SYMBOLIC DYNAMICS IN FREE PROFINITE SEMIGROUPS

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Abstract. This is a survey and announcement of recent results on the structure of free profinite semigroups using techniques and results from symbolic dynamics. The intimate connection between uniformly recurrent infinite words and $\mathcal{J}$-maximal regular $\mathcal{J}$-classes is explored to compute the maximal subgroups of the $\mathcal{J}$-classes associated with Arnoux-Rauzy infinite words, which turn out to be free profinite groups whose rank is the number of letters involved.

1. Introduction

Given a pseudovariety V of semigroups and a finite set A, the associated relatively free profinite semigroup $\overline{\Omega}_A\mathsf{V}$ is an object which encodes in its topological and algebraic structures and the interplay between them the common properties of A-generated members of V. This fact comes basically out of any definition of $\overline{\Omega}_A\mathsf{V}$ and is behind the usefulness of relatively free profinite semigroups in finite semigroup theory and its applications. There are by now several works reviewing various aspects of this role played by such profinite semigroups, even when they are not explicitly mentioned by this name [1, 8, 6, 22, 4].

While this is generally a hard problem, for some pseudovarieties, a complete structural description or at least a substantial knowledge of $\overline{\Omega}_A\mathsf{V}$ has been achieved (see, for instance, [1, 9, 14, 21]). But for most pseudovarieties very little is known about their free profinite semigroups. This is the case in particular for absolutely free profinite semigroups $\overline{\Omega}_A\mathsf{S}$, which are the object of this paper. Since the subsemigroup of $\overline{\Omega}_A\mathsf{S}$ generated by A is the free semigroup on A, whose elements are generally known as *words*, for shortness we call the elements of $\overline{\Omega}_A\mathsf{S}$ *pseudowords*; they have also been called *profinite words* [5, 7].

The author has discovered some promising connections with symbolic dynamics which have already proved to be fruitful [3, 2]. Surely a lot remains to be done in this direction particularly since all developments in this area are rather recent. The present paper is both a research announcement and a survey of results pertaining specifically to the structure of free profinite semigroups. Since several results remain hitherto unpublished and have only been announced in conferences and workshops, the main purpose of this paper is to put in print a collection of statements that may contribute for other researchers to take advantage of the results and perhaps join the research effort made by the author.

This paper contains almost no proofs. For results for which no reference is given, detailed proofs will appear in forthcoming papers. We collect in Section 2 a number of preliminaries which are required for understanding the rest of the paper, including a self-sufficient introduction to relatively free profinite semigroups. Section 3 presents a number of results concerning uniformly recurrent pseudowords, which form maximal $\mathcal{J}$-classes not containing finite words and which are regular. The main results presented in the paper concern the identification of the structure of the maximal subgroups in these $\mathcal{J}$-classes, which turn out to be free profinite groups in some

Work supported, in part, by *Fundação para a Ciência e a Tecnologia* (FCT) through the *Centro de Matemática da Universidade do Porto*, by the FCT and POCTI approved project POCTI/32817/MAT/2000 which is partially funded by the European Community Fund FEDER, and by the INTAS grant #99-1224. Address: Centro de Matemática da Universidade do Porto, Departamento de Matemática Pura, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal.

particularly natural classes of examples. In Section 3 this is stated for minimal complexity non-quasi-periodic pseudowords, called *Sturmian pseudowords*. This is generalized in Section 4 to *Arnoux-Rauzy pseudowords* and the proof that Arnoux-Rauzy maximal subgroups are free profinite groups is sketched through the statement of intermediate results which may be of independent interest.

Section 5 reviews results on the other extreme of free profinite semigroups, namely the minimal ideal, which were obtained jointly with M. V. Volkov using another dynamical systems parameter, the entropy. Finally, we propose in Section 6 some open problems suggested by this research.

2. Preliminaries

2.1. **Relatively free profinite monoids.** We begin with an elementary remark on the topology of products of finite spaces. Recall that an *ultrametric* on a set X is a function d defined on $X \times X$ with nonnegative real values such that

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

Proposition 2.1. *Let $X = \prod_{i \in I} M_i$ be the Cartesian product of a family of finite sets with at least 2 elements and define a real-valued function on $X \times X$ by letting $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2^{-r}$ if r is the minimum size $|M_i|$ of a factor M_i such that x and y differ in the ith component. Then d is an ultrametric on X. The induced topology coincides with the product topology for the discrete topologies on the factors M_i if and only if, for each $r \geq 2$ the set $I_r = \{i \in I : |M_i| < r\}$ is finite.*

Proof. The verification of properties (1) and (2) of the definition of ultrametric is immediate. For (3), just note that if $x, y, z \in X$ are such that x and y coincide in all components i with $|M_i| \leq r$ and y and z coincide in all components i with $|M_i| \leq s$, then x and z certainly coincide in all components i such that $|M_i| \leq \min\{r, s\}$. Hence d is an ultrametric.

Suppose that every subset I_r is finite ($r \geq 1$). Given $r \geq 2$ and $x \in X$, the open ball $B_r(x)$ of radius 2^{-r} centered at x consists of all $y \in X$ which coincide with x in all components i with $i \in I_r$, a finite set of indices, and therefore $B_r(x)$ is open in the product topology. On the other hand, if we fix $i_0 \in I$, choose $a \in M_{i_0}$, and take $r = |M_{i_0}| + 1$ then, by the finiteness assumption on the I_r, there is a finite subset F of X such that the members of F use all possible i-components for every $i \in I_r \setminus \{i_0\}$ and whose i_0-component is a. Since the set of all $x \in X$ whose i_0-component is a coincides with the union $\bigcup_{y \in F} B_r(y)$, it follows that the basic open sets of the product topology are open in the topology induced by d. Hence the two topologies coincide.

Conversely, assume that the two topologies coincide and consider $r \geq 2$. Then the corresponding quotient topologies on the product X_r of all M_i with $i \in I_r$ also coincide. The quotient topology for the product topology on X is the product topology on X_r. In turn the quotient topology for the topology on X induced by the ultrametric d is discrete since balls of the form $B_r(x)$ project to singletons. Hence X_r is finite. Since we have excluded singleton and empty factors, we deduce that I_r is finite. □

Let V be a *pseudovariety of finite monoids*, that is a class of finite monoids which is closed under taking homomorphic images, submonoids and finite direct products. Consider a finite set A, whose members are called *letters*, and say that a monoid M is *A-generated* if a function $\varphi : A \to M$ is fixed whose image generates M. In general we omit reference to the generating function φ. A *homomorphism of A-generated monoids* is a monoid homomorphism $h : M \to N$

such that the image of each letter a in M is mapped to its image in N. Let V_0 be a subset of V which contains exactly one monoid from each isomorphism class of A-generated members of V. Since on a given finite set there are only finitely many monoid structures that one may define, the condition of Proposition 2.1 is verified for the product $\prod_{M\in\mathsf{V}_0} M$ and so the product topology is induced by the corresponding ultrametric defined in the proposition. Since the multiplication is continuous on each finite monoid for the discrete topology, multiplication is continuous in the product monoid $\prod_{M\in\mathsf{V}_0} M$ and so this product is a *topological monoid*. As a product of (Hausdorff) compact and zero-dimensional spaces (that is, spaces which admit bases of clopen sets), $\prod_{M\in\mathsf{V}_0} M$ is a compact zero-dimensional space and therefore so is every closed subspace.

Now consider the free monoid A^* on the set A and the unique homomorphism $\iota : A^* \to \prod_{M\in\mathsf{V}_0} M$ which maps each letter a to the element whose M-component is the corresponding generator of M ($M \in \mathsf{V}_0$). It is a well-known result of Birkhoff [12] that the image of ι is the V-free monoid on the set A, which we denote $\Omega_A\mathsf{V}$. The closure of $\Omega_A\mathsf{V}$ in $\prod_{M\in\mathsf{V}_0} M$ is a compact monoid which we denote $\overline{\Omega}_A\mathsf{V}$. It is the completion of $\Omega_A\mathsf{V}$ with respect to the ultrametric of Proposition 2.1 and it may also be seen directly as the completion of A^* with respect to the pseudo-ultrametric d_V defined by $d_\mathsf{V}(u,v) = 2^{-r}$, where r is the minimum size of a monoid $M \in \mathsf{V}$ for which there exists a homomorphism $\varphi : A^* \to M$ such that $\varphi(u) \neq \varphi(v)$, in case there is such a homomorphism, or $d_\mathsf{V}(u,v) = 0$ otherwise. Note that there exists such a homomorphism φ if and only if $u \neq v$ in $\Omega_A\mathsf{V}$. Note also that the identification of $\overline{\Omega}_A\mathsf{V}$ as a completion provides a characterization of $\overline{\Omega}_A\mathsf{V}$ which is independent of the choice of V_0.

The monoid $\overline{\Omega}_A\mathsf{V}$ is therefore a compact monoid whose topology is zero-dimensional and which is residually in V in the sense that continuous homomorphisms into members of V suffice to separate points. A topological monoid with these properties is called a *pro-*V *monoid*. Note that the finite pro-V monoids are the elements of V. By a *profinite monoid* we mean a pro-V monoid with respect to the pseudovariety of all finite monoids.

It turns out that $\overline{\Omega}_A\mathsf{V}$ is the *free pro-*V *monoid on* A in the sense that the function ι satisfies the following universal property: given any mapping $\varphi : A \to M$ into a pro-V monoid there is a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_A\mathsf{V} \to M$ such that $\hat{\varphi} \circ \iota = \varphi$. In case M is finite, the existence of such a continuous homomorphism is established by taking the projection on the component in the product $\prod_{N\in\mathsf{V}_0} N$ isomorphic to the submonoid of M generated by the image of φ. The general case follows by observing that every pro-V monoid embeds, as a topological monoid, in a product of members of V. We thus obtain another abstract characterization of $\overline{\Omega}_A\mathsf{V}$, although in this way we do not establish the existence of such a structure.

2.2. **Examples of relatively free profinite structures.** In the above we haven chosen to work with monoids because this is close to the main topics of this paper. But we could as well had worked with any finite algebraic signature involving only finitary operations. In particular, we could had considered semigroups instead.

The following are important examples of pseudovarieties and some of their associated free profinite structures.

Example 2.2. Take

$$\mathsf{K} = \bigcup_{n\geq 1} [\![x_1 \cdots x_n y = x_1 \cdots x_n]\!] = [\![ex = e]\!],$$

the pseudovariety consisting of all finite semigroups in which every sufficiently long product is a left-zero or, equivalently, every idempotent is a left-zero. (See [1] for a more thorough explanation of the notation.) Then the K-free semigroup on a finite set A is the free semigroup A^+ and the ultrametric d_K described above is topologically equivalent to the *longest common prefix* metric appearing in symbolic dynamics which is defined by $d(u,v) = 2^{-r}$ if $u \neq v$ and r is the length of the longest common prefix of u and v, with $d(u,v) = 0$ otherwise. Indeed, if s_r is the number of elements of the $[\![x_1 \cdots x_r y = x_1 \cdots x_r]\!]$-free semigroup S_r on A, then $d(u,v) < 2^{-r}$ implies $d_{\mathsf{K}}(u,v) < 2^{-s_r}$ since two words are equal in S_r if and only if they have the same prefixes of length at most r. On the other hand, if $d(u,v) \geq 2^{-r}$, $S \in \mathsf{K}$ has less than r elements, and $\varphi : A^+ \to S$ is a homomorphism, then the images of u and v under φ are both left-zeros in S which are determined by their prefixes of length at most r and so $\varphi(u) = \varphi(v)$, which shows that $d_{\mathsf{K}}(u,v) \geq 2^{-r}$.

Now, it is immediate to show that Cauchy sequences in A^+ with respect to d are in bijective correspondence with infinite words on the alphabet A. Denoting by A^ω the set of all such infinite words, we conclude that $\overline{\Omega}_A\mathsf{K}$ is isomorphic to the topological semigroup $A^+ \cup A^\omega$, where multiplication is concatenation of words except that infinite words are taken to be left-zeros, and the topology is that determined by the longest common prefix metric.

Example 2.3. Let G_p denote the pseudovariety consisting of all finite p-groups, where p is a prime integer. For a singleton alphabet $A = \{a\}$, $\overline{\Omega}_A\mathsf{G}_p$ is the *free cyclic pro-p-group*. The mapping $\varphi : \mathbb{Z} \to \overline{\Omega}_A\mathsf{G}_p$ sending n to a^n is a group embedding of the additive group of integers such that its image is dense in $\overline{\Omega}_A\mathsf{G}_p$. Moreover, the ultrametric induced on $\mathbb{Z}$ is such that, if $m \neq n$ then $d(m,n) = 2^{-r}$ where r is the greatest nonnegative integer such that m and n are congruent modulo p^r. This is one possible way of defining a p-adic metric on $\mathbb{Z}$. Hence $\overline{\Omega}_A\mathsf{G}_p$ is isomorphic to $\mathbb{Z}_p$, the p-adic completion of the integers.

Other relevant examples for this paper are the following:

- The pseudovariety G of all finite groups; there is an extensive theory of free profinite groups $\overline{\Omega}_A\mathsf{G}$ with connections with number theory and logic, among other areas [16, 21].
- The pseudovariety S of all finite semigroups; free profinite semigroups $\overline{\Omega}_A\mathsf{S}$ play an important role in the theory of finite semigroups [1, 8, 4].
- The pseudovariety M of all finite monoids; it is easy to see that the free profinite monoid $\overline{\Omega}_A\mathsf{M}$ is obtained from $\overline{\Omega}_A\mathsf{S}$ by adjoining an identity element which topologically is an isolated point.

The elements of $\overline{\Omega}_A\mathsf{S}$ or of $\overline{\Omega}_A\mathsf{M}$ will be called *pseudowords*. Recall that pseudowords are limits of sequences of (finite) words. Pseudowords from $\overline{\Omega}_A\mathsf{S} \setminus A^+$ are said to be *infinite*. More generally, elements of $\overline{\Omega}_A\mathsf{V} \setminus \Omega_A\mathsf{V}$ will be said to be *infinite*.

Given two pseudovarieties of semigroups V and W with $\mathsf{V} \subseteq \mathsf{W}$, every pro-$\mathsf{V}$ semigroup is also a pro-W semigroup. Hence the natural mapping $\iota_{\mathsf{V}} : A \to \overline{\Omega}_A\mathsf{V}$ induces a unique continuous homomorphism $p : \overline{\Omega}_A\mathsf{W} \to \overline{\Omega}_A\mathsf{V}$ which is onto since the image of ι_{V} generates a dense subsemigroup. We call p the *natural projection*.

In particular, the existence of the natural projection $\overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{K}$ implies that for every infinite pseudoword w there is a unique accumulation point in A^ω of the sequence $(w_n)_n$ of its finite prefixes, which we will call the *infinite prefix* of w.

2.3. **Local structure of semigroups.** We recall from semigroup theory the fundamental *Green relations*. For a semigroup S denote by S^1 either S if S is a monoid or $S \cup \{1\}$ where 1 is an added identity element. For $s,t \in S$, write $s \leq_{\mathcal{J}} t$ if $s \in S^1tS^1$ that is if t can be found as a factor in some factorization of s. Write $s \leq_{\mathcal{R}} t$ if $s \in tS^1$ and $s \leq_{\mathcal{L}} t$ if $s \in S^1t$. The relations $\leq_{\mathcal{J}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are quasi-orders on S. Also consider the associated equivalence relations $\mathcal{J} = {\leq_{\mathcal{J}}} \cap {\geq_{\mathcal{J}}}$, $\mathcal{L} = {\leq_{\mathcal{L}}} \cap {\geq_{\mathcal{L}}}$, $\mathcal{R} = {\leq_{\mathcal{R}}} \cap {\geq_{\mathcal{R}}}$, $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$ and $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. Using associativity one shows that $\mathcal{R}$ and $\mathcal{L}$ commute under relation composition and so $\mathcal{D}$ is given by their composite.

An element s of a semigroup S is said to be *regular* if there exists $t \in S$ such that $sts = s$. A subset of S is said to be *regular* if all its elements are regular. More generally, a subset S is said to have property $\mathcal{P}$ if all its elements have property $\mathcal{P}$. It is well-known that the regular $\mathcal{H}$-classes of S are its maximal subgroups.

If a topological semigroup S is compact then the following facts are well-known:

(1) $\mathcal{D} = \mathcal{J}$;
(2) $s \leq_{\mathcal{R}} t$ and $s \mathrel{\mathcal{J}} t$ implies $s \mathrel{\mathcal{R}} t$;
(3) $s \leq_{\mathcal{L}} t$ and $s \mathrel{\mathcal{J}} t$ implies $s \mathrel{\mathcal{L}} t$;
(4) a regular $\mathcal{D}$-class is isomorphic, as a partial subsemigroup, to a *Rees matrix (partial) semigroup* $\mathcal{M}(I, G, \Lambda; P) = I \times G \times \Lambda$ with multiplication

$$(i_1, g_1, \lambda_1)(i_2, g_2, \lambda_2) = (i_1, g_1 P(\lambda_1, i_2) g_2, \lambda_2)$$

where I and Λ are compact spaces, G is a compact group, and $P : \Lambda \times I \to G$ is a partially-defined continuous function;
(5) S has a unique minimal ideal which has the structure described in (4) where P is a fully-defined function.

2.4. **Iterated substitutions.** For an element s of a finite semigroup S, the sequence $(s^{n!})_n$ is eventually constant and its limit is the unique idempotent power of s. Since profinite monoids embed in products of finite monoids, it follows that if S is a profinite semigroup and $s \in S$ then the sequence $(s^{n!})_n$ converges to an idempotent, namely the unique idempotent in the closed subsemigroup generated by s. This idempotent is denoted s^ω. More generally, any accumulation point in S of the subsemigroup generated by s is denoted in the form s^ν where ν is called an *infinite exponent*. Another example of an infinite exponent is $\omega - 1$ defined by $s^{\omega-1} = \lim s^{n!-1}$ or, in structural terms, the inverse of ss^ω in the maximal subgroup of the closed subsemigroup generated by s.

We say that a profinite monoid M is *finitely generated* if there is a finite subset of M which generates a dense submonoid of M. Recall that the *pointwise convergence topology* of a set $\mathcal{F}$ of functions $Y \to X$, where X is a topological space, is the subspace topology for $\mathcal{F}$ viewed as a subset of the product space X^Y.

For a profinite monoid M, let $\operatorname{End} M$ denote the set of all continuous endomorphisms of M. This is clearly a monoid under composition but it is not immediately apparent which topology should be taken on $\operatorname{End} M$ in order to make it a relevant topological monoid.

Theorem 2.4. *Let M be a profinite monoid.*

(a) If M is finitely generated then $\operatorname{End} M$ *is a profinite monoid under the pointwise convergence topology and the evaluation mapping*

$$\begin{aligned} \eta : \operatorname{End} M \times M &\to M \\ (\varphi, m) &\mapsto \varphi(m) \end{aligned}$$

is continuous.

(b) If $\operatorname{End} M$ *is a profinite monoid with a continuous evaluation mapping η then the topology of* $\operatorname{End} M$ *is the pointwise convergence topology.*

In particular, for a finite set A, $\operatorname{End} \overline{\Omega}_A \mathsf{V}$ is a profinite monoid under the pointwise convergence topology. Hence, if B is another finite alphabet, then any mapping $B \to \operatorname{End} \overline{\Omega}_A \mathsf{V}$ extends to a unique continuous homomorphism $\overline{\Omega}_B \mathsf{S} \to \operatorname{End} \overline{\Omega}_A \mathsf{V}$. Elements of $\operatorname{End} \overline{\Omega}_A \mathsf{V}$ may also be called *substitutions*. A substitution $f \in \operatorname{End} \overline{\Omega}_A \mathsf{V}$ is said to be *primitive* if there exists a positive integer k such that, for every $a \in A$, the word $f^k(a)$ involves all letters from A.

Example 2.5. The *Arnoux-Rauzy* homomorphism

$$\begin{aligned} \rho : \overline{\Omega}_A \mathsf{S} &\to \operatorname{End} \overline{\Omega}_A \mathsf{S} \\ w &\mapsto \rho_w \end{aligned}$$

is defined by the following formula for $a, b \in A$:

$$\rho_a(b) = \begin{cases} a & \text{if } a = b \\ ab & \text{otherwise.} \end{cases}$$

For instance, the infinite prefix of the *Fibonacci pseudoword* $\rho_{(ab)^\omega}(a)$ is the usual *Fibonacci word*, which starts with *abaababaabaababaababaabaababa*..., and the infinite prefix of $\rho_{(abc)^\omega}(a)$ is known as the *Tribonacci word abacabaabacababacabaabacabacabaabacababacaba*... [19, 11], and so we will also call $\rho_{(abc)^\omega}(a)$ the *Tribonacci pseudoword*.

2.5. **Some tools and notions from symbolic dynamics.** Let V be a pseudovariety of semigroups such that $\Omega_A\mathsf{V}$ is isomorphic with A^+. For $w \in \overline{\Omega}_A\mathsf{V}$, we introduce the following sets of finite factors of w, where A^n denotes the set of all words in A^+ of length n:

$$\begin{aligned} F_n(w) &= \{u \in A^n : w \in (\overline{\Omega}_A\mathsf{V})^1 u (\overline{\Omega}_A\mathsf{V})^1\} \\ L_n(w) &= \{u \in A^n : w \in A^* u (\overline{\Omega}_A\mathsf{V})^1\} \\ R_n(w) &= \{u \in A^n : w \in (\overline{\Omega}_A\mathsf{V})^1 u A^*\} \end{aligned}$$

We also write $c(w)$ for $F_1(w)$ and we call it the *content* of w. For $O \in \{F, L, R\}$ we also let $O(w) = \bigcup_{n \geq 1} O_n(w)$. The associated *complexity* functions are given by

$$q_w(n) = |F_n(w)|, \ p_w(n) = |L_n(w)|, \ \tilde{p}_w(n) = |R_n(w)|.$$

Note that $F_n(w)$ is a $\mathcal{J}$-class invariant while $L_n(w)$ is an $\mathcal{R}$-class invariant and, dually, $R_n(w)$ is an $\mathcal{L}$-class invariant. A factor u of w such that $w \in A^* u (\overline{\Omega}_A\mathsf{V})^1$ is said to be *within finite distance from the left*, or simply an *ℓ-factor*.

We say that $w \in \overline{\Omega}_A\mathsf{V}$ is

- *recurrent* if every finite factor of w is a factor of every infinite factor of w;
- *left recurrent* if every finite ℓ-factor of w is an ℓ-factor of every infinite ℓ-factor of w;
- *uniformly recurrent* if, whenever $u \in F(w)$, there exists a positive integer N such that $v \in F_N(w)$ implies $u \in F(v)$, that is, every finite factor can be found within every sufficiently long finite factor;
- *left uniformly recurrent* if, for every $u \in L(w)$, there exists a positive integer N such that $v \in L_N(w)$ implies $u \in L(v)$.

By compactness, (left) uniform recurrence implies (respectively left) recurrence and uniform recurrence implies left uniform recurrence but none of the reverse implications hold. Since no infinite word $w \in A^\omega \subseteq \overline{\Omega}_A\mathsf{K}$ on a non-singleton alphabet A is uniformly recurrent in the above sense, to avoid confusion with the terminology, it should be noted that we will say that an infinite word is left uniformly recurrent when in symbolic dynamics it is said to be uniformly recurrent.

Consider the *shift function* $A^\omega \to A^\omega$ which associates with an infinite word the infinite word which is obtained by dropping the first letter. A *symbolic dynamical system*, also known as *shift space*, *subshift*, or simply *shift*, is a topologically closed subset of A^ω which is also closed under the shift function. A *minimal shift* is a shift which is not properly contained in any other shift. It is well known that minimal shifts are precisely the topological closures of orbits of left uniformly recurrent infinite words under the shift function [13]. They are in bijection with minimal sets of finite words $X \subseteq A^+$ which are closed under taking factors and such that every word in X is a proper prefix of another word in X.

3. Results

We proceed to survey the results on the structure of free profinite semigroups $\overline{\Omega}_A\mathsf{S}$ which have recently been obtained using connections with symbolic dynamics. Most of these results are part of work which is not yet available in written form and so no references can yet be given. When no references are given, the results are due to the author and are being announced here in written form for the first time.

The first result states that left uniformly recurrent pseudowords occupy a rather special place in $\overline{\Omega}_A\mathsf{S}$.

Theorem 3.1. *An infinite pseudoword $w \in \overline{\Omega}_A\mathsf{S}$ is uniformly recurrent if and only if it is $\leq_{\mathcal{J}}$-maximal as an infinite pseudoword.*

Since every infinite pseudoword w admits a factorization of the form $w = xy^\omega z$ [1, Corollary 5.6.2], it follows that uniformly recurrent pseudowords are regular elements of $\overline{\Omega}_A\mathsf{S}$.

In general, the $\mathcal{J}$-class of a pseudoword w is not characterized by its set of finite factors. For instance, $a^\omega ba^\omega$ and $a^\omega ba^\omega ba^\omega$ have the same finite factors, namely the words of the forms a^i and a^iba^j $(i, j \geq 0)$, but they are not $\mathcal{J}$-equivalent since for instance b^2 is a subword of one of them but not of the other (cf. [1]). Since $\mathcal{J}$-maximal infinite pseudowords only have finite pseudowords strictly $\leq_{\mathcal{J}}$-above them, we obtain the following corollary of Theorem 3.1.

Corollary 3.2. *If $v, w \in \overline{\Omega}_A\mathsf{S}$ and w is uniformly recurrent, then $v \mathrel{\mathcal{J}} w$ if and only if $F(v) = F(w)$.*

The connection between uniformly recurrent pseudowords and left uniformly recurrent infinite words is provided by the following easy result.

Theorem 3.3. *The infinite prefix of a uniformly recurrent pseudoword is a left uniformly recurrent infinite word with the same finite factors. Conversely, for a left uniformly recurrent infinite word w, all accumulation points of the sequence of its prefixes are $\mathcal{R}$-equivalent uniformly recurrent pseudowords whose infinite prefix is w and which have the same finite factors as w.*

In terms of dynamical systems, Theorem 3.3 gives the following connection between minimal shifts and the structure of free profinite semigroups.

Corollary 3.4. *Given a minimal shift $\mathcal{S} \subseteq A^\omega$, the accumulation points in $\overline{\Omega}_A\mathsf{S}$ of sequences of finite factors of elements of $\mathcal{S}$ are uniformly recurrent pseudowords in the same $\mathcal{J}$-class $J_{\mathcal{S}}$ and the correspondence $\mathcal{S} \mapsto J_{\mathcal{S}}$ defines a bijection between the set of all minimal shifts $\mathcal{S} \subseteq A^\omega$ and the set of all uniformly recurrent $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{S}$.*

The next result gives criteria for producing uniformly recurrent pseudowords by *iterating ω times* a substitution.

Theorem 3.5. *Let $f \in \operatorname{End}\overline{\Omega}_A\mathsf{S}$ be such that each $f(a)$, with $a \in A$, is a finite word and these words together use up all the letters, that is $c(\prod_{a\in A} f(a)) = A$. Then the following conditions are equivalent:*

1. *f is primitive;*
2. *the $f^\omega(a)$ $(a \in A)$ are all $\mathcal{J}$-equivalent and infinite;*
3. *the $f^\omega(a)$ $(a \in A)$ are all $\mathcal{J}$-equivalent and uniformly recurrent.*

For example, the Fibonacci and Tribonacci pseudowords are uniformly recurrent. Another "classical" example is obtained by considering the substitution $f \in \operatorname{End}\overline{\Omega}_{\{a,b\}}\mathsf{S}$ defined by $f(a) = ab$ and $f(b) = ba$. The pseudoword $f^\omega(a)$ is called the *Prouhet-Thuë-Morse pseudoword*. Its infinite prefix is the usual Prouhet-Thuë-Morse infinite word which starts with $abbabaabbaababbabaababbaabbabaab\ldots$.

The property of being uniformly recurrent turns out to be preserved under finite substitution as stated in the following result.

Theorem 3.6. *Let $f \in \operatorname{End} \overline{\Omega}_A \mathsf{S}$ be such that each $f(a)$ $(a \in A)$ is finite. If $w \in \overline{\Omega}_A \mathsf{S}$ is uniformly recurrent then so is $f(w)$.*

The next result gives a more general way of producing uniformly recurrent pseudowords by using substitutions without just plainly taking ω-powers.

Theorem 3.7. *Let $\sigma : \overline{\Omega}_B \mathsf{S} \to \operatorname{End} \overline{\Omega}_A \mathsf{S}$ be such that, for all $b \in B$ and $a \in A$, $\sigma_b(a)$ is a finite word. Let $w = b_1 b_2 \dots b_n \dots$ be an infinite word over B such that*

$$\text{every finite factor } u \text{ of } w \text{ can be extended, on the right, to a finite factor } u' \text{ of } w \text{ such that } c(\sigma_{u'}(a)) = A \text{ for every } a \in A. \tag{3.1}$$

Furthermore, let $\bar{w}$ be any accumulation point of the sequence $(b_1 \dots b_n)_n$ in $\overline{\Omega}_B \mathsf{S}$. Then the $\sigma_{\bar{w}}(a)$ $(a \in A)$ are all $\mathcal{J}$-equivalent uniformly recurrent pseudowords and their $\mathcal{J}$-class depends only on w and not on $\bar{w}$.

For instance, if $w \in \overline{\Omega}_B \mathsf{K}$ is any infinite word in which every letter occurs infinitely often, then the hypotheses of Theorem 3.7 hold for the Arnoux-Rauzy homomorphism, giving a recipe for producing uniformly recurrent pseudowords which generalizes considerably the construction of the Fibonacci and Tribonacci pseudowords.

To define Sturmian pseudowords, let us consider the complexity functions introduced in Subsection 2.5.

Proposition 3.8 (Almeida and Volkov [7])**.** *The complexity sequences $(p_w(n))_n$, $(q_w(n))_n$ and $(\tilde{p}_w(n))_n$ are all increasing for an infinite pseudoword $w \in \overline{\Omega}_A \mathsf{S}$.*

For a uniformly recurrent pseudoword w, it is easy to show that the three complexity functions coincide.

The following result is the analogue for pseudowords of a classical result of Hedlund and Morse [17] characterizing ultimate periodicity (which is basically the version of the next theorem for $\overline{\Omega}_A \mathsf{K}$). Of course, for infinite pseudowords, which do not just "grow" in one direction, the situation is somewhat more complicated.

Theorem 3.9 (Almeida and Volkov [7])**.** *The following conditions are equivalent for an infinite pseudoword $w \in \overline{\Omega}_A \mathsf{S}$:*

1. *$q_w(n) = q_w(n+1)$ for some n;*
2. *$q_w(n) \leq n$ for some n;*
3. *$\{q_w(n)\}_n$ is bounded;*
4. *$w = xy^\nu z$ for some finite words x, y, z and some infinite exponent ν.*

Thus the minimum complexity for non-*quasi-periodicity* is $q(n) = n+1$. In particular, $q(1) = 2$, that is pseudowords satisfying this condition must involve exactly two letters. In order to avoid uninteresting examples such as $a^\omega b a^\omega$, we further require that the other two complexities are also beyond the minimal threshold for excluding ultimate periodicity from both sides. This leads us to define *Sturmian pseudowords* to be pseudowords w such that $q_w(n) = p_w(n) = \tilde{p}_w(n) = n+1$ for every $n \geq 1$.

The *rank* of a topological semigroup S is the smallest cardinal of a subset A such that the closed subsemigroup generated by A is S. Note that a closed subsemigroup of a profinite group is a subgroup and, therefore, it is a profinite group.

Theorem 3.10. *Sturmian pseudowords are uniformly recurrent. Their $\mathcal{J}$-classes are of the form $\mathcal{M}(C, \overline{\Omega}_{\{a,b\}} \mathsf{G}, C; P)$ where C is the Cantor set and there exist three distinct elements $c_1, c_2, c_3 \in C$ such that $P(c_2, c_1) = P(c_3, c_2) = P(x, x) = 1$ for all $x \in C \setminus \{c_2\}$, and P is undefined otherwise.*

The proof of this result has two kinds of ingredients. One is the identification of the set C and the function P. This is done by defining a *left intercept* for a Sturmian pseudoword as the intercept of its infinite prefix (cf. [19]) and, dually, a *right intercept*. For Sturmian pseudowords in a $\mathcal{J}$-class, the $\mathcal{R}$-class is determined by the left intercept and the $\mathcal{L}$-class is determined by the right intercept. Moreover the product uv of two $\mathcal{J}$-equivalent Sturmian pseudowords u and v lies in the same $\mathcal{J}$-class if and only if the right intercept of u uniquely matches with the left intercept of v, except for some special values which have two matching values, corresponding to so-called *characteristic Sturmian* infinite words [19].

The other ingredient concerns the identification of the structure group, that is of the maximal subgroups of a Sturmian $\mathcal{J}$-class, as free profinite groups of rank 2. In the next section we present extensions of this result as well as a sketch of a proof.

4. Extensions and ingredients in proofs

A finite factor u of a pseudoword $w \in \overline{\Omega}_A\mathsf{S}$ is said to be *right special* of degree n if there are exactly n factors of w of the form ua with $a \in A$. *Left special* factors are defined dually. A *return word* of a finite factor u of a pseudoword w is a finite factor v such that vu is a factor of w and u appears in vu as its prefix and its suffix but nowhere else as a factor. Thus, in a left uniformly recurrent pseudoword w, a return word of u is the factor between the beginning positions of two consecutive occurrences of u in w.

By an *Arnoux-Rauzy pseudoword* we mean a pseudoword $w \in \overline{\Omega}_A\mathsf{S}$ satisfying the following conditions:

- for every $n \geq 1$, has exactly one right special factor and one left special factor of length n, each of degree $|c(w)|$;
- the equalities $q_w(n) = p_w(n) = \tilde{p}_w(n)$ hold for every $n \geq 1$.

This is the natural generalization for pseudowords of the notion of an Arnoux-Rauzy infinite word [10, 11]. The second condition serves to avoid uninteresting examples such as $a^\omega b^\omega$ and also to allow us to extend more or less automatically results about Arnoux-Rauzy infinite words which involve only properties of finite factors to Arnoux-Rauzy pseudowords. The following facts are examples of such results.

Theorem 4.1. *Sturmian pseudowords are precisely the Arnoux-Rauzy pseudowords on two-letter alphabets.*

Theorem 4.2. *In an Arnoux-Rauzy pseudoword $w \in \overline{\Omega}_A\mathsf{S}$ every finite factor u has precisely $|C|$ return words, where $C = c(w)$. Moreover, the return words of u are obtained by cyclic conjugation from the $\rho_{v_n}(a)$ where v_n is the prefix of length $n = |u|$ of an infinite word $v \in C^\omega$ in which every letter occurs infinitely often. In fact, w is an element of the $\mathcal{J}$-class of the $\rho_{\bar{v}}(a)$ ($a \in A$) where $\bar{v}$ is any accumulation point in $\overline{\Omega}_A\mathsf{S}$ of $(v_n)_n$.*

The word v in Theorem 4.2 is uniquely determined by w and is called the *directive word of* w.

By an Arnoux-Rauzy maximal subgroup of $\overline{\Omega}_A\mathsf{S}$ we mean a maximal subgroup of $\overline{\Omega}_A\mathsf{S}$ consisting of Arnoux-Rauzy pseudowords. The following result generalizes the statement about the group component of the structural description of Sturmian $\mathcal{J}$-classes in Theorem 3.10.

Theorem 4.3. *If H is an Arnoux-Rauzy maximal subgroup of $\overline{\Omega}_A\mathsf{S}$ then the restriction $H \to \overline{\Omega}_A\mathsf{G}$ of the natural projection $\overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{G}$ is an isomorphism of profinite groups.*

We proceed to sketch the ingredients in the proof of this result. In the following, we assume $v \in A^\omega$ is an infinite word in which every letter occurs infinitely often, $\bar{v}$ is an accumulation point in $\overline{\Omega}_A\mathsf{S}$ of the sequence of finite prefixes of v, $\sigma : \overline{\Omega}_B\mathsf{S} \to \operatorname{End} \overline{\Omega}_A\mathsf{S}$ is a continuous homomorphism satisfying condition (3.1), H is a maximal subgroup of the $\mathcal{J}$-class of the $\rho_{\bar{v}}(a)$ ($a \in A$), and $p : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{G}$ is the natural projection.

We say that $f \in \operatorname{End} \overline{\Omega}_A \mathsf{S}$ is G-*invertible* if there exists $g \in \operatorname{End} \overline{\Omega}_A \mathsf{S}$ such that $p(fg(u)) = p(u)$ for all $u \in \overline{\Omega}_A \mathsf{S}$, that is if f induces an automorphism of $\overline{\Omega}_A \mathsf{G}$. An endomorphism g satisfying this condition is said to be a G-*inverse* of f. For example, the images of the letters under the Arnoux-Rauzy homomorphism are G-invertible. Indeed, if g is defined by

$$g(b) = \begin{cases} b & \text{if } b = a \\ a^{\omega-1}b & \text{otherwise} \end{cases}$$

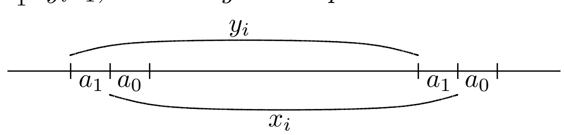
then g is a G-inverse of ρ_a.

The next result is a rather general and easy observation.

Proposition 4.4. *If K is a subgroup of $\overline{\Omega}_A \mathsf{S}$ and $f \in \operatorname{End} \overline{\Omega}_A \mathsf{S}$ are such that the restriction $p|_K : K \to \overline{\Omega}_A \mathsf{G}$ is an isomorphism and f is G-invertible, then $f|_K : K \to f(K)$ and the restriction $p|_{f(K)} : f(K) \to \overline{\Omega}_A \mathsf{G}$ are also isomorphisms.*

Using the preceding proposition, in the case of an idempotent substitution we may now compute some special maximal subgroups. Since, in a $\mathcal{J}$-class, they are all isomorphic, this will achieve our goal.

Proposition 4.5. *Suppose $\bar{v}$ is an idempotent and let $K = \sigma_{\bar{v}}(H)$. If the elements of K start with the letter a_0 and end with the letter a_1, $y_1, \ldots, y_m$ are the return words of $a_1 a_0$ in the elements of K, and $x_i = a_1^{-1} y_i a_1$, according to the picture*



then K is the closure of the subgroup generated by the $\sigma_{\bar{v}}(x_i)$ $(i = 1, \ldots, m)$ and a maximal subgroup of $\overline{\Omega}_A \mathsf{S}$. Moreover, if the σ_b are G-invertible then $p(K)$ is the closure of the subgroup of $\overline{\Omega}_A \mathsf{G}$ generated by the x_i $(i = 1, \ldots, m)$.

Taking into account Theorem 4.2 for the structure of return words, it is now easy to conclude the proof of Theorem 4.3 in case $\bar{v}$ is an idempotent. For the general case, a compactness argument plus Ramsey's Theorem provides a factorization $\bar{v}' = \bar{x}\bar{y}$ where $\bar{v}'$ and $\bar{x}$ are accumulation points of the sequence of finite prefixes of v and $\bar{y}$ is an idempotent accumulation point of a sequence of finite factors of v. Using Proposition 4.4 and some extra work the general case my now be deduced from the idempotent case.

But not all maximal subgroups consisting of uniformly recurrent pseudowords are free profinite groups as stated in the following example.

Proposition 4.6. *Let f be the continuous endomorphism of $\overline{\Omega}_{\{a,b\}} \mathsf{S}$ defined by $f(a) = a^3 b$ and $f(b) = ab$. Then the maximal subgroups of the $\mathcal{J}$-class of $f^\omega(a)$ are non-free profinite groups of rank* 2.

It should be observed that if $|A| = 2$, then any $\sigma : \overline{\Omega}_A \mathsf{S} \to \operatorname{End} \overline{\Omega}_A \mathsf{S}$ such that each σ_a is G-invertible produces Sturmian pseudowords by the scheme described in Theorem 3.7 (see [19]). On the other hand, for $|A| > 3$ it is known that that there are such σ which do not produce Arnoux-Rauzy pseudowords (see [11]).

5. Entropy

This section contains the announcement of some results involving entropy which were obtained jointly with M. V. Volkov [7].

Let $w \in \overline{\Omega}_A \mathsf{V}$. Since a factor of length $m + n$ of w is a product of a factor of length m by one of length n, we obtain the inequality $q_w(m + n) \leq q_w(m) q_w(n)$. From this observation it is an exercise in elementary calculus to show that the following limit exists (see [18]):

$$h(w) = \lim_{n\to\infty} \frac{1}{n} \log_{|A|} q_w(n).$$

The number $h(w)$ is clearly a real number in the interval $[0,1]$ and it is called the *entropy* of w. Similarly, a *left entropy* may be defined as the limit

$$h_\ell(w) = \lim_{n\to\infty} \frac{1}{n} \log_{|A|} p_w(n).$$

Note that $h_\ell(w) = h(w)$ for a uniformly recurrent pseudoword w.

The main connection which has been discovered so far between entropy and the structure of $\overline{\Omega}_A\mathsf{S}$ is given by the following result which is easy to prove taking into account that an element of $\overline{\Omega}_A\mathsf{S}$ lies in the minimal ideal if and only if admits every finite word as a factor.

Theorem 5.1. *A pseudoword $w \in \overline{\Omega}_A\mathsf{S}$ has $h(w) = 1$ if and only if w belongs to the minimal ideal of $\overline{\Omega}_A\mathsf{S}$.*

In case $|A| = 1$, there is nothing else than finite pseudowords and the minimal ideal, in which every element is uniformly recurrent. On the other hand, a recent result of Damanik and Solomyak [15] shows that in general there are left uniformly recurrent infinite words w with arbitrarily large entropy $h_\ell(w) < 1$. Hence, in view of Theorem 3.3, there are uniformly recurrent pseudowords of arbitrarily large entropy less than 1.

Next we examine how entropy is affected by iteration and evaluation.

Theorem 5.2. *Let $\varphi \in \operatorname{End}\overline{\Omega}_A\mathsf{S}$ and assume that $|A| > 1$, $\max_{a\in A} h(\varphi(a)) \le r$, and $w \in \overline{\Omega}_A\mathsf{S}$ also has $h(w) \le r$. Then $h(\varphi(w)) \le r$ and $h(\varphi^\omega(w)) \le r$.*

So, in particular, it is not possible to reach the minimal ideal of $\overline{\Omega}_A\mathsf{S}$ by applying or iterating substitutions whose values on letters do not reach the minimal ideal on pseudowords outside the minimal ideal. This extends considerably an earlier observation of Volkov and the author [5, Corollary 3.4] according to which, for alphabets A with more than one letter, no element of the minimal ideal of $\overline{\Omega}_A\mathsf{S}$ belongs to the subsemigroup generated by A closed under arbitrary infinite powers $s \mapsto s^\nu$. This result was deduced from a more precise result whose proof depends on structural knowledge of free Burnside semigroups.

Thus, it is not so easy to describe elements of the minimal ideal of $\overline{\Omega}_A\mathsf{S}$ for $|A| > 1$. Constructions of idempotents in the minimal ideal were found by Reilly and Zhang [20] and independently by Almeida and Volkov [5]. See also these two papers for applications of such idempotents.

6. Open problems

We conclude this paper by proposing a few open problems which are suggested by the work reported herein.

Problem 6.1. *Are the maximal subgroups of pseudowords in $\overline{\Omega}_A\mathsf{S}$ produced by the recipe in Theorem 3.7, under the extra assumption that σ_b is G-invertible, always free profinite groups of rank $|A|$?*

Note that a lot of the arguments in Section 4 were developed in this more general framework.

Problem 6.2. *Find the structure of Arnoux-Rauzy $\mathcal{J}$-classes.*

While for Sturmian $\mathcal{J}$-classes we have a complete structural result, in Section 4 we only identified the maximal Arnoux-Rauzy subgroups. More generally, it should contribute to a deeper knowledge of the structure of $\overline{\Omega}_A\mathsf{S}$ and its relationships with symbolic dynamics to be able to describe the structure of arbitrary uniformly recurrent maximal subgroups of $\overline{\Omega}_A\mathsf{S}$. Since this is probably a very hard problem to handle at present, a specific example is proposed in the next problem.

Problem 6.3. *Find the structure of the $\mathcal{J}$-class of the Prouhet-Thuë-Morse pseudoword.*

Finally, in view of Corollary 3.4, minimal shifts $\mathcal{S} \subseteq A^\omega$ are naturally associated with uniformly recurrent $\mathcal{J}$-classes of $\overline{\Omega}_A\mathsf{S}$. Thus there is this profinite group associated with a minimal shift which we call the *Schützenberger group* of the system. In general groups mean symmetry, which suggests that there is some hidden symmetry in symbolic dynamical systems and leads to the following problem.

Problem 6.4. *Find the significance of the Schützenberger group of minimal shifts in terms of symmetry of the shift.*

An answer to this problem might open the road for further connections between free profinite semigroups and symbolic dynamics and perhaps to applications to symbolic dynamics.

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