Consiglio Nazionale delle Ricerche

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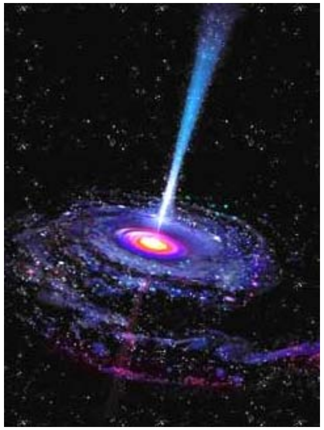
# *Observers, observables and frames in general relativity: applications to light propagation tracing*

**How the measurement process is modeled in the context of General Relativity?**

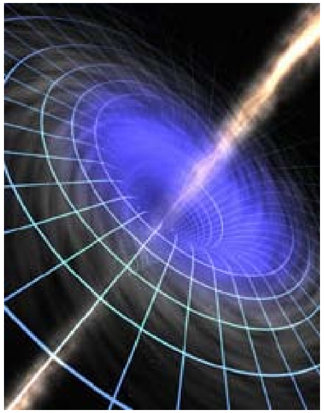
(GR=default)

# 1: Spacetime

→Select the background spacetime metric to model specific astrophysics situations (the physical phenomena, i.e. the "system" under consideration).








**SPACETIME (=4d point of view):**

Absolute formulation of physics laws, unifying aspects of GR,
Geometrical, elegant …

but

not too much useful for our “classical” understanding of physical phenomena:
The latter are in fact something happening somewhere in space and having
a definite duration in time (enough to perform measurements), i.e. it is usually
Meant as having a 3d+1d characterization istead of a 4d characterization.

# 2: <u>Observables</u>

→Select the obervables (specific measurements to be performed).

Path deflections on particle or photon motion

Electromagnetic field strenght

Fluid pressure, energy density, dynamics

Gravitational radiation

Precessions

Scattering

# 3: Observers

→Select the observer family suited for the chosen measurement (specific kinematical status of the observer family with respect to the background spacetime).

u=Observer four velocity vector

$$
\begin{aligned}
T(u) &= -u^\sharp \otimes u^\flat \\
P(u) &= I_{TM} + u^\sharp \otimes u^\flat
\end{aligned}
$$

Spacetime (tangent space) splitting associated with u

Systematic use of projectors for tensor and equations

$$[P(u)S]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_\gamma \cdots P(u)^\delta{}_\beta \cdots S^{\gamma\ldots}{}_{\delta\ldots}$$

$$[T(u)S]^{\alpha\ldots}{}_{\beta\ldots} = T(u)^\alpha{}_\gamma \cdots T(u)^\delta{}_\beta \cdots S^{\gamma\ldots}{}_{\delta\ldots}.$$

# (Geometric) Measurements

1+3 decomposition = geometric measurement

$$S \quad \leftrightarrow \quad \{u \cdot S, [P(u)S]\}$$

Measurement of a vector

$$S^\alpha \quad \leftrightarrow \quad \{u_\gamma S^\gamma, P(u)^\alpha{}_\gamma S^\gamma\}$$

Measurement of a $\binom{1}{1}$ tensor

$$S^\alpha{}_\beta \leftrightarrow \{u^\delta u_\gamma S^\gamma{}_\delta, \; P(u)^\alpha{}_\gamma u^\delta S^\gamma{}_\delta, \; P(u)^\delta{}_\alpha u_\gamma S^\gamma{}_\delta, \; P(u)^\alpha{}_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta\}$$

**<u>Examples: splitting of tensors</u>**

$$g_{\alpha\beta} = P(u)_{\alpha\beta} + T(u)_{\alpha\beta} \qquad \eta_{\alpha\beta\gamma\delta} = -2u_{[\alpha}\eta(u)_{\beta]\gamma\delta} - 2u_{[\gamma}\eta(u)_{\delta]\alpha\beta}$$

$$\eta(u)_{\alpha\beta\gamma} = u^\delta \eta_{\delta\alpha\beta\gamma}$$

# A note:

The “splitting game” with both tensor (familiar) and tensorial differential operators (unfamiliar) is a <u>necessary</u> tool to reproduce formal “3+1” expressions having also a geometric consistency and meaning.

# Splitting of differential operators

(i) The Lie derivative of $T$ along the direction of a given vector field $X$: $[\pounds_X T]$.

(ii) The covariant derivative of $T$: $\nabla T$.

(iii) The absolute derivative of $T$ along a curve with unit tangent vector $X$ and parameterized by $s$: $\nabla_X T \equiv DT/ds$.

(iv) The Fermi-Walker derivative of $T$ along a non-null curve with unit tangent vector $X$ and parameterized by $s$: $D_{(\mathrm{fw},X)}T/ds$.

Finally if $S$ is a $p$-form, one has

(v) The exterior derivative of $S$: $dS$.

4 $\longrightarrow$ 3+1

(i) The spatially projected Lie derivative along a vector field $X$

$$[\pounds(u)_X T]^{\alpha\ldots}{}_{\beta\ldots} \equiv P(u)^\alpha{}_\sigma \ldots P(u)^\rho{}_\beta \ldots [\pounds_X T]^{\sigma\ldots}{}_{\rho\ldots}; \qquad (3.26)$$

when $X = u$ we use also the notation

$$\nabla(u)_{(\rm lie)} T \equiv \pounds(u)_u T \ , \qquad (3.27)$$

and this operation will be termed "spatial-Lie temporal derivative".

(ii) The spatially projected covariant derivative along any $e_\gamma$ frame direction

$$\nabla(u)_\gamma T \equiv P(u) \nabla_\gamma T, \qquad (3.28)$$

namely

$$[\nabla(u)_\gamma T]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_{\alpha_1} \ldots P(u)^{\beta_1}{}_\beta \ldots P(u)^\sigma{}_\gamma \nabla_\sigma T^{\alpha_1\ldots}{}_{\beta_1\ldots}. \qquad (3.29)$$

(iii) The spatially projected absolute derivative along a curve with unit tangent vector $X$

$$[P(u)\nabla_X T]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_{\alpha_1} \ldots P(u)^{\beta_1}{}_\beta \ldots [\nabla_X T]^{\alpha_1\ldots}{}_{\beta_1\ldots} \qquad (3.30)$$

(iv) The spatially projected "Fermi-Walker derivative" along a curve with unit tangent vector $X$ and parameterized by $s$

$$\left[P(u)\frac{D_{({\rm fw},X)}T}{ds}\right]^{\alpha\ldots}{}_{\beta\ldots} = P(u)^\alpha{}_\sigma \ldots P(u)^\rho{}_\beta \ldots \left[\frac{D_{({\rm fw},X)}T}{ds}\right]^{\sigma\ldots}{}_{\rho\ldots} \qquad (3.31)$$

(v) the spatially projected exterior derivative of a $p$-form $S$

$$d(u)S \equiv P(u)dS, \qquad (3.32)$$

namely

$$[d(u)S]_{\alpha_1\ldots\alpha_p\beta} = P(u)^{\beta_1}{}_{\alpha_1} \ldots P(u)^\sigma{}_\beta [dS]_{\beta_1\ldots\sigma} \ . \qquad (3.33)$$

# 3d notation

$$X \cdot_u Y = P(u)_{\alpha\beta} X^\alpha Y^\beta$$

$$[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma} X^\beta Y^\gamma$$

$$\begin{aligned} \mathrm{grad}_u\, f &= \nabla(u) f \ , \\ \mathrm{curl}_u\, X &= \nabla(u) \times_u X \ , \\ \mathrm{div}_u\, X &= \nabla(u) \cdot_u X \ . \end{aligned}$$

… less familiar

$$\begin{aligned} [X \times_u A]^{\alpha\beta} &= \eta(u)^{\gamma\delta(\alpha} X_\gamma A^{\beta)}{}_\delta \\ [A \times_u B]_\alpha &= \eta(u)_{\alpha\beta\gamma} A^\beta{}_\delta\, B^{\delta\gamma} \\ [A \cdot_u B]_\alpha{}^\beta &= A_{\alpha\gamma} B^{\gamma\beta} \ , \\ \mathrm{Tr}[A \cdot_u B] &= A_{\alpha\beta} B^{\alpha\beta} \ . \end{aligned}$$

$$[\mathrm{Scurl}_u\, A]^{\alpha\beta} = \eta(u)^{\gamma\delta(\alpha} \nabla(u)_\gamma\, A^{\beta)}{}_\delta$$

… less familiar

$$[\mathrm{div}_u\, X]^{\alpha\ldots\beta} = \nabla(u)_\sigma X^{\sigma\alpha\ldots\beta}, \quad X = P(u) X$$

# Geometrical properties and kinematics of the observer congruence

**Acceleration, vorticity, expansion (shear)**

$$\begin{aligned} a(u) &= P(u)\nabla_u u \ , \\ k(u) &= -\nabla(u)u = \omega(u) - \theta(u) \ . \end{aligned}$$

$$\begin{aligned} [\omega(u)]_{\alpha\beta} &= -P(u)^{\mu}_{\alpha} P(u)^{\nu}_{\beta} \nabla_{[\mu} u_{\nu]} \ , \\ [\theta(u)]_{\alpha\beta} &= P(u)^{\mu}_{\alpha} P(u)^{\nu}_{\beta} \nabla_{(\mu} u_{\nu)} \\ &= \frac{1}{2}[\pounds(u)_u P(u)]_{\alpha\beta} \ , \end{aligned}$$

The obervers with their timelike world lines fill An (open) region of spacetime.
The boudary of such a region corresponds to causality changings, i.e. marks the location where they are no more “useful.”

# 4: Observer-adapted frames

→Select the most convenient spatial frame for the chosen observer family (specify e.g. the geometrical properties of such spatial axes).

Given a field of observers $u$, a frame $\{e_\alpha\}$ with $\alpha = 0, 1, 2, 3$ (with dual $\omega^\alpha$) is termed adapted to $u$ if $e_0 = u$ and $e_a$ with $a = 1, 2, 3$ are orthogonal to $u$, namely $u \cdot e_a = 0$. From this it follows that $\omega^0 = -u^\flat$. Obviously the components of $u$ relative to the frame are simply $u^\alpha = \delta^\alpha_0$ and the metric tensor reads

$$g^\flat = -u^\flat \otimes u^\flat + P(u)_{ab}\omega^a \otimes \omega^b.$$

# Natural frames: FS

$E_{\hat{0}} = U$ (observer's world line unit tangent vector)

$$\frac{DE_{\hat{0}}}{d\tau_U} = \kappa E_{\hat{1}}, \qquad \frac{DE_{\hat{1}}}{d\tau_U} = \kappa E_{\hat{0}} + \tau_1 E_{\hat{2}},$$

$$\frac{DE_{\hat{2}}}{d\tau_U} = -\tau_1 E_{\hat{1}} + \tau_2 E_{\hat{3}}, \qquad \frac{DE_{\hat{3}}}{d\tau_U} = -\tau_2 E_{\hat{2}},$$

Absolute FS frames, to be distinguished from relative FS frames

$$\begin{aligned}
\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}}\hat{\nu}(U,u) &= k_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}, \\
\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}}\hat{\eta}_{(\mathrm{fw},U,u)} &= -k_{(\mathrm{fw},U,u)}\hat{\nu}(U,u) + \tau_{(\mathrm{fw},U,u)}\hat{\beta}_{(\mathrm{fw},U,u)}, \\
\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}}\hat{\beta}_{(\mathrm{fw},U,u)} &= -\tau_{(\mathrm{fw},U,u)}\hat{\eta}_{(\mathrm{fw},U,u)}.
\end{aligned}$$

$$U = \gamma(U,u)[u + \nu(U,u)]$$

$$\frac{d\tau_{(U,u)}}{d\tau_U} = \gamma(U,u), \qquad \frac{d\ell_{(U,u)}}{d\tau_U} = \gamma(U,u)||\nu(U,u)||$$

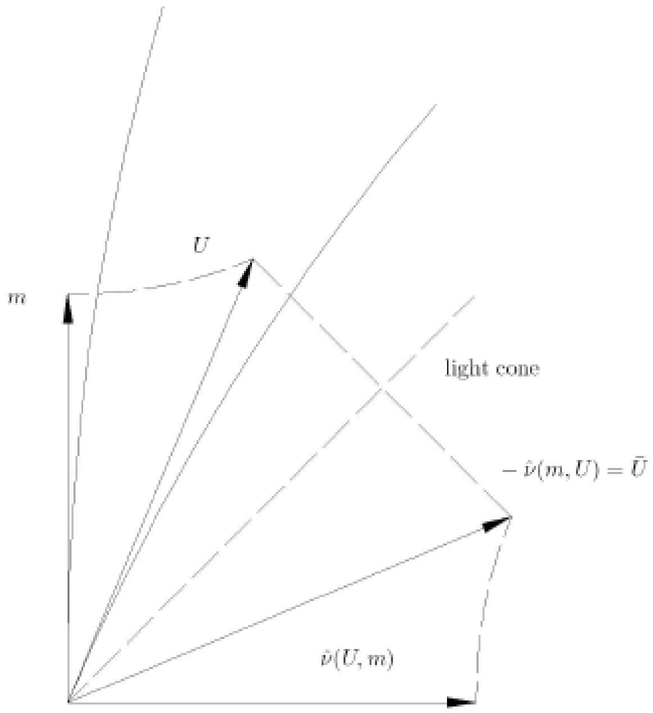
# But also…other similarly natural frames

Adapted frames with spatial axes undergoing special transport laws along the observers world lines:

**Parallel transport law**
**Fermi-Walker transport law**
**Lie transport law**
**…**

Actually the choice of the spatial frame is a free choice!!!
It depends on specific applications only…

# Local Rest Space




A note: among all the steps necessary to characterize the measurement process in GR the choice of a spatial frame adapted to selected family of observers is not so fundamental as the previous ones and it can be (eventually) post-poned. In other words one still has a freedom in the choice

# "1+3" TRASLATION of "4": a tutorial example

$$a(U) = f(U)$$

$$U = \gamma(U,u)[u + \nu(U,u)]$$

$$p(U,u) = \gamma\nu(U,u)$$

$$\frac{D_{(\mathrm{fw},U,u)}p(U,u)}{d\tau_{(U,u)}} = F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}$$

$$\frac{dE(U,u)}{d\tau_{(U,u)}} = \nu(U,u)\cdot[F(U,u) + F^{(G)}_{(\mathrm{fw},U,u)}]$$

$$F^{(G)}_{(\mathrm{fw},U,u)} = -\gamma[a(u) + \omega(u)\times_u \nu(U,u) + \theta(u)\llcorner\nu(U,u)]$$

$$P(u)\frac{Du}{d\tau_U} = -F^{(G)}_{(\mathrm{fw},U,u)} \qquad P(u,U)f(U) \equiv \gamma F(U,u)$$

$$P(U,u) = P(U)P(u) : LRS_u \to LRS_U$$

$$\begin{aligned} P(u)\frac{DX}{d\tau_U} &= \gamma[P(u)\nabla_u X + \nabla(u)_{\nu(U,u)}X] \\ &\equiv \frac{D_{(\mathrm{fw},U,u)}X}{d\tau_U}. \end{aligned}$$

$$\frac{d\tau_{(U,u)}}{d\tau_U} = \gamma(U,u), \qquad \frac{d\ell_{(U,u)}}{d\tau_U} = \gamma(U,u)||\nu(U,u)||$$

# Relative-observer formulation of the physics laws: examples

$$F^\flat = dA = u^\flat \wedge E(u)^\flat + {}^{*(u)}B(u)^\flat \qquad d^2A^\flat = 0, \quad {}^*d^*F = 4\pi J$$

$$\begin{aligned}
&\mathrm{div}_u\, B(u) + 2\omega(u) \cdot_u E(u) = 0\ , \\
&\mathrm{curl}_u\, E(u) + a(u) \times_u E(u) = \\
&\qquad -[\pounds(u)_u + \Theta(u)]B(u) \\
&\mathrm{div}_u\, E(u) - 2\omega(u) \cdot_u B(u) = 4\pi\rho(u)\ , \\
&\mathrm{curl}_u\, B(u) + a(u) \times_u B(u) - [\pounds(u)_u + \Theta(u)]E(u) = \\
&\qquad 4\pi J(u)\ ,
\end{aligned}$$

$$J = \rho(u)u + J(u)$$

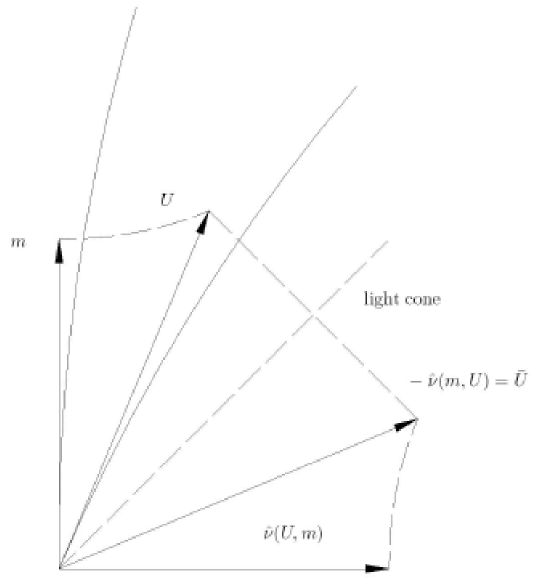
**Changing (boosting) the observers and re-adpting the frames: these are now well established techniques!**

# More than one observer family present?

$$
\begin{aligned}
U &= \gamma(U,u)[u+\nu(U,u)] \\
&= \gamma(U,u)[u+||\nu(U,u)||\hat{\nu}(U,u)]
\end{aligned}
$$

$$
\begin{aligned}
u &= \gamma(u,U)[U+\nu(u,U)] \\
&= \gamma(u,U)[U+||\nu(u,U)||\hat{\nu}(u,U)]
\end{aligned}
$$

$$
\begin{aligned}
B(U,u)u &= U \\
&= \gamma[u+\nu\hat{\nu}(U,u)] \\
B(U,u)\hat{\nu}(U,u) &= -\hat{\nu}(u,U) \\
&= \gamma[\hat{\nu}(U,u)+\nu u]\,.
\end{aligned}
$$



# How to connect two different LRSs

**Projections and Boosts**

The spatial measurements of two observers in relative motion can be compared only relating their respective LRSs. Let $U$ and $u$ be two such observers and $LRS_U$ and $LRS_u$ their LRSs. There exists several maps between these LRSs, as we are going to discuss now.

Combining the projection operators $P(U)$ and $P(u)$ one can form the following "mixed projection" maps:

# Projections

(i) $P(U,u)$ from the $LRS_u$ into $LRS_U$, defined as

$$P(U,u) = P(U)P(u) : LRS_u \to LRS_U \,,$$

with inverse:

$$P(U,u)^{-1} : LRS_U \to LRS_u \,;$$

(ii) $P(u,U)$ from the $LRS_U$ into $LRS_u$, defined as

$$P(u,U) = P(u)P(U) : LRS_U \to LRS_u \,,$$

with inverse:

$$P(u,U)^{-1} : LRS_u \to LRS_U \,.$$

Note that $P(U,u) \neq P(u,U)^{-1}$ as it follows from their representations

$$\begin{aligned} P(U,u) &= P(u) + \gamma\nu U \otimes \hat{\nu}(U,u) \,, \\ P(U,u)^{-1} &= P(U) + \nu U \otimes \hat{\nu}(u,U) \,, \\ P(u,U) &= P(U) + \gamma\nu u \otimes \hat{\nu}(u,U) \,, \\ P(u,U)^{-1} &= P(u) + \nu u \otimes \hat{\nu}(U,u) \,. \end{aligned}$$

$$\begin{aligned} P(U,u)\hat{\nu}(U,u) &= -\gamma\hat{\nu}(u,U) \,, \\ P(u,U)^{-1}\hat{\nu}(U,u) &= -\frac{1}{\gamma}\hat{\nu}(u,U) \,. \end{aligned}$$

# Boosts

$$
\begin{aligned}
B(U,u)u &= U \\
&= \gamma[u + \nu\hat{\nu}(U,u)] \\
B(U,u)\hat{\nu}(U,u) &= -\hat{\nu}(u,U) \\
&= \gamma[\hat{\nu}(U,u) + \nu u]\,.
\end{aligned}
$$

Spacetime map

$$B_{(\mathrm{lrs})}(U,u) \equiv P(U)B(U,u)P(u) : LRS_u \to LRS_U$$

Restriction to LRSs

The representations of the boost and its inverse can be given in terms of the associated tensors

$$B_{(\mathrm{lrs})u}(U,u)\,, \quad B_{(\mathrm{lrs})U}(U,u)\,, \quad B_{(\mathrm{lrs})u}(u,U)\,, \quad B_{(\mathrm{lrs})U}(u,U),$$

defined by:

$$
\begin{aligned}
B_{(\mathrm{lrs})u}(U,u) &= P(U,u)^{-1} \llcorner B_{(\mathrm{lrs})}(U,u)\,, \\
B_{(\mathrm{lrs})U}(U,u) &= B_{(\mathrm{lrs})}(U,u) \llcorner P(U,u)^{-1},
\end{aligned}
$$

Associated tensors needed for its representation

with the corresponding expressions for the inverse boost obtained simply by exchanging the role of $U$ and $u$ and with

$$B_{(\mathrm{lrs})}(U,u) = B_{(\mathrm{lrs})U}(U,u) \llcorner P(U,u) = P(U,u) \llcorner B_{(\mathrm{lrs})u}(U,u)\,.$$

The explicit expression of $B_{(\mathrm{lrs})u}(U,u)$, for example, is given by

$$B_{(\mathrm{lrs})u}(U,u) = P(u) + \frac{1-\gamma}{\gamma}\hat{\nu}(U,u) \otimes \hat{\nu}(U,u).$$

# Composition laws

$$\begin{aligned} U &= \gamma(U,u)[u+\nu(U,u)] \\ &= \gamma(U,u)[u+||\nu(U,u)||\hat{\nu}(U,u)] \end{aligned}$$

$$P(\bar{u},u)^{-1}\gamma(\bar{u},u)\nu(U,\bar{u}) = \frac{\nu(U,u)-\nu(\bar{u},u)}{1-\nu(U,u)\cdot\nu(\bar{u},u)}$$

$$\begin{aligned} a(U) &= \gamma P(u,U)^{-1}F^{(G)}_{(\mathrm{fw},U,u)} + \gamma^2 P(U,u)a_{(\mathrm{fw},U,u)} \\ &= -\gamma^2\left\{P(u,U)^{-1}[a(u)+\omega(u)\times_u \nu(U,u)+\theta(u)\llcorner\nu(U,u)]\right. \\ &\quad \left. -P(U,u)a_{(\mathrm{fw},U,u)}\right\} . \end{aligned}$$

without introducing frames or components….

# and related…

# An explicit example

**Poynting-Robertson effect in bhs (scattering of light by massive particles, deflections, etc.)**

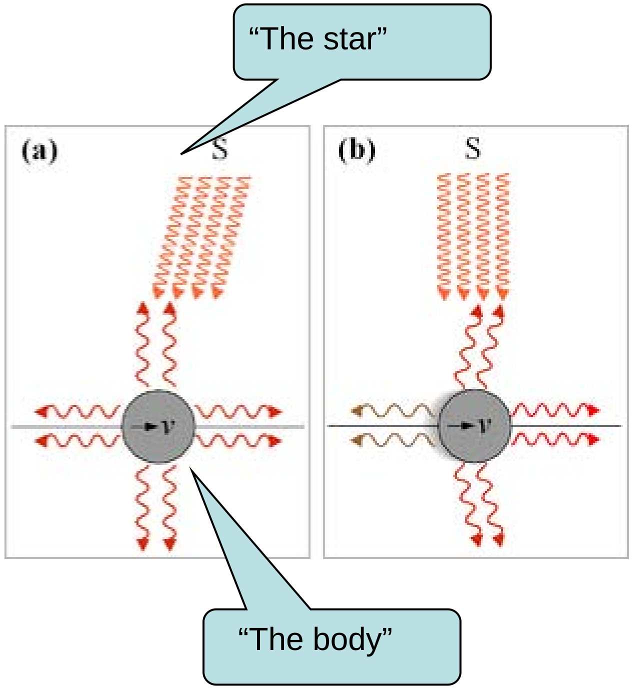
# PR effect: a short introduction

(Poynting 1903, Newtonian gravity;
Robertson 1937, linearized GR).

Consider a small body orbiting a star.

The light emitted by the star exerts a radiation pressure on the body whose direct effect is a drag force, which causes the body to fall into the star, unless the body is so small that it is pushed away from the star itself.

**Radiation from the star (S) and thermal radiation from a particle seen**
**(a) from an observer moving with the particle**
**(b) from an observer at rest with respect to the star.**




The effect can be understood in two ways, depending on the reference frame chosen.

From the perspective of the "body" circling the star, the star's radiation appears to be coming from a slightly forward direction (aberration of light). Therefore the absorption of this radiation leads to a force with a component against the direction of motion.
(The angle of aberration is extremely small since the radiation is moving at the speed of light while the body is moving much slower than that.)

From the perspective of the "star," the body absorbs light entirely in a radial direction, thus the body's angular momentum remains unchanged. However, in absorbing photons, the body acquires added mass via mass-energy equivalence. In order to conserve angular momentum (which is proportional to mass), the body must drop into a lower orbit.

# PR effect

- The drag force is “naively” understood as an aberration effect: if the body is in a circular orbit, for example, the radiation pressure is radially outward from the star, but in the rest frame of the body, the radiation appears to be coming from a direction slightly towards its own direction of motion, and hence a backwards component of force which acts as a drag force is exerted on the body. If the drag force dominates the outward radial force, the body falls into the star.

- For the case in which a body is momentarily at rest, a critical luminosity (similar to the Eddington limit) exists at which the inward gravitational force balances the outward radiation force, that is there exists a critical value separating radial infall from radial escape.

- Similarly for a body initially in a circular orbit, there are two possible solutions: those in which the body spirals inward or spirals outward, depending on the strength of the radiation pressure.

# Modeling the phenomenon, object of current study (PR effect and bh)

- We consider this problem in terms of a test body in orbit first in a spherically symmetric Schwarzschild spacetime without the restriction of slow motion, and then in the larger context of an axially symmetric Kerr spacetime.

- The finite size of the radiating body is ignored.

- The photon flux from the central body is modeled by test photons in outward radial motion with respect to the locally nonrotating observers, namely photons with vanishing conserved angular momentum (later on we will generalize this request).

- The basic equations are developed for a stationary axisymmetric spacetime. Explicit examples then follow for the Schwarzschild and Kerr spacetimes.

# GR setting of the problem

Background spacetime

**Stationary, axisymmetric and reflection-symmetric spacetimes**

Metric

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{\phi\phi}\mathrm{d}\phi^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2$$

Lapse-Shift notation

$$n = N^{-1}(\partial_t - N^\phi \partial_\phi) \qquad N = (-g^{tt})^{-1/2} \qquad N^\phi = g_{t\phi}/g_{\phi\phi}$$

Fiducial observers: ZAMOs
the associated kinematical properties and an OAF

$$e_{\hat{t}} = n\,, \quad e_{\hat{r}} = \frac{1}{\sqrt{g_{rr}}}\partial_r\,, \quad e_{\hat{\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}}\partial_\theta\,, \quad e_{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi$$

$$\begin{aligned}
a(n) &= a(n)^{\hat{r}} e_{\hat{r}} + a(n)^{\hat{\theta}} e_{\hat{\theta}} = \partial_{\hat{r}}(\ln N) e_{\hat{r}} + \partial_{\hat{\theta}}(\ln N) e_{\hat{\theta}}\,,\\
\theta_{\hat{\phi}}(n) &= \theta_{\hat{\phi}}(n)^{\hat{r}} e_{\hat{r}} + \theta_{\hat{\phi}}(n)^{\hat{\theta}} e_{\hat{\theta}} = -\frac{\sqrt{g_{\phi\phi}}}{2N}\left[\partial_{\hat{r}} N^\phi e_{\hat{r}} + \partial_{\hat{\theta}} N^\phi e_{\hat{\theta}}\right],\\
k_{(\mathrm{lie})}(n) &= k_{(\mathrm{lie})}(n)_{\hat{r}} e_{\hat{r}} + k_{(\mathrm{lie})}(n)_{\hat{\theta}} e_{\hat{\theta}} = -[\partial_{\hat{r}}(\ln\sqrt{g_{\phi\phi}}) e_{\hat{r}} + \partial_{\hat{\theta}}(\ln\sqrt{g_{\phi\phi}}) e_{\hat{\theta}}]
\end{aligned}$$

$$T^{\alpha\beta} = \Phi^2 k^\alpha k^\beta, \qquad k^\alpha k_\alpha = 0$$

Photon
(test) field

*Photon field, superposed as a test field to the gravitational background*

where $k$ is assumed to be tangent to an affinely parametrized outgoing null geodesic in the equatorial plane, i.e., $k^\alpha \nabla_\alpha k^\beta = 0$ with $k^\theta = 0$. We will only consider photons in the equatorial plane which are in outward radial motion with respect to the ZAMOs, namely with 4-momentum

$$k = E(n)[n + \hat{\nu}(k,n)], \qquad \hat{\nu}(k,n) = e_{\hat{r}}\ ,$$

where $E(n) = E/N$ is the relative energy of the photon and $E = -k_t$ is the conserved energy associated with the timelike Killing vector field and $L = k_\phi = 0$ is the vanishing conserved angular momentum associated with the rotational Killing vector field, while $\hat{\nu}(k,n)$ defines the unit vector direction of the relative velocity. For the Schwarzschild case, these orbits are radial geodesics with respect to the static observers tied to the coordinate system, but for the Kerr case, they are dragged azimuthally by the rotation of the spacetime with respect to the coordinates.

$$T^{\alpha\beta}{}_{;\beta} = 0 \quad \Rightarrow \quad \Phi = \frac{\Phi_0}{[g_{\theta\theta} g_{\phi\phi}]^{1/4}}$$

Test particle

Consider now a test particle moving in the equatorial plane $\theta = \pi/2$ accelerated by the radiation field, i.e., with 4-velocity

$$U = \gamma(U,n)[n + \nu(U,n)]\,, \quad \nu(U,n) \equiv \nu^{\hat r} e_{\hat r} + \nu^{\hat\phi} e_{\hat\phi} = \nu \sin\alpha e_{\hat r} + \nu\cos\alpha e_{\hat\phi}$$

where $\gamma(U,n) = 1/\sqrt{1 - ||\nu(U,n)||^2}$ is the Lorentz factor and the abbreviated notation $\nu^{\hat a} = \nu(U,n)^{\hat a}$ has been used. In a similarly abbreviated notation, $\nu = ||\nu(U,n)||$ and $\alpha$ are the magnitude of the spatial velocity $\nu(U,n)$ and its polar angle measured clockwise from the positive $\phi$ direction in the $r$-$\phi$ tangent plane, while $\hat\nu = \hat\nu(U,n)$ is the associated unit vector. Note that $\alpha = 0$ corresponds to azimuthal motion with respect to the ZAMOs, while $\alpha = \pm\pi/2$ corresponds to (outward/inward) radial motion with respect to the ZAMOs.

$$\begin{aligned}
a(U)^{\hat t} &= \gamma^2 \nu \sin\alpha \left[a(n)^{\hat r} + 2\nu\cos\alpha\, \theta(n)^{\hat r}{}_{\hat\phi}\right] + \gamma^3 \nu \frac{\mathrm{d}\nu}{\mathrm{d}\tau} \\
a(U)^{\hat r} &= \gamma^2 [a(n)^{\hat r} + k_{(\mathrm{lie})}(n)^{\hat r}\, \nu^2 \cos^2\alpha + 2\nu\cos\alpha\, \theta(n)^{\hat r}{}_{\hat\phi}] \\
&\quad + \gamma \left[\gamma^2 \sin\alpha \frac{\mathrm{d}\nu}{\mathrm{d}\tau} + \nu\cos\alpha \frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right] \\
a(U)^{\hat\theta} &= 0\,, \\
a(U)^{\hat\phi} &= -\gamma^2\nu^2 \sin\alpha\cos\alpha\, k_{(\mathrm{lie})}(n)^{\hat r} + \gamma \left(\gamma^2 \cos\alpha \frac{\mathrm{d}\nu}{\mathrm{d}\tau} - \nu\sin\alpha \frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right)
\end{aligned}$$

4-acceleration (frame components)

Equations of motion

Scattering of radiation as well as the momentum-transfer cross section $\sigma$ (assumed to be a constant) of the particle is independent of the direction and frequency of the radiation; therefore the associated force is given by

$$\mathcal{F}_{(\mathrm{rad})}(U)^{\alpha} = -\sigma P(U)^{\alpha}{}_{\beta} T^{\beta}{}_{\mu} U^{\mu}$$

where $P(U)^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + U^{\alpha}U_{\beta}$ projects orthogonally to $U$. The equation of motion of the particle then becomes

$$ma(U) = \mathcal{F}_{(\mathrm{rad})}(U) \,,$$

where $m$ is the mass of the particle and $a(U) = \nabla_U U$ is its 4-acceleration.

$$\begin{aligned}
\frac{\mathrm{d}\nu}{\mathrm{d}\tau} &= -\frac{\sin\alpha}{\gamma}[a(n)^{\hat{r}} + 2\nu\cos\alpha\,\theta(n)^{\hat{r}}{}_{\hat{\phi}}] + \frac{A}{N^2\sqrt{g_{\theta\theta}g_{\phi\phi}}}(1-\nu\sin\alpha)(\sin\alpha-\nu)\,,\\
\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} &= -\frac{\gamma\cos\alpha}{\nu}[a(n)^{\hat{r}} + 2\nu\cos\alpha\,\theta(n)^{\hat{r}}{}_{\hat{\phi}} + \nu^2 k_{(\mathrm{lie})}(n)^{\hat{r}}]\\
&\quad + \frac{A}{N^2\sqrt{g_{\theta\theta}g_{\phi\phi}}}\frac{(1-\nu\sin\alpha)\cos\alpha}{\nu}\,,\\
\frac{\mathrm{d}r}{\mathrm{d}\tau} &= \frac{\gamma\nu\sin\alpha}{\sqrt{g_{rr}}}\,,
\end{aligned}$$

**Equations of motion,**
**to be integrated numerically.**

where the positive constant $A$ is defined by

$$A = \sigma\Phi_0^2 E^2$$

# Further specification of the background spacetime

The Schwarzschild spacetime is characterized by the metric functions

$$g_{tt} = -N^2\,, \quad g_{t\phi} = 0\,, \quad g_{rr} = 1/N^2\,, \quad g_{\theta\theta} = r^2\,, \quad g_{\phi\phi} = r^2\sin^2\theta\,,$$

where the lapse function is

$$N = \sqrt{1-\frac{2M}{r}} \sim 1 - \frac{M}{r}\,,$$

in which the approximate expression represents the asymptotic value at $r \to \infty$ and to first order in $M$. In this case as well the ZAMOs are aligned with the coordinate time world lines; however, they form an accelerated ($a(n)^{\hat r} = M/(r\sqrt{r^2-2Mr}) \sim M/r^2$) and expansionfree ($\theta(n) = 0$) congruence. The Lie curvature of the $\phi$ loops has only a radial component with value $k_{(\mathrm{lie})}(n)^{\hat r} = -N/r \sim -1/r + M/r^2$. Radially outgoing (geodesic) photons on the equatorial plane have 4-momentum

$$k = E\left[\left(1-\frac{2M}{r}\right)^{-1}\partial_t + \partial_r\right] = E(n)[n + e_{\hat r}]\,.$$

Eqs. (2.29) reduce to

$$\frac{\mathrm{d}\nu}{\mathrm{d}\tau} = \frac{\sin\alpha}{\gamma}k_{(\mathrm{lie})}(n)^{\hat r}\nu_K^2 + \frac{A}{r^2N^2}(1-\nu\sin\alpha)(\sin\alpha-\nu)\,,$$
$$\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = -\frac{\gamma\cos\alpha}{\nu}k_{(\mathrm{lie})}(n)^{\hat r}(\nu^2-\nu_K^2) + \frac{A}{r^2N^2\nu}(1-\nu\sin\alpha)\cos\alpha\,,$$
$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \gamma\nu N\sin\alpha\,,$$

where we have used the relation $a(n)^{\hat r} = -k_{(\mathrm{lie})}(n)^{\hat r}\nu_K^2$,

$$\nu_K = \sqrt{\frac{M}{r-2M}}$$

is the Keplerian speed associated with circular geodesics. If one is only interested in the spatial orbit of the particle, one can choose $r$ or $\phi$ as the parameter along its path, re-expressing the above equations using the chain rule.

For the case $\nu = 0$ of a particle at rest, these equations reduce to the single condition Eq. (2.33) representing the balancing of the gravitational attraction and the radiation pressure at constant $r$ and $\phi$, namely

$$\frac{A}{M} = \left(1-\frac{2M}{r}\right)^{1/2} \quad \to \quad r = r_{(\mathrm{crit})} \equiv \frac{2M}{1-A^2/M^2}\,.$$

**Schwarzschild**

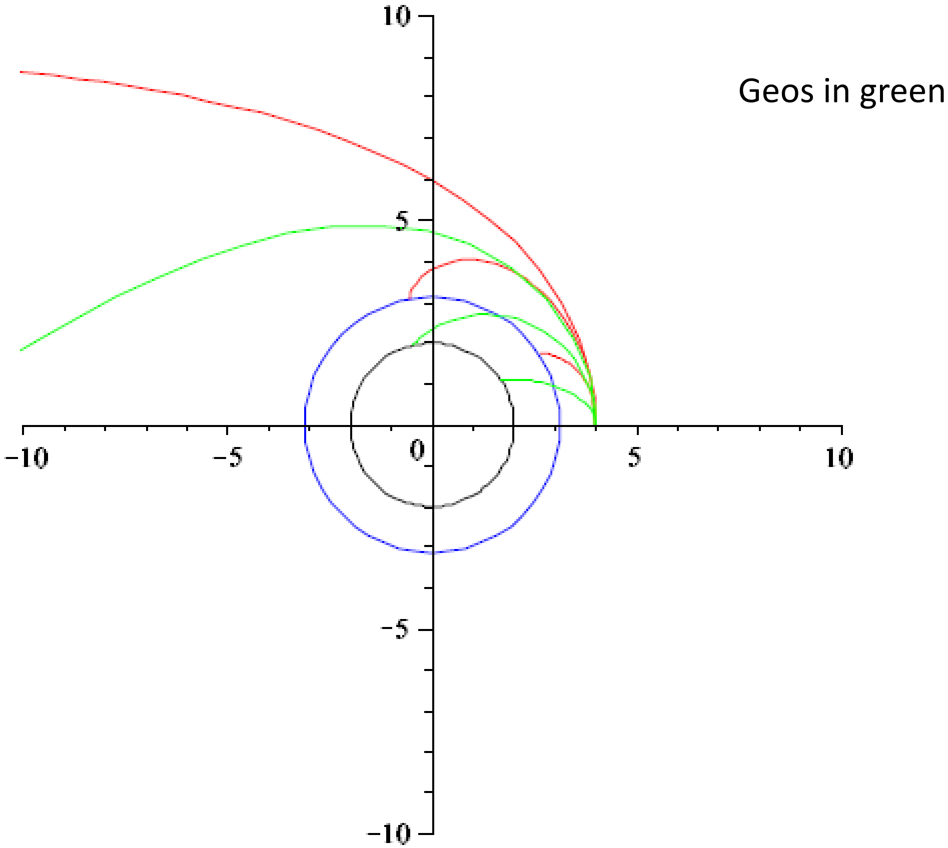



**Figure 1.** The orbit of the particle in the Schwarzschild spacetime with $M = 1$, $A/M = 0.6$, $r_{(\mathrm{crit})} = 3.125M$. The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius which is inside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$. The corresponding geodesics $A/M = 0$ are in gray.

Geos in green

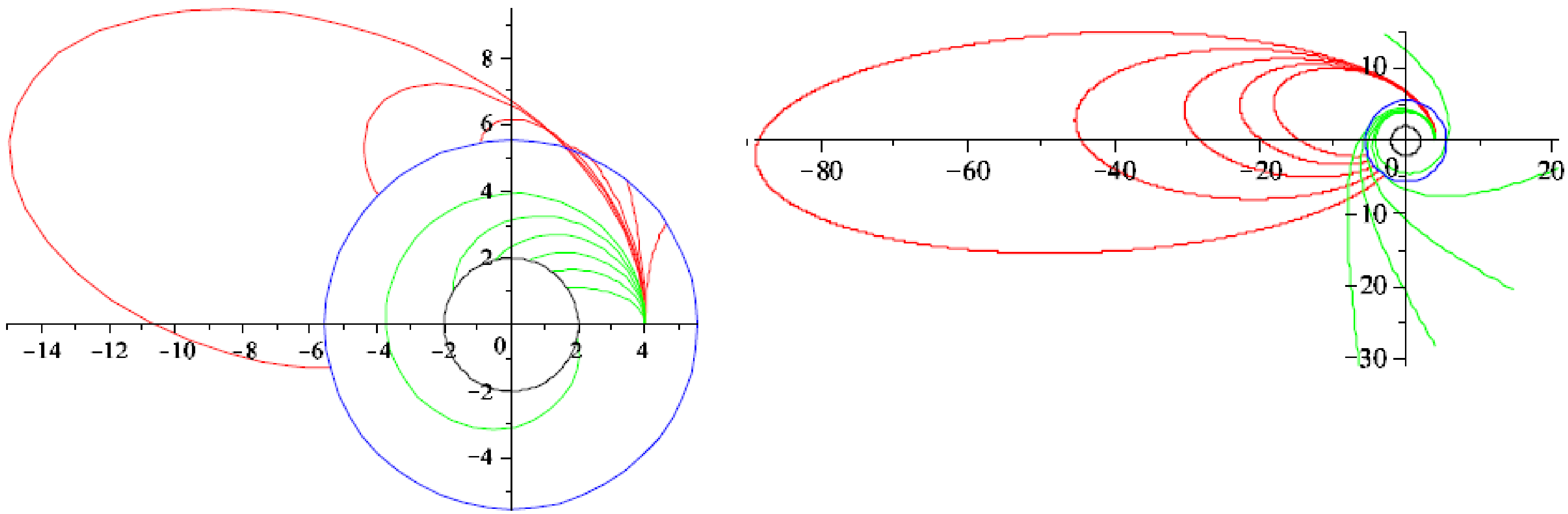



**Figure 2.** The orbit of the particle in the Schwarzschild spacetime with $M = 1$, $A/M = 0.8$, $r_{(\mathrm{crit})} = 5.5M$. The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius which is outside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and for the left figure $\nu(0) = 0.2, 0.3, \ldots, 0.7$ while for the right figure $0.71, 0.72, \ldots, 0.75$. The corresponding geodesics $A/M = 0$ are in gray.

In the equatorial plane of the Kerr metric, the metric is

$$g_{tt} = -\left(1 - \frac{2M}{r}\right), \quad g_{t\phi} = -\frac{2aM}{r}, \quad g_{\phi\phi} = \frac{r^3 + a^2 r + 2a^2 M}{r},$$
$$g_{rr} = \frac{r^2}{\Delta}, \quad g_{\theta\theta} = r^2,$$

so that

$$N = \sqrt{\frac{r\Delta}{r^3 + a^2 r + 2a^2 M}} \sim 1 - \frac{M}{r}, \qquad N^{\phi} = -\frac{2aM}{r^3 + a^2 r + 2a^2 M} \sim -\frac{2aM}{r^3}$$

where $\Delta = r^2 + a^2 - 2Mr$ and the approximate expressions represent their asymptotic values ($r \to \infty$) to first order in $M$. The ZAMOs are timelike outside the horizon $r_+ = M + \sqrt{M^2 - a^2}$. The nonvanishing components of the ZAMO kinematical fields are

$$\begin{aligned}
a(n)^{\hat r} &= \frac{M[(r^2+a^2)^2 - 4a^2 Mr]}{r^2\sqrt{\Delta}(r^3 + a^2 r + 2a^2 M)} \sim \frac{M}{r^2}, \\
\theta(n)^{\hat r}{}_{\hat\phi} &= -\frac{aM(3r^2 + a^2)}{r^2(r^3 + a^2 r + 2a^2 M)} \sim -\frac{3aM}{r^3}, \\
k_{(\mathrm{lie})}(n)^{\hat r} &= -\frac{\sqrt{\Delta}(r^3 - a^2 M)}{r^2(r^3 + a^2 r + 2a^2 M)} \sim -\frac{1}{r} + \frac{M}{r^2}.
\end{aligned}$$

Circular geodesics correspond to orbits

$$U_\pm = \gamma_\pm(n + \nu_\pm e_{\hat\phi}), \qquad \nu_\pm = \frac{a^2 \mp 2a\sqrt{Mr} + r^2}{\sqrt{\Delta}(a \pm r\sqrt{r/M})},$$

and the following relation between $\nu_\pm$ and the ZAMO kinematical fields hold

$$a(n)^{\hat r} = k_{(\mathrm{lie})}(n)^{\hat r}\nu_+\nu_-, \quad -2\theta(n)^{\hat r}{}_{\hat\phi} = k_{(\mathrm{lie})}(n)^{\hat r}(\nu_+ + \nu_-).$$

Equations (2.29) can then be rewritten as

$$\begin{aligned}
\frac{d\nu}{d\tau} &= -\frac{\sin\alpha\, k_{(\mathrm{lie})}(n)^{\hat r}}{\gamma}[\nu_+\nu_- - \nu\cos\alpha\,(\nu_+ + \nu_-)] + \frac{A(1 - \nu\sin\alpha)(\sin\alpha - \nu)}{r\sqrt{g_{\phi\phi}}N^2}, \\
\frac{d\alpha}{d\tau} &= -\frac{\gamma\cos\alpha\, k_{(\mathrm{lie})}(n)^{\hat r}}{\nu}[\nu_+\nu_- - \nu\cos\alpha\,(\nu_+ + \nu_-) + \nu^2] + \frac{A(1 - \nu\sin\alpha)\cos\alpha}{r\sqrt{g_{\phi\phi}}N^2\nu}, \\
\frac{dr}{d\tau} &= \frac{\gamma\nu\sin\alpha}{\sqrt{g_{rr}}}.
\end{aligned}$$

**Kerr**

For the case $\nu = 0$ of a particle remaining at rest with respect to the ZAMOs, these reduce to the single radial force balance condition

$$\frac{A}{M} = \frac{[(r^2 + a^2)^2 - 4a^2 M r]\sqrt{\Delta}}{r^2 (g_{\phi\phi})^{3/2}} \,,$$

which cannot be solved explicitly for $r$. However, the right hand side of this equation takes values between 0 at the horizon $r = r_+$ and 1 when as $r \to \infty$ so a critical radius $r_{(\mathrm{crit})}$ always exists for which this is satisfied for any proper fractional value of $A/M$. If $A/M > 1$ of course no static solutions exist.

**Kerr**

i.e. it rotates wrt to the coords with the ZAMO angular velocity

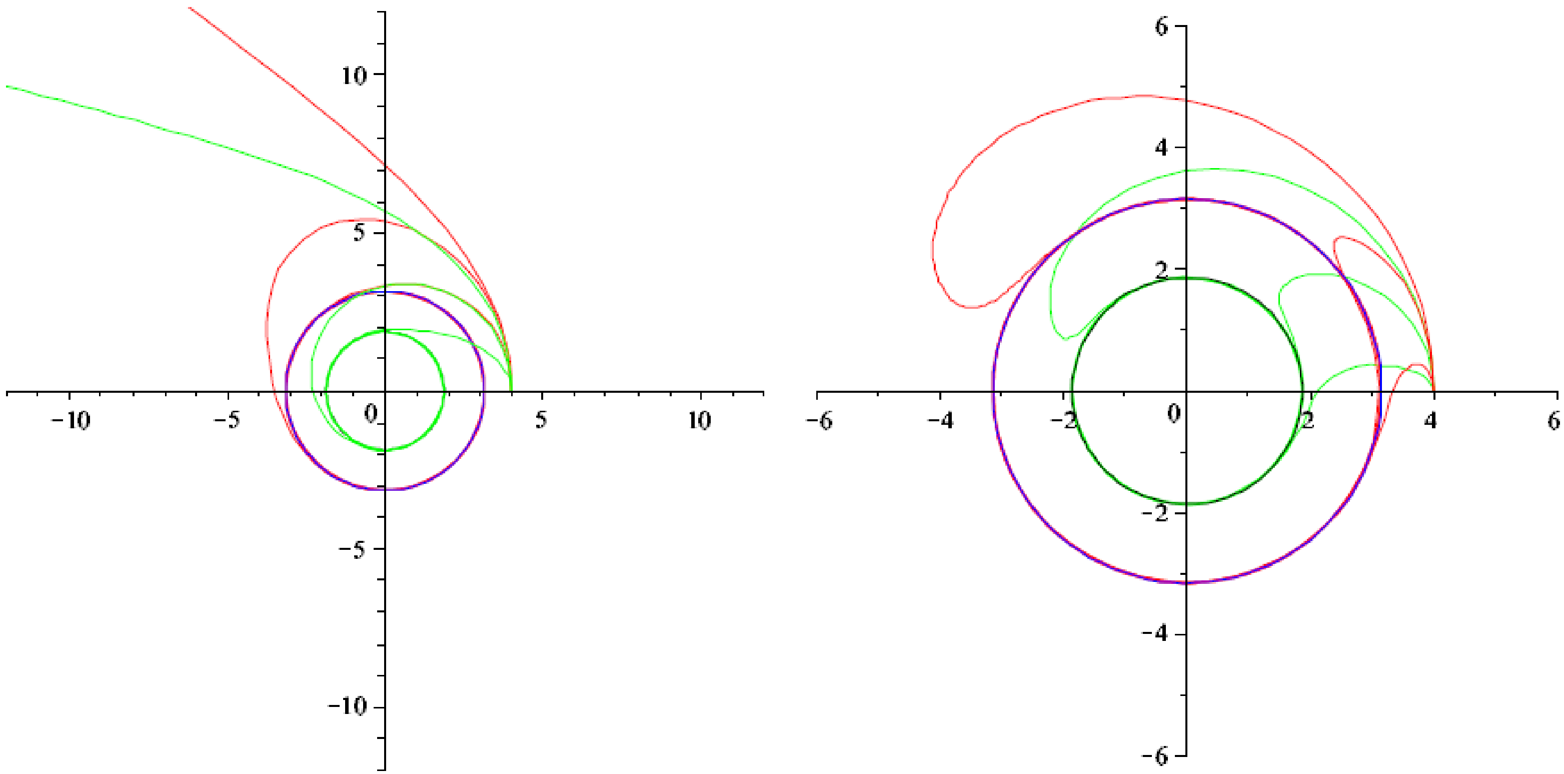



**Figure 3.** The orbit of the particle in the Kerr spacetime with $M = 1$, $a = 0.5$ (left figure), $a = -0.5$ (right figure), $A/M = 0.6$, $r_{(\mathrm{crit})} = 3.154M$. The inner circle is the horizon $r = 1.866M$, while the outer circle is at the critical radius which is inside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$. The corresponding geodesics $A/M = 0$ are in gray.

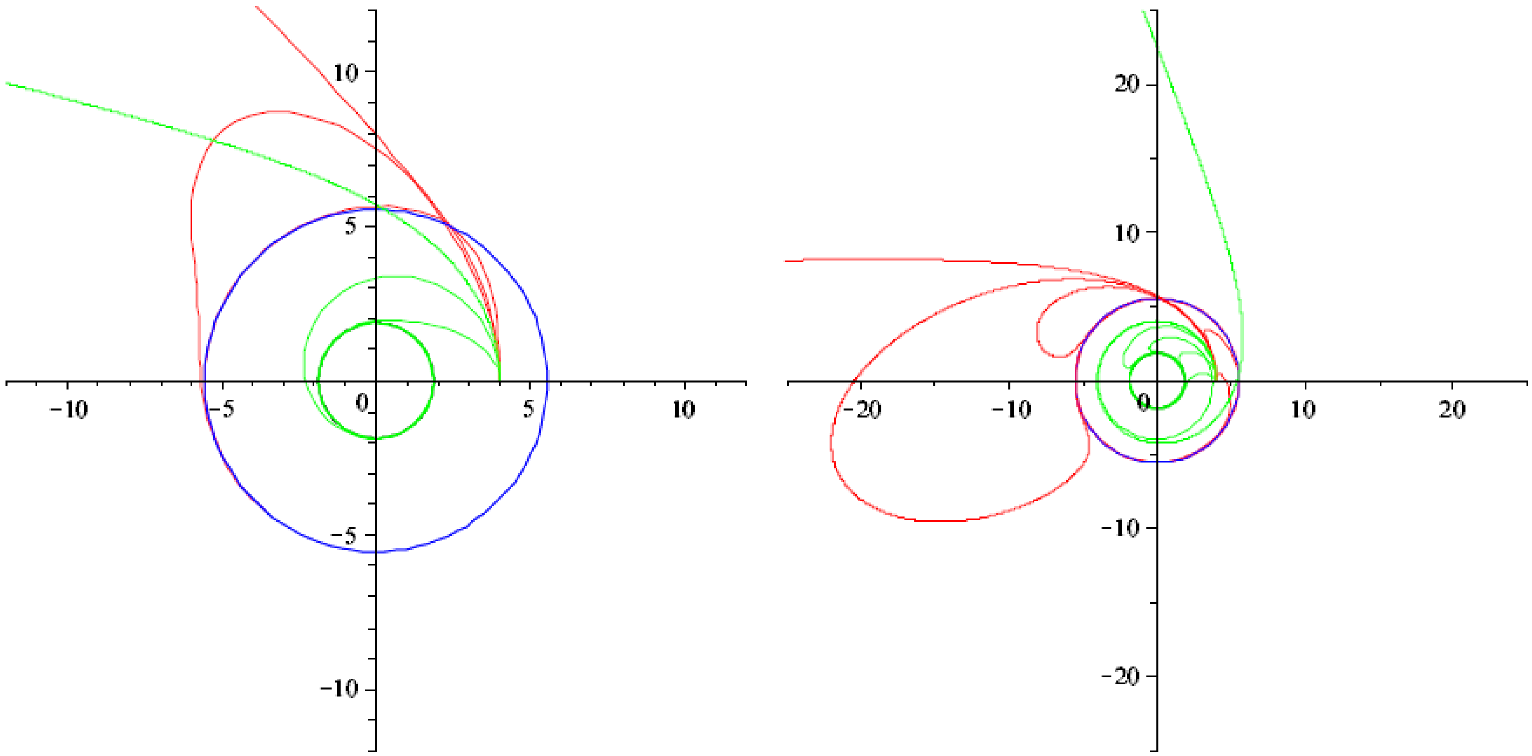



**Figure 4.** The orbit of the particle in the Kerr spacetime with $M = 1$, $a = 0.5$ (left figure), $a = -0.5$ (right figure), $A/M = 0.8$, $r_{(\text{crit})} = 5.551M$. The inner circle is the horizon $r = 1.866M$, while the outer circle is at the critical radius which is outside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$ for the left figure, while in the right figure $\nu(0) = 0.2, 0.5, 0.8, 0.847$ for both the accelerated and geodesic curves and then finally $\nu(0) = 0.9$ for the accelerated curve and $\nu(0) = 0.848$ for the geodesic, both of which escape to infinity. The corresponding geodesics $A/M = 0$ are in gray.

# Generalizations

Photons with nonzero angular momentum

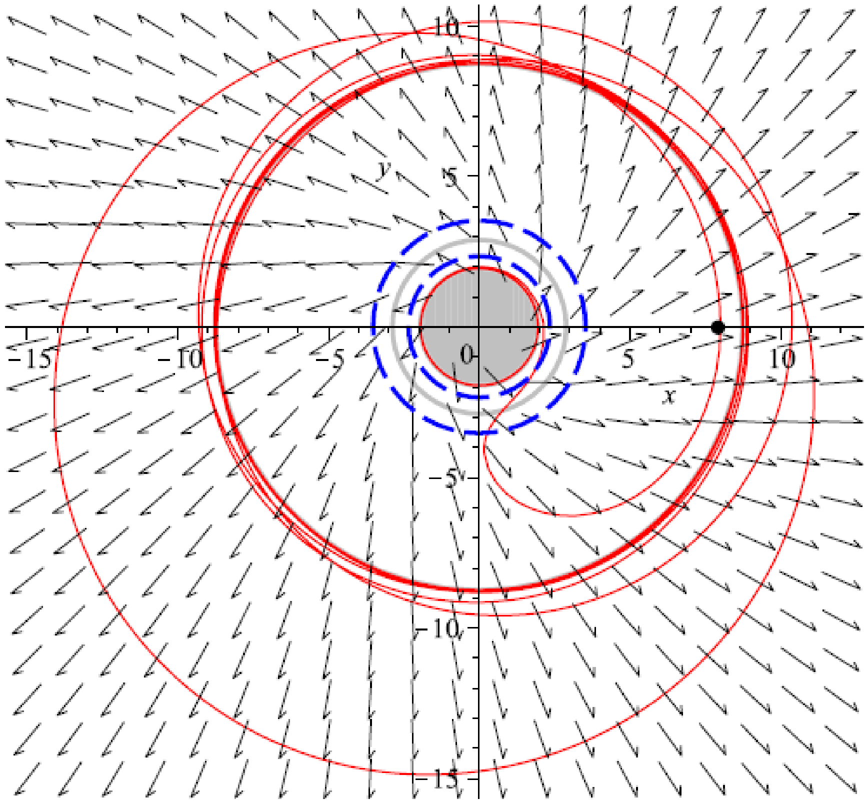
Test body endowed with intrinsic angular momentum

Radiation field as an exact solution of the Einstein’s field equations (Vaidya spacetime)

Energy flux at infinity

Finite size of the emitting star

Etc.




Kerr with superposed radiation field with nonzero angular momentum

b=L/E

The spacetime and photon parameters are $a = 0.5$, $A/M = 0.3$, $b/M = 3$, showing two orbits moving initially from the bullet point in the two azimuthal directions just inside the outer critical radius with 1.2 times the critical speed for that counterclockwise critical orbit. The unit velocity direction field $\hat{\nu}(k, n)$ of the radiation with respect to the ZAMOs is superimposed on the plot. The dashed circles are the two null circular geodesics orbits. The gray filled circle extends to the horizon. The clockwise moving orbit settles down to the critical orbit, while the counterclockwise moving orbit quickly falls into the inner critical orbit near the horizon. The gray circle between the null orbits is the unstable critical orbit. This and all further plots show $r/M$ on the axes.

# Generalizations

Photons with nonzero angular momentum

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Etc.

## Generalization to the case of spinning test particles undergoing PR effect

$$\frac{\mathrm{D}P^\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} \equiv F^{(\mathrm{spin})\mu}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = P^\mu U^\nu - P^\nu U^\mu \,,$$

$$S^{\mu\nu}P_\nu = 0$$

MP- model for spinning test particles

$$\frac{\mathrm{D}P^\mu}{\mathrm{d}\tau} = F^{(\mathrm{spin})\mu} + F^{(\mathrm{rad})\mu}$$

**Explicit calculations in the Schwarzschild background, linearization in spin**

Let the 4-momentum $P = mu$ be fully general, i.e. with

$$u = \gamma_u[n + \nu_u(\sin\alpha_u e_{\hat r} + \cos\alpha_u e_{\hat\phi})]\,, \qquad \gamma_u = \frac{1}{\sqrt{1-\nu_u^2}}\,,$$

and introduce the spin vector associated with $S_{\mu\nu}$ by spatial duality

$$S^\beta = u_\alpha \eta^{\alpha\beta\mu\nu} S_{\mu\nu}\,,$$

where $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$ is the unit volume 4-form and $\epsilon_{\alpha\beta\gamma\delta}$ ($\epsilon_{0123} = 1$) is the Levi-Civita alternating symbol. It is also useful to consider the scalar invariant

$$s^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}\,,$$

constant along $\mathcal{C}_U$ because of Eqs. (1.2) and (1.3). Consistency of the model requires that the length scale $|s|/m$ associated with the spinning particle be much smaller than the one associated with the background spacetime, say $M$, namely

$$|\hat s| \equiv \frac{|s|}{mM} \ll 1\,.$$

Let us consider Eqs. (1.1) and (1.2) with $U^\alpha$ given by Eq. (2.7) and $u^\alpha$ given by Eq. (2.14). In the spinless case $P$ is aligned with $U$, i.e. $u = U$, implying that $\nu = \nu_u$. The presence of the spin causes a change in both $U$ and $u$ according to

$$U = U_0 + \hat s U_{\hat s}\,, \qquad u = U_0 + \hat s u_{\hat s}\,,$$

where

$$U_0 = \gamma_0(n + \nu_0^{\hat r} e_{\hat r} + \nu_0^{\hat\phi} e_{\hat\phi})$$

satisfies Eq. (2.12) and corrections are first order in the spin. Higher order terms in Eqs. (1.1) and (1.2) are neglected. This leads to two different sets of equations for zeroth and first order in spin respectively, which are listed in Appendix A.

The spin force to first order in $\hat s$ is given by

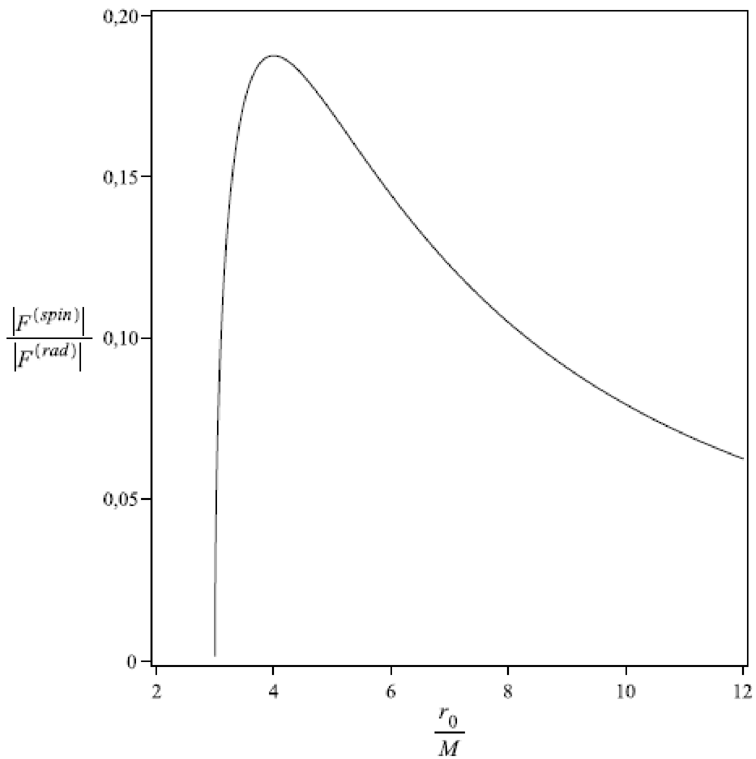
$$F^{(\rm spin)} = -\frac{3mM^2}{r_0^3}\hat s \gamma_0^2 \nu_0^{\hat\phi}(\nu_0^{\hat r} n + e_{\hat r})\,.$$

We find that the mass of the spinning particle $m$ is a constant of motion. Furthermore, from the evolution equations for the spin it follows that the spin vector has a single nonvanishing and constant component along $\theta$ (or $z$), namely

$$S = -S^{\hat\theta} e_{\hat\theta} = s e_{\hat z}\,.$$

It is convenient to introduce a friction parameter $f$, so that the length scale $A$ associated with the radiation field be much smaller than $M$, i.e.

$$f \equiv \frac{A}{M} \ll 1 \,.$$




F spin << F rad

**Figure 4.** The ratio between the magnitudes of spin and radiation forces given by Eq. (3.13) is plotted in units of $|\hat{s}|/f$ as a function of $r_0/M$.

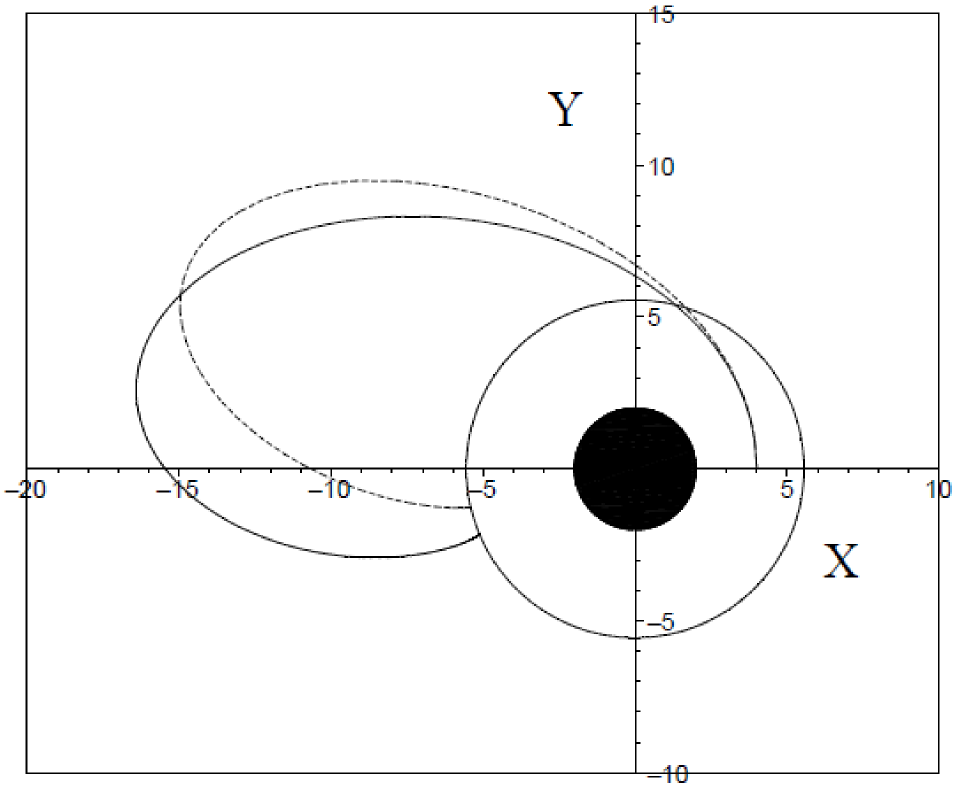



**Figure 1.** The orbit of a spinning particle (solid curve) undergoing Poynting-Robertson effect is shown for the choice of parameters $A/M = 0.8$ and $\hat{s} = 0.5$ ($X = r\cos\phi$ and $Y = r\sin\phi$ are Cartesian-like coordinates). The starting point is located at $r_0(0) = 4M$ and $\phi_0(0) = 0$ with $\nu_{u0}(0) = 0.7$, $\alpha_{u0}(0) = 0$, $t_s(0) = 0$, $r_s(0) = 0$ and $\phi_s(0) = 0$, $\nu_s^{\hat{r}}(0) = 0$ and $\nu_s^{\hat{\phi}}(0) = 0$. The values of the spin parameter has been exaggerated in order to evidentiate the difference from the motion of a spinless particle (dashed curve). The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius $r_{(\mathrm{crit})} = 5.5M$ which is outside the initial data position.

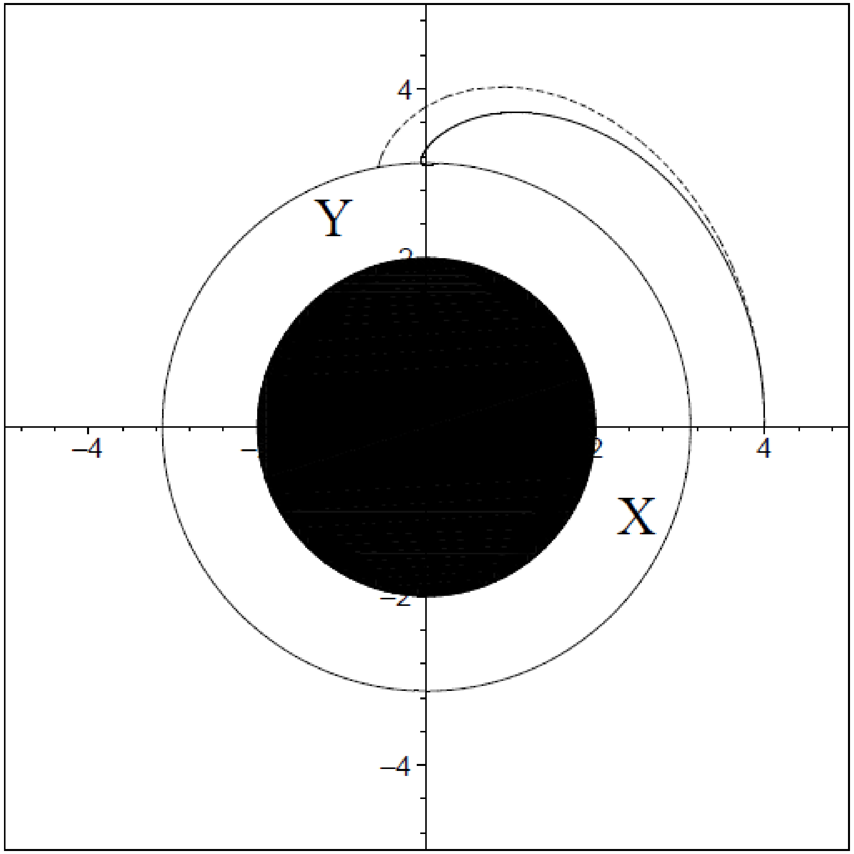



**Figure 2.** The orbit of a spinning particle (solid curve) undergoing Poynting-Robertson effect is shown for the choice of parameters $A/M = 0.6$ and $\hat{s} = 0.5$. The starting point is located at $r_0(0) = 4M$ and $\phi_0(0) = 0$ with $\nu_{u0}(0) = 0.5$, $\alpha_{u0}(0) = 0$, $t_s(0) = 0$, $r_s(0) = 0$ and $\phi_s(0) = 0$, $\nu_s^{\hat{r}}(0) = 0$ and $\nu_s^{\hat{\phi}}(0) = 0$. The corresponding orbit for a spinless particle is also shown (dashed curve). The critical radius $r_{(\mathrm{crit})} = 3.125M$ is inside the initial data position.

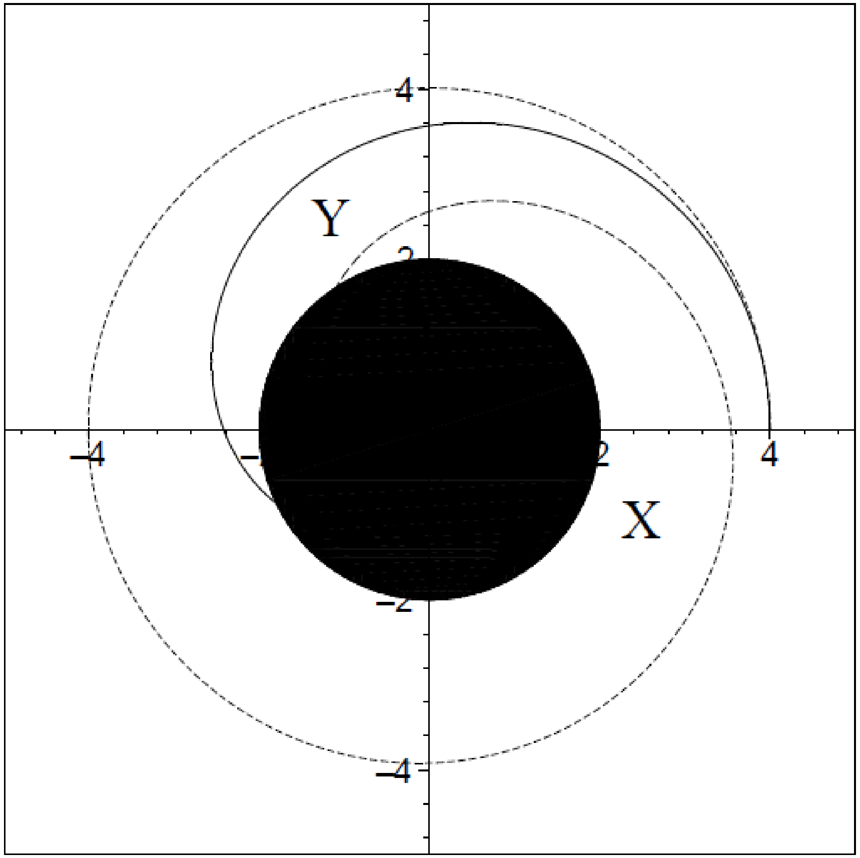



**Figure 3.** The orbit of a spinning particle (solid curve) undergoing Poynting-Robertson effect is shown for the choice of parameters $A/M = 0.01$ and $\hat{s} = 0.5$. The starting point is located at $r_0(0) = 4M$ and $\phi_0(0) = 0$ with $\nu_{u0}(0) = \nu_K \approx 0.7071$, $\alpha_{u0}(0) = 0$, $t_s(0) = 0$, $r_s(0) = 0$ and $\phi_s(0) = 0$, $\nu_s^{\hat{r}}(0) = 0$ and $\nu_s^{\hat{\phi}}(0) = 0$. The corresponding orbit for a spinless particle is also shown (dashed curve). In this case $r_{(\mathrm{crit})} \approx 2M$.

# Details...

### Deviation from the circular geodesic

Consider now the corrections to geodesic circular motion, by taking the effect of the radiation field also small.

In absence of both spin and radiation we assume the geodesic motion of the particle to be circular at $r = r_0$ ($r_0 > 3M$ in order $U_K$ to be timelike), that is

$$U = U_K = \gamma_K (n \pm \nu_K e_{\hat{\phi}}) , \tag{3.1}$$

where the Keplerian value of speed ($\nu_K$) and the associated Lorentz factor ($\gamma_K$) and angular velocity ($\zeta_K$) are given by

$$\nu_K = \sqrt{\frac{M}{r_0 - 2M}} , \qquad \gamma_K = \sqrt{\frac{r_0 - 2M}{r_0 - 3M}} , \qquad \zeta_K = \sqrt{\frac{M}{r_0^3}} . \tag{3.2}$$

The $\pm$ signs in Eq. (3.1) correspond to co-rotating (+) or counter-rotating (−) orbits with respect to increasing values of the azimuthal coordinate $\phi$ (counter-clockwise motion as seen from above). The azimuthal direction in the local rest space of $U_K$ pointing in the direction of relative motion (i.e. the boost of $e_{\hat\phi}$ in the local rest space of $U_K$) is specified by the following unit vector orthogonal to $U_K$ in the $t$-$\phi$ plane

$$\bar U_K = \gamma_K\,(\nu_K n \pm e_{\hat\phi})\,, \qquad (3.3)$$

where the $\pm$ signs are correlated with those in $U_K$.

The parametric equations of $U_K$ are

$$\begin{aligned} t_K &= t_0 + \frac{\gamma_K}{N_0}\tau \equiv t_0 + \Gamma_K \tau\,,\\ r &= r_0\,, \qquad \theta = \frac{\pi}{2}\,,\\ \phi_K &= \phi_0 \pm \frac{\gamma_K \nu_K}{r_0}\tau \equiv \phi_0 \pm \Omega_K \tau\,, \end{aligned} \qquad )$$

where now $t_0$, $r_0$ and $\phi_0$ are constants and

$$\Gamma_K = \sqrt{\frac{r_0}{r_0 - 3M}}\,, \qquad \Omega_K = \frac{1}{r_0}\sqrt{\frac{M}{r_0 - 3M}}\,.$$

It is convenient to introduce a friction parameter $f$, so that the length scale $A$ associated with the radiation field be much smaller than $M$, i.e.

$$f \equiv \frac{A}{M} \ll 1\,. \qquad )$$

Therefore, in the present analysis corrections to geodesic motion will be limited to first order terms in both parameters $\hat s$ and $f$, according to

$$\begin{aligned} &t = t_K + f t_f + \hat s t_{\hat s}\,, \quad r = r_0 + f r_f + \hat s r_{\hat s}\,, \quad \phi = \phi_K + f\phi_f + \hat s \phi_{\hat s}\,,\\ &\nu^{\hat r} = f\nu^{\hat r}_f + \hat s \nu^{\hat r}_{\hat s}\,, \quad \nu^{\hat\phi} = \pm\nu_K + f\nu^{\hat\phi}_f + \hat s \nu^{\hat\phi}_{\hat s}\,,\\ &\nu_u = \pm\nu_K + f\nu_{uf} + \hat s \nu_{u\hat s}\,, \quad \alpha_u = f\alpha_{uf} + \hat s \alpha_{u\hat s}\,, \end{aligned} \qquad )$$

where $t_K$ and $\phi_K$ are given by Eq. (3.4). This implies

$$U = U_K + fU_f + \hat s U_{\hat s}\,, \qquad 8)$$

where

$$\begin{aligned} U_f &= \left(-\nu_K \frac{r_f}{r_0} \pm \gamma_K^2 \nu^{\hat\phi}_f\right)\bar U_K + \gamma_K \nu^{\hat r}_f e_{\hat r}\,,\\ U_{\hat s} &= \left(-\nu_K \frac{r_{\hat s}}{r_0} \pm \gamma_K^2 \nu^{\hat\phi}_{\hat s}\right)\bar U_K + \gamma_K \nu^{\hat r}_{\hat s} e_{\hat r}\,. \end{aligned} \qquad )$$

Similarly

$$u = U + f u_f + \hat s u_{\hat s}\,, \qquad )$$

with $U$ given by Eq. (3.8) and

$$u_f = \gamma_K(\nu_K^2 \nu^{\hat\phi}_f - \nu^{\hat r}_f)e_{\hat r}\,, \qquad u_{\hat s} = \gamma_K(\nu_K^2 \nu^{\hat\phi}_{\hat s} - \nu^{\hat r}_{\hat s})e_{\hat r}\,, \qquad 1)$$

as discussed in Appendix B.

To first order in $\hat s$ and $f$ the spin force and radiation force are given by

$$\begin{aligned} F^{(\rm spin)} &= \mp 3mM\,\hat s\,\gamma_K^2 \zeta_K^2 \nu_K\, e_{\hat r}\,,\\ F^{(\rm rad)} &= -mf\Omega_K \nu_K(\gamma_K \nu_K \bar U_K - e_{\hat\phi})\,, \end{aligned} \qquad 2)$$

$$r = r_0 + f r_f + \hat s r_{\hat s}$$

$$
\begin{aligned}
t_s &= \mp \frac{6M^2}{r_0}\frac{\Omega_K^3}{\Omega_{\rm ep}^3}[\sin(\Omega_{\rm ep}\tau)-\Omega_{\rm ep}\tau]\,,\\
t_f &= 4r_0\zeta_K\nu_K^2\frac{\Omega_K^3}{\Omega_{\rm ep}^4}\left\{[\cos(\Omega_{\rm ep}\tau)-1]+\frac{\Omega_{\rm ep}}{2r_0\zeta_K\Omega_K}[\sin(\Omega_{\rm ep}\tau)-\Omega_{\rm ep}\tau]+\frac{3}{8}\gamma_K^2\Omega_{\rm ep}^2\tau^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
r_s &= \pm\, 3r_0\frac{\Omega_K\zeta_K}{\Omega_{\rm ep}^2}[\cos(\Omega_{\rm ep}\tau)-1]\,,\\
r_f &= -\, r_0\zeta_K\frac{\Omega_K}{\Omega_{\rm ep}^2}\left\{[\cos(\Omega_{\rm ep}\tau)-1]-2r_0\zeta_K\frac{\Omega_K}{\Omega_{\rm ep}}[\sin(\Omega_{\rm ep}\tau)-\Omega_{\rm ep}\tau]\right\}\,,\\
\phi_s &= \pm\,\frac{\zeta_K}{\nu_K^2}t_s\,,\qquad \phi_f = \pm\frac{\zeta_K}{\nu_K^2}t_f\,,
\end{aligned}
$$

where

$$
\Omega_{\rm ep}=\sqrt{\frac{M(r_0-6M)}{r_0^3(r_0-3M)}}
$$

is the well known epicyclic frequency governing the radial perturbations of circular geodesics.

The constant term in $r_s$ represents the slight change in the radius of the circular orbit about which the solution oscillates with proper period $2\pi/\Omega_{\rm ep}$. In contrast, the presence of a secular term in $r_f$ is responsible for the deviation from geodesic motion due to friction, which is measurable in principle. In fact, by taking the mean values over a period of the perturbed radius we can estimate the amount of variation of radial distance

$$
\left\langle\frac{\delta r}{r}\right\rangle \equiv \frac{r-r_0}{r_0}=\Gamma_K\frac{\zeta_K^2}{\Omega_{\rm ep}^2}\left[\left(1-2\pi r_0\zeta_K\frac{\Omega_K}{\Omega_{\rm ep}}\right)f\mp 3M\,\hat{s}\,\gamma_K\,\zeta_K\,N_0\right].
$$

## Numbers

For instance, for the motion of the Earth about the Sun we find

$$\left\langle \frac{\delta r}{r} \right\rangle \approx f \mp 2 \times 10^{-17} \frac{(s/m)_\oplus}{\mathrm{cm}} \approx 3 \times 10^{-5} \mp 4 \times 10^{-15} \approx 3 \times 10^{-5}\,, \tag{3.18}$$

since $r_0 \approx 1.5 \times 10^{13}$ cm, $M = M_\odot \approx 1.5 \times 10^5$ cm and the ratio $(s/m)_\oplus \approx 200$ cm for the Earth; the friction parameter is related to the ratio between the solar luminosity $\mathcal{L}_\odot \approx 3.8 \times 10^{33}$ erg/s and the Eddington luminosity [20, 21] $\mathcal{L}_{\rm Edd} \approx 1.3 \times 10^{38}$ erg/s, and for the Sun is given by $f \approx 3 \times 10^{-5}$. Therefore, in this case the effect of the radiation field on the orbit dominates. Note that the estimate of the contribution due to spin is in agreement with [22].

The effect of the spin may become important when considering the orbiting extended body is a fast rotating object. To illustrate the order of magnitude of the effect, we may consider the binary pulsar system PSR J0737-3039 as orbiting Sgr A*, the supermassive ($M \simeq 10^6\, M_\odot$) black hole located at the Galactic Center [23, 24], at a distance of $r \simeq 10^9$ Km. The PSR J0737-3039 system consists of two close neutron stars (their separation is only $d_{AB} \sim 8 \times 10^5$ Km) of comparable masses $m_A \simeq 1.4\, M_\odot$, $m_B \simeq 1.2\, M_\odot$), but very different intrinsic spin period (23 ms of pulsar A vs 2.8 s of pulsar B) [25]. Its orbital period is about 2.4 hours, the smallest yet known for such an object. Since the intrinsic rotations are negligible with respect to the orbital period, we can treat the binary system as a single object with reduced mass $\mu_{AB} \simeq 0.7\, M_\odot$ and intrinsic rotation equal to the orbital period. The spin parameter thus turns out to be equal to $\hat{s} \approx 1.0 \times 10^{-3}$. The luminosity of Sgr A* is about $10^3 \mathcal{L}_\odot$, whereas its Eddington luminosity is $\mathcal{L}_{\rm Edd} \approx 10^{11} \mathcal{L}_\odot$, so that $f \approx 3 \times 10^{-8}$. Therefore, in this case

$$\left\langle \frac{\delta r}{r} \right\rangle \approx 7.6 \times 10^{-9} \mp 1.8 \times 10^{-7} \approx 1.7 \times 10^{-7}\,. \tag{3.19}$$

# Generalizations

Photons with nonzero angular momentum

Test body endowed with intrinsic angular momentum

Radiation field as an exact solution of the Einstein's field equations (Vaidya spacetime)

Energy flux at infinity

Finite size of the emitting star

Etc.

# Vaidya spacetime

$$ds^2 = -N^2 du^2 \mp 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$N = \sqrt{1 - \frac{2M(u)}{r}} \,.$$

Substituting the Vaidya metric into the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, one finds that the energy-momentum tensor has only one non-zero component

$$T_{uu} = T^{rr} = \mp \frac{M_{,u}}{4\pi r^2} \,.$$

Such a $T^{\mu\nu}$ can be interpreted as "pure radiation," $T^{\mu\nu} = \Phi^2 k^\mu k^\nu$, where $k^\mu$ is a purely radial outgoing/ingoing null vector and $\Phi^2$ is given by normalization chosen for $k^\mu$. For instance, if one takes $k^\mu = \pm\frac{1}{\sqrt{2}}\,\delta^\mu_r$ , then

$$T^{rr} = \frac{\Phi^2}{2} \quad \Longrightarrow \quad \Phi^2 = \mp \frac{M_{,u}}{2\pi r^2} \,.$$

The above choice of $k^\mu$ means $k_\alpha = g_{\alpha\mu} k^\mu = g_{\alpha r} k^r = \mp k^r \delta^u_\alpha = -\frac{1}{\sqrt{2}}\,\delta^u_\alpha$ , in particular, the energy of radiation particles is proportional to $-k_u = \pm k^r = \frac{1}{\sqrt{2}}$ . One also checks easily that $k$ is tangent to a null geodesic,

$$k^\mu{}_{;\nu} k^\nu = \Gamma^\mu{}_{\nu\lambda} k^\nu k^\lambda = \frac{1}{2}\,\Gamma^\mu{}_{rr} = 0,$$

expanding ($k^\mu{}_{;\mu} = \pm\sqrt{2}/r$) and having zero twist and shear.

# Equations of motion

Mass function interpolating between two Schwarzschild

$$M(u) = M_1 + \frac{1}{2}(M_2 - M_1)(1 + \tanh \beta u)$$

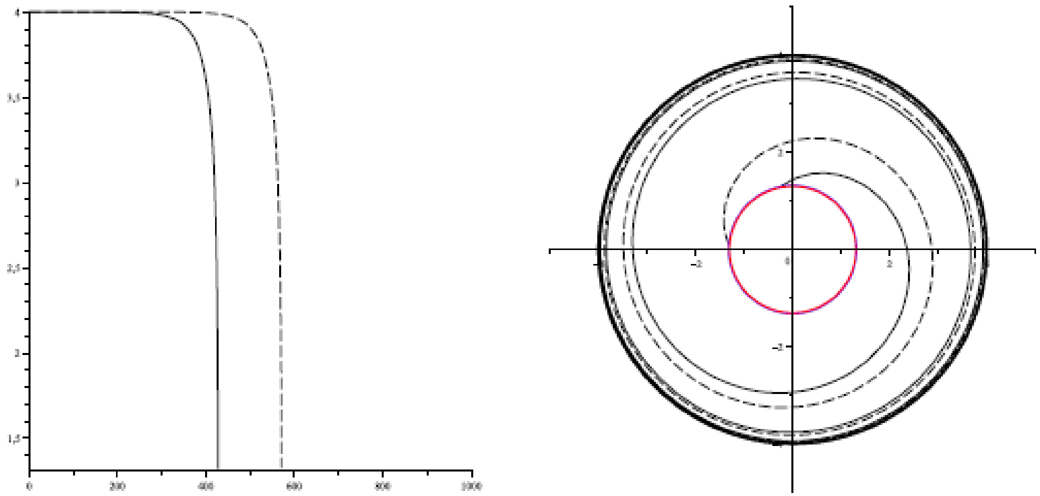
for the outgoing- ($M_1 > M_2$) and ingoing-radiation ($M_1 < M_2$) case, the parameter $\beta$ = const governing the transition between the two asymptotic Schwarzschild spacetimes (the smaller is the value of $\beta$ the smoother is the transition). Another possibility is that the mass decreases/increases linearly with $u$ within certain period (between certain finite limit values), which includes the case of a self-similar spacetime.

$$\begin{aligned}
\frac{\mathrm{d}\nu}{\mathrm{d}\tau} &= -\frac{\sin\alpha}{\gamma N^2}\left[N^2 N_{,r} \pm N_{,u}(1 \mp \nu\sin\alpha)\right] \pm \frac{\tilde{\sigma} T_{uu}}{N^2}(1 \mp \nu\sin\alpha)(\sin\alpha \mp \nu) \\
\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} &= \frac{\gamma\cos\alpha}{\nu N^2}\left[-N^2 N_{,r} \mp N_{,u}(1 \mp \nu\sin\alpha) + \frac{\nu^2 N^3}{r}\right] \\
&\quad \pm \frac{\tilde{\sigma} T_{uu}\cos\alpha}{N^2\nu}(1 \mp \nu\sin\alpha)\,.
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \frac{\gamma}{N}(1 \mp \nu\sin\alpha)\,, \\
\frac{\mathrm{d}r}{\mathrm{d}\tau} &= \gamma\nu N\sin\alpha\,, \\
\frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= \frac{\gamma\nu\cos\alpha}{r}\,,
\end{aligned}$$

as well as the initial conditions $u(0)$, $r(0)$, $\phi(0)$, $\nu(0)$ and $\alpha(0)$.

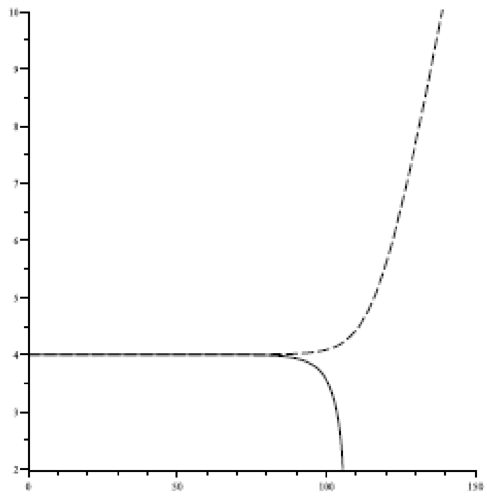
# Vaidya: ingoing radiation



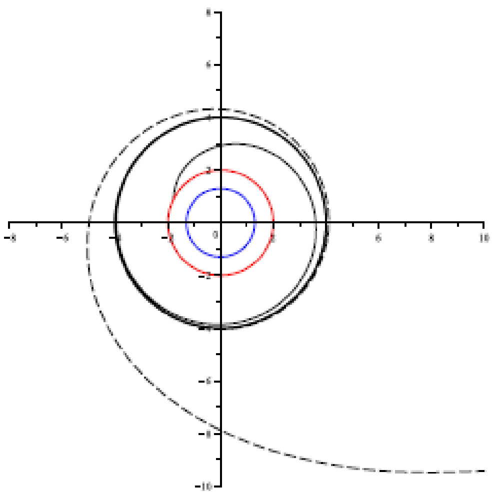
(a) (b)

**Figure 1.** The behavior of $r(\tau)$ is shown in Fig. (a) in the case of ingoing radiation with the following parameter choice: $M_1 = 0.65$, $M_2 = 1$, $\beta = 10^{-2}$ and $\bar{\sigma} = 0$ (geodesic, thick dashed line) and $\bar{\sigma} = 10^4$ (solid line). The initial conditions are for both cases $u(0) = -1000$, $r(0) = 4$, $\phi(0) = 0$, $\nu(0) \approx 0.49$, $\alpha(0) = 0$, which correspond to a circular geodesic in the past asymptotic Schwarzschild spacetime with mass $M_1$. During the accretion by ingoing null dust the orbit initially circular gets spiraling, until the apparent horizon is reached in a finite proper time interval (see Fig. (b)). The coupling with the background radiation field causes the accelerated particle to cross the apparent horizon before the geodesic one. The horizon is located at $r \approx 1.31$ in both cases (solid circle).

# Vaidya: outgoing radiation



(a)



(b)

**Figure 6.** The behavior of $r(\tau)$ is shown in Fig. (a) in the case of outgoing radiation with the following parameter choice: $M_1 = 1$, $M_2 = 0.65$, $\beta = 10^{-2}$ and $\tilde{\sigma} = 0$ (geodesic, thick dashed line) and $\tilde{\sigma} = 10^4$ (solid line). The initial conditions are for both cases $u(0) = -1000$, $r(0) = 4$, $\phi(0) = 0$, $\nu(0) \approx 0.707$, $\alpha(0) = 0$, which correspond to a circular geodesic in the past asymptotic Schwarzschild spacetime with mass $M_1$. During the radiating process the orbit initially circular gets spiraling. In the geodesic case, the orbit escapes outwards after a few loops. In constrast, the accelerated particle spirals towards the apparent horizon, which is reached in a finite proper time interval at $r \approx 2$. The asymptotic inner horizon at $r \approx 1.3$ is also shown.

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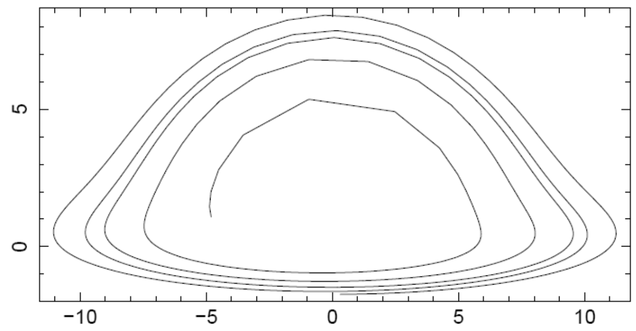
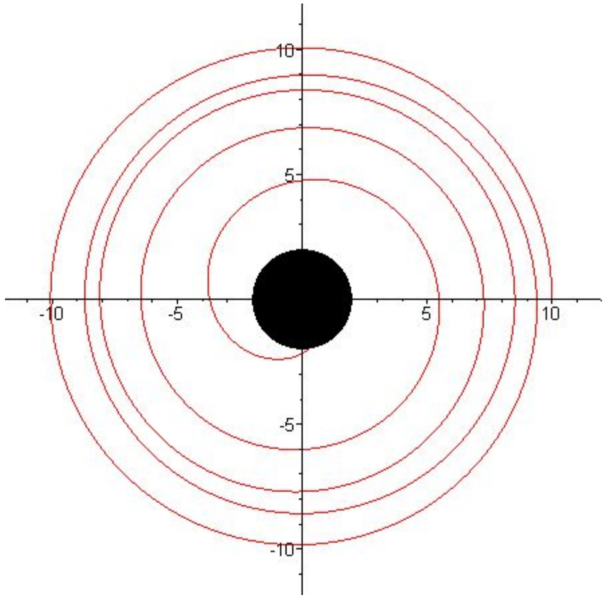
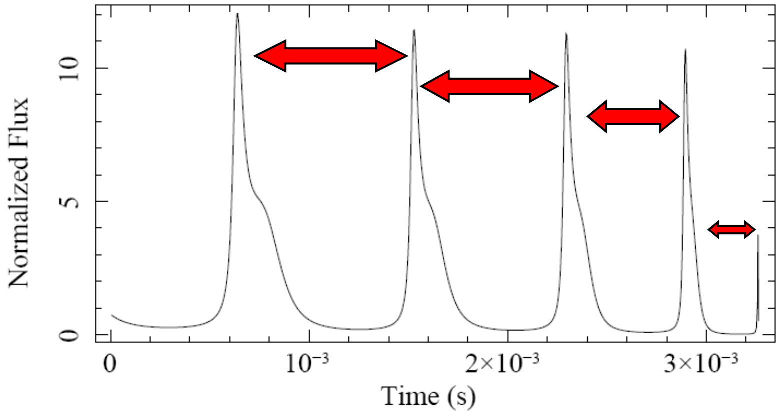
Apparent position of the direct image




Realistic orbit on the equatorial plane of the Schwarzschild spacetime

Parameters: M=1, A/M=0.01, r(0)=10, phi(0)=0, nu(0)=nuK=0.3535, alpha(0)=0.

(Peak= lensing; gobba = redshift)

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in progress



Thanks for your kind attention!