Time delay and propagation direction of light in static, spherically symmetric space-times

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See, e.g., Klioner & Zschocke 2010, and refs. therein.

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  "To understand the n-th order, know the $(n+1)$*-th order"*

  (popular wisdom)

- Illustrated in the context of Gaia mission by a recent analysis taking into account 'enhanced' post-post-Newtonian terms in a 3-parameter family of static, spherically symmetric space-times (Klioner & Zschocke 2010, Zschocke 2011).

Time delay and light direction in static, spherically symmetric (s.s.s) space-times (1)

- We assume that space-time is endowed with a s.s.s. metric:

$$ds^2 = \mathcal{A}(r)(dx^0)^2 - \mathcal{B}^{-1}(r)\,\delta_{ij}dx^i dx^j, \qquad \lim_{r\to\infty}\mathcal{A}(r) = \lim_{r\to\infty}\mathcal{B}(r) = 1$$

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"Time transfer (or time delay) function" $\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B)$

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- Direction of Γ at any of its points $x(\lambda)$:

$$\widehat{\underline{\mathbf{l}}} = (l_i/l_0)\,, \quad l_0 = \mathcal{A}(r)\frac{dx^0}{d\lambda}, \quad l_i = -\frac{1}{\mathcal{B}(r)}\frac{dx^i}{d\lambda},$$

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- Relation with the tangent vector:

$$\widehat{\underline{\mathbf{l}}} = -\frac{1}{\mathcal{A}(r)\mathcal{B}(r)}\frac{d\mathbf{x}}{dx^0}$$

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- Angular separation ϕ_{U_B} between $\mathbf{x}_A$ and $\mathbf{x}_{A'}$ *as measured* by $\mathcal{O}(U_B)$:

$$\sin^2\frac{\phi_{U_B}}{2} = \frac{1}{4}\mathcal{A}(r_B)\mathcal{B}(r_B)(\widehat{\underline{\mathbf{l}}}_B - \widehat{\underline{\mathbf{l}}}'_B)^2$$

(see T. & Le Poncin-Lafitte 2006)

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- Since $\sqrt{\mathcal{A}(r)\mathcal{B}(r)}\,\underline{\widehat{\mathbf{l}}}$ is a unit vector for the *usual Euclidean* norm

$$\phi_{U_B} = \text{Euclidean angle } \angle(\underline{\widehat{\mathbf{l}}}_B, \underline{\widehat{\mathbf{l}}}'_B) \equiv \text{ angle given by } \cos\phi_{U_B} = \frac{\underline{\widehat{\mathbf{l}}}_B . \underline{\widehat{\mathbf{l}}}'_B}{|\underline{\widehat{\mathbf{l}}}_B| . |\underline{\widehat{\mathbf{l}}}'_B|}$$

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    $\underline{\widehat{\mathbf{l}}}_A$, $\underline{\widehat{\mathbf{l}}}_B$ and $\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B)$ are obtained independently (Goicoechea et al. 1992; under current investigation by Linet & T.)

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where $(m = GM/c^2,\ M = \text{mass of the central body})$

$$\mathcal{A}(r) = 1 - \frac{2m}{r} + 2\beta\frac{m^2}{r^2} + \sum_{n=3}^{\infty}\frac{(-1)^n n}{2^{n-2}}\beta_{n-1}\frac{m^n}{r^n}$$

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In GR

$$\beta = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \cdots = 1, \quad \gamma = \gamma_2 = \gamma_3 = \gamma_4 = 1, \quad \gamma_5 = \cdots = 0$$

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$$c^2\delta_{ij}\frac{\partial\mathcal{T}(\mathbf{x},\mathbf{x}_B)}{\partial x^i}\frac{\partial\mathcal{T}(\mathbf{x},\mathbf{x}_B)}{\partial x^j} - \frac{1}{\mathcal{A}(r)\mathcal{B}(r)} = 0$$

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Assuming

$$\mathcal{T}(\mathbf{x}_A,\mathbf{x}_B)=\frac{|\mathbf{x}_B-\mathbf{x}_A|}{c}+\sum_{n=1}^{\infty}G^n\mathcal{T}^{(n)}(\mathbf{x}_A,\mathbf{x}_B)$$

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we find each $\mathcal{T}^{(n)}(\mathbf{x}_A, \mathbf{x}_B)$ by iteration as an integral over the straight line joining $\mathbf{x}_A$ and $\mathbf{x}_B$ (T. & Le Poncin 2008).

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$$\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B) = \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c} + \frac{(\gamma+1)m}{c} \ln\left(\frac{r_A + r_B + |\mathbf{x}_B - \mathbf{x}_A|}{r_A + r_B - |\mathbf{x}_B - \mathbf{x}_A|}\right) + \frac{m^2}{r_A r_B} \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c} \left[\frac{\kappa \arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A \times \mathbf{n}_B|} - \frac{(\gamma+1)^2}{1 + \mathbf{n}_A.\mathbf{n}_B}\right] + \cdots ,$$

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$$\mathbf{n}_A = \frac{\mathbf{x}_A}{r_A}, \quad \mathbf{n}_B = \frac{\mathbf{x}_B}{r_B}, \quad \kappa = \frac{8 - 4\beta + 8\gamma + 3\gamma_2}{4}$$

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$$+\frac{m^2}{r_Ar_B}\frac{|\mathbf{x}_B-\mathbf{x}_A|}{c}\left[\frac{\kappa\arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A\times\mathbf{n}_B|}-\frac{(\gamma+1)^2}{1+\mathbf{n}_A.\mathbf{n}_B}\right]+\cdots,$$

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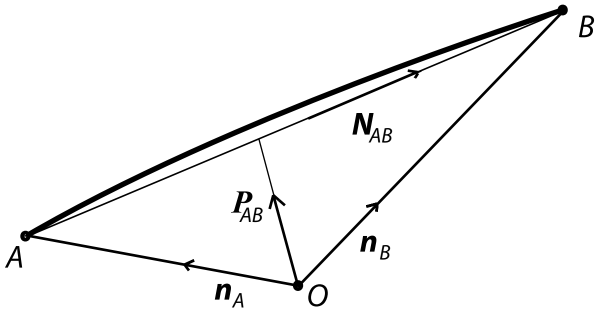
- We use $\widehat{\underline{\mathbf{l}}}_A=c\boldsymbol{\nabla}_{\mathbf{x}_A}\mathcal{T}(\mathbf{x}_A,\mathbf{x}_B)$ and $\widehat{\underline{\mathbf{l}}}_B=-c\boldsymbol{\nabla}_{\mathbf{x}_B}\mathcal{T}(\mathbf{x}_A,\mathbf{x}_B)$.

Schema



Post-post-Newtonian approximation (1)

Introducing

$$\mathbf{N}_{AB} = \frac{\mathbf{x}_B - \mathbf{x}_A}{|\mathbf{x}_B - \mathbf{x}_A|}, \qquad \mathbf{P}_{AB} = \mathbf{N}_{AB} \times \left(\frac{\mathbf{n}_A \times \mathbf{n}_B}{|\mathbf{n}_A \times \mathbf{n}_B|} \right),$$

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we get, e.g.,

$$\begin{aligned} \widehat{\underline{\mathbf{l}}}_B = -\mathbf{N}_{AB} - \frac{m}{r_B}\left\{\gamma + 1 + \frac{m}{r_B}\left[\kappa - \frac{(\gamma+1)^2}{1 + \mathbf{n}_A.\mathbf{n}_B}\right]\right\}\mathbf{N}_{AB} \\ + \frac{m}{r_B}\left\{(\gamma+1)\frac{|\mathbf{n}_A \times \mathbf{n}_B|}{1 + \mathbf{n}_A.\mathbf{n}_B} - \frac{m}{r_c}\left\{\kappa\left[\frac{\arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A \times \mathbf{n}_B|}(\mathbf{N}_{AB}.\mathbf{n}_A)\right.\right.\right. \\ \left.\left.\left. -(\mathbf{N}_{AB}.\mathbf{n}_B)\right] + (\gamma+1)^2 \frac{r_A + r_B}{|\mathbf{x}_B - \mathbf{x}_A|}\frac{1 - \mathbf{n}_A.\mathbf{n}_B}{1 + \mathbf{n}_A.\mathbf{n}_B}\right\}\right\}\mathbf{P}_{AB} \end{aligned}$$

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$$+ \frac{m}{r_B}\left\{ (\gamma+1)\frac{|\mathbf{n}_A \times \mathbf{n}_B|}{1 + \mathbf{n}_A.\mathbf{n}_B} - \frac{m}{r_c}\left\{ \kappa \left[\frac{\arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A \times \mathbf{n}_B|}(\mathbf{N}_{AB}.\mathbf{n}_A) \right.\right.\right.$$
$$\left.\left.\left. -(\mathbf{N}_{AB}.\mathbf{n}_B) \right] + (\gamma+1)^2 \, \frac{r_A + r_B}{|\mathbf{x}_B - \mathbf{x}_A|} \, \frac{1 - \mathbf{n}_A.\mathbf{n}_B}{1 + \mathbf{n}_A.\mathbf{n}_B} \right\} \right\} \mathbf{P}_{AB}$$

$$r_c = \frac{r_A r_B}{|\mathbf{x}_B - \mathbf{x}_A|}|\mathbf{n}_A \times \mathbf{n}_B| = 0^{th}\text{-order “distance of closest approach”}.$$

Post-post-Newtonian approximation (2)

For geodesics in static, spherically symmetric space-times (Chandrasekhar 1983):

$b = |-\mathbf{x} \times \widehat{\underline{\mathbf{l}}}| =$ *impact parameter* of the light ray (*intrinsic quantity*)

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So

$$b = r_c \Bigg\{ 1 + \frac{(\gamma+1)m}{r_c} \frac{|\mathbf{N}_{AB} \times \mathbf{n}_A| + |\mathbf{N}_{AB} \times \mathbf{n}_B|}{1 + \mathbf{n}_A.\mathbf{n}_B} + \frac{m^2}{r_c^2} \Bigg\{ \kappa \left[1 - \frac{\arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A \times \mathbf{n}_B|} (\mathbf{N}_{AB}.\mathbf{n}_A)(\mathbf{N}_{AB}.\mathbf{n}_B) \right] - (\gamma+1)^2 \frac{1 - \mathbf{n}_A.\mathbf{n}_B + |\mathbf{N}_{AB} \times \mathbf{n}_A|.|\mathbf{N}_{AB} \times \mathbf{n}_B|}{1 + \mathbf{n}_A.\mathbf{n}_B} \Bigg\} \Bigg\}$$

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Light deflection

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- The term in blue is currently used in high-accuracy astrometry (VLBI,...)
- $\Delta\chi_B$ is a *coordinate-independent* quantity.

Method of "constrained integration"(1)

We use now spherical coordinates:

$$ds^2 = \mathcal{A}(r)(dx^0)^2 - \mathcal{B}^{-1}(r)(dr^2 + r^2 d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

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Assume

$$b = r_c \left[1 + \sum_{n=1}^{\infty} q_n \left(\frac{m}{r_c} \right)^n \right]$$

Method of "constrained integration"(2)

Then

$$\frac{dx^0}{dr} = \pm \frac{r}{\sqrt{r^2 - r_c^2}} \pm \sum_{n=1}^{\infty} \left(\frac{m}{r_c}\right)^n X_n(r, r_c, q_1, \ldots, q_n)$$

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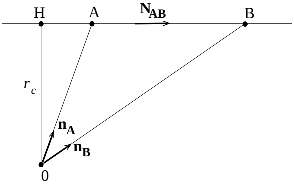

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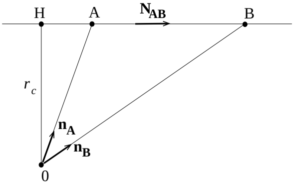

(In the case $\mathbf{N}_{AB}.\mathbf{n}_A < 0$, be careful! We have to introduce the pericenter P...)

Method of "constrained integration" (3)

$$c\mathcal{T}(\mathbf{x}_A,\mathbf{x}_B)=\underbrace{\int_{r_A}^{r_B}\frac{rdr}{\sqrt{r^2-r_c^2}}}_{=|\mathbf{x}_B-\mathbf{x}_A|}+\sum_{n=1}^{\infty}\left(\frac{m}{r_c}\right)^n\int_{r_A}^{r_B}X_n(r,r_c,q_1,\ldots,q_n)dr,$$

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- We know that $r_c = \dfrac{r_A r_B}{|\mathbf{x}_B - \mathbf{x}_A|} |\mathbf{n}_A \times \mathbf{n}_B|$
- We have to determine $q_1 = q_1(\mathbf{x}_A, \mathbf{x}_B)$, $q_2 = q_2(\mathbf{x}_A, \mathbf{x}_B)$, etc.

Method of "constrained integration"(4)

Using the geodesic eq. satisfied by φ

$$\frac{d\varphi}{dr} = \pm' \frac{b}{r} \frac{\sqrt{\mathcal{A}(r)\mathcal{B}(r)}}{\sqrt{r^2 - b^2\mathcal{A}(r)\mathcal{B}(r)}}$$

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we get the expansion

$$\frac{d\varphi}{dr} = \pm' \frac{r_c}{r} \frac{1}{\sqrt{r^2 - r_c^2}} \pm' \sum_{n=1}^{\infty} \left(\frac{m}{r_c}\right)^n Y_n(r, r_c, q_1, \ldots, q_n)$$

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$$\Downarrow$$

$$\varphi_B - \varphi_A = \underbrace{\pm' \int_{r_A}^{r_B} \frac{r_c}{r} \frac{dr}{\sqrt{r^2 - r_c^2}}}_{= \varphi_B - \varphi_A} \underbrace{\pm' \sum_{n=1}^{\infty} \left(\frac{m}{r_c}\right)^n \int_{r_A}^{r_B} Y_n(r, r_c, q_1, \ldots, q_n) dr}_{=0}$$

Method of "constrained integration"(5)

So we have the infinite system

$$\int_{r_A}^{r_B} Y_n(r, r_c, q_1, \ldots, q_n)dr = 0 \quad \text{for} \quad n = 1, 2, \ldots$$

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- $\int_{r_A}^{r_B} Y_2(r, r_c, q_1, q_2)dr = 0 \to q_2$
- etc.

Method of "constrained integration"(6)

Moreover

$$\int_{r_A}^{r_B} X_n(r, r_c, q_1, \ldots, q_n)dr$$

$$= \int_{r_A}^{r_B} \left[X_n(r, r_c, q_1, \ldots, q_n) - \sum_{p=1}^{n} k_{np} Y_p(r, r_c, q_1, \ldots, q_n) \right] dr \;\; \forall n \quad \forall k_{np}$$

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- A judicious choice of the k_{np} simplifies the integrals
- For $n = 2$ and $n = 3$, q_2 and q_3 *are not involved*

  $\Rightarrow$ considerable simplification of the expressions

For the Schwarzschild space-time in isotropic coordinates, this method yields:

$$\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B) = \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c} + \frac{2m}{c} \ln\left(\frac{r_A + r_B + |\mathbf{x}_B - \mathbf{x}_A|}{r_A + r_B - |\mathbf{x}_B - \mathbf{x}_A|}\right)$$
$$+ \frac{m^2}{r_A\, r_B} \frac{|\mathbf{x}_B - \mathbf{x}_A|}{c} \left[\frac{15}{4} \frac{\arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A \times \mathbf{n}_B|} - \frac{4}{1 + \mathbf{n}_A.\mathbf{n}_B}\right]$$

Schwarzschild space-time within the 3PN approximation

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$$
\begin{aligned}
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& + \frac{m^2}{r_A\, r_B}\,\frac{|\mathbf{x}_B-\mathbf{x}_A|}{c}\left[\frac{15}{4}\,\frac{\arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A\times\mathbf{n}_B|} - \frac{4}{1+\mathbf{n}_A.\mathbf{n}_B}\right] \\
& + \frac{1}{2}\,\frac{m^3}{r_A\, r_B}\left(\frac{1}{r_A}+\frac{1}{r_B}\right)\frac{|\mathbf{x}_B-\mathbf{x}_A|}{c}\,\frac{1}{1+\mathbf{n}_A.\mathbf{n}_B} \\
& \times\left[9+\frac{16}{1+\mathbf{n}_A.\mathbf{n}_B} - 15\,\frac{\arccos(\mathbf{n}_A.\mathbf{n}_B)}{|\mathbf{n}_A\times\mathbf{n}_B|}\right] + \cdots
\end{aligned}
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- We confirm the recent discussion of "enhanced post-post-Newtonian terms" in the Gaia context.

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