Portugal. Math. (N.S.)
Vol. xx, Fasc. , 200x, xxx–xxx

Portugaliae Mathematica

Exponential decay of hyperbolic times for Benedicks-Carleson quadratic maps

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Abstract. We consider the Benedicks-Carleson quadratic maps and prove that the tail of Hyperbolic Times, introduced in [5], decays exponentially fast. This improves a previous work [13], where subexponential estimates for this tail were obtained and allows to use the theory developed by Alves et al as another approach to recover statistical properties of these maps like exponential Decay of Correlations, Large Deviations, Central Limit Theorem, Statistical Stability and continuity of metric entropy. *Portugaliae Mathematica.*

Mathematics Subject Classification (2000). Primary 37A10 37C40, 37E05; Secondary 37A25, 37C75.

Keywords. Logistic family, Hyperbolic Times, statistical properties, tail sets.

1. Introduction

In [13], the author studied statistical properties of the quadratic family of maps given by $f_a(x) = 1-ax^2$, where a belongs to the Benedicks-Carleson set of parameters $\mathcal{BC}$, introduced in [9]. Namely, it was obtained the continuous variation of the SRB measure (*Statistical Stability*) and continuity of metric entropy within $\mathcal{BC}$. This was achieved by estimating the tail of Hyperbolic Times. Essentially, this tail was split into two components, the first corresponding to the points that do not reach exponential growth of the derivative sufficiently fast (*Expansion Tail*) and the second corresponding to the points whose early iterates went too close to the critical point (*Recurrence Tail*). The main results in [13] assert that the volume (or Lebesgue measure) of the Expansion Tail decays exponentially fast (Theorem A) and the volume of the Recurrence Tail falls off subexponentially fast (Theorem B). This was enough to obtain the Statistical Stability and continuous variation of metric entropy, since the results by Alves et al require only polynomial (summable) tails.

*Work partially supported by FCT through CMUP

The purpose of this paper is to improve [13, Theorem B] and obtain that the volume of the Recurrence Tail decays exponentially fast, which gives, together with [13, Theorem A], that the volume of the tail of Hyperbolic Times decays exponentially fast. We take the opportunity to correct a problem with the combinatorics in the proof of [13, Proposition 6.1], which estimates the measure of the points whose orbits go near the critical point. This affected in particular [13, Lemma 8.1] and in general the final proofs of [13, Theorems A and B].

Hyperbolic Times were introduced in [1] and have revealed as a useful tool to study non-uniformly hyperbolic systems. They can be seen as check points at which the system presents good hyperbolic behaviour and have been used to study statistical properties such as: the existence of SRB measures ([1, 2]), Decay of Correlations, Central Limit Theorems ([5, 16]), Statistical Stability, continuous variation of metric entropy ([3, 4, 7]), Stochastic Stability Stability, ie, robustness of the SRB measures under small random noise ([6]).

The existence of Hyperbolic Times has been shown for systems that present non-uniformly expanding behaviour in the unstable direction. In the 1-dimensional setting and in this particular case where the source of non-uniform hyperbolic behaviour is the presence of a critical point, their existence is a consequence of the following two conditions almost everywhere (a.e.), with respect to Lebesgue measure (which we denote by Leb):

(NUE) *Non-uniform expansion*: $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\left|f_a'\left(f_a^i(x)\right)\right| > d$, for some $d>0$;

(SRCS) *Slow recurrence to the critical set*: For every $\epsilon>0$, there exists $\gamma>0$ such that $\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \mathrm{dist}_\gamma\left(f_a^j(x),0\right) < \epsilon$,

where $\mathrm{dist}_\gamma(x,y)=|x-y|$ if $|x-y|\le\gamma$ and $\mathrm{dist}_\gamma(x,y)=1$ otherwise.

In [5, 16] the authors used Hyperbolic Times to build inducing schemes like the ones in [23, 24] and showed how the tail of the inducing return times relates with the tail of Hyperbolic Times, introduced in [5] and which we define next. Let

$$\mathcal{E}^a(x) = \min\left\{ N\ge 1:\ \frac{1}{n}\sum_{i=0}^{n-1}\log\left|f_a'\left(f_a^i(x)\right)\right| > d,\ \forall n\ge N\right\}$$
$$\mathcal{R}^a(x) = \min\left\{ N\ge 1:\ \frac{1}{n}\sum_{j=0}^{n-1} -\log \mathrm{dist}_\gamma\left(f_a^j(x),0\right) < \epsilon,\ \forall n\ge N\right\}$$

which are both defined and finite Leb-a.e. under the assumptions that (NUE) and (SRCS) hold Leb-a.e. We define the *Expansion Tail* at time n, as the set of points that, up to this time, resist to present exponential growth of the derivative along their orbits $ET(n)=\{x\in I:\ \mathcal{E}^a(x)>n\}$; and the *Recurrence Tail* at time n, as the set of points that in its early iterates could not be satisfactorily kept way from the critical point $RT(n) = \{x \in I : \mathcal{R}^a(x) > n\}$. The *Tail of Hyperbolic Times* at time n, is just the union of $ET(n)$ and $RT(n)$, ie, the set of points:

$$\Gamma(n) = \{x \in I : \mathcal{E}^a(x) > n \text{ or } \mathcal{R}^a(x) > n\}.$$

We are now in conditions of stating our main results.

Theorem 1.1 (Theorem A of [13]). *Assume that $a \in \mathcal{BC}$. Then f_a satisfies (NUE) Leb-a.e. Moreover, there are positive real numbers C_1 and τ_1 such that $Leb(ET(n)) \leq C_1 e^{-\tau_1 n}$, for all $n \in \mathbb{N}$.*

Theorem 1.2. *Assume that $a \in \mathcal{BC}$. Then f_a satisfies $(SRCS)$ Leb-a.e. Moreover, there are positive real numbers C_2 and τ_2 such that $Leb\,(RT(n)) \leq C_2 e^{-\tau_2 n}$, for all $n \in \mathbb{N}$.*

Remark 1.3. The constants d in (NUE), ϵ, γ in (SRCS), can be chosen uniformly on $\mathcal{BC}$. Moreover, the constants C_1, τ_1 given by theorem 1.1 and the constants C_2, τ_2 given by theorem 1.2 do not depend on the parameter $a \in \mathcal{BC}$. Thus, we may say that $\{f_a\}_{a\in\mathcal{BC}}$ is a *uniform family* in the sense considered in [4].

Both theorems easily imply that $\mathrm{Leb}\,(\Gamma(n)) \leq \mathrm{const}\,\mathrm{e}^{-\tau n}$, for some $\tau > 0$, const > 0 and all $n \in \mathbb{N}$. This allows to deduce immediately the following conclusions, which, despite not being new, illustrate what can be obtained from fitting the Benedicks-Carleson quadratic maps in the theory developed by Alves et al about Hyperbolic Times. This theory has also been applied to infinite modal maps to obtain the same conclusions in [8].

Corollary 1.4. *The Benedicks-Carleson family of quadratic maps has the following properties:*

(1) *each f_a admits a unique SRB invariant measure μ_a which is absolutely continuous with respect to Lebesgue (*[9, 10]*)*
(2) *each f_a has exponential decay of correlations (*[22, 18]*)*
(3) *each f_a satisfies a Central Limit Theorem (*[22, 18]*)*
(4) *each f_a admits an exponential estimate for Large Deviations (*[18]*)*
(5) *is Statistically Stable in the strong sense, meaning that the map $\mathcal{BC} \ni a \mapsto d\mu_a/dLeb$ is continuous in the L^1-norm (*[13]*)*
(6) *the metric entropy with respect to μ_a varies continuously within $\mathcal{BC}$ (*[13]*)*

We say that f_a has exponential decay of correlations if for every observable functions φ, ψ in some appropriate functional spaces, there exists $C > 0$ and $0 < \tau < 1$ such that $\left|\int \varphi\, \psi \circ f_a^n \, d\mu_a - \int \varphi \, d\mu_a \int \psi \, d\mu_a\right| \leq C\,\tau^n$, for all $n \in \mathbb{N}$. The Central Limit Theorem holds for f_a if for every φ in some appropriate functional space and $\sigma > 0$, we have $\mu_a\left(\sqrt{n}\left(\frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^i - \int \varphi \, d\mu_a\right) \leq x\right) \to \Phi(x/\sigma)$, as $n \to \infty$, where Φ is the standard Gaussian distribution function. Finally, f_a admits an exponential estimate for large deviations if for ervery φ in some appropriate functional space and every $\varepsilon > 0$, there exists $C > 0$ and $0 < \tau < 1$ such that

$\mu_a \left(\left|\frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^i - \int \varphi \, d\mu_a\right| > \varepsilon\right) \le C\,\tau^n$, for all $n \in \mathbb{N}$. The precise statements corresponding to (2), (3) and (4) can be found in the cited references.

We remark that the existence of absolutely continuous invariant measures for a positive Lebesgue measure set of parameters had already been shown in [17, 12].

The proofs of the above properties follow from the exponential volume decay of the tail of Hyperbolic Times together with: [5, Theorem 2] for (1) and (3) , [16, Theorem 1.1] for (2), [16, Theorem 1.1] and [19, Theorem 2.1] for (4) [4, Theorem A] for (5) and [7, Corollary C] for (6).

2. Benedicks-Carleson quadratic maps

In this section we describe succinctly the Benedicks-Carleson quadratic maps and its main features. These can be found in [9, 10, 21, 20, 13, 14], just to cite a few, but we refer to [13] for most proofs since the setting and notation is practically the same.

The Benedicks-Carleson Theorem (see [9] or Section 2 of [10]) states that there exists a positive Lebesgue measure set of parameters, $\mathcal{BC}$, verifying

$$\text{there is } c > 0 \ (c \approx \log 2) \text{ such that } |Df_a^n(f_a(0))| \ge \mathrm{e}^{cn} \text{ for all } n \ge 0; \tag{EG}$$

$$\text{there is a small } \alpha > 0 \text{ such that } |f_a^n(0)| \ge \mathrm{e}^{-\alpha n} \text{ for all } n \ge 1. \tag{BA}$$

The condition (EG) is usually known as the Collet-Eckmann condition and it was introduced in [12].

We define the *critical region* as the interval $(-\delta, \delta)$, where $\delta = \mathrm{e}^{-\Delta} > 0$ is chosen small but much larger than $2 - a$. This region is partitioned into the intervals $(-\delta, \delta) = \bigcup_{m \ge \Delta} I_m$, where $I_m = (\mathrm{e}^{-(m+1)}, \mathrm{e}^{-m}]$ for $m > 0$ and $I_m = -I_{-m}$ for $m < 0$; then each I_m is further subdivided into m^2 intervals $\{I_{m,j}\}$ of equal length inducing the partition $\mathcal{P}_0$ of $[-1, 1]$ into $[-1, -\delta) \cup \bigcup_{m,j} I_{m,j} \cup (\delta, 1]$. Given $J \in \mathcal{P}$, let nJ denote the interval n times the length of J centred at J and define $U_m := (-\mathrm{e}^{-m}, \mathrm{e}^{-m})$, for every $m \in \mathbb{N}$.

2.1. Expansion outside the critical region. There is $c_0 > 0$ and $M_0 \in \mathbb{N}$ such that

(1) If $x, \dots, f_a^{k-1}(x) \notin (-\delta, \delta)$ and $k \ge M_0$, then $|Df_a^k(x)| \ge \mathrm{e}^{c_0 k}$;

(2) If $x, \dots, f_a^{k-1}(x) \notin (-\delta, \delta)$ and $f_a^k(x) \in (-\delta, \delta)$, then $|Df_a^k(x)| \ge \mathrm{e}^{c_0 k}$;

(3) If $x, \dots, f_a^{k-1}(x) \notin (-\delta, \delta)$, then $|Df_a^k(x)| \ge \delta \mathrm{e}^{c_0 k}$.

While the orbit goes through a *free period* its iterates are always away from the critical region which means that the above estimates apply and it experiences an exponential growth of the derivative. However, it is inevitable that the orbit of almost every $x \in [-1, 1]$ makes a *return* to the critical region. We say that $n \in \mathbb{N}$ is a *return time* of the orbit of x if $f_a^n(x) \in (-\delta, \delta)$. Every free period of x ends with a *free return* to the critical region. We say that the return has a *depth* $m \in \mathbb{N}$ if $f_a^n(x) \in I_{\pm m}$. Once in the critical region, the orbit of x initiates a binding with the critical point.

2.2. Bound period definition and properties. Let $\beta = 14\alpha$. For $x \in (-\delta, \delta)$ define $p(x)$ to be the largest integer p such that $|f_a^k(x) - f_a^k(0)| < \mathrm{e}^{-\beta k}, \forall k < p$. Then

(1) $\frac{1}{2}|m| \leq p(x) \leq 3|m|$, for each $x \in I_m$;

(2) $|Df_a^p(x)| \geq \mathrm{e}^{c'p}$, where $c' = \frac{1-4\beta}{3} > 0$.

The orbit of x is said to be bound to the critical point during the period $0 \leq k < p$. We may assume that p is constant on each $I_{m,j}$. Note that during the bound period the orbit of x may return to the critical region. These instants are called *bound return times.*

2.3. Bookkeeping, essential and inessential returns. A sequence of partitions $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \ldots$ is built with the following properties (see [13, Section 4]). For Lebesgue almost every $x \in I$, $\{x\} = \cap_{n \geq 0} \omega_n(x)$, where $\omega_n(x)$ is the element of $\mathcal{P}_n$ containing x. For such x there is a sequence $t_1, t_2, \ldots$ corresponding to the instants when the orbit of x experiences a free *essential return situation*, which means $I_{m,k} \subset f_a^{t_i}(\omega_{t_i-1}(x))$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$. We have that $\omega_n(x) = \omega_{t_{i-1}}(x)$, for every $t_{i-1} \leq n < t_i$ and $f_a^{t_i}(\omega_{t_i}(x)) = \omega_0(f^{t_i}(x))$, except for the points at the two ends of $f_a^{t_i}(\omega_{t_{i-1}}(x))$ for which it may occur an adjoining to the neighbouring interval. If t_i is an essential return situation for x, then it is either an *essential return time* for x, which means that there exists $m \geq \Delta$ and $1 \leq k \leq m^2$ such that $I_{m,k} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset 3I_{m,k}$; or an *escaping time* for x, which is to say that $I_{(\Delta-1),1} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset (\delta, 1]$ or $I_{-(\Delta-1),1} \subset f_a^{t_i}(\omega_{t_i}(x)) \subset [-1, -\delta)$, where $I_{\pm(\Delta-1),1}$ is the subinterval of $I_{\pm(\Delta-1)}$ closest to 0.

We remark that every point in $\omega \in \mathcal{P}_n$ has the same history up to n, in the sense that they have the same free periods, return to the critical region simultaneously, with the same depth and their bound periods expire at the same time.

We say that v is a free return time for x of *inessential* type if $f_a^v(\omega_v(x)) \subset 3I_{m,k}$, for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$, but $f_a^v(\omega_v(x))$ is not large enough to contain an interval $I_{m,k}$ for some $|m| \geq \Delta$ and $1 \leq k \leq m^2$.

2.4. Distortion of the derivative. The sequence of partitions described above is designed so that we have bounded distortion in each element of the partition $\mathcal{P}_{n-1}$ up to time n. To be more precise, consider $\omega \in \mathcal{P}_{n-1}$. There exists a constant C independent of ω, n and the parameter a such that for every $x, y \in \omega$,

$$\frac{|Df_a^n(x)|}{|Df_a^n(y)|} \leq C. \tag{1}$$

See [13, Lemma 4.2] for a proof.

2.5. Growth of returning and escaping components. Let t be a return time for $\omega \in \mathcal{P}_t$, with $f_a^t(\omega) \subset 3I_{m,k}$ for some $m \geq \Delta$ and $1 \leq k \leq m^2$. If n is the next free return situation for ω (either essential or inessential) then

$$|f_a^n(\omega)| \geq \mathrm{e}^{c_0 q} \mathrm{e}^{(1-5\beta)|m|} \, |f_a^z(\omega)| \text{ and if } t \text{ is essential then } |f_a^n(\omega)| \geq \mathrm{e}^{c_0 q} \mathrm{e}^{-5\beta|m|}, \tag{2}$$

where $q = n - (t + p)$. See [13, Lemma 4.1].

Suppose that $\omega \in \mathcal{P}_t$ is an escape component. Then, in the next return situation for ω, at time t_1, we have that

$$\left|f_a^{t_1}(\omega)\right| \geq \mathrm{e}^{-\beta\Delta}. \tag{3}$$

See [20] or [14, Lemma 4.2].

3. Depths of bounded and inessential returns

As we have already mentioned, there are three types of returns: essential, bounded and inessential, whose instants of occurrence we denote by t, u and v respectively. The usual picture is the following: we start with an essential return at time t with depth η and bound period $p(\eta)$. After $t + p$ the orbit goes into a free period and then, possibly, enters a cycle of inessential returns, say ι inessential returns at times $v_1, \ldots, v_\iota$, with depths $\eta_1, \ldots \eta_\iota$ with bound periods $p_1(\eta_1), \ldots, p_k(\eta_\iota)$, before a new essential return occurs at time $t' > v_\iota + p_\iota$. Of course that after each essential or inessential return, bounded returns may occur during the respective bound periods.

We mention that by [13, Lemma 5.3] the length of the cycle is bounded by η, namely $t' - t \leq 5|\eta|$. The purpose of this section is to show that the depths of the inessential and bounded returns are also controlled by the depth η of the essential return that initiated the cycle at time t. We start with two more simple observations:

(1) The depth of the inessential returns is less than the depth of the essential return that initiated the cycle, i.e., $|\eta_i| \leq |\eta|$, for all $i = 1, \ldots, \iota$; [13, Lemma 5.1].

(2) The depth of any bounded return is always less than the depth of the return (essential or inessential) that originated the bound period; [13, Lemma 5.2].

The next two propositions are the cornerstone of the improvement on the estimates that allows us to get the exponential decay of the tail of hyperbolic times. In the proof of the following proposition we will use a condition known as the *free assumption* for the critical orbit. This condition, which was proved by a large deviations argument in [10, Section 2] (see also [21, condition FA(n)]), essentially asserts that the set of Benedicks-Carleson parameters is built in such a way that the amount of time spent by the critical orbit in bound periods totally makes up a small fraction of the whole time.

Proposition 3.1. *Let t be a free return time (either essential or inessential) for $\omega \in \mathcal{P}_t$ with $f_a^t(\omega) \subset 3I_{\eta,k}$. Let $p = p(\eta)$ be the bound period associated with this return. Let S_1 denote the sum of the depths of all the bound returns plus the depth of the return that originated the bound period. Then $S_1 \leq C_1\eta$, with constant $C_1 = C_1(\alpha)$.*

Proof. Recall that by Section 2.2 (1) we know that $\frac{1}{2}\eta \leq p \leq 3\eta$. Let $x \in \omega$. We say that a bound return is of *level i* if, at the moment of this bound return, x has already initiated exactly i bindings to the critical point ξ_0 and all of them are still active. By active we mean that the respective bound periods have not finished yet. To illustrate, suppose that u_1 is the first time between t and $t+p$ that the orbit of x enters U_Δ. Obviously, at this moment, the only active binding to ξ_0 is the one initiated at time t. Thus, u_1 is a bounded return of level 1. Now, at time u_1, the orbit of x establishes a new binding to the critical point which ends at the end of the corresponding bound period that we denote by p_1 which depends on the depth η_1 of the bound return in question. During the period from u_1 to $u_1 + p_1$ new returns may occur and their level is at least 2 since there are at least 2 active bindings: the one initiated at t and the one initiated at u_1. If $u_1 + p_1 < t + p$ then new bound returns of level 1 may occur after $u_1 + p_1$.

We may redefine the notion of bound period so that the bound periods are nested (see [10], section 6.2). This means that we may suppose that no binding of level i extends beyond the bound period of level $i-1$ during which it was initiated.

Taking into account the *free assumption* condition for the critical orbit we may assume that in a period of length $n \in \mathbb{N}$, the time spent by the critical orbit in bound periods is at most αn (see [21, condition FA(n)]).

Since, when a point initiates a binding with ξ_0, it shadows the early iterates of the critical point, the same applies to any of these points $x \in \omega$ bounded to ξ_0. Thus in the period of time from t to $t+p$, the orbit of x can spend at most the fraction of time αp in bound periods. So if ℓ denotes the number of bound returns of level 1, $u_1, \ldots, u_\ell$ their instants of occurrence, $\eta_1, \ldots \eta_\ell$ their respective depths and $p_1, \ldots, p_\ell$ their respective bound periods, then we have :

$$\frac{1}{2}\sum_{i=1}^{\ell} \eta_i \leq \sum_{i=1}^{\ell} p_i \leq \alpha p \leq 3\alpha\eta$$

from where we easily obtain $\sum_{i=1}^{\ell} \eta_i \leq 6\alpha\eta$. The same argument applies to the bound returns of level 2 within the i-th bound period of level 1. So if ℓ_i denotes the number of bounded returns of level 2 within the i-th bound period of level 1, $u_{i1}, \ldots, u_{i\ell_i}$ their instants of occurrence, $\eta_{i1}, \ldots \eta_{i\ell_i}$ their respective depths and $p_{i1}, \ldots, p_{i\ell_i}$ their respective bound periods, then we have

$$\frac{1}{2}\sum_{j=1}^{\ell_i} \eta_{ij} \leq \sum_{i=1}^{\ell_i} p_{ij} \leq \alpha p_i \leq 3\alpha\eta_i$$

from where we easily obtain $\sum_{i=1}^{\ell}\sum_{j=1}^{\ell_i}\eta_{ij} \leq (6\alpha)^2\eta$. Observing that by choice of α we have $6\alpha < 1$, a simple induction argument then yields $S_1 \leq \sum_{i=0}^{\infty}(6\alpha)^i\eta \leq C_1\eta$, where $C_1 = \frac{1}{1-6\alpha}$. □

Proposition 3.2. *Let t be an essential return time for $\omega \in \mathcal{P}_t$ with $I_{\eta,k} \subset f_a^t(\omega) \subset 3I_{\eta,k}$. Let p denote the associated bound period. Let S_2 denote the sum of the depths of all the free inessential returns before the next essential return situation. Then $S_2 \leq C_2\eta$, with constant $C_2 = C_2(\beta)$.*

Proof. Suppose that ι is the number of inessential returns before the next essential return situation of ω, which occur at times $v_1, \ldots, v_\iota$, with respective depths $\eta_1, \ldots, \eta_\iota$ and respective bound periods $p_1, \ldots, p_\iota$. Also denote by $v_{\iota+1}$ the next essential return situation of ω. For $j = 1, \ldots \iota$, let $\sigma_j = f_a^{v_j}(\omega)$. By (2) we have

$$|\sigma_1| \geq \mathrm{e}^{c_0 q}\mathrm{e}^{-5\beta|\eta|} \text{ and } \frac{|\sigma_{j+1}|}{|\sigma_j|} \geq \mathrm{e}^{c_0 q_i}\mathrm{e}^{(1-5\beta)|\eta_i|},$$

where $q = v_1 - (t+p)$, $q_j = v_{j+1} - (v_j + p_j)$, for $j = 1, \ldots, \iota$. Since $|\sigma_{\iota+1}| \leq 2$ and

$$|\sigma_{v+1}| = |\sigma_1| \prod_{i=1}^{v} \frac{|\sigma_{i+1}|}{|\sigma_i|},$$

we get $\exp\left\{c_0 q - 5\beta\eta + \sum_{i=1}^{\iota}(c_0 q_i + (1-5\beta)\eta_i)\right\} \leq \mathrm{e}$, which implies that

$$\sum_{i=1}^{\iota}(c_0 q_i + (1-5\beta)\eta_i) \leq 5\beta\eta + 1.$$

Finally, one easily derives that $S_2 \leq C_2\eta$, where $C_2 = \frac{5\beta}{1-5\beta}$. □

4. Probability of an essential return reaching a certain depth

Since, as we have seen in the previous section, the depth of the essential returns plays a prominent role, in this section, we study the probability of these returns hitting very high depths. We call the attention for the fact that there is a problem with the combinatorics in [13, Proposition 6.1] and the correct statement is as follows.

For each $x \in I$, let $u_n(x)$ denote the number of essential return situations of x between 1 and n, $s_n(x)$ be the number of those which are actually essential return times and $\mathfrak{S}_n$ the number of the latter that correspond to *deep essential returns* of the orbit of x, i.e, with return depths above a threshold $\Theta \geq \Delta$. Observe that $u_n(x) - s_n(x)$ is the exact number of escaping situations of the orbit of x, up to

n. Given the integers $0 \leq s \leq 2n/\Theta$, $s \leq u \leq n$ and the s integers $\gamma_1, \dots, \gamma_s$, each greater than or equal to Θ, we define the event:

$$A^{u,s}_{\gamma_1,\dots,\gamma_s}(n) = \left\{ x \in I : \quad \begin{array}{r} u_n(x) = u,\ \mathfrak{S}_n(x) = s \text{ and the depth of the i-th deep} \\ \text{essential return is } \gamma_i \text{ for all } i = 1, \dots, s \end{array} \right\}.$$

Remark 4.1. Observe that the upper bound $2n/\Theta$ for the number of deep essential returns up to time n derives from the fact that each deep essential return originates a bound period of length at least $\Theta/2$ (see Section 2.2) and no essential return can occur during bound periods.

Proposition 4.2. *Given the integers $0 \leq s \leq 2n/\Theta$ and $s \leq u \leq n$, consider s integers $\gamma_1, \dots, \gamma_s$, each greater than or equal to Θ. If Θ is large enough, then*

$$\mathit{Leb}\left(A^{u,s}_{\gamma_1,\dots,\gamma_s}(n)\right) \leq \binom{u}{s} \mathit{Exp}\left\{ -(1-6\beta) \sum_{i=1}^{s} \gamma_i \right\}.$$

See [15, Proposition 5.2] for a proof.

Fix $n \in \mathbb{N}$, the integers $1 \leq s \leq 2n/\Theta$, $s \leq u \leq n$ and $j \leq s$. Given an integer $\rho \geq \Theta$, consider the event

$$A^{u,s}_{\rho,j}(n) = \left\{ x \in I \ : \ u_n(x) = u,\ \mathfrak{S}_n(x) = s, \quad \begin{array}{r} \text{and the depth of the j-th deep} \\ \text{essential return is } \rho \end{array} \right\}.$$

Corollary 4.3. *If Θ is large enough, then*

$$\mathit{Leb}\left(A^{u,s}_{\rho,j}(n)\right) \leq \binom{u}{s} e^{-(1-6\beta)\rho}$$

Proof. Since $A^{u,s}_{\rho,j}(n) = \bigcup_{\substack{\rho_i \geq \Theta \\ i \neq j}} A^{u,s}_{\rho_1,\dots,\rho_{j-1},\rho,\rho_{j+1},\dots,\rho_s}(n)$, then by Proposition 4.2 we have

$$\mathrm{Leb}\left(A^{u,s}_{\rho,j}(n)\right) \leq \binom{u}{s} \mathrm{e}^{-(1-6\beta)\rho} \left(\sum_{\eta=\Theta}^{\infty} \mathrm{e}^{-(1-6\beta)\eta} \right)^{s-1} \leq \binom{u}{s} \mathrm{e}^{-(1-6\beta)\rho},$$

as long as Θ is sufficiently large so that $\sum_{\eta=\Theta}^{\infty} \mathrm{e}^{-(1-6\beta)\eta} \leq 1$. $\square$

Remark 4.4. Observe that the bound for the probability of the event $A^{u,s}_{\rho,j}(n)$ does not depend on the $j \leq s$ chosen.

Remark 4.5. Observe that proposition 4.2 and corollary 4.3 also apply when $\Theta = \Delta$ in which case we have $\mathfrak{S}_n = s_n$.

5. Non-uniform expansion

The proof of Theorem 1.1 follows the one in [13, Theorem A] except for the necessary adjustments due to the changes in the estimates given by Proposition 4.2. So, following the strategy in [13, Section 3], the proof of Theorem 1.1 reduces to show that for some constants $D_1, \tau_1 > 0$, eventually depending on α, β and Δ, we have $\mathrm{Leb}(E_1(n)) \leq D_1 \mathrm{e}^{-\tau_1 n}$, where

$$E_1(n) = \left\{ x \in I : \exists i \in \{1, \ldots, n\},\ |f_a^i(x)| < \mathrm{e}^{-\alpha n} \right\}. \tag{4}$$

In fact, one realizes that $\{x \in I : \mathcal{E}^a(x) > k\} \subset \bigcup_{n \geq k} E_1(n)$. The idea is that we have exponential growth of the derivative during free periods and even at the end of the bound periods. Besides, just after the serious setbacks at the returns we can count on (EG) to regain growth of the derivative. Hence, the crucial thing to do is to make sure that the cut off at the returns is not that serious which is the case for points in $I - E_1(n)$.

In this section we consider that the threshold $\Theta = \Delta$. Also remember that $u_n(x) - s_n(x)$ is the exact number of escaping situations the orbit of x goes through until time n. We define the following events:

$$A_\rho^{u,s}(n) = \left\{ \begin{array}{r} x \in I :\ u_n(x) = u,\ s_n(x) = s \ \text{ and there is one essential return} \\ \text{reaching the depth } \rho \end{array} \right\},$$

for fixed $n \in \mathbb{N}$, $s \leq n$ and $\rho \geq \Delta$;

$$A_\rho(n) = \left\{ x \in I : \exists t \leq n : t \text{ is essential return time and } |f_a^t(x)| \in I_\rho \right\},$$

for fixed n and $\rho \geq \Delta$. Now, because $A_\rho^{u,s}(n) = \bigcup_{j=1}^{s} A_{\rho,j}^{u,s}(n)$, by corollary 4.3, we have

$$\mathrm{Leb}\left(A_\rho^{u,s}(n)\right) \leq \sum_{j=1}^{s} \mathrm{Leb}\left(A_{\rho,j}^{u,s}(n)\right) \leq s \binom{u}{s} \mathrm{e}^{-(1-6\beta)\rho}. \tag{5}$$

Observing that $A_\rho(n) = \bigcup_{s=1}^{2n/\Delta} \bigcup_{u=s}^{n} A_\rho^s(n)$, then by (5) we get

$$\begin{aligned} \mathrm{Leb}\left(A_\rho(n)\right) &\leq \sum_{s=1}^{2n/\Delta} \sum_{u=s}^{n} \mathrm{Leb}\left(A_\rho^{u,s}(n)\right) \leq \sum_{s=1}^{2n/\Delta} \sum_{u=s}^{n} s \binom{u}{s} \mathrm{e}^{-(1-6\beta)\rho} \\ &\leq n\mathrm{e}^{-(1-6\beta)\rho} \sum_{s=1}^{2n/\Delta} s \binom{n}{s} \leq \frac{4n^3}{\Delta} \binom{n}{2n/\Delta} \mathrm{e}^{-(1-6\beta)\rho}. \end{aligned}$$

Using the Stirling formula, if we choose Δ large enough we have

$$\binom{n}{2n/\Delta} \leq \mathrm{const} \left(\left(1 + \frac{\frac{2}{\Delta}}{1-\frac{2}{\Delta}}\right) \left(1 + \frac{1-\frac{2}{\Delta}}{\frac{2}{\Delta}}\right)^{\frac{\frac{2}{\Delta}}{1-\frac{2}{\Delta}}} \right)^{(n-2n/\Delta)} \leq \mathrm{const}\mathrm{e}^{h(\Delta)n},$$

where $h(\Delta) \to 0$, as $\Delta \to \infty$. The last inequality derives from the fact that each factor in the middle expression can be made arbitrarily close to 1 by taking Δ sufficiently large.

Since, from Section 3, we know that the depths of inessential and bounded returns are not greater than the depth of the essential return preceding them, we have for all large n,

$$E_1(n) = \left\{x \in I : \exists i \in \{1, \ldots, n\}, |f_a^i(x)| < \mathrm{e}^{-\alpha n}\right\} \subset \bigcup_{\rho=\alpha n}^{\infty} A_\rho(n).$$

Consequently, taking $\tau_1 = \frac{(1-6\beta)\alpha}{4}$ and Δ large so that $h(\Delta) \leq \frac{(1-6\beta)\alpha}{2}$

$$\mathrm{Leb}(E_1(n)) \leq \mathrm{const}\, \tfrac{4n^3}{\Delta} \mathrm{e}^{h(\Delta)n} \sum_{\rho=\alpha n}^{\infty} \mathrm{e}^{-(1-6\beta)\rho} \leq \mathrm{const}\, \tfrac{4n^3}{\Delta} \mathrm{e}^{h(\Delta)n} \mathrm{e}^{-(1-6\beta)\alpha n} \leq \mathrm{const}\ \mathrm{e}^{-\tau_1 n}.$$

6. Slow recurrence to the critical set

We define for a point $x \in I$ and $n \in \mathbb{N}$,

$$T_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} -\log \mathrm{dist}_\gamma \left(f_a^j(x), 0\right), \tag{6}$$

where $\gamma = \mathrm{e}^{-\Theta}$ is the same of condition (SRCS). We note that the only points of the orbit of x that contribute to the sum in (6) are those considered to be *deep* returns with depth above the threshold $\Theta \geq \Delta$, which is to be determined below. Let $F_n(x) = \sum_{i=1}^{\mathfrak{S}_n} \eta_i$, where $\mathfrak{S}_n$ is the number of essential returns with depths above Θ that occur up to n and η_i their respective depths. Using Propositions 3.1 and 3.2 we get

$$T_n(x) \leq \frac{C_3}{n} F_n(x), \tag{7}$$

where $C_3 = C_3(\alpha, \beta) = (C_1 + C_1 C_2)$.

For every $n \in \mathbb{N}$, let $E_2(n) = \{x \in I : T_n(x) > \epsilon\}$. We will show that for all $n \in \mathbb{N}$ and every given ϵ, we may choose a small $\gamma = \mathrm{e}^{-\Theta}$ such that

$$\mathrm{Leb}\{E_2(n)\} \leq \mathrm{Leb}\left\{x : F_n(x) > \frac{\epsilon n}{C_3}\right\} \leq \mathrm{const}\ \mathrm{e}^{-\tau_2 n},$$

for some $\tau_2 = \tau_2(\epsilon, \Theta) > 0$. We will do this through a large deviation argument for which we start by estimating the moment generating function of F_n. In what follows $\mathbb{E}(\cdot)$ denotes expectation with respect to Leb.

Lemma 6.1. *Take $0 < t \leq \frac{1-6\beta}{3}$. If Θ is sufficiently large, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mathbb{E}\left(e^{tF_n}\right) \leq e^{h(\Theta)n}$. Moreover $h(\Theta) \to 0$, as $\Theta \to \infty$.*

Proof.

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{tF_n}\right) = \mathbb{E}\left(\mathrm{e}^{t\sum_{i=1}^{s}\eta_i}\right) &= \sum_{u,s,(\rho_1,\ldots,\rho_s)} \mathrm{e}^{t\sum_{i=1}^{s}\rho_i}\mathrm{Leb}\left(A^{u,s}_{\rho_1,\ldots,\rho_s}(n)\right)\\ &\le \sum_{u,s,(\rho_1,\ldots,\rho_s)} \mathrm{e}^{t\sum_{i=1}^{s}\rho_i}\binom{u}{s}\mathrm{e}^{-3t\sum_{i=1}^{s}\rho_i}, \text{ by proposition 4.2}\\ &\le \sum_{u,s,R}\binom{u}{s}\zeta(s,R)\mathrm{e}^{-2tR},\end{aligned}$$

where $\zeta(s,R)$ is the number of integer solutions of the equation $x_1+\ldots+x_s=R$ satisfying $x_i\ge\Theta$ for all i. We have

$$\zeta(s,R)\le\#\{\text{solutions of } x_1+\ldots+x_s=R, x_i\in\mathbb{N}_0\}=\binom{R+s-1}{s-1}.$$

Using the Stirling formula, we have

$$\binom{R+s-1}{s-1}\le\mathrm{const}\frac{(R+s-1)^{R+s-1}}{R^R(s-1)^{s-1}}.$$

So, if we choose Θ large enough we have

$$\zeta(s,R)\le\left(\mathrm{const}^{\frac{1}{R}}\left(1+\tfrac{s-1}{R}\right)\left(1+\tfrac{R}{s-1}\right)^{\frac{s-1}{R}}\right)^R\le\mathrm{e}^{tR}.$$

The last inequality derives from the fact that $s\Theta\le R$, and so each factor in the middle expression can be made arbitrarily close to 1 by taking Θ sufficiently large. Hence,

$$\mathbb{E}\left(\mathrm{e}^{tF_n}\right)\le\sum_{u,s,R}\binom{u}{s}\mathrm{e}^{tR}\mathrm{e}^{-2tR}\le\sum_{u,s,R}\binom{u}{s}\mathrm{e}^{-tR}\le\sum_{u,s}\binom{u}{s},$$

for Θ sufficiently large. Now, we have

$$\sum_{u,s}\binom{u}{s}\le\sum_{s=1}^{\frac{2n}{\Theta}}\sum_{u=s}^{n}\binom{u}{s}\le n\sum_{s=1}^{\frac{2n}{\Theta}}\binom{n}{s}\le n\sum_{s=1}^{\frac{2n}{\Theta}}\binom{n}{\frac{2n}{\Theta}}\le\frac{2n^2}{\Theta}\binom{n}{\frac{2n}{\Theta}}.$$

Using the Stirling formula, if we choose Θ large enough we have

$$\binom{n}{2n/\Theta}\le\mathrm{const}\left(\left(1+\tfrac{\frac{2}{\Theta}}{1-\frac{2}{\Theta}}\right)\left(1+\tfrac{1-\frac{2}{\Theta}}{\frac{2}{\Theta}}\right)^{\frac{\frac{2}{\Theta}}{1-\frac{2}{\Theta}}}\right)^{(n-2n/\Theta)}\le\mathrm{const}\ \mathrm{e}^{h^*(\Theta)n},$$

where $h^*(\Theta)\to0$, as $\Theta\to\infty$. The last inequality derives from the fact that each factor in the middle expression can be made arbitrarily close to 1 by taking Θ sufficiently large. This means that we may take $N=N(\Theta)\in\mathbb{N}$ sufficiently large so that for all $n\ge N$ we have $\mathbb{E}\left(\mathrm{e}^{tF_n}\right)\le\mathrm{e}^{h(\Theta)n}$, where $h(\Theta)\to0$, as $\Theta\to\infty$. $\square$

If we take $t = \frac{1-6\beta}{3}$ and Θ large enough so that $\tau_2 = \frac{t\epsilon}{C_3} - h(\Theta) > 0$, then, using Markov-Tchebychev's inequality and Lemma 6.1, we have

$$\mathrm{Leb}\left(F_n > \frac{\epsilon n}{C_3}\right) \leq \mathrm{e}^{-t\frac{\epsilon n}{C_3}}\,\mathbb{E}\left(\mathrm{e}^{tF_n}\right) \leq \mathrm{e}^{-\frac{t\epsilon n}{C_3}}\,\mathrm{e}^{h(\Theta)n} \leq \mathrm{e}^{-\tau_2 n},$$

for any $n > N_2$. Consequently, $\mathrm{Leb}\{E_2(n)\} \leq$ const $\mathrm{e}^{-\tau_2 n}$, which implies that $\sum_{n\geq k}\mathrm{Leb}\left(E_2(n)\right) \leq \mathrm{const}\,\mathrm{e}^{-\tau_2 k}$. Hence, applying Borel Cantelli's lemma, we get $\mathrm{Leb}(E_2) = 0$, where $E_2 = \cap_{k\geq 1}\cup_{n\geq k} E_2(n)$ and finally conclude that (SRCS) holds on the full Lebesgue measure set $I - E_2$. Observe that $\{x \in I :\ \mathcal{R}^a(x) > k\} \subset \bigcup_{n\geq k} E_2(n)$, and thus, for all $n \in \mathbb{N}$,

$$\mathrm{Leb}\left(\{x \in I :\ \mathcal{R}^a(x) > n\}\right) \leq \mathrm{const}\mathrm{e}^{-\tau_2 n}.$$

Acknowledgments. I wish to thank V. Araújo for pointing out the problem in [13, Proposition 6.1] and for very helpful discussions.

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Received

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