SRB MEASURES FOR PARTIALLY HYPERBOLIC ATTRACTORS

by

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Introduction

One of the main goals of Dynamical Systems is to describe the typical behavior of orbits, specially as time goes to infinity. Even in cases of very simple evolution laws the orbits may have a rather complicated behavior, specially because systems may display sensitivity on the initial conditions, i.e. a small variation on the initial state gives rise to a completely different behavior of its orbit. The approach to this kind of systems has been particularly well succeeded through *physical measures*, or *Sinai-Ruelle-Bowen (SRB) measures*, which characterize asymptotically, in time average, a large set of orbits in the phase space.

Systems displaying uniformly expanding/contracting behavior on Riemannian manifolds (uniformly expanding maps and Axiom A attractors for diffeomorphisms and flows) have been exhaustively studied in the last decades; see [Bow75, BR75, KS69, Rue76, Sin68, Sin72]. Systems exhibiting expansion only in asymptotic terms have been considered in [Jak81], where the existence of physical measures for many quadratic transformations of the interval were established; see also [BC85, BY92]. Related to [BC85] is the work [BC91] for Hénon maps exhibiting strange attractors. Motivated by the results for multidimensional non-uniformly expanding systems in [Via97, Alv00], general conclusions for systems exhibiting non-uniformly expanding behavior were drawn in [ABV00].

The aim of these notes is to present an introduction to physical measures for partially hyperbolic attractors in finite dimensional compact Riemannian manifolds. Here we essentially give an overview of the main results in [PS82, BV00] for the mostly expanding case, and [ABV00, ADLP14] for the mostly expanding case. The strategy used in [ADLP14] to build SRB measures for partially hyperbolic attractors whose central direction is weakly expanding will be discussed in more detail in Section 3.

1. SRB measures

We start by presenting some basic results on Ergodic Theory to motivate the definition of SRB measure. Given a probability measure space $(X, \mathcal{A}, \mu)$, we say that a map $f\colon X \to X$ is *measurable* if $f^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$. We say that f *preserves* μ, or μ is *invariant* under f, if $\mu(f^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$. It easily follows from this definition that if μ is an f-invariant probability measure, then the sets $\{x \in M\colon x \in A\}$ and $\{x \in M\colon f^n(x) \in A\}$ have the same μ measure for every $n \in \mathbb{N}$. This means that the probability of finding a point in a measurable set does not depend on the moment we consider.

Let X be a compact metric space. We denote by $\mathbb{P}(X)$ the space of probability measures defined on the Borel σ-algebra of X. We introduce the weak* topology on $\mathbb{P}(X)$ in the following way: a sequence $(\mu_n)_n$ in $\mathbb{P}(X)$ converges to $\mu \in \mathbb{P}(X)$

if and only if

$$\int \varphi d\mu_n \to \int \varphi d\mu, \quad \text{for each continuous } \varphi\colon X \to \mathbb{R}.$$

We associate to a measurable map $f\colon X \to X$ an operator $f_*\colon \mathbb{P}(X) \to \mathbb{P}(X)$, assigning to each $\mu \in \mathbb{P}(X)$ the *push-forward* $f_*\mu$, which is defined as

$$f_*\mu(A) = \mu(f^{-1}(A)), \quad \text{for each } A \in \mathcal{A}.$$

One can easily check that f_* is continuous whenever f is continuous. Note that μ is invariant by f if and only if $f_*\mu = \mu$. If f is continuous, then taking some measure $\mu \in \mathbb{P}(X)$, a Dirac measure for instance, we define a sequence of measures in $\mathbb{P}(X)$,

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \mu.$$

A weak* accumulation point of this sequence is necessarily a fixed point for f_*.

Theorem 1.1 **(Krylov-Bogolyubov)**. — *If $f\colon M \to M$ is continuous map of a compact metric space M, then f has some invariant Borel probability measure.*

Some of the first results on the probabilistic features of dynamical systems with invariant probability measure go back to the work of Poincaré for conservative systems and can be translated to our context in the following way:

Theorem 1.2 **(Poincaré)**. — *Let f preserve a probability measure μ. If A is a measurable set, then for almost every $x \in A$, there are infinitely many $n \in \mathbb{N}$ for which $f^n(x) \in A$.*

The previous result gives no information on the asymptotic *frequency* that typical orbits visit A, i.e.

$$\lim_{n\to\infty} \frac{\#\{0 \le j < n\colon f^j(x) \in A\}}{n}. \tag{1}$$

Does this limit exist? Does it depend on x? Birkhoff Ergodic Theorem gives answers to these questions and, in fact, gives more general conclusions. Before we state it, let us introduce some important concept on this subject.

Assume that f preserves a measure μ. We say that μ is *ergodic* if $\mu(A) = 0$ or $\mu(M \setminus A) = 0$ whenever $A \in \mathcal{A}$ satisfies $f^{-1}(A) = A$. Observing that $f^{-1}(A) = A$ implies that $f(A) \subset A$ and $f(M \setminus A) \subset M \setminus A$, this means that the space cannot be decomposed into two parts which are relevant (positive measure) that do not interact.

Theorem 1.3 **(Birkhoff)**. — *Assume that f preserves a probability measure μ. If φ is integrable, then there is an integrable function φ^* such that for μ almost* *every* $x \in M$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \varphi^*(x).$$

Moreover, if μ *is ergodic, then* $\varphi^*(x) = \int \varphi d\mu$ *for* μ *almost every* $x \in M$*.*

Taking φ as the characteristic function of a measurable set A, we easily deduce that the limit in (1) exists for μ almost every $x \in M$. Furthermore, if μ is ergodic, then that limit is equal $\mu(A)$. This means that the frequency of visits to A coincides with the proportion that A occupies in the phase space.

The results that we have presented so far concern dynamics over a probability measure space with no additional structure on the underlying phase space M. Frequently M has a Riemannian manifold structure and a volume form which gives rise to a Lebesgue measure on the Borel sets of M. As already seen, Birkhoff Ergodic Theorem states that asymptotic time averages exist for almost every point, with respect to an invariant measure μ, and they coincide with the spatial average, provided μ is ergodic. However, an invariant measure can lack of physical meaning, in the sense that sets with full μ measure may have zero Lebesgue measure.

An invariant probability measure μ is called an *Sinai-Ruelle-Bowen (SRB) measure* for $f\colon M \to M$ if, for a positive Lebesgue measure set of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu, \tag{2}$$

for all continuous $\varphi : M \to \mathbb{R}$. This means that the averages of Dirac measures over the orbit of x converge in the weak* topology to the measure μ. We define $B(\mu)$, the *basin* of μ, as the set of points $x \in M$ for which (2) holds for all continuous φ.

It is straightforward to check that a Dirac measure supported on an attracting fixed point (or a periodic orbit, more generally) is an SRB measure. These are examples of SRB measures which are singular with respect to Lebesgue measure on M. However, there may be SRB measures which are absolutely continuous with respect to Lebesgue measure. We leave it as an exercise to see that if μ is an ergodic probability measure μ defined on the Borel sets of M, then $\mu(B(\mu)) = 1$ (to see this, use the fact that the space of continuous functions has a countable dense subset). It follows immediately that if, additionally, μ is absolutely continuous with respect to Lebesgue measure, then μ is an SRB measure.

The systems that we are interested in this work are neither singular nor absolutely continuous with respect to Lebesgue measure. Actually we are interested in measures supported in partially hyperbolic attractors whose volume is zero, hence we cannot expect them to be absolutely continuous with respect to Lebesgue measure on the ambient manifold. On the other hand, as these attractors contain some local unstable manifolds, the SRB measures cannot be supported in periodic attractors, as well. These SRB measures will actually be the so called Gibbs u-states that we introduce in Subsection 2.1.

2. Partially hyperbolic attractors

Let $f\colon M \to M$ be C^k diffeomorphism ($k \geq 2$) defined on a finite dimensional Riemannian manifold M endowed with a normalized volume form on the Borel sets that we denote by Leb and call *Lebesgue measure.* Given a submanifold $\gamma \subset M$ we use Leb_γ to denote the measure on γ induced by the restriction of the Riemannian structure to γ.

A compact invariant set $A \subset M$ is called an *attractor* if there is an open set $U \supset A$ such that

$$f\left(\overline{U}\right) \subset U \quad \text{and} \quad A = \bigcap_{n\geq 0} f^n(\overline{U}). \tag{3}$$

Notice that if there is an open set $U \supset A$ satisfying the second condition in (3), then there is an open set $U' \supset A$ satisfying both conditions in (3); see e.g. [Shu87, Lemma 2.9].

An attractor A is said to be *partially hyperbolic*, if there is an f-invariant splitting $T_A M = E^{cs} \oplus E^{cu}$ such that for some choice of a Riemannian metric on M there is a constant $0 < \lambda < 1$ such that:

1. $E^{cs} \oplus E^{cu}$ is a *dominated splitting*: for all $x \in A$
$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda.$$
2. E^{cs} is *uniformly contracting* or E^{cu} is *uniformly expanding*: for all $x \in A$
$$\|Df \mid E^{cs}_x\| \leq \lambda \quad \text{or} \quad \|Df^{-1} \mid E^{cu}_{f(x)}\|^{-1} \leq \lambda.$$

The attractor A is called *hyperbolic* if both E^{cs} and E^{cu} are uniform. To emphasize the uniform behavior, we shall write E^s instead of E^{cs} in the first case, and E^u instead of E^{cu} in the second case.

Classical results by Sinai, Ruelle and Bowen give a good description of the statistical properties for

– Anosov diffeomorphisms [Sin72];
– Axiom A attractors [Rue76];
– Axiom A flows [BR75].

In particular, they prove the existence of ergodic *Gibbs u-states*, which in the uniformly hyperbolic case happen to be SRB measures.

2.1. Gibbs u-states. — Let $W^u(x)$ be the unstable manifold through a point $x \in A$. Given $\epsilon > 0$ and Σ a C^1 disk through x transverse to $W^u(x)$, let $\Pi(x, \Sigma, \epsilon)$ be a box around $W^u(x)$ made of the union of all local unstable manifolds $W^u_\epsilon(z)$ with $z \in A \cap \Sigma$. Given any invariant probability measure μ supported on Λ, by Rokhlin Disintegration Theorem, for each $z \in A \cap \Sigma$ there are

– a probability measure $\hat{\mu}$ on $A \cap \Sigma$;
– a probability measure μ_z on $W^u_\epsilon(z)$;

such that for any Borel set $B \subset \Pi(x, \Sigma, \epsilon)$ we have

$$\mu(B) = \int \mu_z(B) d\hat{\mu}(z).$$

We say that an invariant probability measure μ supported on Λ is a *Gibbs u-state* if the measures μ_z given by Rokhlin Disintegration Theorem are absolutely continuous with respect to $\mathrm{Leb}_{W^u_\epsilon(z)}$. Moreover, as the unstable direction is uniform, the density of μ_z with respect to Lebesgue measure on each unstable leaf through z is uniformly bounded from above and below.

Notice that in the uniformly hyperbolic case, an ergodic Gibbs u-state is necessarily an SRB measure. To see this, one just have to see that for an ergodic measure Birkhoff's averages are constant almost everywhere on local unstable and local stable disks.

2.2. Mostly contracting case. — Here we consider a partially hyperbolic attractor A whose tangent bundle splits as $T_AM = E^{cs} \oplus E^u$. Notice that in this case, there is a well-defined local unstable manifold through each $x \in A$. The strategy used in [BV00] to prove the existence of SRB measures for partially hyperbolic attractors in this case is based on previous work, where Gibbs u-states have already been obtained.

***Theorem 2.1* (PS82).** — *Let $A \subseteq M$ be a partially hyperbolic attractor for $f \in \mathrm{Diff}^2(M)$. Then there is some Gibbs u-state supported on A.*

This result was proved in [PS82] considering a local unstable manifold $\gamma \subset A$ and showing that any weak* accumulation point of the sequence

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j_* \,\mathrm{Leb}_\gamma$$

has conditional measures on local unstable manifolds which are absolutely continuous with respect to Leb on those manifolds.

To prove that the u-Gibbs states given by the previous theorem are in fact SRB measures, we need some contraction in the E^{cu} direction. Giving a point $x \in A$ we define the largest *Lypaunov exponent* in the E^{cs} direction as

$$\lambda^c_+(x) = \limsup_{n\to\infty} \frac{1}{n} \|Df^n | E^{cs}_x\|.$$

***Theorem 2.2* (BV00).** — *Let $f \in \mathrm{Diff}^2(M)$ have a partially hyperbolic attractor A for which $T_AM = E^{cs} \oplus E^u$, and assume that*

> *for any local unstable manifold $D^u \subset A$ we have $\lambda^c_+(x) < 0$ for x in a set with positive Leb_{D^u} measure.*

Then there exist SRB measures $\mu_1, ..., \mu_\ell$ supported in A such that for Leb *almost every x with $\omega(x) \subset A$ we have $x \in \mathcal{B}(\mu_j)$ for some $1 \leq j \leq \ell$.*

The results in [BV00] give also sufficient conditions for the existence a unique SRB measure. In the remaining of this subsection we give a rough idea on how it is proved in [BV00] that a Gibbs u-state is an SRB measure and, additionally, there is a finite number of ergodic Gibbs u-states. Let R be the set of *regular points* of f, i.e. the set of points $x \in \Lambda$ for which both conditions hold:

1. given any continuous function $\varphi : M \to \mathbb{R}$, both limits
$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \quad \text{and} \quad \lim_{n\to-\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \tag{4}$$
exist and coincide;
2. the largest Lyapunov exponent of f at x
$$\limsup_{n\to\infty} \frac{1}{n} \|Df^n|E^{cs}_x\| = \limsup_{n\to-\infty} \frac{1}{n} \|(Df^n|E^{cs}_x)^{-1}\|. \tag{5}$$
exists and is negative.

By Pesin theory, through each regular point $x \in R$ there is a well defined local stable manifold $W^s_{\text{loc}}(x)$. A strong stable leaf W^u is called regular if Lebesgue almost every point in W^u is regular. We define $S \subset R$ as the set of regular points in regular leaves.

Lemma 2.3. — *The set S has full μ measure.*

Proof. — Actually, considering a box $\Pi(x, \Sigma, \epsilon)$, let $(\mu_z)_z$ be the conditional measures along the unstable manifolds of $\Pi(x, \Sigma, \epsilon)$ and $\bar{\mu}$ the quotient measure. According to Birkhoff and Oseledets Theorems, the limits in (4) and (5) exist for μ almost every point. Thus, these limits exist μ_z almost everywhere for $\bar{\mu}$ almost every z. It remains to see that the limit in (5) is negative. Since the limit in (5) when $n \to -\infty$ is constant on unstable leaves and by assumption there is a positive Lebesgue measure subset of points where it is negative, then it is negative almost everywhere. Consider a covering of S by boxes $\Pi(x, \Sigma, \epsilon)$ to get the result. □

We say that $x, y \in S$ belong in the same accessibility class if there are points $x = z_0, z_1, \ldots, z_n = y$ in S such that at least one of the points y_i or y_{i+1} belongs in a stable or unstable leaf. This defines an equivalence relation, and points in a same equivalence class necessarily have the same Birkhoff averages. Using ideas similar to those in the proof of the previous lemma we are able to deduce:

Lemma 2.4. — *Accessibility classes are open sets of S.*

It follows that there must be an at most countable number of accessibility classes. Discarding those accessibility classes which have zero μ measure, and recalling that Birkhoff averages are constant on accessibility classes, for any accessibility classe C, the measure defined by

$$\mu_C(B) = \frac{\mu(B \cap C}{\mu(C)}$$

is ergodic. Hence, the ergodic components of μ are precisely the normalizations of these measures μ_C. These measures have obviously conditional measures on local unstable leaves absolutely continuous with respect to Lebesgue measures on those leaves.

It remains to see that there are only finitely many accessibility classes. Assuming, by contradiction, that there is an infinite sequence (C_n) of accessibility classes, take for each n a regular strong unstable leaf γ_n such that $S \cap \gamma_n$ is nonempty and contained in C_n. taking a subsequence, if necessary, and using Ascoli-Arzela Theorem, we may assume that the disks γ_n converge in the C^0 norm to some strong unstable disk D^u. By assumption, there is a subset of points x in D^u for which $\lambda_+^c(x) < 0$, and so through each of those points passes a local unstable leave. As the points in those local unstable leaves and D^u belong in the same accessibility class, one necessarily deduces that there are distinct large n and m such that $C_n \cap C_m \neq \emptyset$. This gives a contradiction.

2.3. Mostly expanding case. — Assume now we have a partially hyperbolic attractor A whose tangent bundle splits as $T_A M = E^{cs} \oplus E^u$. In this case we need some some expansion in the E^{cu} direction, particularly ensuring the existence of local unstable manifolds. Reducing the trapping region U, if necessary, we may assume that the fibre bundles E^s and E^{cu} have a (not necessarily invariant) continuous extension to U.

We say that $H \subset U$ is *non-uniformly expanding (NUE)* along E^{cu} if there are $\epsilon > 0$ and a Riemannian metric on M such that for all $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| < -\epsilon.$$

The set $H \subset U$ is *weakly non-uniformly expanding (WNUE)* along E^{cu} if there are $\epsilon > 0$ and some Riemannian metric on M such that for all $x \in H$

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| < -\epsilon.$$

***Theorem 2.5* (ABV00).** — *Let $f \in \mathrm{Diff}^2(M)$ have a partially hyperbolic attractor A for which $T_A M = E^{cs} \oplus E^u$, and assume that*

there is $H \subseteq U$ with $\mathrm{Leb}(H) > 0$ such that f is NUE along E^{cu}.

Then there exist SRB measures $\mu_1, ..., \mu_\ell$ supported on A such that for Lebesgue almost every $x \in H$ we have $x \in \mathcal{B}(\mu_j)$ for some $1 \leq j \leq \ell$.

A key step in the proof of this result in [ABV00] is that for some centre-unstable disk D there are $H_1, H_2, H_3 \cdots \subset D$ such that the weak* accumulation points (which are not necessarily invariant) of the sequence of measures

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\mathrm{Leb}_D\,|H_j), \quad n \geq 1,$$

have the u-Gibbs property. Moreover, there is some uniform constant $\alpha > 0$ such that for all $n \geq 1$

$$\mu_n(M) = \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{Leb}_D(H_j) \geq \alpha.$$

To prove this last property it is essential to have NUE and not just WNUE. Though the proof of this result in [ABV00] is deep and technically intricate, the rough idea is that the above property implies that an accumulation point (which now is an invariant measure) of

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\,\mathrm{Leb}_\gamma$$

must necessarily have some ergodic component whose conditional measures on local unstable manifolds are absolutely continuous with respect to Lebesgue measure on those unstable manifolds.

Using a different approach, in [ADLP14] it is possible to deduce the same conclusion under the assumption WNUE along E^{cu}.

***Theorem 2.6* (ADLP14)**. — *Let $f \in \mathrm{Diff}^2(M)$ have a partially hyperbolic attractor A for which $T_AM = E^{cs} \oplus E^u$, and assume that*

there is $H \subseteq U$ with $\mathrm{Leb}(H) > 0$ such that f is WNUE along E^{cu}.

Then

1. *there exist closed invariant transitive sets $\Omega_1, ..., \Omega_\ell \subset A$ such that for* Leb *almost every $x \in H$ we have $\omega(x) = \Omega_j$ for some $1 \leq j \leq \ell$;*
2. *there exist SRB measures $\mu_1, ..., \mu_\ell$ supported on $\Omega_1, ..., \Omega_\ell$, whose basins have nonempty interior, such that for* Leb *almost every $x \in H$ we have $x \in \mathcal{B}(\mu_j)$ for some $1 \leq j \leq \ell$.*

Notice that the first item is a consequence of the second one, but for the strategy developed in [ADLP14] we need to prove it independently, for it is a fundamental step towards proving the second item. Differently from [ABV00], the strategy used in [ADLP14] to prove the existence of an SRB measure supported on each transitive piece Ω_j is based on the existence of some *geometric structures* and *induced schemes* that we introduce in Subsection 2.5.

It remains an interesting question to know whether these results hold true under the (weaker?) assumption that the map f has all *Lyapunov exponents* along E^{cu} positive: there exists some $\epsilon > 0$ such that for every $x \in H$ and every non-zero vector $v \in E^{cu}_x$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| > \epsilon. \tag{6}$$

Notice that condition (6), unlike NUE or WNUE, does not depend on the choice of the metric. Clearly, if $\dim(E^{cu}) = 1$, then (6) is equivalent to WNUE.

Problem 2.7. — Assume that f has all Lyapunov exponents positive along E^{cu} on a subset of U with positive Lebesgue measure. Is there a Riemannian metric on M such that f is WNUE along E^{cu} on some subset of U with positive Lebesgue measure?

2.4. Examples. — Here we present an open class of diffeomorphisms introduced in [ABV00, Appendix A] defined on the d-dimensional torus $M = T^d$, with $d \geq 4$, whose tangent bundle splits into $TM = E^{cs} \oplus E^{cu}$, having non-uniformly expanding behavior in the E^{cu} direction, and non-uniformly contracting behavior in the E^{cs} direction Lebesugue almost everywhere. Further examples of partially hyperbolic attractors which are not uniformly hyperbolic can be found in [BV00, Section 6].

Our construction allows in particular one of the two fibre bundles to keep the uniform behavior, thus obtaining examples of systems which are not uniformly hyperbolic but satisfy the assumptions of the theorems above.

We start with a linear Anosov diffeomorphism f_0 on $M = T^d$ and let $TM = E^u \oplus E^s$ be the corresponding hyperbolic decomposition. Let V be a small neighborhood of a fixed point of f_0 and let $f : M \to M$ be a C^2 diffeomorphism such that

(a) f admits invariant cone fields C^{cu} and C^{cs} with small width $\alpha > 0$ and containing, respectively, the unstable bundle E^u and the stable bundle E^s of the Anosov diffeomorphism f_0;

(b) there exists $\sigma_1 > 1$ such that

$$|\det Df(x)| > \sigma_1$$

for any $x \in V$;

(c) there exists $\sigma_2 < 1$ satisfying

$$\|(Df \mid T_x\mathcal{D}^{cu})^{-1}\| < \sigma_2 \quad \text{and} \quad \|(Df \mid T_x\mathcal{D}^{cs})\| < \sigma_2$$

for $x \in M \setminus V$ and any disks $\mathcal{D}^{cu}$, $\mathcal{D}^{cs}$ tangent to C^{cu}, C^{cs}, respectively;

(d) there exists some small $\delta_0 > 0$ satisfying

$$\|(Df \mid T_x\mathcal{D}^{cu})^{-1}\| < 1 + \delta_0 \quad \text{and} \quad \|(Df \mid T_x\mathcal{D}^{cs})\| < 1 + \delta_0$$

for any $x \in V$ and any disks $\mathcal{D}^{cu}$, $\mathcal{D}^{cs}$ tangent to C^{cu}, C^{cs}, respectively.

For instance, if f_1 is a torus diffeomorphism satisfying (a), (b), (d), and coinciding with f_0 outside V, then any map f in a C^1 neighbourhood of f_1 satisfies all the previous conditions. The C^1 open classes of transitive non-Anosov diffeomorphisms presented in [BV00, Section 6], as well as other robust examples from [Mañ78], are constructed in this way and they fit in the present setting: both these diffeomorphisms and their inverses satisfy (a)–(d).

Now, using the same arguments as in [ABV00, Lemma A1], we may conclude that for sufficiently small $\delta_0 > 0$ Lebesgue almost every point $x \in M$ spends a positive fraction of the time outside the domain V. Then, using assumptions (c) and (d) above, there exists $c_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df|E^{cu}_{f^j(x)})^{-1}\| \leq -c_0$$

These arguments also show that f is non-uniformly contracting along the centre-stable direction, if it satisfies (a)–(d): Lebesgue almost every $x \in M$ has

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df|E^{cs}_{f^j(x)}\| \leq -c_0 .$$

This in particular implies that it is mostly contracting along the E^{cs} direction.

2.5. Gibbs-Markov-Young structures. — Here we introduce the geometric structures that will enable us to prove the second part of Theorem 2.6. An embedded disk $\gamma \subset M$ is called an *unstable disk* if

$$\operatorname{dist}(f^{-n}(x), f^{-n}(y)) \to 0, \quad \text{as } n \to \infty$$

for every $x, y \in \gamma$. Similarly, γ is called a *stable manifold* if

$$\operatorname{dist}(f^{n}(x), f^{n}(y)) \to 0, \quad \text{as } n \to \infty$$

for every $x, y \in \gamma$.

Let D^u be the unit disk in some Euclidean space and $\operatorname{Emb}^1(D^u, M)$ be the space of C^1 embeddings from D^u into M. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s and $\Phi^u\colon K^s \times D^u \to M$ such that

i) $\gamma^u = \Phi^u(\{x\} \times D^u)$ is an unstable manifold;
ii) Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
iii) $x \mapsto \Phi^u|(\{x\} \times D^u)$ defines a continuous map from K^s into $\operatorname{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined similarly.

We say that $\Lambda \subset M$ has a *hyperbolic product structure* if there exist a continuous family of unstable manifolds $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable manifolds $\Gamma^s = \{\gamma^s\}$ such that

i) $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$;
ii) $\dim \gamma^u + \dim \gamma^s = \dim M$;

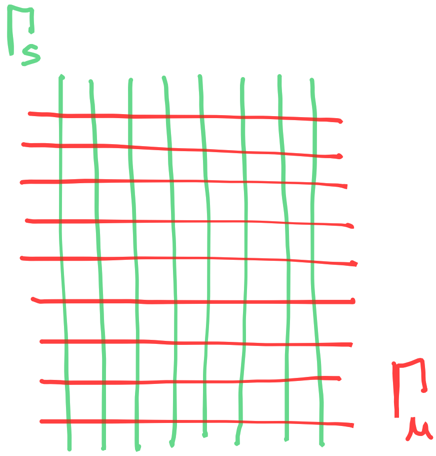

FIGURE 1. Stable and unstable leaves

iii) each γ^s meets each γ^u in exactly one point;
iv) stable and unstable manifolds meet with angles bounded away from 0.

Let $\Lambda \subset M$ have a hyperbolic product structure defined by the families of stable and unstable leaves Γ^s and Γ^u. A subset $\Lambda_0 \subset \Lambda$ is called an *s-subset* if Λ_0 also has a hyperbolic product structure and its defining families Γ_0^s and Γ_0^u can be chosen with $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined analogously.

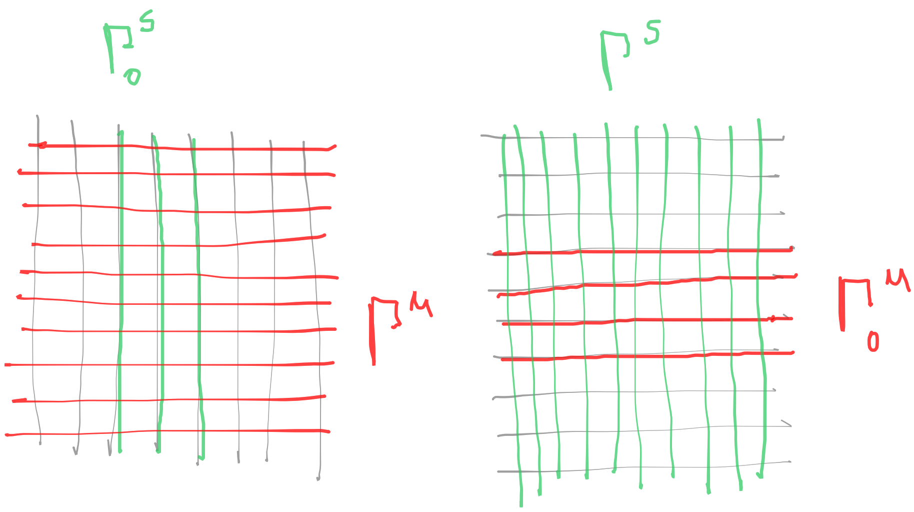

FIGURE 2. *s*- and *u*-subsets

Given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$. For each $n \geq 1$, let $(f^n)^u$ denote the restriction of the map f^n to γ^u-disks, and let $\det D(f^n)^u$ be the Jacobian of $D(f^n)^u$.

We say that f admits a *Gibbs-Markov-Young (GMY) structure* if there exist a set Λ with hyperbolic product structure and constants $C > 0$ and $0 < \alpha < 1$, depending on f and Λ, satisfying the following properties:

(P_0) *Lebesgue detectable*: $\mathrm{Leb}_\gamma(\Lambda) > 0$ for each $\gamma \in \Gamma^u$.

(P_1) *Markov*: there are pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that
  (a) $\mathrm{Leb}_\gamma\big((\Lambda \setminus \cup\Lambda_i) \cap \gamma\big) = 0$ on each $\gamma \in \Gamma^u$;
  (b) for each $i \in \mathbb{N}$ there is $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is u-subset, and for all $x \in \Lambda_i$

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$

(P_2) *Contraction on stable leaves*: for all $\gamma^s \in \Gamma^s$, $x, y \in \gamma^s$ and $n \ge 1$

$$\mathrm{dist}(f^n(y), f^n(x)) \le C\alpha^n.$$

(P_3) *Backward contraction on unstable leaves*: for all $\gamma^u \in \Gamma^u$, $x, y \in \Lambda_i \cap \gamma^u$ and $0 \le n < R_i$

$$\mathrm{dist}(f^n(y), f^n(x)) \le C\alpha^{R_i - n}\,\mathrm{dist}(f^{R_i}(x), f^{R_i}(y)).$$

(P_4) *Bounded distortion*: for all $\gamma^u \in \Gamma^u$ and $x, y \in \Lambda_i \cap \gamma^u$

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \le C\,\mathrm{dist}(f^{R_i}(x), f^{R_i}(y)).$$

(P_5) *Regularity of the stable foliation*: defining $\Theta\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ for $\gamma, \gamma' \in \Gamma^u$ by taking $\Theta(x)$ equal to $\gamma^s(x) \cap \gamma'$, then Θ is absolutely continuous.

This GMY structure allows us to introduce a *return time* function $R : \Lambda \to \mathbb{N}$ and an *induced map* $F : \Lambda \to \Lambda$ defined by

$$F|_{\Lambda_i} = f^{R_i}|_{\Lambda_i} \quad \text{and} \quad R|_{\Lambda_i} = R_i.$$

The following result is standard for piecewise hyperbolic maps and a proof of it can be found in [You98, Section 2].

Theorem 2.8. — *Let $F : \Lambda \to \Lambda$ be the induced map of a GMY structure for f. Then*

1. *the induced map F has a unique SRB measure ν;*
2. *if the return time $R : \Lambda \to \mathbb{N}$ is integrable with respect to ν, then*

$$\mu = \sum_{j=0}^{\infty} f_*^j(\nu|\{R > j\})$$

is a finite measure and its normalization is an SRB measure for f.

The strategy to prove the second part of Theorem 2.6 (existence of SRB measures) is to see that on each transitive component Ω_j given by the first part we may build a GMY structure $\Lambda \subseteq \Omega_j$ with integrable return times, and a then use Theorem 2.8.

3. GMY structures

We dedicate this section to give an idea how to prove the existence of a GMY structure under the assumptions of Theorem 2.6. We essentially describe the geometric construction on a reference leaf. Properties (P_0)-(P_4) follow easily from the construction. Property (P_5) is standard for uniformly hyperbolic attractors and the classical ideas can be adapted to the partially hyperbolic setting; see [ADLP14, Section 6.3] for details.

3.1. Hyperbolic times. — Given $0<\sigma<1$, we say that n is a *σ-hyperbolic time* for $x \in A$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \le \sigma^k, \qquad \text{for all } 1 \le k \le n.$$

For $n \ge 1$ we define

$$H_n = H_n(\sigma) = \{x \in A\colon\ n \text{ is a } \sigma\text{-hyperbolic time for } x\ \}.$$

As a consequence of WNUE we have:

Lemma 3.1. — *There exist $\theta>0$ and $\sigma>0$ such that for every $x \in H$*

$$\limsup_{n\to\infty} \frac{\#\{1 \le j \le n : x \in H_j(\sigma)\}}{n} \ge \theta.$$

The dominated splitting $T_AM = E^s \oplus E^{cu}$ gives rise to Df-invariant conefields $\{C^s_x\}_{x\in V}$ and $\{C^{cu}_x\}_{x\in V}$ in a neighborhood V of A. We say that $D \subset V$ is a *cu-disk* if $T_xD \subset C^{cu}_x$ for all $x \in V$.

Lemma 3.2. — *Let D be a cu-disk. There exists $C>1$ such that for each $x \in H_n \cap D$ there exists a neighborhood $V_n(x)$ of x in D so that:*

1. *f^n maps $V_n(x)$ diffeomorphically onto $B^{cu}(f^n(x), \delta_1)$;*
2. backward contraction*: for all $1 \le k \le n$ and $y, z \in V_n(x)$,*

$$\operatorname{dist}_{f^{n-k}(V_n(x))}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}_{f^n(V_n(x))}(f^n(y), f^n(z));$$

3. bounded distortion*: for all $y, z \in V_n(x)$*

$$\log \frac{|\det Df^n \mid T_yD|}{|\det Df^n \mid T_zD|} \le C \operatorname{dist}_{f^n(D)}(f^n(y), f^n(z))^{\zeta}.$$

The sets $V_n(x)$ are called *hyperbolic pre-disks.*

Lemma 3.3. — *Let D be a cu-disk with $\operatorname{Leb}_D(H)>0$. There are hyperbolic pre-disks $V_1, V_2, \dots \subseteq D$ and integers $n_1<n_2<\cdots$ such that for $B_k = f^{n_k}(V_k)$*

$$\lim_{k\to\infty} \frac{\operatorname{Leb}_{B_k} f^{n_k}(H \cap D)}{\operatorname{Leb}_{B_k}(B_k)} = 1.$$

Using the previous lemma and taking an accumulation disk of the sequence $(B_k)_k$ we can prove the result below, which gives that the attractor can be decomposed into a finite number of transitive pieces, thus giving the first part of Theorem 2.6. See [ADLP14, Section 3] for details.

Proposition 3.4. — *There exist closed invariant sets $\Omega_1, ..., \Omega_\ell \subseteq A$ such that for Lebesgue almost every $x \in H$ we have $\omega(x) = \Omega_j$ for some $1 \le j \le \ell$. Moreover, each Ω_j is transitive and contains a cu-disk Δ_j of radius δ_1 on which f is WNUE along E^{cu} for Leb_{Δ_j} almost every point in Δ_j.*

To prove the second part of Theorem 2.6 it is enough to show that there is a GMY structure $\Lambda \subseteq \Omega_j$ with integrable return times for each Ω_j. Recall that each Ω_j is transitive and contains a *cu*-disk Δ_j of radius δ_1 on which f is WNUE along E^{cu} for Leb_{Δ_j} almost every point in Δ_j. From here on we fix

$$\Omega := \Omega_j \quad \text{and} \quad \Delta := \Delta_j \subset \Omega_j.$$

3.2. Construction on a reference leaf. — In this section we describe an algorithm for the construction of a partition of some subdisk of Δ which is the basis of the construction of the GMY structure. We first fix some arbitrary $1 \le j \le \ell$ and for the rest of the paper we let $\Omega = \Omega_j$ and $\Delta = \Delta_j$ as in Proposition 3.4. We also fix a constant $\delta_s > 0$ so that local stable manifolds $W^s_{\delta_s}(x)$ are defined for all points $x \in K$. For any subdisk $\Delta' \subset \Delta$ we define

$$\mathcal{C}(\Delta') = \bigcup_{x \in \Delta'} W^s_{\delta_s}(x).$$

Let π denote the projection from $\mathcal{C}(\Delta')$ onto Δ' along local stable leaves. We say that a centre-unstable disk $\gamma^u \subset M$ *u-crosses* $\mathcal{C}(\Delta')$ if $\pi(\gamma) = \Delta'$ for some connected component γ of $\gamma^u \cap \mathcal{C}(\Delta')$. The proof of the next lemma can be found

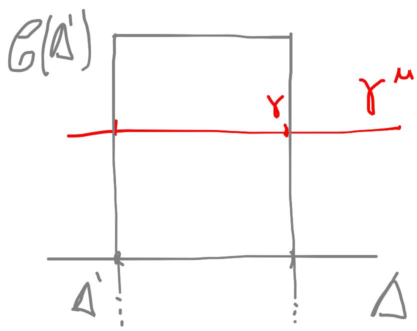

FIGURE 3. Disk u-crossing

in [ADLP14, Lemmas 4.3].

Lemma 3.5. — *There are $p \in \Delta$ and $N_0 \geq 1$ such that for all $\delta_0 > 0$ sufficiently small and each hyperbolic pre-disk $V_n(x) \subseteq \Delta$ there is $0 \leq m \leq N_0$ such that $f^{n+m}(V_n(x))$ intersects $W^s_{\delta_s/2}(p)$ and u-crosses $\mathcal{C}(B^u_{\delta_0}(p))$, where $B^u_{\delta_0}(p)$ is the ball in Δ of radius δ_0 centred at p.*

We now fix $p \in \Delta$, $N_0 \geq 1$ and $\delta_0 > 0$ sufficiently small so that the conclusions of Lemma 3.5 hold. Considering the constant

$$K_0 = \max_{x \in M} \left\{ \|Df^{-1}(x)\|, \|Df(x)\| \right\}, \tag{7}$$

we choose in particular $\delta_0 > 0$ small so that

$$2\delta_0 K_0^{N_0} \sigma^{-N_0} < \delta_1 K_0^{-N_0}. \tag{8}$$

Now we define

$$\Delta_0 = B^u_{\delta_0}(p) \quad \text{and} \quad \mathcal{C}_0 = \mathcal{C}(\Delta_0). \tag{9}$$

We also choose $\delta_0 > 0$ small so that any cu-disk intersecting $W^s_{3\delta_s/4}$ cannot reach the top or bottom parts of $\mathcal{C}_0$, i.e. the boundary points of the local stable manifolds $W^s_{\delta_s}(x)$ through points $x \in \Delta_0$. For every $n \geq 1$ we define

$$H_n = \{x \in \Delta \cap H \colon \; n \text{ is a hyperbolic time for } x\,\}.$$

It follows from Lemma 3.2 that for each $x \in H_n \cap \Delta_0$ there exists a hyperbolic pre-disk $V_n(x) \subset \Delta$. Then, by Lemma 3.5 there are $0 \leq m \leq N_0$ and a centre-unstable disk $\omega^x_n \subseteq \Delta$ such that

$$\pi(f^{n+m}(\omega^x_n)) = \Delta_0. \tag{10}$$

We remark that condition (10) may in principle hold for several values of m.

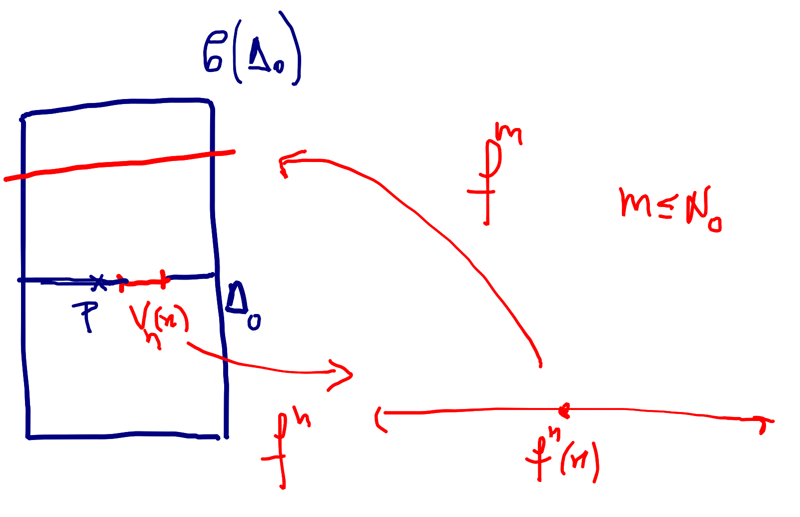

FIGURE 4. A returning disk

For definiteness, we shall always assume that m takes the smallest possible value.

Notice that ω_n^x is associated to x by construction, but does not necessarily contain x.

In the sequel we describe an inductive partitioning algorithm which gives rise to a (Leb mod 0) partition $\mathcal{P}$ of the cu-disk Δ_0.

First step of induction. — Notice that since $\|Df\|$ is uniformly bounded, for any $n \geq 1$, all hyperbolic pre-disks $V_n(x)$ contain a ball of some radius $\tau_n > 0$ depending only on n. In particular, by compactness, the set $H_n \cap \Delta_0$ is covered by a finite number of hyperbolic pre-disks $V_n(x)$. We fix some large $n_0 \in \mathbb{N}$ and ignore any dynamics occurring up to time n_0. Then there exist ℓ_{n_0} and points $z_1, \dots, z_{\ell_{n_0}} \in H_{n_0}$ such that

$$H_{n_0} \cap \Delta_0 \subset V_{n_0}(z_1) \cup \cdots \cup V_{n_0}(z_{\ell_{n_0}}).$$

We now choose a maximal subset of points $x_1, \dots, x_{j_{n_0}} \in \{z_1, \dots, z_{\ell_{n_0}}\}$ such that the corresponding sets $\omega_{n_0}^{x_i}$ of type (10) are pairwise disjoint and contained in Δ_0, and let

$$\mathcal{P}_{n_0} = \{\omega_{n_0}^{x_1}, \dots, \omega_{n_0}^{x_{j_{n_0}}}\}.$$

These are the elements of the partition $\mathcal{P}$ constructed in the n_0-step of the algorithm. Let

$$\Delta_{n_0} = \Delta \setminus \bigcup_{\omega \in \mathcal{P}_{n_0}} \omega.$$

For each $0 \leq i \leq j_{n_0}$, we define the inducing time

$$R|_{\omega_{n_0}^{x_i}} = n_0 + m_i$$

where $0 \leq m_i \leq N$ is the integer associated to $\omega_{n_0}^{x_i}$ as in (10). Let now Z_{n_0} be the set of points in $\{z_1, \dots, z_{\ell_{n_0}}\}$ which were not chosen in the construction of $\mathcal{P}_{n_0}$, i.e.

$$Z_{n_0} = \{z_1, \dots, z_{\ell_{n_0}}\} \setminus \{x_1, \dots, x_{j_{n_0}}\}.$$

We remark that for every $z \in Z_{n_0}$, the set $\omega_{n_0}^z$ associated to z must either intersect some $\omega_{n_0}^{x_i} \in \mathcal{P}_{n_0}$ or intersect the complement of Δ_0 in Δ, since otherwise it would have been included in the set $\mathcal{P}_{n_0}$.

We now introduce some notation to keep track of which one of the above reasons is responsible for the fact that z belongs to Z_{n_0}. We let $\Delta_0^c = \Delta \setminus \Delta_0$ and for each $\omega \in \mathcal{P}_{n_0} \cup \{\Delta_0^c\}$ we define

$$Z_{n_0}^\omega = \left\{x \in Z_{n_0} : \omega_{n_0}^x \cap \omega \neq \emptyset\right\}$$

and the associated n_0-*satellite* set

$$S_{n_0}^\omega = \bigcup_{x \in Z_{n_0}^\omega} V_{n_0}(x).$$

Finally let

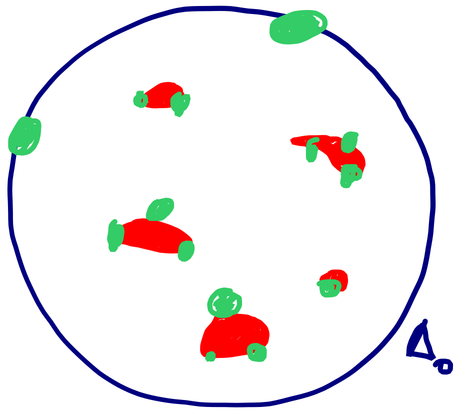

FIGURE 5. First elements and satellites

$$V_{n_0} = \bigcup_{i=1}^{j_{n_0}} V(x_i)$$

and

$$S_{n_0} = \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \{\Delta_0^c\}} S_{n_0}^{\omega} \cup V_{n_0}.$$

Notice that S_{n_0}

$$H_{n_0} \cap \Delta_0 \subset S_{n_0} \cup \bigcup_{\omega \in \mathcal{P}_{n_0}} \omega.$$

General step of induction. — We now proceed inductively and assume that the construction has been carried out up to time $n-1$ for some $n > n_0$. More precisely, for each $n_0 \le k \le n-1$ we have a collection of pairwise disjoint sets $\mathcal{P}_k = \{\omega_k^{x_1}, ..., \omega_k^{x_{j_k}}\}$ which "return" at time $k+m$ with $0 \le m \le N$, and such that for any $k \ne k'$, any two sets $\omega \in \mathcal{P}_k$ and $\omega' \in \mathcal{P}_{k'}$ we have $\omega \cap \omega' = \emptyset$. We also have a set Δ_k which is the set of points which do not yet have an associated return time. To construct all relevant objects at time n, we note first all, as before, that there are $z_1, \dots, z_{\ell_n} \in H_n \cap \Delta_{n-1}$ such that

$$H_n \cap \Delta_{n-1} \subset V_n(z_1) \cup \cdots \cup V_n(z_{\ell_n}),$$

and we choose a maximal subset of points $x_1, \dots, x_{j_n} \in \{z_1, \dots, z_{\ell_n}\}$ such that the corresponding sets of type (10) are pairwise disjoint and contained in Δ_{n-1}. Then we let

$$\mathcal{P}_n = \{\omega_n^{x_1}, \dots, \omega_n^{x_{j_n}}\}$$

These are the elements of the partition $\mathcal{P}$ constructed in the n-step of the algorithm. We also define the set of points of Δ_0 which do not belong to partition elements constructed up to this point:

$$\Delta_n = \Delta_0 \setminus \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n} \omega.$$

For each $0 \le i \le j_n$ we set

$$R|_{\omega_n^{x_i}} = n + m_i,$$

where $0 \le m_i \le N$ is the integer associated to $\omega_{n_0}^{x_i}$ as in (10). Let

$$Z_n = \{z_1, \dots, z_{\ell_n}\} \setminus \{x_1, \dots, x_{j_n}\}$$

and for any $\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n \cup \{\Delta_0^c\}$ define

$$Z_n^\omega = \{z \in Z_n : \omega_n^z \cap \omega \neq \emptyset\}$$

and its *n-satellite*

$$S_n^\omega = \bigcup_{z \in Z_n^\omega} V_n(z).$$

Finally let

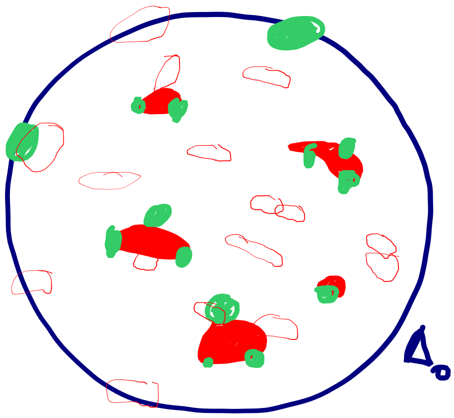

FIGURE 6. Next elements

$$V_n = \bigcup_{i=1}^{j_n} V(x_i)$$

and

$$S_n = \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n \cup \{\Delta_0^c\}} S_n^\omega \cup V_n.$$

Note that for each $n \ge n_0$ one has

$$H_n \cap \Delta_{n-1} \subset S_n \cup \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n} \omega. \tag{11}$$

More specifically we have that $H_n \cap \Delta_{n-1} \subset S_n$, i.e. all points in Δ_{n-1} which have a hyperbolic time at time n are "covered" by S_n while the points which have a hyperbolic time at time n but which are already contained in previously constructed partition elements, are trivially 'covered" by the union of these partition elements. The inclusion (11) will be crucial to prove the integrability of the return times.

This inductive construction allows us to define the family

$$\mathcal{P} = \bigcup_{n \geq n_0} \mathcal{P}_n$$

of pairwise disjoint subsets of Δ_0. At this point there is no guarantee that $\mathcal{P}$ forms a Leb mod 0 partition of Δ_0. This will follow as a consequence of Proposition 3.7 below.

3.3. The measure of satellites. — In this section we state some estimates on the decay of the Lebesgue measure of satellites. This will be useful to show that the elements of $\mathcal{P}$ defined in the previous section form a Leb_Δ mod 0 partition of the disk Δ_0 introduced in (9). The estimates will also be used later to prove the integrability of the return times with respect to Leb_Δ.

Lemma 3.6. — *There is $C > 0$ such that for all $k \geq n_0$, $\omega \in \mathcal{P}_k \cup \{\Delta_0^c\}$ and $n \geq k$, we have*

$$\mathrm{Leb}_D(S_n(\omega)) < C\sigma^{\frac{n-k}{2}} \, \mathrm{Leb}_D(\omega).$$

Using this lemma one can prove the following result.

Proposition 3.7. — $\displaystyle\sum_{n=n_0}^{\infty} \mathrm{Leb}_\Delta(S_n) < \infty.$

For a proof of these results see [ADLP14, Section 5].

3.4. The partition. — We are now ready to show that our inductive construction gives rise to a Leb_Δ mod 0 partition of Δ_0. Recall that $\Delta_0 \supset \Delta_{n_0} \supset \Delta_{n_0+1} \supset$ ..., where Δ_n is the set of points which does not belong to any element of the collection $\mathcal{P}$ constructed up to time n. It is enough to show that

$$\mathrm{Leb}_\Delta \left(\bigcap_n \Delta_n \right) = 0. \tag{12}$$

To prove this, notice that by Proposition 3.7, the sum of the Leb_Δ measures of the sets S_n is finite. It follows from Borel-Cantelli Lemma that Leb_Δ almost every $x \in \Delta_0$ belongs only to finitely many S_n's, and therefore one can find n such that $x \notin S_j$ for $j \geq n$. Since Leb_Δ almost every $x \in \Delta_0$ has infinitely many hyperbolic times, it follows from (11) that $x \in \omega$ for some $\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n$ and therefore (12) holds.

We are now ready to define the GMY structure on Ω as in the beginning of Section 3.2. Consider the center-unstable disk $\Delta_0 \subset \Delta$ as in (9) and the Leb_Δ mod 0 partition $\mathcal{P}$ of Δ_0 defined in Section 3.2. We define

$$\Gamma^s = \left\{ W^s_{\delta_s}(x) : x \in \Delta_0 \right\}.$$

Moreover, we define Γ^u as the set of all local unstable manifolds contained in $\mathcal{C}_0$ which u-cross $\mathcal{C}_0$. Clearly, Γ^u is nonempty because $\Delta_0 \in \Gamma^u$. We need to see that the union of the leaves in Γ^u is compact. This follows ideas that we have already used to prove Proposition 3.4. By the domination property and Ascoli-Arzelà Theorem, any limit leaf γ_∞ of leaves in Γ^u is still a cu-disk u-crossing $\mathcal{C}_0$. Thus, by definition of Γ^u, we have $\gamma_\infty \in \Gamma^u$. We thus define our set Λ with hyperbolic product structure as the intersection of these families of stable and unstable leaves. The cylinders $\{\mathcal{C}(\omega)\}_{\omega\in\mathcal{P}}$ then clearly form a countable collection of s-subsets of Λ that play the role of the sets $\Lambda_1, \Lambda_2, \dots$ in (P_1) with the corresponding return times $R(\omega)$. We refer to [ADLP14, Section 6] to see that (P_1)-(P_5) hold. It remains to check the integrability of the return times.

3.5. Integrability of the return times. — So far, we have proved some result in a general setting for an induced map F over a GMY structure Λ and the respective quotient map $\bar{F}$. We have shown that $\bar{F}$ has an absolutely continuous measure with respect to a reference measure $\bar{m}$ and this measure $\bar{\nu}$ lifts to a measure ν which is an SRB measure for F.

Our goal now is to prove the integrability of the return time for the GMY structure constructed in Section 3 with respect to R, which completes the proof of Theorem 2.6. Noting that R is constant on stable leaves, by Proposition **??** it is enough to show that R is integrable with respect to $\bar{\nu}$. Note also that in this case we may think of Λ as being equal to the disk Δ_0 and $\bar{\nu}$ as being Lebesgue measure on Δ_0.

Lemma 3.8. — *The inducing time function R is $\bar{\nu}$-integrable on Δ_0.*

Proof. — For $x \in \Delta$ we consider the orbit $x, f(x), ..., f^{n-1}(x)$ of the point x under iteration by f for some large value of n. Then we define

$H^{(n)}(x) :=$ number of hyperbolic times for x before time n

$S^{(n)}(x) :=$ number of times x belongs to a satellite before time n

$R^{(n)}(x) :=$ number of returns of x before time n

Each time x has a hyperbolic time, it either has a return within some finite number of iterations, or it belongs to a satellite. Therefore,

$$R^{(n)}(x) + S^{(n)}(x) \geq H^{(n)}(x)$$

Dividing by n we get

$$\frac{R^{(n)}(x)}{n} + \frac{S^{(n)}(x)}{n} \geq \frac{H^{(n)}(x)}{n}.$$

Recall that there exist $\theta > 0$ and arbitrarily large values of n such that $H^{(n)}(x)/n \geq \theta$, and therefore

$$\frac{R^{(n)}(x)}{n}\left(1+\frac{S^{(n)}(x)}{R^{(n)}(x)}\right) \geq \theta.$$

By Birkhoff's Ergodic Theorem we have

$$\frac{S^{(n)}(x)}{R^{(n)}(x)} \longrightarrow \int S d\bar{\nu} \simeq \sum_{n=n_0}^{\infty} \mathrm{Leb}_{\Delta}(S_n) < \infty.$$

(Note that $R^{(n)}(x)$ is the number of iterations under the induced map F). It follows from the last two equations that there is $\kappa > 0$ such that we can choose arbitrarily large values of n for which

$$\frac{R^{(n)}(x)}{n} \geq \kappa. \tag{13}$$

Arguing by contradiction, assume that $\int R d\bar{\nu} = +\infty$. Notice that the average return times before time n is

$$\frac{n}{R^{(n)}(x)} \longrightarrow \int R d\bar{\nu} = +\infty.$$

Therefore

$$\frac{R^{(n)}(x)}{n} \longrightarrow 0.$$

This contradicts (13), and so we must have

$$\int R d\bar{\nu} < +\infty.$$

□

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