POINTLIKE SETS WITH RESPECT TO R AND J

JORGE ALMEIDA, JOSÉ CARLOS COSTA, AND MARC ZEITOUN

Abstract. We present an algorithm to compute the pointlike subsets of a finite semigroup with respect to the pseudovariety R of all finite $\mathcal{R}$-trivial semigroups. The algorithm is inspired by Henckell's algorithm for computing the pointlike subsets with respect to the pseudovariety of all finite aperiodic semigroups. We also give an algorithm to compute J-pointlike sets, where J denotes the pseudovariety of all finite $\mathcal{J}$-trivial semigroups. We finally show that, in contrast with the situation for R, the natural adaptation of Henckell's algorithm to J computes pointlike sets, but not all of them.

1. Introduction

The notion of *pointlike set* in a finite semigroup or monoid has emerged, in a particular case, from the *type II conjecture* of Rhodes [19] proved by Ash [13]. It proposed an algorithm to compute the *kernel* of a finite monoid with respect to finite groups, that is, the submonoid of elements whose image by any relational morphism into a group contains the neutral element of the group. The notion of kernel has then been generalized to other semigroup pseudovarieties: for a pseudovariety V and a semigroup S, a subset X of S is V-*pointlike* if any relational morphism from S into a semigroup of V relates all elements of X with a *single* element of T. The kernel consists in those G-pointlike sets which are related with the neutral element, for any relational morphism into a finite group (where G denotes the pseudovariety of groups).

Ash's theorem has a number of deep consequences. It can be used to derive a decision criterion for Mal'cev products $\mathsf{U} \textcircled{m} \mathsf{V}$ of two pseudovarieties U and V. It is known [22, 23, 15] that this operator does not preserve the decidability of the membership problem. Yet, a semigroup is in $\mathsf{U} \textcircled{m} \mathsf{G}$ if and only if its kernel belongs to U. Hence, Ash's result implies that if U is a decidable pseudovariety, then so is $\mathsf{U} \textcircled{m} \mathsf{G}$. (This also gives the decidability of semidirect products of the form $\mathsf{U} * \mathsf{G}$ for local decidable pseudovarieties U.) Pin and Weil [21] described $\mathsf{U} \textcircled{m} \mathsf{V}$ by a pseudoidentity basis obtained by substituting from a basis of U variables $\{x_1, \ldots, x_n\}$ by pseudowords $\{w_1, \ldots, w_n\}$ such that V satisfies $w_1^2 = w_1 = w_2 = \cdots = w_n$. The projection of such a set $\{w_1, \ldots, w_n\}$ into a finite semigroup by an onto continuous homomorphism is called V-*idempotent pointlike*. It follows that if U is decidable and V has decidable pointlikes, then $\mathsf{U} \textcircled{m} \mathsf{V}$ is decidable.

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There are relatively few results concerning the computation of pointlike sets. Henckell presented algorithms for computing A-pointlike sets [17] and A-idempotent pointlike sets [18] for the pseudovariety A of aperiodic semigroups. As a consequence, the Mal'cev product $\mathsf{V} ⓜ \mathsf{A}$ is decidable for any decidable pseudovariety V. The kernel computation for the pseudovariety of Abelian groups was settled by Delgado [16]. For further properties of pointlike sets, see [24, 23, 25, 14].

This paper presents an algorithm to compute R-pointlike subsets of a given finite semigroup, where R is the pseudovariety of all $\mathcal{R}$-trivial semigroups. Although it is already known that R has decidable pointlikes, the algorithms derived from [10, 9] are not very effective. In contrast, the algorithm presented in the present paper only uses the Green structure of the power semigroup of S. It is adapted from Henckell's construction [17] for the pseudovariety A. The algorithm could be adapted to the computation of idempotent pointlike sets, which would provide a new proof of the decidability of $\mathsf{V} ⓜ \mathsf{R}$ if V is decidable (this also follows from the complete κ-tameness of R [8, 9]).

We also present an algorithm to compute J-pointlike sets, where J is the pseudovariety of all $\mathcal{J}$-trivial semigroups. Perhaps surprisingly, the algorithm inspired by Henckell's construction does not work for J, and a counterexample is exhibited.

The paper is organized as follows: notation is settled in Section 2, the algorithm for computing R-pointlikes is presented in Section 3, and the one for computing J-pointlikes is presented in Section 4. We finally give several examples in Section 5.

2. Notation

We assume that the reader is acquainted with notions concerning semigroup pseudovarieties and profinite semigroups. See [5] for an introduction, and [4, 2] for more details. We recall some notation and terminology.

2.1. **Semigroups.** Let S be a semigroup. The Green equivalence relation $\mathcal{R} \subseteq S \times S$ is defined by $s \mathrel{\mathcal{R}} t$ if $sS^1 = tS^1$, where S^1 is the semigroup S itself if it has a neutral element, or the disjoint union $S \uplus \{1\}$ otherwise, where 1 acts as a neutral element. When T is a subsemigroup of S, we write $s \mathrel{\mathcal{R}^T} t$ for $sT^1 = tT^1$. A semigroup S is $\mathcal{R}$-trivial if the relation $\mathcal{R}$ on S coincides with the equality on S. We also recall that the Green equivalence relation $\mathcal{J} \subseteq S \times S$ is defined by $s \mathrel{\mathcal{J}} t$ if $S^1sS^1 = S^1tS^1$ and call *$\mathcal{J}$-trivial* a semigroup in which this relation is the equality.

The *power semigroup* $\mathcal{P}(S)$ of S is the semigroup of subsets of S under the multiplication defined by $XY = \{xy : x \in X,\ y \in Y\}$, for $X, Y \subseteq S$. Let U be a subsemigroup of $\mathcal{P}(S)$. We define $D_{\mathsf{R}}(U)$ to be the subsemigroup generated by the singleton sets $\{s\}$ $(s \in S)$ together with the subsets of the form $\bigcup R = \bigcup_{X \in R} X$, where R is an $\mathcal{R}$-class of U. We also define $\downarrow U$ to be the set $\bigcup_{X \in U} \mathcal{P}(X)$ and we note that $\downarrow U$ is again a subsemigroup of $\mathcal{P}(S)$. We let $C_{\mathsf{R}}(U) = \downarrow D_{\mathsf{R}}(U)$. We let $C^0_{\mathsf{R}}(S)$ be the subsemigroup of $\mathcal{P}(S)$ consisting of all singleton subsets of S. For $n > 0$, we define, recursively, $C^n_{\mathsf{R}}(S) = C_{\mathsf{R}}(C^{n-1}_{\mathsf{R}}(S))$. Finally, we put $C^\omega_{\mathsf{R}}(S) = \bigcup_{n \geqslant 0} C^n_{\mathsf{R}}(S)$.

2.2. **Pro-V semigroups.** In the following, A denotes a finite set, and V a semigroup pseudovariety. We let S be the pseudovariety of all finite semigroups, R be the pseudovariety of all finite $\mathcal{R}$-trivial semigroups and J be the pseudovariety of all finite $\mathcal{J}$-trivial semigroups. The A-generated relatively V-free profinite semigroup is denoted by $\overline{\Omega}_A\mathsf{V}$. Its elements are called *pseudowords*. We denote by $\Omega_A\mathsf{V}$ the subsemigroup of $\overline{\Omega}_A\mathsf{V}$ generated by A.

2.3. **Relational morphisms and pointlike sets.** Denote by $p_{\mathsf{V}} : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{V}$ the unique continuous homomorphism sending each free generator to itself. Let Sl be the pseudovariety of all finite semilattices (that is, idempotent and commutative semigroups). It is well known that $\overline{\Omega}_A\mathsf{Sl}$ is isomorphic to $\mathcal{P}(A)$, the union-semilattice of subsets of A. The projection p_{Sl} is commonly denoted by c, and called the *content*. For a word $x \in A^+$, the content $c(x)$ of x is the set of letters occurring in x.

A *relational morphism* μ between two semigroups S and T is a subsemigroup of $S \times T$ whose projection on S is onto. For $s \in S$, we let $\mu(s) = \{t \in T : (s,t) \in \mu\}$. A subset X of S is called *μ-pointlike* if $\bigcap_{x \in X} \mu(x) \neq \emptyset$. and V*-pointlike* if it is μ-pointlike for every relational morphism μ between S and a semigroup of V. We denote by $\mathcal{P}_{\mathsf{V}}(S)$ the set of V-pointlike subsets of S. It is easy to check that $\mathcal{P}_{\mathsf{V}}(S)$ is a subsemigroup of $\mathcal{P}(S)$. Given a finite A-generated semigroup S and an onto continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{S} \to S$, we denote by μ_{V} the relational morphism $p_{\mathsf{V}} \circ \varphi^{-1}$ between S and $\overline{\Omega}_A\mathsf{V}$. The morphism μ_{V} is called *universal*, in the sense that it can be used to test whether a subset of an A-generated semigroup is V-pointlike [3, 4].

Proposition 2.1. *Let $\varphi : \overline{\Omega}_A\mathsf{S} \to S$ be a continuous onto homomorphism into an A-generated semigroup, and let $\mu_{\mathsf{V}} = p_{\mathsf{V}} \circ \varphi^{-1}$. Any subset of S is V-pointlike if and only if it is μ_{V}-pointlike.*

In other words, V pointlike sets of an A-generated semigroup are obtained by projecting onto S pseudowords of $\overline{\Omega}_A\mathsf{S}$ whose p_{V}-values coincide.

2.4. **The pseudovariety** R. The pseudovariety R has been extensively studied in [11, 10, 12, 7, 8, 9]. We will use two useful and basic properties of this pseudovariety. For $x \in \overline{\Omega}_A\mathsf{S}$, a factorization of the form $x = x_1ax_2$ with $a \notin c(x_1)$ and $c(x_1a) = c(x)$ is called a *left basic factorization* of x. Using compactness of $\overline{\Omega}_A\mathsf{S}$, continuity of the content function, and the fact that $\Omega_A\mathsf{S}$ is dense in $\overline{\Omega}_A\mathsf{S}$, it is easy to show that every non-empty pseudoword admits at least one left basic factorization. The following result from [6] is the fundamental observation for the identification of pseudowords over R.

Proposition 2.2. *Let $x, y \in \overline{\Omega}_A\mathsf{S}$ and let $x = x_1ax_2$ and $y = y_1by_2$ be left basic factorizations. If $\mathsf{R} \models x = y$, then $a = b$ and R satisfies the pseudoidentities $x_1 = y_1$ and $x_2 = y_2$.*

If the content of x_2 is still the same as the content of x, then one may factorize x_2, taking its left basic factorization. Iterating this process yields the factorization $x \in \overline{\Omega}_A\mathsf{S}$ as

$$x = x_1a_1x_2a_2 \cdots x_ka_kx'_k \tag{2.1}$$

where each $x_i \cdot a_i \cdot (x_{i+1}a_{i+1} \cdots x_ka_kx'_k)$ is a left basic factorization, and $c(x_ia_i)$ is constant. We call (2.1) the *k-iterated left basic factorization* of x. If k is maximum for such a factorization of x (that is, $c(x'_k) \neq c(x)$), then we set $\|x\| = k$. If there is no such maximum, we set $\|x\| = \infty$. The following results can be found in [12].

Proposition 2.3. *Let $x, y \in \overline{\Omega}_A\mathsf{S}$ such that $\mathsf{R} \models x = y$. Then, $c(x) = c(y)$ and $\|x\| = \|y\|$.*

The function $\|\cdot\|$ also characterizes idempotents over R.

Proposition 2.4. *Let $x \in \overline{\Omega}_A\mathsf{S}$. Then $\mathsf{R} \models x = x^2$ if and only if $\|x\| = \infty$.*

From the above propositions, we deduce the following technical result.

Corollary 2.5. *Let $S \in \mathsf{S}$ be an A-generated semigroup, and let $\psi : \overline{\Omega}_A\mathsf{S} \to S$ be an onto continuous homomorphism. Let $x_1, \ldots, x_n \in \overline{\Omega}_A\mathsf{S}$ be such that $\mathsf{R} \models x_i = x_j$ for $1 \leqslant i, j \leqslant n$. Let $B = c(x_1)$ and $k \leqslant \|x_1\|$. Then each x_i has a factorization*

$$x_i = x_{i,1}a_1x_{i,2}a_2 \cdots x_{i,k}a_kz_{i,k}, \tag{2.2}$$

where

$$c(x_{i,\ell}) = B \setminus \{a_\ell\}, \quad \mathsf{R} \models x_{i,\ell} = x_{j,\ell} \quad \text{and} \quad \mathsf{R} \models z_{i,k} = z_{j,k} \quad (1 \leqslant \ell \leqslant k \text{ and } 1 \leqslant i \leqslant n). \tag{2.3}$$

Further, either no $p_{\mathsf{R}}(x_i)$ *is idempotent and* $c(z_{j,k}) \subsetneqq B$ *for* $k = \|x_1\|$*, or all* $p_{\mathsf{R}}(x_i)$ *are idempotents. In the later case, there exist indices* p *and* q *such that* $1 \leqslant p, q \leqslant |S|^n + 1$ *and, for* $i = 1, \dots, n$*, we have*

$$\psi(x_{i,1}a_1 \cdots x_{i,p}a_p) = \psi(x_{i,1}a_1 \cdots x_{i,p}a_p) \cdot \psi(x_{i,p+1}a_{p+1} \cdots x_{i,p+q}a_{p+q})^\omega \tag{2.4}$$

Proof. By Proposition 2.3, $c(x_i)$ and $\|x_i\|$ are constant. By Proposition 2.4, $p_{\mathsf{R}}(x_i)$ are all idempotent, or none of them is. Next, (2.2) and (2.3) simply express properties of the k-iterated left basic factorization (for $k = \|x_i\|$ if $\|x_i\|$ is finite, and for all k otherwise). Finally, $\alpha_k = (\psi(x_{i,1}a_1 \cdots x_{i,k}a_k))_{1 \leqslant i \leqslant n} \in S^n$, so there exist $1 \leqslant p, q \leqslant |S|^n + 1$ such that $\alpha_p = \alpha_{p+q}$, which yields (2.4). □

3. An algorithm to compute R-pointlike sets

The aim of this section is to establish the following result.

Theorem 3.1. *If* S *is a finite semigroup then* $C^\omega_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$.

Observe that $C^\omega_{\mathsf{R}}(S)$ can be computed iteratively, so that Theorem 3.1 establishes an algorithm to compute $\mathcal{P}_{\mathsf{R}}(S)$. It is similar to Henckell's algorithm to compute $\mathcal{P}_{\mathsf{A}}(S)$. We first treat one inclusion of Theorem 3.1.

Lemma 3.2. *Let* S *be a finite semigroup. If* T *is a subsemigroup of* $\mathcal{P}_{\mathsf{R}}(S)$*, then so is* $C_{\mathsf{R}}(T)$.

Proof. Obviously $C_{\mathsf{R}}(T)$ is a subsemigroup of $\mathcal{P}(S)$. Hence, it suffices to show that for $X \in T$, we have $\bigcup_{Y \mathcal{R}^T X} Y \in \mathcal{P}_{\mathsf{R}}(S)$. Let $\{X_1, \dots, X_n\}$ be the $\mathcal{R}$-class of X in T. There exist $Y_1, \dots, Y_n \in T$ such that $X_{i+1} = X_i Y_i$ for $1 \leqslant i < n$ and $X_1 = X_n Y_n$. Therefore, we have $X_1 = X_1(Y_1 \cdots Y_n) = X_1(Y_1 \cdots Y_n)^\omega$, and for $i \geqslant 1$, $X_i = X_1(Y_1 \cdots Y_n)^\omega \prod_{k=1}^{i-1} Y_k$. Hence

$$\bigcup_{Y \mathcal{R}^T X} Y = X_1(Y_1 \cdots Y_n)^\omega \bigcup_{i=1}^{n} \prod_{k=1}^{i-1} Y_k$$

Now, X_1 and all Y_i's are R-pointlike since T is a subsemigroup of $\mathcal{P}_{\mathsf{R}}(S)$. Therefore, there exist $x_1, y_1, \dots, y_n \in \overline{\Omega}_A \mathsf{R}$ such that $X_1 \subseteq \mu_{\mathsf{R}}^{-1}(x_1)$ and for $i = 1, \dots, n$, $Y_i \subseteq \mu_{\mathsf{R}}^{-1}(y_i)$. Since $\mathsf{R} \models x_1(y_1 \cdots y_n)^\omega y_1 \cdots y_{i-1} = x_1(y_1 \cdots y_n)^\omega$, we obtain $\bigcup_{Y \mathcal{R}^T X} Y \subseteq \mu_{\mathsf{R}}^{-1}(x_1(y_1 \cdots y_n)^\omega)$. □

Since $C^0_{\mathsf{R}}(S)$ is a subsemigroup of $\mathcal{P}_{\mathsf{R}}(S)$, we obtain one of the inclusions of Theorem 3.1.

Corollary 3.3. *If* S *is a finite semigroup then* $C^\omega_{\mathsf{R}}(S) \subseteq \mathcal{P}_{\mathsf{R}}(S)$.

In the rest of the section, we complete the proof of Theorem 3.1, which depends on several intermediate results.

3.1. Behaviour of C_{R} and C^ω_{R} under onto homomorphisms.

The following result is crucial in the sequel. It is part of a well-known lifting property of Green's relations under onto homomorphisms [20, Fact 2.1, p. 160].

Lemma 3.4. *Let* $\psi : U \to V$ *be an onto homomorphism between finite semigroups. Then, for every* $\mathcal{R}$*-class* R' *of* V *there is an* $\mathcal{R}$*-class* R *of* U *such that* $\psi(R) = R'$.

Given an homomorphism $\varphi : S \to T$ between finite semigroups, we let $\bar{\varphi} : \mathcal{P}(S) \to \mathcal{P}(T)$ be the associated homomorphism defined by taking subset images. Note that if φ is onto, so is $\bar{\varphi}$.

Proposition 3.5. *Let $\varphi : S \to T$ be an onto homomorphism between finite semigroups. Let U be a subsemigroup of $\mathcal{P}(S)$ and let $V = \bar{\varphi}(U)$ be its image in $\mathcal{P}(T)$. Then $C_{\mathsf{R}}(V) = \bar{\varphi}(C_{\mathsf{R}}(U))$.*

Proof. Obviously, the singleton subsets of S map onto the singleton subsets of T. Since φ respects the Green relations, given an $\mathcal{R}$-class R of U, $\bar{\varphi}(R)$ is contained in some $\mathcal{R}$-class R' of V and so $\bar{\varphi}(\bigcup R) \subseteq \bigcup R'$. It follows that $\bar{\varphi}(D_{\mathsf{R}}(U)) \subseteq C_{\mathsf{R}}(V)$. Moreover, if $X \subseteq S$ is such that $\bar{\varphi}(X) \in C_{\mathsf{R}}(V)$ and $Y \subseteq X$, then the set $\bar{\varphi}(Y)$ is contained in $\bar{\varphi}(X)$ and therefore it also belongs to $C_{\mathsf{R}}(V)$. Hence $\bar{\varphi}(C_{\mathsf{R}}(U)) \subseteq C_{\mathsf{R}}(V)$.

For the converse, suppose that R' is an $\mathcal{R}$-class of V. Then, by Lemma 3.4, there is an $\mathcal{R}$-class R of U such that $\bar{\varphi}(R) = R'$. It follows that $\bar{\varphi}(\bigcup R) = \bigcup R'$. Together with the earlier observation on the behaviour of $\bar{\varphi}$ on singleton sets, this implies that $D_{\mathsf{R}}(V) \subseteq \bar{\varphi}(D_{\mathsf{R}}(U))$. Suppose next that $X' \in D_{\mathsf{R}}(V)$ and $Y' \subseteq X'$. Then there exists $X \in D_{\mathsf{R}}(U)$ such that $\bar{\varphi}(X) = X'$, which implies that $Y' = \bar{\varphi}(Y)$, where $Y = \bar{\varphi}^{-1}(Y') \cap X$, and whence $Y \in C_{\mathsf{R}}(U)$. Hence $C_{\mathsf{R}}(V) \subseteq \bar{\varphi}(C_{\mathsf{R}}(U))$, which completes the proof of the proposition. □

Iterating the application of Proposition 3.5, we obtain the following result.

Corollary 3.6. *If $\varphi : S \to T$ is an onto homomorphism between finite semigroups, then $\bar{\varphi}(C^{\omega}_{\mathsf{R}}(S)) = C^{\omega}_{\mathsf{R}}(T)$.*

We say that a semigroup S has a *content homomorphism* c if there exists an onto continuous homomorphism $\psi : \overline{\Omega}_A\mathsf{S} \to S$ and a homomorphism $c : S \to \mathcal{P}(A)$ into the union-semilattice of subsets of A, such that $c \circ \psi$ sends each $a \in A$ to the singleton subset $\{a\}$. In this case, the *content of* $s \in S$ is $c(s)$.

Corollary 3.7. *If the equality $C^{\omega}_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$ holds for all finite semigroups with a content homomorphism, then it holds for all finite semigroups.*

Proof. Let T be a finite A-generated semigroup, let $\psi : A^+ \to T$ be an onto homomorphism, and let S be the subsemigroup of $T \times \mathcal{P}(A)$ generated by all pairs $(\psi(a), a)$. Then, S has a content homomorphism given by the projection on the second component.

It is easy to see that, for every pseudovariety V and every onto homomorphism $\varphi : S \to T$ between finite A-generated semigroups, $\bar{\varphi}(\mathcal{P}_{\mathsf{V}}(S)) = \mathcal{P}_{\mathsf{V}}(T)$. (If $X \subseteq S$ is V-pointlike and $\mu_T : T \to U \in V$ is a relational morphism, use the relational morphism $\mu_T \circ \varphi : S \to U$ to show that $\varphi(X)$ is μ_T-pointlike; if $Y \subseteq T$ is V-pointlike and $\mu_S : S \to U \in V$ is a relational morphism, use $\mu_S \circ \varphi^{-1} : T \to U$ to find a μ_S-pointlike set $X \subseteq S$ such that $\varphi(X) = Y$.) Since the projection $(\psi(x), x) \mapsto \psi(x)$ from S to T is indeed an onto homomorphism, in view of Corollary 3.6 it suffices to prove the equality $C^{\omega}_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$ to prove that $C^{\omega}_{\mathsf{R}}(T) = \mathcal{P}_{\mathsf{R}}(T)$. □

3.2. **Structure of $\mathcal{P}_{\mathsf{R}}(S)$.** In this subsection, we assume that we are given an A-generated finite semigroup S with an onto homomorphism $\psi : A^+ \to S$ and a content homomorphism.

Lemma 3.8. *Let X be an R-pointlike subset of S which consists of idempotents. Then all elements of X have the same content B, and $X\psi(B^+)$ is an R-pointlike subset of S.*

Proof. Since $X \in \mathcal{P}_{\mathsf{R}}(S)$, there exists, by Proposition 2.1, a function $\delta : X \to \overline{\Omega}_A\mathsf{S}$ such that $p_{\mathsf{R}} \circ \delta$ is a constant function, and $\psi(\delta(e)) = e$ for every $e \in X$. Since e is idempotent, we obtain $\psi(\delta(e)^{\omega}) = e$, and we may as well assume that each $\delta(e)$ is idempotent. Since the semilattice $\mathcal{P}(A)$ belongs to R, the continuous homomorphism $c \circ \psi$ factors through $\overline{\Omega}_A\mathsf{R}$. Hence all elements e of X have indeed the same content $B = c(e)$.

Extend δ to a function $\varepsilon : X\psi(B^+) \to \overline{\Omega}_A\mathsf{S}$ by choosing for each element s of $X\psi(B^+) \setminus X$ a word $w \in B^+$ and $e \in X$ such that $s = e\psi(w)$ and letting $\varepsilon(s) = \delta(e)w$. Then $\psi(\varepsilon(s)) = s$ for every $s \in X\psi(B^+)$ and $p_{\mathsf{R}} \circ \varepsilon$ is a constant function with the same value as $p_{\mathsf{R}} \circ \delta$. Hence $X\psi(B^+)$ belongs to $\mathcal{P}_{\mathsf{R}}(S)$. $\square$

Let U be the subsemigroup of $\mathcal{P}(S)$ generated by the singleton subsets together with the subsets of the form $X\psi(B^+)$, where $X \in \mathcal{P}_{\mathsf{R}}(S)$ consists of idempotents and B is the content of the elements of X.

Proposition 3.9. *We have $\mathcal{P}_{\mathsf{R}}(S) = {\downarrow} U$.*

Proof. By Lemma 3.8, we have the inclusion $U \subseteq \mathcal{P}_{\mathsf{R}}(S)$ and, therefore, also the inclusion ${\downarrow} U \subseteq {\downarrow} \mathcal{P}_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$. For the reverse inclusion, let $X = \{s_1, \ldots, s_n\} \in \mathcal{P}_{\mathsf{R}}(S)$. By Proposition 2.1, there exist $x_1, \ldots, x_n \in \overline{\Omega}_A\mathsf{S}$ such that $\psi(x_i) = s_i$ for $i = 1, \ldots, n$ and $\mathsf{R} \models x_1 = \cdots = x_n$. By Proposition 2.3, all x_i's have the same content B. We show by induction on $|B|$ that $X \in {\downarrow} U$. If $|B| = 0$, then $X = \emptyset \in {\downarrow} U$. For the induction step, by Corollary 2.5 we have a factorization (2.2) for each x_i.

Assume first that no $p_{\mathsf{R}}(x_i)$ is idempotent. Then $k = \|x_i\|$, which does not depend on i by Proposition 2.3, is finite by Proposition 2.4. By Corollary 2.5, we have $c(x_{i,\ell}) \subsetneqq B$ and $c(z_{i,k}) \subsetneqq B$ for $1 \leqslant i \leqslant n$ and $1 \leqslant \ell \leqslant k$, and also $\mathsf{R} \models x_{i,\ell} = x_{j,\ell}$ and $\mathsf{R} \models z_{i,k} = z_{j,k}$. This makes it possible to apply the induction hypothesis to the subsets $X_\ell = \{\psi(x_{i,\ell}) : i = 1, \ldots, n\}$ ($\ell = 1, \ldots, k$) and $Z = \{\psi(z_{i,k}) : 1 \leqslant i \leqslant n\}$ of S, which therefore belong to ${\downarrow} U$. Now, $X \subseteq X_1\{\psi(a_1)\}X_2\{\psi(a_2)\} \cdots X_k\{\psi(a_k)\}Z$, hence $X \in {\downarrow} U$.

Assume next that all x_i's are idempotent over R, so that by Corollary 2.5, there exist indices p and q such that $1 \leqslant p, q \leqslant |S|^n + 1$ and (2.4) holds for all $1 \leqslant i \leqslant n$. Choose $z_i \in B^+$ such that $\psi(z_i) = \psi(z_{i,p})$ and set $e_i = \psi(x_{i,p+1}a_{p+1} \cdots x_{i,p+q}a_{p+q})^\omega$, so that $s_i = \psi(x_{i,1}a_1 \cdots x_{i,p}a_p) \cdot e_i \cdot \psi(z_i)$. By Corollary 2.5, we have $c(x_i, \ell) \subsetneqq B$ and $\mathsf{R} \models x_{i,\ell} = x_{j,\ell}$ for all $1 \leqslant i, j \leqslant n$ and $1 \leqslant \ell \leqslant k$. Therefore, the sets $X_\ell = \{\psi(x_{i,\ell}) : i = 1, \ldots, n\}$ belong to ${\downarrow} U$ by induction hypothesis. Further, $E = \{e_1, \ldots, e_n\}$ is a set of idempotents and is R-pointlike. Hence $E\{\psi(z_i) : 1 \leqslant i \leqslant n\} \subseteq E\psi(B^+)$ belongs to U, by definition of U. Therefore, $X \subseteq X_1\{\psi(a_1)\} \cdots X_p\{\psi(a_p)\}E\psi(B^+)$ also belongs to ${\downarrow} U$. $\square$

3.3. **The algorithm à la Henckell.**

Lemma 3.10. *Let F be a set of idempotents of S and suppose that there are $X, Y, Q \in C^\omega_{\mathsf{R}}(S)$ such that $F \subseteq XQY$. Then $F \cup FQ$ also belongs to $C^\omega_{\mathsf{R}}(S)$.*

Proof. Let W be the union of the $\mathcal{R}$-class of $(XQY)^\omega$ in $C^\omega_{\mathsf{R}}(S)$. Note that $W \in C^\omega_{\mathsf{R}}(S)$. Since F consists of idempotents, certainly F is contained in $(XQY)^\omega$ and therefore also in W. Since $(XQY)^\omega X \;\mathcal{R}^{C^\omega_{\mathsf{R}}(S)}\; (XQY)^\omega$, we deduce that also $FX \subseteq W$. Hence $F \cup FX \in C^\omega_{\mathsf{R}}(S)$. Next, let Z be the union of the $\mathcal{R}$-class of $(WQY)^\omega$ in $C^\omega_{\mathsf{R}}(S)$, which is again an element of $C^\omega_{\mathsf{R}}(S)$. Since $FX \subseteq W$ and $F \subseteq XQY$, we have $F \subseteq (FXQY)^\omega \subseteq (WQY)^\omega \subseteq Z$. Finally, since $F \subseteq W$, we have $FQ \subseteq WQ$. Again since F consists of idempotents, $FQ \subseteq F \cdot (FQ) \subseteq (WQY)^\omega \cdot WQ \;\mathcal{R}^{C^\omega_{\mathsf{R}}(S)}\; (WQY)^\omega$ which implies that also $FQ \subseteq Z$. Hence $F \cup FQ$ is contained in Z, whence it belongs to $C^\omega_{\mathsf{R}}(S)$. $\square$

Lemma 3.11. *Let F be a set of idempotents of S, let $Q_1, \ldots, Q_n \in C^\omega_{\mathsf{R}}(S)$, and suppose that $F \cup FQ_i \in C^\omega_{\mathsf{R}}(S)$ ($i = 1, \ldots, n$). Then $F \cup \bigcup_{i=1}^n FQ_i$ also belongs to $C^\omega_{\mathsf{R}}(S)$.*

Proof. Proceeding by induction, we assume that the set $X = F \cup \bigcup_{i=1}^{n-1} FQ_i$ belongs to $C^\omega_{\mathsf{R}}(S)$ and we let $Y = F \cup FQ_n$. Let Z be the union of the $\mathcal{R}$-class of $(XY)^\omega$ in $C^\omega_{\mathsf{R}}(S)$. Then $Z \in C^\omega_{\mathsf{R}}(S)$ and, since F consists of idempotents and $F \subseteq X \cap Y$, we have $F \subseteq (XY)^\omega \cap (XY)^{\omega-1}X$, which implies that $X \subseteq FX \subseteq (XY)^\omega X \subseteq Z$ and $Y \subseteq FY \subseteq (XY)^{\omega-1}X \cdot Y = (XY)^\omega \subseteq Z$. This shows that $X \cup Y \subseteq Z$ and proves the lemma. $\square$

Lemma 3.12. *Let F be a set of idempotents of S, $Q_1, \ldots, Q_m \in C^\omega_{\mathsf{R}}(S)$, and suppose that there exist $X_i, Y_i \in C^\omega_{\mathsf{R}}(S)$ such that $F \subseteq \bigcap_{i=1}^m X_iQ_iY_i$. Then $F \cup FQ_1 \cdots Q_m$ belongs to $C^\omega_{\mathsf{R}}(S)$.*

Proof. The case $m = 1$ is given by Lemma 3.10. Proceeding by induction on m, we may as well assume that $F \cup FQ_1 \cdots Q_{m-1} \in C^\omega_{\mathsf{R}}(S)$. Since $F \cup FX_m$ is contained in the union of the $\mathcal{R}$-class of $(X_mQ_mY_m)^\omega$, we also have $F \cup FX_m \in C^\omega_{\mathsf{R}}(S)$. By Lemma 3.11, we deduce that $W = F \cup FX_m \cup FQ_1 \cdots Q_{m-1}$ belongs to $C^\omega_{\mathsf{R}}(S)$. Let Z be the union of the $\mathcal{R}$-class of $(WQ_mY_m)^\omega$. Since F consists of idempotents, $F \subseteq X_mQ_mY_m$, and $FX_m \subseteq W$, we have $F \subseteq (WQ_mY_m)^\omega \subseteq Z$. On the other hand, since $FQ_1 \cdots Q_{m-1} \subseteq W$ we also have $FQ_1 \cdots Q_m \subseteq (WQ_mY_m)^\omega WQ_m \subseteq Z$ (since $(WQ_mY_m)^\omega WQ_m \; \mathcal{R}^{C^\omega_{\mathsf{R}}(S)} \; (WQ_mY_m)^\omega$). Hence $F \cup FQ_1 \cdots Q_m$ is contained in Z, which shows that it belongs to $C^\omega_{\mathsf{R}}(S)$. $\square$

Proof of Theorem 3.1. We have $C^\omega_{\mathsf{R}}(S) \subseteq \mathcal{P}_{\mathsf{R}}(S)$ by Corollary 3.3. For the reverse inclusion, we first use Corollary 3.7 to reduce it to the case where S is an A-generated semigroup, under an onto continuous homomorphism $\psi : \overline{\Omega}_A\mathsf{S} \to S$, with a content homomorphism $c : S \to \mathcal{P}(A)$. For $X \subseteq S$, let $\bar{c}(X) = \bigcup_{x \in X} c(x)$. We show, by induction on $|\bar{c}(X)|$, that for all $X \in \mathcal{P}_{\mathsf{R}}(S)$ and for all $a \in \bar{c}(X)$, we have

$$\mathcal{C}(X, a) \qquad \exists X_a, Y_a \in C^\omega_{\mathsf{R}}(S) \text{ such that } X \subseteq X_a\psi(a)Y_a.$$

Note that proving $\mathcal{C}(X, a)$ for all $X \in \mathcal{P}_{\mathsf{R}}(S)$ and $a \in \bar{c}(X)$ entails that $\mathcal{P}_{\mathsf{R}}(S) \subseteq C^\omega_{\mathsf{R}}(S)$. In case $|\bar{c}(X)| = 0$, then $X = \emptyset$ and so certainly $\mathcal{C}(X, a)$ holds. Let $X \in \mathcal{P}_{\mathsf{R}}(S)$ be nonempty, let $\bar{c}(X) = B$, and assume inductively that $\mathcal{C}(Y, a)$ holds for every $Y \in \mathcal{P}_{\mathsf{R}}(S)$ and $a \in \bar{c}(Y)$ with $|\bar{c}(Y)| < |\bar{c}(X)|$. By Proposition 3.9, X is included in a product $U_1 \cdots U_k$, where each U_i is either a singleton, or of the form $F\psi(C^+)$, where F is a pointlike set of idempotents of content C. Replacing such a subset F by $F \cap \psi(B^+)$, and C by $C \cap B$, we may as well assume that $C \subseteq B$, since $\bar{c}(X) = B$. Furthermore, proving $\mathcal{C}(F\psi(C^+), a)$, for such F and C, and $a \in C$, yields in particular $U_i \in C^\omega_{\mathsf{R}}(S)$, which then implies $\mathcal{C}(X, a)$. Therefore, one can assume that X is of the form $F\psi(C^+)$ for an R-pointlike set F of idempotents of content $C \subseteq B$. If $C \subsetneq B$, then the induction hypothesis immediately yields $\mathcal{C}(X, a)$, so we may as well assume that $C = B$.

Let $F = \{s_1, \ldots, s_n\}$. Since $F \in \mathcal{P}_{\mathsf{R}}(S)$, there exist $x_1, \ldots, x_n \in \overline{\Omega}_A\mathsf{S}$ such that $\psi(x_i) = s_i$ and $\mathsf{R} \models x_i = x_j$ $(1 \leqslant i, j \leqslant n)$. Since s_i is idempotent, $\psi(x_i^\omega) = s_i$ and one can assume that x_i is idempotent. Let p, q be the integers given by Corollary 2.5. Consider the k-iterated left basic factorizations (2.2) of x_i for $k \geqslant p + q$, whose factors satisfy (2.3) and (2.4). Choose $z_i \in B^+$ such that $\psi(z_i) = \psi(z_{i,p})$ and let $e_i = \psi(x_{i,p+1}a_{p+1} \cdots x_{i,p+q}a_{p+q})^\omega$ and $E = \{e_1, \ldots, e_n\}$.

By (2.4), we have $s_i = \psi(x_{i,1}a_1 \cdots x_{i,p}a_p)e_i\psi(z_i)$. By (2.3), the set $X_\ell = \{\psi(x_{i,\ell}) : 1 \leqslant i \leqslant n\}$ is R-pointlike for $1 \leqslant \ell \leqslant p + q$, and $|\bar{c}(X_\ell)| < |B|$. By induction hypothesis, $\mathcal{C}(X_\ell, a)$ holds for $a \in \bar{c}(X_\ell)$, and in particular $X_\ell \in C^\omega_{\mathsf{R}}(S)$. Therefore, $Y = X_{p+1}\psi(a_{p+1}) \cdots X_{p+q}\psi(a_{p+q}) \in C^\omega_{\mathsf{R}}(S)$, and $E \subseteq Y^\omega$ also belongs to $C^\omega_{\mathsf{R}}(S)$. Let $Z = \{\psi(z_i) : 1 \leqslant i \leqslant n\}$. We have $F \subseteq X_1\psi(a_1) \cdots X_p\psi(a_p)EZ$ and $EZ \subseteq E\psi(B^+)$, so $F\psi(B^+) \subseteq X_1\psi(a_1) \cdots X_p\psi(a_p).E.E\psi(B^+)$. Since all factors of the right hand side of this inclusion are in $C^\omega_{\mathsf{R}}(S)$, except perhaps $E\psi(B^+)$, and since E itself appears as a factor of content B, to show that $\mathcal{C}(F\psi(B^+), a)$, it is sufficient to verify that:

(i) Property $\mathcal{C}(E, a)$ holds for all $a \in B$, and
(ii) $E\psi(B^+) \in C^\omega_{\mathsf{R}}(S)$.

Clearly $\mathcal{C}(E,a)$ holds for $a \in \{a_{p+1},\ldots,a_{p+q}\}$, since $E \subseteq Y^\omega$, and $X_\ell \in C^\omega_{\mathsf{R}}(S)$. Otherwise, choose $m \in \{p+1,\ldots,p+q\}$ such that $a \in c(x_{i,m})$ for $1 \leqslant i \leqslant n$. By induction hypothesis, there are $X',Y' \in C^\omega_{\mathsf{R}}(S)$ such that $X_m = X'\psi(a)Y'$. Hence $E \subseteq X_a\psi(a)Y_a$ for $X_a = Y^{\omega-1}X_{p+1}\psi(a_{p+1})\cdots X_{m-1}\psi(a_{m-1})X'$ and $Y_a = Y'\psi(a_m)X_{m+1}\psi(a_{m+1})\cdots X_{p+q}\psi(a_{p+q})$. This proves (i) since $X_a, Y_a \in C^\omega_{\mathsf{R}}(S)$.

From Lemma 3.12, we deduce that, if $w \in B^+$, then $E \cup E\psi(w) \in C^\omega_{\mathsf{R}}(S)$. By Lemma 3.11, it follows that $E\psi(B^+) = E \cup \bigcup_{w\in B^+} E\psi(w) \in C^\omega_{\mathsf{R}}(S)$ since $\psi(B^+)$ is a finite set. This shows (ii), completes the induction step and proves the theorem. □

3.4. **Alternative proofs using tameness and canonical forms.** We give alternative proofs of Proposition 3.9 and Theorem 3.1. Recall that the canonical implicit signature κ is $\{\_\cdot\_, \_^{\omega-1}\}$. The V-free κ-semigroup over A is denoted $\Omega^\kappa_A\mathsf{V}$. We use a weak form of κ-tameness for R [7], and the canonical form of κ-terms defined in [12]. These proofs require more knowledge on the pseudovariety R, but are somewhat shorter and more elegant. They rely on the following statement.

Proposition 3.13. *Let $w_1,\ldots,w_n \in \Omega^\kappa_A\mathsf{S}$ be such that $p_{\mathsf{R}}(w_i)$ is independent of i. Then each w_i admits a factorization*

$$w_i = u_0 v_{i,1}^\omega r_{i,1} u_1 \cdots v_{i,p}^\omega r_{i,p} u_p \tag{3.1}$$

where:

- (a) *each u_j is a possibly empty word,*
- (b) *each $v_{i,j}$ and each $r_{i,j}$ is given by a κ-term,*
- (c) *$c(r_{i,j}) \subseteq c(v_{i,j})$,*
- (d) *the first letter of the first nonempty factor after $r_{i,j}$, if there is such a factor, does not belong to $c(v_{i,j})$,*
- (e) *the canonical form $\bar{v}_j$ of $v_{i,j}$ is independent of i,*
- (f) *the ω-term $u_0\bar{v}_1^\omega u_1\cdots\bar{v}_p^\omega u_p$ is in canonical form.*

Proof. Each element w of $\Omega^\kappa_A\mathsf{S}$ has a representation as a term in the signature $\{\_.\_, \_^{\omega-1}\}$, consisting of the multiplication $\_.\_$ and the unary $(\omega-1)$-power. We recall from [12, Theorem 6.1] that we can associate to w a canonical form $\mathsf{cf}(w)$, obtained by rewriting w using the following identities: $(xy)^\omega = (xy)^\omega x = (xy)^\omega x^\omega = x(yx)^\omega, (x^\omega)^\omega = x^\omega, (x^r)^\omega = x^\omega, r \geqslant 2$, and such that two terms have the same projection under p_{R} if and only if their canonical forms are equal. Let $u_0\bar{v}_1^\omega u_1\cdots\bar{v}_p^\omega u_p$ be the common canonical form of $w_1,\ldots,w_n$, where $u_0,\ldots,u_p$ are possibly empty words. This form is obtained using the above identities, which are either valid in $\Omega^\kappa_A\mathsf{S}$, or which add or remove a term u after an idempotent v^ω of larger content than u. One can track back these rewritings, so that each w_i has a factorization (3.1) satisfying properties (a)–(f). Note that we use the identity $x^{\omega-1} = x^\omega.x^{\omega-1}$ to replace an $(\omega-1)$-power by an ω-power followed by a remainder, and that (d) comes from the corresponding property for canonical forms. □

Alternative proof of Proposition 3.9. The inclusion $\downarrow U \subseteq \mathcal{P}_{\mathsf{R}}(S)$ follows from Lemma 3.8. We have to show that $\mathcal{P}_{\mathsf{R}}(S) \subseteq \downarrow U$. Let $X \in \mathcal{P}_{\mathsf{R}}(S)$. Since R is κ-tame for systems of equations of the form $x_1 = \cdots = x_n$ [7], it follows that there exists a function $\delta : X \to \overline{\Omega}_A\mathsf{S}$ such that $\psi(\delta(s)) = s$ for every $s \in X$, $p_{\mathsf{R}} \circ \delta$ is a constant function, and each $\delta(s)$ is given by a κ-term. Let $X = \{s_1,\ldots,s_n\}$ and let $w_i = \delta(s_i)$ $(i = 1,\ldots,n)$. Then there are factorizations (3.1) satisfying conditions (a)–(f) of Proposition 3.13. It follows that for $j = 1,\ldots,p$, each set $X_j = \{\psi(v_{i,j}^\omega) : i = 1,\ldots,n\}$ is an R-pointlike subset of S consisting of idempotents. Moreover, if $B_j = c(v_{i,j})$, which is independent of i by (e), then $\{\psi(v_{i,j}^\omega r_{i,j}) : i = 1,\ldots,n\}$ is contained in $X_j\psi(B_j^+)$. Hence $X \in \downarrow U$, which completes the proof of the proposition. □

Alternative proof of Theorem 3.1. As in the first proof, of Theorem 3.1, we can assume that S has a content homomorphism. We show $\mathcal{C}(X,a)$ by induction on $|\bar{c}(X)|$, for all $X \in \mathcal{P}_{\mathsf{R}}(S)$ and all $a \in \bar{c}(X)$. The case $|\bar{c}(X)| = 0$ is trivial. Let $X = \{s_1, \dots, s_n\}$ and assume inductively that $\mathcal{C}(Y,a)$ holds for every $Y \in \mathcal{P}_{\mathsf{R}}(S)$ with $|\bar{c}(Y)| < |\bar{c}(X)|$ and all $a \in \bar{c}(Y)$. Since R is κ-tame for systems of the form $x_1 = \cdots = x_n$ [7], by Proposition 3.13 there exist κ-terms w_i such that $\psi(w_i) = s_i$ and w_i admits a factorization of the form (3.1) satisfying conditions (a)-(f) of Proposition 3.13. Hence it suffices to show $\mathcal{C}(F\psi(B^+), a)$ for all $a \in B$, where $F = \psi\{v_1^\omega, \dots, v_n^\omega\}$ and the v_i are given by κ-terms such that $\bar{v} = p_{\mathsf{R}}(v_i)$ is independent of i, $\bar{v}^\omega$ is in canonical form, and $B = c(\bar{v})$. Since $F \subseteq F \cdot F\psi(B^+)$, it suffices to show that

(i) Property $\mathcal{C}(F,a)$ holds for all $a \in B$, and
(ii) $F\psi(B^+) \in C_{\mathsf{R}}^{\omega}(S)$.

By definition of canonical form, $\bar{v}$ has the form

$$\bar{v} = \bar{z}_1 a_1 \cdots \bar{z}_m a_m \tag{3.2}$$

for some $\bar{z}_j$ given by ω-terms and some $a_j \in A$ such that $c(\bar{v}) = c(\bar{z}_j a_j) \supsetneqq c(\bar{z}_j)$. By the results of [12], each v_i admits a corresponding factorization $v_i = z_{i,1} a_1 \cdots z_{i,m} a_m$ such that $z_{i,j} \in \Omega_A^\kappa \mathsf{S}$ and $p_{\mathsf{R}}(z_{i,j}) = \bar{z}_j$ $(i = 1, \dots, n;\ j = 1, \dots, m)$. Therefore, for $j = 1, \dots, m$, the sets $X_j = \psi\{z_{1,j}, \dots, z_{n,j}\}$ are R-pointlike, and $|\bar{c}(X_j)| < |B|$. By the induction hypothesis applied to X_j, we conclude that $\mathcal{C}(X_j, a)$ holds for all $a \in \bar{c}(X_j)$. In particular all X_j belong to $C_{\mathsf{R}}^{\omega}(S)$. Now, $F \subseteq X_1\psi(a_1) \cdots X_m\psi(a_m)$, which shows $\mathcal{C}(F,a)$ if $a \in \{a_1, \dots, a_n\}$. Otherwise, let $\ell \in \{1, \dots, m\}$ be such that $a \in c(\bar{z}_\ell)$. Then, by induction hypothesis there are $X', Y' \in C_{\mathsf{R}}^{\omega}(S)$ such that $X_\ell = X'\psi(a)Y'$. Hence $F \subseteq X_a\psi(a)Y_a$ for $X_a = X_1\psi(a_1) \cdots X_{\ell-1}\psi(a_{\ell-1})X'$ and $Y_a = Y'\psi(a_\ell)X_{\ell+1}\psi(a_{\ell+1}) \cdots X_m\psi(a_m)$. This proves (i) since $X_a, Y_a \in C_{\mathsf{R}}^{\omega}(S)$.

From Lemma 3.12, we deduce that, if $w \in B^+$, then $F \cup F\psi(w) \in C_{\mathsf{R}}^{\omega}(S)$. By Lemma 3.11, it follows that $F\psi(B^+) = F \cup \bigcup_{w \in B^+} F\psi(w) \in C_{\mathsf{R}}^{\omega}(S)$ since $\psi(B^+)$ is a finite set. This proves (ii), and by the above reductions, this completes the induction step and proves the theorem. $\square$

4. An algorithm to compute J-pointlike sets

In this section, we describe an algorithm to compute J-pointlike subsets of a finite semigroup S. While the algorithm for R consists in replacing $\mathcal{H}$ by $\mathcal{R}$ in Henckell's construction, replacing $\mathcal{H}$ by $\mathcal{J}$ does not work, as explained in Section 5. Recall from Subsection 3.1 that, taking an expansion if necessary, one can assume that S has a content homomorphism.

A well-known characterization of equality of idempotents over J [1] states that, given two pseudowords $x, y \in \overline{\Omega}_A\mathsf{S}$, x^ω and y^ω have the same projection onto J if and only if $c(x) = c(y)$. Furthermore, for all $z \in \overline{\Omega}_A\mathsf{S}$ such that $c(z) \subseteq c(x)$, we have $\mathsf{J} \models zx^\omega = x^\omega = x^\omega z$. Using these properties, one immediately deduces that a set $F \subseteq S$ of idempotents is J-pointlike if and only if all elements of F have the same content. Moreover, for such a set F, we have $\psi(B^+)F\psi(B^+) \in \mathcal{P}_{\mathsf{J}}(S)$, a result analogous to Lemma 3.8.

We also have a notion of $\mathcal{J}$-canonical factorization of a pseudoword, which plays here the same role as the factorizations of Corollary 2.5 or Proposition 3.13 for R.

Theorem 4.1 ([1], [2, Theorem 8.1.11])**.** *Every pseudoword $x \in \overline{\Omega}_A\mathsf{S}$ has a factorization $x = x_1 \cdots x_k$, called* $\mathcal{J}$-canonical*, satisfying the following properties:*

– *for every $i = 1, \dots, k$, either $x_i \in A^+$ or $p_{\mathsf{J}}(x_i)$ is idempotent;*
– *x_i and x_{i+1} are not both in A^+;*
– *if $p_{\mathsf{J}}(x_i)$ and $p_{\mathsf{J}}(x_{i+1})$ are idempotent, then $c(x_i)$ and $c(x_{i+1})$ are not comparable;*

– *if $p_{\mathsf{J}}(x_i)$ is idempotent and x_{i+1} (resp. x_{i-1}) is in A^+, then the first (resp. the last) letter of x_{i+1} (resp. x_{i-1}) does not belong to $c(x_i)$.*

Moreover, if $x = x_1 \cdots x_k$ and $y = y_1 \cdots y_\ell$ are $\mathcal{J}$-canonical factorizations and if $\mathsf{J} \models x = y$, then $k = \ell$ and $\mathsf{J} \models x_i = y_i$ for all $1 \leqslant i \leqslant k$. This implies that either x_i and y_i are both in A^+, or their projections into J are both idempotent. In the first case, they are equal and in the second case, they have the same content.

Theorem 4.1 makes it possible to repeat for J, *mutatis mutandis*, the proof of Proposition 3.9 to deduce Proposition 4.2 below. Observe however that Proposition 4.2 gives an algorithm to compute J-pointlike sets. This is in contrast with Proposition 3.9, where the candidates for the set F, belonging to $\mathcal{P}_{\mathsf{R}}(S)$, could not be computed directly. The point here is that for J, we know how to characterize and to compute sets of idempotents which are pointlike, just by inspecting the contents of their elements.

Proposition 4.2. *Let S be a finite semigroup with a content homomorphism and $\psi : \overline{\Omega}_A \mathsf{S} \to S$ be an onto continuous homomorphism. Let U be the subsemigroup of $\mathcal{P}(S)$ generated by the singleton subsets together with the sets $\psi(B^+)F\psi(B^+)$, where $B \subseteq A$ and F is a set of idempotents of S, all of them of content B. Then $\mathcal{P}_{\mathsf{J}}(S) = \downarrow U$.*

5. Some examples

5.1. **Behavior of Henckell's construction for J.** For a subsemigroup U of $\mathcal{P}(S)$, denote by $D_{\mathsf{J}}(U)$ the subsemigroup generated by all singleton sets of $\mathcal{P}(S)$ together with the subsets of the form $\bigcup_{X \in J} X$, where J is a $\mathcal{J}$-class of U. Let then $C_{\mathsf{J}}(U) = \downarrow D_{\mathsf{J}}(U)$. Define $C^0_{\mathsf{J}}(S) = \big\{\{s\} : s \in S\big\}$ and, for $n > 0$, let $C^n_{\mathsf{J}}(S) = C_{\mathsf{J}}(C^{n-1}_{\mathsf{J}}(S))$. Finally, let $C^\omega_{\mathsf{J}}(S) = \bigcup_{n \geqslant 0} C^n_{\mathsf{J}}(S)$.

It is tempting to guess that $C^\omega_{\mathsf{J}}(S) = \mathcal{P}_{\mathsf{J}}(S)$. Perhaps surprisingly, this is not the case, as shown by the following counterexample. Let S_1 be the semigroup on two generators a, b given by the following presentation: $(bab)^2 = bab$, $(aba)^2 = aba$, $a^2ba^2 = a^2$, $b^2ab^2 = b^2$, $a^3 = b^3 = (ba)^2 = (ab)^2 = a^2b^2 = b^2a^2 = 0$. Its Green relation structure is summarized in the diagram of Figure 1. Call J_0 and

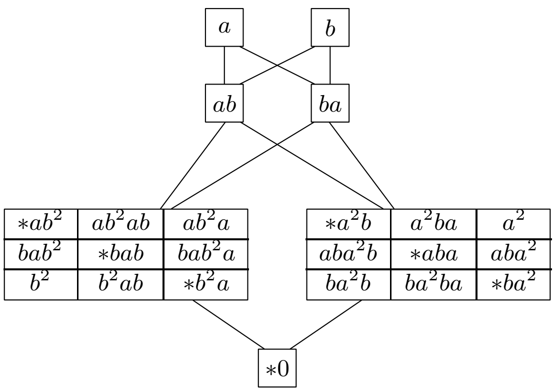

FIGURE 1. The semigroup S_1

J_1 the regular nontrivial $\mathcal{J}$-classes. Then, the subset F of all idempotents of S_1 is J-pointlike since all idempotents have content $\{a, b\}$. Consequently, the subset $X = S_1 \setminus \{a, b, ab, ba\} = J_0 \cup J_1 \cup \{0\}$ is also J-pointlike, because it is obtained by multiplying F by elements of content contained in $\{a, b\}$. On the other hand, one can compute $C^\omega_{\mathsf{J}}(S_1)$. By definition, $D_{\mathsf{J}}(C^0_{\mathsf{J}}(S_1))$ is the subsemigroup of $\mathcal{P}(S_1)$

generated by the singletons and the $\mathcal{J}$-classes of S_1. For $\ell = 0, 1$, multiplying an element from J_ℓ by any element of S_1 yields an element of $J_\ell \cup \{0\}$. Hence $C^1_{\mathsf{J}}(S_1) \subseteq {\downarrow}\big\{\{a\}, \{b\}, \{ab\}, \{ba\}, J_0 \cup \{0\}, J_1 \cup \{0\}\big\}$. For the same reason, no element of $C^1_{\mathsf{J}}(S_1)$ intersecting J_0 can be $\mathcal{J}$-equivalent with an element intersecting J_1. Therefore, we have $C^2_{\mathsf{J}}(S_1) = C^1_{\mathsf{J}}(S_1) = C^\omega_{\mathsf{J}}(S_1)$ and $X = J_0 \cup J_1 \cup \{0\} \in \mathcal{P}_{\mathsf{J}}(S_1) \setminus C^\omega_{\mathsf{J}}(S_1)$.

5.2. **Subsemigroup of $\mathcal{P}(S)$ generated by $\mathcal{P}_{\mathsf{R}}(S)$ and $\mathcal{P}_{\mathsf{L}}(S)$.** Another question is whether $\mathcal{P}_{\mathsf{J}}(S) = {\downarrow}\langle \mathcal{P}_{\mathsf{R}}(S) \cup \mathcal{P}_{\mathsf{L}}(S)\rangle$. The answer is negative, as is again witnessed by the semigroup S_1 of Figure 1. Since $\mathsf{J} \subseteq \mathsf{R} \cap \mathsf{L}$, we have ${\downarrow}\langle \mathcal{P}_{\mathsf{R}}(S) \cup \mathcal{P}_{\mathsf{L}}(S)\rangle \subseteq \mathcal{P}_{\mathsf{J}}(S)$ for all S. On the other hand, we claim that $J_0 \cup J_1 \cup \{0\} \notin {\downarrow}\langle \mathcal{P}_{\mathsf{R}}(S_1) \cup \mathcal{P}_{\mathsf{L}}(S_1)\rangle$. Let indeed $\{s_0, s_1\} \in \mathcal{P}_{\mathsf{R}}(S_1)$ with $s_0 \neq s_1$, and let u_i be an element of $\overline{\Omega}_A\mathsf{S}$ projecting to s_i and such that $p_{\mathsf{R}}(u_0) = p_{\mathsf{R}}(u_1)$. In particular, u_0 and u_1 have the same prefix of length 4. This implies that their images in S_1 lie in the same ideal $J_0 \cup \{0\}$ or $J_1 \cup \{0\}$. Dually, no L-pointlike can intersect both J_0 and J_1. Therefore, this property also holds for elements of ${\downarrow}\langle \mathcal{P}_{\mathsf{R}}(S_1) \cup \mathcal{P}_{\mathsf{L}}(S_1)\rangle$, which proves the claim.

5.3. **Pointlike subsets of a join.** In general, being both V and W-pointlike does not entail being $\mathsf{V} \vee \mathsf{W}$-pointlike [26]. The diagram of Figure 2 gives the Green relation structure of a semigroup S_2 with a subset which is both R and L-pointlike but which is not $\mathsf{R} \vee \mathsf{L}$-pointlike. The semigroup S_2 is given by the following presentation: the set of generators is $A = \{a, b, c, d\}$ and the relations are $aba = a$, $bab = b$, $dab = abc$, $da^2b = ab^2c$, $ac = ad = cb = c^2 = cd = db = dc = d^2 = 0$, $a^3 = b^3 = a^2b^2 = b^2a^2 = a^2bd = cab^2 = cabd = 0$. Then

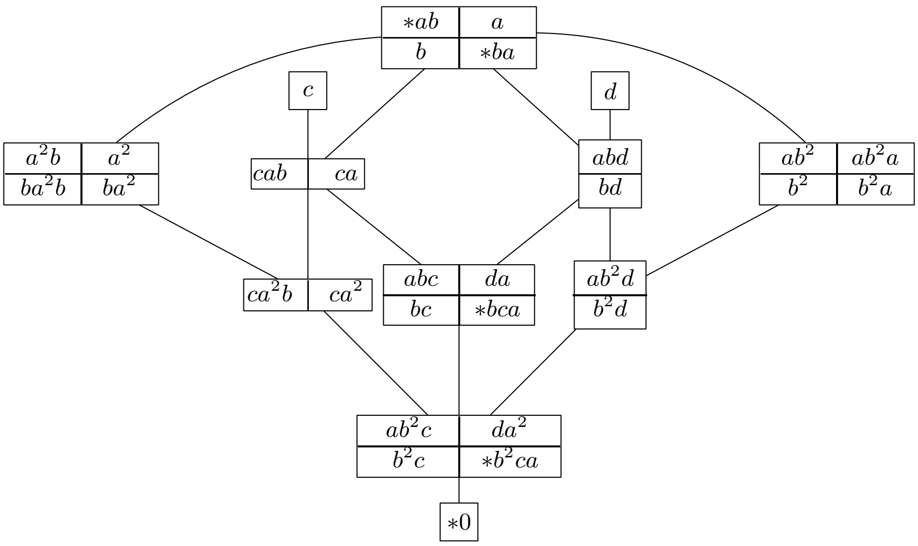

FIGURE 2. The semigroup S_2

$$\{abc, ab^2c\} = \{(ab)^\omega c, (ab)^\omega bc\} \in \mathcal{P}_{\mathsf{R}}(S_2)$$

and

$$\{abc, ab^2c\} = \{dab, da^2b\} = \{d(ab)^\omega, da(ab)^\omega\} \in \mathcal{P}_{\mathsf{L}}(S_2)$$

but $\{abc, ab^2c\} \notin \mathcal{P}_{\mathsf{R}\vee\mathsf{L}}(S_2)$ since, for the natural continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{S} \to S_2$,

$$\varphi^{-1}(abc) = \overline{(ab)^+}\, c\, \overline{(ab)^*} \cup \overline{(ab)^*}\, d\, \overline{(ab)^+}$$
$$\varphi^{-1}(ab^2c) = \overline{(ab)^+}\, bc\, \overline{(ab)^*} \cup \overline{(ab)^*}\, da\, \overline{(ab)^+}$$

where $\overline{L}$ denotes the topological closure of L in $(\overline{\Omega}_A\mathsf{S})^1$. Indeed, by a result of the first author and Azevedo [6] (see [2, Theorem 9.2.13]), there is no pseudoidentity valid in $\mathsf{R}\vee\mathsf{L}$ in which one side belongs to $\varphi^{-1}(abc)$ and the other to $\varphi^{-1}(ab^2c)$, and so the set $\{abc, ab^2c\}$ is not pointlike with respect to the relational morphism $\mu_{\mathsf{R}\vee\mathsf{L}}$.

5.4. **An example where $C^1_{\mathsf{R}}(S)$ differs from $C^{\omega}_{\mathsf{R}}(S)$.** Our algorithm for computing R-pointlike sets does not stop, in general, after the first iteration. An example is given by the semigroup S_3 whose Green relation structure is given in Figure 3. A presentation on $\{a,b\}$ is $a^3 = a$, $b^2 = (ba)^2ab = ba(ab)^2 = 0$, $(ba)^2b = bab = a(ab)^2$, $(ba^2)^2b = ba^2b$.

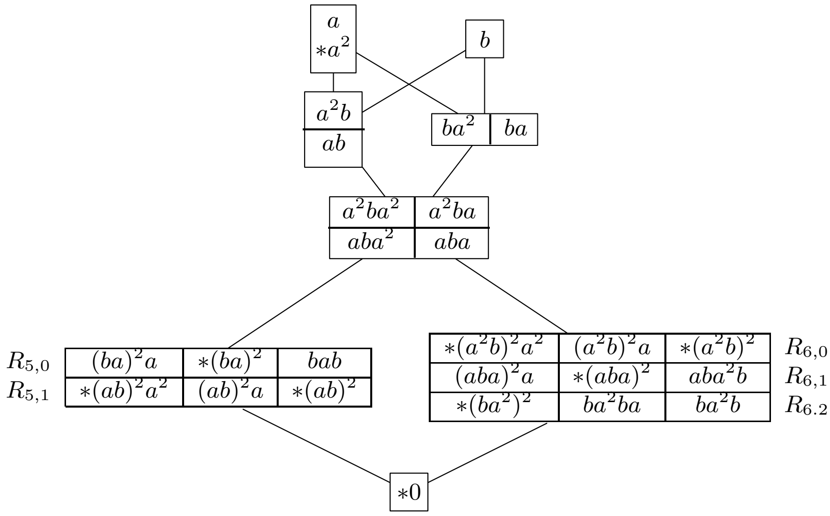

FIGURE 3. The semigroup S_3

By definition, the elements of $C^1_{\mathsf{R}}(S)$ are the subsets of elements of the semigroup generated by the singletons and the $\mathcal{R}$-classes. One can check that it is exactly made up of the subsets of the $\mathcal{R}$-classes and of the following nine subsets of the semigroup S_3:

$$
\begin{aligned}
J_2 &= J_0J_1\\
J_4 &= J_2J_3\\
(J_2R_{6,0})^2 &= \{0\}\cup R_{6,0}\cup R_{6,1}\\
R_{6,2}R_{6,2} &= R_{6,2}\cup\{0\}\\
(J_1J_4)^2 &= \{0, ba^2ba, (ba^2)^2, (ba)^2, (ba)^2a\}\\
X &= (J_4J_1)J_2 = \{0, bab, ba^2b\}\\
J_0X &= \{0, (a^2b)^2, aba^2b, (ab)^2, bab\}\\
J_0XJ_0 &= \{0, (a^2b)^2a, (a^2b)^2a^2, aba^2ba, a(ba^2)^2, (ab)^2a, (ab)^2a^2, (ba)^2, (ba)^2a\}\\
J_0XR_{5,2} &= \{0, (ab)^2, (ab)^2a, (ab)^2a^2, bab, (ba)^2, (ba)^2a\}.
\end{aligned}
$$

However, it does not contain $\{(ab)^2a, (a^2b)^2\}$, which is R-pointlike since $(ab)^2a = (a^{\omega+1}b)^{\omega}a$ and $(a^2b)^2 = (a^{\omega}b)^{\omega}$, and $\mathsf{R}\models(a^{\omega+1}b)^{\omega}a = (a^{\omega}b)^{\omega}$.

It should be possible to use the same idea to show that, for every $n\geqslant 0$ there exists a finite semigroup S for which $C^n_{\mathsf{R}}(S)\neq C^{\omega}_{\mathsf{R}}(S)$, but we have not attempted to prove it.

6. Conclusion

In this paper, we have presented two algorithms which can be used to test whether a subset X of a finite semigroup is R or J-pointlike. Both algorithms work by generating pointlike subsets until X is found or all pointlike subsets have been generated. We do not know whether there are more efficient algorithms, whose complexity would depend also on X. One possible track would be to compute the closures of the preimages of elements of X in $\overline{\Omega}_A\mathsf{S}$ and testing emptiness of the intersection of their projections on R or J.

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Centro de Matemática e Departamento de Matemática Pura, Faculdade de Ciências,, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal.
E-mail address: jalmeida@fc.up.pt

Centro de Matemática, Universidade do Minho, Campus de Gualtar, 4700-320 Braga, Portugal.
E-mail address: jcosta@math.uminho.pt

LaBRI, Université Bordeaux 1 & CNRS UMR 5800. 351 cours de la Libération, 33405 Talence Cedex, France.
E-mail address: mz@labri.fr