SRB measures and Young Towers

José F. Alves

Markov Partitions and Young Towers in Dynamics

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CMUP
Centro de Matemática
Universidade do Porto

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Physical measures

Physical measures

Let $f : M \to M$ be defined on a Riemannian manifold M with Lebesgue measure m. An f-invariant probability measure μ on the Borel sets of M is called a physical measure if, for a positive m measure set of points $x \in M$,

$$\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)} \xrightarrow[n\to\infty]{w^*} \mu, \qquad (*)$$

or equivalently, for all continuous $\varphi : M \to \mathbb{R}$

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) = \int \varphi\, d\mu.$$

We define the basin of μ as

$$\mathcal{B}(\mu) = \left\{x \in M : \ (*) \text{ holds}\right\}$$

Exercise 1.1

Show that the mean of the Dirac measures supported on the points of an attracting periodic orbit is a physical measure.

Absolutely continuous measures

Proposition 1.2

Let M be a compact metric space and $f : M \to M$ a Borel measurable map. If μ is an ergodic f-invariant probability measure, then the basin of μ covers μ almost all of M.

See Proposition 2.12 in [Alves 2020] for a proof.

Exercise 1.3

Prove Proposition 1.2.
Hint: use that $C^0(M)$ has a countable dense subset and Birkhoff Ergodic Theorem.

Corollary 1.4

Any ergodic absolutely continuous (wrt Lebesgue measure) invariant probability measure is a physical measure.

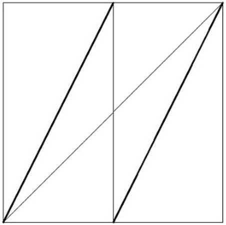
Remark 1.5

A physical does need to be ergodic [Muñoz-Young, Navas, Pujals, and Vásquez 2008].

Toy model I: Doubling map

Consider $f : S^1 \to S^1$ given by

$$f(x) = 2x \pmod 1.$$



It is clear that f preserves the length of intervals, and so (...)
f preserves the Lebesgue (length) measure m on the Borel sets of S^1.

Exercise

Show that m is ergodic.
Hint 1: Use Fourier series and the fact that f is ergodic iff for all $\varphi \in L^2(m)$

$$\varphi \circ f = \varphi \implies \varphi = \text{const}.$$

Hint 2: Use that any interval becomes the whole interval after a finite number iterates, the fact that f preserves proportions and Lebesgue Density Theorem.

Toy model II: Solenoid attractor

Consider the unit disk $D \subset \mathbb{C}$, the map $F : S^1 \times D \to S^1 \times D$ given by

$$F(t, z) = \left(2t \,(\mathrm{mod}\ 1), \frac{z}{4} + \frac{1}{2}e^{2\pi i t}\right), \qquad (1)$$

and the attractor

$$A = \bigcap_{n \geq 0} F^n(S^1 \times D).$$

Some well-known facts:

1. A is a uniformly hyperbolic set;
2. each $x \in A$ has a stable disk $\gamma^s(x)$ and an unstable disk $\gamma^u(x)$;
3. there exists a unique F-invariant ergodic probability measure μ such that $\pi_* \mu = m$.

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\pi"'] & A \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$

Some more facts:

(1) A is foliated by unstable manifolds;

(2) the *conditionals* of μ on unstable disks are equivalent to the conditionals of Lebesgue measure on those disks;

(3) for any continuous $\phi : A \to \mathbb{R}$ and any $\tilde{\xi} \in \gamma^s(\xi)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(F^j(\tilde{\xi})) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(F^j(\xi));$$

(4) the stable foliation is *absolutely continuous*.

It follows that

μ ergodic supported on A

$\Downarrow$ (1)+Prop 1.2

μ_{γ^u} almost every point in an unstable disk γ^u belongs in $B(\mu)$

$\Downarrow$ (2)

m_{γ^u} almost every point in an unstable leaf γ^u belongs in $B(\mu)$

$\Downarrow$ (3)+(4)

μ is a physical measure

Rohlin disintegration

Let X be a compact metric space, $\mathcal{B}$ the σ-algebra of Borel sets and μ a Borel probability measure. Given a partition $\mathcal{P}$ of X into Borel sets, let $\pi : X \to \mathcal{P}$ assign to each $x \in X$ the element $\omega \in \mathcal{P}$ such that $x \in \omega$. Consider the probability measure space $(\mathcal{P}, \pi_*\mathcal{P}, \pi_*\mu)$, where

$$\pi_*\mathcal{P} = \{\mathcal{Q} \subset \mathcal{P} \colon \pi^{-1}(\mathcal{Q}) \in \mathcal{B}\}$$

and $\pi_*\mu$ is given by

$$\pi_*\mu(\mathcal{Q}) = \mu(\pi^{-1}(\mathcal{Q})), \quad \text{for all } \mathcal{Q} \in \pi_*\mathcal{P}.$$

A disintegration of μ with respect to the partition $\mathcal{P}$ is a family $\{\mu_\omega\}_{\omega \in \mathcal{P}}$ of probability measures on X such that

- $\mu_\omega(\omega) = 1$, for $\pi_*\mu$ almost every $\omega \in \mathcal{P}$;
- given any continuous $\varphi : X \to \mathbb{R}$, the function $\mathcal{P} \ni \omega \mapsto \int \varphi d\mu_\omega$ is measurable and

$$\int_X \varphi d\mu = \int_{\mathcal{P}} \left(\int_X \varphi d\mu_\omega \right) d\pi_*\mu.$$

We refer to the measures μ_ω as the conditional measures of μ with respect to $\mathcal{P}$.

Theorem 1.6 (Rohlin 1952)

Every Borel probability measure has a disintegration with respect to any measurable partition.

A partition $\mathcal{P}$ into Borel sets is a measurable partition if there is a sequence $(E_n)_n$ of Borel sets and $X_0 \subset X$ with full μ measure such that, for all $\omega \in \mathcal{P}$,

$$\omega \cap X_0 = X_0 \cap E_1^* \cap E_2^* \cap \cdots,$$

where each E_n^* is either E_n or its complement $X \setminus E_n$.

Lyapunov exponents

Let $f : M \to M$ be a diffeomorphism of a smooth manifold M.
Given $x \in M$ and $v \in T_x M$, set

$$\lambda(x, v) = \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x)v\|,$$

if these limits exist and coincide.

Theorem 1.7 (Oseledec 1968)

Let f preserve an invariant probability measure μ. There exist measurable functions λ_i and a Df-invariant splitting $T_x M = \oplus_i E_i(x)$ with $\lambda(x, v) = \lambda_i(x)$ for μ almost every $x \in M$ and every $v \in E_i(x)$. In addition, if μ is ergodic, then λ_i and $\dim(E_i)$ are constant μ almost everywhere.

Each λ_i is called a Lyapunov exponent of f (with respect to μ). The regular set $\mathcal{R} \subset M$ is the set of points for which the Lyapunov exponents are defined.

Theorem 1.8 (Pesin 1976)

If $x \in \mathcal{R}$ has at least one positive Lyapunov exponent, then there is a small disk $\gamma^u(x) \subset M$ tangent to $\oplus_{\lambda_i > 0} E_i(x)$ such that for all $y \in \gamma^u(x)$

$$\limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(y), f^{-n}(x)) < 0.$$

$\gamma^u(x)$ is called an unstable disk of $x \in \mathcal{R}$. A stable disk $\gamma^s(x)$ can be obtained similarly for a point $x \in \mathcal{R}$ with at least one negative Lyapunov exponent.

SRB measures

Assume $x \in \mathcal{R}$ has at least one positive Lyapunov exponent.
We define the unstable manifold of x

$$W^u(x) = \bigcup_{n \geq 0} f^n(\gamma^u(f^{-n}(x))).$$

A measurable partition $\mathcal{P}$ of a set $X \subset M$ is said to be subordinate to unstable manifolds if f has at least one positive Lyapunov exponent μ almost everywhere and, for μ almost every $x \in X$, the element of $\mathcal{P}$ containing x is a subset of $W^u(x)$.

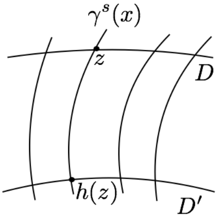
A probability measure μ is called a Sinai-Ruelle-Bowen (SRB) measure if, for any measurable partition $\mathcal{P}$ subordinate to unstable manifolds, the conditionals $\{\mu_\omega\}_{\omega \in \mathcal{P}}$ of μ are absolutely continuous with respect to the conditionals $\{m_\omega\}_{\omega \in \mathcal{P}}$ of the Lebesgue measure m.

Absolute continuity of the stable foliation

Given embedded disks $D, D' \subset M$ intersecting transversally a set $\{\gamma^s(x)\}_x$ of stable disks, define the holonomy map

$$h : \bigcup_x \gamma^s(x) \cap D \longrightarrow \bigcup_x \gamma^s(x) \cap D'$$

assigning to $z \in \gamma^s(x) \cap D$ the unique point in $\gamma^s(x) \cap D'$.



The stable foliation is called absolutely continuous if for any $A \subset \bigcup_x \gamma^s(x) \cap D$, we have

$$m_D(A) = 0 \iff m_{D'}(h(A)) = 0.$$

Theorem 1.9 (Pesin 1976)

Let $f : M \to M$ be a C^2 diffeomorphism having all Lyapunov exponents nonzero with respect to an ergodic invariant probability measure μ. Then the stable foliation is absolutely continuous.

Corollary 1.10

Every ergodic SRB measure with non-zero Lyapunov exponents is a physical measure.

Existence of SRB measures

Endomorphisms:

- Uniformly expanding [Krzyżewski and Szlenk 1969]
- Quadratic [Jakobson 1981]
- Viana [Alves 2000]
- Nonuniformly expanding [Alves, Bonatti, and Viana 2000; Pinheiro 2006]
- Partially hyperbolic [Tsujii 2005]

Diffeomorphisms:

- Anosov [Sinai 1972; Bowen 1975]
- Axiom A [Bowen and Ruelle 1975; Ruelle 1976]
- Hénon [Benedicks and Young 1993]
- Partially hyperbolic $E^u \oplus E^{cs}$ [Pesin and Sinai 1982; Bonatti and Viana 2000]
- Partially hyperbolic $E^{cu} \oplus E^s$ [Alves, Bonatti, and Viana 2000; Alves, Dias, Luzzatto, and Pinheiro 2017]
- Nonuniformly hyperbolic [Climenhaga, Luzzatto, and Pesin 2021; Ben Ovadia 2021]

Methods:

- Lebesgue iteration
- Markov partitions
- Inducing schemes

Decay of correlations

The correlation of observables $\varphi, \psi \colon M \to \mathbb{R}$ is defined as

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

Assuming $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \to 0$ for φ and ψ characteristic functions of Borel sets, we obtain the usual notion of *mixing*:

$$\mu(A \cap f^{-n}(B)) \to \mu(A)\mu(B).$$

We are interested in rates at which $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n)$ converges to zero with n. In general, we need some regularity of the observables.

Remark 1.11

In some cases, μ is equivalent to Lebesgue measure m. Assuming m normalised and taking $\varphi = dm/d\mu$, we have

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \left| \int \psi f_*^n dm - \int \psi d\mu \right|.$$

So, the decay of $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n)$ gives information on the speed at which the push-forwards $f_*^n m$ approach the physical measure μ.

The *push-forward* is defined as $f_*^n m(A) = m(f^{-n}(A))$, for any measurable set A.

Other statistical properties

Here, we will use Young Towers to obtain Decay of Correlations for SRB measures of some specific classes of dynamical systems. The same approach has been used to deduce other statistical properties of those systems:

- Central Limit Theorem [Young 1998; Young 1999];
- Local Limit Theorem [Gouëzel 2005];
- Berry-Esseen Theorem [Gouëzel 2005];
- Almost Sure Invariance Principle [Melbourne and Nicol 2005; Melbourne and Nicol 2009];
- Large Deviations [Melbourne and Nicol 2008; Melbourne 2009; Rey-Bellet and Young 2008];
- Escape rates [Demers and Young 2006; Demers, Wright, and Young 2010; Demers, Wright, and Young 2012]
- Entropy Formula [Alves and Mesquita 2021].

Endomorphisms

Induced maps

Consider $f : M \to M$ and a measure m on M. Given $\Delta_0 \subset M$ with $m(\Delta_0) < \infty$, we say that $F : \Delta_0 \to \Delta_0$ is an induced map for f if there exist

- a countable m mod 0 partition $\mathcal{P}$ of Δ_0
- a function $R : \mathcal{P} \to \mathbb{N}$

such that

$$F|_\omega = f^{R(\omega)}|_\omega, \quad \forall \omega \in \mathcal{P}.$$

We frequently denote the induced map F by f^R.

Proposition 2.1

Let $f^R : \Delta_0 \to \Delta_0$ be an induced map for $f : M \to M$ and $\nu_0 \ll m$ an f^R-invariant probability measure ν_0. If

$$\nu = \sum_{j=0}^{\infty} f_*^j(\nu_0|\{R > j\}),$$

then

1. ν *is an* f*-invariant measure with* $\nu|_{\Delta_0} \geq \nu_0$*;*
2. ν *finite* $\iff \sum_{j=0}^{\infty} \nu_0\{R > j\} < \infty \iff R \in L^1(\nu_0)$*;*
3. ν_0 *ergodic* $\implies \nu$ *ergodic;*
4. $f_* m \ll m \implies \nu \ll m$.

See Theorem 3.18 in [Alves 2020] for a proof.

Gibbs-Markov maps

Let $(\Delta_0, \mathcal{A}, m)$ be a finite measure space and $F : \Delta_0 \to \Delta_0$ a measurable map. Given a mod 0 partition $\mathcal{P}$, set

$$F^{-n}\mathcal{P} = \{F^{-n}(\omega) \colon \omega \in \mathcal{P}\}, \quad n \geq 0.$$

Assuming $F_* m \ll m$, it follows that $F^{-n}\mathcal{P}$ is a mod 0 partition of M, for all $n \geq 1$. In this case, we have mod 0 partitions

$$\bigvee_{j=0}^{n-1} F^{-j}\mathcal{P} = \left\{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots \cap F^{-n+1}(\omega_{n-1}) \colon \omega_0, \ldots, \omega_{n-1} \in \mathcal{P}\right\}$$

and

$$\bigvee_{n=0}^{\infty} F^{-n}\mathcal{P} = \left\{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots : \omega_n \in \mathcal{P} \text{ for all } n \geq 0\right\}.$$

We say that $F : \Delta_0 \to \Delta_0$ is a Gibbs-Markov map if there is an m mod 0 countable partition $\mathcal{P}$ into measurable subsets of Δ_0 such that:

(G$_1$) **Markov**: F maps each $\omega \in \mathcal{P}$ bijectively to Δ_0.

(G$_2$) **Nonsingular**: $\exists\, J_F > 0$ such that for each $A \subset \omega \in \mathcal{P}$

$$m(F(A)) = \int_A J_F \, dm.$$

This in particular implies $F_* m \ll m$.

(G$_3$) **Separable**: the sequence $(\bigvee_{i=0}^{n-1} F^{-i}\mathcal{P})_n$ is a *basis* of Δ_0:

1. $(\bigvee_{i=0}^{n-1} F^{-i}\mathcal{P})_n$ generates $\mathcal{A}$ (mod 0);
2. $(\bigvee_{i=0}^{\infty} F^{-i}\mathcal{P})_n$ is the partition into single points (mod 0).

In particular, m almost all $x, y \in \Delta_0$ have defined the separation time:

$$s(x,y) = \min \left\{ n \geq 0 : F^n(x),\ F^n(y) \text{ lie in distinct elements of } \mathcal{P} \right\}.$$

(G$_4$) **Gibbs**: $\exists\, C > 0$ and $0 < \beta < 1$ such that for all $x, y \in \omega \in \mathcal{P}$

$$\log \frac{J_F(x)}{J_F(y)} \leq C \beta^{s(F(x),F(y))}.$$

Piecewise smooth maps

Lemma 2.2

Let Δ_0 be a manifold (possibly with a boundary) and m Lebesgue measure on the Borel sets of Δ_0. Let $\mathcal{P}$ be a countable m mod 0 partition of Δ_0 into open sets whose closures have smooth boundary, and $F\colon \Delta_0 \to \Delta_0$ be such that, for all $\omega \in \mathcal{P}$, the restriction of F to ω has an extension to the boundary of ω which is a C^1 diffeomorphism onto its image. Then,

1. *F is nonsingular and $J_F(x) = |\det DF(x)|$, for m almost all $x \in \Delta_0$;*
2. *if there is $0 < \alpha < 1$ such that $\|(DF|_\omega)^{-1}(x)\| \le \alpha$, for all $x \in F(\omega)$, then F satisfies the separability property;*
3. *if there are $C, \zeta > 0$ such that, for all $\omega \in \mathcal{P}$ and $x, y \in \omega$,*

$$\log \frac{\det DF(x)}{\det DF(y)} \le C d(F(x), F(y))^\zeta,$$

then F satisfies the Gibbs property.

See Lemma 3.3 in [Alves 2020] for a proof.

A space for invariant densities

Consider the space

$$\mathcal{F}_\beta(\Delta_0) = \left\{ \varphi : \Delta_0 \to \mathbb{R} \;\text{ s.t. }\; |\varphi|_\beta \equiv \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\beta^{s(x,y)}} < \infty \right\}.$$

endowed with the norm

$$|\varphi|_\beta + \|\varphi\|_\infty,$$

and

$$\mathcal{F}_\beta^+(\Delta_0) = \{\varphi \in \mathcal{F}_\beta(\Delta_0) : \varphi \geq c \text{ for some } c > 0\}.$$

Lemma 2.3

$\mathcal{F}_\beta(\Delta_0)$ is relatively compact in $L^1(\Delta_0)$.

See Proposition 3.8 in [Alves 2020] for a proof.

Exercise 2.4

Prove Lemma 2.3.
Hint: mimic the proof of Ascoli-Arzela Theorem in Wikipedia.

An F-invariant probability measure is called exact ($\Rightarrow$ mixing $\Rightarrow$ ergodic) if

$$A \in \bigcap_{n\geq 0} F^{-n}(\mathcal{A}) \quad \text{and} \quad \nu(A) > 0 \quad \Longrightarrow \quad \nu(A) = 1.$$

Theorem 2.5

If $F : \Delta_0 \to \Delta_0$ is a Gibbs-Markov map, then F has a unique absolutely continuous invariant probability measure ν. Moreover, ν is exact, $d\nu/dm$ belongs in $\mathcal{F}_\beta(\Delta_0)$ and there is $K > 0$ such that

$$\frac{1}{K} \leq \frac{d\nu}{dm} \leq K.$$

See Theorem 3.13 in [Alves 2020] for a proof.
The idea is to show that the sequence of densities of the measures

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} F_*^j m.$$

is bounded in $\mathcal{F}_\beta(\Delta_0)$. By Lemma 2.3, it has an accumulation point in $L^1(\Delta_0)$, which is the density of an absolutely continuous F-invariant finite measure. The first item in the separability property **(G$_3$)** is used to obtain the exactness of ν.

Proposition 2.6

Let $f^R : \Delta_0 \to \Delta_0$ be a Gibbs-Markov map and $\nu_0 \ll m$ be its unique f^R-invariant probability measure. If $f_* m \ll m$ and $\nu = \sum_{j=0}^{\infty} f_*^j(\nu_0|\{R > j\})$, then

1. *ν is an f-invariant ergodic measure with $\nu \ll m$ and $\nu|_{\Delta_0} \geq \nu_0$;*
2. *ν is finite if, and only if, R is integrable with respect to m;*
3. *$d\nu/dm|_{\Delta_0}$ is bounded from below by some positive constant;*
4. *if ν is finite, then $\mu = \nu/\nu(M)$ is the unique ergodic f-invariant probability measure such that $\mu \ll m$ and $\mu(\Delta_0) > 0$.*

See Corollary 3.21 in [Alves 2020] for a proof.

Corollary 2.7

Assume M is a Riemannian manifold, m is Lebesgue measure on M and $f : M \to M$ is such that $f_* m \ll m$. If f has an induced Gibbs-Markov map $f^R : \Delta_0 \to \Delta_0$ with $R \in L^1(m|_{\Delta_0})$, then f has a unique SRB measure μ with $\mu(\Delta_0) > 0$.

Decay of correlations

Let $f : M \to M$ have an induced Gibbs-Markov map $f^R : \Delta_0 \to \Delta_0$ with $R \in L^1(m|_{\Delta_0})$ and μ be the unique ergodic f-invariant probability measure such that

$$\mu \ll m \quad \text{and} \quad \mu(\Delta_0) > 0.$$

Our next goal is to obtain estimates on the decay (with $n \to \infty$) of

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

The general idea is that

$$\boxed{\textit{the decay of } \mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \searrow 0 \textit{ is given by the decay of } m\{R > n\} \searrow 0}$$

Unfortunately, this statement cannot be proved with this generality and only in specific cases (polynomial, stretched exponencial, exponential) for observables with some regularity (Hölder continuous φ and $\psi \in L^\infty(m)$) will be proved here. Given $\eta > 0$, we say that $\varphi : M \to \mathbb{R}$ is η-Hölder continuous if

$$|\varphi|_\eta \equiv \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\eta} < \infty.$$

Set

$$\mathcal{H}_\eta = \{\varphi : M \to \mathbb{R} \mid \varphi \text{ is } \eta\text{-Hölder continuous}\}.$$

Extension

The decay of correlations for system (f, μ) with μ will be obtained through an extension (T, ν) of (f, μ): we are going to introduce maps $T : \Delta \to \Delta$ and $\pi : \Delta \to M$ and a T-invariant probability measure ν such that

$$f \circ \pi = \pi \circ T \quad \text{and} \quad \pi_* \nu = \mu. \tag{2}$$

Exercise 2.8

Show that if (2) holds, then

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathrm{Cor}_\nu(\varphi \circ \pi, \psi \circ \pi \circ T^n), \tag{3}$$

for all $\varphi, \psi : M \to \mathbb{R}$ for which the expressions make sense.

We also need a suitable space $\mathcal{F}$ of observables in Δ such that

$$\varphi \in \mathcal{H}_\eta \implies \varphi \circ \pi \in \mathcal{F}. \tag{4}$$

Having (2), (3) and (4), our problem on decay of correlations for (f, μ) with observables $\varphi \in \mathcal{H}_\eta$ will be reduced to a problem with respect to (T, ν) with observables $\varphi \in \mathcal{F}$.

Tower extension

Consider the partition $\mathcal{P} = \{\Delta_{0,i}\}_i$ associated with an induced Gibbs-Markov map $f^R : \Delta_0 \to \Delta_0$. Set the tower

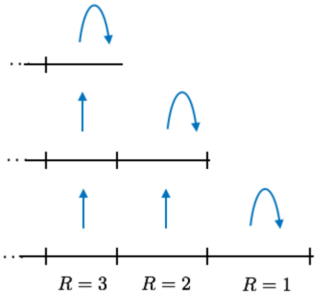
$$\Delta = \big\{(x,\ell)\colon x \in \Delta_0 \text{ and } 0 \le \ell < R(x)\big\},$$

and the tower map $T : \Delta \to \Delta$ given by

$$T(x,\ell) = \begin{cases} (x,\ell+1), & \text{if } \ell < R(x)-1; \\ (f^R(x),0), & \text{if } \ell = R(x)-1. \end{cases}$$

Define $\pi\colon \Delta \to M$ by

$$\pi(x,\ell) = f^\ell(x) \tag{5}$$



Exercise 2.9

Show that π is measurable and $f \circ \pi = \pi \circ T$.

Remark 2.10

The tower construction can be carried out for any Gibbs-Markov map $F : \Delta_0 \to \Delta_0$ (not necessarily an induced map) and any $R : \Delta_0 \to \mathbb{N}$, provided R is constant in the elements of the partition associated with F.

The ℓ^{th} level of the tower is the set

$$\Delta_\ell = \{(x, \ell) \in \Delta\}.$$

The 0^{th} level is naturally identified with the set $\Delta_0 \subset M$. Under this identification we have that $T^R = f^R : \Delta_0 \longrightarrow \Delta_0$ is an induced Gibbs-Markov map for T.

The ℓ^{th} level of Δ is a copy of $\{R > \ell\} \subset \Delta_0$. This allows us to extend to Δ

- the σ-algebra $\mathcal{A}$;
- the reference measure m;
- the m mod 0 partition $\mathcal{P}$ to an m mod 0 partition

$$\mathcal{Q} = \{\Delta_{\ell,i}\}.$$

Finally, we extend the separation time to $\Delta \times \Delta$ in the following way:
if $x, y \in \Delta_\ell$, then there are unique $x_0, y_0 \in \Delta_0$ such that $x = T^\ell(x_0)$ and $y = T^\ell(y_0)$; set

$$s(x, y) = s(x_0, y_0).$$

Set $s(x, y) = 0$ for all other points $x, y \in \Delta$.

Existence of equilibrium

Set

$$\mathcal{F}_\beta(\Delta) = \left\{ \varphi : \Delta \to \mathbb{R} \,|\, \exists C > 0 : |\varphi(x) - \varphi(y)| \le C\beta^{s(x,y)}, \forall x, y \in \Delta \right\}$$

and for $\varphi \in \mathcal{F}_\beta$

$$C_\varphi = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\beta^{s(x,y)}}$$

Set also

$$\mathcal{F}_\beta^+(\Delta) = \{ \varphi \in \mathcal{F}_\beta(\Delta) \,|\, \exists c > 0 : \varphi \ge c \}\,.$$

and for $\varphi \in \mathcal{F}_\beta^+(\Delta)$

$$C_\varphi^+ = \max\left\{ C_\varphi,\, \|\varphi\|_\infty,\, \left\| \frac{1}{\varphi} \right\|_\infty \right\}.$$

Theorem 2.11

If $R \in L^1(m)$, then $T : \Delta \to \Delta$ has a unique invariant probability measure $\nu \ll m$. Moreover, $d\nu/dm \in \mathcal{F}_\beta^+(\Delta)$, ν is ergodic, ν is exact iff $\gcd\{R\} = 1$, and $\pi_*\nu = \mu$.

See Theorems 3.24 & 3.25 and Proposition 3.27 in [Alves 2020] for a proof.
Existence and uniqueness of ν follows from Corollary 2.6.
(Note that $T^R = f^R$ is a Gibbs-Markov induced map for T).

Exercise 2.12

Show that $\gcd\{R\} > 1 \implies (T, \nu)$ not mixing.

Space of observables

Assume now $f : M \to M$ defined on a *metric space* M with a reference measure m. Let $f^R : \Delta_0 \to \Delta_0$ be an induced map and $\mathcal{P}$ the partition of Δ_0 associated with f^R. We say that f^R is expanding if there are $C > 0$ and $0 < \beta < 1$ such that, for all $\omega \in \mathcal{P}$ and $x, y \in \omega$:

- $\operatorname{dist}(f^R(y), f^R(x)) \le C\beta^{s(x,y)}$;
- $\operatorname{dist}(f^j(x), f^j(y)) \le C \operatorname{dist}(f^R(x), f^R(y)),$ for all $0 \le j \le R$.

Lemma 2.13

If $T : \Delta \to \Delta$ is the tower map associated with an expanding Gibbs-Markov induced map, then

$$\varphi \in \mathcal{H}_\eta \implies \varphi \circ \pi \in \mathcal{F}_{\beta^\eta}(\Delta).$$

We need to show that there is $K > 0$ such that, for all $x, y \in \Delta$,

$$|\varphi(\pi(x)) - \varphi(\pi(y))| \le K\beta^{\eta s(x,y)}, \tag{6}$$

If $s(x, y) = 0$ (in particular, if x, y belong in distinct levels of Δ), then (6) is an easy consequence of the fact that φ is bounded. Consider now $\ell \ge 0$ and $x, y \in \Delta_\ell$ with $s(x, y) \ge 1$. By definition, there are $\omega \in \mathcal{P}$ and $x_0, y_0 \in \omega$ such that $s(x_0, y_0) = s(x, y) \ge 1$. This gives in particular $R(x_0), R(y_0) > \ell$.

Using that $\varphi \in \mathcal{H}_\eta$, we get

$$
\begin{aligned}
|\varphi(\pi(x)) - \varphi(\pi(y))| &= |\varphi(\pi(x_0,\ell)) - \varphi(\pi(y_0,\ell))| \\
&= |\varphi(f^\ell(x_0)) - \varphi(f^\ell(y_0))| \\
&\leq |\varphi|_\eta d(f^\ell(x_0), f^\ell(y_0))^\eta. \qquad (7)
\end{aligned}
$$

Since f^R is expanding, there is $C > 0$ such that

$$
\begin{aligned}
d(f^\ell(x_0), f^\ell(y_0)) &\leq Cd(f^R(x_0), f^R(y_0)) \\
&\leq C^2 \beta^{s(f^R(x_0), f^R(y_0))} \\
&= C^2 \beta^{s(x_0,y_0)-1} \\
&= C^2 \beta^{s(x,y)-1}. \qquad (8)
\end{aligned}
$$

It follows from (7) and (8) that

$$
|\varphi(\pi(x)) - \varphi(\pi(y))| \leq |\varphi|_\eta C^{2\eta} \beta^{-\eta} \beta^{\eta s(x,y)}.
$$

This yields (6) with $K = |\varphi|_\eta C^{2\eta}\beta^{-\eta}$. ∎

Decay of correlations for tower maps

At this point, we have reduced the correlation problem on (f, μ) with observables

$$\varphi \in \mathcal{H}_\eta \quad \text{and} \quad \psi \in L^\infty(m)$$

to a problem on the tower system (T, ν) with

$$\varphi \in \mathcal{F}_{\beta^\eta}(\Delta) \quad \text{and} \quad \psi \in L^\infty(m).$$

Recall Exercises 2.8 and 2.9, Theorem 2.11, Lemma 2.13, and note that

$$\psi \in L^\infty(m) \implies \psi \circ \pi \in L^\infty(m).$$

Theorem 2.14 (Young 1999; Gouëzel 2006)

Let $T : \Delta \to \Delta$ be the tower map of a Gibbs-Markov map f^R with $R \in L^1(m)$ and ν the unique ergodic T-invariant probability measure such that $\nu \ll m$. If $\gcd(R) = 1$

1. *if $m\{R > n\} \leq Cn^{-\alpha}$ for some $C > 0$ and $\alpha > 1$, then for all $\varphi \in \mathcal{F}_\beta(\Delta)$ and $\psi \in L^\infty(m)$, there is $C' > 0$ such that*

$$\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \leq C' n^{-a+1};$$

2. *if $m\{R > n\} \leq Ce^{-cn^a}$ for some $C, c > 0$ and $0 < a \leq 1$, given $0 < \beta < 1$ there is $c' > 0$ such that, for all $\varphi \in \mathcal{F}_\beta(\Delta)$ and $\psi \in L^\infty(m)$, there is $C' > 0$ such that*

$$\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \leq C' e^{-c' n^a}.$$

Non-exact case

Corollary 2.15

Let $T : \Delta \to \Delta$ be the tower map of a Gibbs-Markov map f^R with $R \in L^1(m)$ and ν be the unique ergodic T-invariant probability measure such that $\nu \ll m$. If $\gcd(R) = q$, then there are exact T^q-invariant probability measures $\nu_1, \dots, \nu_q$ such that

$$T_*\nu_1 = \nu_2, \dots, T_*\nu_{q-1} = \nu_q, T_*\nu_q = \nu_1 \quad \text{and} \quad \nu = \frac{1}{q}(\nu_1 + \cdots + \nu_q).$$

Moreover, for all $1 \le i \le q$,

1. *if $m\{R > n\} \le Cn^{-\alpha}$ for some $C > 0$ and $\alpha > 1$, then for all $\varphi \in \mathcal{F}_\beta(\Delta)$ and $\psi \in L^\infty(m)$ there is $C' > 0$ such that*

$$\mathrm{Cor}_{\nu_i}(\varphi, \psi \circ T^{qn}) \le C' n^{-a+1};$$

2. *if $m\{R > n\} \le Ce^{-cn^a}$ for some $C, c > 0$ and $0 < a \le 1$, given $0 < \beta < 1$ there is $c' > 0$ such that for all $\varphi \in \mathcal{F}_\beta(\Delta)$ and $\psi \in L^\infty(m)$ there is $C' > 0$ such that*

$$\mathrm{Cor}_{\nu_i}(\varphi, \psi \circ T^{qn}) \le C' e^{-c' n^a}.$$

Reduction to the exact case

Let $T : \Delta \to \Delta$ be the tower map of $f^R : \Delta_0 \to \Delta_0$. Set $q = \gcd(R)$ and, for each $1 \le i \le q$,

$$\Upsilon_i = \bigcup_{\ell \equiv i-1 (\mathrm{mod}\ q)} \Delta_\ell .$$

We have that $\{\Upsilon_1, \dots, \Upsilon_q\}$ is a partition of Δ. Moreover,

$$T(\Upsilon_1) = \Upsilon_2, \dots, T(\Upsilon_{q-1}) = \Upsilon_q \quad \text{and} \quad T(\Upsilon_q) = \Upsilon_1. \tag{9}$$

Since ν is a T-invariant measure, we have that $\nu|_{\Upsilon_i}$ is an invariant measure for $T^q : \Upsilon_i \to \Upsilon_i$, for each $1 \le i \le q$. Setting

$$\nu_i = \frac{1}{\nu(\Upsilon_i)} (\nu|_{\Upsilon_i}),$$

we have $\nu = (\nu_1 + \dots + \nu_q)/q$. Note that

$$\nu_i \ll m, \quad \text{for all } 1 \le i \le q. \tag{10}$$

It follows from (9) and the T-invariance of ν that

$$T_*\nu_1 = \nu_2, \dots, T_*\nu_{q-1} = \nu_q \quad \text{and} \quad T_*\nu_q = \nu_1.$$

Let now $T' : \Delta' \to \Delta'$ be a tower map with base map f^R and recurrence time $R' = R/q$; recall Remark 2.10. By Theorem 2.11, the tower map T' has a unique invariant probability measure $\nu' \ll m$, which is an exact measure, since $\gcd(R') = 1$.

Exercise 2.16

Show that the map $S_i : \Delta' \to \Upsilon_i$, given by $S_i(x, \ell) = (x, q\ell + i - 1)$, is a bimeasurable conjugacy between T' and $T^q|_{\Upsilon_i}$, for all $1 \le i \le q$.

Hence, $S_{i*}\nu'$ is an exact invariant probability measure for $T^q|_{\Upsilon_i}$. Since S_i and S_i^{-1} preserve sets with zero m measure, using (10) we get

$$S_{i*}^{-1}\nu_i \ll m.$$

Uniqueness gives $S_{i*}^{-1}\nu_i = \nu'$, and so $\nu_i = S_{i*}\nu'$ is exact.

Back to the original dynamics

Theorem 2.17

Let $f : M \to M$ be defined on a metric space M with a reference measure m such that $f_* m \ll m$. Let $f^R : \Delta_0 \to \Delta_0$ be a Gibbs-Markov expanding map with $R \in L^1(m)$ and μ be the unique ergodic f-invariant probability measure $\mu \ll m$ such that $\mu(\Delta_0) > 0$. If $\gcd(R) = q$, then f^q has $p \leq q$ exact probability measures $\mu_1, \dots, \mu_p$ with

$$f_*\mu_1 = \mu_2, \dots, f_*\mu_{p-1} = \mu_p, f_*\mu_p = \mu_1 \quad \text{and} \quad \mu = \frac{1}{p}(\mu_1 + \cdots + \mu_p).$$

Moreover, for all $1 \leq i \leq p$,

1. *if $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, then for all $\varphi \in \mathcal{H}_\eta$ and $\psi \in L^\infty(m)$ there exists $C' > 0$ such that*

$$\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \leq C' n^{-a+1};$$

2. *if $m\{R > n\} \leq Ce^{-cn^a}$ for some $C, c > 0$ and $a > 1$, then given $\eta > 0$, there is $c' > 0$ such that, for all $\varphi \in \mathcal{H}_\eta$ and $\psi \in L^\infty(m)$, there is $C' > 0$ for which*

$$\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \leq C' e^{-c' n^a}.$$

Corollary 2.18

Let $f : M \to M$ be defined on a metric space M with a reference measure m such that $f_* m \ll m$. Let $f^R : \Delta_0 \to \Delta_0$ be a Gibbs-Markov expanding map with $R \in L^1(m)$ and $\mu \ll m$ be the unique ergodic f-invariant probability measure such that $\mu(\Delta_0) > 0$. If μ is ergodic for all f^n, then

1. *if $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, then for all $\varphi \in \mathcal{H}_\eta$ and $\psi \in L^\infty(m)$ there exists $C' > 0$ such that*
$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' n^{-a+1};$$
2. *if $m\{R > n\} \leq Ce^{-cn^a}$ for some $C, c > 0$ and $a > 1$, then given $\eta > 0$, there is $c' > 0$ such that, for all $\varphi \in \mathcal{H}_\eta$ and $\psi \in L^\infty(m)$, there is $C' > 0$ for which*
$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' e^{-c' n^a}.$$

Proof. Consider $1 \leq p \leq q$ and probability measures μ and $\mu_1, \ldots, \mu_p$ as in Theorem 2.17. In particular, $\mu_i \ll \mu$, for all $1 \leq i \leq p$.

Exercise 2.19

If ν_0, ν_1 are f-invariant probability measures with ν_0 ergodic and $\nu_1 \ll \nu_0$, then $\nu_1 = \nu_0$.

Since μ is ergodic for f^q, it follows that $\mu_i = \mu$, for all $1 \leq i \leq p$. The expected conclusions for $\mathrm{Cor}_\mu(\varphi, \psi \circ f^{qn})$ then follow from Theorem 2.17. Apply these conclusions to the observables $\psi \circ f, \cdots, \psi \circ f^{q-1}$ in the place of ψ. ∎

Back to the exact tower

Now we sketch the proof of Theorem 2.14. Given $\varphi \in L^\infty(m)$ with $\varphi \neq 0$, set

$$\varphi^* = \frac{1}{\int(\varphi + 2\|\varphi\|_\infty)d\nu}(\varphi + 2\|\varphi\|_\infty). \tag{11}$$

Note that φ^* is strictly positive and its integral with respect to ν is 1.
Next we reduce the proof of Theorem 2.14 to obtain convenient estimates for $|T_*^n\lambda - \nu|$.

Lemma 2.20

For all $\varphi \in \mathcal{F}_\beta(\Delta) \setminus \{0\}$, we have

1. $\varphi^* \in \mathcal{F}_\beta^+(\Delta)$ *and* $1/3 \leq \varphi^* \leq 3$;
2. $\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \leq 3\|\varphi\|_\infty\|\psi\|_\infty|T_*^n\lambda - \nu|$ *for all* $\psi \in L^\infty(m)$, *where λ is the probability measure on Δ such that $d\lambda/d\nu = \varphi^*$.*

Proof. We have

$$\|\varphi\|_\infty \leq \varphi + 2\|\varphi\|_\infty \leq 3\|\varphi\|_\infty. \tag{12}$$

Since ν is a probability measure, we get

$$\frac{1}{3\|\varphi\|_\infty} \leq \frac{1}{\int(\varphi + 2\|\varphi\|_\infty)d\nu} \leq \frac{1}{\|\varphi\|_\infty}. \tag{13}$$

From (12) and (13) we get $1/3 \leq \varphi^* \leq 3$.

For all $x, y \in \Delta$ we have

$$\frac{\varphi^*(x) - \varphi^*(y)}{\beta^{s(x,y)}} = \frac{1}{\int(\varphi + 2\|\varphi\|_\infty)d\nu} \cdot \frac{\varphi(x) - \varphi(y)}{\beta^{s(x,y)}}. \tag{14}$$

Since $\varphi^* \geq 1/3$,

$$\varphi \in \mathcal{F}_\beta(\Delta) \implies \varphi^* \in \mathcal{F}_\beta^+(\Delta).$$

Setting $a = \int(\varphi + 2\|\varphi\|_\infty)d\nu$, we may write

$$\begin{aligned}
\mathrm{Cor}_\mu(\varphi, \psi \circ T^n) &= \left| \int \varphi(\psi \circ T^n)d\nu - \int \varphi d\nu \int \psi d\nu \right| \\
&= a \left| \int \varphi^*(\psi \circ T^n)\, d\nu - \int \varphi^* d\nu \int \psi\, d\nu \right| \\
&= a \left| \int (\psi \circ T^n)\, d\lambda - \int \psi\, d\nu \right| \\
&= a \left| \int \psi\, dT_*^n \lambda - \int \psi\, d\nu \right| \\
&\leq a\|\psi\|_\infty |T_*^n \lambda - \nu|.
\end{aligned}$$

Observing that $a \leq 3\|\varphi\|_\infty$, we obtain Lemma 2.20. ■

Convergence to equilibrium

Theorem 2.21

Assume that $\gcd\{R\} = 1$ *and* λ *is a measure such that* $\varphi = d\lambda/dm \in \mathcal{F}_\beta^+(\Delta)$.

1. *If* $m\{R > n\} \le Cn^{-\zeta}$ *for some* $C > 0$ *and* $\zeta > 1$, *then for some* $C' > 0$
$$|T_*^n\lambda - \nu| \le C' n^{-\zeta+1}.$$
2. *If* $m\{R > n\} \le Ce^{-cn^\eta}$, *then for some* $C, c > 0$ *and* $0 < \eta \le 1$
$$|T_*^n\lambda - \nu| \le C' e^{-c' n^\eta}.$$
for some $C', c' > 0$;

Moreover, c' *does not depend on* φ *and* C' *depends only on* C_φ^+.

The proof of this result uses a *coupling argument* developed in [Young 1999]†, based on a careful study of returns to the base of the tower.

Remark

There are (infinitely many) probability measures λ on Δ with $d\lambda/dm \in \mathcal{F}_\beta^+(\Delta)$ such that for some $c > 0$
$$|T_*^n\lambda - \nu| \ge c \sum_{\ell > n} m\{R > \ell\}.$$

†See also [Gouëzel 2006] for an optimal estimate in the stretched exponencial case.

Coupling

Let λ, λ' be probability measures in Δ whose densities wrt m belong in $\mathcal{F}_\beta^+(\Delta)$. Set

$$\varphi = \frac{d\lambda}{dm} \quad \text{and} \quad \varphi' = \frac{d\lambda'}{dm}.$$

Consider the product map

$$T \times T : \Delta \times \Delta \to \Delta \times \Delta,$$

and the product measure $P = \lambda \times \lambda'$ on $\Delta \times \Delta$. Let $\pi, \pi' : \Delta \times \Delta \to \Delta$ be the projections on the first and second coordinates respectively. Consider also the partition $\mathcal{Q} \times \mathcal{Q}$ of $\Delta \times \Delta$. For each $n \geq 1$, let

$$(\mathcal{Q} \times \mathcal{Q})_n := \bigvee_{i=0}^{n-1} (T \times T)^{-i} (\mathcal{Q} \times \mathcal{Q}),$$

and $(\mathcal{Q} \times \mathcal{Q})_n(x, x')$ be the element of $(\mathcal{Q} \times \mathcal{Q})_n$ containing $(x, x') \in \Delta \times \Delta$.
Since $\gcd\{R\} = 1$, then (T, ν) is mixing. Using that $d\nu/dm$ is bounded, we find $n_0 \in \mathbb{N}$ and $\gamma_0 > 0$ such that

$$m(T^{-n}(\Delta_0) \cap \Delta_0) \geq \gamma_0, \quad \forall n \geq n_0.$$

Consider $\widehat{R} : \Delta \to \mathbb{Z}$ defined as

$$\widehat{R}(x) = \min\{n \geq 0 : T^n(x) \in \Delta_0\}.$$

We introduce a sequence of stopping times $0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots$ in $\Delta \times \Delta$ by

$$\begin{aligned} \tau_1(x,x') &= n_0 + \widehat{R}(T^{n_0}(x)), \\ \tau_2(x,x') &= \tau_1 + n_0 + \widehat{R}(T^{\tau_1+n_0}(x')), \\ \tau_3(x,x') &= \tau_2 + n_0 + \widehat{R}(T^{\tau_2+n_0}(x)), \\ &\vdots \end{aligned}$$

with returns to the ground level Δ_0 alternating between the first and second coordinate and a spacing between returns of at least n_0 iterations. We define the simultaneous return time $S : \Delta \times \Delta \to \mathbb{N}$ by

$$S(x,x') = \min\left\{\tau_i \colon (T^{\tau_i}(x), T^{\tau_i}(x')) \in \Delta_0 \times \Delta_0\right\}.$$

Exercise 2.22

Show that (T,ν) mixing $\implies$ $(T \times T, \nu \times \nu)$ mixing.

In particular, $(T \times T, \nu \times \nu)$ is ergodic, and so S is defined $m \times m$ almost everywhere. Note that

$$S(x,x') = n \implies S|_{(\mathcal{Q}\times\mathcal{Q})_n(x,x')} = n \quad \text{and} \quad (T \times T)^n((\mathcal{Q} \times \mathcal{Q})_n(x,x')) = \Delta_0 \times \Delta_0.$$

A simplified model

Assume that J_T and the densities $d\lambda/dm$ and $d\lambda'/dm$ are constant on each element of $\mathcal{Q}$. We may write

$$\begin{aligned}
|T^n_*\lambda - T^n_*\lambda'| =& |\pi_*(T\times T)^n_* P - \pi'_*(T\times T)^n_* P| \\
\leq& |\pi_*(T\times T)^n_*(P|\{S>n\}) - \pi'_*(T\times T)^n_*(P|\{S>n\})| \\
&+ \sum_{i=1}^{n} |\pi_*(T\times T)^n_*(P|\{S=i\}) - \pi'_*(T\times T)^n_*(P|\{S=i\})| \\
=& \left|\pi_*(T\times T)^n_*(P|\{S>n\}) - \pi'_*(T\times T)^n_*(P|\{S>n\})\right| \\
&+ \sum_{i=1}^{n} \left| T^{n-i}_* \left(\pi_*(T\times T)^i_*(P|\{S=i\}) - \pi'_*(T\times T)^i_*(P|\{S=i\})\right)\right| \qquad (15)
\end{aligned}$$

In the last equality we have used that

$$T^{n-i}\circ\pi = \pi\circ(T\times T)^i \quad \text{and} \quad T^{n-i}\circ\pi' = \pi'\circ(T\times T)^i.$$

Exercise 2.23

Show that if S(x,x')=i, then

$$\pi_*(T\times T)^i_*\left(P|(\mathcal{Q}\times\mathcal{Q})_i(x,x')\right) = \frac{P\left((\mathcal{Q}\times\mathcal{Q})_i(x,x')\right)}{m(\Delta_0)}(m|\Delta_0) = \pi'_*(T\times T)^i_*\left(P|(\mathcal{Q}\times\mathcal{Q})_i(x,x')\right).$$

It follows that the terms in the summation (15) are all equal to zero, and so

$$|T^n_*\lambda - T^n_*\lambda'| \leq 2P\{S>n\}.$$

Taking $\lambda' = \nu$ we have $T^n_*\nu = \nu$, and so we are reduced to find an upper bound for $P\{S>n\}$.

General case

Lemma 2.24 (Young 1998)

There are $\theta < 1$ and $K > 0$ such that for all $n \geq 1$

$$\left|T_*^n\lambda - T_*^n\lambda'\right| \leq 2P\{S > n\} + K\sum_{i=1}^{\infty}\theta^i(i+1)P\left\{S > \frac{n}{i+1}\right\}.$$

See Proposition 3.45 in [Alves 2020] for a proof.
Since $T_*\nu = \nu$, taking $\lambda' = \nu$ we get an upper bound for $|T_*^n\lambda - \nu|$.
In the polynomial and (stretched) exponential cases, the second term in the sum above decays at the same speed of the first one. Hence, Theorem 2.21 follows from

Lemma 2.25 (Young 1998; Gouëzel 2006)

1. If $m\{R > n\} \lesssim n^{-\zeta}$ for some $\zeta > 1$, then $P\{S > n\} \lesssim n^{-\zeta+1}$.
2. If $m\{R > n\} \lesssim e^{-cn^\eta}$ for some $c > 0$ and $0 < \eta \leq 1$, then $P\{S > n\} \lesssim e^{-c'n^\eta}$ for some $c' > 0$.

See Propositions 3.46 & 3.48 in [Alves 2020] for a proof.

Application: Intermittent maps

Here, we exhibit two examples of transformations with indifferent fixed points, associated with phase transitions from stable periodic behaviour to a chaotic one, related in [Pomeau and Manneville 1980] to an intermittent transition to turbulence in convective fluids.

We consider first an interval map, introduced in [Liverani, Saussol, and Vaienti 1999], where it is relatively easy to define a Gibbs-Markov induced map, and then deduce some interesting statistical properties of the original system.

The second example is a circle map introduced in [Young 1999]. The construction of the Gibbs-Markov induced map is a little more intrincate, but using it as the base dynamics, a solenoid with intermittency was built in [Alves and Pinheiro 2008], providing an example of a diffeomorphism with a Young tower with polynomial recurrence times.

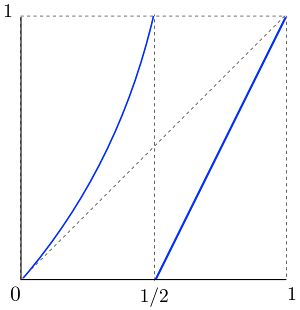
See [Hu 2004; Thaler 1980] for more results on one-dimensional maps with neutral fixed points or [Bahsoun, Bose, and Duan 2014] for a skew product map on a square.

Interval map

Given $\alpha > 0$, consider $I = [0,1]$, the Lebesgue measure m on I and the map $f : I \to I$, given by

$$f(x) = \begin{cases} x + 2^{\alpha} x^{\alpha+1}, & \text{if } 0 \le x \le 1/2; \\ 2x - 1, & \text{if } 1/2 < x \le 1. \end{cases}$$

Note that $f|_{[0,1/2]}$
is C^2 for $\alpha \ge 1$ and $C^{1+\alpha}$ for $0 < \alpha < 1$.



Theorem 2.26

1. For $\alpha < 1$, the map f has a unique SRB measure μ. Moreover, μ is exact, the support of μ coincides with I, its basin covers m almost all of I and, for each Hölder continuous $\varphi : I \to \mathbb{R}$ and $\psi \in L^{\infty}(m)$,

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) \lesssim 1/n^{1/\alpha - 1}.$$

2. For $\alpha \ge 1$, the Dirac measure at 0 is a physical measure for f and its basin covers m almost all of I.

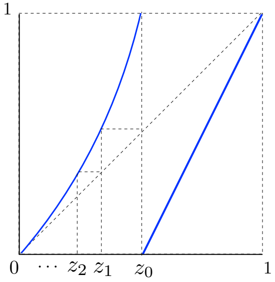
Local behaviour

Note that

(c_1) $f(0) = 0$ and $f'(0) = 1$;

(c_2) $f' > 1$ on $I \setminus \{0, 1/2\}$;

(c_3) f is C^2 on $I \setminus \{0, 1/2\}$ and $xf''(x) \approx |x|^\alpha$, for x close to 0.



It follows from (c_1) and (c_3) that

$$f'(x) - 1 \approx x^\alpha \quad \text{and} \quad f(x) - x \approx x^{\alpha+1}.$$

Let $(z_n)_n$ be the sequence in $[0, 1/2]$ defined recursively for $n \geq 0$ as

$$z_0 = \frac{1}{2} \quad \text{and} \quad f(z_{n+1}) = z_n.$$

Lemma 2.27

$(z_n)_n$ has the same asymptotics of the sequence $(1/n^{1/\alpha})_n$.

See Section 3.5.1 of [Alves 2020] for a proof.
The proof uses only the information given by (c_1)-(c_3).

Induced map

Set

$$J_0 = (1/2, 1) \quad \text{and} \quad J_n = (z_n, z_{n-1}), \quad \text{for } n \geq 1.$$

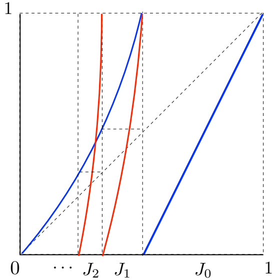
It follows that

$$m(J_n) \approx n^{-(\alpha+1)/\alpha}. \tag{16}$$

Set

$$R|_{J_n} = n + 1, \quad \text{for all } n \geq 0. \tag{17}$$

Since $f^{n+1}(J_n) = (0, 1)$, we have an induced map $f^R : I \to I$. Moreover, Lemma 2.27 gives



$$m\{R > n\} \approx \sum_{k \geq n} m(J_k) \approx \sum_{k \geq n} \left(\frac{1}{k}\right)^{1+1/\alpha} \approx n^{-1/\alpha}. \tag{18}$$

Lemma 2.28

$f^R : I \to I$ is a Gibbs-Markov map with $\gcd(R) = 1$ and $m\{R > n\} \approx n^{-1/\alpha}$.

See Lemma 3.60 in [Alves 2020] for a proof.
Note that f^R is always a Gibbs-Markov map, regardless of the value of $\alpha > 0$.
It follows from (18) that

$$R \in L^1(m) \iff 0 < \alpha < 1.$$

$0 < \alpha < 1$

We know that
$$0 < \alpha < 1 \implies R \in L^1(m).$$

From Corollary 2.7 and Lemma 2.28

$$f \text{ has a unique SRB measure } \mu.$$

Recall that the domain of f^R is the whole interval I. Proposition 2.6 gives that

$$\mu \text{ is equivalent to } m.$$

Together with Proposition 1.2, this implies that

$$\mathcal{B}(\mu) \text{ covers } m \text{ almost all of } I.$$

Since $\gcd(R) = 1$, Theorem 2.17 gives the exactness of μ and the conclusion on the decay of correlations in the first item of Theorem 2.26.

$\alpha \geq 1$

By Theorem 2.5, f^R has a unique ergodic SRB measure ν which is equivalent to m. Moreover, $d\nu/dm$ is bounded from above and below by positive constants. Then,

$$\alpha \geq 1 \overset{(18)}{\Longrightarrow} R \notin L^1(m) \implies R \notin L^1(\nu).$$

Proposition 2.29

Let M be a compact metric space and m a finite measure on the Borel sets of M. Assume that $f : M \to M$ has an induced map $f^R : M \to M$ such that

1. $\exists$ ergodic f^R-invariant probability measure ν equivalent to m with $R \notin L^1(\nu)$;
2. $\exists$ $x_0 \in M$ with $f(x_0) = x_0$ and neighbourhoods $M = U_0 \supset U_1 \supset \cdots$ of x_0 such that $\bigcap_{n\geq 0} U_n = \{x_0\}$ and, for all $n \geq 0$ and $J_n = U_n \setminus U_{n+1}$,

$$f(J_{n+1}) \subset J_n \quad \text{and} \quad R|_{J_n} = n+1.$$

Then, m almost every $x \in M$ belongs in the basin of δ_{x_0}.

See Proposition 3.61 in [Alves 2020] for a proof.
The second item of Theorem 2.26 follows from Proposition 2.29 with $x_0 = 0$ and $U_n = [0, z_{n-1}]$, for each $n \geq 1$.

Circle map

Given $\alpha > 0$, let

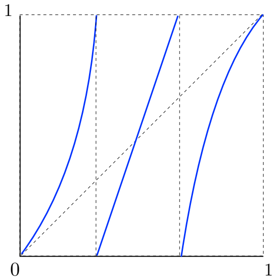
$$f : S^1 \to S^1 \tag{19}$$

be a degree $d \geq 2$ map of $S^1 = \mathbb{R}/\mathbb{Z}$ so that

(c_1) $f(0) = 0$ and $f'(0) = 1$;

(c_2) $f' > 1$ on $S^1 \setminus \{0\}$;

(c_3) f is C^2 on $S^1 \setminus \{0\}$ and $xf''(x) \approx |x|^\alpha$, for x close to 0.



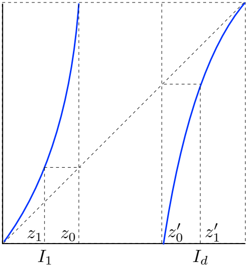
Theorem 2.30

1. For $\alpha < 1$, the map f has a unique SRB measure μ. Moreover, μ is exact, equivalent to m, its basin covers m almost all of S^1 and, for every Hölder continuous $\varphi : S^1 \to \mathbb{R}$ and $\psi \in L^\infty(m)$,

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \lesssim 1/n^{1/\alpha - 1};$$

2. For $\alpha \geq 1$, the Dirac measure at 0 is a physical measure for f and its basin covers m almost all of S^1.

Natural partition



Let $\{I_1, \ldots, I_d\}$ be an m mod 0 partition of S^1 into intervals such that, for $1 \le i \le d$

$$f|_{I_i} : I_i \to S^1 \setminus \{0\} \text{ is a diffeomorphism.}$$

Take these intervals in the natural order, with

$$\inf I_1 = 0 = \sup I_d.$$

Consider

$$z_0 = \sup I_1, \quad z_0' \inf I_d$$

and $(z_n)_n$ and $(z_n')_n$ sequences in I_1 and I_d, respectively, defined for $n \ge 0$ as

$$f(z_{n+1}) = z_n \quad \text{and} \quad f(z_{n+1}') = z_n'.$$

Set for each $n \ge 1$

$$J_n = (z_n, z_{n-1}) \quad \text{and} \quad J_n' = (z_{n-1}', z_n').$$

Note that

$$f(J_{n+1}) = J_n \quad \text{and} \quad f(J_{n+1}') = J_n'. \tag{20}$$

Consider the m mod 0 partition of S^1

$$\mathcal{P} = \{I_2, \ldots, I_{d-1}\} \cup \{J_n, J_n' : n \ge 1\}, \tag{21}$$

ignoring, of course, the first part of the union above for $d = 2$.

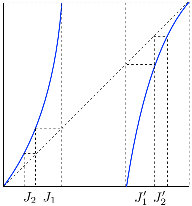
Weak Gibbs-Markov induced map

For each $2 \le i \le d-1$ (if $d \ge 3$) and $n \ge 1$, set

$$R|_{I_i} = 1 \quad \text{and} \quad R|_{J_n} = R|_{J'_n} = n.$$

Up to m mod 0 sets, we have for all $\omega \in \mathcal{P}$

$$f^R(\omega) \in \left\{ S^1,\ I_1 \cup \cdots \cup I_{d-1},\ I_2 \cup \cdots \cup I_d \right\}.$$



See Lemma 3.64 in [Alves 2020] for a proof of the next lemma.

Lemma 2.31

$f^R : S^1 \to S^1$ is a weak Gibbs-Markov map.

By weak Gibbs-Markov we mean **(G_2)**-**(G_4)** hold and **(G_1)** replaced by

(G'_1) *there is $\delta_0 > 0$ such that the image of every $\omega \in \mathcal{P}$ is a union of elements in $\mathcal{P}$ with m measure $\ge \delta_0$.*

The argument used to prove Theorem 2.5 also gives that weak Gibbs-Markov maps also have absolutely continuous invariant probability measures, but not necessarily unique.

Lemma 2.32

$f^R : S^1 \to S^1$ has a unique SRB measure ν. Moreover, ν is ergodic and $d\nu/dm$ is bounded from above and below by positive constants.

See Lemma 3.65 in [Alves 2020] for a proof.

SRB measure

Moreover, by Lemma 2.27

$$m\{R > n\} \approx \sum_{k \geq n} \left(m(J_k) + m(J'_k)\right) \approx \sum_{k \geq n} \left(\frac{1}{k}\right)^{1+1/\alpha} \approx n^{-1/\alpha}.$$

This implies $R \in L^1(m)$ for $0 < \alpha < 1$. Using Proposition 2.1 and Lemma 2.31 we deduce the existence of an ergodic SRB measure μ for f.

Exercise 2.33

Show that the $\mathcal{B}(\mu)$ covers m almost all of S^1, and so μ is the unique SRB measure for f.

Using f^R it is possible to build an induced Gibbs-Markov map f^S with $\gcd(S) = 1$ and

$$m\{S > n\} \lesssim n^{-1/\alpha}.$$

This will provide us with the expected estimate on the decay of correlations and the exactness of μ.

Decay of correlations

Consider functions $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, defined inductively for $i \geq 1$ by

$$\tau_i = \tau_{i-1} + R \circ f^{\tau_{i-1}}.$$

Notice that

$$(f^R)^i = f^{\tau_i}, \quad \text{for all } i \geq 1.$$

Now, consider I_0 the union of two consecutive intervals in the sequence $(J_n)_n$.
The reason for taking two consecutive intervals is to ensure that the greatest common divisor of the recurrence times is equal to one. Set for each $x \in S^1$

$$S(x) = \min_{i \geq 1} \left\{ \tau_i(x) : f^{\tau_i(x)}(x) \in I_0 \right\}.$$

Proposition 2.34

$f^S : I_0 \to I_0$ is a Gibbs-Markov map and $\gcd(S) = 1$.

See Proposition 3.67 in [Alves 2020] for a proof.
The exactness of μ and the conclusion about the decay of correlations in the first item of Theorem 2.30 are finally a consequence of Theorem 2.17 together with Proposition 2.34.
The second item of Theorem 2.30 follows from Proposition 2.29.

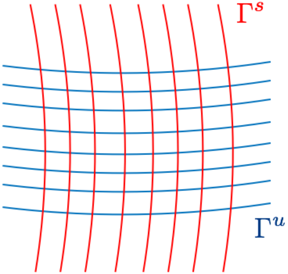
Diffeomorphisms

Young structures

Let M be a Riemannian manifold and $f : M \to M$ a diffeomorphism onto its image. We say that a compact set $\Lambda \subset M$ has a product structure if there exist a family $\Gamma^s = \{\gamma^s\}$ of stable disks and a family $\Gamma^u = \{\gamma^u\}$ of unstable disks in M such that

- $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$;
- $\dim\gamma^u + \dim\gamma^s = \dim M$;
- each γ^s and γ^u meet in exactly one point;

Given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$.



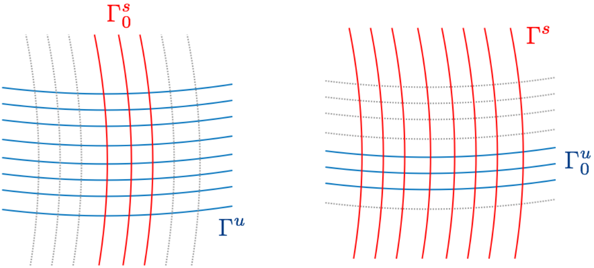
Given disks $\gamma, \gamma' \in \Gamma^u$, define $\Theta_{\gamma,\gamma'} : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ by

$$\Theta_{\gamma,\gamma'}(x) = \gamma^s(x) \cap \gamma, \tag{22}$$

and $\Theta_\gamma : \Lambda \to \gamma \cap \Lambda$ by

$$\Theta_\gamma(x) = \Theta_{\gamma^u(x),\gamma}(x). \tag{23}$$

We say that the hyperbolic product structure is measurable if the maps $\Theta_{\gamma,\gamma'}$ and Θ_γ are measurable, for all $\gamma, \gamma \in \Gamma^u$.



$\Lambda_0 \subset \Lambda$ is called an s-subset if $\Lambda_0 = \Gamma_0^s \cap \Gamma^u$ for some $\Gamma_0^s \subset \Gamma^s$.

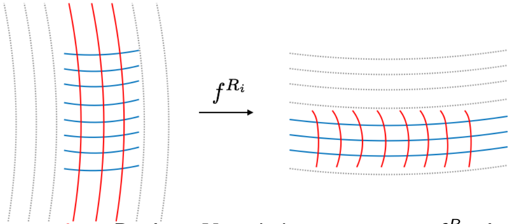
$\Lambda_0 \subset \Lambda$ is called a u-subset if $\Lambda_0 = \Gamma_0^u \cap \Gamma^s$ for some $\Gamma_0^u \subset \Gamma^u$.

A set Λ with a measurable product structure for which (Y_1)-(Y_5)[†] below hold will be called a Young structure.

(Y_1) **Markov:** $\exists$ pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that

- $m_\gamma(\Lambda \cap \gamma) > 0$ and $m_\gamma(\Lambda \setminus \cup_i \Lambda_i) \cap \gamma) = 0$ for all $\gamma \in \Gamma^u$;
- $\forall i \geq 1\ \exists R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is a u-subset and, for all $x \in \Lambda_i$,

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$



We define the recurrence time $R : \Lambda \to \mathbb{N}$ and the return map $f^R : \Lambda \to \Lambda$

$$R|_{\Lambda_i} = R_i \quad \text{and} \quad f^R|_{\Lambda_i} = f^{R_i}.$$

The separation time for $s(x,y)$ for $x, y \in \Lambda$ is the smallest $n \geq 0$ such that $(f^R)^n(x)$ and $(f^R)^n(y)$ lie in distinct Λ_i's.

[†]Originally in [Young 1998], with improvements in [Alves and Pinheiro 2008] and [Korepanov, Kosloff, and Melbourne 2019] (the latter based on an oral communication from Gouëzel).

Let $C > 0$ and $0 < \beta < 1$ be constants depending only on f and Λ .

(Y_2) **Contraction on stable disks**: for all $\gamma \in \Gamma^s$ and $x, y \in \gamma$

- $\operatorname{dist}(f^R(y), f^R(x)) \leq \beta \operatorname{dist}(x, y)$;
- $\operatorname{dist}(f^j(y), f^j(x)) \leq C \operatorname{dist}(x, y)$, for all $1 \leq j < R(x)$.

(Y_3) **Expansion on unstable disks**: for all $\gamma \in \Gamma^u$, all Λ_i and $x, y \in \gamma \cap \Lambda_i$

- $\operatorname{dist}(x, y) \leq \beta \operatorname{dist}(f^R(y), f^R(x))$;
- $\operatorname{dist}(f^j(y), f^j(x)) \leq C \operatorname{dist}(f^R(x), f^R(y))$, for all $1 \leq j < R(x)$.

(Y_4) **Absolute continuity of Γ^s**: for all $\gamma, \gamma' \in \Gamma^u$, we have $(\Theta_{\gamma,\gamma'})_* m_\gamma \ll m_{\gamma'}$. Moreover, letting $\xi_{\gamma,\gamma'} = d(\Theta_{\gamma,\gamma'})_* m_\gamma / dm_{\gamma'}$, we have for all $x, y \in \gamma' \cap \Lambda$

$$\frac{1}{C} \leq \xi_{\gamma,\gamma'}(x) \leq C \quad \text{and} \quad \log \frac{\xi_{\gamma,\gamma'}(x)}{\xi_{\gamma,\gamma'}(y)} \leq C\beta^{s(x,y)}.$$

(Y_5) **Gibbs**: for all $\gamma_0 \in \Gamma^u$ and $x, y \in \gamma_0 \cap \Lambda$

$$\log \frac{\det Df^R | T_x\gamma_0}{\det Df^R | T_y\gamma_0} \leq C\beta^{s(f^R(x), f^R(y))}.$$

The Young structure has integrable recurrence times if for some (hence for all) $\gamma \in \Gamma^u$

$$\int_{\gamma\cap\Lambda} R\, dm_\gamma < \infty.$$

In many cases, the unstable disks $\gamma \in \Gamma^u$ are totally contained in Λ. In such case, we say that Λ has a full Young structure.

SRB measures

Theorem 3.1

The return map f^R of a set with a Young structure has a unique ergodic SRB measure ν. Moreover, the densities of its conditionals with respect to Lebesgue on unstable disks are bounded above and below by constants.

Similar to Theorem 2.5, controlling the densities of the measures

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1}(f^R)^j_* m_{\gamma^u}, \quad \text{for some } \gamma \in \Gamma^u.$$

Theorem 3.2

If f has a Young structure Λ with integrable recurrence times, then f has a unique ergodic SRB measure with $\mu(\Lambda) > 0$.

The measure is given by

$$\mu = \frac{1}{\sum_{j=0}^{\infty}\nu\{R > j\}}\sum_{j=0}^{\infty} f^j_*(\nu|\{R > j\}). \tag{24}$$

Proposition 3.3

If $f : M \to M$ has a set Λ with a full Young structure contained in some compact transitive set $\Omega \subset M$, then the support of μ coincides with Ω.

See Section 4 in [Alves 2020] for proofs of the results above.

Decay of Correlations

Let $\mathcal{H}$ be the space of Hölder continuous functions from M to $\mathbb{R}$.

Theorem 3.4 (Young 1998)

Let f have a Young structure Λ with integrable recurrence time R and μ be the unique ergodic SRB measure of f with $\mu(\Lambda) > 0$. If $\gcd(R) = 1$, then

1. *if $m_\gamma\{R > n\} \leq Cn^{-a}$ for some $\gamma \in \Gamma^u$ and $C > 0, a > 1$, then for all $\varphi, \psi \in \mathcal{H}$ there exists $C' > 0$ such that $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' n^{-a+1}$;*
2. *if $m_\gamma\{R > n\} \leq Ce^{-cn^a}$ for some $\gamma \in \Gamma^u$ and constants $C, c > 0$ and $0 < a \leq 1$, then for all $\varphi, \psi \in \mathcal{H}$ there exists $C' > 0$ such that $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' e^{-c' n^a}$.*

Theorem 3.5

Let f have a Young structure Λ with integrable recurrence time R and μ be the unique ergodic SRB measure of f with $\mu(\Lambda) > 0$. If $\gcd(R) = q$, then f^q has $p \leq q$ exact invariant probability measures $\mu_1, \dots, \mu_p$ with $f_*\mu_1 = \mu_2, \dots, f_*\mu_p = \mu_1$ and $\mu = (\mu_1 + \cdots + \mu_p)/p$. Moreover, for all $1 \leq i \leq p$,

1. *if $m_\gamma\{R > n\} \leq Cn^{-a}$ for some $\gamma \in \Gamma^u$, $C > 0$ and $a > 1$, then for all $\varphi, \psi \in \mathcal{H}_\eta$ there is $C' > 0$ such that $\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \leq C' n^{-a+1}$;*
2. *if $m_\gamma\{R > n\} \leq Ce^{-cn^a}$ for some $\gamma \in \Gamma^u$, $C, c > 0$ and $0 < a \leq 1$, then given $\eta > 0$, there is $c' > 0$ such that, for all $\varphi, \psi \in \mathcal{H}_\eta$, there is $C' > 0$ for which $\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \leq C' e^{-c' n^a}$.*

The proof of Theorem 3.4 will be sketched here using a tower extension. Theorem 3.5 can then be deduced as in Corollary 2.15; see Section 4.4.4 in [Alves 2020].

Tower extension

Let $f : M \to M$ have a Young structure Λ with recurrence time $R : \Lambda \to \mathbb{N}$. As before, we define a tower

$$\hat{\Delta} = \{(x, \ell) \colon x \in \Lambda \text{ and } 0 \le \ell < R(x)\},$$

and a tower map $\hat{T} : \hat{\Delta} \to \hat{\Delta}$ as

$$\hat{T}(x, \ell) = \begin{cases} (x, \ell + 1), & \text{if } \ell + 1 < R(x); \\ (f^R(x), 0), & \text{if } \ell + 1 = R(x). \end{cases}$$

The ℓ-level of the tower is

$$\hat{\Delta}_\ell = \{(x, \ell) \in \hat{\Delta}\}.$$

The 0-level of the tower $\hat{\Delta}_0$ is naturally identified with Λ. We have a partition of $\hat{\Delta}_0$ into subsets $\hat{\Delta}_{0,i} = \Lambda_i$. This gives a partition $\{\hat{\Delta}_{\ell,i}\}_i$ on each level ℓ. Collecting all these sets we obtain a partition $\hat{\mathcal{Q}} = \{\hat{\Delta}_{\ell,i}\}_{\ell,i}$ of $\hat{\Delta}$.
Setting

$$\begin{array}{cccc} \pi & : & \hat{\Delta} & \longrightarrow & M \\ & & (x, \ell) & \longmapsto & f^\ell(x) \end{array}$$

we have $f \circ \pi = \pi \circ \hat{T}$.

Theorem 3.6

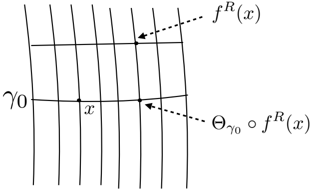
$\hat{T}$ has a unique ergodic SRB measure $\hat{\nu}$. Moreover, $\mu = \pi_* \hat{\nu}$ is the unique ergodic SRB measure of f with $\mu(\Lambda) > 0$.

See Theorem 4.11 in [Alves 2020] for a proof.

Quotient return map

Given $\gamma_0 \in \Gamma^u$, define the quotient map of f^R

$$\begin{array}{cccccc} F & : & \gamma_0 \cap \Lambda & \longrightarrow & \gamma_0 \cap \Lambda \\ & & x & \longmapsto & \Theta_{\gamma_0} \circ f^R(x). \end{array}$$



Proposition 3.7

F is Gibbs-Markov with respect to the m_{γ_0} mod 0 partition $\mathcal{P} = \{\gamma_0 \cap \Lambda_1, \gamma_0 \cap \Lambda_2, \dots\}$ of $\gamma_0 \cap \Lambda$.

See Proposition 4.2 in [Alves 2020] for a proof. Since f^R sends stable disks into stable disks, by (Y_1), it follows that

$$F \circ \Theta_{\gamma_0} = \Theta_{\gamma_0} \circ f^R. \tag{25}$$

Lemma 3.8

If ν is the SRB measure for f^R, then $\nu_0 = (\Theta_{\gamma_0})_* \nu$ is the F-invariant probability measure such that $\nu_0 \ll m_{\gamma_0}$.

See Lemma 4.5 in [Alves 2020] for a proof.

Quotient tower

Consider $\gamma_0 \in \Gamma^u$ and the quotient map

$$F : \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda.$$

Consider the tower map $T : \Delta \to \Delta$ of F with recurrence time R.
Notice that for all $i \geq 1$

$$R|_{\gamma_0 \cap \Lambda_i} = R|_{\Lambda_i} = R_i.$$

Since $\gamma_0 \cap \Lambda \subset \Lambda$, it easily follows that for all $\ell \geq 0$ we have

$$\Delta_\ell \subset \hat{\Delta}_\ell \quad \text{and} \quad T = \hat{T}|_\Delta. \tag{26}$$

Moreover, from (25) we get $T \circ \Theta = \Theta \circ \hat{T}$, where

$$\begin{array}{cccccc} \Theta & : & \hat{\Delta} & \longrightarrow & \Delta \\ & & (x,\ell) & \longmapsto & (\Theta_{\gamma_0}(x),\ell)\,. \end{array} \tag{27}$$

Proposition 3.9

If $\hat{\nu}$ is the ergodic SRB measure of $\hat{T}$, then $\Theta_*\hat{\nu}$ is the unique ergodic T-invariant probability measure absolutely continuous with respect to m_{γ_0}.

See Proposition 4.13 in [Alves 2020] for a proof.

Decay of correlations

We have

$$\pi \circ \hat{T} = f \circ \pi \quad \text{and} \quad \Theta \circ \hat{T} = T \circ \Theta.$$

Given $\varphi, \psi \in \mathcal{H}$, set

$$\hat{\psi} = \psi \circ \pi \quad \text{and} \quad \hat{\varphi} = \varphi \circ \pi.$$

Let

- $\hat{\nu}$ be the unique ergodic SRB measure of $\hat{T}$;
- μ be the unique ergodic SRB measure of f such that $\mu(\Lambda) > 0$;
- ν be the unique ergodic T-invariant measure such that $\nu \ll m_{\gamma_0}$.

$$\begin{tikzcd}
\Delta \arrow[r, "T"] & \Delta & \\
\hat{\Delta} \arrow[u, "\Theta"] \arrow[r, "\hat{T}"] \arrow[d, "\pi"'] & \hat{\Delta} \arrow[u, "\Theta"'] \arrow[d, "\pi"'] \arrow[dr, "\hat{\varphi}"] & \\
M \arrow[r, "f"] & M \arrow[r, "\varphi"] & \mathbb{R}.
\end{tikzcd}$$

By Theorem 3.6 and Proposition 3.9, we have

$$\mu = \pi_* \hat{\nu} \quad \text{and} \quad \nu = \Theta_* \hat{\nu}.$$

By Exercise 2.8,

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathrm{Cor}_{\hat{\nu}}(\hat{\varphi}, \hat{\psi} \circ \hat{T}^n).$$

It is enough to obtain estimates for $\mathrm{Cor}_\nu(\hat{\varphi}, \hat{\psi} \circ \hat{T}^n)$. The idea is to reduce to a problem on the quotient tower $T : \Delta \to \Delta$.

Reducing to the quotient tower

Take an arbitrary $n \geq 1$ and $k \approx n/4$. Set

$$\hat{\mathcal{Q}}_k = \bigvee_{j=0}^{k-1} \hat{T}^{-j} \hat{\mathcal{Q}}. \tag{28}$$

Define the *discretisation* $\varphi_k : \hat{\Delta} \to \mathbb{R}$ of $\hat{\varphi}$, setting for each $Q \in \hat{\mathcal{Q}}_{2k}$

$$\varphi_k|_Q = \inf\{\hat{\varphi} \circ \hat{T}^k(x) : x \in Q\}. \tag{29}$$

Since $Q \in \hat{\mathcal{Q}}_{2k}$ contains full stable disks, φ_k may also be thought of as a function on Δ.

Proposition 3.10

For all $\varphi, \psi \in \mathcal{H}$ and $1 \leq k \leq n$,

$$\begin{aligned} \mathrm{Cor}_{\hat{\nu}}(\hat{\varphi}, \hat{\psi} \circ \hat{T}^n) \leq{} & \mathrm{Cor}_{\nu}(\varphi_k, \psi_k \circ T^n) \\ & + 2\|\varphi\|_0 \|\hat{\psi} \circ \hat{T}^k - \psi_k\|_1 + 2\|\psi\|_0 \|\hat{\varphi} \circ \hat{T}^k - \varphi_k\|_1. \end{aligned}$$

Here, $\| \ \|_1$ denotes the L^1-norm with respect to the probability measure $\hat{\nu}$ on $\hat{\Delta}$.
See Proposition 4.16 in [Alves 2020] for a proof.

We are left to control the correlation term and the L^1-norms on the right hand side of the inequality in Proposition 3.10.

Correlation term

Suppose $\varphi_k \neq 0$ and consider φ_k^* associated with φ_k as in (11). Let λ_k^* be the probability measure on Δ whose density with respect to ν is φ_k^*. It follows from Lemma 2.20 that

$$\mathrm{Cor}_\nu(\varphi_k, \psi_k \circ T^n) \leq 3\|\varphi\|_0\|\psi\|_0 |T_*^n \lambda_k^* - \nu|. \tag{30}$$

Set $\lambda_k = T_*^{2k}\lambda_k^*$ and ϕ_k the density of λ_k with respect to m_{γ_0}. We have

$$\phi_k = \frac{d\lambda_k}{dm_{\gamma_0}} = \frac{dT_*^{2k}\lambda_k^*}{dm_{\gamma_0}} \quad \text{and} \quad \frac{d\lambda_k^*}{dm_{\gamma_0}} = \varphi_k^* \rho, \tag{31}$$

where $\rho = d\nu/dm_{\gamma_0}$. Since $T_*^n\lambda_k^* = T_*^{n-2k}\lambda_k$, it follows from (30) that

$$\mathrm{Cor}_\nu(\varphi_k, \psi_k \circ T^n) \leq 3\|\varphi\|_0\|\psi\|_0 |T_*^{n-2k} \lambda_k - \nu|. \tag{32}$$

Proposition 3.11

There is $C > 0$ such that $\phi_k \in \mathcal{F}_\beta^+(\Delta)$ and $C_{\phi_k}^+ \leq C$, for all $k \geq 1$.

See Proposition 4.18 in [Alves 2020] for a proof.
The conclusion of Proposition 3.11 makes it possible to apply Theorem 2.21† to obtain estimates on the decay of $|T_*^{n-2k}\lambda_k - \nu|$. Note that $k \approx n/4 \implies n - 2k \approx n/2$.

† This is the only place where the assumption $\gcd(R) = 1$ is used.

L^1-norm terms

Set for $x \in \hat{\Delta}$ and $k \geq 1$

$$b_k(x) = \#\{1 \leq j \leq k : \hat{T}^j(x) \in \hat{\Delta}_0\}.$$

Given $x, y \in Q \in \hat{\mathcal{Q}}_{2k}$, we have $b_{2k}(y) = b_{2k}(x)$ and, in particular, $b_k(y) = b_k(x)$.
Taking $z \in Q$ such that $z \in \gamma^u(x) \cap \gamma^s(y)$, we may write

$$\mathrm{dist}(\pi(\widehat{T}^k(x)), \pi(\widehat{T}^k(y))) \leq \mathrm{dist}(\pi(\widehat{T}^k(x)), \pi(\widehat{T}^k(z))) + \mathrm{dist}(\pi(\widehat{T}^k(z)), \pi(\widehat{T}^k(y))).$$

Lemma 3.12

For every $\varphi \in \mathcal{H}$ there are $C > 0$ and $0 < \sigma < 1$ such that for all $k \geq 1$ and $x \in \hat{\Delta}$

$$|\hat{\varphi} \circ \hat{T}^k(x) - \varphi_k(x)| \leq C \left(\sigma^{b_k(x)} + \sigma^{b_k(\hat{T}^k(x))}\right).$$

Since b_k is constant on stable disks, it follows from Proposition 3.9 and (65) that

$$\int \sigma^{b_k} d\hat{\nu} = \int \sigma^{b_k} \circ \Theta d\hat{\nu} = \int \sigma^{b_k} d\Theta_* \hat{\nu} = \int \sigma^{b_k} d\nu.$$

Set $R_k = \sum_{j=0}^{k-1} R \circ F^j$, for each $k \geq 1$.

Proposition 3.13

Given $0 < \sigma < 1$, there exists $C > 0$ such that for all $k \geq 1$ we have

$$\int \sigma^{b_k} d\nu \leq C \sum_{\ell \geq k/3} m_{\gamma_0}\{R \geq \ell\} + Ck \sum_{\ell \geq 1} \sigma^{\ell} m_{\gamma_0} \left\{R_\ell > \frac{k}{3}\right\}.$$

See Section 4.4.1 in in [Alves 2020] for proofs of these results.
They yield the desired estimates in polynomial and (stretched) exponential cases.

Application: Solenoid with intermittency

Now, we consider a diffeomorphism introduced in [Alves and Pinheiro 2008], obtained by replacing in the solenoid map introduced in (1) the expanding map in the base S^1 by a map with a neutral fixed point. Consider an intermittent circle map

$$f\colon S^1 \to S^1$$

as in (19). Recall that f is C^2 on $S^1 \setminus \{0\}$ and depends on a parameter $\alpha > 0$ such that, for x near 0,

$$xf''(x) \approx |x|^\alpha.$$

Consider the solid torus $M = S^1 \times D^2$, where D^2 is the unit disk in $\mathbb{R}^2$. Let $F\colon M \to M$ be defined by

$$F(x,y,z) = \left(f(x), \frac{1}{2}\cos(2\pi x) + \frac{1}{5}y, \frac{1}{2}\sin(2\pi x) + \frac{1}{5}z\right). \tag{33}$$

Note that F is a $C^{1+\alpha}$ diffeomorphism.

F has a compact attractor

$$\Omega = \bigcap_{n \geq 0} F^n(M)$$

It is easily verified that $p_0 = (0, 5/8, 0)$ is a fixed point for F, naturally belonging in Ω, and 1 is an eigenvalue of $DF(p_0)$. Therefore, Ω is not a hyperbolic set for F.

Theorem 3.14

1. *For $\alpha < 1$, the map F has a unique ergodic SRB measure μ. Moreover, μ is exact, the support of μ coincides with Ω, its basin covers m almost all of M and, for all Hölder continuous $\varphi, \psi : M \to \mathbb{R}$,*

$$\mathrm{Cor}_\mu(\varphi, \psi \circ F^n) \lesssim 1/n^{1/\alpha - 1}.$$

2. *For $\alpha \geq 1$, the Dirac measure at p_0 is a physical measure for F and its basin covers m almost all of M.*

Let $\pi : M \to S^1$ be given by $\pi(x, y, z) = x$. Recalling (33), we easily see that

$$f \circ \pi = \pi \circ F. \tag{34}$$

$\alpha \geq 1$

In this case, Theorem 2.30 gives that δ_0 is a physical measure for f and its basin B covers m_1 almost all of S^1. It follows that $\pi^{-1}(B)$ has full m measure in M. For each $n \geq 1$ and $p \in \pi^{-1}(B)$, set

$$\mu_{p,n} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j(p)}.$$

The next result gives the second item of Theorem 3.14.

Lemma 3.15

For each $p \in \pi^{-1}(B)$, any weak accumulation point of $(\mu_{p,n})_n$ is δ_{p_0}.*

Proof. Assume that $(\mu_{p,n_k})_k$ converges to a probability measure μ on M. Since the push-forward π_* is continuous, we have that $\pi_* \mu_{p,n_k} \to \pi_* \mu$, when $k \to \infty$. Using the linearity of π_* and (34), we get

$$\pi_* \mu_{p,n_k} = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \pi_* \delta_{F^j(p)} = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{\pi(F^j(p))} = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{f^j(\pi(p))}.$$

Since $\pi(p) \in B$, this last sequence converges to δ_0, when $k \to \infty$. Therefore, $\pi_* \mu = \delta_0$ and so μ is an F-invariant measure with its support contained in the stable disk $\gamma^s(p_0)$. Since F is a contraction on $\gamma^s(p_0)$, with p_0 its unique fixed point, it must be $\mu = \delta_{p_0}$. ■

$0 < \alpha < 1$

Consider an interval $I_0 \subset S^1$ such that

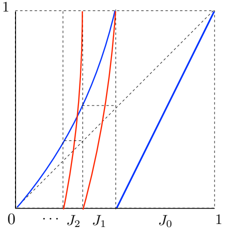
$$I_0 = J_{n_0} \cup J_{n_0+1}$$

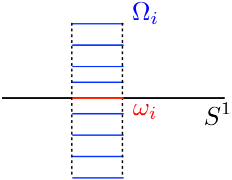
and $f^S : I_0 \to I_0$ the induced Gibbs-Markov map given by Proposition 2.34. Let $\mathcal{Q}$ be partition of I_0 associated with f^S. Considering the elements of $\mathcal{Q}$ listed as $\omega_1, \omega_2, \omega_3 \dots$, set for $i \geq 1$

$$\Omega_i = \{p \in \Omega \colon \pi(p) \in \omega_i\} \quad \text{and} \quad S_i = S(\omega_i).$$

Given $p \in M$, define the cone

$$\mathcal{C}^{cu}_p = \left\{(v_1, v_2, v_3) \in T_pM \colon 15v_1^2 \geq v_2^2 + v_3^2\right\}.$$





Lemma 3.16

For all $p \in \Omega$ we have $DF(p)\mathcal{C}^{cu}_p \subset \mathcal{C}^{cu}_{F(p)}$ and the angle between any two nonzero vectors in $DF(F^{-n}(p))\mathcal{C}^{cu}_{F^{-n}(p)}$ converges to zero as n goes to infinity. Moreover,

1. $\|DF(p)v\| \geq \|v\|/4$, *for all $p \in M$ and $v \in \mathcal{C}^{cu}_p$;*
2. $\|DF^{S_i}(p)v\| \geq 5\|v\|/4$, *for all $p \in \Omega_i$, $v \in \mathcal{C}^{cu}_p$ and $i \geq 1$.*

See Lemma 4.29 in [Alves 2020] for a proof.

Partially hyperbolicity

Corollary 3.17

There is a DF-invariant splitting $T_\Omega M = E^{cu} \oplus E^s$ *such that*

1. $\|DF(p)v\| \leq \|v\|/5$, *for all* $p \in \Omega$ *and* $v \in E^s_p$;
2. $\|DF(p)v\| \geq \|v\|/4$, *for all* $p \in \Omega$ *and* $v \in E^{cu}_p$.

See Corollary 4.30 in [Alves 2020] for a proof.
Thus, Ω is a *partially hyperbolic set* for F. It follows from Theorem IV.1 in [Shub 1987] that each $p \in \Omega$ has an embedded centre-unstable C^1 disk $W^{cu}_\varepsilon(p)$ such that $T_p W^{cu}_\varepsilon(p) = E^{cu}_p$ and

$$F(W^{cu}_\varepsilon(p)) \cap B_\varepsilon(F(p)) \subset W^{cu}_\varepsilon(F(p)), \tag{35}$$

where $B_\varepsilon(F(p))$ denotes the ball of radius $\varepsilon > 0$ around the point $F(p) \in \Omega$. In addition, the disks $W^{cu}_\varepsilon(p)$ depend continuously on $p \in \Omega$ in the C^1 topology. Note that these centre-unstable disks are not unique, in general.

Positive Lyapunov exponent

Let ν be the ergodic SRB measure for f provided by Theorem 2.30. Using Bowen's argument to lift measures we obtain

Lemma 3.18

There exists an F-invariant Borel probability measure $\hat{\nu}$ on M such that $\pi_*\hat{\nu} = \nu$. Moreover, the support of $\hat{\nu}$ coincides with Ω.

See Lemma 4.31 in [Alves 2020] for a proof.

Lemma 3.19

There is a set $A \subset \Omega$ with $\hat{\nu}(A) = 1$ such that F has a positive Lyapunov exponent in the direction of E^{cu}_p, for all $p \in A$. Moreover, the local unstable disk $\gamma^u(p)$ through $p \in A$ is contained in $W^{cu}_\varepsilon(p) \cap \Omega$ and $T_p\gamma^u(p) = E^{cu}_p$.

See Lemma 4.32 in [Alves 2020] for a proof. Since the support of μ coincides with Ω, this lemma provides us with a dense set of points in Ω for which the conclusion holds. Recall that $W^{cu}_\varepsilon(p)$ is the centre-unstable disk through a point $p \in \Omega$ and satisfies (35).

Young structure

Now, we introduce a compact set Λ with a Young structure given by continuous families of C^1 disks Γ^s and Γ^u and deduce the first item of Theorem 3.14. We define Γ^u by mean of an inductive process that we describe below. First, take n_0 sufficiently large so that, for any $p \in \Omega$ with $\pi(p)$ belonging in $I_0 = J_{n_0} \cup J_{n_0+1}$, we have

$$\pi(W^{cu}_\varepsilon(p)) \supset I_0.$$

Lemma 3.20

There exists an unstable disk $\gamma_0 \subset \Omega$ such that π projects γ_0 diffeomorphically to I_0.

Proof. By Lemma 3.19, there is $p_0 \in \Omega$ s.t. $\gamma^u(p_0) \subset W^{cu}_\varepsilon(p_0) \cap \Omega$ and $T_{p_0}\gamma^u(p_0) = E^{cu}_{p_0}$. Choosing $\gamma^u(p_0)$ small enough, we may have

$$T_p\gamma^u(p_0) \subset \mathcal{C}^{cu}_p, \quad \text{for all } p \in \gamma^u(p_0). \tag{36}$$

Hence, π maps $\gamma^u(p_0)$ diffeomorphically onto its image. In particular, $\pi(\gamma^{cu}(p_0))$ contains some open interval in S^1. Since f is topologically conjugate to $x \mapsto dx$ (mod 1), there is an interval $I \subset \pi(\gamma^u(p_0))$ and $n_1 \geq 1$ such that f^{n_1} maps I diffeomorphically to I_0. Take $\gamma \subset \gamma^u(p_0)$ such that π projects γ diffeomorphically to I, and set $\gamma_0 = F^{n_1}(\gamma)$. Notice that γ_0 is an unstable disk contained in Ω and $\pi(\gamma_0) = I_0$. It follows from (36) and Lemma 3.16 that

$$T_p\gamma_0 \subset \mathcal{C}^{cu}_p, \quad \text{for all } p \in \gamma_0. \tag{37}$$

This implies that π projects γ_0 diffeomorphically to I_0. ∎

The set Λ

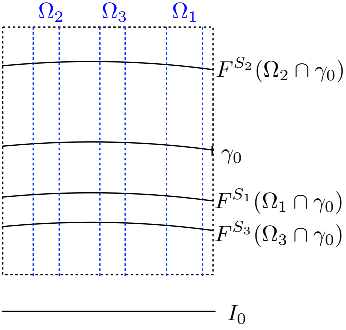
This disk γ_0 given by the previous lemma will be used as the first element of the inductive construction leading to Γ^u. For that, we use some properties of $f \colon S^1 \to S^1$. Recall that, by Proposition 2.34, the map f has an induced Gibbs-Markov map $f^S \colon I_0 \to I_0$.

Consider the sets $(\Omega_i)_i$ in Ω and the respective sequence of times $(S_i)_i$ as in (72). We define inductively families of unstable disks $\Gamma_0, \Gamma_1, \dots$. Set

$$\Gamma_0 = \{\gamma_0\}\,.$$

Assuming Γ_{n-1} is defined for some $n \geq 1$, set

$$\Gamma_n = \bigcup_{i\geq 1}\left\{F^{S_i}(\Omega_i \cap \gamma_{n-1}) :\ \gamma_{n-1} \in \Gamma_{n-1}\right\}.$$



Observe that π maps each $\gamma_n \in \Gamma_n$ diffeomorphically to I_0 and γ_n is the forward iterate of a subset of γ_0. Since the union of all disks in the families Γ_n with $n \geq 0$ is not necessarily a compact set, we still need to take the accumulation points of that union. Set

$$\Lambda = \overline{\bigcup_{n\geq 0}\ \bigcup_{\gamma_n\in\Gamma_n} \gamma_n}.$$

Lemma 3.21

Λ is a union of centre-unstable disks.

Proof. Given any $q \in \Lambda$, there are $n_1 < n_2 < \cdots$, disks $\gamma_{n_k} \in \Gamma_{n_k}$ and points $q_k \in \gamma_{n_k}$ converging to q, when $k \to \infty$. As we have seen above, for each $k \geq 1$, there are $p_k \in \Omega$ and a centre-unstable disk $W^{cu}_\varepsilon(p_k) \supset \gamma_{n_k}$. Taking a subsequence, if necessary, we may assume that $(p_k)_k$ converges to some point $p \in \Omega$. Since the centre-unstable disks $W^{cu}_\varepsilon(p_k)$ depend continuously on $p_k \in \Omega$ in the C^1 topology, then the disks γ_{j_k} converge in the C^1 topology to a disk $\gamma_\infty \subset W^{cu}_\varepsilon(p)$ containing q, when $k \to \infty$. ∎

We define Γ^u as the set of all these disks γ_∞. Note that

$$\Lambda = \bigcup_{\gamma \in \Gamma^u} \gamma.$$

Finally, set

$$\Gamma^s = \left\{ \{x\} \times D^2 : x \in S^1 \right\}. \tag{38}$$

Proposition 3.22

Λ has a full Young structure with recurrence time S. Moreover, $\gcd(S) = 1$ *and* $m_{\gamma_0}\{S > n\} \lesssim 1/n^{1/\alpha}$.

See Proposition 4.33 in [Alves 2020] for a proof. Using Proposition 3.22, Theorem 3.2 and Theorem 3.4, we obtain an exact SRB measure μ such that, for every Hölder continuous $\varphi, \psi : M \to \mathbb{R}$,

$$\mathrm{Cor}_\mu(\varphi, \psi \circ F^n) \lesssim 1/n^{1/\alpha - 1}.$$

Since F is topologically conjugate to the solenoid map, then Ω is a transitive set for F. By Proposition 3.3, the support of μ coincides with Ω.

Lemma 3.23

m almost every point in M belongs in the basin of μ.

Proof. Proposition 1.2 gives that μ almost every point in M belongs in $\mathcal{B}(\mu)$. From Theorem 3.1 and the formula for μ in (24) the density $d\mu_\gamma/dm_\gamma$ is bounded from below by some uniform constant, for almost all $\gamma \in \Gamma^u$. Hence, $\exists \gamma_0 \in \Gamma^u$ such that m_{γ_0} almost every point in γ_0 belongs in $\mathcal{B}(\mu)$. Since f is topologically conjugate to $x \mapsto dx$ (mod 1), there is $n_0 \geq 1$ such that $f^{n_0}(\pi(\gamma_0)) = S^1$. Therefore

$$\pi(F^{n_0}(\gamma_0)) = f^{n_0}(\pi(\gamma_0)) = S^1. \tag{39}$$

Since $\mathcal{B}(\mu)$ is invariant, $m_{F^{n_0}(\gamma_0)}$ almost every point in the curve $F^{n_0}(\gamma_0)$ belongs in $\mathcal{B}(\mu)$. It follows from (39) and the fact that $M = S^1 \times D^2$ is foliated by stable disks that m almost every point in M belongs in $\mathcal{B}(\mu)$. ∎

Inducing schemes

A general framework

Let $f : M \to M$ be a map of a finite dimensional Riemannian manifold M, and Σ be a submanifold of M (possibly equal to M, possibly with a boundary). Consider dist the distance on Σ and m the Lebesgue measure on the Borel sets of Σ, both induced by the Riemannian metric. For each $n \geq 0$, denote

$$\mathrm{dist}_n = \mathrm{dist}_{f^n(\Sigma)} \quad \text{and} \quad m_n = m_{f^n(\Sigma)},$$

where $\mathrm{dist}_{f^n(\Sigma)}$ is the distance in the submanifold $f^n(\Sigma)$ and $m_{f^n(\Sigma)}$ is the Lebesgue measure on the Borel sets of $f^n(\Sigma)$, both induced by the Riemannian metric on M. Assume that there exists a disk $\Delta_0 \subset \Sigma$ with the same dimension of Σ for which conditions (I_1)-(I_3) below hold:

(I_1) There is a sequence $(H_n)_n$ of compact sets in Δ_0 such that m_0 almost every point in Δ_0 belongs in infinitely many H_n's.

(I_2) There is $\delta_1 > 0$ such that every $x \in H_n$ has a neighbourhood $V_n(x)$ of x in Σ that f^n maps diffeomorphically to a disk of radius δ_1 around $f^n(x)$. Moreover, there are $C_0, \eta > 0$ and $0 < \sigma < 1$ such that, for all $V_n(x)$ and $y, z \in V_n(x)$,

- $\mathrm{dist}_{n-k}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^k \, \mathrm{dist}_n(f^n(y), f^n(z))$, for all $1 \leq k \leq n$.
- $\log \dfrac{\det Df^n | T_y\Sigma}{\det Df^n | T_z\Sigma} \leq C_0 \, \mathrm{dist}_n \left(f^n(y), f^n(z)\right)^\eta .$

(I_3) There exist $L, \delta_0 > 0$ such that, for each $x \in H_n$, there are $0 \le \ell \le L$ and

$$\omega_{n,\ell} \subset \widetilde{\omega}_{n,\ell} \subset V_n(x),$$

so that $f^{n+\ell}$ maps $\omega_{n,\ell}$ (resp. $\widetilde{\omega}_{n,\ell}$) diffeomorphically to a disk of radius δ_0 (resp. $2\delta_0$). Moreover, there are $C_1, \eta > 0$ such that, for all $\widetilde{\omega}_{n,\ell}$ and $y, z \in f^n(\widetilde{\omega}_{n,\ell})$,

- $\frac{1}{C_1} \operatorname{dist}_{n+j}(f^j(y), f^j(z)) \le \operatorname{dist}_{n+\ell}(f^\ell(y), f^\ell(z)) \le C_1 d_n(y,z)$, for all $0 \le j \le \ell$;
- $\log \frac{\det Df^\ell | T_y\Sigma}{\det Df^\ell | T_z\Sigma} \le C_1 \operatorname{dist}_{n+\ell}(f^\ell(y), f^\ell(z))^\eta$.

Note that $V_n(x)$ is a neighbourhood of the point x in Σ, but not necessarily $x \in \widetilde{\omega}_{n,\ell}$. For simplicity, we will often denote the sets $\omega_{n,\ell}$ as ω and $\widetilde{\omega}_{n,\ell}$ as $\widetilde{\omega}$. In such case, we also set

$$V_n(\omega) = V_n(x) \quad \text{and} \quad \ell_\omega = \ell. \tag{40}$$

In the applications, $V_n(x)$ will be defined using

- expanding properties for points in H_n;
- transitivity to assure that forward iterates of a part of $V_n(x)$ returns onto
  - Δ_0 (endomorphisms);
  - an unstable disk of a Young structure containing Δ_0 (diffeomorphisms).

In many cases, typical points in Δ_0 belong in $(H_n)_n$ with

- positive frequency: there is $0 < \theta \leq 1$ such that for m_0 almost every $x \in \Delta_0$

$$\limsup_{n\to\infty} \frac{1}{n} \# \{1 \leq j \leq n : x \in H_j\} \geq \theta. \tag{41}$$

- strong positive frequency: there is $0 < \theta \leq 1$ such that for m_0 almost every $x \in \Delta_0$

$$\liminf_{n\to\infty} \frac{1}{n} \# \{1 \leq j \leq n : x \in H_j\} \geq \theta. \tag{42}$$

In the latter case, set $h_\theta(x) = \min\{n_0 : \# \{1 \leq j \leq n : x \in H_j\} > \theta n, \quad \forall n \geq n_0\}$.

Theorem 4.1

If (I_1)-(I_3) hold, then there is an m_0 mod 0 partition $\mathcal{P}$ of Δ_0 into domains $\omega_{n,\ell}$ as in (I_3). Moreover, setting $R(x) = n + \ell$ for each $x \in \omega_{n,\ell} \in \mathcal{P}$,

1. *there are $C > 0$ and an arbitrarily small $0 < \beta < 1$ such that, for all $x, y \in \omega \in \mathcal{P}$,*
   1. $\mathrm{dist}_0(x, y) \leq \beta \, \mathrm{dist}_R(f^R(x), f^R(y))$;
   2. $\mathrm{dist}_j(f^j(x), f^j(y)) \leq C \, \mathrm{dist}_R(f^R(x), f^R(y))$, *for all* $0 \leq j \leq R$;
   3. $\log \dfrac{\det Df^R | T_x\Sigma}{\det Df^R | T_y\Sigma} \leq C \, \mathrm{dist}_R(f^R(x), f^R(y))^\eta$;
2. *there are $S_1, S_2, \dots \subset \Delta_0$ with $\sum_{n\geq 1} m_0(S_n) < \infty$ such that, for all $n \geq 1$,*
$$H_n \cap \{R > L + n\} \subset S_n;$$
3. *if typical points belong in $(H_n)_n$ with strong positive frequency, then there are $E_1, E_2, \dots \subset \Delta_0$ with $m_0(E_n) \to 0$ exponentially fast with n such that, for all $n \geq 1$,*
$$\{R > n + L\} \subset \{h_\theta > n\} \cup E_n.$$

Inductive construction

Now, we describe an inductive process leading to an m_0 mod 0 partition of the disk Δ_0 under the assumptions of Theorem 4.1. Our approach is closely related to constructions performed in [Alves, Dias, and Luzzatto 2013; Alves, Dias, Luzzatto, and Pinheiro 2017; Alves and Li 2015; Pinheiro 2006], based on ideas from [Gouëzel 2006].

We will define inductively sequences $(\mathcal{P}_n)_n$, $(\Delta_n)_n$ and $(S_n)_n$, using the objects given by (I_1)-(I_3), in such a way that

- $\mathcal{P}_n$ is the family of elements of the partition constructed at step n;
- Δ_n is the set of points which do not belong to any element of the partition defined up to time n;
- S_n contains points in H_n not taken by elements of $\mathcal{P}_k$'s constructed until time n.

A key point in our argument is the conclusion of Lemma 4.2 below, which asserts that every point in H_n either belongs to an element of the partition constructed until that moment or to an S_n.

First step

We start our inductive process at time N_0, for some large N_0. Since H_{N_0} is a compact set, there is a finite set $F_{N_0} \subset H_{N_0}$ such that

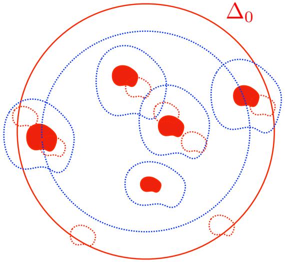
$$H_{N_0} \subset \bigcup_{x \in F_{N_0}} V_{N_0}(x).$$

Consider $x_1, \ldots, x_{j_{N_0}} \in F_{N_0}$ and, for each $1 \le i \le j_{N_0}$, a set $\omega_{N_0,\ell_i} \subset V_{N_0}(x_i)$ as in (I_3), such that $\mathcal{P}_{N_0} = \{\omega_{N_0,\ell_1}, \ldots, \omega_{N_0,\ell_{j_{N_0}}}\}$ is a maximal family of pairwise disjoint sets contained in Δ_0. For each $\omega_{N_0,\ell} \in \mathcal{P}_{N_0}$, set

$$S_n(\omega) = \left\{ y \in \widetilde{\omega}_{N_0,\ell} : 0 < \mathrm{dist}_{N_0+\ell}(f^{N_0+\ell}(y), f^{N_0+\ell}(\omega)) \le 2\delta_1 \sigma^{N_0} \right\}.$$

and for $\Delta_0^c = \Sigma \setminus \Delta_0$

$$S_{N_0}(\Delta_0^c) = \left\{ x \in \Delta_0 : \mathrm{dist}(x, \partial\Delta_0) \right)$$



Finally, set

$$S_{N_0} = \bigcup_{\omega \in \mathcal{P}_{N_0}} S_{N_0}(\omega) \cup S_{N_0}(\Delta_0^c)$$

and

$$\Delta_{N_0} = \Delta_0 \setminus \bigcup_{\omega \in \mathcal{P}_{N_0}} \omega.$$

For
definiteness, set $\Delta_n = S_n = \Delta_0$ for each $1 \le n < N_0$.

General step

Given $n > N_0$, assume that $\mathcal{P}_k$, Δ_k and S_k have already been defined for all k with $N_0 \leq k \leq n-1$. Let F_n be a finite subset of the compact set H_n such that

$$H_n \subset \bigcup_{x \in F_n} V_n(x). \tag{43}$$

Consider $x_1, \ldots, x_{j_n} \in F_n$ and, for each $1 \leq i \leq j_n$, a domain $\omega_{n,\ell_i} \subset V_n(x_i)$ as in (I_3) for which $\mathcal{P}_n = \left\{\omega_{n,\ell_1}, \ldots, \omega_{n,\ell_{j_n}}\right\}$ is a maximal family of pairwise disjoint sets contained in Δ_{n-1}, such that for each $1 \leq i \leq j_n$

$$\omega_{n,\ell_i} \cap \left(\bigcup_{k=N_0}^{n-1} \bigcup_{\omega \in \mathcal{P}_k} \omega \right) = \emptyset. \tag{44}$$

The sets in $\mathcal{P}_n$ are the elements of the partition $\mathcal{P}$ obtained in the n-th step of the construction. Set

$$\Delta_n = \Delta_0 \setminus \bigcup_{k=N_0}^{n} \bigcup_{\omega \in \mathcal{P}_k} \omega. \tag{45}$$

Given $\omega_{k,\ell} \in \mathcal{P}_k$, for some $N_0 \le k \le n$, set

$$S_n(\omega) = \left\{ y \in \widetilde{\omega} : 0 < \mathrm{dist}_{k+\ell}(f^{k+\ell}(y), f^{k+\ell}(\omega)) \le 2\delta_1 \sigma^{n-k} \right\}. \tag{46}$$

Set also

$$S_n(\Delta_0^c) = \{ x \in \Delta_0 : \mathrm{dist}(x, \partial\Delta_0) < \delta_1 \sigma^n \}.$$

and

$$S_n = \bigcup_{k=N_0}^{n} \bigcup_{\omega \in \mathcal{P}_k} S_n(\omega) \cup S_n(\Delta_0^c).$$

Finally, set

$$\mathcal{P} = \bigcup_{n \ge N_0} \mathcal{P}_n$$

By construction, the elements in $\mathcal{P}$ are pairwise disjoint and contained in Δ_0. However, there is still no evidence that the union of these elements covers a full m_0 measure subset of Δ_0.

Metric estimates

Lemma 4.2

$H_n \cap \Delta_n \subset S_n$ *for all* n.

Lemma 4.3

There exists $C > 0$ *such that for all* $n \geq k \geq N_0$ *and* $\omega \in \mathcal{P}_k$ *we have*

$$m_0\left(S_n(\omega)\right) \leq C\sigma^{n-k} m_0(\omega).$$

Lemma 4.4

$$\sum_{n=N_0}^{\infty} m_0\left(S_n\right) < \infty.$$

Corollary 4.5

$\mathcal{P}$ *is an* m_0 *mod 0 partition of* Δ_0.

See Section 5.2 in [Alves 2020] for proofs of these results.

We have constructed an m mod 0 partition $\mathcal{P} = \bigcup_{n \geq N_0} \mathcal{P}_n$ of the disk Δ_0, where each element of $\mathcal{P}_n$ is a domain $\omega_{n,\ell}$ related to some point in H_n as in (I_3). Set

$$R(x) = n + \ell \tag{47}$$

for each $n \geq N_0$ and $x \in \omega_{n,\ell} \in \mathcal{P}_n$. For all $n \geq N_0$, we have

$$H_n \cap \{R > n + L\} \subset S_n. \tag{48}$$

Indeed, if $R(x) > n + L$ for some $x \in H_n$, then by (47) we necessarily have $x \in \Delta_n$, since $\ell \leq L$. It follows from Proposition 4.2 that $x \in S_n$, and therefore (48) holds. Finally,

- the conclusions of first item of Theorem 4.1 follow from the conditions in (I_2) and (I_3), provided N_0 is sufficiently large;
- the second item of Theorem 4.1 is given by Lemma 4.4 and (48);
- for a proof of the third item, see Subsection 5.3.2 in [Alves 2020].

Integrability of the recurrence times

Theorem 4.1 will be useful to build

(a) **Gibbs-Markov maps**: we choose each $\omega \in \mathcal{P}$ in such a way that $f^R(\omega) = \Delta_0$ for all $\omega \in \mathcal{P}$, thus obtaining from Theorem 4.1 that $f^R : \Delta_0 \to \Delta_0$ is a Gibbs-Markov map.

(b) **Young structures**: we take the set Δ_0 in a family of unstable disks Γ^u and each $f^R(\omega)$ an unstable disk in Γ^u. Then, using a quotient map as in (63) with domain Δ_0, we have again a Gibbs-Markov map $F : \Delta_0 \to \Delta_0$.

Set for each $k \geq 1$

$$R_k = \sum_{j=0}^{k-1} R \circ F^j \quad \text{and} \quad R_0 = 0.$$

We say that a sequence $(H_n^*)_n$ of sets in Δ_0 is F-concatenated in $(H_n)_n$ if

$$x \in H_n^* \implies F^i(x) \in H_{n-R_i}, \tag{49}$$

whenever $R_i(x) \leq n < R_{i+1}(x)$, for some $i \geq 0$.[†]

[†]In case (a), we will actually take $H_n^* = H_n$, for all $n \geq 1$. The possibility of having this new sequence $(H_n^*)_n$ will be particularly useful in case (b).

Proposition 4.6

Let $F : \Delta_0 \to \Delta_0$ be a Gibbs-Markov map with respect to a partition $\mathcal{P}$ and $R : \Delta_0 \to \mathbb{N}$ constant in the elements of $\mathcal{P}$. Assume that there exist

1. *a sequence $(H_n^*)_n$ such that typical points in Δ_0 belong in $(H_n^*)_n$ with positive frequency and $(H_n^*)_n$ is F-concatenated in $(H_n)_n$;*
2. *a sequence $(S_n)_n$ such that $\sum_n m_0(S_n) < \infty$ and $L \in \mathbb{N}$ such that $H_n \cap \{R > n + L\} \subset S_n$, for all n.*

Then R is integrable with respect to m_0.

Proof. Consider the ergodic F-invariant probability measure $\nu \ll m_0$ given by Theorem 2.5. Since $d\nu/dm_0$ is bounded above and below by positive constants, it is enough to check the integrability of R with respect to ν. Assume by contradiction that $R \notin L^1(\nu)$. Since R is a positive function, it follows from Birkhoff Ergodic Theorem that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} R(F^i(x)) \to \int R d\nu = \infty, \tag{50}$$

for ν almost every $x \in \Delta_0$. Since $\sum_n m_0(S_n) < \infty$, it follows from Borel-Cantelli Lemma that ν almost every $x \in \Delta_0$ belongs in only a finite number of sets S_n.

Set for $x \in \Delta_0$

$$s(x) = \#\left\{n \geq 1 : x \in S_n\right\}.$$

Using that $d\nu/dm_0$ is bounded above by a positive constant and Birkhoff Ergodic Theorem, we have for ν almost every $x \in \Delta_0$

$$\frac{1}{k}\sum_{i=0}^{k-1} s(F^i(x)) \to \int s d\nu = \sum_n \nu(S_n) < \infty. \tag{51}$$

Exercise 4.7

Show that

$$(H_n^*)_n \textit{ is } F\textit{-concatenated in } (H_n)_n \implies \#\left\{R_i \leq j < R_{i+1} : x \in H_j^*\right\} \leq 1 + s(F^i(x)).$$

Given $n \geq 1$, set $r(n) = \min\{R_i : R_i > n\}$. For each $n \geq 1$, we have

$$\#\{1 \leq j \leq n : x \in H_j^*\} \leq \sum_{i=0}^{r(n)} (1 + s(F^i(x))) \leq r(n) + \sum_{i=0}^{r(n)} s(F^i(x)).$$

Therefore,

$$\frac{1}{n}\#\left\{j \leq n : x \in H_j^*\right\} \leq \frac{r(n)}{n}\left(1 + \frac{1}{r(n)}\sum_{i=0}^{r(n)} s(F^i(x))\right). \tag{52}$$

Observe that if $r(n) = k$, then by definition we have $R_{k-1} \leq n < R_k$. Hence,

$$\frac{R_{k-1}}{k} \leq \frac{n}{r(n)} < \frac{R_k}{k} = \frac{R_k}{k+1}\left(1 + \frac{1}{k}\right),$$

which together with (50) gives

$$\lim_{n\to\infty} \frac{n}{r(n)} = \lim_{k\to\infty} \frac{R_k}{k} = \lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} R(F^i(x)) = \infty. \tag{53}$$

It follows from (51), (52) and (53) that

$$\lim_{n\to\infty} \frac{1}{n}\#\left\{1 \leq j \leq n : x \in H_j^*\right\} = \lim_{n\to\infty} \frac{r(n)}{n} = 0,$$

which clearly contradicts the fact that $(H_n^*)_n$ is a frequent sequence.

Nondegenerate sets

Consider $f\colon M \to M$ a $C^{1+\eta}$ local diffeomorphism out of a set $\mathcal{C} \subset M$, possibly $\mathcal{C} = \emptyset$. In practice, $\mathcal{C}$ can be a set of points where the derivative of f is not an isomorphism (*critical set*), a set of points where the derivative simply does not exist (*singular set*), or even the boundary of M. We say that $\mathcal{C}$ is a nondegenerate set if $m(\mathcal{C}) = 0$ and (C_1)-(C_3) hold:

(C_1) If A is Borel set, then $f(A)$ and $f^{-1}(A)$ are both Borel sets and

$$m(A) = 0 \implies m(f^{-1}(A)) = 0 \quad \text{and} \quad m(f(A)) = 0.$$

We also assume that there exist $K, \alpha > 0$ such that

(C_2) $\|Df(x)\| \le K \operatorname{dist}(x, \mathcal{C})^{-\alpha}$, for every $x \in M \setminus \mathcal{C}$;

(C_3) for every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(y, x) < \operatorname{dist}(x, \mathcal{C})/2$ we have

- $\left| \log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\| \right| \le \dfrac{K}{\operatorname{dist}(x, \mathcal{C})^{\alpha}} \operatorname{dist}(x, y)^{\eta}$;
- $\left| \log |\det Df(x)| - \log |\det Df(y)| \right| \le \dfrac{K}{\operatorname{dist}(x, \mathcal{C})^{\alpha}} \operatorname{dist}(x, y)^{\eta}$.

Nonuniform expansion and slow recurrence

We say that f is nonuniformly expanding (NUE) on H if there exists $\lambda > 0$ such that, for some choice of a Riemannian metric on M and all $x \in H$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < -\lambda. \tag{54}$$

We say that f has slow recurrence to $\mathcal{C}$ (SR$_\mathcal{C}$) on H if, for every $\varepsilon > 0$, there is $r > 0$ such that for all $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_r(f^j(x), \mathcal{C}) < \varepsilon, \tag{55}$$

with $\operatorname{dist}_r(x, \mathcal{C})$ is the *truncated distance*, defined for $x \in M \setminus \mathcal{C}$ as

$$\operatorname{dist}_r(x, \mathcal{C}) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, \mathcal{C}) \geq r; \\ \operatorname{dist}(x, \mathcal{C}), & \text{otherwise.} \end{cases}$$

Hyperbolic times

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C} \subset M$. Fix $b > 0$ small, only depending on the constants in the definition of a nondegenerate set. Given $\sigma \in (0,1)$ and $r > 0$, we say that n is a (σ, r)-hyperbolic time for $x \in M$ if

$$\prod_{j=k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^{n-k} \quad \text{and} \quad \mathrm{dist}_r(f^k(x), \mathcal{C}) \ge \sigma^{b(n-k)}, \quad \forall 0 \le k < n.$$

Proposition 4.8 (Alves, Bonatti, and Viana 2000)

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$. Given $\lambda > 0$, there exist (ε_1, r_1), (ε_2, r_2) and $\theta > 0$ such that if, for $x \in M$ and $i = 1, 2$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < -\lambda \quad \textit{and} \quad \frac{1}{n}\sum_{j=0}^{n-1} -\log \mathrm{dist}_{r_i}(f^j(x), \mathcal{C}) < \varepsilon_i,$$

then x has $(e^{-\lambda/4}, r_2)$-hyperbolic times $1 \le n_1 < \cdots < n_\ell \le n$ with $\ell \ge \theta n$.

See Proposition 6.3 in [Alves 2020] for a proof, which is based on the next result.

Lemma 4.9 (Pliss)

Let $0 < c \le A$ and $a_1, \ldots, a_n \le A$ be such $\sum_{j=1}^{n} a_j \ge cn$. There are $\ell \ge cn/A$ and $1 \le n_1 < \cdots < n_\ell \le n$ such that $\sum_{j=k}^{n_i} a_j \ge 0$, for all $1 \le k \le n_i$ and $1 \le i \le \ell$.

Proposition 4.10 (Alves, Bonatti, and Viana 2000)

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$. Given $\sigma \in (0,1)$ and $r > 0$, there exists $\delta_1 > 0$ such that for all $x \in M$ with a (σ, r)-hyperbolic time n there is a neighbourhood $V_n(x)$ of x such that

1. f^n *maps* $V_n(x)$ *diffeomorphically to* $B_{\delta_1}(f^n(x))$;
2. *for all* $y, z \in V_n(x)$,
    1. $\operatorname{dist}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}(f^n(y), f^n(z))$, *for all* $1 \le k \le n$;
    2. $\log \dfrac{|\det Df^n(y)|}{|\det Df^n(z)|} \le C \operatorname{dist}(f^n(y), f^n(z))^\eta$.

See Proposition 6.6 in [Alves 2020] for a proof.

Theorem 4.11

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$.
If f is NUE and has $SR_{\mathcal{C}}$ on $H \subset M$ with $m(H) > 0$, then there are transitive invariant sets $\Omega_1, \ldots, \Omega_\ell \subset M$ such that, for m almost every $x \in H$, there is $1 \le j \le \ell$ such that $\omega(x) = \Omega_j$. Moreover, for each $1 \le j \le \ell$, there is a ball $\Sigma_j \subset \Omega_j$ such that f is NUE and has $SR_{\mathcal{C}}$ on m almost all of Σ_j.

The proof uses ideas from [Pinheiro 2011]. See Theorem 6.9 in [Alves 2020].

Transitive sets

Let Ω be a transitive set and $\Sigma \subset \Omega$ a ball as in the previous theorem. Consider $H \subset \Sigma$ with $m(H) = m(\Sigma)$ such that f is NUE and has $\mathrm{SR}_{\mathcal{C}}$ on H. By Proposition 4.8 there are $\sigma, r > 0$ such that every point in H has infinitely many (σ, r)-hyperbolic times. Setting for $n \geq 1$

$$H_n = \{x \in H\colon n \text{ is a } (\sigma, r)\text{-hyperbolic time for } x\}, \tag{56}$$

Proposition 4.8 also gives $\theta > 0$ such that for all $x \in H$

$$\limsup_{n\to\infty} \frac{1}{n} \# \{1 \leq j \leq n : x \in H_j\} \geq \theta, \tag{57}$$

In particular, m almost every point in Σ belongs in infinitely many H_n's, thus ensuring the validity of (I_1) in any ball $\Delta_0 \subset \Sigma$. Taking the sets $V_n(x)$ and the constant $\delta_1 > 0$ as in Proposition 4.10 we obtain (I_2). Assuming Ω a transitive set, we find $L > 0$, a ball $\Delta_0 \subset \Sigma$ and sets

$$\omega_{n,\ell} \subset \widetilde{\omega}_{n,\ell} \subset V_n(x), \quad \text{with } \ell \leq L$$

as in (I_3) for which $f^{n+\ell}(\omega_{n,\ell}) = \Delta_0$.

Gibbs-Markov structure and SRB measure

Setting $R|_{\omega_{n,\ell}} = n + \ell$ as in (47), we obtain a Gibbs-Markov map

$$f^R : \Delta_0 \to \Delta_0$$

By the definition of hyperbolic time, for all $0 \le j < n$

$$x \in H_n \implies f^j(x) \in H_{n-j}, \tag{58}$$

and so $(H_n)_n$ is f^R-concatenated in $(H_n)_n$. Moreover, (57) implies that typical points in Δ_0 have positive frequency in $(H_n)_n$. Using Theorem 4.1 and Proposition 4.6, we obtain:

Theorem 4.12

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$ and $\Omega \subset M$ a transitive set. If f is NUE and has $SR_{\mathcal{C}}$ for m almost every point in a ball $\Sigma \subset \Omega$, then f has a Gibbs-Markov induced map with integrable recurrence times defined on some ball $\Delta_0 \subset \Sigma$.

A first version of this result was proved in [Pinheiro 2006] under global assumptions of NUE and has $SR_{\mathcal{C}}$. See Theorem 6.13 in [Alves 2020] for details.

Corollary 4.13

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$ and $\Omega \subset M$ a transitive set. If f is NUE and has $SR_{\mathcal{C}}$ on $H \subset \Omega$ with $m(H) > 0$, then f has a unique ergodic SRB measure μ whose support coincides with Ω.

For the decay of correlations we need a stronger version of the nonuniform expansion.

Strong nonuniform expansion

Let $f\colon M \to M$ be a local diffeomorphism out of a nondegenerate set $\mathcal{C} \subset M$. We say that f is strongly nonuniformly expanding (SNUE) on $H \subset M$, if there exists $\lambda > 0$ such that, for all $x \in H$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda. \tag{59}$$

Clearly SNUE $\implies$ NUE. Fixing $\lambda > 0$ as in (59), we may define for each $x \in H$

$$\mathcal{E}(x) = \min\left\{ N \geq 1 : \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda,\ \forall n \geq N \right\}.$$

Suppose, in addition, that f has $\mathrm{SR}_{\mathcal{C}}$ on H. Considering (ε_1, r_1), (ε_2, r_2) as in Proposition 4.8, we may also define for $x \in H$ and $i = 1, 2$

$$\mathcal{R}_i(x) = \min\left\{ N \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_{r_i}(f^j(x), \mathcal{C}) < \varepsilon_i,\ \forall n \geq N \right\}.$$

Set for each $x \in H$

$$h_H(x) = \max\left\{\mathcal{E}(x), \mathcal{R}_1(x), \mathcal{R}_2(x)\right\}. \tag{60}$$

Obviously, $h_H(x) = \mathcal{E}(x)$ when $\mathcal{C} = \emptyset$, that is, when $f : M \to M$ is a local diffeomorphism.

The Gibbs-Markov map $f^R : \Delta_0 \to \Delta_0$ of the next result was obtained in Theorem 4.12, under the weaker assumption of NUE.

Theorem 4.14

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$ and $\Omega \subset M$ a transitive set. If $\Sigma \subset \Omega$ is a ball such that f is SNUE and has $SR_{\mathcal{C}}$ on $H \subset \Sigma$ with $m(H) = m(\Sigma)$, then there are $L > 0$ and a Gibbs-Markov induced map f^R defined on a ball $\Delta_0 \subset \Sigma$ such that $\{R > n + L\} \subset \{h_H > n\} \cup E_n$, for some sequence of sets $(E_n)_n$ in Δ_0 with $m(E_n) \to 0$ exponentially fast, as $n \to \infty$.

Taking $0 < \theta \leq 1$ as in Proposition 4.8, it follows from the definition of h_H that, for all $x \in H$ and $n \geq h_H(x)$, there are $\ell \geq \theta n$ and (σ, r)-hyperbolic times $1 \leq n_1 < \cdots < n_\ell \leq n$ for x, with $\sigma = e^{-\lambda/4}$ and $r = r_2$. For each $n \geq 1$, set

$$H_n = \{x \in H \colon n \text{ is a } (\sigma, r)\text{-hyperbolic time for } x\}.$$

Clearly,

$$n \geq h_H(x) \implies \frac{1}{n}\#\{1 \leq j \leq n : x \in H_j\} \geq \theta,$$

and so typical points in Δ_0 has strong positive frequency in $(H_n)_n$. Thus, h_θ as in (42) can be defined and, moreover, $h_\theta \leq h_H$. By the third item of Theorem 4.1, we have

$$\{R > n + L\} \subset \{h_\theta > n\} \cup E_n,$$

for a sequence $(E_n)_n$ of sets in Δ_0 such that $m(E_n)$ decays exponentially fast with n. Since $\{h_\theta > n\} \subset \{h_H > n\}$, we are done.

Corollary 4.15

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$ and $\Omega \subset M$ is a transitive set. If $\Sigma \subset \Omega$ is a ball such that f is SNUE and has $SR_{\mathcal{C}}$ on $H \subset \Sigma$ with $m(H) = m(\Sigma)$, then there is a Gibbs-Markov induced map f^R defined on a ball $\Delta_0 \subset \Sigma$ such that

1. *if $m\{h_H > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, then there is $C' > 0$ such that $m\{R > n\} \leq C'n^{-a}$;*
2. *if $m\{h_H > n\} \leq Ce^{-cn^a}$ for some $C, c > 0$ and $a > 1$, then there are $C', c' > 0$ such that $m\{R > n\} \leq C'e^{-c'n^a}$.*

By the previous theorem, there are $L > 0$ and a Gibbs-Markov induced map f^R defined on a ball $\Delta_0 \subset \Sigma \subset \Omega$ for which

$$\{R > n + L\} \subset \{h_H > n\} \cup E_n, \tag{61}$$

where $(E_n)_n$ is a sequence of sets in Δ_0 such that $m(E_n) \to 0$ exponentially fast with n. Since we consider $\{h_H > n\}$ decaying no faster than exponential, it follows from (61) that $m\{R > n\}$ decays (at least) at the same speed of $m\{h_H > n\}$.

Decay of correlations

Let $\mathcal{H}$ be the space of Hölder continuous functions defined on M. Under the assumptions of the next results, the existence of a unique ergodic SRB measure μ whose support coincides with Ω has already been obtained in Theorem 4.13. Actually, under the weaker assumption of NUE.

Corollary 4.16

Let $f : M \to M$ be a $C^{1+\eta}$ local diffeomorphism out of a nondegenerate set $\mathcal{C}$ and $\Omega \subset M$ an transitive set containing a ball Σ such that f is SNUE and has $SR_{\mathcal{C}}$ on a set $H \subset \Sigma$ with full m measure in Σ. If μ is the unique ergodic SRB measure for f whose support coincides with Ω, then there are $1 \le p \le q$ and exact SRB measures $\mu_1, \dots, \mu_p$ for f^q with $f_*\mu_1 = \mu_2, \dots, f_*\mu_p = \mu_1$ and $\mu = (\mu_1 + \cdots + \mu_p)/p$ such that, for all $1 \le i \le p$,

1. *if $m\{h_H > n\} \le Cn^{-a}$ for some $C > 0$ and $a > 1$, for all $\varphi \in \mathcal{H}$ and $\psi \in L^\infty(m)$, there is $C' > 0$ such that $\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \le C' n^{-a+1}$;*
2. *if $m\{h_H > n\} \le Ce^{-cn^a}$ for some $C, c > 0$ and $a > 1$, given $\rho > 0$, there is $c' > 0$ such that, for all $\varphi \in \mathcal{H}$ and $\psi \in L^\infty(m)$, there is $C' > 0$ such that $\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \le C' e^{-c' n^a}$.*

Under assumptions of topologically mixing or ergodicity of the powers of f, we may deduce $p = q = 1$. See Section 6.5 in [Alves 2020].

Viana maps

Let $a_0 \in (1,2)$ be such that $x = 0$ is pre-periodic for $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ be a Morse function, e.g. $b(s) = \sin(2\pi s)$. For small $\alpha > 0$, define

$$\begin{array}{rccc} \hat{f}: & S^1 \times \mathbb{R}, & \longrightarrow & S^1 \times \mathbb{R} \\ & (s,x) & \longmapsto & (\hat{g}(s), \hat{q}(s,x)), \end{array}$$

where $\hat{g} : S^1 \to S^1$ is given by $\hat{g}(s) = ds \pmod 1$, for some $d \geq 16^\dagger$, and

$$\hat{q}(s,x) = a(s) - x^2 \quad \text{with} \quad a(s) = a_0 + \alpha b(s).$$

It is easily verified that, for any small $\alpha > 0$, there exists an interval $I \subset (-2,2)$ for which

$$\hat{f}(S^1 \times I) \subset \text{int}(S^1 \times I).$$

This implies that any f sufficiently close to $\hat{f}$ in the C^0 topology still has $S^1 \times I$ as a forward invariant region. Consider the attractor

$$\Omega_f = \bigcap_{n \geq 0} f^n(S^1 \times I).$$

We define the family $\mathcal{V}$ of Viana maps as the set of C^3 maps in a sufficiently small neighbourhood of $\hat{f} : S^1 \times I \to S^1 \times I$ in the C^2 topology.

†This has been weakened to $d \geq 2$ in [Buzzi, Sester, and Tsujii 2003], but only for open sets of maps in the C^∞ topology.

Theorem 4.17

For every $f \in \mathcal{V}$, there is a set H with full m measure on $S^1 \times I$ such that f is SNUE and has $SR_{\mathcal{C}_f}$ on H. Moreover, there are $C, c > 0$ such that $m\{h_H > n\} \leq Ce^{-c\sqrt{n}}$ and f is topologically mixing on Ω_f.

The strong nonuniform expansion and the measure estimate for $\{h_H > n\}$ have essentially been obtained in [Viana 1997]†; see also [Alves and Araújo 2003]. The fact that Viana maps are locally eventually onto on the attractor was proved in [Alves and Viana 2002] using previous estimates from [Alves 2000].

Corollary 4.18

Each $f \in \mathcal{V}$ has a unique ergodic SRB measure μ whose basin covers m almost all of $S^1 \times I$. Moreover, given $\eta > 0$, there exists $c > 0$ such that, for all $\varphi \in \mathcal{H}_\eta$ and $\psi \in L^\infty(m)$, there is $C > 0$ for which $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq Ce^{-c\sqrt{n}}$.

The existence of an ergodic SRB measure and the stretched exponential decay of correlations with respect to this SRB measure have been obtained in [Alves 2000] and [Gouëzel 2006], respectively. The uniqueness of the SRB measure and the continuity of its density in the L^1-norm with the map $f \in \mathcal{V}$ (*statistical stability*) was obtained in [Alves and Viana 2002].

† It remains an interesting open question to know whether the estimate for $\{h_H > n\}$ is optimal or just a limitation of the method used in [Viana 1997].

Partial hyperbolicity

We say that a *forward invariant compact set* $K \subset M$ for $f \in \mathrm{Diff}^1(M)$ has a dominated splitting if there are a Df-invariant decomposition $T_K M = E^{cs} \oplus E^{cu}$ and $0 < \lambda < 1$ such that, for some choice of a Riemannian metric on M,

$$\|Df|E^{cs}_x\| \cdot \|Df^{-1}|E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for all } x \in K.$$

We say that E^{cs} is *uniformly contracting* if

$$\|Df|E^{cs}_x\| \leq \lambda, \quad \text{for all } x \in K,$$

and E^{cu} is *uniformly expanding* if

$$\|Df^{-1}|E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for all } x \in K.$$

If E^{cs} (resp. E^{cu}) is uniformly contracting (resp. expanding), we simply denote it by E^s (resp. E^u). We say that $K \subset M$ is partially hyperbolic if it has dominated splitting $T_K M = E^{cs} \oplus E^{cu}$ for which

$$E^{cs} \text{ is uniformly contracting or } E^{cu} \text{ is uniformly expanding.}$$

We say that $\Omega \subset M$ is an attractor if there exists some compact neigbourhood U of K such that $f(U) \subset \mathrm{int}\, U$ and

$$\Omega = \bigcap_{n \geq 1} f^n(U).$$

Case $E^s \oplus E^{cu}$

Let $K \subset M$ have a dominated splitting $T_K M = E^{cs} \oplus E^{cu}$. We say that f is nonuniformly expanding along E^{cu} (NUE^{cu}) on a set $H \subset K$ if there exists $c > 0$ such that

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| < -c, \quad \text{for all } x \in H. \tag{62}$$

Given $\sigma < 1$, we say that n is a σ-hyperbolic time for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \le \sigma^k, \quad \text{for all } 1 \le k \le n. \tag{63}$$

The next result is an easy consequence of Pliss Lemma.

Lemma 4.19

Given $c > 0$, there exists $\theta > 0$ such that if for $x \in K$ we have

$$\frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| < -c,$$

then x has $e^{-c/2}$-hyperbolic times $1 \le n_1 < \cdots < n_\ell \le n$ with $\ell \ge \theta n$.

By the continuity of Df, we may fix $\delta_1 > 0$ sufficiently small such that, for all $x \in K$,

$$\|Df^{-1}(f(y))v\| \le \sigma^{-1/4}\|Df^{-1}|E^{cu}_{f(x)}\| \, \|v\|, \tag{64}$$

whenever $\operatorname{dist}(x, y) \le \delta_1$ and $v \in C^{cu}_a(y)$.

Proposition 4.20 (Alves, Bonatti, and Viana 2000)

Given a cu-disk D, there exist $C, \zeta > 0$ such that for all $x \in K \cap D$ with a $\sigma^{3/4}$-hyperbolic time n there exists a neighbourhood $V_n(x)$ of x in D such that

1. *f^n maps $V_n(x)$ to a cu-disk of radius δ_1 around $f^n(x)$;*
2. *n is a $\sigma^{1/2}$-hyperbolic time for all $y \in V_n(x)$;*
3. *for all $y, z \in V_n(x)$ we have*
   1. $\operatorname{dist}_{f^{n-k}(D)}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}_{f^n(D)}(f^n(y), f^n(z))$, *for all* $1 \le k \le n$;
   2. $\log \dfrac{\det Df^n|T_yD}{\det Df^n|T_zD} \le C \operatorname{dist}_{f^n(D)}(f^n(y), f^n(z))^{\zeta}$.

See Proposition 7.3 in [Alves 2020] for a proof.

Theorem 4.21 (Alves, Dias, Luzzatto, and Pinheiro 2017)

Let $f \in \mathrm{Diff}^{1+\eta}(M)$ and $K \subset M$ be such that $f(K) \subset K$ and $T_KM = E^s \oplus E^{cu}$. If f is NUE^{cu} on $H \subset K$ with $m(H) > 0$, then there are transitive sets $\Omega_1, \dots, \Omega_\ell \subset K$ such that, for m almost every $x \in H$, there is $1 \le j \le \ell$ for which $\omega(x) = \Omega_j$. Moreover, each Ω_j contains a cu-disk Σ_j such that f is NUE^{cu} for m_{Σ_j} almost every point in Σ_j.

See Theorem 7.9 in [Alves 2020] for a proof.

Young structure

Let Ω be a transitive set and $\Sigma \subset \Omega$ a *cu*-disk as in the previous theorem. Consider $H \subset \Sigma$ with full m_Σ measure such that f is NUE^{cu} on H. By Lemma 4.19, there are $0 < \sigma < 1$ and $\theta > 0$ such that, for every $x \in H$,

$$\limsup_{n\to\infty} \frac{1}{n} \# \{1 \leq j \leq n : j \text{ is a } \sigma\text{-hyperbolic time for } x\} \geq \theta. \tag{65}$$

Set for each $n \geq 1$

$$H_n = \left\{x \in H \colon n \text{ is a } \sigma^{3/4}\text{-hyperbolic time for } x\right\}. \tag{66}$$

From the definition of hyperbolic time,

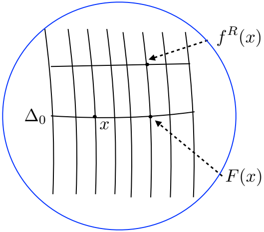
$$x \in H_n \implies f^j(x) \in H_{n-j}, \quad \text{for all } 0 \leq j < n. \tag{67}$$

Since σ-hyperbolic time $\implies$ $\sigma^{3/4}$-hyperbolic time, it follows from (65) that m_Σ almost every point in Σ belongs in infinitely many H_n's. Moreover, by Proposition 4.20, each $x \in H_n$ has a neighbourhood $V_n(x)$ in Σ that grows to a disk of radius δ_1 in n iterates. A priori, we do not know where this image disk is located. Using the transitivity in Ω we bring it close to Σ in a finite number of iterates.

Theorem 4.22 (Alves, Dias, Luzzatto, and Pinheiro 2017)

Let $f \in \mathrm{Diff}^{1+\eta}(M)$ and $\Omega \subset M$ be a transitive set with $T_\Omega M = E^s \oplus E^{cu}$. If there is a cu-disk $\Sigma \subset \Omega$ such that f is NUE^{cu} for m_Σ almost every point in Σ, then f has a set $\Lambda \subset \Omega$ with a full Young structure and integrable recurrence times.

Integrability of recurrence times



Let $F : \Delta_0 \to \Delta_0$ be the Gibbs-Markov quotient map associated with the Young structure. To apply Proposition 4.6, we need that typical points in Δ_0 have positive frequency in a sequence $(H_n^*)_n$ that is F-concatenated in $(H_n)_n$; recall (49) and (41). Set for each $n \geq 1$

$$H_n^* = \{x \in \Delta_0 \colon n \text{ is a } \sigma\text{-hyperbolic time for } x\}.$$

It follows from (65) that, for m_{Δ_0} almost every $x \in \Delta_0$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \# \left\{1 \leq j \leq n : x \in H_j^*\right\} \geq \theta.$$

Thus, $(H_n^*)_n$ is a frequent sequence of sets in Δ_0. Choosing $\varepsilon \ll \delta_1$, from (64) and (67)

$$x \in H_n^*,\ y \in \gamma_\varepsilon^s(f^j(x)) \implies y \in H_{n-j}, \quad \text{for all } 0 \leq j < n;$$

recall that H_n is the set of points x such that n is a $\sigma^{3/4}$-hyperbolic time for x.

Theorem 4.23

Let $f \in \mathrm{Diff}^{1+\eta}(M)$ and $\Omega \subset M$ be a transitive set with $T_\Omega M = E^s \oplus E^{cu}$. If Ω contains some cu-disk Σ such that f is NUE^{cu} on a subset of Σ with positive m_Σ measure, then f has a unique ergodic SRB measure μ whose support coincides with Ω.

See Theorem 6.17 in [Alves 2020] for a proof. A first version of this result was obtained in [Alves, Bonatti, and Viana 2000] under the stronger version of NUE^{cu}.

Decay of correlations

We say that f is strongly nonuniformly expanding along E^{cu} (SNUE^{cu}) on $H \subset K$ if

$$\exists c > 0 : \quad \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| < -c, \quad \text{for all } x \in H.$$

Clearly, SNUE^{cu} $\implies$ NUE^{cu}. Fixing $c > 0$ as above, set for each $x \in H$

$$h_H(x) = \min\left\{ N \geq 1 : \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| < -c, \ \forall n \geq N \right\}.$$

Theorem 4.24 (Alves and Li 2015)

Let $f \in \mathrm{Diff}^{1+\eta}(M)$ and $\Omega \subset M$ be a transitive set with $T_\Omega M = E^s \oplus E^{cu}$ for which there is a cu-disk $\Sigma \subset \Omega$ such that f is SNUE^{cu} on $H \subset \Sigma$ with $m_\Sigma(H) = m_\Sigma(\Sigma)$. If μ is the unique ergodic SRB measure for f with $\mathrm{supp}(\mu) = \Omega$, then there are $1 \leq p \leq q$ and exact SRB measures $\mu_1, \dots, \mu_p$ for f^q with $f_*\mu_1 = \mu_2, \dots, f_*\mu_p = \mu_1$ and $\mu = (\mu_1 + \cdots + \mu_p)/p$ such that, for all $1 \leq i \leq p$,

1. *if $m_\Sigma\{h_H > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, then for all $\varphi, \psi \in \mathcal{H}_\eta$ there is $C' > 0$ such that $\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \leq C' n^{-a+1}$;*
2. *if $m_\Sigma\{h_H > n\} \leq Ce^{-bn^a}$ for some $C, c > 0$ and $0 < a \leq 1$, then given $\eta > 0$, there is $c' > 0$ such that, for all $\varphi, \psi \in \mathcal{H}_\eta$, there is $C' > 0$ for which $\mathrm{Cor}_{\mu_i}(\varphi, \psi \circ f^{qn}) \leq C' e^{-c' n^a}$.*

Under assumptions of topologically mixing or ergodicity of the powers of f, we may deduce $p = q = 1$. See Section 6.5 in [Alves 2020].

Case $E^{cs} \oplus E^u$: SRB measures

Theorem 4.25 (Pesin and Sinai 1982)

Let $f \in \mathrm{Diff}^{1+\eta}(M)$ and $\Omega \subset M$ be an attractor such that $T_\Omega M = E^{cs} \oplus E^u$. Then there are ergodic SRB measures with support contained in Ω.

To prove that the SRB measures are physical measures, some contraction in the E^{cs} direction is needed. Giving a point $x \in A$, consider its largest Lyapunov exponent in the E^{cs} direction:

$$\lambda_+^c(x) = \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n|E_x^{cs}\|.$$

We say that E^{cs} is mostly contracting if for any unstable manifold γ^u we have $\lambda_+^c(x) < 0$ for a positive m_{γ^u} measure set of points $x \in \gamma^u$.

Theorem 4.26 (Bonatti and Viana 2000)

Let $f \in \mathrm{Diff}^{1+\eta}(M)$ and $\Omega \subset M$ be an attractor with $T_\Omega M = E^{cs} \oplus E^u$ with E^{cs} mostly contracting. Then there are ergodic SRB measures $\mu_1, \ldots, \mu_\ell$ supported on Ω such that for m almost every x with $\omega(x) \subset \Omega$ we have $x \in \mathcal{B}(\mu_j)$ for some $1 \le j \le \ell$. Moreover, if the leaves of the unstable foliation are dense in Ω, then there is a unique SRB measure supported on Ω.

Case $E^{cs} \oplus E^u$: decay of correlations

Let $f \in \mathrm{Diff}^{1+\eta}(M)$ and $\Omega \subset M$ be an attractor such that $T_\Omega M = E^{cs} \oplus E^u$. Assume the unstable manifolds of points in Ω are dense in Ω (in particular, the SRB in Ω is unique).

Theorem 4.27 (Dolgopyat 2000)

If $\dim(M) = 3$ and $E^{cs} = E^c \oplus E^s$ with E^c mostly contracting, then the SRB measure has exponential decay of correlations for Hölder continuous observables.

Theorem 4.28 (Castro 2002)

If f has a finite Markov partition and E^{cs} is mostly contracting, then the SRB measure has exponential decay of correlations for Hölder continuous observables.

Theorem 4.29 (Castro 2004)

If $\dim(E^{cs}) = 1$ and E^{cs} is mostly contracting, then the SRB measure has exponential decay of correlations for Hölder continuous observables.

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