Profinite semigroups and applications*

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Notes taken by Alfredo COSTA

Abstract

Profinite semigroups may be described shortly as projective limits of finite semigroups. They come about naturally by studying pseudovarieties of finite semigroups which in turn serve as a classifying tool for rational languages. Of particular relevance are relatively free profinite semigroups which for pseudovarieties play the role of free algebras in the theory of varieties. Combinatorial problems on rational languages translate into algebraic-topological problems on profinite semigroups. The aim of these lecture notes is to introduce these topics and to show how they intervene in the most recent developments in the area.

1 Introduction

With the advent of electronic computers in the 1950's, the study of simple formal models of computers such as automata was given a lot of attention. The aims were multiple: to understand the limitations of machines, to determine to what extent they might come to replace humans, and later to obtain efficient schemes to organize computations. One of the simplest models that quickly emerged is the finite automaton which, in algebraic terms is basically the action of a finitely generated free semigroup on a finite set of states and thus leads to a finite semigroup of transformations of the states [48, 61].

In the 1960's, the connection with finite semigroups was first explored to obtain computability results [79] and in parallel a decomposition theory of finite computing devices inspired in the theory of groups and the complexity of such decompositions [51, 52], again led to develop the theory of finite semigroups [21], which had not previously deserved any specific attention from specialists on semigroups.

In the early 1970's, both trends, the former more combinatorial and more directly concerned with applications in computer science, the latter more algebraic, continued to flourish with various results that nowadays are seen as pioneering. In the mid-1970's, S. Eilenberg, in part with the collaboration of M. P. Schützenberger and B. Tilson [35, 36] laid the foundations for a theory which was already giving signs of being potentially quite rich. One of

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the cornerstones of their work is the notion of a pseudovariety of semigroups and a correspondence between such pseudovarieties and varieties of rational languages which provided a systematic framework and a program for the classification of rational languages.

The next ten years or so were rich in the execution of Eilenberg's program [53, 64, 65] which in turn led to deep problems such as the identification of the levels of J. Brzozowski's concatenation hierarchy of star-free languages [29] while various steps forward were taken in the understanding of the Krohn-Rhodes group complexity of finite semigroups [73, 71, 47].

In the beginning of the 1980's, the author was exploring connections of the theory of pseudovarieties with Universal Algebra to draw information on the lattice of pseudovarieties of semigroups and to compute some operators on pseudovarieties (see [3] for results and references). The heart of the combinatorial work was done at manipulating identities and so when J. Reiterman [70] showed that it was possible to define pseudovarieties by pseudoidentities, which are identities in an enlarged signature whose interpretation in finite semigroups is natural, this immediately appeared to be a powerful tool to explore. Reiterman introduced pseudoidentities as formal equalities of implicit operations, and defined a metric structure on sets of implicit operations but no algebraic structure. There is indeed a natural algebraic structure and the interplay between topological and algebraic structure turns out to be very rich and very fruitful.

Thus, the theory of finite semigroups and applications led to the study of profinite semigroups, particularly those that are free relative to a pseudovariety. These structures play the role of free algebras for varieties in the context of profinite algebras, which already explains the interest in them. When the first concrete new applications of this approach started to appear (see [3] for results and references), other researchers started to consider it too and nowadays it is viewed as an important tool which has found applications across all aspects of the theory of pseudovarieties.

The aim of these notes is to introduce this area of research, essentially from scratch, and to survey a significant sample of the most important recent developments. In Section 2 we show how the study of finite automata and rational languages leads to study pseudovarieties of finite semigroups and monoids, including some of the key historical results.

Section 3 explains how relatively free profinite semigroups are found naturally in trying to construct free objects for pseudovarieties, which is essentially the original approach of B. Banaschewski [26] in his independent proof that pseudoidentities suffice to define pseudovarieties. The theory is based here on projective limits but there are other alternative approaches [3, 7]. Section 3 also lays the foundations of the theory of profinite semigroups which are further developed in Section 4, where the operational aspect is explored. Section 4 also includes the recent idea of using iteration of implicit operations to produce new implicit operations. Subsection 4.3 presents for the first time a proof that the monoid of continuous endomorphisms of a finitely generated profinite semigroup is profinite so that implicit operations on finite monoids also have natural interpretations in that monoid.

The remaining sections are dedicated to a reasonably broad survey, without proofs, of how the general theory introduced earlier can be used to solve problems. Section 5 sketches the proof of I. Simon's characterization of piecewise testable languages in terms of the solution of the word problem for free pro-J semigroups. Section 6 presents an introduction to the notion of tame pseudovarieties which is a sophisticated tool to handle decidability questions which extends the approach of C. J. Ash to the "Type II conjecture" of J. Rhodes, as presented in the seminal paper [22]. The applications of this approach can be found in Sections 7 and 8 in the computation of several pseudovarieties obtained by applying natural operators to known pseudovarieties. The difficulty in this type of calculation is that it is known that those operators do not preserve decidability [1, 72, 24]. The notion of tameness came about precisely in trying to find a stronger form of decidability which would be preserved or at least guarantee decidability of the operator image [15].

Finally, Section 9 introduces some very recent developments in the investigation of connections between free profinite semigroups and Symbolic Dynamics. The idea to explore such connections eventually evolved from the need to build implicit operations through iteration in order to prove that the pseudovariety of finite p-groups is tame [6]. Once a connection with Symbolic Dynamics emerged several applications were found but only a small aspect is surveyed in Section 9 namely that which appears to have a potential to lead to applications of profinite semigroups to Symbolic Dynamics.

2 Automata and languages

An abstraction of the notion of an *automaton* is that of a semigroup S acting on a set Q, whose members are called the *states* of the automaton. The action is given by a homomorphism $\varphi : S \to \mathcal{B}_Q$ into the semigroup of all binary relations on the set Q, which we view as acting on the right. If all binary relations in $\varphi(S)$ have domain Q, then one talks about a *complete* automaton, as opposed to a *partial* automaton in the general situation. If all elements of $\varphi(S)$ are functions, then the automaton is said to be *deterministic*. The semigroup $\varphi(S)$ is called the *transition semigroup* of the automaton. In some contexts it is better to work with monoids, and then one assumes the acting semigroup S to be a monoid and the action to be given by a monoid homomorphism φ.

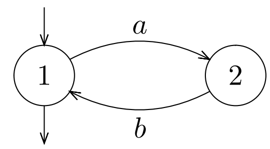
Usually, a set of generators A of the acting semigroup S is fixed and so the action homomorphism φ is completely determined by its restriction to A. In case both Q and A are finite sets, the automaton is said to be *finite*. Of course the restriction that Q is finite is sufficient to ensure that the transition semigroup of the automaton is finite.

To be used as a recognition device, one fixes for an automaton a set I of *initial* states and a set F of *final* states. Moreover, in Computer Science one is interested in recognizing sets of words (or strings) over an alphabet A, so that the acting semigroup is taken to be the semigroup A^+ freely generated by A, consisting of all non-empty words in the letters of the *alphabet* A. The *language recognized* by the automaton is then the following set of words:

$$L = \{w \in A^+ : \varphi(w) \cap (I \times F) \neq \emptyset\}. \tag{2.1}$$

If the empty word 1 is also relevant, then one works instead in the monoid context and one considers the free monoid A^*, the formula (2.1) for the language recognized being then suitably adapted. Whether one works with monoids or with semigroups is often just a matter of personal preference, although there are some instances in which the two theories are not identical. Most results in these notes may be formulated in both settings and we will sometimes switch from one to the other without warning. Parts of the theory may be extended to a much a more general universal algebraic context (see [3, 7] and M. Steinby's lecture notes in this volume).

For an example, consider the automaton described by the following picture



where we have two states, 1 and 2, the former being both initial and final, and two acting letters, a and b, the action being determined by the two partial functions associated with a and b, respectively $\bar{a} : 1 \mapsto 2$ and $\bar{b} : 2 \mapsto 1$. The language of $\{a, b\}^*$ recognized by this automaton consists of all words of the form $(ab)^k$ with $k \geq 0$ which are *labels* of paths starting and ending at state 1. This is the submonoid generated by the word ab, which is denoted $(ab)^*$.

In terms of the action homomorphism, the language L of (2.1) is the inverse image of a specific set of binary relations on Q. We say that a language $L \subseteq A^+$ is *recognized* by a homomorphism $\psi : A^+ \to S$ into a semigroup S if there exists a subset $P \subseteq S$ such that $L = \psi^{-1}P$ or, equivalently, if $L = \psi^{-1}\psi L$. We also say that a language is *recognized* by a finite semigroup S if it is recognized by a homomorphism into S. By the very definition of recognition by a finite automaton, every language which is recognized by such a device is also recognized by a finite semigroup.

Conversely, if $L = \psi^{-1}\psi L$ for a homomorphism $\psi : A^+ \to S$ into a finite semigroup, then one can construct an automaton recognizing L as follows: for the set of states take S^1, the monoid obtained from S by adjoining an identity if S is not a monoid and S otherwise; for the action take the composition of ψ with the *right regular representation*, namely the homomorphism $\varphi : A^+ \to \mathcal{B}_{S^1}$ which sends each word w to right translation by $\psi(w)$, that is the function $s \mapsto s\psi(w)$. This proves the following theorem and, by adding the innocuous assumption that ψ is onto, it also shows that every language which is recognized by a finite automaton is also recognized by a finite complete deterministic automaton with only one initial state (the latter condition being usually taken as part of the definition of deterministic automaton).

2.1 Theorem (Myhill [61]) *A language L is recognized by a finite automaton if and only if it is recognized by a finite semigroup.* □

In particular, the complement $A^+ \setminus L$ of a language $L \subseteq A^+$ recognized by a finite automaton is also recognized by a finite automaton since a homomorphism into a finite semigroup recognizing a language also recognizes its complement.

A language $L \subseteq A^*$ is said to be *rational* (or *regular*) if it may be expressed in terms of the empty language and the languages of the form $\{a\}$ with $a \in A$ by applying a finite number of times the binary operations of taking the union $L \cup K$ of two languages L and K or their concatenation $LK = \{uv : u \in L, v \in K\}$, or the unary operation of taking the submonoid L^* generated by L; such an expression is called a *rational expression* of L. For example, if letters stand for elementary tasks a computer might do, union and concatenation correspond to performing tasks respectively in parallel or in series, while the star operation corresponds to iteration. The following result makes an important connection between this combinatorial concept and finite automata. Its proof can be found in any introductory text to automata theory such as Perrin [63].

2.2 Theorem (Kleene [48]) *A language L over a finite alphabet is rational if and only if it is recognized by some finite automaton.*

An immediate corollary which is not evident from the definition is that the set of rational languages $L \subseteq A^*$ is closed under complementation and, therefore it constitutes a Boolean subalgebra of the algebra $\mathcal{P}(A^+)$ of all languages over A.

Rational languages and finite automata play a crucial role in both Computer Science and current applications of computers, since many very efficient algorithms, for instance for dealing with large texts use such entities [34]. This already suggests that studying finite semigroups should be particularly relevant for Computer Science. We present next one historical example showing how this relevance may be explored.

The *star-free* languages over an alphabet A constitute the smallest Boolean subalgebra closed under concatenation of the algebra of all languages over A which contains the empty language and the languages of the form $\{a\}$ with $a \in A$. In other words, this definition may be formulated as that of rational languages but with the star operation replaced by complementation. *Plus-free* languages $L \subseteq A^+$ are defined similarly.

On the other hand we say that a finite semigroup S is *aperiodic* if all its subsemigroups which are groups (in this context called simply *subgroups*) are trivial. Equivalently, the cyclic subgroups of S should be trivial, which translates in terms of universal laws to stating that S should satisfy some identity of the form $x^{n+1} = x^n$.

The connection between these two concepts, which at first sight have nothing to do with each other, is given by the following remarkable theorem.

2.3 Theorem (Schützenberger [79]) *A language over a finite alphabet is star-free if and only if it is recognized by a finite aperiodic monoid.*

Eilenberg [36] has given a general framework in which Schützenberger's theorem becomes an instance of a general correspondence between families of rational languages and finite monoids. To formulate this correspondence, we first introduce some important notions.

The *syntactic congruence* of a subset L of a semigroup S is the largest congruence ρ_L on S which *saturates* L in the sense that L is a union of congruence classes. The existence of such a congruence may be easily established even for arbitrary subsets of universal algebras [3, Section 3.1]. For semigroups, it is easy to see that it is the congruence ρ_L defined by $u \, \rho_L \, v$ if, for all $x, y \in S^1$, $xuy \in L$ if and only if $xvy \in L$, that is if u and v appear as factors of members of L precisely in the same context. The quotient semigroup S/ρ_L is called the *syntactic semigroup* of L and it is denoted $\operatorname{Synt} L$; the natural homomorphism $S \to S/\rho_L$ is called the *syntactic homomorphism* of L.

The syntactic semigroup $\operatorname{Synt} L$ of a rational language $L \subseteq A^+$ is the smallest semigroup S which recognizes L. Indeed all semigroups of minimum size which recognize L are isomorphic. To prove this, one notes that a homomorphism $\psi : A^+ \to S$ recognizing L may as well be taken to be onto, in which case S is determined up to isomorphism by a congruence on A^+, namely the *kernel* congruence $\ker \psi$ which identifies two words if they have the same image under ψ. The assumption that ψ recognizes L translates in terms of this congruence by stating that $\ker \psi$ saturates L and so $\ker \psi$ is contained in ρ_L. Noting that rationality really played no role in the argument, this proves the following result where we say that a semigroup S *divides* a semigroup T and we write $S \prec T$ if S is a homomorphic image of some subsemigroup of T.

2.4 Proposition *A language $L \subseteq A^+$ is recognized by a semigroup S if and only if* $\operatorname{Synt} L$ *divides S.* □

The syntactic semigroup of a rational language L may be effectively computed from a rational expression for the language. Namely, one can efficiently compute the *minimal automaton* of L [63], which is the complete deterministic automaton recognizing L with the minimum number of states; the syntactic semigroup is then the transition semigroup of the minimal automaton.

Given a finite semigroup S, one may choose a finite set A and an onto homomorphism $\varphi : A^+ \to S$: for instance, one can take $A = S$ and let φ be the homomorphism which extends the identity funtion $A \to S$. For each $s \in S$, let $L_s = \varphi^{-1}s$. Since φ is an onto homomorphism which recognizes L_s, there is a homomorphism $\psi_s : S \to \operatorname{Synt} L_s$ such that the composite function $\psi_s \circ \varphi : A^+ \to \operatorname{Synt} L_s$ is the syntactic homomorphism of L_s. The functions ψ_s induce a homomorphism $\psi : S \to \prod_{s\in S} \operatorname{Synt} L_s$ which is injective since $\psi_s(t) = \psi_s(s)$ means that there exist $u, v \in A^+$ such that $\varphi(u) = s$, $\varphi(v) = t$ and $u\ \rho_{L_s}\ v$, which implies that $v \in L_s$ since $u \in L_s$ and so $t = s$. As we did at the beginning of the section, we may turn $\varphi : A^+ \to S$ into an automaton which recognizes each of the languages L_s and from this any proof of Kleene's Theorem will yield a rational expression for each L_s. Hence we have the following result.

2.5 Proposition *For every finite semigroup S one may effectively compute rational languages $L_1, \ldots, L_n$ over a finite alphabet A which are recognized by S and such that S divides* $\prod_{i=1}^n \operatorname{Synt} L_i$. □

It turns out there are far too many finite semigroups for a classification up to isomorphism to be envisaged [78]. Instead, from the work of Schützenberger and Eilenberg eventually emerged [36, 37] the classification of classes of finite semigroups called *pseudovarieties*. These are the (non-empty) closure classes for the three natural algebraic operators in this context, namely taking homomorphic images, subsemigroups and finite direct products. For example, the classes A, of all finite aperiodic semigroups, and G, of all finite groups, are pseudovarieties of semigroups.

On the language side, the properties of a language may depend on the alphabet on which it is considered. To take into account the alphabet, one defines a *variety of rational languages* to be a correspondence $\mathcal{V}$ associating to each finite alphabet A a Boolean subalgebra $\mathcal{V}(A^+)$ of $\mathcal{P}(A^+)$ such that

(1) if $L \in \mathcal{V}(A^+)$ and $a \in A$ then the *quotient* languages $a^{-1}L = \{w : aw \in L\}$ and $La^{-1} = \{w : wa \in L\}$ belong to $\mathcal{V}(A^+)$ (*closure under quotients*);

(2) if $\varphi : A^+ \to B^+$ is a homomorphism and $L \in \mathcal{V}(B^+)$ then the inverse image $\varphi^{-1}L$ belongs to $\mathcal{V}(A^+)$ (*closure under inverse homomorphic images*).

For example, the correspondence which associates with each finite alphabet the set of all plus-free languages over it is a variety of rational languages. The correspondence between varieties of rational languages and pseudovarieties is easily described in terms of the syntactic semigroup as follows:

- associate with each variety of rational languages $\mathcal{V}$ the pseudovariety V generated by all syntactic semigroups $\operatorname{Synt} L$ with $L \in \mathcal{V}(A^+)$;

- associate with each pseudovariety V of finite semigroups the correspondence

$$\begin{aligned}\mathcal{V} : A \mapsto \mathcal{V}(A^+) &= \{L \subseteq A^+ : \operatorname{Synt} L \in \mathsf{V}\} \\ &= \{L \subseteq A^+ : L \text{ is recognized by some } S \in \mathsf{V}\}\end{aligned}$$

Since intersections of non-empty families of pseudovarieties are again pseudovarieties, pseudovarieties of semigroups constitute a complete lattice for the inclusion ordering. Similarly, one may order varieties of languages by putting $\mathcal{V} \leq \mathcal{W}$ if $\mathcal{V}(A^+) \subseteq \mathcal{W}(A^+)$ for every finite alphabet A. Then every non-empty family of varieties $(\mathcal{V}_i)_{i\in I}$ admits the infimum $\mathcal{V}$ given by $\mathcal{V}(A^+) = \bigcap_{i\in I} \mathcal{V}_i(A^+)$ and so again the varieties of rational languages constitute a complete lattice.

2.6 Theorem (Eilenberg [36]) *The above two correspondences are mutual inverse isomorphisms between the lattice of varieties of rational languages and the lattice of pseudovarieties of finite semigroups.*

Schützenberger's Theorem provides an instance of this correspondence, but of course this by no means says that that theorem follows from Eilenberg's Theorem. See M. V. Volkov's lecture notes in this volume and Section 5 for another important "classical" instance of Eilenberg's correspondence, namely Simon's Theorem relating the variety of so-called piecewise testable languages with the pseudovariety J of finite semigroups in which every principal ideal admits a unique element as a generator. See Eilenberg [36] and Pin [65] for many more examples.

2.7 Example An elementary example which is easy to treat here is the correspondence between the variety $\mathcal{N}$ of finite and cofinite languages and the pseudovariety N of all finite nilpotent semigroups. We say that a semigroup S is *nilpotent* if there exists a positive integer n such that all products of n elements of S are equal; the least such n is called the *nilpotency index* of S. The common value of all sufficiently long products in a nilpotent semigroup must of course be zero. If the alphabet A is finite, the finite semigroup S is nilpotent with nilpotency index n, and the homomorphism $\varphi : A^+ \to S$ recognizes the language L, then either φL does not contain zero, so that L must consist of words of length smaller than n, which implies L is finite, or φL contains zero and then every word of length at least n must lie in L, so that the complement of L is finite.

Since N is indeed a pseudovariety and the correspondence $\mathcal{N}$ associating with a finite alphabet A the set of all finite and cofinite languages $L \subseteq A^+$ is a variety of rational languages, by Eilenberg's Theorem to prove the converse it suffices to show that every singleton language $\{w\}$ over a finite alphabet A is recognized by a finite nilpotent semigroup. Now, given a finite alphabet A and a positive integer n, the set I_n of all words of length greater than n is an ideal of the free semigroup A^+ and the Rees quotient A^+/I_n, in which all words of I_n are identified to a zero element, is a member of N. If $w \notin I_n$, that is if the length $|w|$ of w satisfies $|w| \leq n$, then the quotient homomorphism $A^+ \to A^+/I_n$ recognizes $\{w\}$. Hence we have $\mathsf{N} \leftrightarrow \mathcal{N}$ via Eilenberg's correspondence.

Eilenberg's correspondence gave rise to a lot of research aimed at identifying pseudovarieties of finite semigroups corresponding to combinatorially defined varieties of rational languages and, conversely, varieties of rational languages corresponding to algebraically defined pseudovarieties of finite semigroups.

Another aspect of the research is explained in part by the different character of the two directions of Eilenberg's correspondence. The pseudovariety V associated with a variety $\mathcal{V}$ of rational languages is defined in terms of generators. Nevertheless, Proposition 2.5 shows how to recover from a given semigroup $S \in \mathsf{V}$ an expression for S as a divisor of a product of generators so that a finite semigroup S belongs to V if and only if the languages computed from S according to Proposition 2.5 belong to $\mathcal{V}$.

On the other hand, if we could effectively test membership in V, then we could effectively determine if a rational language $L \subseteq A^+$ belongs to $\mathcal{V}(A^+)$: we would simply compute the syntactic semigroup of L and test whether it belongs to V, the answer being also the answer to the question whether $L \in \mathcal{V}(A^+)$. This raises the most common problem encountered in finite semigroup theory: given a pseudovariety V defined in terms of generators, determine whether it has a decidable membership problem. A pseudovariety with this property is said to be *decidable*. Since for instance for each set P of primes, the pseudovariety consisting of all finite groups G such that the prime factors of $|G|$ belong to P determines P, a simple counting argument shows that there are too many pseudovarieties for all of them to be decidable. For natural constructions of undecidable pseudovarieties from decidable ones see [1, 24].

For the reverse direction, given a pseudovariety V one is often interested in natural and combinatorially simple generators for the associated variety $\mathcal{V}$ of rational languages. These generators are often defined in terms of Boolean operations: for each finite alphabet A a "natural" generating subset for the Boolean algebra $\mathcal{V}(A^+)$ should be identified. For instance, a language $L \subseteq A^+$ is *piecewise testable* if and only if it is a Boolean combination of languages of the form $A^*a_1A^* \cdots a_nA^*$ with $a_1, \ldots, a_n \in A$. We will run again into this kind of question in Subsection 3.3 where it will be given a simple topological formulation.

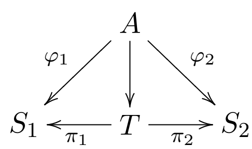
3 Free objects

A basic difficulty in dealing with pseudovarieties of finite algebraic structures is that in general they do not have free objects. The reason is quite simple: free objects tend to be infinite.

As a simple example, consider the pseudovariety N of all finite nilpotent semigroups. For a finite alphabet A and a positive integer n, denoting again by I_n the set of all words of length greater than n, the Rees quotient A^+/I_n belongs to N. In particular, there are arbitrarily large A-generated finite nilpotent semigroups and therefore there can be none which is free among them. In general, there is an A-generated free member of a pseudovariety V if and only if up to isomorphism there are only finitely many A-generated members of V, and most interesting pseudovarieties of semigroups fail this condition.

In universal algebraic terms, we could consider the free objects in the variety generated by V. This variety is defined by all identities which are valid in V and for instance for N there are no such nontrivial semigroup identities: in the notation of the preceding paragraph, A^+/I_n satisfies no nontrivial identities in at most $|A|$ variables in which both sides have length at most n. This means that if we take free objects in the algebraic sense then we loose a lot of information since in particular all pseudovarieties containing N will have the same associated free objects.

Let us go back and try to understand better what is meant by a free object. The idea is to take a structure which is just as general as it needs to be in order to be more general than all A-generated members of a given pseudovariety V. Let us take two A-generated members of V, say given by functions $\varphi_i : A \to S_i$ such that $\varphi_i(A)$ generates S_i $(i = 1, 2)$. Let T be the subsemigroup of the product generated by all pairs of the form $(\varphi_1(a), \varphi_2(a))$ with $a \in A$. Then T is again an A-generated member of V and we have a commutative diagram



where $\pi_i : T \to S_i$ is the projection on the ith component. The semigroup T is therefore more general than both S_1 and S_2 as an A-generated member of V and it is as small as possible to satisfy this property. We could keep going on doing this with more and more A-generated members of V but the problem is that we know by the above discussion concerning N that in general we will never end up with one member of V which is more general than all the others. So we need some kind of limiting process. The appropriate construction is the projective (or inverse) limit which we proceed to introduce in the somewhat wider setting of topological semigroups.

3.1 Profinite semigroups

By a *directed set* we mean a poset in which any two elements have a common upper bound. A subset C of a poset P is said to be *cofinal* if, for every element $p \in P$ there exists $c \in C$ such $p \leq c$.

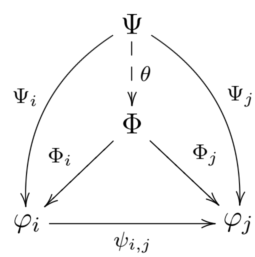
By a *topological semigroup* we mean a semigroup S endowed with a topology such that the semigroup operation $S \times S \to S$ is continuous. Fix a set A and consider the category of *A-generated topological semigroups* whose objects are the mappings $A \to S$ into topological semigroups whose images generate dense subsemigroups, and whose morphisms $\theta : \varphi \to \psi$, from $\varphi : A \to S$ to $\psi : A \to T$, are given by continuous homomorphisms $\theta : S \to T$ such that $\theta \circ \varphi = \psi$. Now, consider a *projective system* in this category, given by a directed set I of indices, for each $i \in I$ an object $\varphi_i : A \to S_i$ in our category of A-generated topological semigroups and, for each pair $i, j \in I$ with $i \geq j$ a *connecting* morphism $\psi_{i,j} : \varphi_i \to \varphi_j$ such that the following conditions hold for all $i, j, k \in I$:

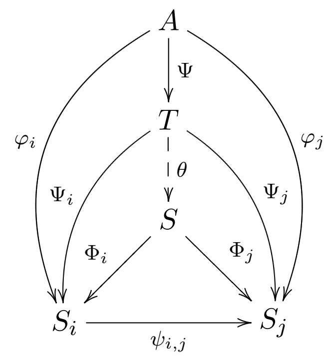
- $\psi_{i,i}$ is the identity morphism on φ_i;
- if $i \geq j \geq k$ then $\psi_{j,k} \circ \psi_{i,j} = \psi_{i,k}$.

The *projective limit* of this projective system is an A-generated topological semigroup $\Phi : A \to S$ together with morphisms $\Phi_i : \Phi \to \varphi_i$ such that for all $i, j \in I$ with $i \geq j$, $\psi_{i,j} \circ \Phi_i = \Phi_j$ and, moreover, the following universal property holds:

> for any other A-generated topological semigroup $\Psi : A \to T$ and morphisms $\Psi_i : \Psi \to \varphi_i$ such that for all $i, j \in I$ with $i \geq j$, $\psi_{i,j} \circ \Psi_i = \Psi_j$ there exists a morphism $\theta : \Psi \to \Phi$ such that $\Phi_i \circ \theta = \Psi_i$ for every $i \in I$.

The situation is depicted in the following two commutative diagrams, respectively of morphisms and mappings:





The uniqueness up to isomorphism of such a projective limit is a standard diagram chasing exercise. Existence may be established as follows.

Consider the subsemigroup S of the product $\prod_{i\in I} S_i$ consisting of all $(s_i)_{i\in I}$ such that, for all $i, j \in I$ with $i \geq j$,

$$\varphi_{i,j}(s_i) = s_j \tag{3.1}$$

endowed with the induced topology from the product topology. To check that S provides a construction of the projective limit, we first claim that the mapping $\Phi : A \to S$ given by $\Phi(a) = (\varphi_i(a))_{i\in I}$ is such that $\Phi(A)$ generates a dense subsemigroup T of S. Indeed, since the system is projective, to find an approximation $(t_i)_{i\in I} \in T$ to an element $(s_i)_{i\in I}$ of S given by $t_{i_j} \in K_{i_j}$ for a clopen set $K_{i_j} \subseteq S_{i_j}$ containing s_{i_j} with $j = 1, \ldots, n$, one may first take $k \in I$ such that $k \geq i_1, \ldots, i_n$. Then, by the hypothesis that the subsemigroup T_k of S_k generated by $\varphi_k(A)$ is dense, there is a word $w \in A^+$ which in T_k represents an element of the open set $\bigcap_{j=1}^n \psi_{k,i_j}^{-1} K_{i_j}$ since this set is non-empty as s_k belongs to it. This word w then represents and element $(t_i)_{i\in I}$ of T which is an approximation as required.

It is now an easy exercise to show that the projections $\Phi_i : S \to S_i$ have the above universal property. Note that since each of the conditions (3.1) only involves two components and $\varphi_{i,j}$ is continuous, S is a closed subsemigroup of the product $\prod_{i\in I} S_i$. So, by Tychonoff's Theorem, if all the S_i are compact semigroups, then so is S. We assume Hausdorff's separation axiom as part of the definition of compactness.

Recall that a topological space if *totally disconnected* if its connected components are singletons and it is *zero-dimensional* if it admits a basis of open sets consisting of clopen (meaning both closed and open) sets. See Willard [93] for background in General Topology.

A finite semigroup is always viewed in this paper as a topological semigroup under the discrete topology. A *profinite semigroup* is defined to be a projective limit of a projective system of finite semigroups in the above sense, that is for some suitable choice of generators. The next result provides several alternative definitions of profinite semigroups.

3.1 Theorem *The following conditions are equivalent for a compact semigroup S:*

(1) *S is profinite;*

(2) *S is residually finite as a topological semigroup;*

(3) *S is a closed subdirect product of finite semigroups;*

(4) *S is totally disconnected;*

(5) *S is zero-dimensional.*

Proof By the explicit construction of the projective limit we have (1)⇒(2) while (2)⇒(3) is easily verified from the definitions. For (3)⇒(1), suppose that $\Phi : S \to \prod_{i\in I} S_i$ is an injective continuous homomorphism from the compact semigroup S into a product of finite semigroups and that the factors are such that, for each component projection $\pi_j : \prod_{i\in I} S_i \to S_j$ the mapping $\pi_j \circ \Phi : S \to S_j$ is onto. We build a projective system of S-generated finite semigroups by considering all onto mappings of the form $\Phi_F : S \to S_F$ where F is a finite subset of I and $\Phi_F = \pi_F \circ \Phi$ where $\pi_F : \prod_{i\in I} S_i \to \prod_{i\in F} S_i$ denotes the natural projection; the indexing set is therefore the directed set of all finite subsets of I, under the inclusion ordering, and for the connecting homomorphisms we take the natural projections. It is now immediate to verify that S is the projective limit of this projective system of finite S-generated semigroups.

Since a product of totally disconnected spaces is again totally disconnected, we have (3)⇒(4). The equivalence (4)⇔(5) holds for any compact space and it is a well-known exercise in General Topology [93].

Up to this point in the proof, the fact that we are dealing with semigroups rather than any other variety of universal algebras really makes no essential difference. To complete the proof we establish the implication (5)⇒(2), which was first proved by Numakura [62]. Given two distinct points $s, t \in S$, by zero-dimensionality they may be separated by a clopen subset $K \subseteq S$ in the sense that s lies in K and t does not. Since the syntactic congruence ρ_K saturates K, the congruence classes of s and t are distinct, that is the quotient homomorphism $\varphi : S \to \operatorname{Synt} K$ sends s and t to two distinct points. Hence, to prove (2) it suffices to show that $\operatorname{Synt} K$ is finite and φ is continuous, which is the object of Lemma 3.3 below. □

As an immediate application we obtain the following closure properties for the class of profinite semigroups.

3.2 Corollary *A closed subsemigroup of a profinite semigroup is also profinite. The product of profinite semigroups is also profinite.* □

The following technical result has been extended in [2] to a universal algebraic setting in which syntactic congruences are determined by finitely many terms. See [32] for the precise scope of validity of the implication (5)⇒(1) in Theorem 3.1 and applications in Universal Algebra.

We say that a congruence ρ on a topological semigroup is *clopen* if its classes are clopen.

3.3 Lemma (Hunter [44]) *If S is a compact zero-dimensional semigroup and K is a clopen subset of S then the syntactic congruence ρ_K is clopen, and therefore it has finitely many classes.*

Proof The proof uses nets, sequences indexed by directed sets which play for general topological spaces the role played by usual sequences for metric spaces [93]. Let $(s_i)_{i\in I}$ be a convergent net in S with limit s. We should show that there exists $i_0 \in I$ such that, whenever $i \geq i_0$, we have $s_i \;\rho_K\; s$. Suppose on the contrary that for every $j \in I$ there exists $i \geq j$ such that s_i is not in the same ρ_K-class as s. The set Λ consisting of all $i \in I$ such that

s_i is not in the ρ_K-class of s is then a cofinal subset of I which determines a subnet $(s_i)_{i\in\Lambda}$ converging to s from outside the ρ_K-class of s.

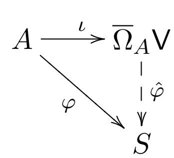
Now we use the characterization of syntactic congruences on semigroups: for each $i \in \Lambda$ there exist $x_i, y_i \in S^1$ such that the products $x_i s_i y_i$ and $x_i s y_i$ do not both lie in K. Note that if a directed set is partitioned into two parts, one of them must contain a cofinal subset. Hence for some cofinal subset of indices M we must have all of the $x_i s_i y_i$ with $i \in M$ lying in K or all of them lying in the complement $S \setminus K$, while the opposite holds for $x_i s y_i$. By compactness and taking subnets, we may as well assume that the nets $(x_i)_{i\in M}$ and $(y_i)_{i\in M}$ converge, say to x and y, respectively. By continuity of multiplication in S, the nets $(x_i s_i y_i)_{i\in M}$ and $(x_i s y_i)_{i\in M}$ both converge to xsy. Since both K and $S \setminus K$ are closed, it follows that xsy must lie in both K and $S \setminus K$, which is absurd. Hence the classes of ρ_K are open and, since they form a partition of a compact space, there can only be finitely many of them and they are also closed. □

3.2 Relatively free profinite semigroups

For a pseudovariety V, we say that a profinite semigroup S is *pro*-V if it is a projective limit of members of V. In view of Theorem 3.1 and its proof, this condition is equivalent to the profinite semigroup being a subdirect product of members of V and also to being *residually* V in the sense that for all distinct $s_1, s_2 \in S$ there exists a continuous homomorphism $\varphi : S \to T$ such that $T \in \mathsf{V}$ and $\varphi(s_1) \neq \varphi(s_2)$.

Let us go back to the construction of free objects for a pseudovariety V. For a generating set A the idea was to take the projective limit of all A-generated members of V. For set theoretical reasons this is inconvenient since there are too many such semigroups but nothing is lost in considering only representatives of isomorphism classes and that is what we do. So, let V_0 be a set containing a representative from each isomorphism class of A-generated members of V. The set V_0 determines a projective system by taking the unique connecting homomorphisms which respect to the choice of generators. The projective limit of this system is denoted $\overline{\Omega}_A\mathsf{V}$.

3.4 Proposition *The profinite semigroup* $\overline{\Omega}_A\mathsf{V}$ *has the following universal property: the natural mapping* $\iota : A \to \overline{\Omega}_A\mathsf{V}$ *is such that, for every mapping* $\varphi : A \to S$ *into a pro*-V *semigroup there exists a unique continuous homomorphism* $\hat{\varphi} : \overline{\Omega}_A\mathsf{V} \to S$ *such that* $\hat{\varphi} \circ \iota = \varphi$ *as depicted in the following diagram:*



Proof Since pro-V semigroups are subdirect products of members of V, it suffices to consider the case when S itself lies in V. Without loss of generality, we may assume that S is generated by $\varphi(A)$. Then S is isomorphic, as an A-generated semigroup, to some member of V_0 and so we may further assume that $S \in \mathsf{V}_0$. But then, from our explicit construction of the projective limit as a closed subsemigroup of a direct product, it suffices to take $\hat{\varphi}$ to be the projection $\overline{\Omega}_A\mathsf{V} \to S$ into the component corresponding to φ. □

Since, by usual diagram chasing there is up to isomorphism at most one A-generated pro-V semigroup with the above universal property, we conclude that $\overline{\Omega}_A\mathsf{V}$ does not depend, up to isomorphism, on the choice of V_0. We call $\overline{\Omega}_A\mathsf{V}$ the *free pro-*V *semigroup on* A. A profinite semigroup is said to be *relatively free* if it is of the form $\overline{\Omega}_A\mathsf{V}$ for some set A and some pseudovariety V.

By construction, $\overline{\Omega}_A\mathsf{V}$ is an A-generated topological semigroup so that the subsemigroup $\Omega_A\mathsf{V}$ generated by the image of ι is dense in $\overline{\Omega}_A\mathsf{V}$, which explains the line over the capital omega. From Proposition 3.4 it follows that $\Omega_A\mathsf{V}$ is the free semigroup in the variety generated by V.

The mapping ι is injective provided V is not the trivial pseudovariety consisting of singleton semigroups. Hence we will often identify the elements of A with their images under ι.

3.3 Recognizable subsets

The following result characterizes the subsets of a pro-V semigroup which are recognized by members of V. The reader may wish to compare it with Hunter's lemma.

3.5 Proposition *Let S be a pro-V semigroup and let $K \subseteq S$. Then the following conditions are equivalent:*

(1) *there exists a continuous homomorphism $\varphi : S \to F$ such that $F \in \mathsf{V}$ and $K = \varphi^{-1}\varphi K$;*

(2) *K is clopen;*

(3) *the syntactic congruence ρ_K is clopen.*

In particular, all these conditions imply that the syntactic semigroup $\operatorname{Synt} K$ *belongs to* V.

Proof Assuming the existence of a function φ satisfying (1), we deduce that K is clopen since it is the inverse image under a continuous function of a clopen set. For the converse, suppose K is clopen and let $S \hookrightarrow \prod_{i\in I} S_i$ be a subdirect product of a family of members of V. Then K may be expressed as $K = S\cap(K_1\cup\cdots\cup K_n)$, where each K_ℓ is a product of the form $\prod_{i\in I} X_i$ with $X_i \subseteq S_i$ and $X_i = S_i$ for all but finitely many indices. Let J be the (finite) set of all exceptional indices with $\ell = 1, \ldots, n$ and consider the projection $\varphi : S \to \prod_{i\in J} S_i$. Then it is routine to check that φ is a continuous homomorphism satisfying the required conditions. This establishes the equivalence (1)⇔(2).

If K is clopen then ρ_K is clopen by Hunter's Lemma. This proves (2)⇒(3) and for the converse it suffices to recall that K is saturated by ρ_K.

Finally, assuming (1), K is recognized by a semigroup from V. Since the syntactic semigroup $\operatorname{Synt} K$ divides every semigroup which recognizes K by Proposition 2.4, $\operatorname{Synt} K$ belongs to V since V is closed under taking divisors. □

We note that the assumption that $\operatorname{Synt} K$ belongs to V for a subset K of a pro-V semigroup S does not suffice to deduce that K is clopen, as the following example shows. Take S to be the Cantor set and consider the left-zero multiplication $st = s$ on S. Although one could easily show it directly, by Theorem 3.1 S is profinite and hence it is pro-LZ for the pseudovariety LZ of all finite left-zero semigroups. Let K be a subset of S. Then a simple calculation shows that the syntactic congruence ρ_K consists of two classes, namely K and its complement. Hence $\operatorname{Synt} K$ belongs to LZ for an arbitrary subset $K \subseteq S$ while K does need to clopen.

We say that a subset L of a semigroup S is V-*recognizable* if there exists a homomorphism $\varphi : S \to F$ into some $F \in \mathsf{V}$ such that $L = \varphi^{-1}\varphi L$. Proposition 3.5 leads to the following topological characterization of V-recognizable subsets of $\Omega_A\mathsf{V}$.

3.6 Theorem *The following conditions are equivalent for a subset $L \subseteq \Omega_A\mathsf{V}$:*

(1) *L is V-recognizable;*

(2) *the closure $K = \overline{L} \subseteq \overline{\Omega}_A\mathsf{V}$ is open and $L = K \cap \Omega_A\mathsf{V}$;*

(3) *$L = K \cap \Omega_A\mathsf{V}$ for some clopen $K \subseteq \overline{\Omega}_A\mathsf{V}$.*

Proof To prove (1)⇒(2), suppose L is recognized by a homomorphism $\varphi : \Omega_A\mathsf{V} \to F$ such that $F \in \mathsf{V}$ and $L = \varphi^{-1}\varphi L$. By the universal property of $\overline{\Omega}_A\mathsf{V}$, there exists a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_A\mathsf{V} \to F$ extending φ. Then $K = \hat{\varphi}^{-1}\varphi L$ is open and satisfies $K \cap \Omega_A\mathsf{V} = L$. Since $\Omega_A\mathsf{V}$ is dense in $\overline{\Omega}_A\mathsf{V}$ so is L dense in K, which shows K has the required properties for (2).

The implication (2)⇒(3) is trivial, so it remains to show (3)⇒(1). Suppose (3) holds. By Proposition 3.5 there exists a continuous homomorphism $\psi : \overline{\Omega}_A\mathsf{V} \to F$ such that $F \in \mathsf{V}$ and $K = \psi^{-1}\psi K$. Let φ be the restriction of ψ to $\Omega_A\mathsf{V}$. Then we have $L = \Omega_A\mathsf{V} \cap K = \Omega_A\mathsf{V} \cap \psi^{-1}\psi K = \varphi^{-1}\psi K$ and so L is V-recognizable. □

Another application of Proposition 3.5 is the following result.

3.7 Proposition *The image of a pro-V semigroup under a continuous homomorphism into a profinite semigroup is pro-V and it belongs to V if it is finite.*

Proof Let $\varphi : S \to T$ be a continuous homomorphism with S pro-V and T profinite. Since T is a subdirect product of finite semigroups, it suffices to consider the case where T is finite and φ is onto and show that $T \in \mathsf{V}$. The sets $K_t = \varphi^{-1}t$, with $t \in T$, are clopen subsets of S. By Proposition 3.5, there is for each $t \in T$ a continuous homomorphism $\psi_t : S \to F_t$ such that $F_t \in \mathsf{V}$ and $\psi_t^{-1}\psi_t K_t = K_t$. The induced homomorphism $\psi : S \to F$, where $F = \prod_{t\in T} F_t$, has a kernel $\ker\psi$ which is contained in $\ker\varphi$ and so T divides F, which shows that indeed $T \in \mathsf{V}$. □

The hypothesis in Proposition 3.7 that the continuous homomorphism assumes values in a profinite semigroup cannot be removed as the following example shows. We take again S to be the Cantor set under left-zero multiplication. It is well-known that the unit interval $T = [0, 1]$ is a continuous image of S and so it is also a continuous homomorphic image if we endow it with the left-zero multiplication. But of course T is not zero-dimensional and therefore it is not profinite.

We have seen in Section 2 that one is often interested in describing a variety of rational languages $\mathcal{V}$ by giving a set of generators for the Boolean algebra $\mathcal{V}(A^+)$ for each finite alphabet A. We now aim to characterize this property in topological terms.

3.8 Proposition *The following conditions are equivalent for a family $\mathcal{F}$ of V-recognizable subsets of $\Omega_A\mathsf{V}$, where $\overline{\mathcal{F}} = \{\overline{L} : L \in \mathcal{F}\}$:*

(1) *$\mathcal{F}$ generates the Boolean algebra of all V-recognizable subsets of $\Omega_A\mathsf{V}$;*

(2) $\overline{\mathcal{F}}$ *generates the Boolean algebra of all clopen subsets of* $\overline{\Omega}_A\mathsf{V}$*;*

(3) $\overline{\mathcal{F}}$ *suffices to separate points of* $\overline{\Omega}_A\mathsf{V}$*.*

Proof Note that, for subsets $L, L_1, L_2 \subseteq \Omega_A\mathsf{V}$, we have $\overline{L_1 \cup L_2} = \overline{L_1} \cup \overline{L_2}$ and, by Theorem 3.6, in case L is V-recognizable, the closure $K = \overline{L}$ is clopen with $K \cap \Omega_A\mathsf{V} = L$ and so $\overline{\Omega_A\mathsf{V} \setminus L} = \overline{\Omega}_A\mathsf{V} \setminus \overline{L}$. Hence a Boolean expression for L in terms of elements of $\mathcal{F}$ gives rise to a Boolean expression for $\overline{L}$ in terms of elements of $\overline{\mathcal{F}}$ and vice versa. This proves the equivalence (1)⇔(2).

Assume (2) and let $s, t \in S$ be two distinct points. Since the topology of $\overline{\Omega}_A\mathsf{V}$ is zero-dimensional, there exists a clopen subset $K \subseteq \overline{\Omega}_A\mathsf{V}$ such that $s \in K$ and $t \notin K$. By assumption, K admits a Boolean expression in terms of the closures of the elements of $\mathcal{F}$ and therefore it admits an expression as a finitary union of finitary intersections of members of $\overline{\mathcal{F}}$ and their complements. At least one term of the union must contain s and none of them can contain t, and so we may avoid taking the union. Similarly, we may avoid taking the intersection, which shows that there is an element of $\overline{\mathcal{F}}$ which contains one of s and t but not the other. This proves (2)⇒(3).

It remains to show that (3)⇒(2). Let $\overline{\mathcal{F}}'$ be the family of all elements of $\overline{\mathcal{F}}$ together with their complements. Given a closed subset $C \subseteq \overline{\Omega}_A\mathsf{V}$ and $s \in \overline{\Omega}_A\mathsf{V} \setminus C$, for each $t \in C$ there exists $K_t \in \overline{\mathcal{F}}'$ such that $s \notin K_t$ and $t \in K_t$. Then the K_t constitute an open cover of the closed set C and so there are $t_1, \ldots, t_n \in C$ such that $K = K_{t_1} \cup \cdots \cup K_{t_n}$ contains C but not s. This shows that we may separate points from closed sets using finitary unions of elements of $\overline{\mathcal{F}}'$.

Let now C be a clopen subset of $\overline{\Omega}_A\mathsf{V}$. For each $s \in C$ we can find a member K_s of the Boolean algebra generated by $\overline{\mathcal{F}}$ containing s with empty intersection with the closed set $\overline{\Omega}_A\mathsf{V} \setminus C$, that is such that $K_s \subseteq C$. Then the compact set C is covered by the open sets K_s with $s \in C$ and so there exist $s_1, \ldots, s_m$ such that $K = K_{s_1} \cup \cdots \cup K_{s_m}$ covers C and $K \subseteq C$, that is $C = K$. This shows that C belongs to the Boolean algebra generated by $\overline{\mathcal{F}}'$. □

3.4 Metric structure

We end this section with a brief reference to a natural metric on finitely generated profinite semigroups. Let S be a profinite semigroup. Define, for $u, v \in S$,

$$d(u,v) = \begin{cases} 2^{-r(u,v)} & \text{if } u \neq v \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where $r(u, v)$ denotes the minimum cardinality of a finite semigroup T such that there exists a continuous homomorphism $\varphi : S \to T$ with $\varphi(u) \neq \varphi(v)$. Note that d is an *ultrametric* in the sense that $d : S \times S \to [0, +\infty)$ is a function satisfying the following conditions:

- $d(u, v) = 0$ if and only if $u = v$;
- $d(u, v) = d(v, u)$;
- $d(u, w) \leq \max\{d(u, v), d(v, w)\}$.

The latter condition is trivial if any two of the three elements $u, v, w \in S$ coincide and otherwise, taking logarithms, we deduce that it is equivalent to the inequality $r(u,w) \geq \min\{r(u,v),\, r(v,w)\}$ which follows from the trivial fact that, if $\varphi(u) = \varphi(v)$ and $\varphi(v) = \varphi(w)$ for a function $\varphi : S \to T$, then $\varphi(u) = \varphi(w)$. We call d the *natural metric* on S.

3.9 Proposition *For a profinite semigroup S, the topology of S is contained in the topology induced by the natural metric and the two topologies coincide in case S is finitely generated.*

Proof We denote by $B_\varepsilon(u)$ the open ball $\{v \in S : d(u,v) < \varepsilon\}$. Given a clopen subset K of S, by Proposition 3.5 there exists a continuous homomorphism $\varphi : S \to T$ into a finite semigroup T such that $K = \varphi^{-1}\varphi K$. Now, for $t \in T$, the ball $B_{2^{-|T|}}(t)$ is contained in $\varphi^{-1}(t)$ and so K is a finite union of open balls.

Next assume that S is finitely generated. Consider the open ball $B = B_{2^{-n}}(u)$. Observe that up to isomorphism there are only finitely many semigroups with at most n elements. Since S is finitely generated, there are only finitely many kernels of continuous homomorphisms from S into semigroups with at most n elements and so their intersection is a clopen congruence on S. It follows that there exists a continuous homomorphism $\varphi : S \to T$ into a finite semigroup T such that $\varphi(u) = \varphi(v)$ if and only if $r(u,v) > n$. Hence $B = \varphi^{-1}\varphi B$ so that B is open in the topology of S. $\square$

We observe that the natural metric d is such that the multiplication is contracting in the sense that the following additional condition is satisfied:

- $d(uv, wz) \leq \max\{d(u,w), d(v,z)\}$.

The completion $\hat{S}$ of a topological semigroup S whose topology is induced by a contracting metric inherits a semigroup structure where the product of two elements $s, t \in \hat{S}$ is defined by taking any sequences $(s_n)_n$ and $(t_n)_n$ converging respectively to s and t and noting that $(s_n t_n)_n$ is a Cauchy sequence whose limit st does not depend on the choice of the two sequences. This gives $\hat{S}$ the structure of a topological semigroup.

In the case of a relatively free profinite semigroup $\overline{\Omega}_A\mathsf{V}$, by Proposition 3.7 the finite continuous homomorphic images of $\overline{\Omega}_A\mathsf{V}$ are the A-generated members of V. Moreover, by Proposition 3.4 every homomorphism from $\Omega_A\mathsf{V}$ to a member of V has a unique continuous homomorphic extension to $\overline{\Omega}_A\mathsf{V}$. We define the *natural metric* on $\Omega_A\mathsf{V}$ to be the restriction to $\Omega_A\mathsf{V}$ of the natural metric on $\overline{\Omega}_A\mathsf{V}$ and we observe that, by the preceding remarks, this is equivalent to defining the natural metric directly on $\Omega_A\mathsf{V}$ by the formula (3.2) where now $r(u,v)$ denotes the minimum cardinality of a semigroup $T \in \mathsf{V}$ such that there exists a homomorphism $\varphi : \Omega_A\mathsf{V} \to T$ with $\varphi(u) \neq \varphi(v)$.

We are thus led to an alternative construction of $\overline{\Omega}_A\mathsf{V}$.

3.10 Theorem *For a finite set A, the completion of the semigroup $\Omega_A\mathsf{V}$ with respect to the natural metric is a profinite semigroup isomorphic with $\overline{\Omega}_A\mathsf{V}$.*

Proof By Proposition 3.9, $\overline{\Omega}_A\mathsf{V}$ is a metric space under the natural metric, and its restriction to the dense subspace $\Omega_A\mathsf{V}$ is the natural metric of $\Omega_A\mathsf{V}$. By results in General Topology [93, Theorem 24.4], it follows that the metric space $(\overline{\Omega}_A\mathsf{V}, d)$ is the completion of $(\Omega_A\mathsf{V}, d)$. It remains to show that the multiplication of the completion as defined above coincides with the multiplication of $\overline{\Omega}_A\mathsf{V}$ which follows from continuity of multiplication in $\overline{\Omega}_A\mathsf{V}$. $\square$

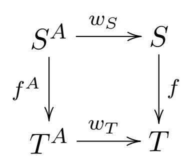
4 The operational point of view

It is well known from Universal Algebra that a term w in a free algebra F on n generators (the variables) in a variety $\mathcal{V}$ induces an n-ary operation on the members S of $\mathcal{V}$ basically by substituting the arguments for the variables and operating in S [31]. This may be formulated by taking the unique extension of the variable evaluation to a homomorphism $\varphi : F \to S$ and then computing the image $\varphi(w)$. This formulation can be suitably applied to relatively free profinite semigroups, which is the starting point for this section.

4.1 Implicit operations

Given $w \in \overline{\Omega}_A\mathsf{V}$ and a pro-V semigroup S, there is a natural interpretation of w as an operation on S namely the mapping $w_S : S^A \to S$ which sends a function $\varphi : A \to S$ to $\hat{\varphi}(w)$ where $\hat{\varphi} : \overline{\Omega}_A\mathsf{V} \to S$ is the unique continuous homomorphism such that $\hat{\varphi} \circ \iota = \varphi$, where in turn $\iota : A \to \overline{\Omega}_A\mathsf{V}$ denotes the generating function associated with $\overline{\Omega}_A\mathsf{V}$ as the free pro-V semigroup on A.

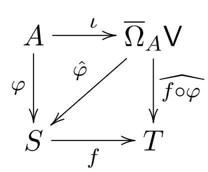
4.1 Proposition *The function w_S as defined above is continuous and if $f : S \to T$ is a continuous homomorphism between two pro-V semigroups then the following diagram commutes*



(4.1)

where $f^A(\varphi) = f \circ \varphi$ for $\varphi \in S^A$.

Proof We first prove the commutativity of the diagram (4.1). Consider the diagram



By the universal property of $\overline{\Omega}_A\mathsf{V}$ there is a unique continuous homomorphism $\widehat{f \circ \varphi}$ such that the diagram commutes. Since $f \circ \hat{\varphi}$ also has this property, it follows that $\widehat{f \circ \varphi} = f \circ \hat{\varphi}$. Hence we have

$$w_T(f^A(\varphi)) = w_T(f \circ \varphi) = \widehat{f \circ \varphi}(w) = (f \circ \hat{\varphi})(w) = f(\hat{\varphi}(w)) = f(w_S(\varphi)).$$

which establishes commutativity of diagram (4.1).

To prove continuity of the natural interpretation w_S, let $K \subseteq S$ be a clopen subset. By Proposition 3.5 there exists a continuous homomorphism $f : S \to T$ such that $T \in \mathsf{V}$ and $K = f^{-1}fK$. By commutativity of diagram (4.1) we have

$$w_S^{-1}K = w_S^{-1}f^{-1}fK = (f^A)^{-1}w_T^{-1}fK.$$

Since w_T is continuous, as T is finite, and f^A is also continuous, we conclude that $w_S^{-1}K$ is clopen. Hence w_s is continuous. □

We say that the operation w *commutes with the homomorphism* $f : S \to T$ if the diagram (4.1) commutes. An operation $w = (w_S)_{S \in \mathsf{V}}$ with an interpretation $w_S : S^A \to A$ on each $S \in \mathsf{V}$ is called an *A-ary implicit operation on* V if it commutes with every homomorphism $f : S \to T$ between members of V. The natural interpretation provides a representation

$$\begin{aligned} \Theta : \overline{\Omega}_A\mathsf{V} &\to \{A\text{-ary implicit operations on } \mathsf{V}\} \\ w &\mapsto (w_S)_{S \in \mathsf{V}} \end{aligned}$$

Since $\overline{\Omega}_A\mathsf{V}$ is residually V, given distinct $u, v \in \overline{\Omega}_A\mathsf{V}$ there exists $S \in \mathsf{V}$ and a continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{V} \to S$ such that $\varphi(u) \neq \varphi(v)$, that is $u_S(\varphi \circ \iota) \neq v_S(\varphi \circ \iota)$ and so we have $u_S \neq v_S$, which shows that Θ is injective.

4.2 Theorem *The mapping Θ is a bijection.*

Proof Let $w = (w_S)_{S \in \mathsf{V}}$ be an A-ary implicit operation on V. We exhibit an element $s \in \overline{\Omega}_A\mathsf{V}$ such that $\Theta(s) = w$. For this purpose, we take a specific representation of $\overline{\Omega}_A\mathsf{V}$ as a projective limit of members of V namely as the projective limit of a projective system containing one representative from each isomorphism class of A-generated members of V: $\varphi_i : A \to S_i$ $(i \in I)$ with connecting morphisms $\psi_{i,j} : \varphi_i \to \varphi_j$ $(i \geq j)$. Let $s_i = w_{S_i}(\varphi_i)$. Since w is an implicit operation, a simple calculation shows that $\psi_{i,j}(s_i) = s_j$ whenever $i \geq j$. Hence $(s_i)_{i \in I}$ determines an element s of the projective limit $\overline{\Omega}_A\mathsf{V}$.

It remains to show that $\Theta(s) = w$, that is $s_T = w_T$ for every $T \in \mathsf{V}$. Let $\varphi \in T^A$. Since both s and w are implicit operations, we may as well assume that $\varphi(A)$ generates T. Hence up to isomorphism $\varphi : A \to T$ is one of the $\varphi_i : A \to S_i$ and so we may assume that the two mappings coincide. Then we have

$$s_T(\varphi) = s_{S_i}(\varphi_i) = \widehat{\varphi}_i(s) = s_i = w_{S_i}(\varphi_i) = w_T(\varphi)$$

where the middle step comes from the observation that $\widehat{\varphi}_i$ is the projection on the ith component. This completes the proof of the equality $\Theta(s) = w$. $\square$

In view of Theorem 4.2 we will from hereon identify members of $\overline{\Omega}_A\mathsf{V}$ with A-ary implicit operations on V. It is this operational point of view that explains the capital omega Ω in the notation for free pro-V semigroups. Starting from an implicit operation on V we realize that it has a natural extension to an operation on all pro-V semigroups which commutes with continuous homomorphisms. Treating a positive integer n as a set with n elements, we may speak of *n-ary implicit operations*.

Recall that $\Omega_A\mathsf{V}$ denotes the subsemigroup of $\overline{\Omega}_A\mathsf{V}$ generated by the image of the natural generating mapping $\iota : A \to \overline{\Omega}_A\mathsf{V}$. Note that natural interpretation of $\iota(a)$ for $a \in A$ is given by $\varphi \in S^A \mapsto \varphi(a)$, that is the projection on the a-component if we view S^A as a product. Hence $\Omega_A\mathsf{V}$ corresponds under the above mapping Θ to the semigroup of A-ary implicit operations generated by these component projections, that is the semigroup terms over A as interpreted in V. The elements of $\Omega_A\mathsf{V}$ are also called *explicit operations*.

For a class $\mathcal{C}$ of finite semigroups, denote by $\mathsf{V}(\mathcal{C})$ the pseudovariety generated by $\mathcal{C}$. In case $\mathcal{C} = \{S_1, \ldots, S_n\}$, we may write $\mathsf{V}(S_1, \ldots, S_n)$ instead of $\mathsf{V}(\mathcal{C})$. Note that $\mathsf{V}(S_1, \ldots, S_n) = \mathsf{V}(S_1 \times \cdots \times S_n)$.

4.3 Proposition *Let S be a finite semigroup, $\mathsf{V} = \mathsf{V}(S)$, and let A be a finite set. Then there is an embedding $\overline{\Omega}_A\mathsf{V} \hookrightarrow S^{S^A}$ and so $\overline{\Omega}_A\mathsf{V}$ is finite and $\overline{\Omega}_A\mathsf{V} = \Omega_A\mathsf{V}$.*

Proof Define the mapping $\Phi : \overline{\Omega}_A\mathsf{V} \to S^{S^A}$ by sending each $w \in \overline{\Omega}_A\mathsf{V}$ to its natural interpretation $w_S : S^A \to S$. Since implicit operations commute with homomorphisms, if two implicit operations $u, v \in \overline{\Omega}_A\mathsf{V}$ coincide in S then they must also coincide in all of V, which consists of divisors of finite products of copies of S. Hence our mapping is injective and the rest of the statement follows immediately. □

If V and W are pseudovarieties with $\mathsf{V} \subseteq \mathsf{W}$, then an implicit operation $w \in \overline{\Omega}_A\mathsf{W}$ determines an implicit operation $w|_\mathsf{V} \in \overline{\Omega}_A\mathsf{V}$ by restriction: $(w_S)_{S\in\mathsf{W}} \mapsto (w_S)_{S\in\mathsf{V}}$. The mapping

$$\begin{aligned} \overline{\Omega}_A\mathsf{W} &\to \overline{\Omega}_A\mathsf{V} \\ w &\mapsto w|_\mathsf{V} \end{aligned}$$

is called the *natural projection*. In terms of the construction of the projective limit, this is indeed a projection which is obtained by disregarding all components in the product $\prod_{i\in I} S_i$ corresponding to A-generated members of W which are not in V. This proves the following result.

4.4 Proposition *For pseudovarieties V and W with $\mathsf{V} \subseteq \mathsf{W}$, the natural projection $\overline{\Omega}_A\mathsf{W} \to \overline{\Omega}_A\mathsf{V}$ is an onto continuous homomorphism.* □

4.2 Pseudoidentities

By a V-*pseudoidentity* we mean a formal equality $u = v$, with $u, v \in \overline{\Omega}_A\mathsf{V}$ for some finite set A. If $u_S = v_S$ then we say that the pseudoidentity $u = v$ *holds* in a given pro-V semigroup S, or that S *satisfies* $u = v$, and we write $S \models u = v$. Note that $S \models u = v$ for $u, v \in \overline{\Omega}_A\mathsf{V}$ if and only if, for every continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{V} \to S$, the equality $\varphi(u) = \varphi(v)$ holds. For a subclass $\mathcal{C} \subseteq \mathsf{V}$, we also write $\mathcal{C} \models u = v$ if $S \models u = v$ for every $S \in \mathcal{C}$. The following result is immediate from the definitions.

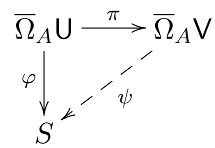
4.5 Lemma *Let V and W be pseudovarieties with $\mathsf{V} \subseteq \mathsf{W}$ and let $\pi : \overline{\Omega}_A\mathsf{W} \to \overline{\Omega}_A\mathsf{V}$ be the natural projection. Then, for $u, v \in \overline{\Omega}_A\mathsf{W}$, we have $\mathsf{V} \models u = v$ if and only if $\pi(u) = \pi(v)$.* □

For a set Σ of V-pseudoidentities, we denote by $[\![\Sigma]\!]_\mathsf{V}$ (or simply $[\![\Sigma]\!]$ if V is understood from the context) the class of all $S \in \mathsf{V}$ which satisfy all pseudoidentities from Σ. From the fact that implicit operations on V commute with homomorphisms between members of V, it follows easily that $[\![\Sigma]\!]$ is a pseudovariety contained in V. The converse is also true.

4.6 Theorem (Reiterman [70]) *A subclass V of a pseudovariety W is a pseudovariety if and only if it is of the form $\mathsf{V} = [\![\Sigma]\!]_\mathsf{W}$ for some set Σ of W-pseudoidentities.*

Proof Let V be a pseudovariety contained in W and let Σ denote the set of all W-pseudoidentities $u = v$ satisfied by V with $u, v \in \overline{\Omega}_A\mathsf{W}$ and $A \subseteq X$, where X is a fixed countably infinite set. Then we have $\mathsf{V} \subseteq [\![\Sigma]\!]$ and we claim equality holds. Let $\mathsf{U} = [\![\Sigma]\!]$ and let $S \in \mathsf{U}$. Then there exists $A \subseteq X$ and an onto continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{U} \to S$.

Let $\pi : \overline{\Omega}_A\mathsf{U} \to \overline{\Omega}_A\mathsf{V}$ be the natural projection. By Lemma 4.5, if $u, v \in \overline{\Omega}_A\mathsf{U}$ are such that $\pi(u) = \pi(v)$ then $\mathsf{V} \models u = v$ and so $u = v$ is a pseudoidentity from Σ so that $S \models u = v$ and $\varphi(u) = \varphi(v)$. This show that $\ker \pi \subseteq \ker \varphi$ and therefore there exists a unique homomorphism $\psi : \overline{\Omega}_A\mathsf{V} \to S$ such that the following diagram commutes:



We claim that ψ is continuous. Indeed, given a subset $K \subseteq S$, by continuity of φ the set $\varphi^{-1}K$ is closed and therefore by continuity of π, $\psi^{-1}K = \pi\varphi^{-1}K$ is closed. Hence ψ is an onto continuous homomorphism. It follows that $S \in \mathsf{V}$ by Proposition 3.7. Hence $\mathsf{V} = [\![\Sigma]\!]_{\mathsf{W}}$. □

There are by now many proofs of this result. It is only fair to mention Banaschewski's proof [26] which was obtained independently of Reiterman's proof and which suggested to look at sets of implicit operations as algebraic-topological structures, a viewpoint which proved to be very productive.

In these notes from hereon we will always take the W of Theorem 4.6 to be the pseudovariety S of all finite semigroups, that is all pseudoidentities will be S-pseudoidentities. A set Σ of pseudoidentities such that $\mathsf{V} = [\![\Sigma]\!]$ is called a *basis of pseudoidentities* for V. The pseudovariety V will be called *finitely based* if it admits a finite basis of pseudoidentities.

To give examples illustrating Reiterman's Theorem, we now describe some important unary implicit operations on finite semigroups. There are several equivalent ways to describe them so we will choose one which is economical in the sense that it requires essentially no verification. For a finite semigroup S, $s \in S$, and $k \in \mathbb{Z}$, the sequence $(s^{n!+k})_n$ becomes constant for n sufficiently large, namely $n > \max\{|k|, |S|\}$ suffices. Hence, in a profinite semigroup S, for $s \in S$ and $k \in \mathbb{Z}$, the sequence $(s^{n!+k})_n$ converges; we denote its limit $s^{\omega+k}$. In particular, we have implicit operations $x^{\omega+k} \in \overline{\Omega}_1\mathsf{S}$ where x is the free generator of $\overline{\Omega}_1\mathsf{S}$.

Note that, in a finite semigroup S, for given $s \in S$, there must be some repetition in the powers $s, s^2, s^3, \dots$ and so there exist minimal positive integers k, ℓ such that $s^k = s^{k+\ell}$. Let n be the unique integer such that $k \leq n < k + \ell$ and ℓ divides n. Then the powers $s^k, s^{k+1}, \dots, s^{k+\ell-1}$ constitute a cyclic group with idempotent s^n and generator s^{n+1}, whose inverse is s^{2n-1}. Since $s^{m!} = s^n$ for all $m \geq n$, it follows that s^ω is the unique idempotent which is a power of s (with positive exponent) and $s^{\omega-1}$ is the inverse of $s^{\omega+1}$ in the maximal subgroup with idempotent s^ω. Hence, in a profinite group G, we have the equality $g^{\omega-1} = g^{-1}$.

From the above unary implicit operations and multiplication one may already easily construct lots of implicit operations such as $x^\omega y^\omega$, $(x^{\omega+1}y^{\omega+1})^\omega$, and the *commutator* $[x, y] = x^{\omega-1}y^{\omega-1}x^{\omega+1}y^{\omega+1}$. These examples illustrate how implicit operations are composed: given m-ary implicit operations $w_1, \dots, w_n \in \overline{\Omega}_m\mathsf{V}$ and an n-ary implicit operation $v \in \overline{\Omega}_n\mathsf{V}$, the natural interpretation of v in $\overline{\Omega}_m\mathsf{V}$ allows us to define $v(w_1, \dots, w_n) \in \overline{\Omega}_m\mathsf{V}$ to be the operation $v_{\overline{\Omega}_m\mathsf{V}}(w_1, \dots, w_n)$. In particular, multiplication of implicit operations is obtained by applying the binary explicit operation x_1x_2 to two given operations. An important property of composition is that it is continuous.

4.7 Proposition *Composition of implicit operations of fixed arity as defined above is a continuous function*

$$\begin{aligned} \overline{\Omega}_n\mathsf{V} \times (\overline{\Omega}_m\mathsf{V})^n &\to \overline{\Omega}_m\mathsf{V} \\ (v, w_1, \ldots, w_n) &\mapsto v(w_1, \ldots, w_n) \end{aligned}$$

Proof Given a clopen subset $K \subseteq \overline{\Omega}_m\mathsf{V}$, by Proposition 3.5 there exists a continuous homomorphism $\varphi : \overline{\Omega}_m\mathsf{V} \to S$ such that $S \in \mathsf{V}$ and $K = \varphi^{-1}\varphi K$. Let $\mathsf{W} = \mathsf{V}(S)$. Then φ factors through the natural projection $\pi_m : \overline{\Omega}_m\mathsf{V} \to \overline{\Omega}_m\mathsf{W}$ and so we may as well assume that $S = \overline{\Omega}_m\mathsf{W}$. Consider the following diagram where $\pi_n : \overline{\Omega}_n\mathsf{V} \to \overline{\Omega}_n\mathsf{W}$ is also a natural projection and the horizontal arrows represent composition of implicit operations defined for each of the pseudovarieties V and W as above:

$$\begin{array}{ccc} \overline{\Omega}_n\mathsf{V} \times (\overline{\Omega}_m\mathsf{V})^n & \longrightarrow & \overline{\Omega}_m\mathsf{V} \\ {\scriptstyle \pi_n \times (\pi_m)^n}\big\downarrow & & \big\downarrow{\scriptstyle \pi_m} \\ \overline{\Omega}_n\mathsf{W} \times (\overline{\Omega}_m\mathsf{W})^n & \longrightarrow & \overline{\Omega}_m\mathsf{W} \end{array}$$

The commutativity of the diagram follows from the fact that implicit operations commute with continuous homomorphisms between pro-V semigroups:

$$\begin{aligned} \pi_m(v(w_1, \ldots, w_n)) &= \pi_m(v_{\overline{\Omega}_m\mathsf{V}}(w_1, \ldots, w_n)) \\ &= v_{\overline{\Omega}_m\mathsf{W}}(\pi_m(w_1), \ldots, \pi_m(w_n)) = \pi_n(v)(\pi_m(w_1), \ldots, \pi_m(w_n)) \end{aligned}$$

Since the bottom line is continuous, as it is a mapping between discrete spaces, it follows that the inverse image by the top line of the clopen set K is again clopen. □

The calculus of implicit operations of the form $x^{\omega+k}$ under the operations of multiplication and composition is quite simple.

4.8 Lemma *The set of unary implicit operations of the form $x^{\omega+k}$, with $k \in \mathbb{Z}$ constitutes a ring whose addition is multiplication of implicit operations and whose multiplication is composition of implicit operations. It is isomorphic with the ring $\mathbb{Z}$ of integers.*

Proof The result is immediate from the following equalities where we use continuity of composition as given by Proposition 4.7:

$$\begin{aligned} x^{\omega+k}x^{\omega+\ell} &= \lim_{n\to\infty} x^{n!+k}x^{n!+\ell} = \lim_{n\to\infty} x^{2(n!)+k+\ell} = \lim_{n\to\infty} x^{n!+k+\ell} = x^{\omega+k+\ell} \\ (x^{\omega+k})^{\omega+\ell} &= \lim_{n\to\infty} (x^{n!+k})^{n!+\ell} = \lim_{n\to\infty} x^{(n!)^2+(k+\ell)(n!)+k\ell} = \lim_{n\to\infty} x^{n!+k\ell} = x^{\omega+k\ell} \quad \square \end{aligned}$$

4.9 Remark Let $\hat{\mathbb{Z}}$ be the profinite completion of the ring $\mathbb{Z}$ of integers. It may be obtained as the completion of $\mathbb{Z}$ with respect to the metric d of Subsection 3.4 defined similarly in the language of rings. Since $\mathbb{Z}$ is the free commutative ring on one generator and it is residually finite, $\hat{\mathbb{Z}}$ is isomorphic to $\overline{\Omega}_1\mathsf{R}$ for the pseudovariety R of finite commutative rings. It follows that the non-explicit unary implicit operations constitute a ring under multiplication and composition which is isomorphic to the completion $\hat{\mathbb{Z}}$. This completion can be easily seen to be the direct product of the p-adic completions $\mathbb{Z}_p$ of the ring $\mathbb{Z}$ as p runs over all prime numbers.

We are now ready to present some examples regarding Reiterman's Theorem.

4.10 Examples (1) The pseudovariety A of all finite aperiodic semigroups is defined by the pseudoidentity $x^{\omega+1} = x^{\omega}$ since all subgroups of a semigroup are trivial if and only if all its cyclic subgroups are trivial.

(2) The pseudovariety N of all finite nilpotent semigroups is defined by the pseudoidentities $x^{\omega}y = yx^{\omega} = x^{\omega}$ which we abbreviate as $x^{\omega} = 0$.

(3) The pseudovariety G of all finite groups is defined by the pseudoidentities $x^{\omega}y = yx^{\omega} = y$ which we naturally abbreviate as $x^{\omega} = 1$.

(4) For a prime p, the pseudovariety G_p of all finite p-groups cannot be defined by pseudoidentities involving only products and the operation $x^{\omega-1}$. Indeed, since $\mathsf{G} \models x^{\omega-1} = x^{-1}$, all such pseudoidentities may be viewed as ordinary identities of group words and it is well known that free groups are residually G_p [28].

4.3 Iteration of implicit operations

The last example from the previous subsection shows that one needs a richer language of implicit operations than that provided by the multiplication plus the unary operations of the form $x^{\omega+k}$ to define some pseudovarieties in terms of pseudoidentities. A powerful tool for constructing implicit operations is infinite iteration of composition. In this subsection, we develop a more general framework where not only infinite iteration but arbitrary implicit operations may be performed, namely we show that the monoid of continuous endomorphisms of a finitely generated profinite semigroup is profinite. Our arguments are not the most economical but the end result is the best which is presently known.

For a topological semigroup S, denote by $\operatorname{End} S$ the monoid of its continuous endomorphisms. Note that $\operatorname{End} S$ is a subset of the set of functions S^S. In general it is a delicate question which topology to consider on a function space. For S^S the two most natural alternatives are the product topology, also known as the *topology of pointwise convergence*, and the *compact-open topology*, for which a subbase consists of the sets of functions of the form

$$V(K,U) = \{f \in S^S : f(K) \subseteq U\}$$

where $K \subseteq S$ is compact and $U \subseteq S$ is open [93]. These topologies retain their names when the induced topologies are considered on subspaces of S^S. Note that a subbase for the product topology is given by the sets of the form $V(\{s\},U)$ with $s \in S$ and $U \subseteq S$ open and so the pointwise convergence topology is contained in the compact-open topology.

For the sequel, we require the following very simple yet very useful observation.

4.11 Lemma *Let S be a profinite semigroup and let d be the natural metric on S. Then every $f \in \operatorname{End} S$ is a contracting function in the sense that, for all $s_1, s_2 \in S$,*

$$d(f(s_1), f(s_2)) \leq d(s_1, s_2). \tag{4.2}$$

Proof If $s_1 = s_2$ then the inequality (4.2) is obvious since both sides are zero. Otherwise, let $n = r(s_1, s_2)$, as defined in Subsection 3.4. Given a continuous homomorphism $\varphi : S \to T$ into a finite semigroup with $|T| < n$, the composite $\varphi \circ f : S \to T$ is again a continuous homomorphism and so, by definition of $r(s_1, s_2)$, we have $\varphi(f(s_1)) = \varphi(f(s_2))$. Hence $r(f(s_1), f(s_2)) \geq n$, which proves (4.2). □

The pointwise convergence topology has the advantage of being easier to handle in terms of convergence since a net converges in it if and only if it converges pointwise. The following result is an illustration of this fact.

4.12 Lemma *If S is a finitely generated profinite semigroup then* $\operatorname{End} S$ *is compact with respect to the pointwise convergence topology.*

Proof For each $s, t \in S$, the set $\{\varphi \in S^S : \varphi(st) = \varphi(s)\varphi(t)\}$ is closed since the equation that defines it only involves three components of the product, namely the components indexed by st, s, and t and the prescription on those three components, as a subset of S^3, is the graph of multiplication, which is assumed to be continuous. Hence the monoid of (not necessarily continuous) endomorphisms of S is closed in S^S with respect to the pointwise convergence topology.

Next, let $(f_i)_{i\in I}$ be a net in $\operatorname{End} S$ converging to some $f \in S^S$. Given $s, t \in S$, then by continuity of the natural metric d, we have $d(f(s), f(t)) = \lim_{i\in I} d(f_i(s), f_i(t))$ while $d(f_i(s), f_i(t)) \leq d(s, t)$ by Lemma 4.11. Hence $d(f(s), f(t)) \leq d(s, t)$ which, in view of Proposition 3.9, shows that f is continuous. Since f is an endomorphism of S by the preceding paragraph, we conclude that $f \in \operatorname{End} S$. Thus $\operatorname{End} S$ is a closed subset of the compact space S^S and, therefore, $\operatorname{End} S$ is compact. □

On the other hand, it is well known that for instance for locally compact S, the compact-open topology on $\operatorname{End} S$ implies continuity of the evaluation mapping

$$\begin{aligned} \varepsilon : (\operatorname{End} S) \times S &\to S \\ (f, s) &\mapsto f(s) \end{aligned}$$

Indeed, for an open subset $U \subseteq S$ and $(f, s) \in (\operatorname{End} S) \times S$ such that $f(s) \in U$, since f is continuous and S is locally compact there is some compact neighbourhood K of s such that $f(K) \subseteq U$. Then $(V(K, U) \cap \operatorname{End} S) \times K$ is a neighbourhood of (f, s) in $(\operatorname{End} S) \times S$ which is contained in $\varepsilon^{-1}(U)$. See [93] for more general results and background.

The comparison between the two topologies on $\operatorname{End} S$ is given by the following result.

4.13 Proposition *Let S be a finitely generated profinite semigroup. Then the pointwise convergence and compact-open topologies coincide on* $\operatorname{End} S$.

Proof It remains to show that every set of the form $V(K, U) \cap \operatorname{End} S$ with $K \subseteq S$ compact and $U \subseteq S$ open is open in the pointwise convergence topology of $\operatorname{End} S$. Without loss of generality, since S is zero-dimensional we may assume that U is clopen. Consider the open cover $U \cup (S \setminus U)$. By Proposition 3.9, this is also an open cover in the topology induced by the natural metric d. By Lebesgue's Covering Lemma [93, Theorem 22.5], there exists $\delta > 0$ such that every subset of S of diameter less than δ is contained in either U or $S \setminus U$.

Consider the cover of K by the open balls $B_\delta(s)$ of radius δ centered at points $s \in K$. By Proposition 3.9, these balls are open in the topology of S and so, since K is compact, there are finitely many points $s_1, \dots, s_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} B_\delta(s_i). \tag{4.3}$$

Let $f \in V(K,U) \cap \operatorname{End} S$ and let

$$W = \bigcap_{i=1}^{n} V(\{s_i\}, B_\delta(f(s_i))) \cap \operatorname{End} S.$$

Then W is an open set of $\operatorname{End} S$ in the pointwise convergence topology which contains f. Hence it suffices to show that $W \subseteq V(K,U)$. Let $g \in W$ and $s \in K$. By (4.3), there exists $i \in \{1, \dots, n\}$ such that $d(s, s_i) < \delta$. By Lemma 4.11, we have $d(g(s), g(s_i)) < \delta$. On the other hand, since $g \in V(\{s_i\}, B_\delta(f(s_i)))$, we obtain $d(g(s_i), f(s_i)) < \delta$. Since d is an ultrametric, it follows that $d(g(s), f(s_i)) < \delta$. Finally, as $f(s_i) \in U$ since $f \in V(K,U)$, we conclude by the choice of δ that $g(s) \in U$, which shows that $g \in V(K,U)$. □

For the case of finitely generated relatively free profinite semigroups, the following result was already observed in [19] as a consequence of a result from [6]. We provide here a direct proof of the more general case without the assumption of relative freeness.

4.14 Theorem *Let S be a finitely generated profinite semigroup. Then* $\operatorname{End} S$ *is a profinite monoid under the pointwise convergence topology and the evaluation mapping is continuous.*

Proof By Proposition 4.13, the pointwise convergence and compact-open topologies coincide on $\operatorname{End} S$. Hence the evaluation mapping $\varepsilon : (\operatorname{End} S) \times S \to S$ is continuous.

Since S is totally disconnected, so is the product space S^S and its subspace $\operatorname{End} S$. Moreover, $\operatorname{End} S$ is compact by Lemma 4.12. Hence, by Theorem 3.1, to prove that $\operatorname{End} S$ is a profinite monoid it suffices to sow that it is a topological monoid, that is that the composition $\mu : (\operatorname{End} S) \times (\operatorname{End} S) \to \operatorname{End} S$ is continuous, for which we show that, for every convergent net $(f_i, g_i) \to (f, g)$ in $(\operatorname{End} S) \times (\operatorname{End} S)$, we have $f_i \circ g_i \to f \circ g$. For the pointwise convergence topology the latter means $f_i(g_i(s)) \to f(g(s))$ for every $s \in S$. This follows from $f_i \to f$ together with $g_i(s) \to g(s)$ since the evaluation mapping is continuous. □

Thus, if S is a finitely generated profinite semigroup then the group $\operatorname{Aut} S$ of its continuous automorphisms is a profinite group since it is a closed subgroup of $\operatorname{End} S$ (cf. Corollary 3.2). In particular, the group $\operatorname{Aut} G$ of continuous automorphisms of a finitely generated profinite group G is profinite for the pointwise convergence topology, a result which is useful in profinite group theory [76]. Moreover, since one can find in [76] examples of profinite groups whose groups of continuous automorphisms are not profinite, we see that the hypothesis that S is finitely generated cannot be removed from Theorem 4.14.

Now, for a finite set A, $\overline{\Omega}_A \mathsf{S}$ is a finitely generated profinite semigroup and so $\operatorname{End} \overline{\Omega}_A \mathsf{S}$ is a profinite monoid. Since $\overline{\Omega}_A \mathsf{S}$ is a free profinite semigroup on the generating set A, continuous endomorphisms of $\overline{\Omega}_A \mathsf{S}$ are completely determined by their restrictions to A. For $\overline{\Omega}_n \mathsf{S}$ we may then choose to represent an element $\varphi \in \operatorname{End} \overline{\Omega}_n \mathsf{S}$ by the n-tuple $(\varphi(x_1), \dots, \varphi(x_n))$ where $x_1, \dots, x_n$ are the component projections. When n is small, we may write $x, y, z, t, \dots$ or $a, b, c, d, \dots$ instead of $x_1, x_2, x_3, x_4, \dots$ respectively.

In the case $n = 1$, take $\varphi \in \operatorname{End}\overline{\Omega}_1\mathsf{S}$ determined by the 1-tuple (x^m). Then φ has an ω-power in $\operatorname{End}\overline{\Omega}_1\mathsf{S}$ and we may consider the operation $\varphi^\omega(x)$ which we denote x^{m^ω} since

$$x^{m^\omega} = \varphi^\omega(x) = \lim_{k\to\infty} \varphi^{k!}(x) = \lim_{k\to\infty} x^{m^{k!}}.$$

4.15 Examples (1) It is now an easy exercise, which we leave to the reader, to show that, for a prime p, $\mathsf{G}_p = [\![x^{p^\omega} = 1]\!]$.

(2) Let $\mathsf{G}_{\mathrm{nil}}$ denote the pseudovariety of all finite nilpotent groups. Consider the endomorphism $\varphi \in \operatorname{End}\overline{\Omega}_2\mathsf{S}$ defined by the pair $([x, y], y)$ where $[x, y]$ denotes the commutator defined earlier. Denote by $[x, {}_\omega y]$ the implicit operation $\varphi^\omega(x)$. Then it follows from a theorem of Zorn [94, 77] that $\mathsf{G}_{\mathrm{nil}} = [\![\, [x, {}_\omega y] = 1]\!]$.

(3) Let $\mathsf{G}_{\mathrm{sol}}$ denote the pseudovariety of all finite solvable groups. Let $\varphi \in \operatorname{End}\overline{\Omega}_3\mathsf{S}$ be defined by the triple $([yxy^{\omega-1}, zxz^{\omega-1}], y, z)$ and let $w = \varphi^\omega(x)$, which is a ternary implicit operation. Using J. Thompson's list of minimal non-solvable simple groups, arithmetic geometry and computer algebra and geometry, Bandman et al [27] have recently established that $\mathsf{G}_{\mathrm{sol}} = [\![w(x^{\omega-2}y^{\omega-1}x, x, y) = 1]\!]$. Since this provides a two-variable pseudoidentity basis for $\mathsf{G}_{\mathrm{sol}}$, as an immediate corollary one obtains the Thompson-Flavell Theorem stating that a finite group is solvable if and only if its 2-generated subgroups are solvable. The fact that the pseudovariety $\mathsf{G}_{\mathrm{sol}}$ is finitely based, and therefore it may be defined by a single pseudoidentity of the form $v = 1$, was previously proved by Lubotzky [56].

Having established the foundations of the theory of profinite semigroups, the remainder of these notes is dedicated to surveying some results in the area which are meant to introduce the reader to recent developments and reveal the richness and depthness of the already existing theory. Most proofs will be omitted. Naturally, this survey will not be exhaustive and it unavoidably reflects the author's personal preferences and tastes.

5 Free pro-J semigroups and Simon's Theorem

For details and proofs of the results in this section, see [3].

Recall that J denotes the pseudovariety consisting of all finite semigroups in which every principal ideal admits a unique element as its generator. The letter J comes from Green's relation $\mathcal{J}$ which relates two elements of a semigroup if they generate the same principal ideal. Hence J consists of all finite $\mathcal{J}$-trivial semigroups, that is finite semigroups in which the relation $\mathcal{J}$ is trivial. It is an exercise to show that

$$\mathsf{J} = [\![(xy)^\omega = (yx)^\omega, x^{\omega+1} = x^\omega]\!] = [\![(xy)^\omega x = (xy)^\omega = y(xy)^\omega]\!]. \tag{5.1}$$

Since $\mathsf{J} \supseteq \mathsf{N}$, J satisfies no nontrivial semigroup identities and so $\Omega_A\mathsf{J} \simeq A^+$ is the free semigroup on A.

We recall next a theorem which was already mentioned in Section 2. A language $L \subseteq A^+$ is said to be *piecewise testable* if it is a Boolean combination of languages of the form

$$A^*a_1A^* \cdots a_nA^*. \tag{5.2}$$

The word $a_1 \cdots a_n$ is said to be a *subword* of an element $w \in \overline{\Omega}_A\mathsf{S}$ if w belongs to the closure of the language (5.2).

By taking Boolean combinations of languages of the form (5.2) we may obtain, for each positive integer N, the classes of the congruence $\sim_N$ on A^+ which identifies two words if they have precisely the same subwords of length at most N. Conversely, a language of the form (5.2) is saturated by the congruence $\sim_N$. Hence a language $L \subseteq A^+$ is piecewise testable if and only if it is saturated by some congruence $\sim_N$.

The language (5.2) is J-recognizable: in view of the preceding paragraph, this can be proved by noting that for words u and v, the words $(uv)^N$ and $(vu)^N$ are in the same $\sim_N$-class, and the same holds for the words u^{N+1} and u^N, which establishes that the quotient semigroup $A^+/{\sim_N}$ satisfies the first set of pseudoidentities in (5.1).

5.1 Theorem (Simon [80]) *A language $L \subseteq A^+$ over a finite alphabet is piecewise testable if and only if it is J-recognizable.*

In terms of Eilenberg's correspondence, Simon's Theorem says that the variety of languages associated with J is the variety of piecewise testable languages. In topological terms, Simon's Theorem says that the closures in $\overline{\Omega}_A\mathsf{J}$ of the languages of the form (5.2) suffice to separate points (cf. Proposition 3.8). In other words, implicit operations over J can be distinguished by looking at their subwords. This is certainly not true for implicit operations over S as for instance $(xy)^\omega$ and $(yx)^\omega$ have the same subwords.

So, it should be possible to prove Simon's Theorem by developing a good understanding of the structure of the finitely generated free pro-J semigroups $\overline{\Omega}_A\mathsf{J}$. This program was carried out by the author in the mid-1980's motivated by a question raised by I. Simon as to whether $\overline{\Omega}_A\mathsf{J}$ is countable.

Recall that an element s of a semigroup S is said to be *regular* if there exists $t \in S$ such that $sts = s$; such an element t is called a *weak inverse* of s. Then $u = tst$ is such that $sus = s$ and $usu = u$; an element $u \in S$ satisfying these two equalities is called an *inverse* of s. Semigroups in which every element has a unique inverse are called *inverse semigroups*. They are precisely the regular semigroups of partial bijections of sets and play an important role in various applications [54].

If s and t are inverses in a semigroup S then st is an idempotent and $s \mathcal{J} st$ and so regular elements are $\mathcal{J}$-equivalent to idempotents. Hence in a $\mathcal{J}$-trivial semigroup the regular elements are the idempotents.

For words $u, v \in A^+$, one can use the pseudoidentities in (5.1) to deduce that J satisfies the pseudoidentity $u^\omega = v^\omega$ if and only if u and v contain the same letters.

Consider the semilattice $\mathcal{P}(A)$ of subsets of A under union. Since it is $\mathcal{J}$-trivial, the function $A^+ \to \mathcal{P}(A)$ which associates with a word u the set of letters occurring in it extends uniquely to a continuous homomorphism $c : \overline{\Omega}_A\mathsf{J} \to \mathcal{P}(A)$. We call it the *content* function. The result in the preceding paragraph may then be stated as saying that an idempotent in $\overline{\Omega}_A\mathsf{J}$ of the form u^ω is completely characterized by its content $c(u^\omega) = c(u)$. Since every element $v \in \overline{\Omega}_A\mathsf{J}$ is the limit of a sequence of words $(v_n)_n$ and we may assume that $(c(v_n))_n$ is constant, if v is idempotent then

$$v = v^n = v^{n!} = \lim_{n\to\infty} v^{n!} = v^\omega = \lim_{n\to\infty} v_n^\omega = \lim_{n\to\infty} u^\omega = u^\omega$$

where u is any word with $c(u) = c(v)$.

Now, idempotents of $\overline{\Omega}_A\mathsf{J}$ may be characterized by the property that whenever a word u is a subword then so is u^n for every $n \geq 1$. This leads to the following result.

5.2 Theorem *Every element of $\overline{\Omega}_A\mathsf{J}$ admits a factorization of the form $u_0v_1^\omega u_1 \cdots v_n^\omega u_n$ where the u_i, v_i are words, with the u_i possibly empty and each v_i with no repeated letters. Furthermore, one may arrange for the following to hold:*

- *no u_i ends with a letter occurring in v_{i+1};*
- *no u_i starts with a letter occurring in v_i;*
- *if u_i is the empty word, then the contents $c(v_i)$ and $c(v_{i+1})$ are incomparable under inclusion.*

Theorem 5.2 answers Simon's question: for A finite, the semigroup $\overline{\Omega}_A\mathsf{J}$ is countable and it is in fact generated by A together with its $2^{|A|} - 1$ idempotents. This suggests viewing $\overline{\Omega}_A\mathsf{J}$ as an algebra of type $(2,1)$ under multiplication and the ω-power. The semigroup $\overline{\Omega}_A\mathsf{J}$ becomes a free algebra in the variety of algebras of this type generated by J and Theorem 5.2 suggests a canonical form for terms in this free algebra. First, Theorem 5.2 already implies that every such term is equal in $\overline{\Omega}_A\mathsf{J}$ to a term of the form described in the theorem. To distinguish terms in canonical form, one may use subwords and thus prove at the same time Simon's Theorem.

5.3 Theorem *Two terms in the form described in Theorem 5.2 are distinct in $\overline{\Omega}_A\mathsf{J}$ if and only if they have distinct sets of subwords.*

One may more precisely bound the length of the subwords which are necessary to distinguish two such terms. Recently, in work whose precise connection with the above remains to be determined, Simon [81] has proposed a very efficient algorithm to distinguish two words by their subwords. For more on the significance in Mathematics and Computer Science of Simon's Theorem, see the lecture notes of M. V. Volkov in this volume.

6 Tameness

This section goes deeper into decidability problems for pseudovarieties. It grows out of [16, 15, 7]. The reader is referred to those publications for details.

Let Σ be a finite system of equations of the form $u = v$ with $u, v \in X^+$. For example such a system might be

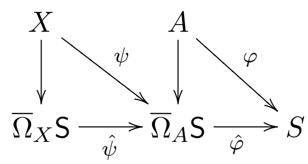
$$\begin{cases} xy &= z \\ zx &= ty \end{cases}$$

We impose on the variables *rational constraints*: for each $x \in X$, we choose a rational language $L_x \subseteq A^+$ where A is another alphabet. A *solution* of the system in an A-generated profinite semigroup $\varphi : A \to S$ is a mapping $\psi : X \to \overline{\Omega}_A\mathsf{S}$ such that

(1) $\psi(x) \in \overline{L_x}$ for every variable $x \in X$,

(2) $\hat{\varphi} \circ \hat{\psi}(u) = \hat{\varphi} \circ \hat{\psi}(v)$ for every equation $(u = v) \in \Sigma$,

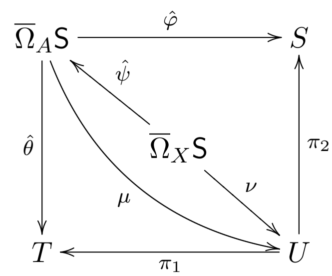
where the various mappings are depicted in the following commutative diagram:



One may compute from the constraints a finite semigroup T and a homomorphism $\theta : A^+ \to T$ recognizing all of them. Let U be the closed subsemigroup of $T \times S$ generated by all elements of the form $\mu(a) = (\theta(a), \varphi(a))$ with $a \in A$, and let $\mu : \overline{\Omega}_A \mathsf{S} \to U$ be the induced continuous homomorphism. Then the above conditions (1) and (2) may be formulated in terms of the composite $\nu = \mu\hat{\psi}$ by stating the following, where $\pi_1 : U \to T$ and $\pi_2 : U \to S$ are the component projections:

(1) $\pi_1\nu(x) \in \pi_1\mu L_x$ for every variable $x \in X$;

(2) $\pi_2\nu(u) = \pi_2\nu(v)$ for every equation $(u = v) \in \Sigma$.



In particular, if S is finite then one can test effectively whether a solution exists in S.

The semigroup U is an example of what is called a “relational morphism”. More generally, a *relational morphism* between two topological semigroups S and T is a relation $\tau : S \to T$ with domain S which is a closed subsemigroup of the product $S \times T$. A continuous homomorphism and the inverse image of an onto continuous homomorphism are relational morphisms and every relational morphism may be obtained by composition of two such relational morphisms. Relational morphisms for monoids are defined similarly.

The following is a compactness result whose proof may be obtained by following basically the same lines as in the proof of the equivalence (1)⇔(2) in Proposition 3.5.

6.1 Theorem *The following conditions are equivalent for a finite system Σ of equations with rational constraints over the finite alphabet A:*

(1) Σ *has a solution in every A-generated semigroup from* V*;*

(2) Σ *has a solution in every A-generated pro-*V *semigroup;*

(3) Σ *has a solution in* $\overline{\Omega}_A \mathsf{V}$*.*

A system satisfying the conditions of Theorem 6.1 is said to be V-*inevitable*. A decidability property for a pseudovariety V with respect to a given recursively enumerable set $\mathcal{C}$ of such systems is whether there is an algorithm to decide whether a given $\Sigma \in \mathcal{C}$ is V-inevitable. Before examining the relevance of such a property, we introduce a few more precise notions.

An *implicit signature* is a set σ of implicit operations (over S) which contains multiplication. It is viewed as an enlarged algebraic language for which profinite semigroups immediately inherit a natural structure by giving the chosen implicit operations their natural interpretation. Note that the subalgebra $\Omega_A^\sigma \mathsf{V}$ of $\overline{\Omega}_A \mathsf{V}$ generated by A is precisely the free σ-algebra in the variety generated by V.

The signature $\kappa = \{\_ \cdot \_,\ \_^{\omega-1}\}$ is called the *canonical signature* since most implicit operations which are commonly used are terms in its language. For finite groups, it becomes the natural signature, with multiplication and inversion. In particular, since free groups are residually finite, $\Omega_A^\kappa \mathsf{G}$ is the free group on A.

Say V is $\mathcal{C}$-*tame* if there is an implicit signature σ such that

(1) σ is recursively enumerable;

(2) the operations in σ are computable;

(3) the word problem for $\Omega_A^\sigma \mathsf{V}$ is decidable;

(4) for every V-inevitable Σ there is a solution $\psi : X \to \overline{\Omega}_A \mathsf{S}$ for $\overline{\Omega}_A \mathsf{V}$ which takes its values in $\Omega_A^\sigma \mathsf{S}$.

Under the above conditions, we may also say that V is $\mathcal{C}$-tame *with respect to* σ. In case $\mathcal{C}$ consists of all finite systems over a fixed countable alphabet, then we say that V is *completely tame* if it is $\mathcal{C}$-tame.

6.2 Theorem *Let $\mathcal{C}$ be a recursively enumerable set of finite systems of equations with rational constraints and suppose V is a $\mathcal{C}$-tame pseudovariety. Then it is decidable whether a given $\Sigma \in \mathcal{C}$ is V-inevitable.*

Proof Let V be $\mathcal{C}$-tame with respect to an implicit signature σ. To prove the theorem it suffices to effectively enumerate those $\Sigma \in \mathcal{C}$ that are V-inevitable and those that are not.

One can start by enumerating all systems $\Sigma \in \mathcal{C}$. Since V is $\mathcal{C}$-tame with respect to σ, if Σ is V-inevitable then there is a solution $\psi : X \to \overline{\Omega}_A \mathsf{S}$ for Σ in $\overline{\Omega}_A \mathsf{V}$ that takes its values in $\Omega_A^\sigma \mathsf{S}$. The candidates for such solutions can be effectively enumerated in parallel with the systems, as σ is recursively enumerable, the constraints can be effectively tested by computing operations in their syntactic semigroups, and the equations can be effectively tested by using a solution of the word problem for $\Omega_A^\sigma \mathsf{V}$. This provides an effective enumeration of all V-inevitable systems in $\mathcal{C}$.

To enumerate those systems in $\mathcal{C}$ that are not V-inevitable, we try out pairs of systems Σ in $\mathcal{C}$ together with candidates for A-generated semigroups $S \in \mathsf{V}$. We already observed that under these conditions one can test effectively whether a solution exists in S and if turns out it does not, we output the system as one that is not V-inevitable. $\square$

One class of systems which the author has introduced for the study of semidirect products of pseudovarieties (cf. Section 7) consists of systems associated with finite directed graphs: to each vertex and edge in the graph one associates a variable and an equation $xy = z$ is written if y is the variable corresponding to an edge which goes from the vertex corresponding to x to the vertex corresponding to z. We will call a pseudovariety *graph-tame* if it is tame with respect to systems of equations arising in this way.

Here are some examples of tame pseudovarieties.

6.3 Example The pseudovarieties N and J are completely tame with respect to the canonical signature κ. Both results follow from the knowledge of the structure of the corresponding relatively free profinite semigroups and the solution of their word problems. In the case of N this is quite simple. In fact, $\overline{\Omega}_A\mathsf{N}$ is obtained from the free semigroup $\Omega_A\mathsf{N} \simeq A^+$ by adjoining a zero element, which is topologically the one-point compactification for the discrete topology on $\Omega_A\mathsf{N}$. Hence a κ-term is zero in $\overline{\Omega}_A\mathsf{N}$ if and only if it involves the $(\omega-1)$-power. Assuming a finite system has a solution in $\overline{\Omega}_A\mathsf{N}$, given by a function $\psi : X \to \overline{\Omega}_A\mathsf{S}$, we modify ψ on those variables x for which $\psi(x)$ is not explicit as follows. From Theorem 6.1, it follows that there exists a factorization $\psi(x) = uv^\omega w$ with $u, v, w \in \overline{\Omega}_A\mathsf{S}$. Since the constraint for x translates into a condition of the form $\psi(x)$ *belongs to a given clopen subset of* $\overline{\Omega}_A\mathsf{S}$ and $\psi(x)$ is zero in $\overline{\Omega}_A\mathsf{N}$, we may replace u, v, w by words. This changes $\psi(x)$ to a κ-term by maintaining a solution in $\overline{\Omega}_A\mathsf{N}$. See [7] for details.

6.4 Theorem (Almeida and Delgado [13]) *The pseudovariety* Ab *of all finite Abelian groups is completely tame with respect to* κ*.*

The proof of Theorem 6.4 amounts to linear algebra over the profinite completion $\hat{\mathbb{Z}}$ of the ring of integers, which we have already observed to be isomorphic with $\overline{\Omega}_1\mathsf{G}$ under multiplication and composition. From work of Steinberg [83] it follows that the pseudovariety Com of finite commutative semigroups is also competely tame with respect to κ.

6.5 Theorem (Ash [22]) *The pseudovariety* G *of all finite groups is graph-tame with respect to* κ*.*

Theorem 6.5 is considered one of the deepest results in finite semigroup theory. In its original version, it was proved by algebraic-combinatorial methods in a somewhat different language; see [5, 12] for a translation to the language of these notes. An independent proof of the case of 1-vertex graphs, which is already quite nontrivial was obtained using profinite group theory where the result translates to a conjecture which had been proposed by Pin and Reutenauer [66], namely the following statement, where by the *profinite topology* of the free group we mean the induced topology from the free profinite group or equivalently the topology whose open subgroups are those of finite index.

6.6 Theorem (Ribes and Zalesskiĭ [74]) *The product of finitely many finitely generated subgroups of the free group is closed in the profinite topology of the free group.*

Theorem 6.6 in turn generalizes a theorem of M. Hall [38] which is the case of just one subgroup. The interest in these results will be explained in more detail in Section 7. Finally, it follows from a result of Coulbois and Khélif [33] that G is not completely tame with respect to κ. At present it is not known whether G is completely tame with respect to some implicit signature.

There are some surprising connections of graph-tameness of G with other areas of Mathematics and in fact the result was rediscovered in disguise in Model Theory. We say that a class $\mathcal{R}$ of relational structures of the same type has the *finite extension property for partial automorphisms* (*FEPPA* for shortness) if for every finite $R \in \mathcal{R}$ and every set P of partial automorphisms of R, if there exists an extension $S \in \mathcal{R}$ of R for which every $f \in P$ extends to a total automorphism of S, then there exists such an extension $S \in \mathcal{R}$ which is finite. For a class $\mathcal{R}$ of relational structures, let $\operatorname{Excl}\mathcal{R}$ denote the class of all structures S for which there is no homomorphism $R \to S$ with $R \in \mathcal{R}$ where by a *homomorphism* of relational structures of the same type we mean a function that preserves the relations in the forward direction. Now, we have the following remarkable statement.

6.7 Theorem (Herwig and Lascar [41]) *For every finite set $\mathcal{R}$ of finite structures of a finite relational language, Excl$\mathcal{R}$ satisfies the FEPPA.*

Herwig and Lascar recognized this result as extending Ribes and Zalesskiĭ's Theorem and provided a general translation into a property of the free group. Delgado and the author [11, 12] in turn recognized that property of the free group as being equivalent to Ash's Theorem.

The following two examples provide another two applications of Ash's Theorem which additionally use results from the theory of regular semigroups. See the quoted papers for appropriate references.

6.8 Example Let OCR be the class consisting of all finite semigroups S which are unions of their subgroups (in which case we say S is *completely regular*) and in which the products of idempotents are again idempotents (in which case S is said to be *orthodox*). In terms of pseudoidentities, we have $\mathsf{OCR} = [\![x^{\omega+1} = x,\, (x^\omega y^\omega)^2 = x^\omega y^\omega]\!]$. Trotter and the author [17] have used graph-tameness of G to show that OCR is also graph-tame with respect to the canonical signature κ.

6.9 Example Let $\mathsf{CR} = [\![x^{\omega+1} = x]\!]$ be the pseudovariety of all finite completely regular semigroups. Trotter and the author [18] have reduced the graph-tameness of CR to a property which was apparently stronger than graph-tameness of G but K. Auinger has observed that the methods of [11, 12] apply to show that the property in question follows from graph-tameness of G.

Our final example comes from [6] which led the author to explore connections with dynamical systems.

The pseudovariety G_p of all finite p-groups is not graph-tame with respect to the signature κ since Steinberg and the author [15] have observed that if it were then G_p would be definable by identities in the signature κ, which we have already observed to be impossible. Nevertheless, based on work of Ribes and Zalesskiĭ [75], Margolis, Sapir and Weil [57], and Steinberg [84], the author has proved the following.

6.10 Theorem *It is possible to enlarge κ to an infinite signature σ so that G_p is graph-tame with respect to this signature [6].*

The added implicit operations are those of the form $\varphi^{\omega-1}(v_i)$ where $\varphi \in \mathrm{End}\,\overline{\Omega}_n\mathsf{S}$ is defined by the n-tuple $(w_1, \ldots, w_n)$ and the v_i and w_i are κ-terms such that $\mathsf{G}_p \models v_i = w_i$ and the determinant of the matrix $(|w_i|_{x_j})_{i,j}$ is invertible in $\mathbb{Z}/p\mathbb{Z}$. Here for a κ-term w and a letter x, $|w|_x$ is the integer obtained by viewing w as a group word and counting the signed number of occurrences of x in w. So, for example, if φ is given by the pair $((xy)^{\omega-1}yx^\omega, x^3yx^{\omega-1})$ then we get $\det \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = -1$. We do not know if G_p is completely tame with respect to this signature.

7 Categories, semigroupoids and semidirect products

Tilson [90] introduced pseudovarieties of categories as the foundations of an approach to the calculation of semidirect products of pseudovarieties of semigroups which had emerged earlier from work of Knast [50], Straubing [86], and Thérien and Weiss [88, 92]. A *pseudovariety of categories* is defined to be a class of finite categories which is closed under taking finite products and "divisors". A *divisor* of a category C is a category D for which there exists a category E and two functors: $E \to C$, which is injective on Hom-sets, and $E \to D$, which is onto and also injective when restricted to objects.

Jones [45] and independently Weil and the author [20] have extended the profinite approach to the realm of pseudovarieties of categories. Thus one can talk of relatively free profinite categories, implicit operations, and pseudoidentities. Instead of an unstructured set, to generate a category one takes a directed graph. Thus relatively free profinite categories are freely generated by directed graphs, implicit operations act on graph homomorphisms from fixed directed graphs into categories, and pseudoidentities are written over finite directed graphs. The free profinite category on a graph Γ will be denoted $\overline{\Omega}_\Gamma\mathsf{Cat}$. A pseudoidentity $(u = v; \Gamma)$ over the graph Γ is given by two coterminal morphisms $u, v \in \overline{\Omega}_\Gamma\mathsf{Cat}$. Examples will be presented shortly.

The morphisms from an object in a category C into itself constitute a monoid which is called a *local submonoid* of C. On the other hand, every monoid M may be viewed as a category by adding a virtual object and considering the elements of M as the morphisms, which are composed as they multiply in M. Note that the notion of division of categories applied to monoids is equivalent to the notion of division of monoids as introduced earlier.

For a pseudovariety V of finite monoids, $g\mathsf{V}$ denotes the (*global*) pseudovariety of categories generated by V and $\ell\mathsf{V}$ denotes the class of all finite categories whose local submonoids lie in V. Note that $g\mathsf{V}$ and $\ell\mathsf{V}$ are respectively the smallest and the largest pseudovarieties of categories whose monoids are those of V. The pseudovariety V is said to be *local* if $g\mathsf{V} = \ell\mathsf{V}$.

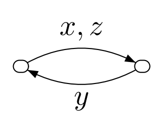
If Σ is a basis of monoid pseudoidentities for V, then the members of Σ may be viewed as pseudoidentities over (virtual) 1-vertex graphs; the resulting set of category pseudoidentities defines $\ell\mathsf{V}$. So, $\ell\mathsf{V}$ is easy to "compute" in terms of basis of pseudoidentities. In general $g\mathsf{V}$ is much more interesting in applications and also much harder to compute. Since the problem of computing $g\mathsf{V}$ becomes simple if V is local, this explains the interest in locality results.

With appropriate care, the theory of pseudovarieties of categories may be extended to pseudovarieties of *semigroupoids*, meaning categories without the requirement for local identities [20]. Again we will move from one context to the other without further warning.

7.1 Examples (1) The pseudovariety $\mathsf{Sl} = [\![xy = yx,\ x^2 = x]\!]$ of finite semilattices is local as was proved by Brzozowski and Simon [30].

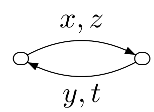
(2) Every pseudovariety of groups is local [86, 90].

(3) The pseudovariety $\mathsf{Com} = [\![xy = yx]\!]$ is not local and its global is defined by the pseudoidentity $xyz = zyx$ over the graph



This was proved by Thérien and Weiss [88].

(4) The pseudovariety J of finite $\mathcal{J}$-trivial semigroups is not local. Its global is defined by the pseudoidentity $(xy)^\omega xt(zt)^\omega = (xy)^\omega (zt)^\omega$ over the graph



This result which has many important applications is also considered to be rather difficult. It was discovered and proved by Knast [50]. A proof using the structure of the free pro-J semigroups can be found in [3]. The application that motivated the calculation of $g\,\mathsf{J}$ was the identification of dot-depth one languages [49] according to a natural hierarchy of plus-free languages introduced by Brzozowski [29]. The work of Straubing [86] was also concerned with the same problem. The computation of levels 2 and higher of this hierarchy remains an open problem.

(5) The pseudovariety $\mathsf{DA} = [\![((xy)^\omega x)^2 = (xy)^\omega x]\!]$ of all finite semigroups whose regular elements are idempotents is local [4]. This result, which was proved using profinite techniques, turns out to have important applications in temporal logic [89]. See [87] for further relevance of the pseudovariety DA in various aspects of Computer Science.

See also [4] for further references to locality results.

Let S and T be semigroups and let $\varphi : T^1 \to \operatorname{End} S$ be a monoid homomorphism. For $t \in T^1$ and $s \in S$, denote $\varphi(t)(s)$ by ${}^t s$. Then the formula

$$(s_1, t_1)(s_2, t_2) = (s_1\ {}^{t_1}s_2, t_1 t_2)$$

defines an associative multiplication on the set $S \times T$; the resulting semigroup is called a *semidirect product* of S and T and it is denoted $S *_\varphi T$ or simply $S * T$. Given two pseudovarieties of semigroups V and W, we denote by $\mathsf{V} * \mathsf{W}$ the pseudovariety generated by all semidirect products of the form $S * T$ with $S \in \mathsf{V}$ and $T \in \mathsf{W}$, which we also call the *semidirect product* of V and W. It is well known that the semidirect product of pseudovarieties is associative, see for instance [90] or [3].

The semidirect product is a very powerful operation. The following is a decomposition result which deeply influenced finite semigroup theory.

7.2 Theorem (Krohn and Rhodes [51]) *Every finite semigroup lies in one of the alternating semidirect products*

$$\mathsf{A} * \mathsf{G} * \mathsf{A} * \cdots * \mathsf{G} * \mathsf{A}. \tag{7.1}$$

Since the pseudovarieties of the form (7.1) form a chain, every finite semigroup belongs to most of them. The least number of factors G which is needed for the pseudovariety (7.1) to contain a given semigroup S is called the *complexity* of S. Although various announcements have been made of proofs that this complexity function is computable, at present there is yet no correct published proof. This has been over the past 40 years a major driving force to the development of finite semigroup theory, the original motivations being again closely linked with Computer Science namely aiming at the effective decomposition of automata and other theoretical computing devices [35, 36].

One approach to compute complexity is to study more generally the semidirect product operation and try to devise a general method to "compute" $\mathsf{V} * \mathsf{W}$ from V and W. What is meant by *computing a pseudovariety* is to exhibit an algorithm to test membership in it; in case such an algorithm exists, we will say, as we have done earlier, that the pseudovariety is *decidable*. So a basic question for the semidirect product and other operations on pseudovarieties defined in terms of generators is whether they preserve decidability. The difficulty in studying such operations lies in the fact that the answer is negative for most natural operations, including the semidirect product [1, 72, 24].

Let $\tau : S \to T$ be a relational morphism of monoids. Tilson [90] defined an associated category D_τ as follows: the objects are the elements of T; the morphisms from t to tt' are equivalence classes $[t, s', t']$ of triples $(t, s', t') \in T \times \tau$ under the relation which identifies the triples (t_1, s_1', t_1') and (t_2, s_2', t_2') if $t_1 = t_2$, $t_1 t_1' = t_2 t_2'$, and for every s such that $(s, t_1) \in \tau$ we have $ss_1' = ss_2'$; composition of morphisms is defined by the formula

$$[t, s_1, t_1]\,[tt_1, s_2, t_2] = [t, s_1 s_2, t_1 t_2].$$

The *derived semigroupoid* of a relational morphism of semigroups is defined similarly. The following is the well-known *Derived Category Theorem*.

7.3 Theorem (Tilson [90]) *A finite semigroup S belongs to $\mathsf{V}*\mathsf{W}$ if and only if there exists a relational morphism $\tau : S \to T$ with $T \in \mathsf{W}$ and $D_\tau \in g\mathsf{V}$.*

By applying the profinite approach, Weil and the author [20] have used the Derived Category Theorem to describe a basis of pseudoidentities for $\mathsf{V} * \mathsf{W}$ from a basis of semigroupoid pseudoidentities for $g\mathsf{V}$. This has come to be known as the *Basis Theorem*. Unfortunately, there is a gap in the argument which was found by J. Rhodes and B. Steinberg in trying to extend the approach to other operations on pseudovarieties and which makes the result only known to be valid in case W is locally finite or $g\mathsf{V}$ has *finite vertex-rank* in the sense that it admits a basis of pseudoidentities in graphs using only a bounded number of vertices. Although a counterexample was at one point announced for the Basis Theorem, at present it remains open whether it is true in general. Here is the precise statement of the "Basis Theorem":

> *Let V and W be pseudovarieties of semigroups and let $\{(u_i = v_i; \Gamma_i) : i \in I\}$ be a basis of semigroupoid pseudoidentities for $g\mathsf{V}$. For each pseudoidentity $u_i = w_i$ over the finite graph Γ_i one considers a labeling $\lambda : \Gamma_i \to (\overline{\Omega}_A \mathsf{S})^1$ of the graph Γ_i such that*
>
> (1) *the labels of edges belong to $\overline{\Omega}_A \mathsf{S}$;*
>
> (2) *for every edge $e : v_1 \to v_2$, W satisfies the pseudoidentity $\lambda(v_1)\lambda(e) = \lambda(v_2)$.*
>
> *The labeling λ extends to a continuous category homomorphism $\hat{\lambda} : \overline{\Omega}_\Gamma \mathsf{Cat} \to (\overline{\Omega}_A \mathsf{S})^1$. Let z be the label of the initial vertex for the morphisms u_i, w_i and consider the semigroup pseudoidentity $z\,\hat{\lambda}(u_i) = z\,\hat{\lambda}(v_i)$. Then the "Basis Theorem" is the assertion that the set of all such pseudoidentities constitutes a basis for $\mathsf{V} * \mathsf{W}$.*

In turn Steinberg and the author [15] have used the Basis Theorem to prove the following result which explains the interest in establishing graph-tameness of pseudovarieties. Say that a pseudovariety is *recursively definable* if it admits a recursively enumerable basis of pseudoidentities in which all the intervening implicit operations are computable.

7.4 Theorem (Almeida and Steinberg [15]) *If* V *is recursively enumerable and recursively definable and* W *is graph-tame, then* $\mathsf{V} * \mathsf{W}$ *has decidable membership problem provided* $g\mathsf{V}$ *has finite vertex-rank or* W *is locally finite.*

Moreover, we have the following result which was meant to handle the iteration of semidirect product. Let B_2 denote the syntactic semigroup of the language $(ab)^+$ over the alphabet $\{a, b\}$.

7.5 Theorem (Almeida and Steinberg [15]) *Let* $\mathsf{V}_1, \ldots, \mathsf{V}_n$ *be recursively enumerable pseudovarieties such that* $B_2 \in \mathsf{V}_1$*, and each* V_i *is tame. If the Basis Theorem holds then the semidirect product* $\mathsf{V}_1 * \cdots * \mathsf{V}_n$ *is decidable through a "uniform" algorithm depending only on algorithms for the factor pseudovarieties.*

Since B_2 belongs to A, in view Ash's Theorem this would prove computability of the Krohn-Rodes complexity of finite semigroups once the Basis Theorem would be settled and a proof that A is graph-tame would be obtained. The latter has been announced by J. Rhodes but a written proof has been withdrawn since the gap in the proof of the Basis Theorem has been found.

8 Other operations on pseudovarieties

We make a brief reference in this section to another two famous results as well as some problems involving other operations on pseudovarieties of semigroups.

Given a semigroup S, one may extend the multiplication to an associative operation on subsets of S by putting $PQ = \{st : s \in P, \ t \in Q\}$. The resulting semigroup is denoted $\mathcal{P}(S)$. For a pseudovariety V of semigroups, let $\mathcal{P}\mathsf{V}$ denote the pseudovariety generated by all semigroups of the form $\mathcal{P}(S)$ with $S \in \mathsf{V}$. The operator $\mathcal{P}$ is called the *power operator* and it has been extensively studied. If V consists of finite semigroups each of which satisfies some nontrivial permutation identity or, equivalently, if V is contained in the pseudovariety

$$\mathsf{Perm} = [\![x^\omega yzt^\omega = x^\omega zyt^\omega]\!],$$

then one can find in [3] a formula for $\mathcal{P}\mathsf{V}$. Otherwise, it is also shown in [3] that $\mathcal{P}^3\mathsf{V} = \mathsf{S}$. These results were preceded by similar results of Margolis and Pin [58] in the somewhat easier case of monoids, where permutativity becomes commutativity.

One major open problem involving the power operator is the calculation of $\mathcal{P}\mathsf{J}$: it is well-known that this pseudovariety corresponds to the variety of dot-depth 2 languages in Straubing's hierarchy of star-free languages [67].

Another value of the operator $\mathcal{P}$ which has deserved major attention is $\mathcal{P}\mathsf{G}$. Before presenting results about this pseudovariety, we introduce another operator. Given two pseudovarieties V and W, their *Mal'cev product* $\mathsf{V} \circledm \mathsf{W}$ is the class of all finite semigroups S such that there exists a relational morphism $\tau : S \to T$ into $T \in \mathsf{W}$ such that, for every idempotent $e \in T$, we have $\tau^{-1}(e) \in \mathsf{V}$. It is an exercise to show that $\mathsf{V} \circledm \mathsf{W}$ is a pseudovariety.

The analog of the "Basis Theorem" for the Mal'cev product is the following.

8.1 Theorem (Pin and Weil [68]) *Let* V *and* W *be two pseudovarieties of semigroups and let* $\{u_i(x_1,\ldots,x_{n_i}) = v_i(x_1,\ldots,x_{n_i}) : i \in I\}$ *be a basis of pseudoidentities for* V. *Then* $\mathsf{V} \textcircled{m} \mathsf{W}$ *is defined by the pseudoidentities of the form* $u_i(w_1,\ldots,w_{n_i}) = v_i(w_1,\ldots,w_{n_i})$ *with* $i \in I$ *and* $w_j \in \overline{\Omega}_A\mathsf{S}$ *such that* $\mathsf{W} \models w_1 = \cdots = w_{n_i} = w_{n_i}^2$.

Call a pseudovariety W *idempotent-tame* if it is $\mathcal{C}$-tame for the set $\mathcal{C}$ of all systems of the form $x_1 = \cdots = x_n = x_n^2$. Applying the same approach as for semidirect products, we deduce that if V is decidable and W is idempotent-tame, then $\mathsf{V} \textcircled{m} \mathsf{W}$ is decidable [7].

For a pseudovariety H of groups, let BH denote the pseudovariety consisting of all finite semigroups in which regular elements have a unique inverse and whose subgroups belong to H. Finally, for a pseudovariety V, let $\mathcal{E}\,\mathsf{V}$ denote the pseudovariety consisting of all finite semigroups whose idempotents generate a subsemigroup which belongs to V.

We have the following chain of equalities

$$\mathcal{P}\,\mathsf{G} = \mathsf{J} * \mathsf{G} = \mathsf{J} \textcircled{m} \mathsf{G} = \mathsf{BG} = \mathcal{E}\,\mathsf{J} \tag{8.1}$$

The last equality is elementary as is the inclusion ($\subseteq$) in the second equality even for any pseudovariety of groups H in the place of G. The first and third equalities were proved by Margolis and Pin [59] using language theory. Using Knast's pseudoidentity basis for $g\,\mathsf{J}$, the inclusion ($\supseteq$) in the second equality was reduced by Henckell and Rhodes [40] to what they called the *pointlike conjecture* which is equivalent to the statement that G is $\mathcal{C}$-tame with respect to the signature κ where $\mathcal{C}$ is the class of systems associated with finite directed graphs with only two vertices and all edges coterminal. Hence Ash's Theorem implies the pointlike conjecture and the sequence of equalities (8.1) is settled.

Another problem which led to Ash's Theorem was the calculation of Mal'cev products of the form $\mathsf{V} \textcircled{m} \mathsf{G}$. For a finite semigroup S, define the *group-kernel* of S to consist of all elements $s \in S$ such that, for every relational morphism $\tau : S \to G$ into a finite group, we have $(s, 1) \in \tau$. Then it is easy to check that S belongs to $\mathsf{V} \textcircled{m} \mathsf{G}$ if and only if $K(S) \in \mathsf{V}$. J. Rhodes conjectured that there should be an algorithm to compute $K(S)$ and proposed a specific procedure that should produce $K(S)$: start with the set E of idempotents of S and take the closure under multiplication in S and *weak conjugation*, namely the operation that, for a pair of elements $a, b \in S$, one of which is a weak inverse of the other, sends s to asb. This came to be known as the *Type II conjecture* and it is equivalent to Ribes and Zalesskiĭ's Theorem. Therefore, as was already observed in Section 6, Ash's Theorem also implies the Type II conjecture. See [39] for further information on the history of this conjecture.

B. Steinberg later joined by K. Auinger have done extensive work on generalizing the equalities (8.1) to other pseudovarieties of groups. This culminated in their recent papers [23, 25] where they completely characterize the pseudovarieties H of groups for which respectively the equalities $\mathcal{P}\,\mathsf{H} = \mathsf{J} * \mathsf{H}$ and $\mathsf{J} * \mathsf{H} = \mathsf{J} \textcircled{m} \mathsf{H}$ hold. On the other hand, Steinberg [85] has observed that results of Margolis and Higgins [42] imply that the inclusion $\mathsf{J} \textcircled{m} \mathsf{H} \subsetneqq \mathsf{BH}$ is strict for every proper subpseudovariety $\mathsf{H} \subsetneqq \mathsf{G}$ such that $\mathsf{H} * \mathsf{H} = \mathsf{H}$. Going in another direction, Escada and the author [14] have used profinite methods to show that several pseudovarieties V satisfy the equation $\mathsf{V} * \mathsf{G} = \mathcal{E}\,\mathsf{V}$. Hence the cryptic line (8.1) has been a source and inspiration for a lot of research.

Another result involving the two key pseudovarieties J and G is the decidability of their join $\mathsf{J} \vee \mathsf{G}$. This was proved independently by Steinberg [83] and Azevedo, Zeitoun and the author [10] using Ash's Theorem and the structure of free pro-J semigroups. Previously, Trotter and Volkov [91] had shown that $\mathsf{J} \vee \mathsf{G}$ is not finitely based.

9 Symbolic Dynamics and free profinite semigroups

We have seen that relatively free profinite semigroups are an important tool in the theory of pseudovarieties of semigroups. Yet very little is known about them in general, in particular for the finitely generated free profinite semigroups $\overline{\Omega}_A\mathsf{S}$. In this section we survey some recent results the author has obtained which reveal strong ties between Symbolic Dynamics and the structure of free profinite semigroups. See [9, 8] for more detailed surveys and [19] for related work.

Throughout this section let A be a finite alphabet. The additive group $\mathbb{Z}$ of integers acts naturally on the set $A^{\mathbb{Z}}$ of functions $f : \mathbb{Z} \to A$ by translating the argument: $(n \cdot f)(m) = f(m+n)$. The elements of $A^{\mathbb{Z}}$ may be viewed as *bi-infinite words* on the alphabet A. Recall that a *symbolic dynamical system (or subshift) over* A is a non-empty subset $\mathcal{X} \subseteq A^{\mathbb{Z}}$ which is topologically closed and stable under the natural action of $\mathbb{Z}$ in the sense that it is a union of orbits.

The *language* $L(\mathcal{X})$ of a subshift $\mathcal{X}$ consists of all finite factors of members of $\mathcal{X}$, that is words of the form $w[n, n+k] = w(n)w(n+1)\cdots w(n+k)$ with $n, k \in \mathbb{Z}$, $k \geq 0$, and $w \in \mathcal{X}$. It is easy to characterize the languages $L \subseteq A^*$ that arise in this way: they are precisely the *factorial* (closed under taking factors) and *extensible* languages ($w \in L$ implies that there exist letters $a, b \in A$ such that $aw, wb \in L$). We say that the subshift $\mathcal{X}$ is *irreducible* if for all $u, v \in L(\mathcal{X})$ there exists $w \in A^*$ such that $uwv \in L(\mathcal{X})$.

A subshift $\mathcal{X}$ is said to be *sofic* if $L(\mathcal{X})$ is a rational language. The subshift $\mathcal{X}$ is called a *subshift of finite type* if there is a finite set W of *forbidden words* which characterize $L(\mathcal{X})$ in the sense that $L(\mathcal{X}) = A^* \setminus (A^*WA^*)$; equivalently, the syntactic semigroup $\operatorname{Synt} L(\mathcal{X})$ is finite and satisfies the pseudoidentities $x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega$ and $x^\omega y x^\omega y x^\omega = x^\omega y x^\omega$ [3, Section 10.8].

The mapping $\mathcal{X} \mapsto L(\mathcal{X})$ transfers structural problems on subshifts to combinatorial problems on certains types of languages. But, from the algebraic-structural point of view, the free monoid A^* is a rather limited entity where combinatorial problems have often to be dealt in an ad hoc way. So, why not going a step forward to the profinite completion $\overline{\Omega}_A\mathsf{M} = (\overline{\Omega}_A\mathsf{S})^1$, where the interplay between algebraic and topological properties is expected to capture much of the combinatorics of the free monoid? We propose therefore to take this extra step and associate with a subshift $\mathcal{X}$ the closed subset $\overline{L(\mathcal{X})} \subseteq \overline{\Omega}_A\mathsf{M}$.

For example, in the important case of sofic subshifts, by Theorem 3.6 we recover $L(\mathcal{X})$ by taking $\overline{L(\mathcal{X})} \cap A^*$. It turns out that the same is true for arbitrary subshifts so that the extra step does not loose information on subshifts but rather provides a richer structure in which to work.

Here is a couple of recent preliminary results following this approach.

9.1 Theorem *A subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$ is irreducible if and only if there is a unique minimal ideal $J(\mathcal{X})$ among those principal ideals of $\overline{\Omega}_A\mathsf{M}$ generated by elements of $\overline{L(\mathcal{X})}$ and its elements are regular.*

By a *topological partial semigroup* we mean a set S endowed with a continuous partial associative multiplication $D \to S$ with $D \subseteq S \times S$. Such a partial semigroup is said to be *simple* if every element is a factor of every other element. The structure of simple compact partial semigroups is well known: they are described by topological *Rees matrix semigroups* $\mathcal{M}(I, G, \Lambda, P)$, where I and Λ are compact sets, G is a compact group, and $P : Q \to G$ is a continuous function with $Q \subseteq \Lambda \times I$ a closed subset; as a set, $\mathcal{M}(I, G, \Lambda, P)$ is the Cartesian product $I \times G \times \Lambda$; the partial multiplication is defined by the formula

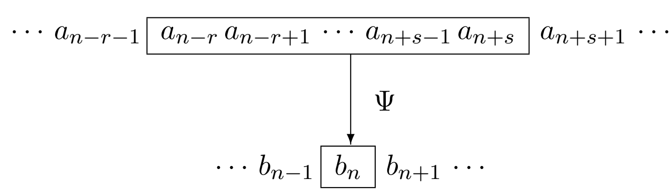
$$(i, g, \lambda)\,(j, h, \mu) = (i, g\,P(\lambda, j)\,h, \mu)$$

in case $P(\lambda, j)$ is defined and the product is left undefined otherwise. The group G is called the *structure group* and the function P is seen as a partial $\Lambda \times I$-matrix which is called the *sandwich matrix.*

It is well known that a regular $\mathcal{J}$-class J of a compact semigroup is a simple compact partial semigroup [43]. The structure group of J is a profinite group which is isomorphic to all maximal subgroups that are contained in J. In particular, for an irreducible subshift $\mathcal{X}$, there is an associated simple compact partial subsemigroup $J(\mathcal{X})$ of $\overline{\Omega}_A\mathsf{M}$. We denote by $G(\mathcal{X})$ the corresponding structure group which is a profinite group by Corollary 3.2.

Let $\mathcal{X} \subseteq A^{\mathbb{Z}}$ and $\mathcal{Y} \subseteq B^{\mathbb{Z}}$ be subshifts over two finite alphabets. A *conjugacy* is a function $\varphi : \mathcal{X} \to \mathcal{Y}$ which is a topological homeomorphism that commutes with the action of $\mathbb{Z}$ in the sense that for all $f \in A^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we have $\varphi(n \cdot f) = n \cdot \varphi(f)$. If there is such a conjugacy, then we say that $\mathcal{X}$ and $\mathcal{Y}$ are *conjugate.* By a *conjugacy invariant* we mean a structure $I(\mathcal{X})$ associated with each subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$ from a given class such that, if $\varphi : \mathcal{X} \to \mathcal{Y}$ is a conjugacy, then $I(\mathcal{X})$ and $I(\mathcal{Y})$ are isomorphic structures.

Let $\mathcal{X} \subseteq A^{\mathbb{Z}}$ and $\mathcal{Y} \subseteq B^{\mathbb{Z}}$ be subshifts. By a *sliding block code* we mean a function $\psi : \mathcal{X} \to \mathcal{Y}$ such that $\psi(w)(n) = \Psi(w[n-r, n+s])$ where $\Psi : A^{r+s+1} \cap L(\mathcal{X}) \to B$ is any function. The following diagram gives a pictorial description of this property and explains its name: each letter in the image $\psi(w)$ of $w \in \mathcal{X}$ is obtained by sliding a window of length $r + s + 1$ along w.



A sliding block code is said to be *invertible* if it is a bijection, which implies its inverse is also a sliding block code. It is well known from Symbolic Dynamics that the conjugacies are the invertible sliding block codes (see for instance [55]). This implies that if a subshift $\mathcal{X}$ is conjugate to an irreducible subshift then $\mathcal{X}$ is also irreducible.

A major open problem in Symbolic Dynamics is if it is decidable whether two subshifts of finite type are conjugate and it is well known that it suffices to treat the irreducible case. Hence, the investigation of invariants seems to be worthwhile.

9.2 Theorem *For irreducible subshifts, the profinite group $G(\mathcal{X})$ is a conjugacy invariant.*

By a *minimal subshift* we mean one which is minimal with respect to inclusion. Minimal subshifts constitute another area of Symbolic Dynamics which has deserved a lot of attention. It is easy to see that minimal subshifts are irreducible.

Say that an implicit operation w is *uniformly recurrent* if every factor $u \in A^+$ of w is also a factor of every sufficiently long factor $v \in A^+$ of w.

9.3 Theorem *A subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$ is minimal if and only if the set $\overline{L(\mathcal{X})}$ meets only one nontrivial $\mathcal{J}$-class. Such a $\mathcal{J}$-class is then regular and it is completely contained in $\overline{L(\mathcal{X})}$. (This $\mathcal{J}$-class is then $J(\mathcal{X})$.) The $\mathcal{J}$-classes that appear in this way are those that contain uniformly recurrent implicit operations or, equivalently the $\mathcal{J}$-classes that contain non-explicit implicit operations and all their regular factors.*

To gain further insight, it seems worthwhile to compute the profinite groups of specific subshifts. One way to produce a wealth of examples is to consider *substitution subshifts*. We say that a continuous endomorphism $\varphi \in \operatorname{End} \overline{\Omega}_A \mathsf{S}$ is *primitive* if, for all $a, b \in A$, there exists n such that a is a factor of $\varphi^n(b)$, and that it is *finite* if $\varphi(A) \subseteq A^*$.

Given $\varphi \in \operatorname{End} \overline{\Omega}_A \mathsf{S}$ and a subpseudovariety $\mathsf{V} \subseteq \mathsf{S}$, φ induces a continuous endomorphism $\varphi' \in \operatorname{End} \overline{\Omega}_A \mathsf{V}$ namely the unique extension to a continuous endomorphism of the mapping which sends each $a \in A$ to $\pi\varphi(a)$, where $\pi : \overline{\Omega}_A \mathsf{S} \to \overline{\Omega}_A \mathsf{V}$ is the natural projection. In case $\varphi(A) \subseteq \Omega^\sigma_A \mathsf{S}$ for an implicit signature σ, the restriction of φ' to $\Omega^\sigma_A \mathsf{V}$ is an endomorphism of this σ-algebra. So, in particular, if φ is finite then it induces an endomorphism of the free group $\Omega^\kappa_A \mathsf{G}$.

The first part of the following result is well known in Symbolic Dynamics [69].

9.4 Theorem *Let $\varphi \in \operatorname{End} \overline{\Omega}_A \mathsf{S}$ be a finite primitive substitution and let $\mathcal{X}_\varphi \subseteq A^{\mathbb{Z}}$ be the subshift whose language $L(\mathcal{X}_\varphi)$ consists of all factors of $\varphi^n(a)$ ($a \in A$, $n \geq 0$). Then the following properties hold:*

(1) *the subshift $\mathcal{X}_\varphi$ is minimal;*

(2) *if φ induces an automorphism of the free group, then $G(\mathcal{X}_\varphi)$ is a free profinite group on $|A|$ free generators.*

To test whether an endomorphism ψ of the free group $\Omega^\kappa_A \mathsf{G}$ is an automorphism, it suffices to check whether the subgroup generated by $\psi(A)$ is all of $\Omega^\kappa_A \mathsf{G}$. There is a well-known algorithm to check this property, namely Stallings' folding algorithm applied to the "flower automaton", whose petals are labeled with the words $\psi(A)$ [82, 46].

9.5 Example The *Fibonacci substitution* φ given by the pair (ab, a) is finite and primitive and therefore it determines a subshift $\mathcal{X}_\varphi$. Moreover, φ is invertible in the free group $\Omega^\kappa_2 \mathsf{G}$ since we may easily recover the generators a and b from their images ab and a using group operations: the substitution given by the pair $(b, b^{\omega-1}a)$ is the inverse of φ in the free group. By Theorem 9.4, the group $G(\mathcal{X}_\varphi)$ is a free profinite group on 2 free generators. At present we do not know the precise structure of the compact partial semigroup $J(\mathcal{X}_\varphi)$. This example has been considerably extended to minimal subshifts which are not generated by substitutions, namely to Sturmian subshifts and even to Arnoux-Rauzy subshifts [9, 8].

9.6 Example For the substitution φ given by the pair (ab, a^3b), one can show that the group $G(\mathcal{X}_\varphi)$ is not a free profinite group.

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