STATISTICAL STABILITY AND CONTINUITY OF SRB ENTROPY FOR SYSTEMS WITH GIBBS-MARKOV STRUCTURES

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ABSTRACT. We present conditions on families of diffeomorphisms that guarantee statistical stability and SRB entropy continuity. They rely on the existence of horseshoe-like sets with infinitely many branches and variable return times. As an application we consider the family of Hénon maps within the set of Benedicks-Carleson parameters.

CONTENTS

1. INTRODUCTION

A *physical measure* for a smooth map $f : M \to M$ on a manifold M is a Borel probability measure μ on M for which there is a positive Lebesgue measure set of points $x \in M$, called the *basin* of μ, such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \stackrel{n\to\infty}{\longrightarrow} \mu \tag{1.1}$$

in the weak* topology, where δ_z stands for the Dirac measure on $z \in M$. Sinai, Ruelle and Bowen showed the existence of physical measures for Axiom A smooth dynamical systems. These were obtained as *equilibrium states* for the logarithm of the Jacobian along the unstable direction. Besides, such probability measures exhibit positive Lyapunov exponents and conditionals which are absolutely continuous with respect to Lebesgue measure on

Date: May 8, 2009.
2000 *Mathematics Subject Classification.* 37A35, 37C40, 37C75, 37D25.
Key words and phrases. Hénon attractor, SRB measure, SRB entropy, statistical stability.

local unstable leaves; probability measures with the latter properties are nowadays known as *Sinai-Ruelle-Bowen measures* (*SRB measures*, for short).

Statistical properties and their stability have met with wide interest, particularly in the context of dynamical systems which do not satisfy classical structural stability. This may be checked through the continuous variation of the SRB measures, referred in [AV] as *statistical stability*. Another characterization of stability addresses the continuity of the metric entropy of SRB measures. Although an old issue, going back to [N] and [Y1] for example, this continuity (topological or metric) is in general a hard problem. Notice that for families of smooth diffeomorphisms verifying the *entropy formula*, see [LY2], and whose Jacobian along the unstable direction depends continuously on the map, the entropy continuity is an immediate consequence of the statistical stability. This holds for instance in the setting of Axiom A attractors whose statistical stability was established in [R] and [M]. The regularity of the SRB entropy for Axiom A flows was proved in [C]. Analiticity of metric entropy for Anosov diffeomorphisms was proved in [P].

More recently, statistical stability for families of partially hyperbolic diffeomorphisms with non-uniformly expanding centre-unstable direction was established in [V]. Due to the continuous variation of the centre-unstable direction in the partial hyperbolicity context, the entropy continuity follows as in the Axiom A case. Statistical stability for Hénon maps within Benedicks-Carleson parameters have been proved in [ACF]; the entropy continuity for this family is a more delicate issue, since the lack of partial hyperbolicity, mostly due to the presence of "critical" points, originates a highly irregular behavior of the unstable direction. In the endomorphism setting, many advances have been obtained for important families of maps, for instance in [RS, T2, T1, AV, A, F, FT] concerning statistical stability, and in [AOT] for the entropy continuity. Actually, our main theorem may be regarded as a version for diffeomorphisms of the entropy continuity result in [AOT].

In this work we give sufficient conditions on families of smooth diffeomorphisms for the statistical stability and the continuous variation of the SRB entropies. The families we study here, though having directions of non-uniform expansion, do not allow the approach of the hyperbolic case, since no continuity assumptions on these directions with the map will be assumed. Instead, we consider diffeomorphisms admitting Gibbs-Markov structures as in [Y2] that may be thought as "horseshoes" with infinitely many branches and variable return times. This is mainly motivated by the important class of Hénon maps presented in the next paragraph. Our assumptions, which have a geometrical and dynamical nature, ensure in particular the existence of SRB measures. Gibbs-Markov structures were used in [Y2] to derive decay of correlations and the validity of the Central Limit Theorem for the SRB measure. Here we prove that under some additional uniformity requirements on the family we obtain statistical stability and SRB entropy continuity.

The major application of our main result concerns the Benedicks-Carleson family of Hénon maps,

$$\begin{array}{rccc} f_{a,b}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (1-ax^2+y, bx). \end{array} \tag{1.2}$$

For small $b>0$ values, $f_{a,b}$ is strongly dissipative, and may be seen as an "unfolded" version of a quadratic interval map. It is known that for small b there is a trapping region whose topological attractor coincides with the closure of the unstable manifold W of a fixed point $z^*_{a,b}$ of $f_{a,b}$. In [BC] it was shown that for each sufficiently small $b>0$ there is a positive Lebesgue measure set of parameters $a\in[1,2]$ for which $f_{a,b}$ has a dense orbit in $\overline{W}$ with a positive Lyapunov exponent, which makes this a non-trivial and strange attractor. We denote by $\mathcal{BC}$ the set of those parameters (a,b) and call it the *Benedicks-Carleson family* *of Hénon maps.* As shown in [BY1], each of these non-hyperbolic attractors supports a unique SRB measure $\mu_{a,b}$, whose main features were further studied in [BY2, BV1, BV2]. In [BY2] a Gibbs-Markov structure was built for each $f_{a,b}$ with $(a,b) \in \mathcal{BC}$, which has been used to obtain statistical behavior of Hölder observables. These structures have also been used in [ACF] to deduce the statistical stability of this family. In this work we add the metric entropy continuity with respect to these measures.

1.1. **Gibbs-Markov structure.** Let $f\colon M \to M$ be C^k diffeomorphism ($k \geq 2$) defined on a finite dimensional Riemannian manifold M, endowed with a normalized volume form on the Borel sets that we denote by Leb and call *Lebesgue measure.* Given a submanifold $\gamma \subset M$ we use Leb_γ to denote the measure on γ induced by the restriction of the Riemannian structure to γ.

An embedded disk $\gamma \subset M$ is called an *unstable manifold* if $\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \to 0$ as $n \to \infty$ for every $x, y \in \gamma$. Similarly, γ is called a *stable manifold* if $\mathrm{dist}(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ for every $x, y \in \gamma$.

Definition 1. Let D^u be the unit disk in some Euclidean space and $\mathrm{Emb}^1(D^u, M)$ be the space of C^1 embeddings from D^u into M. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s and $\Phi^u\colon K^s \times D^u \to M$ such that

i) $\gamma^u = \Phi^u(\{x\} \times D^u)$ is an unstable manifold;
ii) Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
iii) $x \mapsto \Phi^u|(\{x\} \times D^u)$ defines a continuous map from K^s into $\mathrm{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined similarly.

Definition 2. We say that $\Lambda \subset M$ has a *hyperbolic product structure* if there exist a continuous family of unstable manifolds $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable manifolds $\Gamma^s = \{\gamma^s\}$ such that

i) $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$;
ii) $\dim \gamma^u + \dim \gamma^s = \dim M$;
iii) each γ^s meets each γ^u in exactly one point;
iv) stable and unstable manifolds meet with angles larger than some $\theta > 0$.

Let $\Lambda \subset M$ have a hyperbolic product structure, whose defining families are Γ^s and Γ^u. A subset $\Upsilon_0 \subset \Lambda$ is called an *s-subset* if Υ_0 also has a hyperbolic product structure and its defining families Γ_0^s and Γ_0^u can be chosen with $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined analogously. Given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$. For each $n \geq 1$, let $(f^n)^u$ denote the restriction of the map f^n to γ^u-disks, and let $\det D(f^n)^u$ be the Jacobian of $D(f^n)^u$. In the sequel $C > 0$ and $0 < \beta < 1$ are constants, and we require the following properties from the hyperbolic product structure Λ:

($\mathbf{P}_0$) *Positive measure*: for every $\gamma \in \Gamma^u$ we have $\mathrm{Leb}_\gamma(\Lambda \cap \gamma) > 0$.

($\mathbf{P}_1$) *Markovian*: there are pairwise disjoint s-subsets $\Upsilon_1, \Upsilon_2, \dots \subset \Lambda$ such that
(a) $\mathrm{Leb}_\gamma\big((\Lambda \setminus \cup\Upsilon_i) \cap \gamma\big) = 0$ on each $\gamma \in \Gamma^u$;
(b) for each $i \in \mathbb{N}$ there is $\tau_i \in \mathbb{N}$ such that $f^{\tau_i}(\Upsilon_i)$ is a u-subset, and for all $x \in \Upsilon_i$
$$f^{\tau_i}(\gamma^s(x)) \subset \gamma^s(f^{\tau_i}(x)) \quad \text{and} \quad f^{\tau_i}(\gamma^u(x)) \supset \gamma^u(f^{\tau_i}(x));$$
(c) for each $n \in \mathbb{N}$ there are finitely many i's with $\tau_i = n$.

($\mathbf{P}_2$) *Contraction on stable leaves*: for each $\gamma^s \in \Gamma^s$ and each $y \in \gamma^s(x)$
$$\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^n, \quad \forall n \geq 1.$$

For the last two properties we introduce the *return time* $R\colon \Lambda \to \mathbb{N}$ and the *induced map* $F = f^R\colon \Lambda \to \Lambda$, which are defined for each $i \in \mathbb{N}$ as

$$R|_{\Upsilon_i} = \tau_i \quad \text{and} \quad f^R|_{\Upsilon_i} = f^{\tau_i}|_{\Upsilon_i},$$

and, for each $x, y \in \Lambda$, the *separation time* $s(x,y)$ is given by

$$s(x,y) = \min\left\{n \geq 0\colon (f^R)^n(x) \text{ and } (f^R)^n(y) \text{ lie in distinct } \Upsilon_i' s\right\}.$$

($\mathbf{P}_3$) *Regularity of the stable foliation*:

(a) for $y \in \gamma^s(x)$ and $n \geq 0$

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n;$$

(b) given $\gamma, \gamma' \in \Gamma^u$, we define $\Theta\colon \gamma' \cap \Lambda \to \gamma \cap \Lambda$ by $\Theta(x) = \gamma^s(x) \cap \gamma$. Then Θ is absolutely continuous and

$$\frac{d(\Theta_* \operatorname{Leb}_{\gamma'})}{d \operatorname{Leb}_{\gamma}}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\Theta^{-1}(x)))};$$

(c) letting $v(x)$ denote the density in item (b), we have

$$\log \frac{v(x)}{v(y)} \leq C\beta^{s(x,y)}, \quad \text{for } x, y \in \gamma' \cap \Lambda.$$

($\mathbf{P}_4$) *Bounded distortion*: for $\gamma \in \Gamma^u$ and $x, y \in \Lambda \cap \gamma$

$$\log \frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} \leq C\beta^{s(f^R(x), f^R(y))}.$$

Remark 1.1. We do not assume uniform backward contraction along unstable leaves as (P4)(a) in [Y2]. Properties ($\mathbf{P}_3$)(c) and ($\mathbf{P}_4$) are new if comparing our setup to that in [Y2]. However, these are consequence of (P4) and (P5) of [Y2] as done in [Y2, Lemma 1].

In spite of the uniform contraction on stable leaves demanded in ($\mathbf{P}_2$), this is not too restrictive in systems having regions where the contraction fails to be uniform, since we are allowed to remove stable leaves, provided a subset with positive measure of leaves remains in the end. This has been carried out for Hénon maps in [BY2].

1.2. **Uniform families.** Let $\mathcal{F}$ be a a family of C^k maps ($k \geq 2$) from the finite dimensional Riemannian manifold M into itself, and endow $\mathcal{F}$ with the C^k topology. Assume that each map $f \in \mathcal{F}$ admits a Gibbs-Markov structure Λ_f as described in Section 1.1. Let $\Gamma_f^u = \{\gamma_f^u\}$ and $\Gamma_f^s = \{\gamma_f^s\}$ be its defining families of unstable and stable curves. Denote by $R_f : \Lambda_f \to \mathbb{N}$ the corresponding return time function.

Given $f_0 \in \mathcal{F}$, take a sequence $f_n \in \mathcal{F}$ such that $f_n \to f_0$ in the C^1 topology as $n \to \infty$. For the sake of notational simplicity, for each $n \geq 0$ we will indicate the dependence of the previous objects on f_n just by means of the index or supra-index n. If $\gamma_n^u \in \Gamma_n^u$ is sufficiently close to $\gamma_0^u \in \Gamma_0^u$ in the C^k topology, we may define a projection by sliding through the stable manifolds of Λ_0

$$\begin{array}{rccc} H_n: & \gamma_n^u \cap \Gamma_0^s & \longrightarrow & \gamma_0^u \\ & z & \longmapsto & \gamma_0^s(z) \cap \gamma_0^u \end{array}$$

and set

$$\Omega_0 = \gamma_0^u \cap \Lambda_0, \quad \Omega_n^0 = H_n^{-1}(\Omega_0), \quad \Omega_n = \gamma_n^u \cap \Lambda_n, \quad \Omega_0^n = H_n(\Omega_n \cap \Omega_n^0). \qquad (1.3)$$

Given $k \in \mathbb{N}$ and positive integers $i_1, \dots, i_k$, we denote by $\Upsilon^n_{i_1,\dots,i_k}$ the s-sublattice that satisfies $F^j_n(\Upsilon^n_{i_1,\dots,i_k}) \subset \Upsilon^n_{i_j}$ for every $1 \le j < k$ and $F^k_n(\Upsilon^n_{i_1,\dots,i_k}) = \Upsilon^n_{i_k}$.

Definition 3. $\mathcal{F}$ is called a *uniform family* if the conditions $(\mathbf{U}_0)$–$(\mathbf{U}_5)$ below hold:

- $(\mathbf{U}_0)$ *Absolute constants:* the constants C and β in $(\mathbf{P}_2)$,$(\mathbf{P}_3)$ and $(\mathbf{P}_4)$ can be chosen the same for all $f \in \mathcal{F}$.
- $(\mathbf{U}_1)$ *Proximity of unstable leaves:* there are unstable leaves $\hat\gamma_0 \in \Gamma^u_0$ and $\hat\gamma_n \in \Gamma_n$ such that $\hat\gamma_n \to \hat\gamma_0$ in the C^1 topology as $n \to \infty$.
- $(\mathbf{U}_2)$ *Matching of structures:* defining the objects of (1.3) with $\hat\gamma_n$ replacing γ^u_n, we have
$$\mathrm{Leb}_{\hat\gamma_n}\left(\Omega_n \triangle \Omega^0_n\right) \to 0, \quad \text{as } n \to \infty.$$
- $(\mathbf{U}_3)$ *Proximity of stable directions:* for every $z \in \Omega^n_0 \cap \Omega_0$ we have $\gamma^s_n(z) \to \gamma^s_0(z)$ in the C^1 topology as $n \to \infty$.
- $(\mathbf{U}_4)$ *Matching of s-sublattices:* given $N, k \in \mathbb{N}$ and $\Upsilon^0_{i_1,\dots,i_k}$ with $R_0\big(\Upsilon^0_{i_j}\big) \le N$ for $1 \le j \le k$, there is $\Upsilon^n_{\ell_1,\dots,\ell_k}$ such that $R_n\big(\Upsilon^n_{\ell_j}\big) = R_0\big(\Upsilon^0_{i_j}\big)$ for $1 \le j \le k$ and
$$\mathrm{Leb}_{\hat\gamma_0}\left(H_n\big(\Upsilon^n_{\ell_1,\dots,\ell_k} \cap \hat\gamma_n\big) \triangle \big(\Upsilon^0_{i_1,\dots,i_k} \cap \hat\gamma_0\big)\right) \to 0, \quad \text{as } n \to \infty.$$
- $(\mathbf{U}_5)$ *Uniform tail*: given $\varepsilon > 0$, there are $N = N(\varepsilon)$ and $J = J(\varepsilon, N)$ such that
$$\sum_{j=N}^{\infty} j\, \mathrm{Leb}_{\hat\gamma_n}\{R_n = j\} < \varepsilon, \quad \forall n > J.$$

This last property ensures in particular that $\int_{\hat\gamma_n} R_n \, d\,\mathrm{Leb}_{\hat\gamma_n} < \infty$ for large n, which by [Y2, Theorem 1] implies the existence of an SRB measure for each f_n.

Remark 1.2. Using that stable and unstable manifolds of f_0 meet with angles uniformly bounded away from zero at points in Λ_0, and the proximities given by $(\mathbf{U}_1)$ and $(\mathbf{U}_3)$, it follows that there is some $\theta > 0$ such that, for n large enough, the stable manifolds through points in Ω^0_n meet $\hat\gamma_n$ with an angle bigger than θ. Together with $(\mathbf{P}_3)$ and $(\mathbf{U}_1)$, this implies that:

- *i)* $(H_n)_* \mathrm{Leb}_{\hat\gamma_n} \ll \mathrm{Leb}_{\hat\gamma_0}$ with uniformly bounded density;
- *ii)* $\frac{d(H_n)_* \mathrm{Leb}_{\hat\gamma_n}}{\mathrm{Leb}_{\hat\gamma_0}} \to 1$ on $L^1(\mathrm{Leb}_{\hat\gamma_0})$, as $n \to \infty$.

1.3. **Statement of results.** Consider a family $\mathcal{F}$ such that each $f \in \mathcal{F}$ admits a unique SRB measure μ_f. Letting $\mathbb{P}(M)$ denote the space of probability measures on M endowed with the weak* topology, we say that $\mathcal{F}$ is *statistically stable* if the map

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathbb{P}(\mathrm{M}) \\ f & \longmapsto & \mu_f, \end{array}$$

is continuous. In the sequel h_{μ_f} denotes the metric entropy of f with respect to the measure μ_f.

Theorem A. *Let $\mathcal{F}$ be a uniform family such that each $f \in \mathcal{F}$ admits a unique SRB measure. Then*

1. *$\mathcal{F}$ is statistically stable;*
2. *$\mathcal{F} \ni f \mapsto h_{\mu_f}$ is continuous.*

Corollary B. *The family $\mathcal{BC}$ is statistically stable and the map $\mathcal{BC} \ni (a, b) \mapsto h_{\mu_{a,b}}$ is continuous.*

This corollary follows immediately after building Gibbs-Markov structures satisfying $(\mathbf{P}_0)$–$(\mathbf{P}_4)$, as was done in [BY2], and verifying the uniformity conditions $(\mathbf{U}_0)$–$(\mathbf{U}_5)$, as in [ACF]. For the sake of clearness, the following list specifies exactly where each property is obtained.

$(\mathbf{P}_0)$	[BY2, Proposition A(3)]
$(\mathbf{P}_1)$	[BY2, Proposition A(1),(2)]
$(\mathbf{P}_2)$	[BY2, Proposition A(2)]
$(\mathbf{P}_3)$(a)	[BY2, Sublemma 8]
$(\mathbf{P}_3)$(b)	[BY2, Sublemma 10]
$(\mathbf{P}_3)$(c)	[BY2, Sublemma 11]
$(\mathbf{P}_4)$	[BY2, Sublemma 9]
$(\mathbf{U}_0)$	[ACF, Sections 6,7,8]
$(\mathbf{U}_1)$	Hyperbolicity of the fixed point z^*
$(\mathbf{U}_2)$	[ACF, Section 6 in particular Corollary 6.4]
$(\mathbf{U}_3)$	[ACF, Section 7 in particular Proposition 7.3]
$(\mathbf{U}_4)$	[ACF, Section 8 in particular Proposition 8.9]
$(\mathbf{U}_5)$	[BY2, Proposition A(4)]

Concerning $(\mathbf{U}_0)$ and $(\mathbf{U}_5)$, observe that the constants depend exclusively on the maximum value for $b > 0$ and the minimum for $a < 2$ in the choice of Benedicks-Carleson parameters.

2. Quotient dynamics and lifting back

In this section we shall analyze some dynamical features of a diffeomorphism f admitting Λ with a Gibbs-Markov structure that verifies properties $(\mathbf{P}_0)$-$(\mathbf{P}_4)$. Consider a quotient space $\bar{\Lambda}$ obtained by collapsing the stable curves of Λ; i.e. $\bar{\Lambda} = \Lambda/\sim$, where $z \sim z'$ if and only if $z' \in \gamma^s(z)$. Since by $(\mathbf{P}_1)$(b) the induced map $F = f^R : \Lambda \to \Lambda$ takes γ^s leaves to γ^s leaves, then the *quotient induced map* $\overline{F} : \bar{\Lambda} \to \bar{\Lambda}$ is well defined and if $\bar{\Upsilon}_i$ is the quotient of Υ_i, then $\overline{F}$ takes the sets $\bar{\Upsilon}_i$ homeomorphically onto $\bar{\Lambda}$. Given an unstable leaf γ, the set $\gamma \cap \Lambda$ suits as a model for $\bar{\Lambda}$ through the canonical projection $\bar{\pi} : \Lambda \to \bar{\Lambda}$. We will see in Section 2.1 that we may define a natural reference measure $\bar{m}$ on $\bar{\Lambda}$. Besides, $\overline{F}$ is an expanding Markov map (see Lemma 2.1), thus having an absolutely continuous (w.r.t $\bar{m}$), $\overline{F}$-invariant probability measure $\bar{\mu}$. Moreover, if $\tilde{\mu}$ denotes the F-invariant measure supported on Λ then $\bar{\mu} = \bar{\pi}_*(\tilde{\mu})$.

To build an SRB measure μ out of $\tilde{\mu}$ is just a matter of saturating the measure $\tilde{\mu}$. The existence of the measures $\bar{\mu}$, $\tilde{\mu}$ and the fact that $\bar{\mu} = \bar{\pi}_*(\tilde{\mu})$ follows from standard methods, which can be found for instance in [Y2]. For the sake of completeness we will present the construction of the SRB measure, also having in mind how some properties can be carried up through the lifting. We will accomplish this by adapting some ideas used in the construction of Gibbs states; see [B].

2.1. **The natural measure.**

The purpose of this subsection is to introduce a natural probability measure $\bar{m}$ on $\bar{\Lambda}$ and establish some properties of the Jacobian of $\overline{F}$ with respect to $\bar{m}$. Moreover, we show the existence of an $\overline{F}$-invariant density $\bar{\rho}$ with respect to the measure $\bar{m}$.

Fix an arbitrary $\hat{\gamma} \in \Gamma^u$. The restriction of $\bar{\pi}$ to $\hat{\gamma} \cap \Lambda$ gives a homeomorphism that we denote by $\hat{\pi} : \hat{\gamma} \cap \Lambda \to \bar{\Lambda}$. Given $\gamma \in \Gamma^u$ and $x \in \gamma \cap \Lambda$ let $\hat{x}$ be the point in $\gamma^s(x) \cap \hat{\gamma}$.

Defining for $x \in \gamma \cap \Lambda$

$$\hat{u}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\hat{x}))} \tag{2.1}$$

we have that $\hat{u}$ satisfies the bounded distortion property $(\mathbf{P}_3)$(c). For each $\gamma \in \Gamma^u$ let m_γ be the measure in γ such that

$$\frac{dm_\gamma}{d\,\mathrm{Leb}_\gamma} = \hat{u}\,\mathbf{1}_{\gamma\cap\Lambda},$$

where $\mathbf{1}_{\gamma\cap\Lambda}$ is the characteristic function of the set $\gamma \cap \Lambda$. These measures have been defined in such a way that if $\gamma, \gamma' \in \Gamma^u$ and Θ is obtained by sliding along stable leaves from $\gamma \cap \Lambda$ to $\gamma' \cap \Lambda$, then

$$\Theta_* m_\gamma = m_{\gamma'}. \tag{2.2}$$

To verify this let us show that the densities of these two measures with respect to Leb_γ coincide. Take $x \in \gamma \cap \Lambda$ and $x' \in \gamma' \cap \Lambda$ such that $\Theta(x) = x'$. By $(\mathbf{P}_3)$(b) one has

$$\frac{d\Theta_*\,\mathrm{Leb}_\gamma}{d\,\mathrm{Leb}_{\gamma'}}(x') = \frac{\hat{u}(x')}{\hat{u}(x)},$$

which implies that

$$\frac{d\Theta_* m_\gamma}{d\,\mathrm{Leb}_{\gamma'}}(x') = \hat{u}(x)\frac{d\Theta_*\,\mathrm{Leb}_\gamma}{d\,\mathrm{Leb}_{\gamma'}}(x') = \hat{u}(x') = \frac{dm_{\gamma'}}{d\,\mathrm{Leb}_{\gamma'}}(x').$$

Conditions $(\mathbf{P}_0)$ and (2.2) allow us to define the reference probability measure $\bar{m}$ whose representative in each unstable leaf $\gamma \in \Gamma^u$ is exactly $\frac{1}{\mathrm{Leb}_{\hat{\gamma}}(\Lambda)} m_\gamma$.

Let $T : (X_1, m_1) \to (X_2, m_2)$ be a measurable bijection between two probability measure spaces. T is called *nonsingular* if it maps sets of zero m_1 measure to sets of zero m_2 measure. For a nonsingular transformation T we define the Jacobian of T with respect to m_1 and m_2, denoted by $J_{m_1,m_2}(T)$, as the Radon-Nikodym derivative $\frac{dT_*^{-1}(m_2)}{dm_1}$. By assertion (1) of the following lemma it makes sense to consider the Jacobian of the quotient map $\overline{F} : (\overline{\Lambda}, \overline{m}) \to (\overline{\Lambda}, \overline{m})$ that we simply denote $J\overline{F}$.

Lemma 2.1. *Assuming that $F(\gamma \cap \Upsilon_i) \subset \gamma'$ for $\gamma, \gamma' \in \Gamma^u$, let $JF(x)$ denote the Jacobian of F with respect to the measures m_γ and $m_{\gamma'}$. Then*

(1) *$JF(x) = JF(y)$ for every $y \in \gamma^s(x)$;*

(2) *there is $C_0 > 0$ such that for every $x, y \in \gamma \cap \Upsilon_i$*

$$\left| \frac{JF(x)}{JF(y)} - 1 \right| \le C_0 \beta^{s(F(x),F(y))};$$

(3) *for every $k \in \mathbb{N}$ and any k positive integers $i_1, \ldots i_k$, there is $C_1 > 0$ such that for every $x, y \in \Upsilon_{i_1,\ldots,i_k} \cap \gamma$*

$$\left| \frac{JF^k(x)}{JF^k(y)} \right| \le C_1.$$

Proof. (1) For Leb_γ almost every $x \in \gamma \cap \Lambda$ we have

$$JF(x) = |\det DF^u(x)| \cdot \frac{\hat{u}(F(x))}{\hat{u}(x)}. \tag{2.3}$$

Denoting $\varphi(x) = \log|\det Df^u(x)|$ we may write

$$\begin{aligned}\log JF(x) &= \sum_{i=0}^{R-1}\varphi(f^i(x)) + \sum_{i=0}^{\infty}\Big(\varphi(f^i(F(x))) - \varphi(f^i(\widehat{F(x)})\Big) \\ &\quad - \sum_{i=0}^{\infty}\big(\varphi(f^i(x)) - \varphi(f^i(\hat{x})\big) \\ &= \sum_{i=0}^{R-1}\varphi(f^i(\hat{x})) + \sum_{i=0}^{\infty}\Big(\varphi(f^i(F(\hat{x}))) - \varphi(f^i(\widehat{F(x)})\Big).\end{aligned}$$

Thus we have shown that $JF(x)$ can be expressed just in terms of $\hat{x}$ and $\widehat{F(x)}$, which is enough for proving the first part of the lemma.

(2) It follows from (2.3) that

$$\log\frac{JF(x)}{JF(y)} = \log\frac{\det DF^u(x)}{\det DF^u(y)} + \log\frac{\hat{u}(F(x))}{\hat{u}(F(y))} + \log\frac{\hat{u}(y)}{\hat{u}(x)}.$$

Observing that $s(x,y) > s(F(x),F(y))$ the conclusion follows from $(\mathbf{P}_3)$(c) and $(\mathbf{P}_4)$.

(3) Again, from (2.3), we obtain

$$\log\frac{JF^k(x)}{JF^k(y)} = \log\frac{\det D\left(F^k\right)^u(x)}{\det D\left(F^k\right)^u(y)} + \log\frac{\hat{u}(F^k(x))}{\hat{u}(F^k(y))} + \log\frac{\hat{u}(y)}{\hat{u}(x)}.$$

By $(\mathbf{P}_4)$ we have

$$\log\frac{\det D\left(F^k\right)^u(x)}{\det D\left(F^k\right)^u(y)} \le \sum_{l=1}^{k} C\beta^{s(F^l(x),F^l(y))} \le C\sum_{l=0}^{\infty}\beta^l < \infty.$$

The remaining terms are easily controlled once again due to $(\mathbf{P}_3)$(c). □

Lemma 2.2. *The map $\overline{F}:\bar{\Lambda}\to\bar{\Lambda}$ has an invariant probability measure $\bar{\mu}$ with $d\bar{\mu} = \bar{\rho}d\bar{m}$, where $K^{-1}\le\bar{\rho}\le K$, for some $K = K(C_1,\beta) > 0$.*

Proof. We construct $\bar{\rho}$ as the density with respect to $\bar{m}$ of an accumulation point of $\bar{\mu}^{(n)} = 1/n\sum_{i=0}^{n-1}\overline{F}^i_*(\bar{m})$. Let $\bar{\rho}^{(n)}$ denote the density of $\bar{\mu}^{(n)}$ and $\bar{\rho}^i$ the density of $\overline{F}^i_*(\bar{m})$. Also, let $\bar{\rho}^i = \sum_j\bar{\rho}^i_j$, where $\bar{\rho}^i_j$ is the density of $\overline{F}^i_*(\bar{m}|\sigma^i_j)$ and the σ^i_j's range over all components of $\bar{\Lambda}$ such that $\overline{F}^i(\sigma^i_j) = \bar{\Lambda}$.

Consider the normalized density $\tilde{\rho}^i_j = \bar{\rho}^i_j/\bar{m}(\sigma^i_j)$. We have for $\bar{x}'\in\sigma^i_j$ such that $\bar{x} = \overline{F}^i(\bar{x}')$ and for some $\bar{y}'\in\sigma^i_j$

$$\tilde{\rho}^i_j(\bar{x}) = \frac{J\overline{F}^i(\bar{y}')}{J\overline{F}^i(\bar{x}')}(\bar{m}(\bar{\Lambda}))^{-1} = \prod_{k=1}^{i}\frac{J\overline{F}(\overline{F}^{k-1}(\bar{y}'))}{J\overline{F}(\overline{F}^{k-1}(\bar{x}'))}.$$

By Lemma 2.1(2) we have for every $k = 1,\dots,i$

$$\frac{J\overline{F}(\overline{F}^{k-1}(\bar{y}'))}{J\overline{F}(\overline{F}^{k-1}(\bar{x}'))} \le \exp\left\{C_1\beta^{s\left(\overline{F}^k(\bar{y}'),\overline{F}^k(\bar{x}')\right)}\right\} \le \exp\left\{C_1\beta^{(i-k)+s(\bar{x},\bar{y})}\right\},$$

from where we conclude that

$$\tilde{\rho}^i_j(\bar{x}) \le \exp\left\{C_1\beta^{s(\bar{x},\bar{y})}\sum_{j\ge 0}\beta^j\right\} \le \exp\{C_1/(1-\beta)\} = K.$$

Observe that we also get

$$\frac{1}{\tilde{\rho}^i_j(\bar{x})} = \frac{J\overline{F}^i(\bar{x}')}{J\overline{F}^i(\bar{y}')}(\bar{m}(\bar{\Lambda})) \leq K,$$

which yields $\tilde{\rho}^i_j(\bar{x}) \geq K^{-1}$. Now, since $\bar{\rho}^i = \sum_j \bar{m}(\sigma^i_j)\tilde{\rho}^i_j$, we have $K^{-1} \leq \bar{\rho}^i \leq K$ which implies that $K^{-1} \leq \bar{\rho}^{(n)} \leq K$, from where we obtain that $K^{-1} \leq \bar{\rho} \leq K$. □

2.2. **Lifting to the Gibbs-Markov structure.** We now adapt standard techniques for lifting the $\overline{F}$- invariant measure on the quotient space to an F- invariant measure on the initial Gibbs-Markov structure.

Given an $\overline{F}$-invariant probability measure $\bar{\mu}$, we define a probability measure $\tilde{\mu}$ on Λ as follows. For each bounded $\phi : \Lambda \to \mathbb{R}$ consider its discretizations $\phi^\bullet : \hat{\gamma} \cap \Lambda \to \mathbb{R}$ and $\phi^* : \bar{\Lambda} \to \mathbb{R}$ defined by

$$\phi^\bullet(x) = \inf\{\phi(z) : z \in \gamma^s(x)\}, \quad \text{and} \quad \phi^* = \phi^\bullet \circ \hat{\pi}^{-1}. \tag{2.4}$$

If ϕ is continuous, as its domain is compact, we may define

$$\operatorname{var}\phi(k) = \sup\left\{|\phi(z) - \phi(\zeta)| : |z - \zeta| \leq C\beta^k\right\},$$

in which case $\operatorname{var}\phi(k) \to 0$ as $k \to \infty$.

Lemma 2.3. *Given any continuous $\phi : \Lambda \to \mathbb{R}$, for all $k, l \in \mathbb{N}$ we have*

$$\left|\int (\phi \circ F^k)^* d\bar{\mu} - \int (\phi \circ F^{k+l})^* d\bar{\mu}\right| \leq \operatorname{var}\phi(k).$$

Proof. Since $\bar{\mu}$ is $\overline{F}$-invariant

$$\left|\int (\phi \circ F^k)^* d\bar{\mu} - \int (\phi \circ F^{k+l})^* d\bar{\mu}\right| = \left|\int (\phi \circ F^k)^* \circ \overline{F}^l d\bar{\mu} - \int (\phi \circ F^{k+l})^* d\bar{\mu}\right|$$
$$\leq \int \left|(\phi \circ F^k)^* \circ \overline{F}^l - (\phi \circ F^{k+l})^*\right| d\bar{\mu}.$$

By definition of the discretization we have

$$(\phi \circ F^k)^* \circ \overline{F}^l(x) = \inf\left\{\phi(z) : z \in F^k\left(\gamma^s(\overline{F}^l(x))\right)\right\}$$

and

$$(\phi \circ F^{k+l})^*(x) = \inf\left\{\phi(\zeta) : \zeta \in F^{k+l}\left(\gamma^s(x)\right)\right\}.$$

Observe that $F^{k+l}\left(\gamma^s(x)\right) \subset F^k\left(\gamma^s(\overline{F}^l(x))\right)$ and by $(\mathbf{P}_2)$

$$\operatorname{diam} F^k\left(\gamma^s(\overline{F}^l(x))\right) \leq C\beta^k.$$

Thus, $\left|(\phi \circ F^k)^* \circ \overline{F}^l - (\phi \circ F^{k+l})^*\right| \leq \operatorname{var}\phi(k)$. □

By the Cauchy criterion the sequence $\left(\int (\phi \circ F^k)^* d\bar{\mu}\right)_{k\in\mathbb{N}}$ converges. Hence, Riesz Representation Theorem yields a probability measure $\tilde{\mu}$ on Λ

$$\int \phi \, d\tilde{\mu} := \lim_{k\to\infty} \int (\phi \circ F^k)^* d\bar{\mu} \tag{2.5}$$

for every continuous function $\phi : \Lambda \to \mathbb{R}$.

Proposition 2.4. *The probability measure $\tilde{\mu}$ is F-invariant and has absolutely continuous conditional measures on γ^u leaves. Moreover, given any continuous $\phi : \Lambda \to \mathbb{R}$ we have*

(1) $\left|\int \phi \, d\tilde{\mu} - \int (\phi \circ F^k)^* d\bar{\mu}\right| \leq \operatorname{var}\phi(k)$;

(2) *if ϕ is constant in each γ^s, then $\int \phi d\tilde{\mu} = \int \bar{\phi} d\bar{\mu}$, where $\bar{\phi} : \bar{\Lambda} \to \mathbb{R}$ is defined by $\bar{\phi}(x) = \phi(z)$, where $z \in \bar{\pi}^{-1}(x)$;*

(3) *if ϕ is constant in each γ^s and $\psi : \Lambda \to \mathbb{R}$ is continuous, then*

$$\left| \int \psi.\phi d\tilde{\mu} - \int (\psi \circ F^k)^* (\phi \circ F^k)^* d\bar{\mu} \right| \le \|\phi\|_1 \operatorname{var} \psi(k).$$

Proof. Regarding the F-invariance property, note that for any continuous $\phi : \Lambda \to \mathbb{R}$,

$$\int \phi \circ F d\tilde{\mu} = \lim_{k\to\infty} \int \left(\phi \circ F^{k+1}\right)^* d\bar{\mu} = \int \phi d\tilde{\mu},$$

by Lemma 2.3. Assertion (1) is an immediate consequence of Lemma 2.3. Property (2) follows from

$$\int \phi d\tilde{\mu} = \lim_{k\to\infty} \int \left(\phi \circ F^k\right)^* d\bar{\mu} = \lim_{k\to\infty} \int \bar{\phi} \circ \bar{F}^k d\bar{\mu} = \int \bar{\phi} d\bar{\mu},$$

which holds by definition of $\tilde{\mu}$, ϕ^* and the $\bar{F}$-invariance of $\bar{\mu}$. For statement (3) let $\bar{\phi} : \bar{\Lambda} \to \mathbb{R}$ be defined by $\bar{\phi}(x) = \phi(z)$, where $z \in \bar{\pi}^{-1}(x)$. For any k, l positive integers observe that

$$\int (\psi.\phi \circ F^k)^* d\bar{\mu} = \int (\psi \circ F^k)^* (\phi \circ F^k)^* d\bar{\mu}$$

and

$$\begin{aligned}\left| \int (\psi\phi \circ F^{k+l})^* d\bar{\mu} - \int (\psi\phi \circ F^k)^* d\bar{\mu} \right| &= \left| \int (\psi \circ F^{k+l})^* \bar{\phi} \circ \bar{F}^{k+l} d\bar{\mu} - \int (\psi \circ F^k)^* \bar{\phi} \circ \bar{F}^k d\bar{\mu} \right| \\ &\le \int \left| (\psi \circ F^{k+l})^* - (\psi \circ F^k)^* \circ \bar{F}^l \right| |\phi \circ \bar{F}^{k+l}| d\bar{\mu} \\ &\le \operatorname{var} \psi(k) \|\phi\|_1.\end{aligned}$$

Inequality (3) follows letting l go to ∞.

We are then left to verify the absolute continuity. While the properties proved above are intrinsic to the lifting technique, the disintegration into absolutely continuous conditional measures on unstable leaves depends on the definition of the reference measure $\bar{m}$ and the fact that $\bar{\mu} = \bar{\rho}\bar{m}$. Fix an unstable leaf $\gamma^u \in \Gamma^u$. Denote by λ_{γ^u} the conditional Lebesgue measure on γ^u. Consider a set $E \subset \gamma^u$ such that $\lambda_{\gamma^u}(E) = 0$. We will show that $\tilde{\mu}_{\gamma^u}(E) = 0$, where $\tilde{\mu}_{\gamma^u}$ denotes the conditional measure of $\tilde{\mu}$ on γ^u, except for a few choices of γ^u. To be more precise, the family of curves Γ^u induces a partition of Λ into unstable leaves which we denote by $\mathcal{L}$. Let $\pi_{\mathcal{L}} : \Lambda \to \mathcal{L}$ be the natural projection on the quotient space $\mathcal{L}$, i.e. $\pi_{\mathcal{L}}(z) = \gamma^u(z)$. We say that $Q \subset \mathcal{L}$ is measurable if and only if $\pi_{\mathcal{L}}^{-1}(Q)$ is measurable. Let $\hat{\mu} = (\pi_{\mathcal{L}})_*(\tilde{\mu})$, which means that $\hat{\mu}(Q) = \tilde{\mu}\left(\pi_{\mathcal{L}}^{-1}(Q)\right)$. We assume that by definition of Γ^u there is a non-decreasing sequence of finite partitions $\mathcal{L}_1 \prec \mathcal{L}_2 \prec \ldots \prec \mathcal{L}_n \prec \ldots$ such that $\mathcal{L} = \bigvee_{i=1}^{\infty} \mathcal{L}_n$. Thus, by Rokhlin disintegration theorem (see [BDV, Appendix C.6]) there is a system $\left(\tilde{\mu}_{\gamma^u}\right)_{\gamma^u \in \mathcal{L}}$ of conditional probability measures of $\tilde{\mu}$ with respect to $\mathcal{L}$ such that

- $\tilde{\mu}_{\gamma^u}(\gamma^u) = 1$ for $\hat{\mu}$- almost every $\gamma^u \in \mathcal{L}$;
- given any bounded measurable map $\phi : \Lambda \to \mathbb{R}$, the map $\gamma^u \mapsto \int \phi d\tilde{\mu}_{\gamma^u}$ is measurable and $\int \phi d\tilde{\mu} = \int \left(\int \phi d\tilde{\mu}_{\gamma^u}\right) d\hat{\mu}$.

Let $\bar{E} = \bar{\pi}(E)$. Since the reference measure $\bar{m}$ has a representative m_{γ^u} on γ^u which is equivalent to λ_{γ^u}, we have $m_{\gamma^u}(E) = 0$ and $\bar{m}(\bar{E}) = 0$. As $\bar{\mu} = \bar{\rho}\bar{m}$, then $\bar{\mu}(\bar{E}) = 0$. Let $\bar{\phi}_n : \bar{\Lambda} \to \mathbb{R}$ be a sequence of continuous functions such that $\bar{\phi}_n \to \mathbf{1}_{\bar{E}}$ as $n \to \infty$. Consider also the sequence of continuous functions $\phi_n : \Lambda \to \mathbb{R}$ given by $\phi_n = \bar{\phi}_n \circ \bar{\pi}$.

Clearly ϕ_n is constant in each γ^s stable leaf and $\phi_n \to \mathbf{1}_{\bar{E}} \circ \bar{\pi} = \mathbf{1}_{\bar{\pi}^{-1}(\bar{E})}$ as $n \to \infty$. By Lebesgue dominated convergence theorem we have $\int \phi_n d\tilde{\mu} \to \int \mathbf{1}_{\bar{\pi}^{-1}(\bar{E})} d\tilde{\mu} = \tilde{\mu}\left(\bar{\pi}^{-1}(\bar{E})\right)$ and $\int \bar{\phi}_n d\bar{\mu} \to \int \mathbf{1}_{\bar{E}} d\bar{\mu} = \bar{\mu}(\bar{E}) = 0$. By (2) we have $\int \phi_n \tilde{\mu} = \int \bar{\phi}_n d\bar{\mu}$. Hence, we must have $\tilde{\mu}\left(\bar{\pi}^{-1}(\bar{E})\right) = 0$. Consequently,

$$0 = \int \mathbf{1}_{\bar{\pi}^{-1}(\bar{E})} d\tilde{\mu} = \int \left(\int \mathbf{1}_{\bar{\pi}^{-1}(\bar{E})} d\tilde{\mu}_{\gamma^u} \right) d\hat{\mu}(\gamma^u),$$

which implies that $\tilde{\mu}_{\gamma^u}\left(\bar{\pi}^{-1}(\bar{E}) \cap \gamma^u\right) = 0$ for $\hat{\mu}$-almost every γ^u. □

Remark 2.5. Since the continuous functions are dense in L^1, properties (2) and (3) also hold when $\phi \in L^1$, by dominated convergence.

2.3. **Entropy formula.** Let $\tilde{\mu}$ be the SRB measure for F obtained from $\bar{\mu} = \bar{\rho}\bar{m}$ as in (2.5). We define the saturation of $\tilde{\mu}$ by

$$\mu^* = \sum_{l=0}^{\infty} f_*^l \left(\tilde{\mu} | \{R > l\} \right). \tag{2.6}$$

It is well known that μ^* is f-invariant and that the finiteness of μ^* is equivalent to $\int R \, d\tilde{\mu} = \int R \, d\bar{\mu} < \infty$. By construction of and $\bar{m}$ and $\bar{\mu}$, the finiteness of μ^* is also equivalent to $\int_{\gamma \cap \Lambda} R \, d\,\mathrm{Leb}_\gamma < \infty$. Clearly, each $f_*^l \left(\tilde{\mu} | \{R > l\} \right)$ has absolutely continuous conditional measures on $\{f^l \gamma^u\}$, which are Pesin unstable manifolds. Consequently

$$\mu = \frac{1}{\mu^*(M)} \mu^*$$

is an SRB measure for f.

Lemma 2.6. *If λ is a Lyapunov exponent of $\tilde{\mu}$, then λ/σ is a Lyapunov exponent of μ, where $\sigma = \int_\Lambda R d\tilde{\mu}$.*

Proof. As μ is obtained by saturating $\tilde{\mu}$ in (2.6), one easily gets $\mu^*(\Lambda) \geq \tilde{\mu}(\Lambda) = 1$, and so $\mu(\Lambda) > 0$. By ergodicity, it is enough to compare the Lyapunov exponents for points $z \in \Lambda$. Let n be a positive integer. We have for each $z \in \Lambda$

$$F^n(z) = f^{S_n(z)}(z), \quad \text{where } S_n(z) = \sum_{i=0}^{n-1} R(F^i(z)).$$

As $S_n(z) = S_n(\zeta)$ for Lebesgue almost every $z \in \Lambda$ and ζ close to z, we have for $v \in T_z M$

$$\frac{1}{S_n(z)} \log \|Df^{S_n(z)}(z)v\| = \frac{n}{nS_n(z)} \log \|DF^n(z)v\|. \tag{2.7}$$

Since $\tilde{\mu}$ is ergodic, Birkhoff ergodic theorem yields

$$\lim_{n\to\infty} \frac{S_n(z)}{n} = \int_\Lambda R \, d\tilde{\mu} = \sigma \tag{2.8}$$

for $\tilde{\mu}$ almost every $z \in \Lambda$. □

Proposition 2.7. *Let $J\bar{F}$ be the Jacobian of $\bar{F}$ with respect to the measure $\bar{m}$ on $\bar{\Lambda}$. Then*

$$h_\mu = \sigma^{-1} \int_{\bar{\Lambda}} \log J\bar{F} d\bar{m}.$$

Proof. By [LY2, Corollary 7.4.2] we have

$$h_\mu = \sum_{\lambda_i > 0} \lambda_i \dim E_i, \tag{2.9}$$

where λ_i are Lyapunov exponents of μ and E_i the corresponding linear spaces given by Oseledets' decomposition. By Lemma 2.6 we have

$$h_\mu = \sigma^{-1} \sum_{\tilde\lambda_i > 0} \tilde\lambda_i \dim E_i,$$

where $\tilde\lambda_i$ are Lyapunov exponents of $\tilde\mu$. As a consequence of Oseledets theorem we may also write

$$\sum_{\tilde\lambda_i > 0} \tilde\lambda_i \dim E_i = \int_\Lambda \log\det DF^u d\tilde\mu.$$

According to (2.3),

$$\begin{aligned} \int_\Lambda \log JF d\tilde\mu &= \int_\Lambda \log\det DF^u d\tilde\mu + \int_\Lambda \log \hat u \circ F d\tilde\mu - \int_\Lambda \log \hat u d\tilde\mu \\ &= \int_\Lambda \log\det DF^u d\tilde\mu, \end{aligned}$$

where the last equality follows from the F-invariance of $\tilde\mu$. Finally, since by Lemma 2.1 JF is constant in each γ^s-leaf it follows from Proposition 2.4 (2) that

$$\int_\Lambda \log JF d\tilde\mu = \int_{\bar\Lambda} \log J\bar F d\bar m.$$

□

3. Statistical Stability

Let $\mathcal{F}$ be a uniform family of maps. Fix $f_0 \in \mathcal{F}$ and take any sequence $(f_n)_{n\ge1}$ in $\mathcal{F}$ such that $f_n \to f_0$, as $n \to \infty$, in the C^k topology. For each $n \ge 0$, let μ_n denote the (unique) SRB measure for f_n. Given $n \ge 0$, the map $f_n \in \mathcal{F}$ admits a Gibbs-Markov structure Λ_n with $\Gamma^u_n = \{\gamma^u_n\}$ and $\Gamma^s_n = \{\gamma^s_n\}$ its defining families of unstable and stable leaves. Consider $R_n : \Lambda_n \to \mathbb{N}$ the return time, $F_n : \Lambda_n \to \Lambda_n$ the induced map, $\hat\gamma_n$ the special unstable leaf given by condition $(\mathbf{U}_1)$ and $H_n : \hat\gamma_n \cap \Gamma^s_0 \to \hat\gamma_0$ obtained by sliding through the stable leaves of Λ_0. Recall that $\Omega^n_0 = H_n(\hat\gamma_n \cap \Lambda_n)$ and $\Omega_0 = \hat\gamma_0 \cap \Lambda_0$.

Remark 3.1. Since $f_n \to f_0$, as $n \to \infty$, in the C^k topology and $(\mathbf{U}_1)$ holds, then for every $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there exists $N_0 \in \mathbb{N}$ such that for every $n \ge N_0$ we have

$$\|\hat\gamma_n - \hat\gamma_0\|_1 < \varepsilon,$$

$$\max_{x\in\Omega_0\cap\Omega_0^n} \left\{ |(f_n \circ H_n^{-1} - f_0)(x)|, \ldots, |(f_n^\ell \circ H_n^{-1} - f_0^\ell)(x)| \right\} < \varepsilon,$$

and

$$\max_{x\in\Omega_0\cap\Omega_0^n} \left\{ \left|\log \frac{\det Df_n^u(f_n \circ H_n^{-1}(x))}{\det Df_0^u(f_0(x))}\right|, \ldots, \left|\log \frac{\det Df_n^u(f_n^\ell \circ H_n^{-1}(x))}{\det Df_0^u(f_0^\ell(x))}\right| \right\} < \varepsilon.$$

Our goal is to show that $\mu_n \to \mu_0$ in the weak* topology, i.e. for each continuous function $g : M \to \mathbb{R}$ the sequence $\int g\, d\mu_n$ converges to $\int g\, d\mu_0$. We will show that given any continuous $g : M \to \mathbb{R}$, each subsequence of $\int g\, d\mu_n$ admits a subsequence converging to $\int g\, d\mu_0$.

3.1. **Convergence of the densities on the reference leaf.** In Section 2.1 we built a family of holonomy invariant measures on unstable leaves that gives rise to a measure $\bar{m}_n$ on $\bar{\Lambda}_n$. Moreover,

$$(\hat{\pi}_n)_* m_{\hat{\gamma}_n} = \bar{m}_n \quad \text{and} \quad m_{\hat{\gamma}_n} = \mathbf{1}_{\hat{\gamma}_n \cap \Lambda_n} \operatorname{Leb}_{\hat{\gamma}_n}, \tag{3.1}$$

where $\mathbf{1}_{(\cdot)}$ stands for the indicator function. By Lemma 2.2, for each $n \geq 0$ there is an $\bar{F}_n$-invariant measure $\bar{\mu}_n = \bar{\rho}_n \bar{m}_n$ with $\|\bar{\rho}_n\|_\infty \leq K$ for all $n \geq 0$. We define the sequence $(\varrho_n)_{n\geq 0}$ of functions in $\hat{\gamma}_0$ as

$$\varrho_n = \bar{\rho}_n \circ \hat{\pi}_n \circ H_n^{-1} \cdot \mathbf{1}_{\Omega_0^n}, \tag{3.2}$$

which in particular gives

$$\varrho_0 = \bar{\rho}_0 \circ \hat{\pi}_0.$$

The main purpose of this section is to prove that *the sequence $(\varrho_n)_{n\in\mathbb{N}}$ converges to ϱ_0 in the weak* topology.* By Banach-Alaoglu theorem there is a subsequence $(\varrho_{n_i})_{i\in\mathbb{N}}$ converging to some $\varrho_\infty \in L^\infty(\operatorname{Leb}_{\hat{\gamma}_0})$ in the weak* topology, i.e.

$$\int \phi \varrho_{n_i} d\operatorname{Leb}_{\hat{\gamma}_0} \xrightarrow[i\to\infty]{} \int \phi \varrho_\infty d\operatorname{Leb}_{\hat{\gamma}_0}, \quad \forall \phi \in L^1(\operatorname{Leb}_{\hat{\gamma}_0}). \tag{3.3}$$

The following lemma establishes that integration with respect to $\bar{m}_n$ is close to integration with respect to $\varrho_n \operatorname{Leb}_{\hat{\gamma}_0}$, up to a small error.

Lemma 3.2. *Let $\bar{\phi} \in L^\infty(\bar{m}_n)$. If n is sufficiently large, then*

$$\left| \int_{\bar{\Lambda}_n} \bar{\phi}\bar{\rho}_n \, d\bar{m}_n - \int_{\Omega_0^n} (\bar{\phi} \circ \hat{\pi}_n \circ H_n^{-1}) \varrho_n \, d\operatorname{Leb}_{\hat{\gamma}_0} \right| \leq K \|\bar{\phi}\|_\infty Q_n,$$

where $Q_n = \operatorname{Leb}_{\hat{\gamma}_n}(\Omega_n^0 \triangle \Omega_n) + \left| \int_{\Omega_0^n} d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n} - \int_{\Omega_0^n} d\operatorname{Leb}_{\hat{\gamma}_0} \right|$.

Proof. By (3.1), we have $\int_{\bar{\Lambda}_n} \bar{\phi}\bar{\rho}_n \, d\bar{m}_n = \int_{\Omega_n} (\bar{\phi} \circ \hat{\pi}_n)(\bar{\rho}_n \circ \hat{\pi}_n) \, d\operatorname{Leb}_{\hat{\gamma}_n}$. It follows that

$$\begin{aligned}
\left| \int_{\bar{\Lambda}_n} \bar{\phi}\bar{\rho}_n \, d\bar{m}_n - \int_{\Omega_0^n} (\bar{\phi} \circ \hat{\pi}_n \circ H_n^{-1}) \varrho_n \, d\operatorname{Leb}_{\hat{\gamma}_0} \right| &\leq \left| \int_{\Omega_n^0 \triangle \Omega_n} (\bar{\phi} \circ \hat{\pi}_n)(\bar{\rho}_n \circ \hat{\pi}_n) \, d\operatorname{Leb}_{\hat{\gamma}_n} \right| \\
&+ \left| \int_{\Omega_n^0 \cap \Omega_n} (\bar{\phi} \circ \hat{\pi}_n)(\bar{\rho}_n \circ \hat{\pi}_n) \, d\operatorname{Leb}_{\hat{\gamma}_n} - \int_{\Omega_0^n} (\bar{\phi} \circ \hat{\pi}_n \circ H_n^{-1}) \varrho_n \, d\operatorname{Leb}_{\hat{\gamma}_0} \right| \\
&\leq K \|\bar{\phi}\|_\infty \operatorname{Leb}_{\hat{\gamma}_n}(\Omega_n^0 \triangle \Omega_n) \\
&+ \left| \int_{\Omega_0^n} (\bar{\phi} \circ \hat{\pi}_n \circ H_n^{-1}) \varrho_n \, d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n} - \int_{\Omega_0^n} (\bar{\phi} \circ \hat{\pi}_n \circ H_n^{-1}) \varrho_n \, d\operatorname{Leb}_{\hat{\gamma}_0} \right| \\
&\leq K \|\bar{\phi}\|_\infty \operatorname{Leb}_{\hat{\gamma}_n}(\Omega_n^0 \triangle \Omega_n) + K \|\bar{\phi}\|_\infty \left| \int_{\Omega_0^n} d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n} - \int_{\Omega_0^n} d\operatorname{Leb}_{\hat{\gamma}_0} \right|.
\end{aligned}$$

□

Consider the maps $G_0 : \hat{\gamma}_0 \to \hat{\gamma}_0$ and $G_n : \hat{\gamma}_0 \to \hat{\gamma}_n$ defined by

$$G_0 = \hat{\pi}_0^{-1} \circ \bar{F}_0 \circ \hat{\pi}_0 \quad \text{and} \quad G_n = \hat{\pi}_n^{-1} \circ \bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}.$$

Lemma 3.3. *For every $\varepsilon > 0$, $n \in \mathbb{N}$ sufficiently large and $\operatorname{Leb}_{\hat{\gamma}_0}$-almost every $x \in \Omega_0 \cap \Omega_0^n \cap \{R_n = \ell\} \cap \{R_0 = \ell\}$ we have $|G_n(x) - G_0(x)| < \varepsilon$.*

Proof. Consider a point $x \in \Omega_0 \cap \Omega_0^n \cap \{R_n = \ell\} \cap \{R_0 = \ell\}$. We may assume that $G_n(x)$ is a Lebesgue density point of Ω_n. Then, using $(\mathbf{U}_2)$ and the continuity of the stable foliation (see Definition 1 (iii)), for sufficiently large $n \in \mathbb{N}$ we may guarantee the existence of a point $\tilde{y} \in \Omega_n^0 \cap \Omega_n$ such that $\gamma_n^s(\tilde{y})$ is at most $\varepsilon \sin(\theta)/4$ apart from $\gamma_n^s(G_n(x))$ in the C^1-norm; recall Remark 1.2. Using $(\mathbf{U}_3)$ we may assume that $n \in \mathbb{N}$ is also sufficiently large so that the distance in the C^1 norm between $\gamma_n^s(\tilde{y})$ and $\gamma_0^s(\tilde{y})$ is at most $\varepsilon \sin(\theta)/4$.

Taking into account Remark 3.1 and the continuity of the stable foliation, we may assume that $n \in \mathbb{N}$ is large enough so that $|f_n^l(H_n^{-1}(x)) - f_0^l(x)|$ is sufficiently small in order to $\gamma_0^s(f_0^l(x))$ belong to a $\varepsilon \sin(\theta)/4$-neighborhood of $\gamma_0^s(\tilde{y})$, in the C^1-norm. It follows that $\gamma_n^s(f_n^l(H_n^{-1}(x)))$ and $\gamma_0^s(f_0^l(x))$ are at most $3\varepsilon \sin(\theta)/4$ apart, in the C^1-norm. Finally, observing that $G_n(x) = \gamma_n^s(f_n^l(H_n^{-1}(x))) \cap \gamma_n^u$, $G_0(x) = \gamma_0^s(f_0^l(x)) \cap \gamma_0^u$ and γ_n^u can be made arbitrarily close to γ_0^u, in the C^1-norm (by $(\mathbf{U}_1)$), then, as long as n is sufficiently large, we have $|G_n(x) - G_0(x)| < \varepsilon$. □

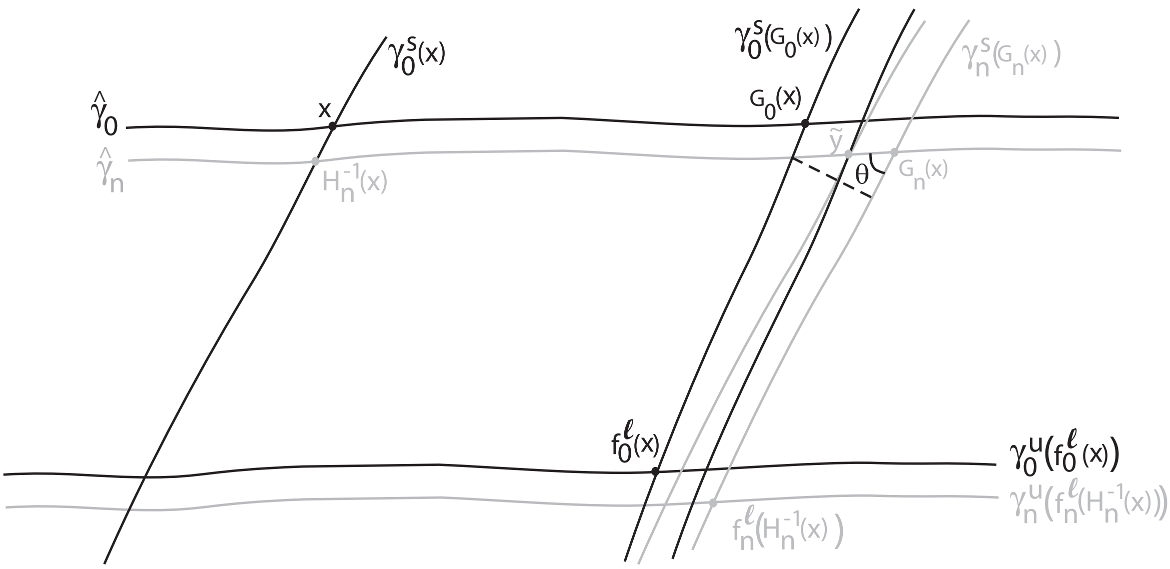

FIGURE 1

Proposition 3.4. *The measure* $(\varrho_\infty \circ \hat{\pi}_0^{-1})\bar{m}_0$ *is* $\bar{F}_0$*-invariant.*

Proof. We just have to verify that for every continuous $\varphi : \bar{\Lambda}_0 \to \mathbb{R}$

$$\int (\varphi \circ \bar{F}_0)(\varrho_\infty \circ \hat{\pi}_0^{-1}) d\bar{m}_0 = \int \varphi(\varrho_\infty \circ \hat{\pi}_0^{-1}) d\bar{m}_0$$

Given such φ, consider a continuous function $\phi : M \to \mathbb{R}$ such that $\|\phi\|_\infty \le \|\varphi\|_\infty$ and $\phi|_{\Omega_0} = \varphi \circ \hat{\pi}_0$. Since $\bar{\mu}_{n_i} = \bar{\rho}_{n_i} d\bar{m}_{n_i}$ is $\bar{F}_{n_i}$-invariant we have

$$\int (\phi \circ \hat{\pi}_{n_i}^{-1} \circ \bar{F}_{n_i})\bar{\rho}_{n_i} d\bar{m}_{n_i} = \int (\phi \circ \hat{\pi}_{n_i}^{-1})\bar{\rho}_{n_i} d\bar{m}_{n_i} \tag{3.4}$$

Recalling definitions (3.1),(3.2), the fact that ϱ_{n_i} is supported on $\Omega_0^{n_i} \subset \Omega_0$ and applying Lemmas 3.2 and 2.2 we get

$$\begin{aligned}
&\left|\int (\phi \circ \hat{\pi}_{n_i}^{-1})\bar{\rho}_{n_i}\ d\bar{m}_{n_i} - \int \varphi(\varrho_\infty \circ \hat{\pi}_0^{-1})\ d\bar{m}_0\right| \le \\
&\quad \le \left|\int (\phi \circ H_{n_i}^{-1})\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int (\varphi \circ \hat{\pi}_0)\varrho_\infty d\,\mathrm{Leb}_{\hat{\gamma}_0}\right| + Q_{n_i} \\
&\quad = \left|\int (\phi \circ H_{n_i}^{-1})\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int \phi\varrho_\infty d\,\mathrm{Leb}_{\hat{\gamma}_0}\right| + Q_{n_i} \\
&\quad \le \left|\int (\phi \circ H_{n_i}^{-1})\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int \phi\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0}\right| + \\
&\qquad + \left|\int \phi\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int \phi\varrho_\infty d\,\mathrm{Leb}_{\hat{\gamma}_0}\right| + Q_{n_i} \\
&\quad \le K \int \left|\phi \circ H_{n_i}^{-1} - \phi\right| d\,\mathrm{Leb}_{\hat{\gamma}_0} + \left|\int \phi\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int \phi\varrho_\infty d\,\mathrm{Leb}_{\hat{\gamma}_0}\right| + Q_{n_i}
\end{aligned}$$

Therefore, using $(\mathbf{U}_1)$ for the first term on the right, (3.3) for the second and $(\mathbf{U}_2)$ plus Remark 1.2 for the Q term, we conclude that

$$\int (\phi \circ \hat{\pi}_{n_i}^{-1})\bar{\rho}_{n_i} d\bar{m}_{n_i} \xrightarrow[i\to\infty]{} \int \varphi(\varrho_\infty \circ \hat{\pi}_0^{-1}) d\bar{m}_0. \tag{3.5}$$

Once we prove the next claim, then equality (3.4), the limit (3.5) and the uniqueness of the limit give the desired result.

Claim 3.1. $\displaystyle\int (\phi \circ \hat{\pi}_{n_i}^{-1} \circ \bar{F}_{n_i})\bar{\rho}_{n_i} d\bar{m}_{n_i} \xrightarrow[i\to\infty]{} \int \varphi \circ \bar{F}_0(\varrho_\infty \circ \hat{\pi}_0^{-1}) d\bar{m}_0.$

Let

$$E_1 := \left|\int (\phi \circ \hat{\pi}_{n_i}^{-1} \circ \bar{F}_{n_i})\bar{\rho}_{n_i} d\bar{m}_{n_i} - \int \varphi \circ \bar{F}_0(\varrho_\infty \circ \hat{\pi}_0^{-1}) d\bar{m}_0\right|.$$

Again, using definitions (3.1),(3.2) and applying Lemma 3.2 we get

$$E_1 \le \left|\int (\phi \circ G_{n_i})\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int (\phi \circ G_0)\varrho_\infty d\,\mathrm{Leb}_{\hat{\gamma}_0}\right| + Q_{n_i}$$

Now, observe that by $(\mathbf{U}_2)$ and Remark 1.2 the term Q_{n_i} can be made arbitrarily small for large i. This leaves us with the first term on the right that we denotee by E_2. Using Lemma 2.2 we have

$$\begin{aligned}
E_2 &\le \int \left|\phi \circ G_{n_i} - \phi \circ G_0\right| \varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} + \left|\int (\phi \circ G_0)\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int (\phi \circ G_0)\varrho_\infty d\,\mathrm{Leb}_{\hat{\gamma}_0}\right| \\
&\le K \int \left|\phi \circ G_{n_i} - \phi \circ G_0\right| d\,\mathrm{Leb}_{\hat{\gamma}_0} + \left|\int (\phi \circ G_0)\varrho_{n_i} d\,\mathrm{Leb}_{\hat{\gamma}_0} - \int (\phi \circ G_0)\varrho_\infty d\,\mathrm{Leb}_{\hat{\gamma}_0}\right|
\end{aligned}$$

According to equation (3.3) it is clear that the last term on the right can be made arbitrarily small provided i is large enough. So, denote by E_3 the first term on the right. Recalling the fact that ϱ_{n_i} is supported on $\Omega_0^{n_i} \subset \Omega_0$, we have for any N

$$\begin{aligned} E_3 \le K\|\phi\|_\infty & \sum_{\ell=N+1}^{\infty} \left(\mathrm{Leb}_{\hat\gamma_0}(\{R_{n_i}=\ell\}) + \mathrm{Leb}_{\hat\gamma_0}(\{R_0=\ell\})\right) \\ & + K\|\phi\|_\infty \sum_{\ell=1}^{N} \mathrm{Leb}_{\hat\gamma_0}(\{R_{n_i}=\ell\} \triangle \{R_0=\ell\}) \\ & + K \sum_{\ell=1}^{N} \int_{\{R_{n_i}=\ell\}\cap\{R_0=\ell\}\cap\Omega_0\cap\Omega_0^{n_i}} |\phi\circ G_{n_i} - \phi\circ G_0| \, d\,\mathrm{Leb}_{\hat\gamma_0}\,. \end{aligned}$$

Denote by E_4, E_5 and E_6 respectively the terms in the last sum. Having in mind $(\mathbf{U}_5)$ and Remark 1.2, we may choose $N \in \mathbb{N}$ sufficiently large so that E_4 is small for large i. For this choice of N, by $(\mathbf{U}_4)$, we also have that E_5 is small for large i. We now turn our attention to E_6. For $\ell = 1, \dots, N$, let

$$E_6^\ell = \int_{\{R_{n_i}=\ell\}\cap\{R_0=\ell\}} |\phi\circ G_{n_i} - \phi\circ G_0| \, \mathbf{1}_{\Omega_0\cap\Omega_0^{n_i}} d\,\mathrm{Leb}_{\hat\gamma_0}\,.$$

Since ϕ is continuous and M is compact then each E_6^ℓ can be made arbitrarily small by Lemma 3.3. □

Corollary 3.5. *Given $\phi \in L^1(\mathrm{Leb}_{\hat\gamma_0})$, we have*

$$\int \phi\varrho_n d\,\mathrm{Leb}_{\hat\gamma_0} \xrightarrow[n\to\infty]{} \int \phi\varrho_0 d\,\mathrm{Leb}_{\hat\gamma_0}\,.$$

Proof. By uniqueness of the absolutely continuous invariant measure for $\bar F$, it follows from Proposition 3.4 that $\bar\rho_0 = \varrho_\infty \circ \hat\pi_0^{-1}$, which immediately yields $\varrho_\infty = \varrho_0$. Hence

$$\int \phi\varrho_{n_i} d\,\mathrm{Leb}_{\hat\gamma_0} \xrightarrow[i\to\infty]{} \int \phi\varrho_0 d\,\mathrm{Leb}_{\hat\gamma_0}, \quad \text{for all } \phi \text{ continuous.} \tag{3.6}$$

The same argument proves that any subsequence of $(\varrho_n)_n$ has a weak* convergent subsequence with limit also equal to ϱ_0. This shows that $(\varrho_n)_n$ itself converges to ϱ_0 in the weak* topology. Since continuous functions are dense in $L^1(\mathrm{Leb}_{\hat\gamma_0})$, using that the densities ϱ_n are uniformly bounded, by Lemma 2.2, the result follows easily from (3.6). □

3.2. **Continuity of the SRB measures.** For each $n \ge 0$ let $\tilde\mu_n$ be the F_n- invariant measure lifted from $\bar\mu_n$ as in (2.5), μ_n^* the saturation of $\tilde\mu_n$ as in (2.6), and $\mu_n = \mu_n^*/\mu_n^*(M)$ the SRB measure. The main goal of this section is to prove the following result.

Proposition 3.6. *For every continuous $g : M \to \mathbb{R}$,*

$$\int g d\mu_n^* \xrightarrow[i\to\infty]{} \int g d\mu_0^*.$$

Proof. As M is compact, then g is uniformly continuous and $\|g\|_\infty < \infty$. Recalling (2.6) we may write for all $n \in \mathbb{N}_0$ and every integer N_0

$$\mu_n^* = \sum_{\ell=0}^{N_0-1} \mu_n^\ell + \eta_n,$$

where $\mu_n^\ell = f_*^\ell(\tilde{\mu_n}|\{R_n > \ell\})$ and $\eta_n = \sum_{\ell\ge N_0} f_*^\ell(\tilde{\mu_n}|\{R_n > l\})$. By $(\mathbf{U}_5)$, we may choose N_0 so that $\eta_n(M)$ is as small as we want, for all $n \in \mathbb{N}_0$. We are left to show that for every

$\ell < N_0$, if n is large enough then

$$\left| \int (g \circ f_n^\ell) \mathbf{1}_{\{R_n > \ell\}} d\tilde{\mu}_n - \int (g \circ f_0^\ell) \mathbf{1}_{\{R_0 > \ell\}} d\tilde{\mu}_0 \right|$$

is arbitrarily small. We fix $\ell < N_0$ and take $k \in \mathbb{N}$ large so that $\operatorname{var}(g(k))$ is sufficiently small. Then, we use Proposition 2.4 (3) and its Remark 2.5 to reduce our problem to controlling the following error term:

$$E := \left| \int (g \circ f_n^\ell \circ F_n^k)^* (\mathbf{1}_{\{R_n > \ell\}} \circ F_n^k)^* d\bar{\mu}_n - \int (g \circ f_0^\ell \circ F_0^k)^* (\mathbf{1}_{\{R_0 > \ell\}} \circ F_0^k)^* d\bar{\mu}_0 \right|.$$

Let $\varrho_0 : \hat{\gamma}_0 \to \mathbb{R}$ be such that $\varrho_0 = \bar{\rho}_0 \circ \hat{\pi}_0 \cdot \mathbf{1}_{\Omega_0}$ and define

$$E_0 = \left| \int \left(\left((g \circ f_n^\ell \circ F_n^k)^\bullet (\mathbf{1}_{\{R_n > \ell\}} \circ F_n^k)^\bullet \right) \circ H_n^{-1} \right) \varrho_n d \operatorname{Leb}_{\hat{\gamma}_0} \right.$$
$$\left. - \int \left((g \circ f_0^\ell \circ F_0^k)^\bullet (\mathbf{1}_{\{R_0 > \ell\}} \circ F_0^k)^\bullet \right) \varrho_0 \, d \operatorname{Leb}_{\hat{\gamma}_0} \right|.$$

By Lemma 3.2, we have $E \leq E_0 + K \|g\|_\infty Q_n$. Observe that by $(\mathbf{U}_2)$ and Remark 1.2 we may consider n large enough so that $K\|g\|_\infty Q_n$ is negligible. Applying the triangular inequality we get

$$E_0 \leq K \int \left| (g \circ f_n^\ell \circ F_n^k)^\bullet \circ H_n^{-1} - (g \circ f_0^\ell \circ F_0^k)^\bullet \right| \mathbf{1}_{\Omega_0 \cap \Omega_0^n} d \operatorname{Leb}_{\hat{\gamma}_0}$$
$$+ K \|g\|_\infty \int \left| (\mathbf{1}_{\{R_n > \ell\}} \circ F_n^k)^\bullet \circ H_n^{-1} - (\mathbf{1}_{\{R_0 > \ell\}} \circ F_0^k)^\bullet \right| \mathbf{1}_{\Omega_0 \cap \Omega_0^n} d \operatorname{Leb}_{\hat{\gamma}_0}$$
$$+ \left| \int (g \circ f_0^\ell \circ F_0^k)^\bullet (\mathbf{1}_{\{R_0 > \ell\}} \circ F_0^k)^\bullet \, \mathbf{1}_{\Omega_0 \cap \Omega_0^n} \, (\varrho_n - \varrho_0) \, d \operatorname{Leb}_{\hat{\gamma}_0} \right|.$$

By Corollary 3.5 the term

$$\left| \int (g \circ f_0^\ell \circ F_0^k)^\bullet (\mathbf{1}_{\{R_0 > \ell\}} \circ F_0^k)^\bullet \, \mathbf{1}_{\Omega_0 \cap \Omega_0^n} \, (\varrho_n - \varrho_0) \, d \operatorname{Leb}_{\hat{\gamma}_0} \right|$$

is as small as we want as long as n is large enough. The analysis of the remaining terms

$$\int \left| (g \circ f_n^\ell \circ F_n^k)^\bullet \circ H_n^{-1} - (g \circ f_0^\ell \circ F_0^k)^\bullet \right| \mathbf{1}_{\Omega_0 \cap \Omega_0^n} d \operatorname{Leb}_{\hat{\gamma}_0}$$

and

$$\int \left| (\mathbf{1}_{\{R_n > \ell\}} \circ F_n^k)^\bullet \circ H_n^{-1} - (\mathbf{1}_{\{R_0 > \ell\}} \circ F_0^k)^\bullet \right| \mathbf{1}_{\Omega_0 \cap \Omega_0^n} d \operatorname{Leb}_{\hat{\gamma}_0}$$

is left to Lemmas 3.8 and 3.9, respectively. □

In the proofs of Lemmas 3.8 and 3.9 we have to produce a suitable positive integer N so that returns that take longer than N iterations are negligible. The next lemma provides the tools for an adequate choice. We consider the sequence of consecutive return times for $z \in \Lambda$

$$R^1(z) = R(z) \quad \text{and} \quad R^n(z) = R\left(f^{R^1 + R^2 + \ldots + R^{n-1}}(z) \right). \tag{3.7}$$

Lemma 3.7. *Given $k, N \in \mathbb{N}$*

$$\bar{m}\left(\left\{ z \in \Lambda : \, \exists t \in \{1, \ldots, k\} \text{ such that } R^t(z) > N \right\} \right) \leq k C_1 \, \bar{m}(\{R > N\}).$$

Proof. We may write

$$\left\{z \in \Lambda :\ \exists t \in \{1, \dots, k\} \text{ such that } R^t(z) > N\right\} = \bigcup_{t=0}^{k-1} B_t,$$

where

$$B_t = \left\{z \in \Lambda :\ R(z) \le N, \dots, R^t(z) \le N, R^{t+1}(z) > N\right\}.$$

If $R(z) \le N, \dots, R^t(z) \le N$ then there exist $j_1, \dots j_t \le N$ with $R(\Upsilon_{j_l}) \le N$ for every $l = 1, \dots, t$ and $z \in \Upsilon_{j_1,\dots,j_t}$. Observe that $\bar F^t\left(\Upsilon_{j_1,\dots,j_t}\right) = \bar\Lambda$ and there is $y \in \Upsilon_{j_1,\dots,j_t}$ such that $\bar m(\bar\Lambda) \le J\bar F^t(y).\bar m(\Upsilon_{j_1,\dots,j_t})$. Also, there exists $x \in \Upsilon_{j_1,\dots,j_t} \cap \bar F^{-t}(\{R > N\})$ such that $\bar m(\{R > N\}) \ge J\bar F^t(x).\bar m(\Upsilon_{j_1,\dots,j_t} \cap \bar F^{-t}(\{R > N\})$. Then, using bounded distortion we obtain

$$\frac{\bar m(\Upsilon_{j_1,\dots,j_t} \cap \bar F^{-t}(\{R > N\})}{\bar m(\Upsilon_{j_1,\dots,j_t})} \le \frac{J\bar F^t(y)}{J\bar F^t(x)} \frac{\bar m(\{R > N\})}{\bar m(\bar\Lambda)} \le C_1\, \bar m(\{R > N\}),$$

Finally, we conclude that

$$\begin{aligned} |B_t| &= \sum_{j_1,\dots,j_t:\, R(\Upsilon_{j_l}) \le N,\, l=1\dots t} \bar m(\Upsilon_{j_1,\dots,j_t} \cap \bar F^{-t}(\{R > N\}) \\ &\le C_1\, \bar m(\{R > N\}) \sum_{j_1,\dots,j_t:\, R(\Upsilon_{j_l}) \le N,\, l=1\dots t} \bar m(\Upsilon_{j_1,\dots,j_t}) \\ &\le C_1\, \bar m(\{R > N\}). \end{aligned}$$

□

Lemma 3.8. *Given $\ell, k \in \mathbb{N}$ and $\varepsilon > 0$ there is $J \in \mathbb{N}$ such that for every $n > J$*

$$\int \left|(g \circ f_n^\ell \circ F_n^k)^\bullet \circ H_n^{-1} - (g \circ f_0^\ell \circ F_0^k)^\bullet\right| \mathbf{1}_{\Omega_0 \cap \Omega_0^n} d\,\mathrm{Leb}_{\hat\gamma_0} < \varepsilon.$$

Proof. We split the argument into three steps:

(1) We appeal to Lemma 3.7 to choose $N \in \mathbb{N}$ sufficiently large so that the set

$$L := \left\{x \in \Omega_0 \cap \Omega_0^n :\ \exists t \in \{1, \dots, k\}\, R_0^t(x) > N \,\text{or}\, R_n^t(x) > N\right\}$$

has sufficiently small mass.

(2) We pick $J \in \mathbb{N}$ large enough to guarantee that, according to condition $(\mathbf{U}_4)$, for every k positive integers $j_1, \dots, j_k$ such that $R_0(\Upsilon^0_{j_l}) \le N$, for all $i = 1, \dots, k$, each set $\Upsilon^0_{j_1,\dots,j_k}$ and its corresponding $\Upsilon^n_{j_1,\dots,j_k}$ satisfy the condition: $\Upsilon^0_{j_1,\dots,j_k} \triangle H_n\left(\Upsilon^n_{j_1,\dots,j_k}\right)$ has sufficiently small conditional Lebesgue measure.

(3) Finally, in each set $\Upsilon^0_{j_1,\dots,j_k} \cap H_n\left(\Upsilon^n_{j_1,\dots,j_k}\right)$ we control

$$\left|(g \circ f_n^\ell \circ F_n^k)^\bullet \circ H_n^{-1} - (g \circ f_0^\ell \circ F_0^k)^\bullet\right|.$$

Step (1): From Lemma 3.7 we have $|L| \le kC_1.\left(\mathrm{Leb}_{\hat\gamma_0}(\{R_0 > N\}) + \mathrm{Leb}_{\hat\gamma_n}(\{R_n > N\})\right)$. So, by assumption $(\mathbf{U}_5)$, we may choose N and J large enough so that

$$2\|g\|_\infty kC_1.\left(\mathrm{Leb}_{\hat\gamma_0}(\{R_0 > N\}) + \mathrm{Leb}_{\hat\gamma_n}(\{R_n > N\})\right) < \frac{\varepsilon}{3},$$

which implies that

$$\int_L \left|(g \circ f_n^\ell \circ F_n^k)^\bullet \circ H_n^{-1} - (g \circ f_0^\ell \circ F_0^k)^\bullet\right| \mathbf{1}_{\Omega_0 \cap \Omega_0^n} d\,\mathrm{Leb}_{\hat\gamma_0} < \frac{\varepsilon}{3}.$$

Step (2): By $(\mathbf{P}_1)$(c) it is possible to define $V = V(N, k)$ as the total number of sets $\Upsilon_{j_1,\dots,j_k}$ such that $R(\Upsilon_{j_l}) \le N$ for all $i = 1, \dots, k$. Now, using $(\mathbf{U}_4)$, we may choose J so that for every $n > J$ and $\Upsilon^0_{j_1,\ldots,j_k}$ such that $R_0(\Upsilon^0_{j_l}) \le N$ for all $i = 1,\ldots,k$ then the corresponding $\Upsilon^n_{j_1,\ldots,j_k}$ is such that

$$\mathrm{Leb}_{\hat\gamma_0}\left(\Upsilon^0_{j_1,\ldots,j_k} \triangle H_n\left(\Upsilon^n_{j_1,\ldots,j_k}\right)\right) < \frac{\varepsilon}{3}V^{-1}\left(2\max\{1,\|g\|_\infty\}\right)^{-1}.$$

Under these circumstances we have

$$\sum_{\substack{j_1,\ldots,j_k:\\ R_0(\Upsilon^0_{j_l})\le N\\ l=1,\ldots,k}} \int_{\Upsilon^0_{j_1,\ldots,j_k}\triangle H_n\left(\Upsilon^n_{j_1,\ldots,j_k}\right)} \left|(g\circ f_n^\ell\circ F_n^k)^\bullet\circ H_n^{-1} - (g\circ f_0^\ell\circ F_0^k)^\bullet\right| \mathbf{1}_{\Omega_0\cap\Omega_0^n}\, d\,\mathrm{Leb}_{\hat\gamma_0} < \frac{\varepsilon}{3}.$$

Step (3): For each $i = 1,\ldots,k$, let $\tau_{j_i} = R_0(\Upsilon^0_{j_i})$. In each set $\Upsilon^0_{j_1,\ldots,j_k}\cap\Upsilon^n_{j_1,\ldots,j_k}$ we have that $F_0^k = f_0^{\tau_1+\ldots+\tau_k}$ and $F_n^k = f_n^{\tau_1+\ldots+\tau_k}$. Since M is compact, each f_n is C^k and $f_n\to f_0$, as $n\to\infty$, in the C^k topology then

- there exists $\vartheta > 0$ such that $|z-\zeta| < \vartheta \Rightarrow |g(z)-g(\zeta)| < \frac{\varepsilon}{3}V^{-1}$;
- there exists J_1 such that for all $n > J_1$ and $z\in M$ we have

$$\max\left\{|f_0(z)-f_n(z)|,\ldots,|f_0^{kN+l}(z)-f_n^{kN+l}(z)|\right\} < \tfrac{\vartheta}{2};$$

- there exists $\eta > 0$ such that for all $z,\zeta\in M$ and $f\in\mathcal{F}$

$$|z-\zeta|<\eta \;\Rightarrow\; \max\left\{|f(z)-f(\zeta)|,\ldots,|f^{kN+l}(z)-f^{kN+l}(\zeta)|\right\} < \tfrac{\vartheta}{2}.$$

Furthermore, according to $(\mathbf{U}_3)$,

- there is J_2 such that for every $n > J_2$ and $x\in\Omega_0\cap\Omega_0^n$ we have

$$|\gamma_0^s(x)-\gamma_n^s(x)|_{C^1} < \eta.$$

Let $n > \max\{J_1,J_2\}$, $z\in\gamma_0^s(x)$ and take $\zeta\in\gamma_n^s(x)$ such that $|z-\zeta|<\eta$. This together with the choices of η and J_1 implies

$$\begin{aligned}\left|f_0^\ell\circ F_0^k(z) - f_n^\ell\circ F_n^k(\zeta)\right| &\le \left|f_0^{\tau_1+\ldots+\tau_k+l}(z) - f_0^{\tau_1+\ldots+\tau_k+l}(\zeta)\right|\\ &\quad+\left|f_0^{\tau_1+\ldots+\tau_k+l}(\zeta) - f_n^{\tau_1+\ldots+\tau_k+l}(\zeta)\right|\\ &< \vartheta/2+\vartheta/2 = \vartheta.\end{aligned}$$

Finally, the above considerations and the choice of ϑ allow us to conclude that for every $n > \max\{J_1,J_2\}$, $x\in\Omega_0\cap\Omega_0^n$ and $z\in\gamma_0^s(x)$, there exists $\zeta\in\gamma_n^s(x)$ such that

$$\left|g(f_n^\ell\circ F_n^k(\zeta)) - g(f_0^\ell\circ F_0^k(z))\right| < \frac{\varepsilon}{3}V^{-1}. \tag{3.8}$$

Attending to (2.4), (3.8) and the fact that we can interchange the roles of z and ζ in the latter, we obtain that for every $n > \max\{J_1,J_2\}$

$$\left|(g\circ f_n^\ell\circ F_n^k)^\bullet\circ H_n^{-1} - (g\circ f_0^\ell\circ F_0^k)^\bullet\right| < \frac{\varepsilon}{3}V^{-1},$$

from where we deduce that

$$\sum_{\substack{j_1,\ldots,j_k\\ R_0(\Upsilon^0_{j_l})\le N\\ 1\le l\le k}} \int_{\Upsilon^0_{j_1,\ldots,j_k}\triangle H_n\left(\Upsilon^n_{j_1,\ldots,j_k}\right)} \left|(g\circ f_n^\ell\circ F_n^k)^\bullet\circ H_n^{-1} - (g\circ f_0^\ell\circ F_0^k)^\bullet\right| \mathbf{1}_{\Omega_0\cap\Omega_0^n}\, d\,\mathrm{Leb}_{\hat\gamma_0} < \frac{\varepsilon}{3}.$$

□

Lemma 3.9. *Given $l, k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that for every $n > J$*

$$\int \left|(\mathbf{1}_{\{R_n>\ell\}} \circ F_n^k)^\bullet \circ H_n^{-1} - (\mathbf{1}_{\{R_0>\ell\}} \circ F_0^k)^\bullet\right| \mathbf{1}_{\Omega_0\cap\Omega_0^n} d\,\mathrm{Leb}_{\hat{\gamma}_0} < \varepsilon.$$

Proof. As in the proof of Lemma 3.8, we divide the argument into three steps.

(1) The condition on N: Consider the set

$$L_1 = \left\{x \in \Omega_0 \cap \Omega_0^n : \exists t \in \{1, \dots, k+1\} \text{ such that } R_0^t(x) > N \text{ or } R_n^t(x) > N\right\}.$$

From Lemma 3.7 we have $|L_1| \le (k+1)C_1.(\mathrm{Leb}_{\hat{\gamma}_0}(\{R_0 > N\}) + \mathrm{Leb}_{\hat{\gamma}_n}(\{R_n > N\}))$. So we choose N large enough so that

$$2\|g\|_\infty (k+1)C_1.(\mathrm{Leb}_{\hat{\gamma}_0}(\{R_0 > N\}) + \mathrm{Leb}_{\hat{\gamma}_n}(\{R_n > N\})) < \frac{\varepsilon}{3},$$

which implies that

$$\int_{L_1} \left|(\mathbf{1}_{\{R_n>\ell\}} \circ F_n^k)^\bullet \circ H_n^{-1} - (\mathbf{1}_{\{R_0>\ell\}} \circ F_0^k)^\bullet\right| \mathbf{1}_{\Omega_0\cap\Omega_0^n} d\,\mathrm{Leb}_{\hat{\gamma}_0} < \frac{\varepsilon}{3}.$$

(2) Let as before $V = V(N, k+1)$ be the total number of sets $\Upsilon_{j_1,\dots,j_{k+1}}$ such that $R(\Upsilon_{j_i}) \le N$ for all $i = 1, \dots, k+1$. Now, using $(\mathbf{U}_4)$, we may choose J so that for every $n > J$ and $\Upsilon^0_{j_1,\dots,j_{k+1}}$ such that $R_0(\Upsilon^0_{j_i}) \le N$ for all $i = 1, \dots, k+1$ then the corresponding $\Upsilon^n_{j_1,\dots,j_{k+1}}$ is such that

$$\mathrm{Leb}_{\hat{\gamma}_0}\left(\Upsilon^0_{j_1,\dots,j_{k+1}} \triangle H_n\left(\Upsilon^n_{j_1,\dots,j_{k+1}}\right)\right) < \frac{\varepsilon}{3} V^{-1} \left(2\max\{1, \|g\|_\infty\}\right)^{-1}.$$

Let $L_2 = \Upsilon^0_{j_1,\dots,j_{k+1}} \triangle H_n\left(\Upsilon^n_{j_1,\dots,j_{k+1}}\right)$ and observe that

$$\sum_{\substack{j_1,\dots,j_{k+1}:\\ R_0(\Upsilon^0_{j_l}) \le N\\ l=1,\dots,k+1}} \int_{L_2} \left|(\mathbf{1}_{\{R_n>\ell\}} \circ F_n^k)^\bullet \circ H_n^{-1} - (\mathbf{1}_{\{R_0>\ell\}} \circ F_0^k)^\bullet\right| \mathbf{1}_{\Omega_0\cap\Omega_0^n} d\,\mathrm{Leb}_{\hat{\gamma}_0} < \frac{\varepsilon}{3}.$$

(3) At last, notice that in each set $\Upsilon^0_{j_1,\dots,j_{k+1}} \cap H_n\left(\Upsilon^n_{j_1,\dots,j_{k+1}}\right)$ we have

$$\left|(\mathbf{1}_{\{R_n>l\}} \circ F_n^k)^\bullet \circ H_n^{-1} - (\mathbf{1}_{\{R_0>l\}} \circ F_0^k)^\bullet\right| = 0,$$

which gives the result. □

4. Entropy continuity

In Proposition 2.7 we have seen that the SRB entropy can be written just in terms of the quotient dynamics. Our aim now is to show that the integrals appearing in that formula are close for nearby dynamics, and this is the content of Proposition 4.4. Notice that since the integrands are not necessarily continuous functions, the continuity of the integrals is not an immediate consequence of the statistical stability.

4.1. Auxiliary results.

Lemma 4.1. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a bounded sequence of m-measurable functions defined on M belonging to $L^\infty(m)$. If $\varphi_n \to \varphi$ in the $L^1(m)$-norm and $\psi \in L^1(m)$, then*

$$\int \psi(\varphi_n - \varphi)dm \to 0, \quad \text{when } n \to \infty.$$

Proof. Take any $\varepsilon > 0$. Let $C > 0$ be an upper bound for $\|\varphi_n\|_\infty$. Since $\psi \in L^1(m)$, there is $\delta > 0$ such that for any Borel set $B \subset M$

$$m(B) < \delta \quad \Rightarrow \quad \int_B |\psi| dm < \frac{\varepsilon}{4C}. \tag{4.1}$$

Define for each $n \geq 1$

$$B_n = \left\{ x \in M : |\varphi_n(x) - \varphi_0(x)| > \frac{\varepsilon}{2\|\psi\|_1} \right\}.$$

Since $\|\varphi_n - \varphi_0\|_1 \to 0$ when $n \to \infty$, then there is $n_0 \in \mathbb{N}$ such that $m(B_n) < \delta$ for every $n \geq n_0$. Taking into account the definition of B_n, we may write

$$\begin{aligned}\int |\psi||\varphi_n - \varphi_0| dm &= \int_{B_n} |\psi||\varphi_n - \varphi_0| dm + \int_{M\setminus B_n} |\psi||\varphi_n - \varphi_0| dm \\ &\leq 2C \int_{B_n} |\psi| dm + \frac{\varepsilon}{2\|\psi\|_1} \int_{M\setminus B_n} |\psi| dm.\end{aligned}$$

Then, using (4.1), this last sum is upper bounded by ε, as long as $n \geq n_0$. □

Lemma 4.2. *There is $C_2 > 0$ such that $\log J\bar{F}_n \leq C_2 \, R_n$ for every $n \geq 0$.*

Proof. Define $L_n = \max_{x\in M}\{|\det Df_n^u(x)|\}$, for each $n \geq 0$. By the compactness of M and the continuity on the first order derivative, there is $L > 1$ such that $L_n \leq L$ for all $n \geq 0$. We have

$$|\det D(F_n)^u(x)| = \prod_{j=0}^{R_n(x)-1} |\det Df_n^u(f_n^j(x))| \leq L^{R_n(x)}.$$

By (2.3) it follows that

$$\log J(F_n)(x) = \log|\det DF_n^u(x)| + \log \hat{u}(F_n(x)) - \log \hat{u}(x).$$

Observing that by $(\mathbf{P}_3)$(a) it follows that $|\log \hat{u}(F_n(x)) - \log \hat{u}(x)| \leq 2C\beta^0 = 2C$, we have

$$\log J(F_n)(x) \leq R_n(x) \log L + 2C.$$

To conclude, we take $C_2 = \log L + 2C$. □

Lemma 4.3. *Given $\varepsilon > 0$, there is $J \in \mathbb{N}$ such that for all $n > J$*

$$\int_{\Omega_0^n \cap \Omega_0} |R_n - R_0| \, d\,\mathrm{Leb}_{\hat{\gamma}_0} \leq \varepsilon$$

Proof. Let $\varepsilon > 0$ be given. Using condition $(\mathbf{U}_5)$ and Remark 1.2, take $N \geq 1$ and $J = J(N, \varepsilon) > 0$ in such a way that $\sum_{j=N}^\infty j \,\mathrm{Leb}_{\hat{\gamma}_0}\{R_n = j\} < \varepsilon/3$ and $\sum_{j=N}^\infty j \,\mathrm{Leb}_{\hat{\gamma}_0}\{R_0 = j\} < \varepsilon/3$. Since

$$R_n = \sum_{j=0}^\infty \mathbf{1}_{\{R_n > j\}},$$

we may write

$$\begin{aligned}\|R_n - R_0\|_1 &= \Big\|R_n - \sum_{j=0}^{N-1} \mathbf{1}_{\{R_n>j\}} + \sum_{j=0}^{N-1} \big(\mathbf{1}_{\{R_n>j\}} - \mathbf{1}_{\{R_0>j\}}\big) + \sum_{j=0}^{N-1} \mathbf{1}_{\{R_0>j\}} - R_0\Big\|_1 \\ &\le \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_n>j\}}\Big\|_1 + \sum_{j=0}^{N-1} \|\mathbf{1}_{\{R_n>j\}} - \mathbf{1}_{\{R_0>j\}}\|_1 + \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_0>j\}}\Big\|_1 \\ &= \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_n>j\}}\Big\|_1 + \sum_{j=0}^{N-1} \|\mathbf{1}_{\{R_n\le j\}} - \mathbf{1}_{\{R_0\le j\}}\|_1 + \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{R_0>j\}}\Big\|_1.\end{aligned}$$

By the choices of N and J, the first and third terms in this last sum are smaller than $\varepsilon/3$. By $(\mathbf{U}_4)$, increasing J if necessary, we can make $\mathrm{Leb}_{\hat\gamma_0}(\{R_n = j\}\triangle\{R_0 = j\})$ sufficiently small in order to have the second term smaller than $\epsilon/3$. □

4.2. **Convergence of metric entropies.** Our aim is to show that $h_{\mu_n} \to h_{\mu_0}$ as $n \to \infty$, which by Proposition 2.7 can be rewritten as

$$\sigma_n^{-1}\int_{\bar\Lambda_n} \log J\bar F_n \, d\bar\mu_n \longrightarrow \sigma_0^{-1}\int_{\bar\Lambda_0} \log J\bar F_0 \, d\bar\mu_0, \quad \text{as } n\to\infty. \tag{4.2}$$

Observing that $\sigma_n = \int_{\Lambda_n} R_n d\tilde\mu_n = \mu_n^*(M)$, then by Proposition 3.6 we have $\sigma_n \to \sigma_0$, as $n \to \infty$. Hence, (4.2) is a consequence of the next result.

Proposition 4.4. $\displaystyle\int_{\bar\Lambda_n} \log J\bar F_n \, d\bar\mu_n \longrightarrow \int_{\bar\Lambda_0} \log J\bar F_0 \, d\bar\mu_0$ *as* $n\to\infty$.

Proof. The convergence above will follow if we show that the following term is arbitrarily small for large $n \in \mathbb{N}$.

$$E := \left|\int_{\Omega_n} (\log J\bar F_n \circ \hat\pi_n)(\bar\rho_n \circ \hat\pi_n)\, d\,\mathrm{Leb}_{\hat\gamma_n} - \int_{\Omega_0} (\log J\bar F_0 \circ \hat\pi_0)\varrho_0 \, d\,\mathrm{Leb}_{\hat\gamma_0}\right|.$$

Recall that $\varrho_0 = \bar\rho_0 \circ \hat\pi_0$ and $\varrho_n = \bar\rho_n \circ \hat\pi_n \circ H_n^{-1}$, for every $n \in \mathbb{N}$. Define

$$E_0 := \left|\int_{\Omega_0^n\cap\Omega_0} (\log J\bar F_n \circ \hat\pi_n \circ H_n^{-1})\varrho_n \, d(H_n)_*\,\mathrm{Leb}_{\hat\gamma_n} - \int_{\Omega_0^n\cap\Omega_0} (\log J\bar F_0 \circ \hat\pi_0)\varrho_0 \, d\,\mathrm{Leb}_{\hat\gamma_0}\right|.$$

By Lemmas 2.2 and 4.2 we have

$$E \le E_0 + KC_2 \int_{\Omega_n\setminus\Omega_n^0} R_n d\,\mathrm{Leb}_{\hat\gamma_n} + KC_2 \int_{\Omega_0\setminus\Omega_0^n} R_0 d\,\mathrm{Leb}_{\hat\gamma_0}.$$

Since $R_0 \in L^1(\mathrm{Leb}_{\hat\gamma_0})$, then, by $(\mathbf{U}_2)$ and Remark 1.2, for large n, we may have $\mathrm{Leb}_{\hat\gamma_0}(\Omega_0\triangle\Omega_0^n)$ small so that $\int_{\Omega_0\setminus\Omega_0^n} R_0 d\,\mathrm{Leb}_{\hat\gamma_0}$ becomes negligible. Now, for each $N \in \mathbb{N}$

$$\int_{\Omega_n\setminus\Omega_n^0} R_n d\,\mathrm{Leb}_{\hat\gamma_n} \le N \int_{\Omega_n\setminus\Omega_n^0} d\,\mathrm{Leb}_{\hat\gamma_n} + \int_{\{R_n>N\}} R_n d\,\mathrm{Leb}_{\hat\gamma_n}.$$

Using condition $(\mathbf{U}_5)$ we may choose N so that for all $n \in \mathbb{N}$ large enough the quantity $\int_{\{R_n>N\}} R_n d\,\mathrm{Leb}_{\hat\gamma_n} = \sum_{j=N+1} j\,\mathrm{Leb}_{\hat\gamma_0}\{R_n = j\}$ is arbitrarily small. Again, using $(\mathbf{U}_2)$, if $n \in \mathbb{N}$ is sufficiently large then $\int_{\Omega_0^n\setminus\Omega_0} d\,\mathrm{Leb}_{\hat\gamma_0}$ is as small as we want. Therefore, we are reduced to estimating E_0.

Note that by definition $\Omega_0^n \subset \Omega_0$. Having this in mind, we split E_0 into the next three terms that we call E_1, E_2, E_3 respectively.

$$E_0 \le \left| \int_{\Omega_0^n} (\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) \varrho_n \, d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n} - \int_{\Omega_0^n} (\log J\bar{F}_0 \circ \hat{\pi}_0) \varrho_n \, d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n} \right|$$
$$+ \left| \int_{\Omega_0^n} (\log J\bar{F}_0 \circ \hat{\pi}_0) \varrho_n \, d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n} - \int_{\Omega_0^n} (\log J\bar{F}_0 \circ \hat{\pi}_0) \varrho_n \, d\operatorname{Leb}_{\hat{\gamma}_0} \right|$$
$$+ \left| \int_{\Omega_0^n} (\log J\bar{F}_0 \circ \hat{\pi}_0) \varrho_n \, d\operatorname{Leb}_{\hat{\gamma}_0} - \int_{\Omega_0^n} (\log J\bar{F}_0 \circ \hat{\pi}_0) \varrho_0 \, d\operatorname{Leb}_{\hat{\gamma}_0} \right| .$$

Concerning E_2, using Lemma 2.2 and Lemma 4.2 we have

$$E_2 \le \int_{\Omega_0^n} |\log J\bar{F}_0| |\varrho_n| \left| \frac{d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n}}{d\operatorname{Leb}_{\hat{\gamma}_0}} - 1 \right| d\operatorname{Leb}_{\hat{\gamma}_0}$$
$$\le KC_2 \int_{\Omega_0^n} R_0 \left| \frac{d(H_n)_* \operatorname{Leb}_{\hat{\gamma}_n}}{d\operatorname{Leb}_{\hat{\gamma}_0}} - 1 \right| d\operatorname{Leb}_{\hat{\gamma}_0} .$$

Now, Remark 1.2 and Lemma 4.1 guarantee that E_2 can be made arbitrarily small for large $n \in \mathbb{N}$. Using Corollary 3.5, E_3 can also be made small for large n. We are left with E_1. By Lemma 2.2 and Remark 1.2 we only need to control

$$\int_{\Omega_0^n \cap \Omega_0} \left| (\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) - (\log J\bar{F}_0 \circ \hat{\pi}_0) \right| \, d\operatorname{Leb}_{\hat{\gamma}_0}$$

whose estimation we leave to Lemma 4.6. □

Remark 4.5. Assume that γ_n is a compact unstable manifold of the map f_n for $n \ge 0$ and $\gamma_n \to \gamma_0$, in the C^1 topology. The convergence of f_n to f_0 in the C^1 topology ensures that given $\ell \in \mathbb{N}$ and $\epsilon > 0$ there exist $\delta = \delta(\ell, \epsilon) > 0$ and $J = J(\delta) \in \mathbb{N}$ such that for every $n > J$, $x \in \gamma_0$ and $y \in \gamma_n$ with $|x - y| < \delta$

$$\max_{j=1,\dots,\ell} \left\{ |f_n^j(y) - f_0^j(x)|, \ |\log\det(Df_n^j)^u(y) - \log\det(Df_0^j)^u(x)| \right\} < \epsilon.$$

Lemma 4.6. *Given any $\varepsilon > 0$ there exists $J \in \mathbb{N}$ such that for every $n > J$*

$$\int_{\Omega_0^n \cap \Omega_0} \left| (\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) - (\log J\bar{F}_0 \circ \hat{\pi}_0) \right| \, d\operatorname{Leb}_{\hat{\gamma}_0} < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. For $n, N \in \mathbb{N}$ define $A_{n,N} = \{R_n \le N\} \cap \{R_0 \le N\}$ and $A_{n,N}^c = \{R_n > N\} \cup \{R_0 > N\}$. By Lemma 4.2 we have

$$\int_{\Omega_0^n \cap A_{n,N}^c} \left| (\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) - (\log J\bar{F}_0 \circ \hat{\pi}_0) \right| \, d\operatorname{Leb}_{\hat{\gamma}_0} \le C_2 \int_{\Omega_0^n \cap A_{n,N}^c} R_n \, d\operatorname{Leb}_{\hat{\gamma}_0}$$
$$+ C_2 \int_{\Omega_0^n \cap A_{n,N}^c} R_0 \, d\operatorname{Leb}_{\hat{\gamma}_0} .$$

Since $R_0 \in L^1(\operatorname{Leb}_{\hat{\gamma}_0})$, there is $\delta > 0$ such that if a measurable set A has $\operatorname{Leb}_{\hat{\gamma}_0}(A) < \delta$, then $\int_A R_0 d\operatorname{Leb}_{\hat{\gamma}_0} < \varepsilon/(4C_2)$. According to $(\mathbf{U}_5)$, we may pick $N \in \mathbb{N}$ and choose $J \in \mathbb{N}$ such that for every $n > J$ we get $\operatorname{Leb}_{\hat{\gamma}_0}(A_{n,N}^c) < \delta$. This implies that the second term on the right hand side of the inequality above is smaller than $\varepsilon/4$. The same argument and

Lemma 4.3 allow us to conclude that for a convenient choice of $N \in \mathbb{N}$ and for $J \in \mathbb{N}$ sufficiently large

$$C_2 \int_{\Omega_0^n \cap A_{n,N}^c} R_n \, d\,\mathrm{Leb}_{\hat{\gamma}_0} \leq C_2 \int_{\Omega_0^n \cap A_{n,N}^c} R_0 \, d\,\mathrm{Leb}_{\hat{\gamma}_0} + C_2 \int_{\Omega_0^n} |R_n - R_0| \, d\,\mathrm{Leb}_{\hat{\gamma}_0} \leq \frac{\varepsilon}{4}.$$

So, assuming that N has been chosen and J is sufficiently large so that

$$\int_{\Omega_0^n \cap A_{n,N}^c} \left|(\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) - (\log J\bar{F}_0 \circ \hat{\pi}_0)\right| \, d\,\mathrm{Leb}_{\hat{\gamma}_0} \leq \varepsilon/2,$$

we are left do deal with

$$\int_{\Omega_0^n \cap A_{n,N}} \left|(\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) - (\log J\bar{F}_0 \circ \hat{\pi}_0)\right| \, d\,\mathrm{Leb}_{\hat{\gamma}_0} \leq$$
$$\sum_{i: R_0(\Upsilon_i^0) \leq N} \int_{\Upsilon_i^0 \cap \Upsilon_i^n} \left|(\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) - (\log J\bar{F}_0 \circ \hat{\pi}_0)\right| \mathbf{1}_{\Omega_0^n \cap \Omega_0} \, d\,\mathrm{Leb}_{\hat{\gamma}_0}$$
$$+ \sum_{i: R_0(\Upsilon_i^0) \leq N} \int_{\Upsilon_i^0 \triangle \Upsilon_i^n} \left|(\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1}) - (\log J\bar{F}_0 \circ \hat{\pi}_0)\right| \mathbf{1}_{\Omega_0^n \cap \Omega_0 \cap A_{n,N}} \, d\,\mathrm{Leb}_{\hat{\gamma}_0}.$$

Denote by S_1 and S_2 respectively the first and second sums above, and v the number of terms in S_1 and S_2. By Lemma 4.2 we have

$$S_2 \leq C_2 \int_{\Upsilon_i^0 \triangle \Upsilon_i^n} (R_n + R_0) \mathbf{1}_{\Omega_0^n \cap \Omega_0 \cap A_{n,N}} \, d\,\mathrm{Leb}_{\hat{\gamma}_0} \leq 2C_2 N \,\mathrm{Leb}_{\hat{\gamma}_0}(\Upsilon_i^0 \triangle \Upsilon_i^n).$$

Hence, using ($\mathbf{U}_4$) we consider $J \in \mathbb{N}$ large enough to have $\mathrm{Leb}_{\hat{\gamma}_0}(\Upsilon_i^0 \triangle \Upsilon_i^n) < \varepsilon/(8C_2 N v)$, and so $S_2 \leq \varepsilon/4$.

Let $\tau_i = R_0(\Upsilon_i^0) = R_n(\Upsilon_i^n) \leq N$. We want to see that for all n large enough and all $x \in \Upsilon_i^0 \cap \Upsilon_i^n$ with $\tau_i \leq N$

$$\left|(\log J\bar{F}_n \circ \hat{\pi}_n \circ H_n^{-1})(x) - (\log J\bar{F}_0 \circ \hat{\pi}_0)(x)\right| \leq \varepsilon/4v, \tag{4.3}$$

which yields $S_1 \leq \varepsilon/4$. Using (2.3) and observing that the curves $\hat{\gamma}_n, \hat{\gamma}_0$ are the leaves we chose to define the reference measures $\bar{m}_n, \bar{m}_0$, then we easily get for $y = H_n^{-1}(x)$

$$\begin{aligned}\left|\log J\bar{F}_n \circ \hat{\pi}_n(y) - \log J\bar{F}_0 \circ \hat{\pi}_0(x)\right| \leq &\left|\log \det(Df_n^{\tau_i})^u(y) - \log \det(Df_0^{\tau_i})^u(x)\right| \\ &+ \left|\log \hat{u}_n(f_n^{\tau_i}(y)) - \log \hat{u}_0(f_0^{\tau_i}(x))\right|.\end{aligned}$$

Using Remark 4.5 with $\ell = N$ and $\varepsilon/8v$ instead of ϵ, and recalling that $\tau_i \leq N$, we may find $\delta > 0$ and $J \in \mathbb{N}$ so that for all $n > J$

$$\left|\log \det(Df_n^{\tau_i})^u(y) - \log \det(Df_0^{\tau_i})^u(x)\right| < \varepsilon/8v. \tag{4.4}$$

Observe that $|x - y| < \delta$ as long as J is sufficiently large, since $x = H_n(y)$.

For every $n, k \in \mathbb{N}_0$ and $t \in \Lambda_n$, let

$$\hat{u}_n^k(t) = \prod_{j=0}^{k} \frac{\det Df_n^u(f_n^j(t))}{\det Df_n^u(f_n^j(\hat{t}))}.$$

By definition of $\hat{u}_n$ (see (2.1)) and by $(\mathbf{P}_3)$(a), there is $k \in \mathbb{N}$ such that for every $n \in \mathbb{N}_0$ and $t \in \Lambda_n$ we have $|\log \hat{u}_n(t) - \log \hat{u}_n^k(t)| < \varepsilon/(48v)$. Thus,

$$
\begin{aligned}
|\log \hat{u}_n(f_n^{\tau_i}(y)) - \log \hat{u}_0(f_0^{\tau_i}(x))| \leq\ & |\log \hat{u}_n(f_n^{\tau_i}(y)) - \log \hat{u}_n^k(f_n^{\tau_i}(y))| \\
& +|\log \hat{u}_n^k(f_n^{\tau_i}(y)) - \log \hat{u}_0^k(f_0^{\tau_i}(x))| \\
& +|\log \hat{u}_0^k(f_0^{\tau_i}(x)) - \log \hat{u}_0(f_0^{\tau_i}(x))| \\
\leq\ & \sum_{j=0}^{k} \left|\log\det Df_n^u(f_n^j(\zeta)) - \log\det Df_0^u(f_0^j(z))\right| \\
& + \sum_{j=0}^{k} \left|\log\det Df_n^u(f_n^j(\hat{\zeta})) - \log\det Df_0^u(f_0^j(\hat{z}))\right| \\
& + \frac{\varepsilon}{24v},
\end{aligned}
$$

where $z = f_0^{\tau_i}(x)$, $\zeta = f_n^{\tau_i}(y)$, $\hat{z}$ is the only point on the set $\gamma_0^s(z) \cap \hat{\gamma}_0$ and $\hat{\zeta}$ is the unique point on the set $\gamma_n^s(\zeta) \cap \hat{\gamma}_n$.

Observe that since $\hat{\gamma}_n \to \hat{\gamma}_0$ and $f_n \to f_0$ in the C^1 topology, and $\tau_i \leq N$, then $\gamma_n^u(\zeta) \to \gamma_0^u(z)$, in the C^1 topology. Besides, using Lemma 3.3 we also have $|\hat{z} - \hat{\zeta}|$ as small as we want for J large enough. Consequently, by Remark 4.5, we may find $J \in \mathbb{N}$ sufficiently large so that for all $n > J$, we have

$$\sum_{j=0}^{k} \left|\log\det Df_n^u(f_n^j(\zeta)) - \log\det Df_0^u(f_0^j(z))\right| < \varepsilon/(24v). \tag{4.5}$$

and

$$\sum_{j=0}^{k} \left|\log\det Df_n^u(f_n^j(\hat{\zeta})) - \log\det Df_0^u(f_0^j(\hat{z}))\right| < \varepsilon/(24v). \tag{4.6}$$

Estimates (4.4),(4.5) and (4.6) yield (4.3). □

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