share the same properties studied by Benedicks and Carleson for the original Hénon family. This means that chaotic attractors arise abundantly in dynamical phenomena.

Continuing the study of the dynamical properties of Hénon maps, Benedicks and Young [**BY93**] developed the machinery even further to obtain that every attractor occurring for each parameter $(a,b) \in \mathcal{BC}$ supports a unique SRB measure $\nu_{a,b}$ which they also proved to be a physical measure. Afterwards, Benedicks and Viana [**BV01**] showed that the basins of these SRB measures have no holes, that is Lebesgue - a.e. point whose orbit converges to the attractor is $\nu_{a,b}$ - generic , meaning that the time average of any continuous function evaluated along its orbit equals the space average of the continuous function computed with respect to $\nu_{a,b}$.

The statistical properties of the Hénon maps of the Benedicks-Carleson type were carried on by Benedicks and Young [**BY00**] who built "Markov extensions" of these maps to prove the *exponential decay of correlations*. In fact, they demonstrate that for Hölder continuous observables $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}$ and for every $(a, b) \in \mathcal{BC}$, there is a real number $\tau < 1$ and a constant $C = C(\phi, \psi)$ such that

$$\left| \int \phi(\psi \circ f_{a,b}^n) d\nu_{a,b} - \int \phi d\nu_{a,b} \int \psi d\nu_{a,b} \right| \leq C\tau^n,$$

for all $n \in \mathbb{N}$. Moreover, the stochastic process $\phi, \phi \circ f_{a,b}, \phi \circ f_{a,b}^2, \ldots$ satisfies the Central Limit Theorem, i.e.

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left(\phi \circ f_{a,b}^i - \int \phi d\nu_{a,b} \right)$$

converges in distribution to the Gaussian law $\mathcal{N}(0, \sigma)$. Young [**Yo98**] extended these results to a wider setting, namely, to dynamical systems that admit a horseshoe with infinitely many branches and variable return times. In doing so she provided a general scheme that unifies the proofs of this kind of results in several situations, like billiards with convex scatterers, Axiom A attractors, piecewise hyperbolic maps, logistic maps and Hénon maps.

Despite being metrically significant, the strange attractors appearing for the Benedicks-Carleson parameters are very fragile under small perturbations of the map. In fact, Ures [**Ur95**] showed that the Benedicks-Carleson parameters can be approximated by other parameters for which the Hénon map has a homoclinic tangency associated to the fixed point z^*. Hence, according to Newhouse's famous results [**Ne74, Ne79**], under small perturbations one may force the appearance of infinitely many attractors in the neighborhood of W. Nevertheless, Benedicks and Viana [**BV06**] showed that the Hénon maps in $\mathcal{BC}$ are remarkably stable under small random noise (*stochastic stability*). Let us elaborate on this by giving a heuristic explanation of the result. Let $(a, b) \in \mathcal{BC}$, $f = f_{a,b}$ and z be a point in the basin of $\nu = \nu_{a,b}$. Now, every time we iterate we consider that a small random mistake is committed. This way, we obtain a "pseudo"- trajectory $\{z_j\}_{j=0}^{\infty}$ where $z_0 = z$ and for all $j \geq 1$ each z_j is a random variable supported on a small neighborhood of $f(z_{j-1})$ (one may suppose that z_j is uniformly distributed in a ball of radius ϵ around $f(z_{j-1})$, for small $\epsilon > 0$). *Stochastic stability* means that as long as the noise level is small (i.e. ϵ is small) then for every continuous $\phi : \mathbb{R}^2 \to \mathbb{R}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(z_j) \quad \text{is close to} \quad \int \phi d\nu.$$

At this point we emphasize the resemblance between the statistical properties proved for Hénon maps and the logistic family. This aspect is certainly related with the fact that the b values are extremely small which make the diffeomorphisms $f_{a,b}$ to present 1-dimensional behavior. This parallelism is evident when we compare the results in the papers concerning the 1-dimensional case [**BC85, BY92, Yo92, BV96**] and their 2-dimensional versions in [**BC91, BY93, BY00, BV06**], which are considerably much harder to prove. Regarding the convergence of physical measures, Thunberg [**Th01**] proved that, in the quadratic family, there are sequences of parameters a_n converging to a Benedicks-Carleson parameter a for which the systems exhibit attracting periodic orbits such that the Dirac measures supported on them converge in the weak* topology to the SRB measure correspondent to the parameter a. Moreover, there are also sequences of the same type such that the Dirac measures do not converge to the SRB measure associated to a. Since both the Dirac measures supported on the attracting periodic orbits and the SRB measures of the Benedicks-Carleson quadratic maps are physical measures, it means that one cannot expect *statistical stability* on a full Lebesgue measure set of parameters. However, the results in [**Fr05**] show that within the Benedicks-Carleson parameter set, which has positive Lebesgue measure, there is strong statistical stability (see also [**Ts96**] and [**RS97**] for similar results). In the 2-dimensional case, Ures [**Ur96**] proved a partial analogue of Thunberg's result. Namely, he showed that the SRB measures $\nu_{a,b}$ corresponding to $(a,b) \in \mathcal{BC}$ can be approximated by Dirac measures supported on sinks. The existence of a 2-dimensional analogous result to the second part of Thunberg's work is unknown to us. However, regarding the *statistical stability* within the Benedicks-Carleson parameters we prove here that for every $f_{a,b}$ with $(a,b) \in \mathcal{BC}$, if we perturb a, b within the Benedicks-Carleson parameter set then time averages of continuous functions keep close. More precisely

THEOREM F. *For each $(a,b) \in \mathcal{BC}$ let $\nu_{a,b}$ denote the SRB measure of $f_{a,b}$. Consider the set $M(\mathbb{R}^2)$ of the Borel probability measures defined in $\mathbb{R}^2$ with the weak* topology. Then the map*

$$\begin{array}{ccc} \mathcal{BC} & \longrightarrow & M(\mathbb{R}^2) \\ (a,b) & \longmapsto & \nu_{a,b} \end{array}$$

is continuous.

2. Insight into the reasoning

We consider a sequence of parameters $(a_n, b_n)_{n\in\mathbb{N}} \in \mathcal{BC}$ converging to $(a_0, b_0) \in \mathcal{BC}$. Let $(\nu_n)_{n\in\mathbb{N}}$ and ν_0 denote the respective SRB measures. Our goal is to show that ν_n converges to ν_0 in the weak* topology. We prove this by showing that every subsequence $(\nu_{n_i})_{i\in\mathbb{N}}$ contains a subsequence convergent to ν_0. Let us give some details on how to find this convergent subsequence.

The main problem we have to overcome is the need of comparing measures supported on different attractors. Our strategy is to look for a common ground where the construction of the SRB measure for every parameter is rooted. To do so, we start by noting that each of these maps admits a horseshoe $\Lambda_{a,b}$ with infinitely many branches and variable return times (we will drop the indices when we refer to properties that apply to all these objects) obtained by intersecting two transversal families of local stable and unstable curves. Besides, Λ intersects each local unstable curve in a positive Lebesgue measure Cantor set, and for each $z \in \Lambda$ it is possible to assign a positive integer $R(z)$ defining the return time function $R : \Lambda \to \mathbb{N}$ which indicates that z returns to Λ after $R(z)$ iterates. The hyperbolic properties of Λ and the good behavior of R allow us to build a Markov extension that organizes the dynamics of these Hénon maps. Thus, one needs to show first that for nearby parameters the corresponding horseshoes are also close. We remark that for each parameter there is not a unique horseshoe with the required properties. Therefore, what we can establish is that for a given parameter (a, b) and a chosen horseshoe $\Lambda_{a,b}$, if we consider a small perturbation (a', b'), then it is possible to build a horseshoe $\Lambda_{a',b'}$ with the desired hyperbolic properties and which is close to $\Lambda_{a,b}$.

These horseshoes play an important role in a construction of the SRB measures that suits our purposes. Actually, $f^R : \Lambda \to \Lambda$ preserves a measure $\tilde{\nu}$ with absolutely continuous conditional measures on local unstable curves with respect to the Lebesgue measure on each curve; the good behavior of the function R ensures that the saturation of $\tilde{\nu}$ is an SRB measure, and by uniqueness it follows that the saturation of $\tilde{\nu}$ is the SRB measure. To prove the continuous dependence of these SRB measures on the parameter, Λ is collapsed along stable curves yielding a quotient space $\bar{\Lambda}$, which can be thought inside a fixed local unstable curve $\hat{\gamma}^u$, and whose elements are represented by the intersection of the corresponding stable curve with $\hat{\gamma}^u$. This way our task is reduced to analyze $\overline{f^R} : \bar{\Lambda} \to \bar{\Lambda}$. This map is piecewise uniformly expanding and its Perron-Frobenius operator has a spectral gap under the usual aperiodicity conditions; so there is an $\overline{f^R}$-invariant density with respect to Lebesgue measure on $\hat{\gamma}^u$. As $\hat{\gamma}^u$ is nearly horizontal, we can think of $\bar{\rho}$ as a function defined on a subset of the x-axis. The advantage of this perspective is that it gives us the desired common domain for these densities, providing the first step in the verification of the continuity.

Therefore, the steps for the construction of the convergent subsequence are the following:

- Fix a parameter $(a_0, b_0) \in \mathcal{BC}$ and a respective horseshoe Λ_0.
- Pick any sequence of parameters $(a_n, b_n) \in \mathcal{BC}$ such that $(a_n, b_n) \to (a_0, b_0)$ as $n \to \infty$ and consider $f_n = f_{a_n,b_n}$ for all $n \in \mathbb{N}_0$.
- Construct for every $n \in \mathbb{N}$ an horseshoe Λ_n adequate to f_n and such that it gets closer to Λ_0 as $n \to \infty$.
- Collapse Λ_n and consider the $\overline{f_n^R}$-invariant densities $\bar{\rho}_n$. Realize them as functions defined on an interval of the x-axis and belonging to a closed disk of L^∞. Apply Banach-Alaoglu Theorem to derive a convergent subsequence $\bar{\rho}_{n_i} \to \bar{\rho}_\infty$.
- Employ a technique used by Bowen in [**Bo75**] to lift the $\overline{f^R}$-invariant measure from the quotient space $\bar{\Lambda}$ to an f^R-invariant measure on the horseshoe Λ. This way we obtain measures $\tilde{\nu}_{n_i}$ and $\tilde{\nu}_\infty$, defined on Λ_{n_i} and Λ_0, respectively.
- Verify that all the measures $\tilde{\nu}_{n_i}$ and $\tilde{\nu}_\infty$ desintegrate into conditional absolutely continuous measures on unstable leaves.
- Saturate the measures $\tilde{\nu}_{n_i}$ and $\tilde{\nu}_\infty$. These saturations are f_{n_i}-invariant and f_0-invariant, respectively, and have absolutely continuous conditional measures on unstable leaves. The uniqueness of the SRB measures ensures that the saturation of $\tilde{\nu}_{n_i}$ is ν_{n_i} (the f_{n_i}-invariant SRB measure) and that of $\tilde{\nu}_\infty$ is ν_0 (the f_0-invariant SRB measure).
- Finally, show that this construction yields $\nu_{n_i} \to \nu_0$ in the weak* topology.

3. Dynamics of Hénon maps on Benedicks-Carleson parameters

In this section we provide information regarding the dynamical properties of the Hénon maps $f = f_{a,b}$, corresponding to the Benedicks-Carleson parameters $(a, b) \in \mathcal{BC}$. We do not intend to give an exhaustive description but rather a brief summary of the most relevant features whose main ideas are scattered through the papers [**BC91, BY93, MV93, BY00**]. We recommend the summary in [**BY93**] and Chapter 4 of [**BDV05**] where the reader can find a comprehensive description of the techniques and results regarding Hénon-like maps, including a revision of the referred papers; both texts inspired our summary. The survey [**LV03**] provides a deep discussion about the exclusion of parameters which are the basis of Benedicks-Carleson results. Concerning the 1-dimensional case we also refer the paper [**Fr05**] in which a description of the Benedicks-Carleson techniques in the phase space setting can be found.

3.1. One-dimensional model. The pioneer work of Jakobson [**Ja81**] establishing the existence of a positive Lebesgue measure set of parameters where the logistic family presents chaotic behavior paved the way for a better understanding of the dynamics beyond the non-hyperbolic case. The analysis of the Hénon maps made by Benedicks and Carleson, triggered by the work of Collet-Eckmann [**CE80a, CE80b**] and Benedicks-Carleson [**BC85**] themselves, was a major breakthrough in that direction. A key idea is the exponential growth of the derivative along the critical orbit, introduced in [**CE83**]. In their remarkable paper [**BC91**], Benedicks and Carleson manage to establish, in a very creative fashion, a parallelism between the estimates for the 1-dimensional quadratic maps and the Hénon maps. This connection supports the use of 1-dimensional language in the present paper and compels us to remind the results in Section 2 of [**BC91**]. In there, it is proved the existence of a positive Lebesgue measure set of parameters, say $\mathcal{B}C_1$, within the family $f_a : [-1,1] \to [-1,1]$, given by $f_a(x) = 1 - ax^2$ verifying

(1) there is $c > 0$ ($c \approx \log 2$) such that $|Df_a^n(f_a(0))| \geq \mathrm{e}^{cn}$ for all $n \geq 0$;
(2) there is a small $\alpha > 0$ such that $|f_a^n(0)| \geq \mathrm{e}^{-\alpha n}$ for all $n \geq 1$.

The idea, roughly speaking, is that while the orbit of the critical point is outside a critical region we have expansion (see Subsection 3.1.1); when it returns we have a serious setback in the expansion but then, by continuity, the orbit repeats its early history regaining expansion on account of (1). To arrange for (1) one has to guarantee that the losses at the returns are not too drastic hence, by parameter elimination, (2) is imposed. The argument is mounted in a very intricate induction scheme that guarantees both the conditions for the parameters that survive the exclusions.

We focus on the maps corresponding to Benedicks-Carleson parameters and study the growth of $Df_a^n(x)$ for $x \in [-1,1]$ and $a \in \mathcal{B}C_1$. For that matter we split the orbit in free periods and bound periods. During the former we are certain that the orbit never visits the critical region. The latter begin when the orbit returns to the critical region and initiates a bound to the critical point, accompanying its early iterates. We describe the behavior of the derivative during these periods in Subsections 3.1.1 and 3.1.2.

The critical region is the interval $(-\delta, \delta)$, where $\delta = \mathrm{e}^{-\Delta} > 0$ is chosen small but much larger than $2 - a$. This region is partitioned into the intervals

$$(-\delta, \delta) = \bigcup_{m \geq \Delta} I_m,$$

where $I_m = (\mathrm{e}^{-(m+1)}, \mathrm{e}^{-m}]$ for $m > 0$ and $I_{-m} = -I_m$ for $m < 0$; then each I_m is further subdivided into m^2 intervals $\{I_{m,j}\}$ of equal length inducing the partition $\mathcal{P}$ of $[-1, 1]$ into

$$[-1, -\delta) \cup \bigcup_{m,j} I_{m,j} \cup (-\delta, 1]. \tag{3.1}$$

Given $J \in \mathcal{P}$, we let nJ denote the interval n times the length of J centered at J.

3.1.1. *Expansion outside the critical region.* There is $c_0 > 0$ and $M_0 \in \mathbb{N}$ such that

(1) If $x, \ldots, f_a^{k-1}(x) \notin (-\delta, \delta)$ and $k \geq M_0$, then $|Df_a^k(x)| \geq \mathrm{e}^{c_0 k}$;
(2) If $x, \ldots, f_a^{k-1}(x) \notin (-\delta, \delta)$ and $f_a^k(x) \in (-\delta, \delta)$, then $|Df_a^k(x)| \geq \mathrm{e}^{c_0 k}$;
(3) If $x, \ldots, f_a^{k-1}(x) \notin (-\delta, \delta)$, then $|Df_a^k(x)| \geq \delta \mathrm{e}^{c_0 k}$.

3.1.2. *Bound period definition and properties.* Let $\beta = 14\alpha$. For $x \in (-\delta, \delta)$ define $p(x)$ to be the largest integer p such that

$$|f_a^k(x) - f_a^k(0)| < \mathrm{e}^{-\beta k}, \qquad \forall k < p. \tag{3.2}$$

Then

(1) $\frac{1}{2}|m| \leq p(x) \leq 3|m|$, for each $x \in I_m$;
(2) $|Df_a^p(x)| \geq \mathrm{e}^{c' p}$, where $c' = \frac{1-4\beta}{3} > 0$.

The orbit of x is said to be bound to the critical point during the period $0 \leq k < p$. We may assume that p is constant on each $I_{m,j}$.

3.1.3. *Distortion of the derivative.* The partition $\mathcal{P}$ is designed so that if $\omega \subset [-1, 1]$ is such that, for all $k < n$, $f^k(\omega) \subset 3J$ for some $J \in \mathcal{P}$, then there exists a constant C independent of ω, n and the parameter so that for every $x, y \in \omega$,

$$\frac{|Df_a^n(x)|}{|Df_a^n(y)|} \leq C.$$

3.1.4. *Derivative estimate.* Suppose that

$$|f_a^j(x)| \geq \delta \mathrm{e}^{-\alpha j}, \qquad \forall j < n. \tag{3.3}$$

Then there is a constant $c_2 > 0$ such that

$$|Df_a^n(x)| \geq \delta \mathrm{e}^{c_2 n}. \tag{3.4}$$

A proof of this fact can be found in [**Fr05**, Section 3] where it is also shown that there is $\kappa > 0$ such that

$$|\{x \in [-1, 1] : |f_a^j(x)| \geq \mathrm{e}^{-\alpha j}, \forall j < n\}| \geq 2 - \mathrm{const}\, \mathrm{e}^{-\kappa n}.$$

As an easy consequence, it is deduced that Lebesgue almost every x has a positive Lyapunov exponent. Moreover, we have a positive Lebesgue measure set of points $x \in [-1, 1]$ satisfying (3.3), and so (3.4), for all $n \in \mathbb{N}$.

3.2. General description of the Hénon attractor. The following facts are elementary for $f = f_{a,b}$ with (a, b) inside an open set of parameters.

Each f has a unique fixed point in the first quadrant $z^* \approx \left(\frac{1}{2}, \frac{1}{2}b\right)$. This fixed point is hyperbolic with an expanding direction presenting a slope of order $-b/2$ and a contractive direction with a slope of approximately 2. The respective eigenvalues are approximately -2 and $b/2$. In [**BC91**] it is shown that if we choose $a_0 < a_1 < 2$ with a_0 sufficiently near 2, then there exists b_0 sufficiently small when compared to $2-a_0$ such that for all $(a,b) \in [a_0,a_1]\times(0,b_0]$, the unstable manifold of z^*, say W, never leaves a bounded region. Moreover, its closure $\overline{W}$ is an attractor in the sense that there is an open neighborhood U of $\overline{W}$ such that for every $z \in U$ we have $f^n(z) \to \overline{W}$ as $n \to \infty$.

3.2.1. *Hyperbolicity outside the critical region.* Let δ be at least as small as in our 1-dimensional analysis and assume that $b_0 \ll 2-a_0 \ll \delta$. The critical region is now $(-\delta,\delta)\times\mathbb{R}$. A simple calculation shows that outside the critical region Df preserves the cones $\{|s(v)| \le \delta\}$ (see [**BY93**] Subsection 1.2.3), where $s(v)$ denotes the slope of the vector v. For $z=(x,y) \notin (-\delta,\delta)\times\mathbb{R}$ and a unit vector v with $s(v) \le \delta$, we have essentially the same estimates as in 1-dimension. That is, there is $c_0>0$ and $M_0 \in \mathbb{N}$ such that

(1) If $z,\dots,f^{k-1}(z) \notin (-\delta,\delta)\times\mathbb{R}$ and $k \ge M_0$ then $|Df^k(z)v| \ge \mathrm{e}^{c_0 k}$;
(2) If $z,\dots,f^{k-1}(z) \notin (-\delta,\delta)\times\mathbb{R}$ and $f^k(z) \in (-\delta,\delta)\times\mathbb{R}$ then $|Df^k(z)v| \ge \mathrm{e}^{c_0 k}$;
(3) If $z,\dots,f^{k-1}(z) \notin (-\delta,\delta)\times\mathbb{R}$ then $|Df^k(z)v| \ge \delta\mathrm{e}^{c_0 k}$.

3.3. The contractive vector field. For $A \in \mathrm{GL}(2,\mathbb{R})$ and a unit vector v, if $v \mapsto |Av|$ is not constant, let $e(A)$ denote the unit vector maximally contracted by A. We will write $e_n(z) := e(Df^n(z))$ whenever it makes sense. Observe that if we have some sort of expansion in z, say $|Df^n(z)v| > 1$ for some vector v, then $e_n(z)$ is defined and $|Df^n(z)e_n(z)| \le b^n$ since $\det(Df^n(z)) = (-b)^n$.

The following general perturbation lemma is stated in [**BY00**] and clarifies the assertions of Lemma 5.5 and Corollary 5.7 in [**BC91**], where the proofs can be found. Given $A_1, A_2, \dots$, we write $A^n := A_n \dots A_1$; all the matrices below are assumed to have determinant equal to b.

Lemma 3.1 (Matrix Perturbation Lemma). *Given $\kappa \gg b$, exists λ with $b \ll \lambda < \min(1,\kappa)$ such that if $A_1 \dots, A_n$, $A_1' \dots, A_n' \in GL(2,\mathbb{R})$ and $v \in \mathbb{R}^2$ satisfy*

$$|A^i v| \ge \kappa^i \qquad \text{and} \qquad \|A_i - A_i'\| < \lambda^i \quad \forall i \le n,$$

then we have, for all $i \le n$:

- $|A'^i v| \ge \frac{1}{2}\kappa^i$;
- $\sphericalangle(A^i v, A'^i v) \le \lambda^{\frac{i}{4}}$.

From the Matrix Perturbation Lemma, it follows that if for some κ and v, we have $|Df^j(z_0)v| \ge \kappa^j$ for all $j \in \{0,\dots,n\}$, then there is a ball of radius $(\lambda/5)^n$ about z_0 on which e_n is defined and $|Df^n e_n| \le 2(b/\kappa)^n$. Assuming that κ is fixed and e_n is defined in a ball B_n around z_0 the following facts hold (see [**BC91**, Section 5], [**BY93**, Section 1.3.4] or [**BY00**, Section 1.5]):

(1) e_1 is defined everywhere and has slope equal to $2ax + \mathcal{O}(b)$;
(2) there is a constant $C>0$ such that for all $z_1, z_2 \in B_n$,

$$|e_n(z_1) - e_n(z_2)| \le C|z_1 - z_2|;$$

(3) for $z_1 = (x_1,y_1)$, $z_2 = (x_2,y_2) \in B_n$ with $|y_1-y_2| \le |x_1-x_2|$

$$|e_n(z_1) - e_n(z_2)| = (2a + \mathcal{O}(b))|x_1 - x_2|;$$

(4) for $m<n$, $|e_n - e_m| \le \mathcal{O}(b^m)$ on B_n.

From this point onward we restrict ourselves to the Hénon maps of the Benedicks-Carleson type, that is, we are considering $f = f_{a,b}$ for $(a, b) \in \mathcal{BC}$.

3.4. Critical points. The cornerstone of Benedicks-Carleson strategy is the critical set in W denoted by $\mathcal{C}$, that plays the role of the critical point 0 in the 1-dimensional model. The critical points correspond to homoclinic tangencies of Pesin stable and unstable manifolds. For $z \in W$, let $\tau(z) \in T_z\mathbb{R}^2$ denote a unit vector tangent to W at z. For each $\zeta \in \mathcal{C}$, the vector $\tau(\zeta)$ is contracted by both forward and backward iterates of the derivative. In fact, we have $\lim_{n\to\infty} e_n(\zeta) = \tau(\zeta)$, which can be thought as the moral equivalent to $Df(0) = 0$ in 1-dimension. The following subsections refer to [**BC91**], mostly Sections 5 and 6 (see also [**BY93**, Section 1.3.1]).

3.4.1. *Rules for the construction of the critical set.* The critical set $\mathcal{C}$ is located in $W \cap (-10b, 10b) \times \mathbb{R}$. There is a unique $z_0 \in \mathcal{C}$ on the roughly horizontal segment of W containing the fixed point z^*. The part of W between $f^2(z_0)$ and $f(z_0)$ is denoted by W_1 and called the leaf of generation 1. Leaves of generation $g \geq 2$ are defined by $W_g := f^{g-1}W_1 \backslash \bigcup_{j\leq g-1} W_j$. We assume that (a, b) is sufficiently near $(2, 0)$ so that $\bigcup_{g\leq 27} W_g$ consists of 2^{26} roughly horizontal segments linked by sharp turns near $x = \pm 1$, $y = 0$, and that $\bigcup_{g\leq 27} W_g \cap (-\delta, \delta) \times \mathbb{R}$ consists of 2^{26} curves whose slope and curvature are $\leq 10b$ – in [**BC91**] such a curve is called $C^2(b)$. In each of them there is a unique critical point

For $g > 27$, assume that all critical points of generation $\leq g - 1$ are already defined. Consider a maximal piece of $C^2(b)$ curve $\gamma \subset W_g$. If γ contains a segment of length $2\varrho^g$ centered at $z = (x, y)$, where ϱ verifies $b \ll \varrho \ll \mathrm{e}^{-72}$, and there is a critical point $\tilde{z} = (\tilde{x}, \tilde{y})$ of generation $\leq g-1$ with $x = \tilde{x}$ and $|y-\tilde{y}| \leq b^{g/540}$, then a unique critical point $z_0 \in \mathcal{C}\cap\gamma$ of generation g is created satisfying the condition $|z_0 - z| \leq |y - \tilde{y}|^{1/2}$. These are the only critical points of generation g.

Observe that the exact position of a critical point is unaccessible since its definition depends on the limiting relation $\lim_{n\to\infty} e_n(\zeta) = \tau(\zeta)$. So the strategy in [**BC91**] is to produce approximate critical points ζ^n of increasing order which are solutions of the equation $e_n(z) = \tau(z)$. Once an approximate critical point is born, parameters are excluded to ensure that a critical point $\zeta \in \mathcal{C}$ is created nearby. Moreover, $|\zeta^n - \zeta| = \mathcal{O}(b^n)$.

3.4.2. *Dynamical properties of the critical set.* The parameter exclusion procedure leading to $\mathcal{BC}$ is designed so that every $z \in \mathcal{C}$ has the following properties:

- there is $c \approx \log 2$ and C independent of b such that for all $n \in \mathbb{N}_0$,
$$|Df^n(z)\binom{0}{1}| \geq \mathrm{e}^{cn} \quad \text{and} \quad |Df^n(z)\tau| \leq (Cb)^n; \tag{UH}$$
- there is a small number $\alpha > 0$, say $\alpha = 10^{-6}$, such that for all $n \in \mathbb{N}$
$$\mathrm{dist}(f^n(z), \mathcal{C}) \geq \mathrm{e}^{-\alpha n}. \tag{BA}$$

The precise meaning of "dist" in the last equation will be described in Section 3.5.1. The uniform hyperbolicity expressed in (UH) is analogous to condition (1) in Subsection 3.1 while the basic assumption stated in (BA) is the surface analogue to condition (2) of the 1-dimensional model.

One of the reasons why the Benedicks-Carleson proof is so involved is that in order to e_n be defined in the vicinity of critical points, one has to require some amount of hyperbolicity which is exactly what one wants to achieve (see (UH) above). This difficulty is overcome by working with finite time approximations and imposing slow recurrence in a very intricate induction scheme. Once an approximate critical point ζ^n of order n is designated one studies its orbit. When it comes near $\mathcal{C}$, there is a near-interchange of stable and unstable directions – hence a setback in hyperbolicity. But then the orbit of ζ^n follows for some time the orbit of some $\tilde{\zeta} \in \mathcal{C}$ of earlier generation and regains hyperbolicity on account of (UH) for $\tilde{\zeta}$. To arrange (UH) at time $n+1$ for ζ^n, it is necessary to keep the orbits from switching stable and unstable directions too fast, so by parameter exclusion we impose (BA). At this stage it is possible to define e_{n+1} and thus find a critical point approximation of order $n+1$, denoted by ζ^{n+1}. The information is updated and the process is repeated. Fortunately, a positive Lebesgue measure set of parameters survives the exclusions.

3.5. Binding to critical points. The critical point 0 in the 1-dimensional context plays a dual role. Firstly, the distance to the critical point is a measure of the norm of the derivative, which is the reason why a recurrence condition like (2) of Subsection 3.1 can be used to bound the loss of expansion when an orbit comes near the critical point and to obtain the exponential growth expressed in (1) of Subsection 3.1. Secondly, during the bound, period information of the early iterates of the critical point is passed through continuity to the points returning to the critical region. In order to replicate this in the Hénon family, for every return time n of the orbit of $z \in W$ (z may belong to $\mathcal{C}$) we must associate a suitable binding critical point for $f^n(z)$ so that we can have some meaning of the distance of $f^n(z)$ to the critical set. The suitability depends on the validity of two requirements: tangential position and correct splitting.

3.5.1. *Tangential position and distance to the critical set.* Let $z \in W$ and n be one of its return time to the critical region. Let $\zeta \in \mathcal{C}$. Essentially we say that $f^n(z)$ is in tangential position with respect to ζ if its horizontal distance to ζ is much larger than the vertical distance. In fact we will use the notion of generalized tangential positions introduced in [**BY93**, Section 1.6.2] instead of the original one from [**BC91**] (see [**BY93**, Section 1.4.1]). For $z \in W$ we say that (x', y') is the natural coordinate system at z if $(0,0)$ is at z, the x' axis is aligned with $\tau(z)$ and the y' axis with $\tau(z)^{\perp}$.

Definition 3.2. Let $c > 0$ be a small number much less than $2a$, say $c = 10^{-2}$, and let $\zeta \in \mathcal{C}$. A point z is said to be in *tangential position* with respect to ζ, if $z = (x', y')$ with $|y'| \leq cx'^2$, in the natural coordinate system at ζ.

In [**BC91**, Section 7.2] it is arranged that for every $\zeta \in \mathcal{C}$ and any n-th return to the critical region, there is a critical point $\hat{\zeta}$ of earlier generation with respect to which $f^n(\zeta)$ is in tangential position. This is done through an argument known as the capture procedure (see also [**BY93**, Section 2.2.2]) which essentially consists in showing that when a critical orbit $\zeta \in \mathcal{C}$ experiences a free return at time n, then $f^n(\zeta)$ is surrounded by a fairly regular collection of $C^2(b)$ segments $\{\gamma_j\}$ of W which are relatively long and of earlier generations. In fact, we have $\mathrm{gen}(\gamma_j) \approx 3^j$, $\mathrm{length}(\gamma_j) \approx \varrho^{3^j}$ and $\mathrm{dist}(f^n(z), \gamma_j) \approx b^{3^j}$, where $3^j < \theta n$ and $\theta \approx \frac{1}{|\log b|}$. Some (maybe all) of these captured segments will have critical points and most locations of $f^n(\zeta)$ will be in tangential position with respect to one of these critical points. Bad locations of $f^n(\zeta)$ correspond to deleted parameters. This is another subtlety of Benedicks-Carleson proof: every time a critical point is created it causes a certain amount of parameters to be discarded so we cannot afford to have too many critical points; however, we must have enough critical points so that a convenient one, in tangential position, may be found every time a return occurs.

In [**BY93**] it is shown that this kind of control when a critical orbit returns can be extended to all points in W. Thus, for any return of the orbit of $z \in W$ to the critical region there is an available binding critical point with respect to which the tangential position requirement holds. In fact [**BY93**, Lemma 7] guarantees that one can systematically assign to each maximal free segment $\gamma \subset W$ intersecting the critical region a critical point $\tilde{z}(\gamma)$ with respect to which each $z \in \gamma$ are in tangential position. When the orbit of $z \in W$ returns to the critical region, say at time n, we denote by $z(f^n(z)) \in \mathcal{C}$ a critical point with respect to which $f^n(z)$ is in tangential position.

These facts lead us to the notion of distance to the critical set. We do not intend to give a formal definition but rather introduce a concept that gives an indication of closeness to the critical set. In [**BC91**] and [**BY93**] two different perspectives of distance to the critical set have been introduced. In [**BY00**, Section 2] this notion is cleaned up and these two different perspectives are seen to translate essentially the same geometrical facts. Let $z \in W$. If $z = (x, y) \notin (-\delta, \delta) \times \mathbb{R}$ we consider that $\mathrm{dist}(z, \mathcal{C}) = |x|$; if $z \in (-\delta, \delta) \times \mathbb{R}$ then we pick any critical point $\zeta \in \mathcal{C}$ with respect to which z is in tangential position and let $\mathrm{dist}(z, \mathcal{C}) = |z - \zeta|$. In order to this notion make sense one has to verify that if $\hat{\zeta} \in \mathcal{C}$ is a different critical point with respect to which z is also in tangential position then $|z - \zeta| \approx |z - \hat{\zeta}|$. This is exactly the content of [**BY00**, Lemma 1'], where it is proved that $|z - \zeta|/|z - \hat{\zeta}| = 1 + \mathcal{O}(\max(b, d^2))$, for $d = \min(|z - \zeta|, |z - \hat{\zeta}|)$. As observed in [**BY00**] for a better understanding of the distance of a given point $z \in W \cap (-\delta, \delta) \times \mathbb{R}$ to the critical set, one should look at the angle between $\tau(z)$ and $e_m(z)$, the most contracted vector at z of a convenient order m. The reason for this is that, at the critical points, this angle is extremely close to 0; actually it tends to 0 if we let m go to infinity.

3.5.2. *Bound period and fold period.* Let $z \in W \cap (-\delta, \delta) \times \mathbb{R}$ be in tangential position with respect to $\zeta \in \mathcal{C}$. Then z initiates a binding to ζ of length p, where $p = p(z, \zeta)$ is the largest k such that

$$|f^j(z) - f^j(\zeta)| < \mathrm{e}^{-\beta j}, \qquad \forall j < k$$

where $\beta = 14\alpha$. We say that in the next p iterates, z is bounded to ζ. It is convenient to modify slightly the above definition of p so that the bound periods become nested. This means that if the orbit of z returns to the critical region before p then the bound period initiated at that time must cease before the end of the bound relation to ζ. This is done in [**BC91**, Section 6.2]. It is further required that if the bound relation between z and ζ is still in effect at time n, which is a return time for both, then $z(f^n(\zeta)) = z(f^n(z))$.

An additional complication arises in the Hénon maps: the folding. To illustrate it, let $\gamma \subset W$ be a $C^2(b)$ segment containing a critical point ζ. The practically horizontal vector $\tau(\zeta)$ will be sent by Df into an approximately vertical direction, which is the typical contracting direction of the system, and will be contracted forever. After few iterations γ develops very sharp bends at the iterates of ζ, which induce an unstable setting near the bends. In fact, if we pick a point $z \in \gamma$ very close to ζ, its iterates diverge very fast from the bends which means that after some time, say n, depending on how close z and ζ are, the vector $\tau(f^n(z))$ will be practically aligned with the horizontal direction again, which, on the contrary, is the typical expanding direction of the system. The interval of time that the tangent direction takes to be horizontal again is called the fold period.

The actual definition of fold period is given in [**BC91**, Sections 6.2 and 6.3]; here, we stick to the previous heuristic motivation and to the following properties. If $z \in W$ has a return at time n, the fold period of $f^n(z)$ with respect to $z(f^n(z)) \in \mathcal{C}$ is a positive integer $l = l(f^n(z), z(f^n(z)))$ such that

(1) $2m \leq l \leq 3m$, where $(5b)^m \leq |f^n(z) - z(f^n(z))| \leq (5b)^{m-1}$;
(2) $l/p \leq \text{const}/|\log b|$, that is the fold period associated to a return is very short when compared to the bound period initiated at that time.

3.5.3. *Correct splitting and controlled orbits.* In order to duplicate the 1-dimensional behavior not only one assigns a binding critical point every time a return to the critical region occurs but also one would like to guarantee that the loss of hiperbolicity due to the return is in some sense proportional to the distance to the critical set. This is achieved through the notion of correct splitting.

Definition 3.3. Let $z \in W$, $v \in T_z\mathbb{R}^2$, $n \in \mathbb{N}$ be a return time for z and consider $z(f^n(z)) \in \mathcal{C}$ with respect to which $f^n(z)$ is in tangential position. We say that the vector $Df^n(z)v$ *splits correctly* with respect to $z(f^n(z)) \in \mathcal{C}$ if and only if we have that

$$3|f^n(z) - z(f^n(z))| \leq \sphericalangle(Df^n(z)v, e_l(f_n(z))) \leq 5|f^n(z) - z(f^n(z))|,$$

where l is the fold period associated to the return.

Now we are in condition of defining controlled orbits.

Definition 3.4. Let $z \in W$ and $v \in T_z\mathbb{R}^2$ and $N \in \mathbb{N}$. We say that the pair (z, v) is controlled on the time interval $[0, N)$ if for every return $n \in [0, N)$ of the orbit of z to the critical region, there is $z(f^n(z)) \in \mathcal{C}$ with respect to which $f^n(z)$ is in tangential position and $Df^n(z)v$ splits correctly with respect to $z(f^n(z)) \in \mathcal{C}$. We say that the pair (z, v) is *controlled* during the time interval $[0, \infty)$ if it is controlled on $[0, N)$ for every $N \in \mathbb{N}$.

One of the most important properties of f proved in [**BC91**] is that for every $\zeta \in \mathcal{C}$, the pair $(\zeta, \binom{0}{1})$ is controlled during the time interval $[0, \infty)$. This fact supports the validity of the 1-dimensional estimates in the surface case.

We say that the pair (z, v) is controlled on $[j, 0)$ with $-\infty < j < 0$, if $(f^j(z), Df^j(z)v)$ is controlled on $[0, -j)$ and that (z, v) is controlled on $(-\infty, 0)$ if it is controlled on $[j, 0)$ for all $j < 0$. In [**BY93**, Proposition 1] it is proved that if the orbit of $z \in W$ never hits the critical set $\mathcal{C}$ then the pair $(z, \tau(z))$ is controlled in the time interval $(-\infty, \infty)$.

3.6. Dynamics in W. As referred, [**BY93**, Proposition 1] shows that every orbit of $z \in W$ can be controlled using those of $\mathcal{C}$, just as it was done for critical orbits in [**BC91**]. This means that each orbit in W can be organized into free periods and bound periods. To illustrate, consider z belonging to a small segment of W around the fixed point z^*. By definition z is considered to be free at this particular time. The first forward iterates of z are also in a free state , until the first return to the critical region occurs, say at time n. Then since the pair $(z, \tau(z))$ is controlled there is $z(f^n(z))$ with respect to which $f^n(z)$ is in tangential position and $Df^n(z)\tau$ splits correctly. During the next p iterates we say that z is bound to the critical point $z(f^n(z))$. If $f^n(z) \in \mathcal{C}$ then the bound period is infinite;

otherwise, after the time $n+p$ the iterates of z are said to be in free state once again and history repeats itself.

This division of the orbits into free periods, bound periods and the special design of the control of orbits through the tangential position and correct splitting requirements allowed [**BY93**] to recover the one dimensional estimates. In fact, the loss of expansion at the returns is somehow proportional to the distance to the binding critical point and it is completely overcome at the end of the bound period.

The following estimates, unless otherwise mentioned, are proved in [**BY93**, Corollary 1].

(1) Free period estimates.
   (a) Every free segment γ has slope less than $2b/\delta$, and $\gamma \cap (-\delta,\delta)\times\mathbb{R}$ is a $C^2(b)$ curve (Lemmas 1 and 2 of [**BY93**]);
   (b) There is $c_0 > 0$ and $M_0 \in \mathbb{N}$ such that if z is free and $z,\dots,f^{k-1}(z) \notin (-\delta,\delta)\times\mathbb{R}$ with $k \geq M_0$ then $|Df^k(z)\tau| \geq \mathrm{e}^{c_0 k}$;
   (c) There is $c_0 > 0$ such that if z is free, $z,\dots,f^{k-1}(z) \notin (-\delta,\delta)\times\mathbb{R}$ and $f^k(z) \in (-\delta,\delta)\times\mathbb{R}$ then $|Df^k(z)\tau| \geq \mathrm{e}^{c_0 k}$.
(2) Bound period estimates.
   There is $c \approx \log 2$ such that if $z \in (-\delta,\delta)\times\mathbb{R}$ is free and initiates a binding to $\zeta \in \mathcal{C}$ with bound period p, then
   (a) If $\mathrm{e}^{-m-1} \leq |z-\zeta| \leq \mathrm{e}^{-m}$, then $\frac{1}{2}m \leq p \leq 5m$;
   (b) $|Df^j(z)\tau| \geq |z-\zeta|\mathrm{e}^{cj}$ for $0 < j < p$.
   (c) $|Df^p(z)\tau| \geq \mathrm{e}^{c\frac{p}{3}}$.
(3) Orbits ending in free states.
   There exists $c_1 > \frac{1}{3}\log 2$ such that if $z \in W \cap (-\delta,\delta)\times\mathbb{R}$ is in a free state, then $|Df^{-j}(z)\tau| \leq \mathrm{e}^{-c_1 j}$, for all $j \geq 0$ ([**BY93**, Lemma 3]).

3.6.1. *Derivative estimate.* The next derivative estimate can be found in [**BY00**, Section 1.4]. It is the 2-dimensional analogue to the 1-dimensional derivative estimate expressed in Subsection 3.1.4. Consider $n \in \mathbb{N}$ and a point z belonging to a free segment of W and satisfying, for every $j < n$

$$\operatorname{dist}(f^j(z),\mathcal{C}) \geq \delta\mathrm{e}^{-\alpha j}. \tag{SA}$$

Then there is a constant $c_2 > 0$ such that

$$|Df^n(z)\tau| \geq \delta\mathrm{e}^{c_2 n}. \tag{EE}$$

Essentially this estimate is saying that if we have slow approximation to the critical set (or, in other words, a (BA) type property), then we have exponential expansion along the tangent direction to W.

3.6.2. *Bookkeeping and bounded distortion.* For $x_0 \in \mathbb{R}$, we let $\mathcal{P}_{[x_0]}$ denote the partition $\mathcal{P}$ defined in (3.1) after being translated from 0 to x_0. Similarly, if γ is a roughly horizontal curve in $\mathbb{R}^2$ and $z_0 = (x_0,y_0) \in \gamma$, we let $\mathcal{P}_{[z_0]}$ denote the partition of γ that projects vertically onto $\mathcal{P}_{[x_0]}$ on the x-axis. Once γ and z_0 are specified, we will use $I_{m,j}$ to denote the corresponding subsegment of γ.

Let $\gamma \subset (-\delta,\delta)\times\mathbb{R}$ be a segment of W. We assume that the entire segment has the same itinerary up to time n in the sense that:

- all $z \in \gamma$ are bound or free simultaneously at any moment;

- if $0 = t_0 < t_1 < \ldots < t_q$ are the consecutive free return times before n, then for all $j \le q$ the entire segment $f^{t_j}\gamma$ has a common binding point $\zeta_j \in \mathcal{C}$ and $f^{t_j}\gamma \subset 5I^j_{m,k}$ for some $I^j_{m,k} \in \mathcal{P}_{[\zeta_j]}$.

Then there exists $C_1 > 0$ independent of γ and n such that for all $z_1, z_2 \in \gamma$

$$\frac{|Df^n(z_1)\tau|}{|Df^n(z_2)\tau|} \le C_1.$$

This result can be found in [**BY93**, Proposition 2].

3.7. Dynamical and geometric description of the critical set. The construction of the critical set seems to be done according to a quite discretionary set of rules. However, as observed in [**BY93**] there are certain intrinsic characterizations of $\mathcal{C}$. Corollary 1 of [**BY93**] gives the following dynamical description of $\mathcal{C}$. Let $z \in W$. Then

$$z \text{ lies on a critical orbit} \quad \Leftrightarrow \quad \limsup_{n\to\infty} |Df^n(z)\tau| < \infty \quad \Leftrightarrow \quad \limsup_{n\to\infty} |Df^n(z)\tau| = 0.$$

In fact, $z \in \mathcal{C}$ if and only if $|Df^j(z)\tau| \le \mathrm{e}^{-c_1|j|}$, for all $j \in \mathbb{Z}$, i.e. the critical points correspond to the tangencies of Pesin stable manifolds with W which endow an homoclinic type behavior.

The critical set $\mathcal{C}$ has also a nice geometric characterization. Given $\zeta \in W$, $\kappa(\zeta)$ denotes the curvature of W at ζ. From the curvature computations in [**BC91**, Section 7.6] (see also [**BY93**, Section 2.1.3]) one gets that

$$z \in \mathcal{C} \quad \Leftrightarrow \quad \kappa(z) \ll 1 \quad \text{and} \quad \kappa(f^n(z)) > b^{-n},\ \forall n \in \mathbb{N}.$$

This means that one can look at the critical points as the points that are sent into the folds of W.

3.8. SRB measures. We begin by giving a formal definition of *Sinai-Ruelle-Bowen measures* (SRB *measures*). Let $f : M \to M$ be an arbitrary C^2 diffeomorphism of a finite dimensional manifold and let μ be an f invariant probability measure on M with compact support. We will assume that μ-a.e. point, there is a strictly positive Lyapunov exponent. Under these conditions, the unstable manifold theorem of Pesin [**Pe78**] or Ruelle [**Ru79**] asserts that passing through μ-a.e. z there is an unstable manifold which we denote by $\gamma^u(z)$.

A measurable partition $\mathcal{L}$ of M is said to be *subordinate* to γ^u (with respect to the measure μ) if at μ-a.e. z, $\mathcal{L}(z)$ is contained in $\gamma^u(z)$ and contains an open neighborhood of z in $\gamma^u(z)$, where $\mathcal{L}(z)$ denotes the atom of $\mathcal{L}$ containing z. By Rokhlin's desintegration theorem there exists a family $\{\mu_z^{\mathcal{L}}\}$ of conditional measures of μ with respect to the partition $\mathcal{L}$ (see for example [**BDV05**, Appendixes C.4 and C.6]).

DEFINITION 3.5. Let $f : M \to M$ and μ be as above. We say that μ is an SRB probability measure if for every measurable partition $\mathcal{L}$ subordinate to γ^u, we have that $\{\mu_z^{\mathcal{L}}\}$ is absolutely continuous with respect to Lebesgue measure in $\gamma^u(z)$ for μ-a.e. z.

In [**BY93**] it is proved that $f_{a,b}$ admits an SRB measure $\nu_{a,b}$, for every $(a, b) \in \mathcal{BC}$. Moreover, $\nu_{a,b}$ is unique (hence ergodic), it is a physical measure, its support is $\overline{W}_{a,b}$ and $(f_{a,b}, \nu_{a,b})$ is isomorphic to a Bernoulli shift.

4. A horseshoe with positive measure

In order to obtain decay of correlations for Hénon maps of the Benedicks-Carleson type, Benedicks and Young build, in [**BY00**], a set Λ of positive SRB-measure with good hyperbolic properties. Λ has hyperbolic product structure and it may be looked at as a horseshoe with infinitely many branches and unbounded return times; it is obtained by intersecting two families of C^1 stable and unstable curves. Dynamically, Λ can be decomposed into a countable union of s-sublattices, denoted Ξ_i, crossing Λ completely in the stable direction, with a Markov type property: for each Ξ_i there is $R_i \in \mathbb{N}$ such that $f^{R_i}(\Xi_i)$ is an u-sublattice of Λ, crossing Λ completely in the unstable direction. The intersection of Λ with every unstable leaf is a positive 1-dimensional Lebesgue measure set. Before continuing with an overview of the construction of such horseshoes, we mention that Young [**Yo98**] has extended the argument in [**BY00**] to a wider setting and observed that similar horseshoes can be found in other situations. We will refer to [**Yo98**] for certain facts not specific to Hénon maps.

Let Γ^u and Γ^s be two families of C^1 curves in $\mathbb{R}^2$ such that

- the curves in Γ^u, respectively Γ^s, are pairwise disjoint;
- every $\gamma^u \in \Gamma^u$ meets every $\gamma^s \in \Gamma^s$ in exactly one point;
- there is a minimum angle between γ^u and γ^s at the point of intersection.

Then we define the lattice associated to Γ^u and Γ^s by

$$\Lambda := \{\gamma^u \cap \gamma^s : \gamma^u \in \Gamma^u, \gamma^s \in \Gamma^s\}.$$

For $z \in \Lambda$ let $\gamma^u(z)$ and $\gamma^s(z)$ denote the curves in Γ^u and Γ^s containing z, respectively.

We say that Ξ is an *s-sublattice* (resp. *u-sublattice*) of Λ if Λ and Ξ have a common defining family Γ^u (resp. Γ^s) and the defining family Γ^s (resp. Γ^u) of Λ contains that of Ξ. A subset $Q \subset \mathbb{R}^2$ is said to be the *rectangle spanned* by Λ if $\Lambda \subset Q$ and ∂Q is made up of two curves from Γ^s and two from Γ^u.

Next, we state Proposition A from [**BY00**] which asserts the existence of two lattices Λ^+ and Λ^- with essentially the same properties; for notation simplicity statements about Λ apply to both Λ^+ and Λ^-.

PROPOSITION 4.1. *There are two lattices Λ^+ and Λ^- in $\mathbb{R}^2$ with the following properties.*

(1) *(Topological Structure) Λ is the disjoint union of s-sublattices Ξ_i, $i = 1, 2\ldots$, where for each i, exists $R_i \in \mathbb{N}$ such that $f^{R_i}(\Xi_i)$ is a u-sublattice of Λ^+ or Λ^-.*

(2) *(Hyperbolic estimates)*
   (a) *Every $\gamma^u \in \Gamma^u$ is a $C^2(b)$ curve; and exists $\lambda_1 > 0$ such that for all $z \in \gamma^u \cap Q_i$,*

$$|Df^{R_i}(z)\tau| \geq \lambda_1^{R_i},$$

   where τ is the unit tangent vector to γ^u at z and Q_i is the rectangle spanned by Ξ_i.

   (b) *For all $z \in \Lambda$, $\zeta \in \gamma^s(z)$ and $j \geq 1$ we have*

$$|f^j(z) - f^j(\zeta)| < Cb^j,$$

(3) *(Measure estimate) $Leb(\Lambda \cap \gamma^u) > 0$, $\forall \gamma^u \in \Gamma^u$.*

(4) *(Return time estimates) Let* $R : \Lambda \to \mathbb{N}$ *be defined by* $R(z) = R_i$ *for* $z \in \Xi_i$. *Then there are* $C_0 > 0$ *and* $\theta_0 < 1$ *such that on every* γ^u

$$Leb\{z \in \gamma^u : R(z) \geq n\} \leq C_0\theta_0^n, \qquad \forall n \geq 1.$$

The proof of Proposition 4.1 can be found in Sections 3 and 4 of [**BY00**]. Since we will need to prove the closeness of these horseshoes for nearby Benedicks-Carleson parameters and this involves slight modifications in the construction of the horseshoes itselves, we will include, for the sake of completeness, the basic ideas of the major steps leading to Λ.

Consider the leaf of first generation W_1 and the unique critical point $z_0 = (x_0, y_0) \in \mathcal{C}$ on it. Take the two outermost intervals of the partition $\mathcal{P}_{[x_0]}$ as in Subsection 3.6.2 and denote them by Ω_0^+ and Ω_0^-; they support the construction of the lattices Λ^+ and Λ^-, respectively. Again we use Ω_0 to simplify notation and statements regarding to it apply to both Ω_0^+ and Ω_0^-.

Let $h : \Omega_0 \to \mathbb{R}$ be a function whose graph is the leaf of first generation W_1, when restricted to the set $\Omega_0 \times \mathbb{R}$ and $H : \Omega_0 \to W_1$ be given by $H(x) = (x, h(x))$.

4.1. Leading Cantor sets. The first step is to build the Cantor set that constitutes the intersection of Λ with the leaf of first generation W_1. We build a sequence $\Omega_0 \supset \Omega_1 \supset \Omega_2 \ldots$ such that for every $z \in H(\Omega_n)$, $\mathrm{dist}(f^j(z), \mathcal{C}) \geq \delta e^{-\alpha j}$, for all $j \in \{1, 2, \ldots, n\}$. This is done by excluding from Ω_{n-1} the points that at step n fail to satisfy the condition $\mathrm{dist}(f^n(H(x)), \mathcal{C}) \geq \delta e^{-\alpha n}$. Then we define the Cantor set $\Omega_\infty = \bigcap_{n\in\mathbb{N}} \Omega_n$. By the derivative estimate in Subsection 3.6.1, on $H(\Omega_\infty)$, the condition (SA) holds and thus $|Df^n(z)\tau(z)| > e^{c_1 n}$, for all $n \in \mathbb{N}$.

Remark 4.2. We observe that there is a difference in the notation used in [**BY00**]: in here, the sets Ω_n (with $n = 0, 1, \ldots, \infty$) are the vertical projections in the x-axis of the corresponding sets in [**BY00**].

Remark 4.3. We note that the procedure leading to Ω_∞ is not unique. Ω_∞ is obtained by successive exclusions of points from the set Ω_0. These exclusions are made according to the distance to a suitable binding critical point every time we have a free return to $[-\delta, \delta] \times \mathbb{R}$. Certainly, the choice for the binding critical point in not unique which leads to different exclusions. However, by the results referred in Subsection 3.5.1 all suitable binding points are essentially the same and these possible differences in the exclusions are insignificant in terms of the properties we want Ω_∞ to have: slow approximation to the critical set and expansion along the tangent direction to W.

4.2. Construction of long stable leaves. The next step towards building Λ involves the construction of long stable curves, $\gamma^s(z)$, at every $z \in H(\Omega_\infty)$. This is done in Lemma 2 of [**BY00**]; let us review the inductive procedure used there.

The contracting vector field of order 1, e_1, is defined everywhere so we may consider the rectangle $Q_0(\omega_0) = \cup_{z\in\omega_0}\gamma_1(z)$, where $\gamma_1(z)$ denotes the e_1-integral curve segment $10b$ long to each side of $z \in \omega_0$ and $\omega_0 = H(\Omega_0)$. Let also $Q_0^1(\omega_0)$ denote the Cb-neighborhood of $Q_0(\omega_0)$ in $\mathbb{R}^2$. We observe that by (1) of Section 3.3 the γ_1 curves in $Q_0(\omega_0)$ have slopes $\approx \pm 2a\delta$ depending on whether Ω_0 refers to Ω_0^+ or Ω_0^-.

Suppose that for every connected component $\omega \in H(\Omega_{n-1})$ we have a strip foliated by integral curves of e_n, $Q_{n-1}(\omega) = \cup_{z\in\omega}\gamma_n(z)$, where $\gamma_n(z)$ denotes the e_n-integral curve segment $10b$ long to each side of $z \in \omega$. From [**BY00**, Section 3.3] one deduces that the vector field e_{n+1} is defined on a $3(Cb)^n$ neighborhood of each curve $\gamma_n(z)$, if $z \in H(\Omega_n)$. Consider the $(Cb)^n$- neighborhood of $Q_{n-1}(\omega)$ in $\mathbb{R}^2$, denoted by $Q^1_{n-1}(\omega)$. If $\tilde{\omega} \subset \omega$ is a connected component of $H(\Omega_n)$ then $Q_n(\tilde{\omega}) = \cup_{z\in\tilde{\omega}}\gamma_{n+1}(z)$ is defined and

$$Q^1_n(\tilde{\omega}) \subset Q^1_{n-1}(\omega), \tag{4.1}$$

where $Q^1_n(\tilde{\omega})$ is a $(Cb)^{n+1}$- neighborhood of $Q_n(\tilde{\omega})$ in $\mathbb{R}^2$.

To fix notation, for some $\omega \subset H(\Omega_0)$ and $n \in \mathbb{N}$, when defined, $Q_n(\omega) = \cup_{z\in\omega}\gamma_{n+1}(z)$ denotes a rectangle foliated by integral curves of e_{n+1} passing through $z \in \omega$ and $10b$ long to each side of z. Besides, $Q^1_n(\omega)$ is a $(Cb)^{n+1}$- neighborhood of $Q_n(\omega)$ in $\mathbb{R}^2$.

To finish the construction of $\gamma^s(z)$, for each $z \in H(\Omega_\infty)$, take the sequence of connected components $\omega_i \subset H(\Omega_i)$ containing z. We have $\{z\} = \cap_i\omega_i$. Let z_n denote the right end point of ω_{n-1}. Then $\gamma_n(z_n)$ converges in the C^1- norm to a C^1-curve $\gamma^s(z)$ with the properties stated in Proposition 4.1. The curve $\gamma_n(z_n)$ acts as an approximate long stable leaf of order n. Note that the choice of the right end point is quite arbitrary; in fact any curve $\gamma_n(\zeta)$ with $\zeta \in \omega_{n-1}$ suits as an approximate stable leaf of order n.

4.3. The families Γ^u and Γ^s. The final step in the construction of Λ is to specify the families Γ^u and Γ^s. Set

$$\Gamma^s := \{\gamma^s(z) : z \in \Omega_\infty\},$$

where $\gamma^s(z)$ is obtained as described in Subsection 4.2. Consider $\tilde{\Gamma}^u := \{\gamma \subset W : \gamma \text{ is a } C^2(b) \text{ segment connecting } \partial^s Q_0\}$, where Q_0 is the rectangle spanned by the family of curves Γ^s, i.e., $Q_0 \supset \bigcup_{z\in H(\Omega_\infty)} \gamma^s(z)$ and ∂Q_0 is made up from two curves of Γ^s. Set

$$\Gamma^u := \{\gamma : \gamma \text{ is the pointwise limit of a sequence in } \tilde{\Gamma}^u\}.$$

4.4. The s-sublattices and the return times. Recall that we are interested in two lattices Λ^+ and Λ^-. Therefore, when we refer to return times we mean return times from the set $\Lambda^+ \cup \Lambda^-$ to itself; in particular, a point in Λ^+ may return to Λ^+ or Λ^-. However, in order to simplify we just write Λ.

We anticipate that the return time function $R : \Lambda \to \mathbb{N}$ is constant in each $\gamma^s \in \Gamma^s$, so R needs only to be defined in $\Lambda \cap H(\Omega_0) = H(\Omega_\infty)$. Moreover, since $H : \Omega_0 \to W_1$ is a bijection we may also look at R as being defined on Ω_∞. We will build partitions on subsets of Ω_0 and use 1-dimensional language. For example, $f^n(z) = \zeta$ for $z, \zeta \in H(\Omega_\infty)$ means that $f^n(z) \in \gamma^s(\zeta)$; similarly, for subsegments $\omega, \omega^* \subset H(\Omega_0)$, $f^n(\omega) = \omega^*$ means that $f^n(\omega) \cap \Lambda$, when slid along γ^s curves back to $H(\Omega_0)$, gives exactly $\omega^* \cap \Lambda$. For an interval $I \subset \Omega_{n-1}$ such that $f^n(H(I))$ intersects the critical region, $\mathcal{P}|f^n(H(I))$ refers to $\mathcal{P}_{[\tilde{z}]}$ where $\tilde{z} \in \mathcal{C}$ is a suitable binding critical point for all $f^n(H(I))$ whose existence is a consequence of Lemma 7 from [**BY93**], mentioned in Subsection 3.5.1.

We will construct sets $\tilde{\Omega}_n \subset \Omega_n$ and partitions $\tilde{\mathcal{P}}_n$ of $\tilde{\Omega}_n$ so that $\tilde{\Omega}_0 \supset \tilde{\Omega}_1 \supset \tilde{\Omega}_2 \ldots$ and $z \in H(\tilde{\Omega}_{n-1} \setminus \tilde{\Omega}_n)$ if and only if $R(z) = n$. Let $\hat{\mathcal{P}}$ be the partition of $H(\Omega_0 \setminus \Omega_\infty)$ into connected components. In what follows $\mathcal{A} \vee \mathcal{B}$ is the join of the partitions $\mathcal{A}$ and $\mathcal{B}$, that is $\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

Definition 4.4. An interval $I \in \Omega_n$ is said to make a *regular return* to Ω_0 at time n if

(i) all of $f^n(H(I))$ is free;
(ii) $f^n(H(I)) \supset 3H(\Omega_0)$.

Remark 4.5. The constant 3 in the definition of regular return is quite arbitrary. In fact its purpose is to guarantee that $f^n(H(I))$ traverses Q_0 by wide margins. When n is a regular return of a certain segment I for a fixed parameter it may happen that n does not classify as a regular return of a perturbed parameter even though the image of I after n iterates by the perturbed dynamics crosses Q_0 by wide margins. We overcome this detail simply by considering that if (ii) holds with 2 instead of 3 for any perturbed parameter then we consider n as a regular return for the perturbed dynamics. Observe that no harm results from making this assumption since it is still guaranteed that Q_0 is traversed by wide margins.

4.4.1. *Rules for defining $\tilde{\Omega}_n$, $\tilde{\mathcal{P}}_n$ and R.*

(0) $\tilde{\Omega}_0 = \Omega_0$, $\tilde{\mathcal{P}}_0 = \{\tilde{\Omega}_0\}$.
Consider $I \in \tilde{\mathcal{P}}_{n-1}$.
(1) If I does not make a regular return to Ω_0 at time n, put $I \cap \Omega_n$ into $\tilde{\Omega}_n$ and set $\tilde{\mathcal{P}}_n|(I \cap \Omega_n) = H^{-1}\left((f^{-n}\mathcal{P})|(H(I \cap \Omega_n))\right)$.
(2) If I makes a regular return at time n, we put $\tilde{I} = H^{-1}\left(H(I) \setminus f^{-n}(H(\Omega_\infty))\right) \cap \Omega_n$ in $\tilde{\Omega}_n$, and let $\tilde{\mathcal{P}}_n|\tilde{I} = H^{-1}\left((f^{-n}\mathcal{P} \vee f^{-n}\hat{\mathcal{P}})|H(\tilde{I})\right)$. For $z \in H(I)$ such that $f^n(z) \in H(\Omega_\infty)$, we define $R(z) = n$.
(3) For $z \in H(\cap_{n\in\mathbb{N}_0}\tilde{\Omega}_n)$, set $R(z) = \infty$.

4.4.2. *Definition of the s-sublattices.* Each Ξ_i in Proposition 4.1 is a sublattice corresponding to a subset of $\Lambda \cap W_1$ of the form $f^{-n}(H(\Omega_\infty)) \cap \Lambda \cap H(I)$, where $I \in \tilde{\mathcal{P}}_{n-1}$ makes a regular return at time n. We will use the notation $\Upsilon_{n,j} = H^{-1}\left(f^{-n}(H(\Omega_\infty)) \cap \Lambda \cap H(I)\right)$. Note that $R(H(\Upsilon_{n,j})) = n$ and $\Upsilon_{n,j}$ determines univocally the corresponding s-sublattice. For this reason we allow some imprecision by referring ourselves to $\Upsilon_{n,j}$ as an s-sublattice.

In order to prove the assertions (1) and (2) of Proposition 4.1 one needs to verify that $f^{R_i}(\Xi_i)$ is an u-sublattice which requires to demonstrate that $f^{R_i}(\Xi_i)$ matches completely with Λ in the horizontal direction. If Ξ_i corresponds to some $\Upsilon_{n,j}$, then the matching of the Cantor sets will follow from the inclusion

$$f^n(H(I \cap \Omega_\infty)) \supset H(\Omega_\infty). \tag{4.2}$$

It is obvious that $H(\Omega_\infty) \subset f^n(H(I))$ by definition of regular return. Nevertheless, (4.2) is saying that if $z \in H(I)$ and $f^n(z)$ hits $H(\Omega_\infty)$, after sliding along a γ^s curve, then $z \in H(I) \cap H(\Omega_\infty)$. This is proved in Lemma 3 of [**BY00**]. In particular, we may write $\Upsilon_{n,j} = H^{-1}\left(f^{-n}(H(\Omega_\infty)) \cap H(I)\right)$.

4.5. Reduction to an expanding map. The Hénon maps considered here are perturbations of the map $f_{2,0}(x,y) = (1-2x^2, 0)$ whose action is horizontal. Also, as we have seen, the horizontal direction is typically expanding. This motivates considering the quotient space $\bar{\Lambda}$ obtained by collapsing the stable curves of Λ; i.e. $\bar{\Lambda} = \Lambda/\sim$, where $z \sim z'$ if and only if $z' \in \gamma^s(z)$. We define the natural projection $\bar{\pi} : \Lambda \to \bar{\Lambda}$ given by

$\bar{\pi}(z) = \gamma^s(z)$. As implied by assertion (1) of Proposition 4.1, $f^R : \Lambda \to \Lambda$ takes γ^s leaves to γ^s leaves (see Lemma 2 of [**BY00**] for a proof). Thus, we may define the quotient map $\overline{f^R} : \bar{\Lambda} \to \bar{\Lambda}$. Observe that each $\bar{\Xi}_i$ is sent by $\overline{f^R}$ homeomorphically onto $\bar{\Lambda}$. Besides we may define a *reference measure* $\bar{m}$ on $\bar{\Lambda}$, whose representative on each $\gamma^u \in \Gamma^u$ is a finite measure equivalent to the restriction of the 1-dimensional Lebesgue measure on $\gamma^u \cap \Lambda$ and denoted by m_{γ^u}.

One can look at $\overline{f^R}$ as an expanding Markov map (see Proposition B of [**BY00**] for precise statements and proofs). Moreover, the corresponding transfer operator, relative to the reference measure $\bar{m}$, has a spectral gap (see Section 3 of [**Yo98**], specially Proposition A). It follows that $\overline{f^R}$ has an absolutely continuous invariant measure given by $\bar{\nu} = \bar{\rho} d\bar{m}$, with $M^{-1} \leq \bar{\rho} \leq M$ for some $M > 0$ (see [**Yo98**, Lemma 2]).

5. Proximity of critical points

In this section we show that up to a fixed generation we have closeness of the critical points for nearby Benedicks-Carleson parameters. This is the content of Proposition 5.3 which summarizes this section. Its proof involves a finite step induction scheme on the generation level. We prepare it by proving first the closeness of critical points of generation 1 in Lemma 5.1. Afterwards, in Lemma 5.2 we obtain the closeness of critical points of higher generations using the information available for lower ones.

Recall that since $f_{a,b}$ is C^∞, then the unstable manifold theorem ensures that W is C^r for any $r > 0$. Moreover, W varies continuously in the C^r topology with the parameters in compact parts. As we are only considering parameters in $\mathcal{BC}$, for each of these dynamics there is a unique critical point $\hat{z}$ of generation 1 situated on the roughly horizontal segment of W containing the fixed point z^*.

LEMMA 5.1. *Let $(a, b) \in \mathcal{BC}$, $\varepsilon > 0$ be given and $\hat{z}$ be the critical point of generation 1 of $f_{a,b}$. There exists a neighborhood $\mathcal{U}$ of (a, b) such that, if $(a', b') \in \mathcal{U} \cap \mathcal{BC}$ and $\hat{z}'$ denotes the critical point of $f_{a',b'}$ of generation 1, then $|\hat{z} - \hat{z}'| < \varepsilon$. Moreover, if $\tau(\hat{z})$ and $\tau(\hat{z}')$ are the unit vectors tangent to W and W' at $\hat{z}$ and $\hat{z}'$ respectively, then $|\tau(\hat{z}) - \tau(\hat{z'})| < \varepsilon$.*

PROOF. Consider the disk $\gamma = W_1 \cap [-10b, 10b] \times \mathbb{R}$. There is a neighborhood $\mathcal{U}$ of (a, b) such that for every $(a', b') \in \mathcal{U}$ there exists a disk $\gamma' \subset W'$ which is ε^2-close to γ in the C^r topology. It is clear that both γ and γ' are $C^2(b)$ curves and there are $\hat{z} \in \gamma$ and $\hat{z}' \in \gamma'$ critical points of $f_{a,b}$ and $f_{a',b'}$ respectively. Our goal is to show that $|\hat{z} - \hat{z}'| < \varepsilon$. The strategy is to pick an approximate critical point $\hat{z}_M$ of $\hat{z}$ and then prove the existence of an approximate critical point $\hat{z}'_M$ of $\hat{z}'$ sufficiently close to $\hat{z}_M$ in order to conclude that, if we choose M large enough, we get the desired closeness of $\hat{z}$ and $\hat{z}'$ (see Figure 1). Take

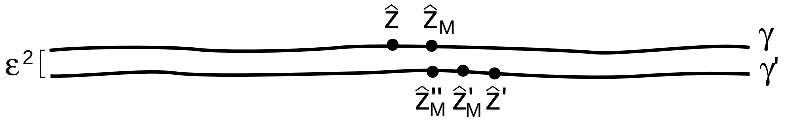

FIGURE 1. Possible configuration of the critical points and their approximates

$M \in \mathbb{N}$ so that $b^M < \varepsilon^2 \leq b^{M-1}$. Let $\hat{z}_M \in \gamma$ be such that $e_M(\hat{z}_M) = \tau(\hat{z}_M)$. Note that

$|\hat{z}-\hat{z}_M| < Cb^M$. Let $\hat{z}''_M \in \gamma'$ be such that $|\hat{z}_M - \hat{z}''_M| < \varepsilon^2$ and $|\tau(\hat{z}_M) - \tau(\hat{z}''_M)| < \varepsilon^2$. Now, $\hat{z}''_M$ may not be the approximate critical point $\hat{z}'_M$ we are looking for, but we will show that it is very close to $\hat{z}'_M$. In fact, we assert that the angle between $e'_M(\hat{z}''_M)$ and $\tau(\hat{z}''_M)$ is of order ε^2, which allows us to find a nearby $\hat{z}'_M$ as a solution of $e'_M(z') = \tau(z')$, which ultimately is very close to the critical point $\hat{z}'$.

Before we prove this last assertion we must guarantee that the vector field e'_M is defined in a neighborhood of $\hat{z}''_M$ and for that we must have some expansion. Since $\hat{z}$ is a critical point of $f_{a,b}$, then $\left|Df^M_{a,b}(\hat{z})\binom{0}{1}\right| > \mathrm{e}^{cM}$. If necessary we tighten $\mathcal{U}$ so that for every z in a compact set of $\mathbb{R}^2$, $\left|Df^M_{a,b}(z)\binom{0}{1} - Df^M_{a',b'}(z)\binom{0}{1}\right|$ is small enough for having $\left|Df^M_{a',b'}(\hat{z})\binom{0}{1}\right| > \mathrm{e}^{cM/2}$, which implies that e'_M is well defined in a ball of radius $3Cb^{M-1} > 3C\varepsilon^2$ around $\hat{z}$. Note that $b \ll \lambda$ and the Matrix Perturbation Lemma applies.

We take $\mathcal{U}$ sufficiently small so that $|e'_M(\hat{z}''_M) - e_M(\hat{z}''_M)| < \varepsilon^2$. This is possible because $e'_M(z)$ and $e_M(z)$ are the maximally contracted vectors of $Df^M_{a',b'}(z)$ and $Df^M_{a,b}(z)$, respectively. Thus it is only a matter of making $Df^M_{a',b'}(z)$ very close to $Df^M_{a,b}(z)$, for every z in a compact set. Hence

$$\begin{aligned}|e'_M(\hat{z}''_M) - \tau(\hat{z}''_M)| &< |e'_M(\hat{z}''_M) - e_M(\hat{z}''_M)| + |e_M(\hat{z}''_M) - e_M(\hat{z}_M)| + |e_M(\hat{z}_M) - \tau(\hat{z}_M)| \\ &\quad + |\tau(\hat{z}_M) - \tau(\hat{z}''_M)| \\ &< \varepsilon^2 + C|\hat{z}_M - \hat{z}''_M| + 0 + \varepsilon^2 \\ &< C\varepsilon^2\end{aligned}$$

Writing $z = (x, y)$ and taking into account that γ' is nearly horizontal we may think of it as the graph of $\gamma'(x)$. Let us also ease on the notation so that $\tau(x)$ and $e'_M(x)$ denote the slopes of the respective vectors at $z = \gamma'(x)$. We know that $|d\tau/dx| < 10b$, $|de'_M/dx| = 2a + \mathcal{O}(b)$ and $|d^2e'_M/dx^2| < C$. As a consequence we obtain $\hat{z}'_M$ such that $e'_M(\hat{z}'_M) = \tau(\hat{z}_M)$ and

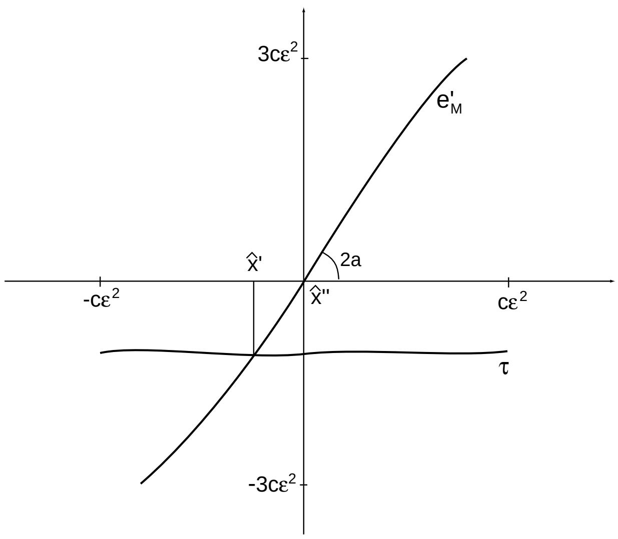

FIGURE 2. Solution of $e'_M(z) = \tau(z)$

$|\hat{z}'_M - \hat{z}''_M| < C\varepsilon^2/3$ (see Figure 2). Now since there is a unique critical point $\hat{z}'$ in γ' we must have $|\hat{z}' - \hat{z}'_M| < C\varepsilon^2$, which yields

$$|\hat{z} - \hat{z}'| \le |\hat{z} - \hat{z}_M| + |\hat{z}_M - \hat{z}''_M| + |\hat{z}''_M - \hat{z}'_M| + |\hat{z}'_M - \hat{z}| < C\varepsilon^2 < \varepsilon,$$

as long as ε is sufficiently small.

Concerning the inequality $|\tau(\hat{z}) - \tau(\hat{z}')| < \varepsilon$, simply observe that since γ and γ' are $C^2(b)$ curves we have

$$\begin{aligned}|\tau(\hat{z}) - \tau(\hat{z}')| &< |\tau(\hat{z}) - \tau(\hat{z}_M)| + |\tau(\hat{z}_M) - \tau(\hat{z}''_M)| + |\tau(\hat{z}''_M) - \tau(\hat{z}'_M)| + |\tau(\hat{z}'_M) - \tau(\hat{z}')| \\ &< 10b|\hat{z} - \hat{z}_M| + \varepsilon^2 + 10b|\hat{z}''_M - \hat{z}'_M| + 10b|\hat{z}'_M - \hat{z}'| \\ &< \varepsilon.\end{aligned}$$

□

As a consequence of Lemma 5.1 we have that for a sufficiently small $\mathcal{U}$ we manage to make W'_1 (the leaf of W' of generation 1) to be as close to W_1 (the leaf of W of generation 1) as we want. This is important because the leaves of higher generations are defined by successive iterations of the first generation leaf. We also remark that by the rules of construction of the critical set we may use the argument of Lemma 5.1 to obtain proximity of the critical points up to generation 27. For higher generations we need the following lemma.

Lemma 5.2. *Let $N \in \mathbb{N}$, $(a,b) \in \mathcal{BC}$ and $\varepsilon > 0$ be given. Assume there exists a neighborhood $\mathcal{U}$ of (a,b) such that for each $(a',b') \in \mathcal{U} \cap \mathcal{BC}$ and any critical point $\hat{z}$ of $f_{a,b}$ of generation $g < N$, there is a critical point $\hat{z}'$ of $f_{a',b'}$ of the same generation with $|\hat{z} - \hat{z}'| < \varepsilon$. If a critical point $\hat{z}$ of $f_{a,b}$ is created at step $g+1$, then we may tighten $\mathcal{U}$ so that a critical point $\hat{z}'$ of generation $g+1$ is created for $f_{a',b'}$ and $|\hat{z} - \hat{z}'| < \varepsilon$. Moreover, if $\tau(\hat{z})$ and $\tau(\hat{z}')$ are the unit vectors tangent to W and W' at $\hat{z}$ and $\hat{z}'$ respectively, then $|\tau(\hat{z}) - \tau(\hat{z'})| < \varepsilon$.*

Proof. As we are only interested in arbitrarily small ε, we may assume that $\varepsilon < b^N$. Suppose that a critical point $\hat{z}$ of generation $g+1$ is created for $f_{a,b}$. Then, by the rules of construction of critical points, there are $z = (x,y)$ lying in a $C^2(b)$ segment $\gamma \subset W$ of generation $g+1$ with γ extending beyond $2\varrho^{g+1}$ to each side of z and a critical point $\tilde{z} = (x, \tilde{y})$ of generation not greater than g such that $|z - \tilde{z}| < b^{(g+1)/540}$. Moreover, $|\hat{z} - z| < |z - \tilde{z}|^{1/2}$.

Taking γ as a compact disk of W, there is a neighborhood $\mathcal{U}$ of (a,b) such that for every $(a',b') \in \mathcal{U}$ we can find a disk $\gamma' \subset W'$ of generation $g+1$ which is ε^2-close to γ in the C^r-topology. It is clear that γ' is a $C^2(b)$ curve. Our aim is to show that a critical point $\hat{z}'$ of $f_{a',b'}$ and generation $g+1$ is created in the segment γ' with $|\hat{z} - \hat{z}'| < \varepsilon$.

By the inductive hypothesis there is $\tilde{z}' = (\tilde{x}', \tilde{y}')$ a critical point of $f_{a',b'}$ such that $|\tilde{z} - \tilde{z}'| < \varepsilon$. Let $z' = (\tilde{x}', y')$ belonging to γ'. Since γ' is ε^2-close to γ in the C^r topology and $\varepsilon < b^N$, which is completely insignificant when compared to $\varrho^{g+1} < \varrho^N$ (recall that $\varrho \gg b$), we may assume that γ' extends more than $2\varrho^{g+1}$ to both sides of z'. Moreover, letting $\zeta' = (x, \eta') \in \gamma'$ we have

$$\begin{aligned} |\tilde{z}' - z'| &< |\tilde{z}' - \tilde{z}| + |\tilde{z} - z| + |z - \zeta'| + |\zeta' - z'| \\ &< \varepsilon + b^{\frac{g+1}{540}} + 2\varepsilon^2 + 2\varepsilon \\ &\lesssim b^{\frac{g+1}{540}}, \end{aligned}$$

where we used the fact that $\varepsilon < b^N \ll b^{\frac{N}{540}} < b^{\frac{g+1}{540}}$ (see Figure 3). By the rules of

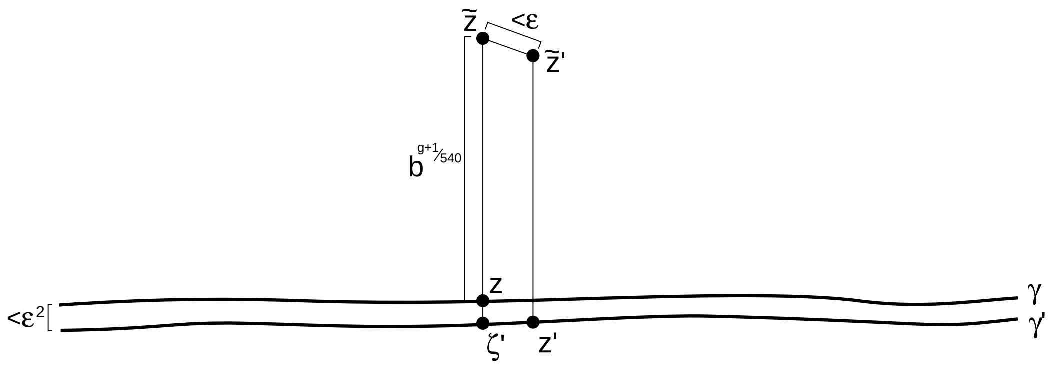

FIGURE 3. Possible relative position of the critical points

construction of critical points, a unique critical point $\hat{z}'$ of generation $g+1$ is created in the segment γ'. We are left to show that $|\hat{z} - \hat{z}'| < \varepsilon$. For that we repeat the argument in the proof of Lemma 5.1. □

COROLLARY 5.3. *Let $N \in \mathbb{N}$, $(a, b) \in \mathcal{BC}$ and $\varepsilon > 0$ be given. There is a neighborhood $\mathcal{U}$ of (a, b) such that if $(a', b') \in \mathcal{U} \cap \mathcal{BC}$ then, for any critical point $\hat{z}$ of $f_{a,b}$ of generation smaller than N, there is a critical point $\hat{z}'$ of $f_{a',b'}$ of the same generation such that $|\hat{z} - \hat{z}'| < \varepsilon$. Moreover if $\tau(\hat{z})$ and $\tau(\hat{z}')$ are the unit vectors tangent to W and W' at $\hat{z}$ and $\hat{z}'$ respectively, then $|\tau(\hat{z}) - \tau(\hat{z}')| < \varepsilon$.*

PROOF. The proof is just a matter of collecting the information in the Lemmas 5.1 and 5.2 and organize it in a finite step induction scheme.

(1) First obtain the proximity of the critical points of generation 1, which has already been done in Lemma 5.1.
(2) Then realize that the same argument in the proof of Lemma 5.1 also gives the proximity of the 2^{26} critical points of generation smaller than 27. (See the rules of construction of critical points in Subsection 3.4.1).
(3) Apply the inductive step stated in Lemma 5.2 to obtain the proximity of critical points of higher and higher generation.
(4) Stop the process when the proximity of all critical points of generation smaller than N is achieved.

Naturally every time we apply Lemma 5.2 to increase the generation level for which the conclusion of the proposition holds, we may need to decrease the size of the neighborhood $\mathcal{U}$. However, because the number of critical points of a given generation is finite and the statement of the proposition is up to generation N, at the end we still obtain a neighborhood containing a non-degenerate ball around (a, b) where the proposition holds. $\square$

6. Proximity of leading Cantor sets

Attending to Lemma 5.1, we may assume that $\Omega_0 = \Omega_0'$. Let $h, h' : \Omega_0 \to \mathbb{R}$ be functions whose graphs are the leaves of first generation W_1 and W_1' respectively, when restricted to the set $\Omega_0 \times \mathbb{R}$. Given an interval $I \subset \Omega_0$ the segments $\omega = H(I)$ and $\omega' = H'(I)$ are respectively the subsets of W_1 and W_1' which correspond to the images in the graph of h and h' of the interval I. Accordingly, if $x \in \Omega_0$ then $z = H(x) = (x, h(x))$ and $z' = H'(x) = (x, h'(x))$. See Figure 4.

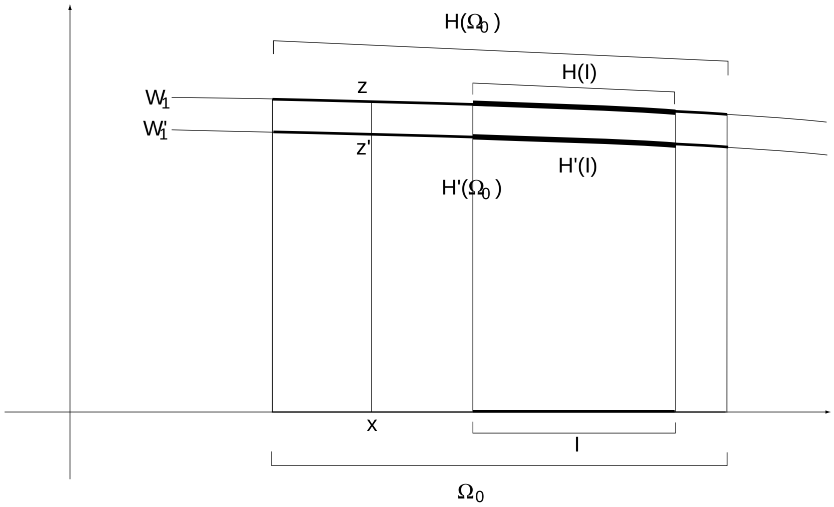

FIGURE 4

Our goal in this section is to show the proximity of the Cantor sets Ω_∞ for close Benedicks-Carleson parameters. More precisely, given any $\varepsilon > 0$ we will exhibit a neighborhood $\mathcal{U}$ of (a, b) such that $|\Omega_\infty \bigtriangleup \Omega_\infty'| < \varepsilon$ for all $(a', b') \in \mathcal{U} \cap \mathcal{BC}$, where $\bigtriangleup$ represents symmetric difference between two sets. In the process, we make a modification in the first steps of the procedure described in Subsection 4.1 to build Ω_∞', which carries only minor differences with respect to the set we would obtain if we were to follow the rules strictly. Ultimately, this affects the construction of the horseshoes Λ'. However, the horseshoes are not uniquely determined and we will evince that the modifications introduced leave unchanged the properties that they are supposed to have.

LEMMA 6.1. *Given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $|\Omega_n \setminus \Omega_\infty| < \varepsilon$ for every $(a, b) \in \mathcal{BC}$ and $n \geq N_1$.*

PROOF. This is a consequence of [**BY00**, Lemma 4] where it is proved that

$$\frac{|\Omega_{n-1} \setminus \Omega_n|}{|\Omega_{n-1}|} \leq C_1 \delta^{1-3\beta} \mathrm{e}^{-\alpha(1-3\beta)n}. \tag{6.1}$$

This inequality follows from the fact that any connected component $\omega \in H(\Omega_{n-1})$ grows to reach a length $|f^n(\omega)| \geq \delta^{3\beta} \mathrm{e}^{-3\alpha\beta n}$, while the subsegment of $f^n(\omega)$ to be deleted in the construction of Ω_n has length at most $4\delta e^{-\alpha n}$; then, simply take bounded distortion into consideration.

From (6.1) one easily gets

$$\begin{aligned}|\Omega_n \setminus \Omega_\infty| &= \sum_{j=0}^{+\infty} |\Omega_{n+j} \setminus \Omega_{n+j+1}| \\ &\leq C_1 \delta^{1-3\beta} \sum_{j=1}^{+\infty} e^{-\alpha(1-3\beta)(n+j)} |\Omega_{n+j-1}| \\ &\leq C_1 \delta^{1-3\beta} |\Omega_n| \sum_{j=1}^{+\infty} e^{-\alpha(1-3\beta)(n+j)} \\ &\leq C_1 \delta^{1-3\beta} \frac{e^{-\alpha(1-3\beta)(n+1)}}{1-e^{-\alpha(1-3\beta)}}\end{aligned}$$

Hence, choose N_1 sufficiently large so that

$$C_1 \delta^{1-3\beta} \frac{e^{-\alpha(1-3\beta)(N_1+1)}}{1-e^{-\alpha(1-3\beta)}} < \varepsilon.$$

□

Observe that, as a consequence of the unstable manifold theorem, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists a neighborhood $\mathcal{U}$ of (a, b) such that for every $(a', b') \in \mathcal{U}$ we have

$$\max \left\{ \|H - H'\|_r, \|f_{a,b} \circ H - f_{a',b'} \circ H'\|_r, \dots, \left\| f_{a,b}^n \circ H - f_{a',b'}^n \circ H' \right\|_r \right\} < \varepsilon, \tag{6.2}$$

where $r \geq 2$ and $\|\cdot\|_r$ is the C^r-norm in Ω_0. In what follows $\Omega_\infty = \cap_{n\in\mathbb{N}} \Omega_n$ is built as described in Section 4.1 for $f = f_{a,b}$.

LEMMA 6.2. *Let $n \in \mathbb{N}$ and $(a, b) \in \mathcal{BC}$ be given and I be a connected component of Ω_{n-1}. Suppose $f_{a,b}^n(H(I))$ intersects $(-\delta, \delta) \times \mathbb{R}$. There is a neighborhood $\mathcal{U}$ of (a, b) such that for every $(a', b') \in \mathcal{U} \cap \mathcal{BC}$ and $x \in I \cap \Omega_n$, if $f_{a,b}^n(H(x)) \in (-\delta, \delta) \times \mathbb{R}$ and $\hat{z}$ is a suitable binding critical point, then there exists a binding critical point $\hat{z}'$ of $f_{a',b'}$ close to $\hat{z}$ suitable for $f_{a',b'}^n(H'(x))$ and $|f_{a',b'}^n(H'(x)) - \hat{z}'| \gtrsim \delta e^{-\alpha n}$.*

PROOF. Let $\tilde{I} = I \cap \Omega_n$ and $\mathcal{U}$ be a neighborhood of (a, b) such that Corollary 5.3 applies up to n with b^{2n} in the place of ε and equation (6.2) also holds with b^{4n} in the place of ε. Then there is a critical point $\hat{z}'$ of $f_{a',b'}$ such that $|\hat{z} - \hat{z}'| < b^{2n}$ and $\left\| f_{a,b}^n \circ H|_{\tilde{I}} - f_{a',b'}^n \circ H'|_{\tilde{I}} \right\|_r < b^{4n}$. We only need to prove that this $\hat{z}'$ is a suitable binding point for $f_{a',b'}^n(z')$ and that $|f_{a',b'}^n(z') - \hat{z}'| \gtrsim \delta e^{-\alpha n}$. In order to verify the suitability of $\hat{z}'$ we have to check that

(1) $f_{a',b'}^n(z')$ is in tangential position with respect to $\hat{z}'$;
(2) $Df_{a',b'}^n(z')\tau(z')$ splits correctly with respect to the contracting field around $\hat{z}'$.

The strategy is to show that $|f_{a,b}^n(z) - \hat{z}| = |f_{a',b'}^n(z') - \hat{z}'| + \mathcal{O}(b^{2n})$. Then, because $f^n(z)$ is in tangential position with respect to $\hat{z}$ and $b^{2n} \ll \delta e^{-\alpha n} \leq |f^n(z) - \hat{z}|$, we conclude the tangential position for $f_{a',b'}^n(z')$ with respect to $\hat{z}'$. As to the correct splitting, we know that $|Df^n(z)\tau(z) - (Df_{a',b'})^n(z')\tau(z')| < b^{4n}$ and $Df^n(z)\tau(z)$ makes an angle with the relevant contracting field of approximately $(2a \pm 1)|f^n(z) - \hat{z}|$. Finally, since $|f^n(z) - \hat{z}| = |f^n_{a',b'}(z') - \hat{z}'| + \mathcal{O}(b^{2n})$ and $b^{2n} \ll (2a \pm 1)|f^n(z) - \hat{z}|$ we obtain the desired result.

Let us start by proving (1). Observe that

$$\begin{aligned}
|f^n_{a,b}(z) - \hat{z}| &\leq |f^n_{a,b}(z) - f^n_{a',b'}(z')| + |f^n_{a',b'}(z') - \hat{z}'| + |\hat{z} - \hat{z}'| \\
&\leq \left\| f^n_{a,b} \circ H|_{\tilde{I}} - f^n_{a',b'} \circ H'|_{\tilde{I}} \right\|_r + |f^n_{a',b'}(z') - \hat{z}'| + b^{2n} \\
&\leq |f^n_{a',b'}(z') - \hat{z}'| + 2b^{2n}.
\end{aligned}$$

Interchanging z with z' we easily get $|f^n_{a',b'}(z') - \hat{z}'| \leq |f^n_{a,b}(z) - \hat{z}| + 2b^{2n}$ which allows us to write $|f^n_{a',b'}(z') - \hat{z}'| = |f^n_{a,b}(z) - \hat{z}| + \mathcal{O}(b^{2n})$. Consider now s and s' the lines through $\hat{z}$ and $\hat{z}'$ with slopes $\tau(\hat{z})$ and $\tau(\hat{z}')$ respectively. By Corollary 5.3 we have $|\hat{z} - \hat{z}'| < b^{2n}$ and also $|\tau(\hat{z}) - \tau(\hat{z}')| < b^{2n}$. Thus, when restricted to the set $[-1, 1] \times \mathbb{R}$ we have $\|s - s'\|_r < \mathcal{O}(b^{2n})$. Let $\operatorname{dist}(z, s)$ denote the distance from the point z to the segment $s \cap [-1, 1] \times \mathbb{R}$. Then

$$\begin{aligned}
\operatorname{dist}(f^n_{a,b}(z), s) &\leq |f^n_{a,b}(z) - f^n_{a',b'}(z')| + \operatorname{dist}(f^n_{a',b'}(z'), s) \\
&\leq |f^n_{a,b}(z) - f^n_{a',b'}(z')| + \|s - s'\|_r + \operatorname{dist}(f^n_{a',b'}(z'), s') \\
&\leq \operatorname{dist}(f^n_{a',b'}(z'), s') + \mathcal{O}(b^{2n})
\end{aligned}$$

Similarly we get $\operatorname{dist}(f^n_{a',b'}(z'), s') \leq \operatorname{dist}(f^n_{a,b}(z), s) + \mathcal{O}(b^{2n})$, and so

$$\operatorname{dist}(f^n_{a',b'}(z'), s') = \operatorname{dist}(f^n_{a,b}(z), s) + \mathcal{O}(b^{2n}).$$

Now, since $f^n(z)$ is in tangential position with respect to $\hat{z}$, then

$$\operatorname{dist}(f^n_{a,b}(z), s) < c|f^n_{a,b}(z) - \hat{z}|^2,$$

where $c \ll 2a$. Besides, $|f^n_{a',b'}(z') - \hat{z}'|^2 = \left(|f^n_{a,b}(z) - \hat{z}| + \mathcal{O}(b^{2n})\right)^2 = |f^n_{a,b}(z) - \hat{z}|^2 + \mathcal{O}(b^{2n})$ because $b^{2n} \ll \delta e^{-\alpha n} \leq |f^n_{a,b}(z) - \hat{z}|$. Consequently

$$\operatorname{dist}(f^n_{a',b'}(z'), s') < c|f^n_{a',b'}(z') - \hat{z}'|^2 + \mathcal{O}(b^{2n}),$$

which again by the insignificance of b^{2n} relative to $|f^n_{a',b'}(z') - \hat{z}'|$ implies that $f^n_{a',b'}(z')$ is in tangential position with respect to $\hat{z}'$.

Concerning (2), notice that if (a', b') is sufficiently close to (a, b), then

$$\left|Df^n_{a,b}(z)\tau(z) - Df^n_{a',b'}(z')\tau(z')\right| \leq \left\| f^n_{a,b} \circ H|_{\tilde{I}} - f^n_{a',b'} \circ H'|_{\tilde{I}} \right\|_r < b^{4n}.$$

Let l and l' denote the lengths of the fold periods for z and z'. Take m and m' such that $(5b)^m \leq |z - \hat{z}| \leq (5b)^{m-1}$ and $(5b)^{m'} \leq |z' - \hat{z}'| \leq (5b)^{m'-1}$ respectively. Since $|z' - \hat{z}'| = |z - \hat{z}| + \mathcal{O}(b^{2n})$ and b^{2n} is negligible when compared to $|z - \hat{z}|$, we may assume that $m = m'$. We know that $|\tau(f^n_{a,b}(z)) - e_l(z)| \approx (2a \pm 1)|z - \hat{z}|$. Since $l \geq 2m$, property (4) of Section 3.3 leads to $|e_l(z) - e_{2m}(z)| = \mathcal{O}(b^{2m})$. As a consequence we have

$$|\tau(f^n_{a,b}(z)) - e_{2m}(z)| = |\tau(f^n_{a,b}(z)) - e_l(z)| + \mathcal{O}(b^{2m}) \approx (2a \pm 1)|z - \hat{z}|,$$

because $|z - \hat{z}| \geq (5b)^m \gg b^m \gg b^{2m}$.

Observe that $|\tau(f^n_{a,b}(z)) - \tau(f^n_{a',b'}(z'))| < b^{2n}$ because $\left|Df^n_{a,b}(z)\tau(z)\right| > \delta e^{c_2 n}$, by (EE), and $\left|Df^n_{a,b}(z)\tau(z) - Df^n_{a',b'}(z')\tau(z')\right| < b^{4n}$. If necessary, we tighten $\mathcal{U}$ in order to guarantee

$|e_{2m}(z) - e'_{2m}(z')| < b^{2n}$. Since $b^{2n} \ll |z' - \hat{z}'|$ we conclude that

$$|\tau(f^n_{a',b'}(z')) - e'_{2m}(z')| = |\tau(f^n_{a,b}(z)) - e_{2m}(z)| + \mathcal{O}(b^{2n}) \approx (2a \pm 1)|z' - \hat{z}'|.$$

Finally, a similar argument allows us to obtain

$$|\tau(f^n_{a',b'}(z')) - e'_{l'}(z)| = |\tau(f^n_{a',b'}(z')) - e'_{2m}(z')| + \mathcal{O}(b^{2m}) \approx (2a \pm 1)|z' - \hat{z}'|,$$

which gives the correct splitting of the vector $(Df_{a',b'})^n(z')\tau(z')$ with respect to the critical point $\hat{z}'$. □

Now we will show that if we change the rules of construction of Ω'_∞ in the first N iterates by choosing a convenient binding critical point at each return happening before N we manage to have $\Omega_N = \Omega'_N$ as long as (a', b') is sufficiently close to (a, b).

Before proceeding let us clarify the equality $\Omega'_n = \Omega_n$ for $n \leq N$. As mentioned in Remark 4.3, the procedure leading to Ω_∞ is not unique. Thus, we have some freedom in the construction of Ω'_∞ as long as we guarantee the slow approximation to the critical set and the expansion along the tangent direction to W.

Take $(a', b') \in \mathcal{U} \cap \mathcal{BC}$, where $\mathcal{U}$ is a small neighborhood of (a, b). Applying the procedure of [**BY00**] described in Section 4.1 we may build a sequence of sets $\Omega'_0 \supset \Omega'_1 \supset \dots$ to obtain $\Omega'_\infty = \bigcap_{j \in \mathbb{N}_0} \Omega'_j$. From Lemmas 5.1 and 6.2 we know that, given N and $j \leq N$, the set Ω_j is a good approximation of Ω'_j. We propose a modification on the first N steps in the construction of Ω'_∞: consider $\Omega'_n = \Omega_n$ for all $n \leq N$; afterwards make the exclusions of points from Ω_N according to the original procedure. This way, we produce a sequence of sets $\Omega_0 \supset \dots \supset \Omega_N \supset \Omega'_{N+1} \supset \dots$ which we intersect to obtain Ω'_∞. We will show that the points in Ω'_∞ have slow approximation to the critical set and expansion along the tangent direction of W' for the dynamics $f_{a',b'}$.

When we perturb a parameter $(a, b) \in \mathcal{BC}$ and change the rules of construction of Ω'_n for a close parameter $(a', b') \in \mathcal{U} \cap \mathcal{BC}$, in the sense mentioned above, we may need to weaken the condition (SA) and introduce condition (SA)' which is defined as (SA) except for the replacement of δ by $\delta/2$. This way we guarantee the validity of (SA)' for every (a', b') in a sufficiently small neighborhood $\mathcal{U}$ of (a, b) as stated in next lemma.

Lemma 6.3. *Let $(a, b) \in \mathcal{BC}$ and $n \in \mathbb{N}$ be given. There is a neighborhood $\mathcal{U}$ of (a, b) such that for all $(a', b') \in \mathcal{U} \cap \mathcal{BC}$ we may take $\Omega'_j = \Omega_j$ for all $j \leq n$ and ensure that (SA)' holds for all $j \leq n$, for the dynamics $f_{a',b'}$.*

Proof. If $\mathcal{U}$ is sufficiently small, then by Corollary 5.3 we have that (SA)' holds for $n = 0$, in $H'(\Omega_0)$, for the dynamics $f_{a',b'}$. Let us suppose that (SA)' holds in $H'(\Omega_{n-1})$, for $f_{a',b'}$ and $j \leq n - 1 < N$. This is to say that for all $x \in \Omega_{n-1}$ the $f_{a',b'}$ orbit of $z' = H'(x)$ is controlled up to $n - 1$ and at each return $k \leq n - 1$, if $\hat{z}'$ denotes a suitable binding critical point, then $|f^k_{a',b'}(z') - \hat{z}'| \geq \delta \mathrm{e}^{-\alpha k}/2$.

Our aim is to show that by tightening $\mathcal{U}$, if necessary, this last statement remains true for n. Let $I \subset \Omega_{n-1}$ be a connected component and $\tilde{I} = I \cap \Omega_n$. Then, by Lemma 6.2, we can tighten $\mathcal{U}$, so that for all $x \in \Omega_{n-1}$, the orbit of $z' = H'(x)$ under $f_{a',b'}$ is controlled up to n. Moreover, if n is a return time for z', and $\hat{z}'$ is a suitable binding point for $f^n_{a',b'}(z')$, then $|f^n_{a',b'}(z') - \hat{z}'| \geq \delta \mathrm{e}^{-\alpha n}/2$. Since each Ω_n has a finite number of connected components and we only wish to carry on this procedure up to N, then at the end we still obtain a neighborhood $\mathcal{U}$ of (a, b). □

Thus, for every $(a',b')\in\mathcal{U}\cap\mathcal{BC}$, where $\mathcal{U}$ is given by Lemma 6.3, we have a sequence of sets $\Omega_0\supset\ldots\supset\Omega_N$ such that (SA)' holds for every $z'=H'(x)$ with $x\in\Omega_N$ and $n\le N$. At this point we proceed with the method described in Section 4 and make exclusions out of Ω_N to obtain a sequence $\Omega_0\supset\ldots\supset\Omega_N\supset\Omega'_{N+1}\supset\ldots$ whose intersection we denote by Ω'_∞. Hence, every point in $H'(\Omega'_\infty)$ satisfies (SA) for every $n>N$.

Corollary 6.4. *Let $(a,b)\in\mathcal{BC}$ and $\varepsilon>0$ be given. There exists a neighborhood $\mathcal{U}$ of (a,b) so that $|\Omega_\infty\bigtriangleup\Omega'_\infty|<\varepsilon$ for each $(a',b')\in\mathcal{U}\cap\mathcal{BC}$.*

Proof. We appeal to Lemma 6.1 and find $N_1=N_1(\varepsilon)$ such that $|\Omega_{N_1}\setminus\Omega_\infty|<\varepsilon/2$. Observe that, using Lemma 6.3, the same N_1 allows us to write that $|\Omega_{N_1}\setminus\Omega'_\infty|<\varepsilon/2$ for all $(a',b')\in\mathcal{U}\cap\mathcal{BC}$. So, we have $|\Omega_\infty\bigtriangleup\Omega'_\infty|\le|\Omega_{N_1}\bigtriangleup\Omega_\infty|+|\Omega_{N_1}\bigtriangleup\Omega'_\infty|<\varepsilon$. □

7. Proximity of stable curves

So far we have managed to prove proximity of the horseshoes in the horizontal direction. The goal of this section is to show the closeness of the stable curves. The main result of this section is Proposition 7.3.

Recall that each long stable curve is obtained as a limit of “temporary stable curves”, γ_n, as described in Section 4.2. In order to obtain proximity of long stable curves for close Benedicks-Carleson dynamics we must produce first an integer N_2 such that the approximate stable curves γ_{N_2} are sufficiently close to the corresponding stable curves γ^s, regardless of the parameter $(a,b)\in\mathcal{BC}$. This is accomplished through Lemma 7.1. Therefore, in Proposition 7.3 we obtain the proximity of the “temporary stable curves” γ_{N_2} for close Benedicks-Carleson parameters and deduce in this way the desired proximity of the long stable curves.

We use the notation $\gamma_n(\zeta)(t)$ or its shorter version, $\gamma_n^t(\zeta)$, for the solution of the equation $\dot z=e_n(z)$ with initial condition $\gamma_n(\zeta)(0)=\gamma_n^0(\zeta)=\zeta$. Recall that $\|e_n\|=1$ and $\gamma_n(\zeta)$ is an e_n-integral curve of length $20b$ centered at ζ. So the natural range of values for t is $[-10b,10b]$.

Lemma 7.1. *Let $(a,b)\in\mathcal{BC}$ and $n\in\mathbb{N}$ be given. Consider a connected component $\omega\subset H(\Omega_{n-1})$ and the rectangle $Q_{n-1}(\omega)$ foliated by the curves γ_n. Then the width of the rectangle $Q_{n-1}(\omega)$ is at most $4\delta^{-1}e^{-c_2n}$.*

Proof. By the derivative estimate in Subsection 3.6.1, for all $z\in\omega$ we have

$$|Df^n(z)\tau(z)|>\delta\mathrm{e}^{c_2n}.$$

Since ω is a connected component of $H(\Omega_{n-1})$ we have that $|f^n(\omega)|<2$. As a consequence, $|\omega|<2\delta^{-1}\mathrm{e}^{-c_2n}$. Observe that this argument also gives that if $z\in H(\Omega_\infty)$ and ω_j denotes the connected component of $H(\Omega_j)$ containing z then $\cap_j\omega_j=\{z\}$. Let z^+ and z^- denote respectively the right and left endpoints of ω. Given $t \in [-10b, 10b]$

$$\begin{aligned}
\left|\gamma_n^t(z^+) - \gamma_n^t(z^-)\right| &\leq \left|z^+ + \int_0^t e_n\left(\gamma_n^r(z^+)\right) dr - z^- - \int_0^t e_n\left(\gamma_n^r(z^-)\right) dr\right| \\
&\leq |z^+ - z^-| + \int_0^t \left|e_n\left(\gamma_n^r(z^+)\right) - e_n\left(\gamma_n^r(z^-)\right)\right| dr \\
&\leq |z^+ - z^-| + 5\int_0^t \left|\gamma_n^r(z^+) - \gamma_n^r(z^-)\right| dr, \text{ by (3) of Section 3.3} \\
&\leq |z^+ - z^-| \mathrm{e}^{5|t|}, \text{ by a Gronwall type inequality} \\
&\leq |z^+ - z^-| \mathrm{e}^{50b} < 2|z^+ - z^-| = 2|\omega|
\end{aligned}$$

Thus, the width of the rectangle $Q_{n-1}(\omega)$ is at most $4\delta^{-1}\mathrm{e}^{-c_2 n}$. □

We will use the following notation for parameters (a', b') close to (a, b). For any $n \in \mathbb{N}$ and $z' \in \omega'_n \subset H'(\Omega'_n)$, we denote by $\gamma'_{n+1}(z')$ the e'_{n+1}- integral curve of length $20b$ centered at z'. Given $n \in \mathbb{N}$, for any connected component $\omega' \subset H'(\Omega'_n)$ we denote by $Q_n(\omega') = \cup_{z' \in \omega'} \gamma'_{n+1}(z')$ the rectangle foliated by the curves $\gamma'_{n+1}(z')$. We define $Q^1_n(\omega')$ as a $(Cb)^{n+1}$- neighborhood of $Q_n(\omega')$ in $\mathbb{R}^2$. Finally, given $n \in \mathbb{N}$ and any interval $\omega \subset H(\Omega_n)$, we denote by $Q^2_n(\omega)$ a $2(Cb)^{n+1}$- neighborhood of $Q_n(\omega)$.

LEMMA 7.2. *Let $(a, b) \in \mathcal{BC}$, $n \in \mathbb{N}$, $\varepsilon > 0$ be given, and fix a connected component I of Ω_{n-1}. Then there is a neighborhood $\mathcal{U}$ of (a, b) such that e_n, e'_n are defined in $Q^2_{n-1}(H(I))$ and for every $x \in I$*

$$\|\gamma_n(H(x)) - \gamma'_n(H'(x))\|_0 < \varepsilon.$$

Moreover, for every interval $J \subset I$ we have that $Q^2_{n-1}(H(J))$ contains $Q^1_{n-1}(H'(J))$.

PROOF. As we are only interested in arbitrarily small ε, we may assume that $\varepsilon < b^{2n}$. Take the neighborhood $\mathcal{U}$ of (a, b) given by Lemma 6.3 applied to n. Within $\mathcal{U} \cap \mathcal{BC}$, the set Ω'_∞ is built out of Ω_n, in the usual way.

Consider the sequence $I_0 \supset \ldots I_j \supset \ldots \supset I_n = I$ of the connected components (intervals) I_j of Ω_j containing I. For every $j \leq n$, let $\omega_j = H(I_j)$ and $\omega'_j = H'(I_j)$. We will use a finite inductive scheme such that at step j, under the hypothesis that e_j and e'_j are both defined in $Q_{j-2}(\omega_{j-1})$, we tighten $\mathcal{U}$ (if necessary) so that for all $x \in I_{j-1}$ we have $\gamma_j(z)$ ε-close to $\gamma'_j(z')$ in the C^0 topology, where $z = H(x)$ and $z' = H'(x)$, which implies that $Q^2_{j-1}(\omega_j)$ contains $Q^1_{j-1}(\omega'_j)$. This way we conclude that both e_{j+1} and e'_{j+1} are defined in the set $Q^2_{j-1}(\omega_j)$, which makes our hypothesis true for step $j+1$. After n steps we still have a vicinity $\mathcal{U}$ of (a, b) and $\gamma_n(z)$ is ε C^0-close to $\gamma'_n(z')$.

We know that e_1 and e'_1 are defined everywhere in $\mathbb{R}^2$, which makes our hypothesis true at the first step.

Suppose now, by induction, that at step j we know that e_j and e'_j are both defined in $Q^2_{j-2}(\omega_{j-1})$, which contains both $Q^1_{j-2}(\omega_{j-1})$ and $Q^1_{j-2}(\omega'_{j-1})$. Let $\mathcal{U}$ be sufficiently small so that for all $(a', b') \in \mathcal{U} \cap \mathcal{BC}$, we have $\|H - H'\|_r < \varepsilon^3$ and $|e_j(z) - e'_j(z)| < \varepsilon$, for every $z \in Q^2_{j-2}(\omega_{j-1})$. Since $Q^1_{j-1}(\omega_{j-1}) \subset Q^1_{j-2}(\omega_{j-1})$ and $Q^1_{j-1}(\omega'_{j-1}) \subset Q^1_{j-2}(\omega'_{j-1})$ (see (4.1)), the curves $\gamma_j(z)$ and $\gamma_j(z')$ never leave the set $Q^2_{j-2}(\omega_{j-1})$, for every $z \in \omega_{j-1}$ and $z' \in \omega'_{j-1}$.

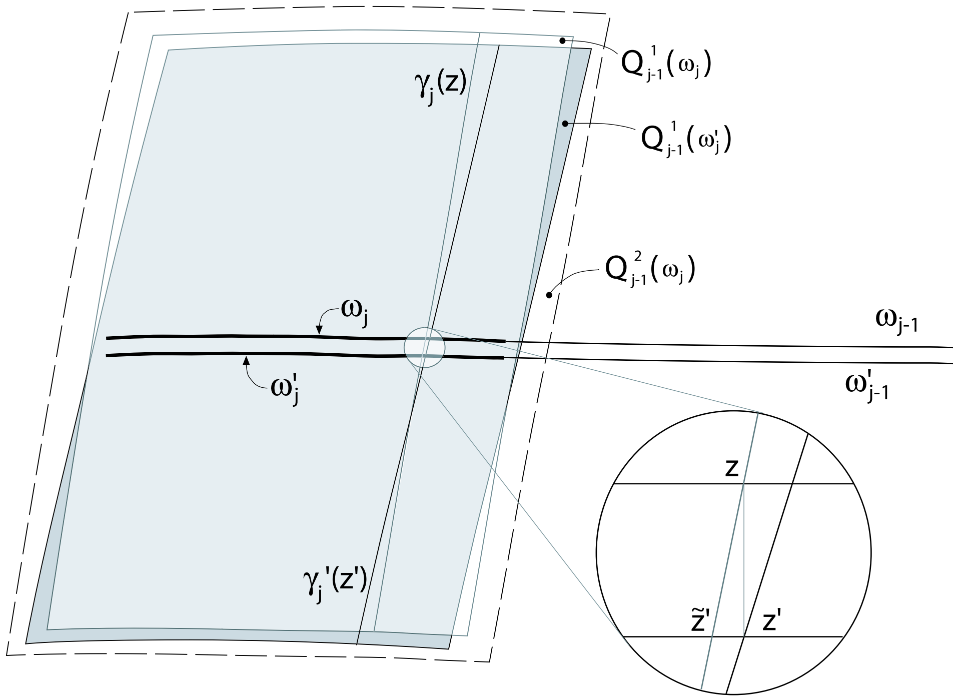

FIGURE 5

Let $\{\tilde z'\} = \gamma_j(z) \cap W_1'$; since $\|H - H'\|_r < \varepsilon^3$ then $|\tilde z' - z| < \varepsilon^2$, $|\tilde z' - z'| < \varepsilon^2$ (see Figure 5). Using the Lipschitzness of the fields e_j and e_j' (property (3) in Section 3.3), the continuity of flows with initial conditions and the continuity of flows as functions of the vector field (see for example [**HS74**]) we have for all t

$$\begin{aligned}\left|\gamma_j(\tilde z')(t) - \gamma_j'(z')(t)\right| &\le \left|\gamma_j(\tilde z')(t) - \gamma_j'(\tilde z')(t)\right| + \left|\gamma_j'(\tilde z')(t) - \gamma_j'(z')(t)\right| \\ &\le \frac{\varepsilon}{2a + \mathcal{O}(b)}(\mathrm{e}^{5|t|} - 1) + |\tilde z' - z'|\mathrm{e}^{5|t|} \\ &\le \frac{\varepsilon}{3}\mathrm{e}^{50b}50b + 2\varepsilon^2 < \varepsilon\end{aligned}$$

Thus $\|\gamma_j(z) - \gamma_j'(z')\|_0 < \varepsilon$. Moreover, since $\varepsilon \ll (Cb)^j$, we easily get that for any interval $J \subset I_{j-1}$, the rectangle $Q_{j-1}^2(H(J))$ contains both $Q_{j-1}^1(H(J))$ and $Q_{j-1}^1(H'(J))$.

From [**BY00**, Section 3.3] we know e_{j+1} is defined in a $3(Cb)^j$-neighborhood in $\mathbb{R}^2$ of $\gamma_j(z)$, for every $z \in \omega_j$. Since the same applies to $\gamma_j'(z')$ where $z' \in \omega_j'$ and clearly $\gamma_j(z)$ lies inside a $(Cb)^j$-neighborhood in $\mathbb{R}^2$ of $\gamma_j'(z')$ ($\varepsilon \ll (Cb)^j$) then e_{j+1}' is defined in all points of $\gamma_j(z)$. This also implies that e_{j+1}' is defined in $Q_{j-1}^2(\omega_j)$.

Thus applying the argument above n times we get that e_n and e_n' are defined in $Q_{n-2}^2(\omega_{n-1})$ and for every $z \in \omega_{n-1}$, $z' = H'\left(H^{-1}(z)\right) \in \omega_{n-1}'$,

$$\|\gamma_n(z) - \gamma_n'(z')\|_0 < \varepsilon,$$

which gives that for any interval $J \subset \Omega_{n-1}$ we have that $Q_{n-1}^2(H(J))$ contains both $Q_{n-1}^1(H(J))$ and $Q_{n-1}^1(H'(J))$, since $\varepsilon \ll b^n$.

$\square$

PROPOSITION 7.3. *Let $(a,b) \in \mathcal{BC}$ and $\varepsilon > 0$ be given. There is a neighborhood $\mathcal{U}$ of (a,b) such that for all $(a',b') \in \mathcal{U} \cap \mathcal{BC}$ and $x \in \Omega_\infty \cap \Omega'_\infty$, we have that $\gamma^s(H(x))$ and $\gamma'^s(H'(x))$ are ε-close in the C^1 topology.*

PROOF. Choose $N_2 \in \mathbb{N}$ large enough so that

$$4\delta^{-1}\mathrm{e}^{-c_2 N_2} + 4(Cb)^{N_2} < \frac{\varepsilon}{3} \tag{7.1}$$

By Lemma 7.1 the width of the rectangle $Q^2_{N_2-1}(\omega_{N_2-1})$ is less than $\frac{\varepsilon}{3}$. This means that for every $\zeta \in \omega_{N_2}$, the curve $\gamma_{N_2}(\zeta)$ is at least $\frac{\varepsilon}{3}$-close to $\gamma^s(z)$ in the C^0 topology. Note that the choice of N_2 does not depend on the point $z \in H(\Omega_\infty)$ taken, neither on the parameter $(a,b) \in \mathcal{BC}$ in question.

Take the neighborhood $\mathcal{U}$ of (a,b) to be such that Lemma 6.3 applies up to N_2 and Lemma 7.2 applies with N_2 replacing n. In particular, for parameters $\mathcal{U} \cap \mathcal{BC}$, the set Ω'_∞ is built out of Ω_{N_2}, in the usual way and $Q^2_{N_2-1}(H(I))$ contains $Q^1_{N_2-1}(H'(I))$ for every connected component $I \subset \Omega_{N_2-1}$. Moreover, for any $x \in I$, $\|\gamma_{N_2}(H(x)) - \gamma'_{N_2}(H'(x))\|_0 < b^{2N_2}$.

Let $x \in \Omega_\infty \cap \Omega'_\infty$ and consider the sequence $I_0 \supset I_1 \supset \dots I_j \supset \dots$ of the connected components (intervals) I_j of Ω_j containing x. Let $z = H(x)$, $z' = H'(x)$ and, for every $j < N_2$, set $\omega_j = H(I_j)$ and $\omega'_j = H'(I_j)$. Collecting all the information we get for any $\zeta \in \omega_{N_2-1}$, $\zeta' = H'\left(H^{-1}(\zeta)\right) \in \omega'_{N_2-1}$

$$\|\gamma^s(z) - \gamma'^s(z')\|_0 \leq \|\gamma^s(z) - \gamma_{N_2}(\zeta)\|_0 + \left\|\gamma_{N_2}(\zeta) - \gamma'_{N_2}(\zeta')\right\|_0 + \left\|\gamma'_{N_2}(\zeta') - \gamma'^s(z')\right\|_0 < \varepsilon.$$

So far we have proved C^0-closeness of the stable leaves. The fact that the fields e_n and e'_n are Lipschitz with uniform Lipschitz constant $3 < 2a + \mathcal{O}(b) < 5$ allows us to improve the previous C^0-estimates to obtain C^1-estimates with little additional effort. $\square$

8. Proximity of s-sublattices and return times

The purpose of this section is to obtain the proximity, for close Benedicks-Carleson dynamics, of the sets of points with the same history, in terms of free and bound periods up to a fixed time. In Subsection 8.1 we accomplish this, up to the first regular return. In Subsection 8.2 we realize that the same result may be achieved even if we consider the itineraries up to a some other return.

8.1. Proximity after the first return. Recall that the return time function R is constant on each s-sublattice and, in particular, on each γ^s. Thus, the return time function R needs only to be defined in $\Lambda \cap W_1$ or in its vertical projection in the x-axis Ω_∞. Let $(\Upsilon_{n,j})_j$ denote the family of subsets of Ω_0 for which $\bar{\pi}^{-1}(H(\Upsilon_{n,j})) \cap \Lambda$ correspond to the s-sublattices of Λ given by [**BY00**, Proposition A] and such that $R(H(\Upsilon_{n,j})) = n$. Observe that $\Upsilon_{n,j}$ determines univocally the corresponding s-sublattice and we allow some imprecision by referring ourselves to $\Upsilon_{n,j}$ as an s-sublattice. The advantage of looking at the s-sublattices as projected subsets on the x-axis is that we can compare these projections of the s-sublattices of different dynamics since all of them live in the same interval, Ω_0, of the x-axis. In Proposition 8.7 we obtain proximity of all the s-sublattices $\Upsilon_{n,j}$, with $n \leq N$, for a fixed integer N and sufficiently close Benedicks-Carleson parameters.

Let us give some insight into the argument. We consider $(a,b) \in \mathcal{BC}$ and Ω_∞ built according to Section 4. Let $N \in \mathbb{N}$ be given. We make some modifications in the procedure described in Subsection 4.4.1 where the s-sublattices are defined so that for each $\Upsilon_{n,j}$, where $n \leq N$, we obtain an approximation $\Upsilon^*_{n,j} \supset \Upsilon_{n,j}$ whose accuracy depends on the choice of a large integer N_3. Moreover, using Lemmas 6.3 and 7.2 we realize that, by construction, $\Upsilon^*_{n,j}$ also suits as an approximation of $\Upsilon'_{n,j} \subset \Upsilon^*_{n,j}$, which is an s-sublattice corresponding to $\Upsilon_{n,j}$ for a sufficiently close $(a',b') \in \mathcal{BC}$. The result follows once we verify that $|\Upsilon^*_{n,j} - \Upsilon_{n,j}| \approx |\Upsilon^*_{n,j} - \Upsilon'_{n,j}|$. Recall that, by construction, for each $\Upsilon_{n,j}$ there are $I \in \tilde{\mathcal{P}}_{n-1}$, $\omega = H(I)$ and n a regular return time for ω such that

$$\Upsilon_{n,j} = H^{-1}\left(f^{-n}(H(\Omega_\infty)) \cap \omega \cap \Lambda\right) = H^{-1}\left(f^{-n}(H(\Omega_\infty)) \cap \omega\right).$$

Observe that since $f^n(\omega) \geq 3|\Omega_0|$ then ω has a minimum length $|\omega| \geq 5^{-n}3|\Omega_0|$. This means that for n fixed there can only be a finite number of $\Upsilon_{n,j}$'s. In fact, if $v(n)$ denotes the number of $\Upsilon_{n,j}$ with $R(H(\Upsilon_{n,j})) = n$, then

$$v(n) \leq \frac{|\Omega_0|}{5^{-n}3|\Omega_0|} \leq 5^n. \tag{8.1}$$

Let $N \in \mathbb{N}$ be given and let $N_3 > 2N$ be a large integer whose choice will be specified later. Let $\varepsilon < b^{2N_3}$ be small. Consider $\mathcal{U}$ small enough so that condition (6.2) holds for such an ε and $\Omega_j = \Omega'_j$ for all $j \in \{0,\dots,N_3\}$ (recall Lemma 6.3), while Ω'_∞ is built in usual way out of Ω'_{N_3}.

For $n \leq N$ we carry out an inductive construction of sets $\tilde{\Omega}^*_n \subset \Omega_n$ and partitions $\tilde{\mathcal{P}}^*_n$ of $\tilde{\Omega}^*_n$ that will coincide for all $(a',b') \in \mathcal{U}$, for every $n \leq N$. This process must ensure that for every $n \leq N$ we have $\tilde{\Omega}^*_n \subset \tilde{\Omega}_n$, and if $\omega^* \in H(\tilde{\mathcal{P}}^*_n)$, then there is $\omega \in H(\tilde{\mathcal{P}}_n)$ such that $\omega \supset \omega^*$. Moreover, by choice of N_3 we will have that $\omega \setminus \omega^*$, when not empty, occupies the tips of ω and it corresponds to such a small part that if ω has a regular return at time $n < j \leq N$ then $f^j(\omega^*) \supset 2\Omega_0$ still traverses Q_0 by wide margins (see Lemma 8.1).

8.1.1. *Rules for defining $\tilde{\Omega}^*_n$, $\tilde{\mathcal{P}}_n^{\,*}$ and R^*.*

(0*) $\tilde{\Omega}^*_0 = \tilde{\Omega}'^*_0 = \Omega_0$, $\tilde{\mathcal{P}}^*_0 = \tilde{\mathcal{P}}'^*_0 = \{\Omega^*_0\}$.

Assume that $\tilde{\Omega}^*_{n-1} = \tilde{\Omega}'^*_{n-1}$ and that for each $I^* \in \tilde{\mathcal{P}}^*_{n-1}$ there is $I \in \tilde{\mathcal{P}}_{n-1}$ such that $I \supset I^*$. Take $I^* \in \tilde{\mathcal{P}}^*_{n-1} = \tilde{\mathcal{P}}'^*_{n-1}$. We denote $\omega = H(I)$, $\omega^* = H(I^*)$ and $\omega^{*\prime} = H'(I^*)$.

(1*) If $\omega \in \tilde{\mathcal{P}}_{n-1}$ does not make a regular return to $H(\Omega_0)$ at time n, put $\tilde{I}^* = I^* \cap \Omega_n$ into $\tilde{\Omega}^*_n$ and let $\tilde{\mathcal{P}}^*_n|_{\tilde{I}^*} = H^{-1}\left(f^{-n}_{a,b}\mathcal{P}\,\Big|_{H(\tilde{I}^*)}\right)$ with the usual adjoining of intervals.

We remark that if we were to apply this rule directly to $(a',b') \in \mathcal{U} \cap \mathcal{BC}$, where $\mathcal{U}$ is sufficiently small so that Corollary 5.3, Lemma 6.3 and equation (6.2) hold for such ε and N_3, then $\tilde{\Omega}'^*_n$ and $\tilde{\mathcal{P}}'^*_n$ would have discrepancies of $\mathcal{O}(\varepsilon)$ relative to $\tilde{\Omega}^*_n$ and $\tilde{\mathcal{P}}^*_n$ built for (a,b), respectively. But $\varepsilon < \mathrm{e}^{-2N_3}$ is negligible when compared to $\mathrm{e}^{-\alpha N}$ or $\mathrm{e}^{-\alpha N}/N^2$. Observe that the points of $H(\tilde{I}^*)$ never get any closer than $\mathrm{e}^{-\alpha N}$ from the critical set, up to time n, and $\mathrm{e}^{-\alpha N}/N^2$ is the minimum size of the elements of the partition $\mathcal{P}$ whose distance to the critical set is larger than $\mathrm{e}^{-\alpha N}$. Hence, there is no harm in setting $\tilde{\Omega}'^*_n = \tilde{\Omega}^*_n$ and $\tilde{\mathcal{P}}'^*_n = \tilde{\mathcal{P}}^*_n$.

Let $\mathcal{S}_{N_3}$ be the partition of Ω_{N_3} into connected components. We clearly have $\#\mathcal{S}_{N_3} \leq 2^{N_3}$. We write $f^n(z) \in H(\Omega_{N_3})$ if there exists $\sigma \in \mathcal{S}_{N_3}$ such that $f^n(z) \in Q^2_{N_3-1}(H(\sigma))$

where, as before, $Q^2_{N_3-1}(H(\sigma))$ is a $2(Cb)^{N_3}$-neighborhood of $Q_{N_3-1}(H(\sigma))$ in $\mathbb{R}^2$. This way let $f^{-n}(H(\Omega_{N_3}))$ have its obvious meaning. Observe that by definition of $Q^2_{N_3-1}(H(\sigma))$ and the construction of the long stable curves (namely (4.1)), then

$$f^{-n}(H(\Omega_{N_3})) \supset f^{-n}(H(\Omega_\infty)), \tag{8.2}$$

where we write $f^n(z) \in H(\Omega_\infty)$ when $f^n(z) \in \gamma^s(\zeta)$ for some $\zeta \in H(\Omega_\infty)$.

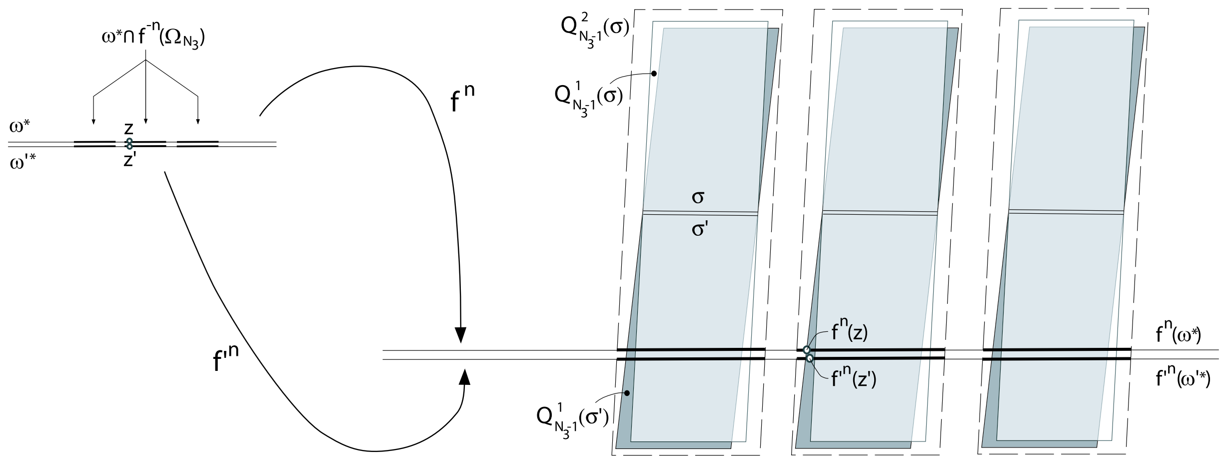

FIGURE 6

(2*) If $\omega \in \tilde{\mathcal{P}}_{n-1}$ makes a regular return at time n, we put

$$\tilde{I}^* = H^{-1}\left(\omega^* \setminus f^{-n}(H(\Omega_{N_3}))\right) \cap \Omega_n$$

into $\tilde{\Omega}^*_n$. Let $\mathcal{S}^*$ be the partition of $\tilde{I}^*$ into connected components. We define $\tilde{\mathcal{P}}^*_n|_{\tilde{I}^*} = H^{-1}\left(f^{-n}\mathcal{P}\Big|_{H(\tilde{I}^*)}\right) \bigvee \mathcal{S}^*$. For $z \in \omega^*$ such that $f^n(z) \in H(\Omega_{N_3})$ we define $R^*(z) = n$.

Suppose that $\mathcal{U}$ is sufficiently small so that as in Lemma 7.2 we have $Q^2_{N_3-1}(H(\sigma)) \supset Q^1_{N_3-1}(H'(\sigma))$ and, as before, Corollary 5.3 and condition (6.2) hold for the considered ε and N_3. Then, the smallness of $\varepsilon < b^{2N_3}$ when compared to the sizes of the elements $f^n\big(H(\tilde{\mathcal{P}}^*_n)\big)$ for $n \le N$ allows us to consider $\tilde{\Omega}^*_n = \tilde{\Omega}'^*_n$ and $\tilde{\mathcal{P}}^*_n = \tilde{\mathcal{P}}'^*_n$.

Essentially in this construction we substitute Ω_∞ by its finite approximation Ω_{N_3} in order to relate the partitions built for (a, b) with the ones built for $(a', b') \in \mathcal{U} \cap \mathcal{BC}$. The sets $\{R = n\} = \tilde{\Omega}_{n-1} \setminus \tilde{\Omega}_n$ were defined as the sets of points that at time n had their first regular return to $H(\Omega_\infty)$ (after sliding along γ^s stable curves). Now $\{R^* = n\} = \tilde{\Omega}^*_{n-1} \setminus \tilde{\Omega}^*_n$ is defined as the set of points that at time n have their first regular return to $H(\Omega_{N_3})$, where the sliding is made along the stable curve approximates, γ_{N_3}.

Let us make clear some aspects related to the previous rules. When we apply rule (2*) at step n, we ensure that for every $z \in \tilde{\Omega}^*_n$ we have $z \notin f^{-n}(H(\Omega_\infty))$. Let us verify that the same applies to $f_{a',b'}$, ie, since we are considering $\mathcal{U}$ sufficiently small so that Lemma 7.2, Corollary 5.3 and condition (6.2) hold for ε and N_3 in question, then for every $z' \in \tilde{\Omega}'^*_n$ we have $z' \notin f^{-n}_{a',b'}(H'(\Omega'_\infty))$ Since ε is irrelevant when compared to $2(Cb)^{N_3}$ we have for all $z' \in H'(\tilde{\Omega}^*_n)$ and for every $\sigma \in \mathcal{S}_{N_3}$, $\mathrm{dist}(f^n_{a',b'}(z'), Q_{N_3-1}(H(\sigma))) > 2(Cb)^{N_3} - \varepsilon$ which implies that $\operatorname{dist}(f^n_{a',b'}(z'), Q_{N_3-1}(H'(\sigma))) > (Cb)^{N_3}$, since by Lemma 7.2 we may assume that $Q^1_{N_3-1}(H'(\sigma)) \subset Q^2_{N_3-1}(H(\sigma))$ and $\operatorname{dist}(Q_{N_3-1}(H'(\sigma)), Q_{N_3-1}(H(\sigma))) \leq \varepsilon$. We have used "dist" to denote the usual distance between two sets.

In next lemma take into account that since $\Omega_{N_3} \supset \Omega_\infty$, then the gaps of Ω_∞ contain those of Ω_{N_3}, and so for all $n \leq N$ and $\omega^* \in H(\tilde{\mathcal{P}}^*_{n-1})$ there exists $\omega \in H(\tilde{\mathcal{P}}_{n-1})$ such that $\omega^* \subset \omega$.

LEMMA 8.1. *Let $n \leq N$, $\omega^* \in H(\tilde{\mathcal{P}}^*_{n-1})$ and consider $\omega \in H(\tilde{\mathcal{P}}_{n-1})$ such that $\omega^* \subset \omega$. If N_3 is large enough then $f^n(\omega) \setminus f^n(\omega^*)$, when not empty, occupies one or both tips of $f^n(\omega)$ and $|f^n(\omega) \setminus f^n(\omega^*)| < |\Omega_0|^2$.*

PROOF. Let $N_3 \in \mathbb{N}$ be sufficiently large so that

$$5^N \left(C_1 \delta^{1-3\beta} \frac{\mathrm{e}^{-\alpha(1-3\beta)(N_3+1)}}{1 - \mathrm{e}^{-\alpha(1-3\beta)}} + 2(Cb)^{N_3} \right) < |\Omega_0|^2. \tag{8.3}$$

For every $i \leq n-1$, let $\omega^*_i \in H(\tilde{\mathcal{P}}^*_i)$ be such that $\omega^* \subset \omega^*_i$ and let $\omega_i \in H(\tilde{\mathcal{P}}_i)$ be such that $\omega \subset \omega_i$. If $\omega \setminus \omega^* \neq \emptyset$ then at some time before $n-1$, rule (2*) was applied. Let $j \leq n-1$ be the last moment in the history of ω^* that rule 2* was applied. Then, $f^j(\omega^*_j)$ hits a gap of Ω_{N_3} while $f^j(\omega_j)$ hits a gap of Ω_∞. According to Lemma 6.1 the difference $f^j(\omega_j) \setminus f^j(\omega^*_j)$ has length of at most

$$C_1 \delta^{1-3\beta} \frac{\mathrm{e}^{-\alpha(1-3\beta)(N_3+1)}}{1 - \mathrm{e}^{-\alpha(1-3\beta)}} + 2(Cb)^{N_3},$$

where the last term results from the fact that we are using $2(Cb)^{N_3}$- neighborhoods of the rectangles spanned by the approximate stable curves. Moreover, $f^j(\omega_j) \setminus f^j(\omega^*_j)$ clearly occupies the tips of $f^j(\omega_j)$.

Now, for simplicity suppose that $\omega = \omega_j$ and $\omega^* = \omega^*_j$. We have that $f^n(\omega) \setminus f^n(\omega^*)$ occupies the tips of $f^n(\omega)$. This geometric property is inherited since by construction we are away from the folds and f is a diffeomorphism. Also, up to time n, $|f^j(\omega_j) \setminus f^j(\omega^*_j)|$ can grow no more than 5^{n-j}. Consequently, by choice of N_3 we must have $|f^n(\omega) \setminus f^n(\omega^*)| < |\Omega_0|^2$.

In the case that $\omega \neq \omega_j$ it means that ω_j will suffer exclusions or subdivisions. Nevertheless, the points of $(\omega_j - \omega^*_j) \cap \omega$ still occupy the tip of $f^n(\omega)$. $\square$

REMARK 8.2. Observe that by choice of N_3 we have that if $\omega^* \in H(\tilde{\mathcal{P}}^*_{n-1})$ and $f^n(\omega)$ makes a regular return then $f^n(\omega^*) \supset (3 - |\Omega_0|^2)\Omega_0$. This means that for $\mathcal{U}$ sufficiently small $f^n_{a',b'}(\omega'^*) \supset (3 - |\Omega_0|^2 - \varepsilon)\Omega_0$.

When at step n we have to apply rule (2*) we make more exclusions from $\tilde{\Omega}^*_{n-1}$ than we would if we were to apply rule (2) as in [**BY00**]. Essentially we are excluding the points that hit $H(\Omega_{N_3})$ instead of only removing the points that hit $H(\Omega_\infty)$ ($\Omega_\infty \subset \Omega_{N_3}$). We argue that by adequate choice of N_3 this over exclusion will not affect the sets $\{R^* = j\}$ with $j \in \{n+1, \ldots, N\}$.

LEMMA 8.3. *Suppose that x is a point that at step n should be excluded by rule (2*) but is not excluded according to rule (2). If N_3 is large enough, then $H(x)$ does not have a regular return to Ω_0 before N.*

Proof. Let $N_3 \in \mathbb{N}$ be sufficiently large so that (8.3) holds and take $\sigma \in \mathcal{S}_{N_3}$. When we apply rule (2*) at step n we remove from $\tilde{\Omega}^*_{n-1}$ all the points hitting $H(\sigma)$, while if we had applied rule (2) instead we would have only removed the points hitting $H(\sigma \cap \Omega_\infty)$. Consider a gap ϖ of $H(\sigma \cap \Omega_\infty)$. We know that the length of ϖ is less than

$$C_1 \delta^{1-3\beta} \frac{e^{-\alpha(1-3\beta)N_3}}{1-e^{-\alpha(1-3\beta)}}.$$

If $\partial\varpi \cap \partial H(\sigma) = \emptyset$ then $\varpi \in \tilde{\mathcal{P}}_n$ and in N iterations it would grow to reach at most the length

$$5^N C_1 \delta^{1-3\beta} \frac{e^{-\alpha(1-3\beta)N_3}}{1-e^{-\alpha(1-3\beta)}} < |\Omega_0|^2 \ll 3|\Omega_0|.$$

Thus, ϖ would not have any regular return to Ω_0 before N.

If $\partial\varpi \cap \partial H(\sigma) \neq \emptyset$, then there is a gap $\hat{\varpi}$ of $H(\Omega_\infty)$ so that $\hat{\varpi} \in \tilde{\mathcal{P}}_n$ and ϖ occupies a tip of $\hat{\varpi}$. Clearly, $\hat{\varpi}$ could have a regular return at $j \in \{n+1, \ldots, N\}$, say. However, by construction $f^j(\varpi)$ will occupy one tip of $f^j(\hat{\varpi})$. Since

$$\left|f^j(\varpi)\right| < 5^N C_1 \delta^{1-3\beta} \frac{e^{-\alpha(1-3\beta)N_3}}{1-e^{-\alpha(1-3\beta)}} < |\Omega_0|^2$$

and $|f^j(\hat{\varpi})| \gtrsim 3|\Omega_0|$ we still have that $f^j(\varpi)$ does not hit Ω_0. We remark that $\hat{\varpi}$ could have suffered subdivisions and exclusions according to rule (1*) before time j. Nevertheless, the points from ϖ that survive the exclusions still occupy the tip of the piece that will contain them at the time of its regular return and the argument applies again. □

By the rules in Subsection 4.4.1, for every s-sublattice $\Upsilon_{n,j}$ there is a segment $\omega_{n,j} \in H(\tilde{\mathcal{P}}_{n-1})$ such that n is a regular return time for $\omega_{n,j}$ and

$$\Upsilon_{n,j} = H^{-1}\left(\omega_{n,j} \cap f^{-n}(\Omega_\infty)\right). \tag{8.4}$$

Lemmas 8.1 and 8.3 allow us to conclude that if $\omega_{n,j} \in \tilde{\mathcal{P}}_{n-1}$ and $n \leq N$ is a regular return time for $\omega_{n,j}$ then there is $\omega^*_{n,j} \in \tilde{\mathcal{P}}^*_{n-1}$ such that $\omega^*_{n,j} \subset \omega_{n,j}$ and $|f^n(\omega_{n,j})| = |f^n(\omega^*_{n,j})| + \mathcal{O}(|\Omega_0|^2)$. Moreover, because the difference between $\omega_{n,j}$ and $\omega^*_{n,j}$ is only in their tips we may write

$$\Upsilon_{n,j} = H^{-1}\left(\omega^*_{n,j} \cap f^{-n}(\Omega_\infty)\right). \tag{8.5}$$

Attending to the procedure above and equation (8.5), given an s-sublattice $\Upsilon_{n,j}$, with $n \leq N$ we define its approximation

$$\Upsilon^*_{n,j} = H^{-1}\left(\omega^*_{n,j} \cap f^{-n}(\Omega_{N_3})\right). \tag{8.6}$$

Taking into consideration (8.2) we have that $\Upsilon_{n,j} \subset \Upsilon^*_{n,j}$, from where we conclude that $\forall n \in \{1, \ldots, N\}$,

$$\{R = n\} = \bigcup_{j \leq v(n)} \Upsilon_{n,j} \subset \bigcup_{j \leq v(n)} \Upsilon^*_{n,j} = \{R^* = n\}.$$

We wish to verify that this substitution of Ω_∞ by Ω_{N_3} does not produce significant changes. In fact, we will show in the next lemma that $\Upsilon_{n,j}$ and $\Upsilon^*_{n,j}$ are very close for all $n \leq N$ and $j \leq v(n)$.

LEMMA 8.4. *Let $\varepsilon > 0$, $N \in \mathbb{N}$ and an s-sublattice $\Upsilon_{n,j}$ with $n \le N$ be given. If N_3 is large enough, then*

$$\left|\Upsilon^*_{n,j} \setminus \Upsilon_{n,j}\right| < \varepsilon \quad \text{and} \quad |\{R^* = n\} \setminus \{R = n\}| < \varepsilon.$$

PROOF. Choose N_3 large enough so that

$$C_1|\Omega_{N_3} \setminus \Omega_\infty| < \varepsilon. \tag{8.7}$$

Let $\omega^*_{n,j}$ be such that $H(\Upsilon_{n,j}) = \omega^*_{n,j} \cap f^{-n}(H(\Omega_\infty))$ and $H(\Upsilon^*_{n,j}) = \omega^*_{n,j} \cap f^{-n}(H(\Omega_{N_3}))$. By bounded distortion we have

$$\frac{|H(\Upsilon^*_{n,j}) \setminus H(\Upsilon_{n,j})|}{|\omega^*_{n,j}|} \le C_1 \frac{\left|f^n\left(H(\Upsilon^*_{n,j}) \setminus H(\Upsilon_{n,j})\right)\right|}{|f^n(\omega^*_{n,j})|} \le C_1 \frac{|\Omega_{N_3} \setminus \Omega_\infty|}{2|\Omega_0|}.$$

Attending to (8.7) this gives that $\left|\Upsilon^*_{n,j} \setminus \Upsilon_{n,j}\right| < \varepsilon$. Besides,

$$\begin{aligned}
|\{R^* = n\} \setminus \{R = n\}| &= \sum_{j \le v(n)} |\Upsilon^*_{n,j} \setminus \Upsilon_{n,j}| \le \sum_{j \le v(n)} C_1 \frac{|\Omega_{N_3} \setminus \Omega_\infty|}{|\Omega_0|} |\omega^*_{n,j}| \\
&\le C_1|\Omega_{N_3} \setminus \Omega_\infty| \\
&< \varepsilon,
\end{aligned}$$

by the choice of N_3. □

REMARK 8.5. By definition of $f^{-n}(H(\Omega_{N_3}))$, in the estimates above we should have considered $\left|\Omega^2_{N_3} \setminus \Omega_\infty\right|$, where $\Omega^2_{N_3}$ is a $2(Cb)^{N_3}$-neighborhood of Ω_{N_3}. However, since Ω_{N_3} has at most 2^{N_3} connected components, then the difference to the estimates above would be at most $2^{N_3+1}(Cb)^{N_3}$, which is as small as we want if we choose N_3 large enough.

REMARK 8.6. The estimates in the proof were used taking $H(\Omega_{N_3})$ and $H(\Omega_\infty)$ as subsets of W_1. According to [**BY00**, Remark 5], upon re-scaling the estimates still work if we consider them as subsets of $\gamma^u \in \Gamma^u$, due to Lemma 2 of [**BY00**].

PROPOSITION 8.7. *Let $(a, b) \in \mathcal{BC}$, $N \in \mathbb{N}$ and $\varepsilon > 0$ be given. There is a neighborhood $\mathcal{U}$ of (a, b) such that for all $(a', b') \in \mathcal{U} \cap \mathcal{BC}$ given any s-sublattice $\Upsilon_{n,j} \subset \Omega_\infty$, with $n \le N$ and $j \le v(n)$, then the corresponding s-sublattice $\Upsilon'_{n,j} \subset \Omega'_\infty$ is such that*

$$|\Upsilon_{n,j} \triangle \Upsilon'_{n,j}| < \varepsilon \quad \text{and} \quad |\{R = n\} \triangle \{R' = n\}| < \varepsilon.$$

PROOF. By Lemma 6.3 we are assuming that Ω'_∞ is built out of Ω_{N_3} in the usual way for $f_{a',b'}$ with $(a', b') \in \mathcal{U} \cap \mathcal{BC}$. Lemma 7.2 assures that if $\mathcal{U}$ is small enough then $Q^2_{N_3-1}(H(\sigma))$, which is a $2(Cb)^{N_3}$- neighborhood of $Q_{N_3-1}(H(\sigma))$, contains $Q^1_{N_3-1}(H'(\sigma))$ for every $\sigma \in \mathcal{S}_{N_3}$. Moreover, for any $x \in \sigma$

$$\|\gamma_{N_3}(H(x)) - \gamma'_{N_3}(H'(x))\|_0 < b^{N_2+1}.$$

Let N_3 be chosen according to equations (8.3) and (8.7) so that Lemmas 8.1 and 8.4 hold. Let $\Upsilon_{n,j}$, with $n \le N$, be a given s-sublattice of $H(\Omega_\infty)$. Let $I^*_{n,j} \in \tilde{\mathcal{P}}^*_{n-1}$ be such that

$$\Upsilon_{n,j} = H^{-1}\left(\omega^*_{n,j} \cap f^{-n}(H(\Omega_\infty))\right),$$

where $\omega^*_{n,j} = H(I^*_{n,j})$. Suppose that $\mathcal{U}$ is sufficiently small so that the construction of the partition is carried out simultaneously for the dynamics $f_{a',b'}$ correspondent to any

$(a',b') \in \mathcal{U} \cap \mathcal{BC}$ and so that $\tilde{\mathcal{P}}^*_m = \tilde{\mathcal{P}}'^*_m$, for all $m \leq N$, as it has been described in the procedure above. Then, $f^n_{a',b'}(\omega'^*_{n,j}) = f^n_{a',b'}(H'(I^*_{n,j}))$ crosses Q_0 by wide margins and we may define

$$\Upsilon'_{n,j} = H'^{-1}\left(\omega'^*_{n,j} \cap f^{-n}_{a',b'}(H'(\Omega'_\infty))\right).$$

Consider the approximation $\Upsilon^*_{n,j}$ built in (8.6) for $\Upsilon_{n,j}$. We have seen that $\Upsilon_{n,j} \subset \Upsilon^*_{n,j}$ and using Lemma 8.4 we may suppose that $\left|\Upsilon^*_{n,j} \setminus \Upsilon_{n,j}\right| < \varepsilon/2$. Now, we shall see that $\Upsilon^*_{n,j}$ is also a good approximation for $\Upsilon'_{n,j}$ if $\mathcal{U}$ is sufficiently small.

First, we verify that $\Upsilon'_{n,j} \subset \Upsilon^*_{n,j}$. Let $x \in \Upsilon'_{n,j}$, $z = H(x)$ and $z' = H'(x)$. We need to check that if $f^n_{a',b'}(z') \in \Lambda'$, then $f^n(z) \in Q^2_{N_3-1}(H(\sigma))$ for some $\sigma \in \mathcal{S}_{N_3}$. We are supposing that $\mathcal{U}$ is sufficiently small so that (6.2) holds for $\varepsilon < b^{2N_3}$ up to N_3, which implies that $|f^n(z) - f^n_{a',b'}(z')| < b^{2N_3}$. Since $\Lambda' \subset \bigcup_{\sigma \in \mathcal{S}_{N_3}} Q^1_{N_3-1}(H'(\sigma))$, we have $f^n_{a',b'}(z') \in Q^1_{N_3-1}(H'(\sigma))$ for some $\sigma \in \mathcal{S}_{N_3}$. Under the assumptions described in the procedure above (namely that $Q^1_{N_3-1}(H'(\sigma)) \subset Q^2_{N_3-1}(H(\sigma))$) and attending to equation (7.2) we get that $\operatorname{dist}\left(f^n_{a',b'}(z'), Q_{N_3-1}(H(\sigma))\right) < 3/2(Cb)^{N_3}$, and thus $\operatorname{dist}\left(f^n(z), Q_{N_3-1}(H(\sigma))\right) < 2(Cb)^{N_3}$.

Additionally, since the upper bound used for $|\Omega_{N_3} \setminus \Omega_\infty|$ also works for $|\Omega_{N_3} \setminus \Omega'_\infty|$ and the width of $Q^1_{N_3-1}(H'(\sigma))$ differs from the width of $Q^2_{N_3-1}(H(\sigma))$ by $\mathcal{O}((Cb)^{N_3})$ we observe that the argument used in Lemma 8.4 gives us that $\left|\Upsilon^*_{n,j} \setminus \Upsilon'_{n,j}\right| < \varepsilon/2$. Therefore

$$|\Upsilon_{n,j} \triangle \Upsilon'_{n,j}| \leq \left|\Upsilon^*_{n,j} \setminus \Upsilon_{n,j}\right| + \left|\Upsilon^*_{n,j} \setminus \Upsilon'_{n,j}\right| < \varepsilon,$$

which gives the first part of the conclusion.

Suppose now that Lemma 8.4 holds and $|\{R^* = n\} \setminus \{R = n\}| < \varepsilon/2$. Observing that $\{R' = n\} = \cup_{j \leq v(n)} \Upsilon'_{n,j}$, then arguing as in Lemma 8.4, we have $|\{R^* = n\} \setminus \{R' = n\}| < \varepsilon/2$, as long as $\mathcal{U}$ is sufficiently small. Finally,

$$|\{R = n\} \triangle \{R' = n\}| \leq |\{R = n\} \triangle \{R^* = n\}| + |\{R^* = n\} \triangle \{R' = n\}| < \varepsilon.$$

$\square$

8.2. Proximity after k returns. Given $z \in H(\Omega_\infty)$ we define

$$R^1(z) = R(z) \quad \text{and} \quad R^{i+1}(z) = R\left(f^{R^1+\ldots+R^i}(z)\right), \quad \text{for } i \geq 1.$$

Observe that $R^1 \equiv n$ in $\Upsilon_{n,j}$. Since $f^R(H(\Upsilon_{n,j}))$ hits each stable leaf of Λ, it makes sense to partition $f^R(H(\Upsilon_{n,j}))$ using again the levels $H(\Upsilon_{n,j})$, and set

$$\Upsilon_{(n_1,j_1)(n_2,j_2)} = \Upsilon_{n_1,j_1} \cap H^{-1}\left(f^{-n_1}(H(\Upsilon_{n_2,j_2}))\right).$$

In general, given $k \in \mathbb{N}$, we consider

$$\Upsilon_{(n_1,j_1)\ldots(n_k,j_k)} = \Upsilon_{n_1,j_1} \cap H^{-1}\left(f^{-n_1}(H(\Upsilon_{n_2,j_2}))\right) \cap \ldots \cap H^{-1}\left(f^{-(n_1+\cdots+n_{k-1})}(H(\Upsilon_{n_k,j_k}))\right).$$

Notice that for every $z \in H(\Upsilon_{(n_1,j_1)\ldots(n_k,j_k)})$ we have $R^i(z) = n_i$ for $1 \leq i \leq k$.

The main result in this subsection (Proposition 8.9) states that if we fix a parameter $(a,b) \in \mathcal{BC}$ and $N \in \mathbb{N}$, then there is a neighborhood $\mathcal{U}$ of (a,b) in $\mathbb{R}^2$ such that for any set $\Upsilon_{(n_1,j_1)\ldots(n_k,j_k)}$ considered, with $n_1,\ldots,n_k \leq N$, it is possible to build a shadow set $\Upsilon'_{(n_1,j_1)\ldots(n_k,j_k)}$ close to the original one, for any $(a',b') \in \mathcal{U} \cap \mathcal{BC}$.

Recall that each $H(\Upsilon_{n,j}) = \omega_{n,j} \cap f^{-n}(\Omega_\infty)$ may also be written as $H(\Upsilon_{n,j}) = \omega^*_{n,j} \cap f^{-n}(H(\Omega_\infty))$, where $\omega_{n,j} \supset \omega^*_{n,j}$ and n is a regular return time for $\omega_{n,j}$. The next result claims that something similar holds for $\Upsilon_{(n_1,j_1)\dots(n_k,j_k)}$. We say that $z \in f^{-\ell}(\omega^*_{n,j})$ whenever $f^\ell(z) \in Q^2_n(\omega^*_{n,j})$, while, as usual, $z \in f^{-\ell}(H(\Omega_\infty))$ means that $f^\ell \in \gamma^s(\zeta)$ for some $\zeta \in H(\Omega_\infty)$.

LEMMA 8.8. *Taking $n_0 = 0$ and $n_1, \dots, n_k$ with $n_i \le N$, we have*

$$H(\Upsilon_{(n_1,j_1)\dots(n_k,j_k)}) = \bigcap_{i=0}^{k-1} f^{-(n_0+\dots+n_i)}(\omega^*_{n_{i+1},j_{i+1}}) \cap f^{-(n_1+\dots+n_k)}(H(\Omega_\infty)). \tag{8.8}$$

PROOF. We begin with the easier inclusion

$$H(\Upsilon_{(n_1,j_1)\dots(n_k,j_k)}) \subset \bigcap_{i=0}^{k-1} f^{-(n_0+\dots+n_i)}(\omega^*_{n_{i+1},j_{i+1}}) \cap f^{-(n_1+\dots+n_k)}(H(\Omega_\infty)).$$

Observe that $Q(H(\Upsilon_{n_i,j_i})) \subset Q^2_{n_i}(\omega^*_{n_i,j_i})$, where $Q(H(\Upsilon_{n_i,j_i}))$ is the rectangle spanned by $\bar\pi^{-1}(H(\Upsilon_{n_i,j_i}))$. If $z \in f^{-(n_1+\dots+n_{k-1})}(H(\Upsilon_{n_k,j_k}))$, then $f^{n_1+\dots+n_{k-1}}(z) \in \gamma^s(\zeta)$ for some $\zeta \in H(\Upsilon_{n_k,j_k})$. By definition of Υ_{n_k,j_k} we have $f^{n_k}(\zeta) \in \gamma^s(\hat\zeta)$ for some $\hat\zeta \in H(\Omega_\infty)$. Then, [**BY00**, Lemma 2(3)] gives that $f^{n_1+\dots+n_k}(z) \in \gamma^s(\hat\zeta)$, which implies that $z \in f^{-(n_1+\dots+n_k)}(H(\Omega_\infty))$.

Let us consider now the other inclusion. Since $H(\Upsilon_{n_i,j_i}) = \omega^*_{n_i,j_i} \cap f^{-n_i}(H(\Omega_\infty))$ we only need to verify that for every $i \in \{0, \dots, k-1\}$

$$z \in \bigcap_{i=0}^{k-1} f^{-(n_0+\dots+n_i)}(\omega^*_{n_{i+1},j_{i+1}}) \cap f^{-(n_1+\dots+n_k)}(H(\Omega_\infty)) \quad \Rightarrow \quad f^{n_1+\dots+n_i}(z) \in H(\Omega_\infty).$$

By [**BY00**, Lemma 3] we have

$$\left(\bigcup_{\zeta\in\Omega_\infty} \gamma^s(\zeta)\right) \bigcap f^{n_{i+1}}\left(Q^2_{n_{i+1}}(\omega^*_{n_{i+1},j_{i+1}})\right) \subset \bigcup_{\zeta\in\Omega_\infty} f^{n_{i+1}}\left(\gamma^s(\zeta)\right).$$

As $f^{n_1+\dots+n_{i+1}}(z) \in \left(\bigcup_{\zeta\in\Omega_\infty} \gamma^s(\zeta)\right) \bigcap f^{n_{i+1}}\left(Q^2_{n_{i+1}}(\omega^*_{n_{i+1},j_{i_1}})\right)$, then there exists $\zeta \in H(\Omega_\infty)$ such that $f^{n_1+\dots+n_{i+1}}(z) \in f^{n_{i+1}}\left(\gamma^s(\zeta)\right)$, which is equivalent to say that $f^{n_1+\dots+n_i}(z) \in \gamma^s(\zeta)$. This means that $f^{n_1+\dots+n_i}(z) \in H(\Omega_\infty)$. $\square$

PROPOSITION 8.9. *Let $(a,b) \in \mathcal{BC}$, $N \in \mathbb{N}$, $k \in \mathbb{N}$ and $\varepsilon > 0$ be given. There is an open neighborhood $\mathcal{U}$ of (a,b) such that for each $(a',b') \in \mathcal{U} \cap \mathcal{BC}$ and $\Upsilon_{(n_1,j_1)\dots(n_k,j_k)}$ there is $\Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}$ such that in $H'(\Upsilon'_{(n_1,j_1)\dots(n_k,j_k)})$ we have $R'^1 = n_1, \dots, R'^k = n_k$ and*

$$\left|\Upsilon_{(n_1,j_1)\dots(n_k,j_k)} \bigtriangleup \Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}\right| < \varepsilon.$$

PROOF. The idea is to build for each $\Upsilon_{(n_1,j_1)\dots(n_k,j_k)}$, with $n_1, \dots, n_k \le N$, an approximation $\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)} \supset \Upsilon_{(n_1,j_1)\dots(n_k,j_k)}$ such that

$$\left|\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)} \setminus \Upsilon_{(n_1,j_1)\dots(n_k,j_k)}\right| < \frac{\varepsilon}{2}$$

and realize that $\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)}$ also suits as an approximation for $\Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}$, as long as $\mathcal{U}$ is sufficiently small. We obtain an approximation of $\Upsilon_{(n_1,j_1)\dots(n_k,j_k)}$ simply by substituting Ω_∞ by Ω_{N_4} in (8.8) for some large N_4. As before we say that $f^n(z) \in H(\Omega_{N_4})$ whenever there is $\sigma \in \mathcal{S}_{N_4}$ such that $f^n(z) \in Q^2_{N_4-1}(H(\sigma))$, which is a $2(Cb)^{N_4}$- neighborhood of $Q_{N_4-1}(H(\sigma))$ in $\mathbb{R}^2$.

Define

$$\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)} = H^{-1}\left(\bigcap_{i=0}^{k-1} f^{-(n_0+\dots+n_i)}(\omega^*_{n_{i+1},j_{i+1}}) \cap f^{-(n_1+\dots+n_k)}(H(\Omega_{N_4}))\right).$$

Since $\Omega_\infty \subset \Omega_{N_4}$ we clearly have that $\Upsilon_{(n_1,j_1)\dots(n_k,j_k)} \subset \Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)}$. Let us now obtain an estimate of $\left|\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)} \setminus \Upsilon_{(n_1,j_1)\dots(n_k,j_k)}\right|$. Considering

$$\omega = \bigcap_{i=0}^{k-1} f^{-(n_0+\dots+n_i)}(\omega^*_{n_{i+1},j_{i+1}}), \quad \omega^* = H(\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)}), \quad \tilde{\omega} = H(\Upsilon_{(n_1,j_1)\dots(n_k,j_k)})$$

we get

$$\frac{|\omega^* \setminus \tilde{\omega}|}{|\omega|} \le C_1 \frac{\left|f^{n_1+\dots+n_k}(\omega^*) \setminus f^{N_1+\dots+n_k}(\tilde{\omega})\right|}{f^{n_1+\dots+n_k}(\omega)} \le \frac{C_1}{2|\Omega_0|}\left(|\Omega_{N_4} \setminus \Omega_\infty| + 4(Cb)^{N_4}\right)$$

Thus, if N_4 is sufficiently large we have

$$\left|\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)} \setminus \Upsilon_{(n_1,j_1)\dots(n_k,j_k)}\right| < \frac{\varepsilon}{2}. \tag{8.9}$$

Suppose now that we take a sufficiently small neighborhood $\mathcal{U}$ of (a,b) so that if $(a',b') \in \mathcal{U} \cap \mathcal{BC}$, then the following conditions hold:

(1) Ω'_∞ is built out of $\Omega'_{N_4} = \Omega_{N_4}$ in the usual way, as in Lemma 6.3;
(2) $Q^2_{N_4-1}(H(\sigma)) \supset Q^1_{N_4-1}(H'(\sigma))$ for each $\sigma \in \mathcal{S}_{N_4}$ and, as in Lemma 7.2,
$$\operatorname{dist}\left(Q_{N_4-1}(H(\sigma)), Q_{N_4-1}(H'(\sigma))\right) < b^{N_4+1};$$
(3) the procedure in Subsection 8.1 leads to $\tilde{\Omega}^*_n = \tilde{\Omega}'^*_n$ and $\tilde{\mathcal{P}}^*_n = \tilde{\mathcal{P}}'^*_n$, for all $n \le N$;
(4) equation (6.2) holds for b^{2N_4} up to kN.

Within $\mathcal{U}$ it makes sense to define

$$\Upsilon'_{(n_1,j_1)\dots(n_k,j_k)} = H'^{-1}\left(\bigcap_{i=0}^{k-1} f_{a',b'}^{-(n_0+\dots+n_i)}(\omega^*_{n_{i+1},j_{i+1}}) \cap f_{a',b'}^{-(n_1+\dots+n_k)}(H'(\Omega'_\infty))\right).$$

Moreover, one realizes that $\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)}$ is a good approximation of $\Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}$. In fact, we have that $\Upsilon'_{(n_1,j_1)\dots(n_k,j_k)} \subset \Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)}$. To see this, observe first that the discrepancies of order b^{2N_4} in the tips of the intervals $H^{-1}\left(f^{-(n_1+\dots+n_i)}(\omega_{n_{i+1},ji+1}) \cap H(\Omega_0)\right)$ and $H'^{-1}\left(f_{a',b'}^{-(n_1+\dots+n_i)}(\omega_{n_{i+1},j_{i+1}}) \cap H'(\Omega_0)\right)$ are negligible since we are only interested in the points of the center of this intervals that hit Ω_0 at their last regular return. Finally, note that by conditions (1), (2) and (4) above, we must have $x \in \Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)}$ whenever $x \in \Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}$. Otherwise, we would have an $x \in \Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}$ such that $z' = f_{a',b'}^{n_1+\dots+n_k}(H'(x)) \in Q^1_{N_4-1}(H'(\sigma))$ for some $\sigma \in \mathcal{S}_{N_4}$ and $z = f^{n_1+\dots+n_k}(H(x)) \notin Q^2_{N_4-1}(H(\sigma))$, for all $\sigma \in \mathcal{S}_{N_4}$. But $z \notin Q^2_{N_4-1}(H(\sigma))$ implies that $\operatorname{dist}(z, Q_{N_4-1}(H(\sigma))) > 2(Cb)^{N_4}$, from where one derives by (2) that

$$\operatorname{dist}(z, Q_{N_4-1}(H'(\sigma))) > 2(Cb)^{N_4} - b^{N_4+1} > \frac{3}{2}(Cb)^{N_4}$$

and

$$\operatorname{dist}\left(z, Q^1_{N_4-1}(H'(\sigma))\right) > \frac{1}{2}(Cb)^{N_4}.$$

However, by (4), $\operatorname{dist}(z,z') < b^{2N_4}$ yields $\operatorname{dist}\left(z, Q^1_{N_4-1}(H'(\sigma))\right) < b^{2N_4}$.

The argument used above to obtain the estimate (8.9) also gives that, for N_4 large enough and $\mathcal{U}$ sufficiently small, $\left|\Upsilon^*_{(n_1,j_1)\dots(n_k,j_k)} \setminus \Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}\right| < \varepsilon/2$, from where one easily deduces that

$$\left|\Upsilon_{(n_1,j_1)\dots(n_k,j_k)} \triangle \Upsilon'_{(n_1,j_1)\dots(n_k,j_k)}\right| < \varepsilon.$$

□

9. Statistical stability

Fix a parameter $(a_0,b_0) \in \mathcal{BC}$ and a horseshoe Λ_0 given by Proposition 4.1. Consider a sequence $(a_n,b_n) \in \mathcal{BC}$ converging to (a_0,b_0). For each $n \geq 0$ set $f_n = f_{a_n,b_n}$ and assign an adequate horseshoe Λ_n in the sense of Proposition 4.1. Let W_1^n denote the leaf of first generation of the unstable manifold through z_n^*, the unique fixed point of f_n in the first quadrant, and a parametrization $H_n : \Omega_0 \to W_1^n$ of the segment of W_1^n that projects vertically onto Ω_0 as in Section 6. Setting $\Omega_\infty^n = H_n^{-1}(\Lambda_n \cap H_n(\Omega_0))$ let $R_n : \Lambda_n \to \mathbb{N}$ denote the return time function and $F_n = f_n^{R_n} : \Lambda_n \to \Lambda_n$. For every $z \in \Lambda_n$ we denote by $\gamma_n^s(z)$ the long stable curve through z.

According to Corollary 6.4 and Propositions 7.3 and 8.7, we assume that all these objects have been constructed in such a way that:

(1) $|\Omega_\infty^n \triangle \Omega_\infty^0| \to 0$ as $n \to \infty$;
(2) $\gamma_n^s(H_n(x)) \to \gamma_0^s(H_0(x))$ as $n \to \infty$ in the C^1-topology;
(3) for $N \in \mathbb{N}$ and $1 \leq j \leq N$ we have $|\{R_n = j\} \triangle \{R_0 = j\}| \to 0$ as $n \to \infty$.

As mentioned is Section 3.8, we know that for all $n \in \mathbb{N}_0$ there is a unique SRB measure ν_n. Our goal is to show that $\nu_n \to \nu_0$ in the weak* topology, i.e. for all continuous functions $g : \mathbb{R}^2 \to \mathbb{R}$ the integrals $\int g d\nu_n$ converge to $\int g d\nu_0$. We will show that given any continuous $g : \mathbb{R}^2 \to \mathbb{R}$, each subsequence of $\int g d\nu_n$ admits a subsequence converging to $\int g d\nu_0$.

9.1. A subsequence in the quotient horseshoe. We begin by considering for each $n \in \mathbb{N}_0$, the quotient horseshoes $\bar{\Lambda}_n$ obtained from Λ_n by collapsing stable curves, as in Section 4.5, and the quotient map $\bar{F}_n = \overline{f_n^{R_n}} : \bar{\Lambda}_n \to \bar{\Lambda}_n$. Every unstable leaf γ_n^u in the definition of Λ_n suits as a model for $\bar{\Lambda}_n$, through the identification of each point $z \in \gamma_n^u \cap \Lambda_n$ with its equivalence class, $\gamma_n^s(z) \in \bar{\Lambda}_n$. We have seen in Section 4.5 that there exists a well defined reference measure in $\bar{\Lambda}_n$, denoted by $\bar{m}_n$. From here and henceforth, for each $n \in \mathbb{N}_0$ we fix the unstable leaf $H_n(\Omega_0)$ and take $H_n(\Omega_0) \cap \Lambda_n = H_n(\Omega_\infty^n)$ as our model for $\bar{\Lambda}_n$. The measure whose density with respect to Lebesgue measure on $H_n(\Omega_0)$ is $\mathbf{1}_{H_n(\Omega_\infty^n)}$ will be our representative for the reference measure $\bar{m}_n$, where $\mathbf{1}_{(\cdot)}$ is the indicator function. In fact we will allow some imprecision by identifying $\bar{\Lambda}_n$ with $H_n(\Omega_\infty^n)$ and $\bar{m}_n$ with its representative on $H_n(\Omega_0)$.

As referred in Section 4.5, for each $n \in \mathbb{N}_0$ there is an $\bar{F}_n$-invariant density $\bar{\rho}_n$, with respect to the reference measure $\bar{m}_n$. We may assume that each $\bar{\rho}_n$ is defined in the interval

Ω_0 and $\bar{\rho}_n(x) = \mathbf{1}_{\Omega_\infty^n}(x)\bar{\rho}_n(H_n(x))$ for every $x \in \Omega_0$. This way we have the sequence $(\bar{\rho}_n)_{n\in\mathbb{N}_0}$ defined on the same interval Ω_0.

LEMMA 9.1. *There is $M > 0$ such that $\|\bar{\rho}_n\|_\infty \leq M$ for all $n \geq 0$.*

PROOF. We follow the proof of [**Yo98**, Lemma 2] and construct $\bar{\rho}$ as the density with respect to $\bar{m}$ of an accumulation point of $\bar{\nu}^n = 1/n\sum_{i=0}^{n-1}\bar{F}_*^i(\bar{m})$. Let $\bar{\rho}^n$ denote the density of $\bar{\nu}^n$ and $\bar{\rho}^i$ the density of $\bar{F}_*^i(\bar{m})$. Also, let $\bar{\rho}^i = \sum_j \bar{\rho}_j^i$, where $\bar{\rho}_j^i$ is the density of $\bar{F}_*^i(\bar{m}|\sigma_j^i)$ and the σ_j^i's range over all components of $\bar{\Lambda}$ such that $\bar{F}^i(\sigma_j^i) = \bar{\Lambda}$.

Consider the normalized density $\tilde{\rho}_j^i = \bar{\rho}_j^i/\bar{m}(\sigma_j^i)$. Let $J\bar{F}$ denote the Radon-Nikodym derivative $\frac{d(\bar{F}_*^{-1}\bar{m})}{d\bar{m}}$. Observing that $\bar{m}(\sigma_j^i) = \bar{F}_*^i\bar{m}(\bar{F}^i(\sigma_j^i))$ we have for $\bar{x}' \in \sigma_j^i$ such that $\bar{x} = \bar{F}^i(\bar{x}')$ and for some $\bar{y}' \in \sigma_j^i$

$$\tilde{\rho}_j^i(\bar{x}) \lesssim \frac{J\bar{F}^i(\bar{y}')}{J\bar{F}^i(\bar{x}')}(\bar{m}(\bar{\Lambda}))^{-1} = \prod_{k=1}^{i}\frac{J\bar{F}(\bar{F}^{k-1}(\bar{y}'))}{J\bar{F}(\bar{F}^{k-1}(\bar{x}'))}(\bar{m}(\bar{\Lambda}))^{-1} \leq M(\bar{m}(\bar{\Lambda}))^{-1}.$$

To obtain the inequality above we appeal to [**Yo98**, Lemma 1(3)] or [**BY00**, Lemma 6]. A careful look at [**BY00**, Lemma 6] allows us to conclude that M does not depend on the parameter in question. Now, $\bar{\rho}_j^i \leq M(\bar{m}(\bar{\Lambda}))^{-1}\sum_j \bar{m}(\sigma_j^i) \leq M$ which implies that $\bar{\rho}^n \leq M$, from where we obtain that $\bar{\rho} \leq M$. □

The starting point in construction of the desired convergent subsequence is to apply the Banach-Alaoglu Theorem to the sequence $\bar{\rho}_n$ to obtain a subsequence $(\bar{\rho}_{n_i})_{i\in\mathbb{N}}$ convergent to $\bar{\rho}_\infty \in L^\infty$ in the weak* topology, i.e.

$$\int \phi\bar{\rho}_{n_i}dx \xrightarrow[i\to\infty]{} \int \phi\bar{\rho}_\infty dx, \quad \forall\phi \in L^1. \tag{9.1}$$

9.2. Lifting to the original horseshoe. At this point we adapt a technique used in [**Bo75**] for the construction of Gibbs states to lift an $\bar{F}$- invariant measure on the quotient space $\bar{\Lambda}$ to an F- invariant measure on the initial horseshoe Λ.

Given an $\bar{F}$-invariant probability measure $\bar{\nu}$, we define a probability measure $\tilde{\nu}$ on Λ as follows. For each bounded $\phi : \Lambda \to \mathbb{R}$ consider its discretization $\phi^* : \bar{\Lambda} \to \mathbb{R}$ defined by

$$\phi^*(x) = \inf\{\phi(z) : z \in \gamma^s(H(x))\}. \tag{9.2}$$

If ϕ is continuous, as its domain is compact, we may define

$$\text{var}\phi(k) = \sup\left\{|\phi(z) - \phi(\zeta)| : |z - \zeta| \leq Cb_0^k\right\},$$

in which case $\text{var}\phi(k) \to 0$ as $k \to \infty$.

LEMMA 9.2. *Given any continuous $\phi : \Lambda \to \mathbb{R}$, for all $k, l \in \mathbb{N}$ we have*

$$\left|\int(\phi\circ F^k)^*d\bar{\nu} - \int(\phi\circ F^{k+l})^*d\bar{\nu}\right| \leq var\phi(k),$$

.

PROOF. Since $\bar\nu$ is $\bar F$-invariant

$$\begin{aligned}\left|\int(\phi\circ F^k)^* d\bar\nu-\int(\phi\circ F^{k+l})^* d\bar\nu\right| &= \left|\int(\phi\circ F^k)^*\circ\bar F^l d\bar\nu-\int(\phi\circ F^{k+l})^* d\bar\nu\right|\\ &\le \int\left|(\phi\circ F^k)^*\circ\bar F^l-(\phi\circ F^{k+l})^*\right| d\bar\nu.\end{aligned}$$

By definition of the discretization we have

$$(\phi\circ F^k)^*\circ\bar F^l(x)=\min\left\{\phi(z): z\in F^k\left(\gamma^s(H(\bar F^l(x)))\right)\right\}$$

and

$$(\phi\circ F^{k+l})^*(x)=\min\left\{\phi(\zeta):\zeta\in F^{k+l}\left(\gamma^s\left(H(x)\right)\right)\right\}.$$

Observe that $F^{k+l}\left(\gamma^s\left(H(x)\right)\right)\subset F^k\left(\gamma^s(H(\bar F^l(x)))\right)$ and by Proposition 4.1

$$\operatorname{diam} F^k\left(\gamma^s(H(\bar F^l(x)))\right)\ \le Cb_0^k.$$

Thus, $\left|(\phi\circ F^k)^*\circ\bar F^l-(\phi\circ F^{k+l})^*\right|\le \operatorname{var}\phi(k)$. □

By the Cauchy criterion the sequence $\left(\int(\phi\circ F^k)^* d\bar\nu\right)_{k\in\mathbb N}$ converges. Hence, Riesz Representation Theorem yields a probability measure $\tilde\nu$ on Λ

$$\int\phi d\tilde\nu:=\lim_{k\to\infty}\int(\phi\circ F^k)^* d\bar\nu \tag{9.3}$$

for every continuous function $\phi:\Lambda\to\mathbb R$.

PROPOSITION 9.3. *The probability measure $\tilde\nu$ is F-invariant and has absolutely continuous conditional measures on γ^u leaves. Moreover, given any continuous $\phi:\Lambda\to\mathbb R$ we have*

1. $\left|\int\phi d\tilde\nu-\int(\phi\circ F^k)^* d\bar\nu\right|\le var\phi(k)$;
2. *If ϕ is constant in each γ^s, then $\int\phi d\tilde\nu=\int\bar\phi d\bar\nu$, where $\bar\phi:\bar\Lambda\to\mathbb R$ is defined by $\bar\phi(x)=\phi(H(x))$.*
3. *If ϕ is constant in each γ^s and $\psi:\Lambda\to\mathbb R$ is continuous then*

$$\left|\int\psi.\phi d\tilde\nu-\int(\psi\circ F^k)^*(\phi\circ F^k)^* d\bar\nu\right|\le\|\phi\|_\infty var\psi(k).$$

PROOF. Regarding the F-invariance property, note that for any continuous $\phi:\Lambda\to\mathbb R$,

$$\int\phi\circ F d\tilde\nu=\lim_{k\to\infty}\int\left(\phi\circ F^{k+1}\right)^* d\bar\nu=\int\phi d\tilde\nu,$$

by Lemma 9.2. Assertion (1) is an immediate consequence of Lemma 9.2. Property (2) follows from

$$\int\phi d\tilde\nu=\lim_{k\to\infty}\int\left(\phi\circ F^k\right)^* d\bar\nu=\lim_{k\to\infty}\int\bar\phi\circ\bar F^k d\bar\nu=\int\bar\phi d\bar\nu,$$

which holds by definition of $\tilde\nu$, ϕ^* and the $\bar F$-invariance of $\bar\nu$. For statement (3) let $\bar\phi:\bar\Lambda\to\mathbb R$ be defined by $\bar\phi(x)=\phi(H(x))$, k,l any positive integers and observe that

$$\int(\psi.\phi\circ F^k)^* d\bar\nu=\int(\psi\circ F^k)^*(\phi\circ F^k)^* d\bar\nu$$

and

$$\begin{aligned}\left|\int(\psi\phi\circ F^{k+l})^*d\bar{\nu}-\int(\psi\phi\circ F^{k})^*d\bar{\nu}\right| &= \left|\int(\psi\circ F^{k+l})^*\bar{\phi}\circ\bar{F}^{k+l}d\bar{\nu}-\int(\psi\circ F^{k})^*\bar{\phi}\circ\bar{F}^{k}d\bar{\nu}\right| \\ &\le \int\left|(\psi\circ F^{k+l})^*-(\psi\circ F^{k})^*\circ\bar{F}^{l}\right|\left|\phi\circ\bar{F}^{k+l}\right|d\bar{\nu} \\ &\le \|\phi\|_\infty \mathrm{var}\psi(k);\end{aligned}$$

inequality (3) follows letting l go to ∞.

Remark 9.4. Since the continuous functions are a dense subset of L^1- functions, then properties (2) and (3) also hold, through Lebesgue Dominated Convergence Theorem, when $\phi\in L^1$.

We are then left to verify the absolute continuity property. While the properties proved above are intrinsic to Bowen's raising technique, the disintegration into absolutely continuous conditional measures on unstable leaves depends heavily on the definition of the reference measure $\bar{m}$ and the fact that $\bar{\nu}=\bar{\rho}d\bar{m}$. Fix an unstable leaf $\gamma^u\in\Gamma^u$. Denote the 1-dimensional Lebesgue measure on γ^u by λ_{γ^u}. Consider a set $E\subset\gamma^u$ such that $\lambda_{\gamma^u}(E)=0$. We will show that $\tilde{\nu}_{\gamma^u}(E)=0$, where $\tilde{\nu}_{\gamma^u}$ denotes the conditional measure of $\tilde{\nu}$ on γ^u, except for a few choices of γ^u. To be more precise, the family of curves Γ^u induces a partition of Λ into unstable leaves which we denote by $\mathcal{L}$. Let $\pi_{\mathcal{L}}:\Lambda\to\mathcal{L}$ be the natural projection on the quotient space $\mathcal{L}$, i.e. $\pi_{\mathcal{L}}(z)=\gamma^u(z)$. We say that $Q\subset\mathcal{L}$ is measurable if and only if $\pi_{\mathcal{L}}^{-1}(Q)$ is measurable. Let $\hat{\nu}=(\pi_{\mathcal{L}})_*(\tilde{\nu})$, which means that $\hat{\nu}(Q)=\tilde{\nu}\left(\pi_{\mathcal{L}}^{-1}(Q)\right)$. By definition of Γ^u there is a non-decreasing sequence of finite partitions $\mathcal{L}_1\prec\mathcal{L}_2\prec\ldots\prec\mathcal{L}_n\prec\ldots$ such that $\mathcal{L}=\bigvee_{i=1}^{\infty}\mathcal{L}_n$; see [**BY93**, Sublemma 7]. Thus, by Rokhlin Disintegration Theorem (see [**BDV05**, Appendix C.6] for an exposition on the subject) there is a system $\left(\tilde{\nu}_{\gamma^u}\right)_{\gamma^u\in\mathcal{L}}$ of conditional probability measures of $\tilde{\nu}$ with respect to $\mathcal{L}$ such that

- $\tilde{\nu}_{\gamma^u}(\gamma^u)=1$ for $\hat{\nu}$- almost every $\gamma^u\in\mathcal{L}$;
- given any bounded measurable map $\phi:\Lambda\to\mathbb{R}$, the map $\gamma^u\mapsto\int\phi d\tilde{\nu}_{\gamma^u}$ is measurable and $\int\phi d\tilde{\nu}=\int\left(\int\phi d\tilde{\nu}_{\gamma^u}\right)d\hat{\nu}$.

Let $\bar{E}=\bar{\pi}(E)$. Since the reference measure $\bar{m}$ has a representative m_{γ^u} on γ^u which is equivalent to λ_{γ^u}, we have $m_{\gamma^u}(E)=0$ and $\bar{m}(\bar{E})=0$. As $\bar{\nu}=\bar{\rho}d\bar{m}$, then $\bar{\nu}(\bar{E})=0$. Let $\bar{\phi}_n:\bar{\Lambda}\to\mathbb{R}$ be a sequence of continuous functions such that $\bar{\phi}_n\to\mathbf{1}_{\bar{E}}$ as $n\to\infty$. Consider also the sequence of continuous functions $\phi_n:\Lambda\to\mathbb{R}$ given by $\phi_n=\bar{\phi}_n\circ\bar{\pi}$. Clearly ϕ_n is constant in each γ^s stable leaf and $\phi_n\to\mathbf{1}_{\bar{E}}\circ\bar{\pi}=\mathbf{1}_{\bar{\pi}^{-1}(\bar{E})}$ as $n\to\infty$. By Lebesgue Dominated Convergence Theorem we have $\int\phi_n d\tilde{\nu}\to\int\mathbf{1}_{\bar{\pi}^{-1}(\bar{E})}d\tilde{\nu}=\tilde{\nu}\left(\bar{\pi}^{-1}(\bar{E})\right)$ and $\int\bar{\phi}_n d\bar{\nu}\to\int\mathbf{1}_{\bar{E}}d\bar{\nu}=\bar{\nu}(\bar{E})=0$. By (2) we have $\int\phi_n\tilde{\nu}=\int\bar{\phi}_n d\bar{\nu}$. Hence, we must have $\tilde{\nu}\left(\bar{\pi}^{-1}(\bar{E})\right)=0$. Consequently,

$$0=\int\mathbf{1}_{\bar{\pi}^{-1}(\bar{E})}d\tilde{\nu}=\int\left(\int\mathbf{1}_{\bar{\pi}^{-1}(\bar{E})}d\tilde{\nu}_{\gamma^u}\right)d\hat{\nu}(\gamma^u),$$

which implies that $\tilde{\nu}_{\gamma^u}\left(\bar{\pi}^{-1}(\bar{E})\cap\gamma^u\right)=0$ for $\hat{\nu}$-almost every γ^u. $\square$

Observe that while $\bar{\nu}_{n_i}$ is $\bar{F}_{n_i}$-invariant we are not certain that $\bar{\nu}_\infty = \bar{\rho}_\infty d\bar{m}_0$ is $\bar{F}_0$-invariant; thus we are not yet in condition to apply Lemma 9.2 to the measure $\bar{\nu}_\infty$. This invariance can be derived from the fact that $\bar{\nu}_{n_i}$ is $\bar{F}_{n_i}$-invariant and equation (9.1).

Lemma 9.5. *The measure $\bar{\nu}_\infty = \bar{\rho}_\infty d\bar{m}_0$ is $\bar{F}_0$-invariant.*

Proof. We just have to verify that for every continuous $\varphi : \bar{\Lambda}_0 \to \mathbb{R}$

$$\int \varphi \circ \bar{F}_0 . \bar{\rho}_\infty d\bar{m}_0 = \int \varphi . \bar{\rho}_\infty d\bar{m}_0$$

Up to composing with H_0 we can think of φ as a function defined in Ω^0_∞. Clearly, there is a continuous function $\phi : \Omega_0 \to \mathbb{R}$ such that $\phi|_{\Omega^0_\infty}(x) = \varphi(x)$. Similarly, we can think of ϕ as being defined in any set $H_{n_i}(\Omega_0)$. So, let us consider a continuous function $\phi : \Omega_0 \to \mathbb{R}$. Having this considerations in mind and the fact that $\bar{\nu}_{n_i}$ is $\bar{F}_{n_i}$-invariant we have

$$\int \phi \circ \bar{F}_{n_i} . \bar{\rho}_{n_i} d\bar{m}_{n_i} = \int \phi . \bar{\rho}_{n_i} d\bar{m}_{n_i}. \tag{9.4}$$

Observing that

$$\int \phi . \bar{\rho}_{n_i} d\bar{m}_{n_i} = \int \phi(x) . \bar{\rho}_{n_i}(x) . \|\tfrac{dH_{n_i}}{dx}\| dx$$

we conclude that

$$\int \phi(x) . \bar{\rho}_{n_i}(x) . \|\tfrac{dH_{n_i}}{dx}\| dx \xrightarrow[i\to\infty]{} \int \phi(x) . \bar{\rho}_\infty(x) . \|\tfrac{dH_0}{dx}\| dx \tag{9.5}$$

due to

$$\left| \int \phi . \bar{\rho}_{n_i} . \|\tfrac{dH_{n_i}}{dx}\| dx - \int \phi . \bar{\rho}_\infty . \|\tfrac{dH_0}{dx}\| dx \right| \leq \left| \int \phi . \bar{\rho}_{n_i} . \|\tfrac{dH_{n_i}}{dx}\| dx - \int \phi . \bar{\rho}_{n_i} . \|\tfrac{dH_0}{dx}\| dx \right| + \left| \int \phi . \bar{\rho}_{n_i} . \|\tfrac{dH_0}{dx}\| dx - \int \phi . \bar{\rho}_\infty . \|\tfrac{dH_0}{dx}\| dx \right|$$

and the fact that the first term in the right side goes to 0 by the unstable manifold theorem, while the second goes to 0 by (9.1).

The convergence (9.5) may be rewritten as

$$\int \phi . \bar{\rho}_{n_i} d\bar{m}_{n_i} \xrightarrow[i\to\infty]{} \int \phi . \bar{\rho}_\infty d\bar{m}_0.$$

Once we prove that

$$\int \phi \circ \bar{F}_{n_i} . \bar{\rho}_{n_i} d\bar{m}_{n_i} \xrightarrow[i\to\infty]{} \int \phi \circ \bar{F}_0 . \bar{\rho}_\infty d\bar{m}_0,$$

equality (9.4) and the uniqueness of the limit give the desired result.

Claim. *$\int \phi \circ \bar{F}_{n_i} . \bar{\rho}_{n_i} d\bar{m}_{n_i}$ converges to $\int \phi \circ \bar{F}_0 \bar{\rho}_\infty d\bar{m}_0$ when $i \to \infty$.*

Given $\varepsilon > 0$, we want to find $J \in \mathbb{N}$ such that for every $i > J$

$$E_1 := \left| \int \phi \circ \bar{F}_{n_i}(x) . \bar{\rho}_{n_i}(x) . \|\tfrac{dH_{n_i}}{dx}\| dx - \int \phi \circ \bar{F}_0(x) . \bar{\rho}_\infty(x) . \|\tfrac{dH_0}{dx}\| dx \right| < \varepsilon.$$

Since $\|\rho_{n_i}\|_\infty, \|\rho_\infty\|_\infty \le M$ and $\|\frac{dH_{n_i}}{dx}\|, \|\frac{dH_0}{dx}\| \le \sqrt{1+(10b)^2}$ we have

$$E_1 \le \left| \int \phi \circ \bar{F}_{n_i}.\bar{\rho}_{n_i}.\|\tfrac{dH_{n_i}}{dx}\| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx - \int \phi \circ \bar{F}_0.\bar{\rho}_\infty.\|\tfrac{dH_0}{dx}\| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx \right|$$
$$+ 2M\sqrt{1+(10b)^2}\|\phi\|_\infty \left|\Omega_\infty^0 \bigtriangleup \Omega_\infty^{n_i}\right|$$

Taking

$$E_2 = \left| \int \phi \circ \bar{F}_{n_i}.\bar{\rho}_{n_i}.\|\tfrac{dH_{n_i}}{dx}\| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx - \int \phi \circ \bar{F}_0.\bar{\rho}_\infty.\|\tfrac{dH_0}{dx}\| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx \right|$$

we have

$$E_1 \le E_2 + 2M\sqrt{1+(10b)^2}\|\phi\|_\infty \left|\Omega_\infty^0 \bigtriangleup \Omega_\infty^{n_i}\right|.$$

By Corollary 6.4, we may take $J \in \mathbb{N}$ sufficiently large so that for $i > J$

$$2M\sqrt{1+(10b)^2}\|\phi\|_\infty \left|\Omega_\infty^0 \bigtriangleup \Omega_\infty^{n_i}\right| < \frac{\varepsilon}{2}.$$

Besides

$$E_2 \le \left| \int \phi \circ \bar{F}_{n_i}.\bar{\rho}_{n_i}.\left[\|\tfrac{dH_{n_i}}{dx}\| - \|\tfrac{dH_0}{dx}\|\right] \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx \right|$$
$$+ \left| \int \phi \circ \bar{F}_0.\left[\bar{\rho}_{n_i} - \bar{\rho}_\infty\right].\|\tfrac{dH_0}{dx}\| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx \right|$$
$$+ \left| \int \left[\phi \circ \bar{F}_{n_i} - \phi \circ \bar{F}_0\right].\bar{\rho}_\infty.\|\tfrac{dH_0}{dx}\| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx \right|.$$

Denote by E_3, E_4 and E_5 respectively the terms in the last sum. Attending to the unstable manifold theorem and equation (9.1) it is clear that E_3 and E_4 can be made arbitrarily small. Noting that $\sqrt{1+(10b)^2} < 2$, we have for any N

$$E_5 \le 2M\|\phi\|_\infty \sum_{l=N+1}^{\infty} \left(|\{R_{n_i} = l\}| + |\{R_0 = l\}|\right)$$
$$+ 2M\|\phi\|_\infty \sum_{l=1}^{N} |\{R_{n_i} = l\} \bigtriangleup \{R_0 = l\}|$$
$$+ 2M \sum_{l=1}^{N} \left| \int_{\{R_{n_i}=l\}\cap\{R_0=l\}} \left[\phi \circ \bar{F}_{n_i} - \phi \circ \bar{F}_0\right] \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx \right|.$$

Denote by E_6, E_7 and E_8 respectively the terms in the last sum. According to Proposition 4.1 we may choose N sufficiently large so that E_6 is small enough. For this choice of N we appeal to Proposition 8.7 to find $J \in \mathbb{N}$ sufficiently large so that E_7 is also small enough. At this point we are left to deal with E_8. Let

$$E_8^l = \left| \int_{\{R_{n_i}=l\}\cap\{R_0=l\}} \left[\phi \circ \bar{F}_{n_i} - \phi \circ \bar{F}_0\right] \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx \right|.$$

The result will follow once we prove that E_8^l is arbitrarily small, which is achieved by showing that given $\varsigma > 0$, there exists $J \in \mathbb{N}$ such that if $i > J$, then $\left|\phi \circ \bar{f}_{n_i}^l - \phi \circ \bar{f}_0^l\right| < \varsigma$.

Suppose that ς is small enough for our purposes. Since ϕ is continuous and Ω_0 is compact then there exists $\eta > 0$ such that $|\phi(x_1) - \phi(x_2)| < \varsigma$, for every x_1, x_2 belonging to any subset of Ω_0 with diameter less than η. We use Lemma 7.1 to choose $N_2 \in \mathbb{N}$ sufficiently large so that if ω is any connected component of $H_0(\Omega_{N_2})$ then the maximum horizontal width of $Q^2_{N_2}(\omega)$ is $\eta/2$. We take $J \in \mathbb{N}$ sufficiently large so that $\Omega^{n_i}_{N_2} = \Omega^0_{N_2}$ and by Lemma 7.2, for every connected component I of $\Omega^0_{N_2}$ we have $Q^1_{N_2}(H_{n_i}(I)) \subset Q^2_{N_2}(H_0(I))$. We also want $J \in \mathbb{N}$ large enough to guarantee (6.2) with b^{2N_2} instead of ε up to N.

Now, since $f^l_0(H_0(x)) \in \Lambda_0$, there exists a connected component I of $\Omega^0_{N_2}$ such that $f^l_0(H_0(x)) \in Q^1_{N_2}(H_0(I))$. As $\left|f^l_{n_i}(H_{n_i}(x)) - f^l_0(H_0(x))\right| < b^{2N_2}$, then clearly $f^l_{n_i}(H_{n_i}(x)) \in Q^2_{N_2}(H_0(I))$. Moreover, since $f^l_{n_i}(H_{n_i}(x)) \in \Lambda_{n_i}$ and we know that $Q^2_{N_2}(H_0(I))$ intersects only one rectangle $Q^1_{N_2}(H_{n_i}(L))$ with L representing any connected component of $\Omega^{n_i}_{N_2}$, then $f^l_{n_i}(H_{n_i}(x)) \in Q^1_{N_2}(H_{n_i}(I))$. Thus we have $\bar{f}^l_0(H_0(x)) \in H_0(\Omega_0) \cap Q^2_{N_2}(H_0(I))$ and $\bar{f}^l_{n_i}(H_{n_i}(x)) \in H_{n_i}(\Omega_0) \cap Q^2_{N_2}(H_0(I))$. Finally, observe that $H_0^{-1}\left(H_0(\Omega_0) \cap Q^2_{N_2}(H_0(I))\right)$ and $H_{n_i}^{-1}\left(H_{n_i}(\Omega_0) \cap Q^2_{N_2}(H_0(I))\right)$ are both intervals containing I with length of at most $\eta/2$ which means that $\left|\phi\left(\bar{f}^l_0(H_0(x))\right) - \phi\left(\bar{f}^l_{n_i}(H_{n_i}(x))\right)\right| < \varsigma$. See Figure 7.

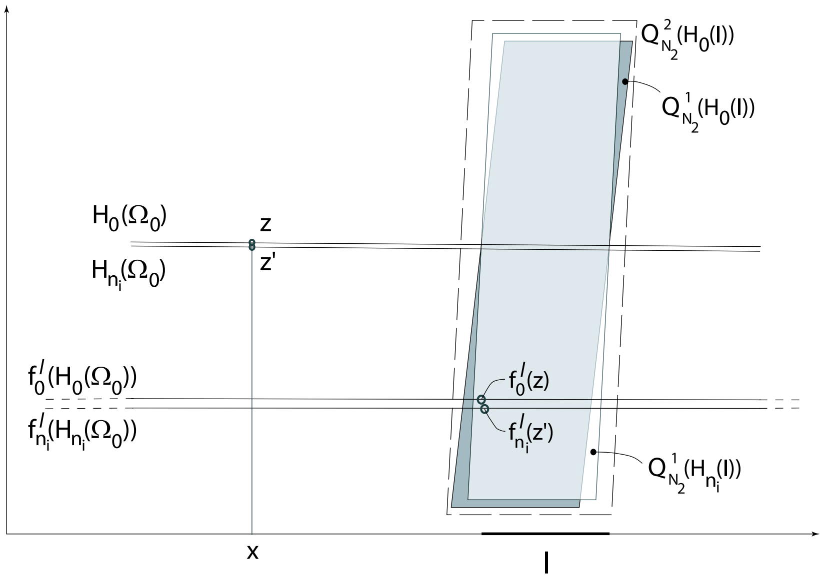

FIGURE 7

□

Then we lift the measure $\bar{\nu}_{n_i}$ to an F_{n_i}- invariant measure $\tilde{\nu}_{n_i}$ defined according to equation (9.3). Lemma 9.5 allows us to apply (9.3) to the measure $\bar{\nu}_\infty$ and generate $\tilde{\nu}_\infty$. We observe that by Proposition 9.3 the measures $\tilde{\nu}_\infty$ and $\tilde{\nu}_{n_i}$ are SRB measures.

9.3. Saturation and convergence of the measures. Now we saturate the measures $\tilde{\nu}_\infty$ and $\tilde{\nu}_{n_i}$. Let $\tilde{\nu}$ be an SRB measure for f^R obtained from $\bar{\nu} = \bar{\rho} d\bar{m}$ as in (9.3). We define the saturation of $\tilde{\nu}$ by

$$\nu^* = \sum_{l=0}^{\infty} f_*^l \left(\tilde{\nu}|\{R > l\}\right) \tag{9.6}$$

It is well known that ν^* is f-invariant and that the finiteness of ν^* is equivalent to $\int R d\tilde{\nu} < \infty$. Since $\|\bar{\rho}\|_\infty < M$ and $\bar{m}$ is equivalent to the 1-dimensional Lebesgue measure with uniformly bounded density, see [**BY00**, Section 5.2], then by Proposition 9.3(2) and Proposition 4.1 we easily get that $\tilde{\nu}(\{R > l\}) \lesssim C_0\theta_0^l$ for some $\theta_0 < 1$. Since $\int R d\tilde{\nu} = \sum_{l=0}^{\infty} \tilde{\nu}(\{R > l\})$, the finiteness of ν^* is assured. Clearly, each $f_*^l\left(\tilde{\nu}|\{R > l\}\right)$ has absolutely continuous conditional measures on $\{f^l\gamma^u\}$, which are Pesin's unstable manifolds, and so ν^* is an SRB measure.

Using (9.6) we define the saturations of the measures $\tilde{\nu}_\infty$ and $\tilde{\nu}_{n_i}$ to obtain ν_∞^* and $\nu_{n_i}^*$ respectively. By construction, we know that ν_∞^* and $\nu_{n_i}^*$ are SRB measures, which implies that $\nu_\infty^* = \nu_0$ and $\nu_{n_i}^* = \nu_{n_i}$, by the uniqueness of the SRB measure.

To complete the argument we just need to the following result.

Proposition 9.6. *For every continuous $g : \mathbb{R}^2 \to \mathbb{R}$,*

$$\int g d\nu_{n_i}^* \xrightarrow[i\to\infty]{} \int g d\nu_\infty^*.$$

Proof. First observe that there is a compact $D \subset \mathbb{R}^2$ containing the attractors corresponding to the parameters (a_n, b_n) for all $n \geq 0$. As the supports of the measures ν_∞^* and $\nu_{n_i}^*$ are contained in D we may assume henceforth that g is uniformly continuous and $\|g\|_\infty < \infty$.

Let ε be given. We look forward to find $J \in \mathbb{N}$ sufficiently large so that for every $i > J$

$$\left| \int g d\nu_{n_i}^* - \int g d\nu_\infty^* \right| < \varepsilon.$$

Recalling (9.6) we may write for any integer N_0

$$\nu^* = \sum_{l=0}^{N_0-1} \nu^l + \eta$$

where $\nu^l = f_*^l(\tilde{\nu}|\{R > l\})$ and $\eta = \sum_{l \geq N_0} f_*^l(\tilde{\nu}|\{R > l\})$. Since $\tilde{\nu}(\{R > l\}) \lesssim C_0\theta_0^l$ for some $\theta_0 < 1$, we may choose N_0 so that $\eta(\mathbb{R}^2) < \varepsilon/3$. We are reduced to find for every $l < N_0$ a sufficiently large J so that for every $i > J$

$$\left| \int (g \circ f_{n_i}^l) \mathbf{1}_{\{R_{n_i} > l\}} d\tilde{\nu}_{n_i} - \int (g \circ f_0^l) \mathbf{1}_{\{R_0 > l\}} d\tilde{\nu}_\infty \right| < \frac{\varepsilon}{3N_0}.$$

Fix $l < N_0$ and take $k \in \mathbb{N}$ large so that $\operatorname{var}(g(k)) < \frac{\varepsilon}{9N_0}$. Attending to Proposition 9.3 (3) and its Remark 9.4, our problem will be solved if we exhibit $J \in \mathbb{N}$ such that for every $i > J$

$$E := \left| \int (g \circ f_{n_i}^l \circ F_{n_i}^k)^* (\mathbf{1}_{\{R_{n_i} > l\}} \circ F_{n_i}^k)^* d\bar{\nu}_{n_i} - \int (g \circ f_0^l \circ F_0^k)^* (\mathbf{1}_{\{R_0 > l\}} \circ F_0^k)^* d\tilde{\nu}_\infty \right| < \frac{\varepsilon}{9N_0}.$$

Defining

$$E_0=\bigg|\int (g\circ f_{n_i}^l\circ F_{n_i}^k)^*(\mathbf{1}_{\{R_{n_i}>l\}}\circ F_{n_i}^k)^*\,\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}\,\bar\rho_{n_i}\|\tfrac{dH_{n_i}}{dx}\|dx \\ -\int (g\circ f_0^l\circ F_0^k)^*(\mathbf{1}_{\{R_0>l\}}\circ F_0^k)^*\,\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}\,\bar\rho_\infty\|\tfrac{dH_0}{dx}\|dx\bigg|$$

we have $E\le E_0+4M\|g\|_\infty\,|\Omega_\infty^0\bigtriangleup\Omega_\infty^{n_i}|$. Using Corollary 6.4 we may find $J\in\mathbb{N}$ so that for $i>J$

$$4M\|g\|_\infty\left|\Omega_\infty^0\bigtriangleup\Omega_\infty^{n_i}\right|<\frac{\varepsilon}{18N_0}.$$

Applying the triangular inequality we get

$$\begin{aligned}E_0\le{}& M\|g\|_\infty\int\Big|\|\tfrac{dH_{n_i}}{dx}\|-\|\tfrac{dH_0}{dx}\|\Big|\,dx\\ &+\bigg|\int (g\circ f_0^l\circ F_0^k)^*(\mathbf{1}_{\{R_0>l\}}\circ F_0^k)^*\,\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}\,\left[\bar\rho_{n_i}-\bar\rho_\infty\right]\|\tfrac{dH_0}{dx}\|dx\bigg|\\ &+2M\int\left|(g\circ f_{n_i}^l\circ F_{n_i}^k)^*-(g\circ f_0^l\circ F_0^k)^*\right|\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}dx\\ &+2M\|g\|_\infty\int\left|(\mathbf{1}_{\{R_{n_i}>l\}}\circ F_{n_i}^k)^*-(\mathbf{1}_{\{R_0>l\}}\circ F_0^k)^*\right|\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}dx.\end{aligned}$$

By the unstable manifold theorem

$$\int\Big|\|\tfrac{dH_{n_i}}{dx}\|-\|\tfrac{dH_0}{dx}\|\Big|\,dx$$

can be made arbitrarily small by choosing a sufficiently large $J\in\mathbb{N}$. The term

$$\bigg|\int (g\circ f_0^l\circ F_0^k)^*(\mathbf{1}_{\{R_0>l\}}\circ F_0^k)^*\,\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}\,\left[\bar\rho_{n_i}-\bar\rho_\infty\right]\|\tfrac{dH_0}{dx}\|dx\bigg|$$

can also be easily controlled attending to (9.1). The analysis of the remaining terms

$$\int\left|(g\circ f_{n_i}^l\circ F_{n_i}^k)^*-(g\circ f_0^l\circ F_0^k)^*\right|\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}dx$$

and

$$\int\left|(\mathbf{1}_{\{R_{n_i}>l\}}\circ F_{n_i}^k)^*-(\mathbf{1}_{\{R_0>l\}}\circ F_0^k)^*\right|\mathbf{1}_{\Omega_\infty^0\cap\Omega_\infty^{n_i}}dx$$

is left to Lemmas 9.8 and 9.9 below. □

In the proofs of Lemmas 9.8 and 9.9 we have to produce a suitable positive integer N so that returns that take longer than N iterations are negligible. The next lemma provides the tools for an adequate choice.

LEMMA 9.7. *Given $k,N\in\mathbb{N}$ we have*

$$\left|\left\{z\in H(\Omega_\infty):\ \exists t\in\{1,\dots,k\}\ \text{ such that } R^t(z)>N\right\}\right|\le k\frac{C_1^2}{|\Omega_0|}|\{R>N\}|.$$

Proof. We may write

$$\left\{z \in H(\Omega_\infty) : \exists t \in \{1,\dots,k\} \text{ such that } R^t(z) > N\right\} = \bigcup_{t=0}^{k-1} B_t,$$

where

$$B_t = \left\{z \in H(\Omega_\infty) : R(z) \le N, \dots, R^t(z) \le N, R^{t+1}(z) > N\right\}.$$

Let us show that $|B_t| \le \frac{C_1^2}{|\Omega_0|}|\{R > N\}|$ for every $t \in \{0,\dots,k-1\}$. Indeed, if $R(z) \le N, \dots, R^t(z) \le N$ then there exist $m_1,\dots m_t \le N$ and $j_1 \le v(m_1),\dots,j_t \le v(m_t)$ such that $z \in H\left(\Upsilon_{(m_1,j_1)\dots(m_t,j_t)}\right)$. Besides, for every $l \in \{1,\dots,t\}$ there is $\omega_{m_l,j_l} \in \tilde{\mathcal{P}}_{m_l-1}$ such that m_l is a regular return time for ω_{m_l,j_l} and, according to Lemma 8.8,

$$H(\Upsilon_{(m_1,j_1)\dots(m_t,j_t)}) = \omega_{m_1,j_1} \cap \dots \cap f^{-(m_1+\dots+m_{t-1})}(\omega_{m_t,j_t}) \cap f^{-(m_1+\dots+m_t)}(H(\Omega_\infty)).$$

Let $\omega = \omega_{m_1,j_1} \cap \dots \cap f^{-(m_1+\dots+m_{t-1})}(\omega_{m_t,j_t})$. Consider the set

$$\tilde\omega = \{z \in H(\Upsilon_{(m_1,j_1)\dots(m_t,j_t)}) : R^{t+1}(z) > N\} = \omega \cap f^{-(m_1+\dots+m_t)}(\{R > N\}).$$

Using bounded distortion we obtain

$$\frac{|\tilde\omega|}{|\omega|} \le C_1 \frac{|f^{m_1+\dots+m_t}(\tilde\omega)|}{|f^{m_1+\dots+m_t}(\omega)|} \le C_1 \frac{|\{R > N\}|}{2|\Omega_0|},$$

and

$$\frac{|H(\Upsilon_{(m_1,j_1)\dots(m_t,j_t)})|}{|\omega|} \ge C_1^{-1} \frac{\left|f^{m_1+\dots+m_t}(H(\Upsilon_{(m_1,j_1)\dots(m_t,j_t)}))\right|}{\left|f^{m_1+\dots+m_t}(\omega)\right|} \ge C_1^{-1}\frac{|\Omega_\infty|}{2},$$

from which we get

$$\frac{|\tilde\omega|}{|H(\Upsilon_{(m_1,j_1)\dots(m_t,j_t)})|} \le \frac{C_1^2}{|\Omega_0|}\frac{|\{R > N\}|}{|\Omega_\infty|}.$$

Finally, we conclude that

$$|B_t| = \sum_{\substack{m_l \le N \\ j_l \le v(m_l) \\ l \in \{1,\dots,t\}}} |\tilde\omega| \le \frac{C_1^2}{|\Omega_0|}\frac{|\{R > N\}|}{|\Omega_\infty|} \sum_{\substack{m_l \le N \\ j_l \le v(m_l) \\ l \in \{1,\dots,t\}}} |H(\Upsilon_{(m_1,j_1)\dots(m_t,j_t)})| \le \frac{C_1^2}{|\Omega_0|}|\{R > N\}|.$$

□

Lemma 9.8. *Given $l,k \in \mathbb{N}$ and $\varepsilon > 0$ there is $J \in \mathbb{N}$ such that for every $i > J$*

$$\int \left|(g \circ f_{n_i}^l \circ F_{n_i}^k)^* - (g \circ f_0^l \circ F_0^k)^*\right| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx < \varepsilon.$$

Proof. We split the argument into three steps:

(1) We appeal to Lemma 9.7 to choose $N_5 \in \mathbb{N}$ sufficiently large so that the set

$$L := \left\{x \in \Omega_\infty^0 \cap \Omega_\infty^{n_i} : \exists t \in \{1,\dots,k\}\, R_0^t(x) > N_5 \text{ or } R_{n_i}^t(x) > N_5\right\}$$

has sufficiently small mass.

(2) We pick $J \in \mathbb{N}$ large enough to guarantee that we are inside the neighborhood of (a_0, b_0) given by Proposition 8.9 when applied to N_5 and a convenient fraction of ε. Namely, we have that for all $m_1, \ldots, m_k \leq N_5$ and all $j_1 \leq v(m_1), \ldots, j_k \leq v(m_k)$, each set $\Upsilon^0_{(m_1,j_1)\ldots(m_k,j_k)} \triangle \Upsilon^{n_i}_{(m_1,j_1)\ldots(m_k,j_k)}$ has small Lebesgue measure.

(3) Finally, in each set $\Upsilon^0_{(m_1,j_1)\ldots(m_k,j_k)} \cap \Upsilon^{n_i}_{(m_1,j_1)\ldots(m_k,j_k)}$ we control
$$\left|(g \circ f^l_{n_i} \circ F^k_{n_i})^* - (g \circ f^l_0 \circ F^k_0)^*\right|$$
for a better choice of $J \in \mathbb{N}$.

Step (1): From Lemma 9.7 we have $|L| \leq \frac{2C_1^2}{|\Omega_0|} k C_0 \theta_0^{N_5}$. So, we choose N_5 large enough such that
$$2\|g\|_\infty \frac{2C_1^2}{|\Omega_0|} k C_0 \theta_0^{N_5} < \frac{\varepsilon}{3},$$
which implies that
$$\int_L \left|(g \circ f^l_{n_i} \circ F^k_{n_i})^* - (g \circ f^l_0 \circ F^k_0)^*\right| \mathbf{1}_{\Omega^0_\infty \cap \Omega^{n_i}_\infty} dx < \frac{\varepsilon}{3}.$$

Step (2): By Proposition 8.9, we may choose J so that for every $i > J$, $m_1, \ldots, m_k \leq N_5$ and $j_1 \leq v(m_1), \ldots, j_k \leq v(m_k)$ we have that
$$\left|\Upsilon^0_{(m_1,j_1)\ldots(m_k,j_k)} \triangle \Upsilon^{n_i}_{(m_1,j_1)\ldots(m_k,j_k)}\right| < \frac{\varepsilon}{3}\, 5^{-k(N_5+2)} \left(2\max\{1, \|g\|_\infty\}\right)^{-1}.$$
Observe that by (8.1) we have that $\sum_{m_1=1}^{N_5} v(m_1) \leq 5^{N_5+2}$ which means that the number of sets $\Upsilon^0_{(m_1,j_1)\ldots(m_k,j_k)}$ is less than $5^{k(N_5+2)}$. Consequently we have
$$\sum_{\substack{m_T \leq N_5 \\ j_T \leq v(m_T) \\ T=1,\ldots,k}} \int_{\Upsilon^0_{(m_1,j_1)\ldots(m_k,j_k)} \triangle \Upsilon^{n_i}_{(m_1,j_1)\ldots(m_k,j_k)}} \left|(g \circ f^l_{n_i} \circ F^k_{n_i})^* - (g \circ f^l_0 \circ F^k_0)^*\right| \mathbf{1}_{\Omega^0_\infty \cap \Omega^{n_i}_\infty} dx < \frac{\varepsilon}{3}.$$

Step (3): In each set $\Upsilon^0_{(m_1,j_1)\ldots(m_k,j_k)} \cap \Upsilon^{n_i}_{(m_1,j_1)\ldots(m_k,j_k)}$ we have that $F^k_0 = f_0^{m_1+\ldots+m_k}$ and $F^k_{n_i} = f_{n_i}^{m_1+\ldots+m_k}$. Since we are restricted to a compact set D and $|Df| \leq 5$ for every $f = f_{a,b}$ with $(a,b) \in \mathbb{R}^2$, then

- there exists $\vartheta > 0$ such that $|z - \zeta| < \vartheta \Rightarrow |g(z) - g(\zeta)| < \frac{\varepsilon}{3}\, 5^{-k(N_5+2)}$;
- there exists J_1 such that for all $i > J_1$ and $z \in D$ we have
$$\max\left\{|f_0(z) - f_{n_i}(z)|, \ldots, |f_0^{kN_5+l}(z) - f_{n_i}^{kN_5+l}(z)|\right\} < \tfrac{\vartheta}{2};$$
- there exists $\eta > 0$ such that for all $z, \zeta \in D$ and $f = f_{a,b}$ with $(a,b) \in \mathbb{R}^2$
$$|z - \zeta| < \eta \;\Rightarrow\; \max\left\{|f(z) - f(\zeta)|, \ldots, |f^{kN_5+l}(z) - f^{kN_5+l}(\zeta)|\right\} < \tfrac{\vartheta}{2}.$$

Furthermore, according to Proposition 7.3,

- there is J_2 such that for every $i > J_2$ and $x \in \Omega^0_\infty \cap \Omega^{n_i}_\infty$ we have
$$\max_{t \in [-10b, 10b]} \left|\gamma^s_0(H_0(x))(t) - \gamma^s_{n_i}(H_{n_i}(x))(t)\right| < \eta.$$

Let $i > \max\{J_1, J_2\}$, $z \in \gamma_0^s(H_0(x))$ and $t \in [-10b, 10b]$ be such that $z = \gamma_0^s(H_0(x))(t)$. Take $\zeta = \gamma_{n_i}^s(H_{n_i}(x))(t)$. Then, by the choice of J_2, it follows that $|z - \zeta| < \eta$. This together with the choices of η and J_1 implies

$$\begin{aligned}\left|f_0^l \circ F_0^k(z) - f_{n_i}^l \circ F_{n_i}^k(\zeta)\right| &\le \left|f_0^{m_1+\ldots+m_k+l}(z) - f_0^{m_1+\ldots+m_k+l}(\zeta)\right| \\ &\quad + \left|f_0^{m_1+\ldots+m_k+l}(\zeta) - f_{n_i}^{m_1+\ldots+m_k+l}(\zeta)\right| \\ &< \vartheta/2 + \vartheta/2 = \vartheta.\end{aligned}$$

Finally, the above considerations and the choice of ϑ allow us to conclude that for every $i > \max\{J_1, J_2\}$, $x \in \Omega_\infty^0 \cap \Omega_\infty^{n_i}$ and $z \in \gamma_0^s(H_0(x))$, there exists $\zeta \in \gamma_{n_i}^s(H_{n_i}(x))$ such that

$$\left|g(f_{n_i}^l \circ F_{n_i}^k(\zeta)) - g(f_0^l \circ F_0^k(z))\right| < \frac{\varepsilon}{3}\, 5^{-k(N_5+2)}. \tag{9.7}$$

Attending to (9.2), (9.7) and the fact that we can interchange the roles of z and ζ in the latter, we obtain that for every $i > \max\{J_1, J_2\}$

$$\left|(g \circ f_{n_i}^l \circ F_{n_i}^k)^* - (g \circ f_0^l \circ F_0^k)^*\right| < \frac{\varepsilon}{3}\, 5^{-k(N_5+2)},$$

from where we deduce that

$$\sum_{\substack{m_T \le N_5 \\ j_T \le v(m_T) \\ T \in \{1,\ldots,k\}}} \int_{\Upsilon^0_{(m_1,j_1)\ldots(m_k,j_k)} \cap \Upsilon^{n_i}_{(m_1,j_1)\ldots(m_k,j_k)}} \left|(g \circ f_{n_i}^l \circ F_{n_i}^k)^* - (g \circ f_0^l \circ F_0^k)^*\right| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx < \frac{\varepsilon}{3}.$$

$\square$

LEMMA 9.9. *Given $l, k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that for every $i > J$*

$$\int \left|(\mathbf{1}_{\{R_{n_i} > l\}} \circ F_{n_i}^k)^* - (\mathbf{1}_{\{R_0 > l\}} \circ F_0^k)^*\right| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx < \varepsilon.$$

PROOF. As in the proof of Lemma 9.8, we divide the argument into three steps.

(1) The condition on N_5: Consider the set

$$L_1 = \left\{x \in \Omega_\infty^0 \cap \Omega_\infty^{n_i} : \exists t \in \{1, \ldots, k+1\} \text{ such that } R_0^t(x) > N_5 \text{ or } R_{n_i}^t(x) > N_5\right\}.$$

From Lemma 9.7 we have $|L_1| \le \frac{2C_1^2}{|\Omega_0|}(k+1)C_0\theta_0^{N_5}$. So we choose N_5 large enough so that

$$\frac{4C_1^2}{|\Omega_0|}(k+1)C_0\theta_0^{N_5} < \frac{\varepsilon}{3},$$

which implies that

$$\int_{L_1} \left|(\mathbf{1}_{\{R_{n_i} > l\}} \circ F_{n_i}^k)^* - (\mathbf{1}_{\{R_0 > l\}} \circ F_0^k)^*\right| \mathbf{1}_{\Omega_\infty^0 \cap \Omega_\infty^{n_i}} dx < \frac{\varepsilon}{3}.$$

(2) Let us choose J large enough so that, by Proposition 8.9, for all $m_1, \ldots, m_{k+1} \le N_5$ and $j_1 \le v(m_1), \ldots, j_{k+1} \le v(m_{k+1})$ we get

$$\left|\Upsilon^0_{(m_1,j_1)\ldots(m_{k+1},j_{k+1})} \triangle \Upsilon^{n_i}_{(m_1,j_1)\ldots(m_{k+1},j_{k+1})}\right| < \frac{\varepsilon}{3}\, 5^{-(k+1)(N_5+2)}\, 2^{-1}.$$

Observe that by (8.1) we have $\sum_{m_1=1}^{N_5} v(m_1) \leq 5^{N_5+2}$ which means that the number of sets $\Upsilon^0_{(m_1,j_1)\dots(m_{k+1},j_{k+1})}$ is less than $5^{(k+1)(N_5+2)}$. Let

$$L_2 = \Upsilon^0_{(m_1,j_1)\dots(m_{k+1},j_{k+1})} \triangle \Upsilon^{n_i}_{(m_1,j_1)\dots(m_{k+1},j_{k+1})}$$

and observe that

$$\sum_{\substack{m_T \leq N_5 \\ j_T \leq v(m_T) \\ T \in \{1,\dots,k+1\}}} \int_{L_2} \left|(\mathbf{1}_{\{R_{n_i}>l\}} \circ F^k_{n_i})^* - (\mathbf{1}_{\{R_0>l\}} \circ F^k_0)^*\right| \mathbf{1}_{\Omega^0_\infty \cap \Omega^{n_i}_\infty} dx < \frac{\varepsilon}{3}.$$

(3) At last, notice that in each set $\Upsilon^0_{(m_1,j_1)\dots(m_{k+1},j_{k+1})} \cap \Upsilon^{n_i}_{(m_1,j_1)\dots(m_{k+1},j_{k+1})}$ we have

$$\left|(\mathbf{1}_{\{R_{n_i}>l\}} \circ F^k_{n_i})^* - (\mathbf{1}_{\{R_0>l\}} \circ F^k_0)^*\right| = 0,$$

which gives the result. $\square$

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