DECIDABILITY AND TAMENESS IN THE THEORY OF FINITE SEMIGROUPS

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ABSTRACT. This is a survey of results of the profinite approach to decidability problems in the theory of finite semigroups.

1. Introduction

This is an expanded version of an invited talk presented at the 38^{th} Iranian International Conference on Mathematics, held on 3-6 September 2007, at the University of Zanjan, Zanjan, Iran. The purpose of the talk was to give a short survey of the profinite approach to decidability problems in the theory of finite semigroups.

The main motivation for studying finite semigroups comes from the theories of formal languages and automata. In the framework set up by S. Eilenberg in the 1970's, under a strong influence of M.-P. Schützenberger, the theory has evolved mainly in the direction of the classification in so-called *pseudovarieties*. The typical problems consist in, for a certain recursively enumerable set of finite semigroups, solving the membership problem in the pseudovariety it generates, that is given a finite semigroup, to determine whether it is a homomorphic image of some subsemigroup of some finite direct product of members of the set. While, even for some naturally constructed sets it is known that the answer is negative, there are many positive examples, and even classes of examples which can be treated more or less uniformly.

In the following sections, we introduce and motivate more extensively the general problem, the profinite method, and how it has been used to obtain positive results. This survey is meant as a quick introduction to the subject and not as an exhaustive coverage of results to date. As this area of research is quite active, perhaps it is does not even make sense to try to elaborate an exhaustive survey, as it will probably already be outdated at the time it is made available. It is hoped that, nevertheless, it will be of use both to researchers with an interest in the area as well to those already initiated or even participating in the joint endeavor of development of the subject.

The reader seaking further details and bibliography on the subject is referred to the books [44, 4], respectively for an elementary and more advanced introductions to the theory of pseudovarieties and its applications. For the profinite approach, see [19, 64, 18, 10, 9, 50]. More specific references will be given in the remainder of the text.

2. Semigroups via automata

Finite automata can be viewed as simple recognition devices for formal languages. In this paper, A will always denote a finite *alphabet*, that is a finite set whose members are called *letters*. The set of all finite words on the alphabet A is denoted A^*. By a *word* we simply mean a finite sequence of letters, normally written consecutively, in the form $a_1a_2\cdots a_n$. We include in A^* the empty

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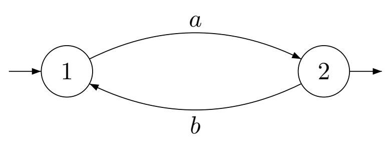
sequence, denoted 1. Under the operation of concatenation of sequences, $a_1 \cdots a_m \cdot b_1 \cdots b_n = a_1 \cdots a_n b_1 \cdots b_n$, the set A^* is the *free monoid on the set* A in the sense that every mapping from A^* to a monoid M extends uniquely to a homomorphism $A \to M$. The set $A^* \setminus \{1\}$ is denoted A^+ and is similarly the *free semigroup on* A.

By an A-*automaton*, we mean a finite directed (multi)graph whose edges are labeled by the elements of A and whose vertices are called *states*, in which subsets of *initial* and *final* states are distinguished. The *language recognized* by such an automaton $\mathfrak{A}$ is the set $L(\mathfrak{A})$ of all words which are obtained by concatenating the successive labels of the edges in some (directed) path from an initial to a terminal state. An automaton is called *deterministic* if it has a unique initial state and there are no two edges leaving from the same state with the same label. An A-automaton is *complete* if, for every state q and every letter $a \in A$ there is an outgoing edge from q labeled a.

For example, consider the $\{a, b\}$-automaton described by the following diagram



where an incoming arrow which does not start at a state means that the state where it ends is an initial state and dually for terminal states. The language recognized by it is $L = (ab)^*a$, where, in general, with an abuse of notation which pervades the literature, for a language $K \subseteq A^*$, K^* denotes the submonoid of A^* generated by K. Similarly, for $K \subseteq A^+$, K^+ denotes the subsemigroup of A^+ generated by L. In other words, in our example, L consists of all words alternating a's and b's, which start and end with a. The automaton in this example is *minimum* in the sense that it is the unique (up to isomorphism) deterministic (perhaps incomplete) automaton recognizing L with a minimum number of states.

Note that, if Q is the set of states of an A-automaton $\mathfrak{A}$, then each letter defines a binary relation on the set Q whose members are the pairs (p, q) such that there is an edge from p to q labeled a. The mapping thus defined extends uniquely to a homomorphism $\varphi : A^+ \to \mathcal{B}_Q$ into the semigroup of binary relations on Q, under the composition of relations. The image $\varphi(A^+)$ is called the *transition semigroup* of $\mathfrak{A}$ and denoted $T(\mathfrak{A})$. In case $\mathfrak{A}$ is deterministic, the action of the alphabet A on Q is by partial transformations. If, further, the automaton is complete then the action is by full transformations. In case $\mathfrak{A}$ is deterministic, the language $L(\mathfrak{A})$ consists precisely of those transformations which map the unique initial state to some final state. In particular, $L(\mathfrak{A}) = \varphi^{-1}\varphi(L(\mathfrak{A}))$, a property that is expressed by saying that the homomorphism φ *recognizes* L. If $\mathfrak{A}$ is the minimum automaton of the language L then the relation $\ker \varphi = \{(u, v) : u, v \in A^+,\ \varphi(u) = \varphi(v)\}$ is the largest congruence on A^+ which *saturates* L, in the sense that L is a union of its classes. The transition semigroup $T(\mathfrak{A}) \simeq A^+/\ker \varphi$ is then the unique (up to isomorphism) smallest semigroup S for which there exists a homomorphism $\psi : A^+ \to S$ that recognizes L. It is called the *syntactic semigroup* of the language L and it is denoted $\mathsf{Synt}(L)$. In general, given an automaton recognizing a certain language, one may apply the power set construction to determinize the automaton and then a minimization procedure to compute the minimum automaton of the language. Hence, the syntactic semigroup of such a language is effectively computable.

Kleene proved in 1956 that a language is recognized by some (finite) A-automaton if and only if it may be expressed in terms of the languages $\emptyset$ and $\{a\}$ $(a \in A)$ by using only the operations $\_ \cup \_$, $\_ \cdot \_$, and $\_^+$ [35]. Such an expression is called a *rational expression* for the language, which in case it admits one is called a *rational language*. The above discussion shows that every rational language is recognized by a homomorphism onto a finite semigroup. The converse is easily established by considering the Cayley graph of the semigroup with respect to the alphabet.

Two remarkable examples of application of the syntactic semigroup are the following two theorems. We say that a semigroup S is *aperiodic* if all its subgroups (that is subsemigroups which are groups) are trivial.

Theorem 2.1 (Schützenberger [54]). *A rational language $L \subseteq A^+$ may be expressed in terms of the languages $\emptyset$ and $\{a\}$ $(a \in A)$ by using only the operations $\_ \cup \_$, $A^+ \setminus \_$, and $\_ \cdot \_$ if and only if its syntactic semigroup is aperiodic.*

By a *subword* of a word $a_1 a_2 \cdots a_m$ we mean a word of the form $a_{i_1} \cdots a_{i_r}$, with $1 \leq i_1 < \cdots < i_r \leq m$ and $r \geq 0$. Call a language $L \subseteq A^+$ *piecewise testable* if there exists a positive integer n and a set P such that a word belongs to L if and only if all its subwords of length at most n belong to P.

Say that two elements s and t of a semigroup S are *$\mathcal{J}$-equivalent (in S)*, and write $s \mathcal{J} t$ if they are factors of each other. Similarly, we write $s \mathcal{R} t$ if s and t are left factors of each other and $s \mathcal{L} t$ if s and t are right factors of each other.

Theorem 2.2 (Simon [55]). *A rational language $L \subseteq A^+$ is piecewise testable if and only if its syntactic semigroup is $\mathcal{J}$-trivial.*

These two results translate combinatorial—expressability—problems concerning rational languages to algebraic properties of their syntactic semigroups. While testing directly whether a given rational language has such a property seems a daunting task, the properties in question of the syntactic semigroups are easily tested once the syntactic semigroup has been computed, although there are some complexity issues related with the fact that in general the minimum automaton may be much (exponentially) smaller than the syntactic semigroup [58].

Eilenberg [29] gave the general framework (which is not to be confused with a generalization) for such results that we proceed to describe.

A *pseudovariety* (of finite semigroups) is a class V of finite semigroups that is closed under taking homomorphic images, subsemigroups, and finite direct products. We include the empty product $\{1\} = \prod \emptyset$, so that pseudovarieties are necessarily nonempty classes. For example, the class G of all finite groups, the class A of all finite aperiodic semigroups, the class J of all finite $\mathcal{J}$-trivial semigroups, and the class R of all finite $\mathcal{R}$-trivial semigroups, are pseudovarieties.

A *variety of rational languages* is a correspondence $\mathcal{V}$ which associates with each finite alphabet A a Boolean algebra $\mathcal{V}(A)$ of languages over A such that:

(1) if $L \in \mathcal{V}(A)$ and $a \in A$ then the languages

$$a^{-1}L = \{w \in A^+ : aw \in L\} \quad \text{and} \quad La^{-1} = \{w \in A^+ : wa \in L\}$$

also belong to $\mathcal{V}(A)$;

(2) if $\varphi : A^+ \to B^+$ is a homomorphism and $L \in \mathcal{V}(B)$, then $\varphi^{-1}(L) \in \mathcal{V}(A)$.

For example, the correspondences that associate with each finite alphabet A the set of all languages $L \subseteq A^+$ which admit a *$+$-free expression* (that is an expression as in Theorem 2.1) or which are piecewise testable are varieties of rational languages.

Theorem 2.3 (Eilenberg [29]). *Consider the correspondence that associates with each pseudovariety V the variety of rational languages $\mathcal{V}$ such that, for a finite alphabet A and $L \subseteq A^+$, $L \in \mathcal{V}(A)$ if and only if $\mathsf{Synt}(L) \in \mathsf{V}$. It is a bijection between the sets of all pseudovarieties of finite semigroups and all varieties of rational languages.*

There have been many extensions of this result to various contexts and in different directions. The interested reader should do a bibliographic search with the keywords Eilenberg and correspondence.

Naturally, Eilenberg's correspondence also provides a translation of operators on varieties of rational languages to operators on pseudovarieties of finite semigroups. Among such operators, on either side, deserve particular attention those that derive from natural constructions, be it on automata, languages or semigroups. For example, the parallel composition of automata corresponds to taking the smallest variety of rational languages containing two given such varieties (which is called their *join*). On the semigroup side, we have the construction of the direct product which leads similarly to the join $\vee$ in the lattice of pseudovarieties of finite semigroups.

The cascade composition of automata is similarly associated with the semidirect product of finite semigroups and of their pseudovarieties. The *semidirect product* $\mathsf{V} * \mathsf{W}$ of two pseudovarieties V and W is the pseudovariety generated by (that is the smallest containing) all semidirect products of the form $S * T$, with $S \in \mathsf{V}$ and $T \in \mathsf{W}$. The semidirect product on pseudovarieties turns out to be an associative operation (see [4, Section 10.1]). For a pseudovariety V, we write V^n for the n-fold semidirect product in which all factors are V.

Performing substitutions in rational languages is in turn associated with the construction of the semigroup $\mathcal{P}(S)$ of subsets of a given semigroup S, in which the product of two subsets X and Y is given by $XY = \{xy : x \in X,\ y \in Y\}$. The corresponding operator on pseudovarieties is known as the *power operator*. It associates with each pseudovariety V the pseudovariety PV generated by all semigroups of the form $\mathcal{P}(S)$ with $S \in \mathsf{V}$. See [44, 4].

Certain naturally defined hierarchies of rational languages also motivate strongly the study of the *Mal'cev product* $\mathsf{V} \circledm \mathsf{W}$, which is defined to be the pseudovariety generated by all finite semigroups S for which there exists a homomorphism $\varphi : S \to T$ into some semigroup $T \in \mathsf{W}$ such that $\varphi^{-1}(e) \in \mathsf{V}$ for every idempotent $e \in T$.

Note that all the above examples of operators on pseudovarieties are defined in terms of generators. This raises the central decision problem for a pseudovariety V, namely the *membership problem*:

Given a finite semigroup S, determine whether $S \in \mathsf{V}$.

We say that a pseudovariety is *decidable* if its membership problem admits an algorithmic solution.

It turns out that none of the above operators on pseudovarieties preserves decidability [1, 23]. Yet, in many useful instances, one obtains decidable pseudovarieties. This raises the question of whether one can find fairly general conditions under which a pseudovariety that is obtained by applying the natural operators is decidable.

Here are a couple of examples of specific open problems of our general kind.

- Krohn and Rhodes [38] proved that every finite semigroup *divides* (that is, it is a homomorphic image of a subsemigroup of) an alternating wreath product of finite permutation groups and finite aperiodic transformation semigroups. The minimum number of group factors is called the *Krohn-Rhodes complexity* of the finite semigroup. In terms of pseudovarieties, the complexity of S is the least $n \geq 0$ such that $S \in \mathsf{A} * (\mathsf{G} * \mathsf{A})^n$. Is the Krohn-Rhodes complexity computable? This is equivalent to asking whether each of the pseudovarieties $\mathsf{A} * (\mathsf{G} * \mathsf{A})^n$ is decidable.
- It turns out that PJ is the pseudovariety corresponding to the level two of the Straubing-Thérien (concatenation) hierarchy of +-free rational languages (see [46]). Is PJ decidable?

3. Profinite semigroups

A *topological semigroup* is a semigroup S endowed with a topology such that the basic semigroup multiplication $S \times S \to S$ is continuous. A *compact semigroup* is a topological semigroup in which the topology is compact, a property in which we include the Hausdorff separation axiom. In particular, finite semigroups are viewed as compact semigroups under the discrete topology.

We say that a topological semigroup S is *A-generated* if a mapping $\varphi : A \to S$ is given such that $\varphi(A)$ generates a dense subsemigroup of S.

Given a class $\mathcal{C}$ of topological semigroups, we say that a semigroup S is *residually in* $\mathcal{C}$ if, for every two distinct points $s, t \in S$, there exists a continuous homomorphism $\varphi : S \to T$ into some member $T \in \mathcal{C}$ such that $\varphi(s) \neq \varphi(t)$. In case $\mathcal{C}$ is the class of all finite semigroups, we then simply say that S is *residually finite*.

A *profinite semigroup* is a compact semigroup which is residually finite. Equivalently, a profinite semigroup is a compact totally disconnected semigroup [43]. More generally, if V is a pseudovariety of finite semigroups, then by a *pro-*V *semigroup* we mean a compact semigroup which is residually in V. Equivalently, a pro-V semigroup is a projective (or inverse) limit of semigroups from V.

Considering only representatives up to isomorphism, the A-generated semigroups from V form a directed system, where $\varphi : A \to S$ is larger than $\psi : A \to T$ if there exists a homomorphism (which must be unique) $h : S \to T$ such that $h \circ \varphi = \psi$. The projective limit of this system is denoted $\overline{\Omega}_A\mathsf{V}$ and is the most general A-generated pro-V semigroup, in the sense that it comes naturally equipped with a generating mapping $\iota : A \to \overline{\Omega}_A\mathsf{V}$ and, for every mapping $\varphi : A \to S$ into a pro-V semigroup S, there exists a unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}_A\mathsf{V} \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & \overline{\Omega}_A\mathsf{V} \\ & \searrow^{\varphi} & \downarrow{\scriptstyle\hat{\varphi}} \\ & & S \end{array}$$

For this reason, $\overline{\Omega}_A\mathsf{V}$ is called the *free pro-*V *semigroup on* A.

The above diagram suggests a way of defining a *natural interpretation* of each $w \in \overline{\Omega}_A\mathsf{V}$ as an A-ary operation $w_S : S^A \to S$: given an argument $\varphi \in S^A$, that is a function $\varphi : A \to S$, define $w_S(\varphi) = \hat{\varphi}(w)$. This interpretation is easily seen to commute with continuous homomorphisms between pro-V semigroups $h : S \to T$ in the sense that the following diagram commutes:

$$\begin{array}{ccc} S^A & \xrightarrow{w_S} & S \\ {\scriptstyle h\circ\_}\downarrow & & \downarrow{\scriptstyle h} \\ T^A & \xrightarrow{w_T} & T \end{array}$$

Operations with this property are called *A-ary implicit operations*. One can show that they are all obtained by natural interpretation of members of $\overline{\Omega}_A\mathsf{V}$. Since pro-$\mathsf{V}$ semigroups are projective limits of semigroups from V, each $w \in \overline{\Omega}_A\mathsf{V}$ is completely determined by the implicit operation $(w_S)_{S\in\mathsf{V}}$.

Given a finite semigroup S, $s \in S$, and $k \in \mathbb{Z}$, the sequence $(s^{n!+k})_n$, which is only defined for n sufficiently large, is eventually constant, that is it converges in S. It follows that, if instead S is a profinite semigroup, then the sequence still converges. The limit is denoted $s^{\omega+k}$. In particular, $s^\omega = s^{\omega+0}$ is an idempotent, and $s^{\omega-1}$ is the inverse of $s^{\omega+1}$ in the maximal subgroup of S containing s^ω. Here are some properties of these operations:

- if $k > \ell$ then $s^{\omega+k} = s^{\omega+\ell}s^{k-\ell}$;
- $s^{\omega+k}s^{\omega+\ell} = s^{\omega+k+\ell}$;
- $(s^{\omega+k})^{\omega+\ell} = s^{\omega+k\ell}$;
- if $k > 0$ then $(s^{\omega-1})^k = s^{\omega-k}$.

Let S be the pseudovariety of all finite semigroups. If $A = \{a\}$, then $a^{\omega+k}$ is a well-defined element of $\overline{\Omega}_A\mathsf{S}$ and therefore determines a unary implicit operation. It is an easy exercise to show that the correspondence $s \mapsto s^{\omega+k}$ is precisely the interpretation of this implicit operation on each profinite semigroup. Elements of $\overline{\Omega}_A\mathsf{S}$ are sometimes also called *profinite words* or pseudowords since A^+ embeds naturally in $\overline{\Omega}_A\mathsf{S}$ as a dense subsemigroup.

A *pseudoidentity* is a formal equality $u = v$ with u and v in some $\overline{\Omega}_A\mathsf{S}$. We say that a profinite semigroup S *satisfies* the pseudoidentity $u = v$ and write $S \models u = v$ if $u_S = v_S$. For a set Σ of pseudoidentities, denote by $[\![\Sigma]\!]$ the class of all finite semigroups which satisfy all pseudoidentities from Σ. It is immediate to show that such a class is a pseudovariety and Reiterman [49] proved that every pseudovariety is of this form. In other words, pseudovarieties may always be described by some set of defining pseudoidentities, also known as a *basis of pseudoidentities.* Here are some examples of bases of pseudoidentities:

$$\begin{aligned}\mathsf{G} &= [\![x^\omega = 1]\!] \quad \text{(as an abbreviation of } x^\omega y = yx^\omega = y)\\ \mathsf{A} &= [\![x^{\omega+1} = x^\omega]\!]\\ \mathsf{J} &= [\![(xy)^{\omega+1} = (yx)^\omega]\!]\end{aligned}$$

More examples of implicit operations may be obtained using the following result from [10]. Recall that, for a topological space X, on a function space $\mathcal{S} \subseteq X^X$, there are several natural topologies. The *pointwise convergence topology* is the subspace topology of the product space X^X. The *compact-open topology* has as basis of open sets the sets of the form $\{f \in \mathcal{S} : f(K) \subseteq U\}$, where $K \subseteq X$ is a compact subset and $U \subseteq X$ is an open subset. In general, these two topologies are different.

Theorem 3.1. *If S is a finitely generated profinite semigroup, then its monoid of continuous endomorphisms $\mathsf{End}(S)$ is a profinite semigroup with respect to the pointwise convergence topology, which coincides with the compact-open topology, so that the evaluation mapping $(f, s) \mapsto f(s)$ is continuous.*

Let $\varphi \in \mathsf{End}(\overline{\Omega}_{\{x\}}\mathsf{S})$ send x to x^p. Then the implicit operation $x^{p^\omega} = \varphi^\omega(x) = \lim \varphi^{n!}(x) = \lim x^{p^{n!}}$ may be used to define the pseudovariety of all finite p-groups: $\mathsf{G}_p = [\![x^{p^\omega} = 1]\!]$.

Let $\varphi \in \mathsf{End}(\overline{\Omega}_{\{x,y\}}\mathsf{S})$ send x to $[x, y] = x^{\omega-1}y^{\omega-1}xy$ and fix y. Then $[x, {}_\omega y] = \varphi^\omega(x) = \lim \varphi^{n!}(x) = [x, {}_{n!}y]$, where the iterated "commutator" is defined recursively by $[x, {}_1y] = [x, y]$ and $[x, {}_{n+1}y] = [[x, {}_ny], y]$. The implicit operation $[x, {}_\omega y]$ may be used to define the pseudovariety of all finite nilpotent groups: $\mathsf{G}_{\mathsf{nil}} = [\![\, [x, {}_\omega y] = 1]\!]$ [65].

Let $\varphi \in \mathsf{End}(\overline{\Omega}_{\{x,y\}}\mathsf{S})$ send x to xy and y to yx, that is the extension of the usual Thue-Morse transformation to the profinite world. Then the implicit operation $\tau(x, y) = \varphi^\omega(x)$ is such that $\mathsf{B}(\mathsf{G}_{\mathsf{nil}} * \mathsf{G}_2) = [\![\tau(x, y) = \tau(y, x)]\!]$ consists of all finite semigroups S such that

(1) for every $s \in S$, there is at most one element $t \in S$ such that $sts = s$ and $tst = t$;
(2) every subgroup in S is an extension of a nilpotent group by a 2-group.

This follows from results of Širšov [63] (see [7]).

Let $\varphi \in \mathsf{End}(\overline{\Omega}_{\{x,y,z\}}\mathsf{S})$ send x to $[yxy^{\omega-1}, zxz^{\omega-1}]$ and fix y and z. Let $w(x, y, z) = \varphi^\omega(x)$. Then $\mathsf{G}_{\mathsf{sol}} = [\![\, w(x^{\omega-2}y^{\omega-1}x, x, y) = 1]\!]$ is the pseudovariety of all finite solvable groups [24]. An alternative description of the same pseudovariety is as follows [25]. Let $\varphi \in \mathsf{End}(\overline{\Omega}_{\{x,y\}}\mathsf{S})$ send x to $[y^{\omega-1}x^{\omega-1}y, x]$ and fix y. Let $v(x, y) = \varphi^\omega(x)$. Then $\mathsf{G}_{\mathsf{sol}} = [\![\, v(x, y) = 1]\!]$.

4. The role of profinite methods

Although pseudovarieties are defined by pseudoidentities, it is not clear how to use them to obtain decidability results. A difficulty in this direction is that the free profinite semigroup $\overline{\Omega}_A\mathsf{S}$ is uncountable even for $|A| = 1$. Yet many natural algebraic properties admit descriptions in terms of pseudoidentities which involve only very simple elements of $\overline{\Omega}_A\mathsf{S}$. In spite of the examples at the end of the preceding section, most often one only uses elements of the subalgebra $\Omega_A^\kappa\mathsf{S}$ generated by A with respect to the algebraic operations of multiplication and $\_^{\omega-1}$.

Here are some examples of results in this direction. Even though the pseudovarieties $\mathsf{J} \vee \mathsf{G}$ and $\mathsf{A} \vee \mathsf{G}$ are not finitely based [62], the former is known to be decidable [12, 56], while for the latter this remains an open problem. Both proofs of the decidability result, which follow similar lines, depend on two main ingredients: a structure theorem for $\overline{\Omega}_A \mathsf{J}$ [2] and a form of "tameness" for G [22]. The sense of the word tameness in this context is explained in Section 5.

Another famous result is the following: the pseudovariety PG is decidable. More precisely, it admits a simple algebraic characterization, namely as the class BG of all finite semigroups S such that, for every element s, there exists at most one $t \in S$ such that $sts = s$ and $tst = t$. This combines the work of several researchers [40, 31, 22]. There are essentially no known general decidability results concerning the power operator.

For the Mal'cev product, the following result is sometimes quite useful.

Theorem 4.1 ([48]). *If* $\mathsf{V} = [\![\Sigma]\!]$ *then* $\mathsf{V} \circledcirc \mathsf{W}$ *is defined by the pseudoidentities of the form* $u(w_1, \ldots, w_n) = v(w_1, \ldots, w_n)$ *where* $\big(u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)\big) \in \Sigma$, $w_1, \ldots, w_n \in \overline{\Omega}_A \mathsf{S}$, *and* $\mathsf{W} \models w_1 = \cdots = w_n = w_n^2$.

To test whether a given finite semigroup S belongs to $\mathsf{V} \circledcirc \mathsf{W}$ by using the basis of pseudoidentities given by Theorem 4.1 we need to be able to figure out for $s_1, \ldots, s_n \in S$, and a given evaluation $\varphi : A \to S$, if there is a *solution modulo* W $(w_1, \ldots, w_n)$ of the system of equations $x_1 = \cdots = x_n = x_n^2$ (that is $\mathsf{W} \models w_1 = \cdots = w_n = w_n^2$) such that $\hat{\varphi}(w_i) = s_i$ $(i = 1, \ldots, n)$. More generally, for a class of arbitary finite systems of equations, if there is an algorithm to solve this problem, then we say that W is *hyperdecidable* with respect to the class in question. A general decidability result about Mal'cev products may now be easily derived from Theorem 4.1.

Theorem 4.2. *If* V *is decidable and* W *is hyperdecidable with respect systems of equations of the form* $x_1 = \cdots = x_n = x_n^2$ *then* $\mathsf{V} \circledcirc \mathsf{W}$ *is decidable.*

There is a similar but incomplete approach to semidirect products, based on ideas of J. Rhodes (late 1960's), which were developed and formalized by Tilson [61]. Small categories (and, more generally, *semigroupoids*, categories without the requirement of local identities) are viewed as generalizations of monoids and semigroups. A similar theory of pseudovarieties has been developed for such structures. The profinite approach also extends to this context, including descriptions by pseudoidentities [33, 20]. The variables in pseudoidentities must now come from a set with a non-trivial structure, namely they are edges on a finite directed graph. The two sides of a pseudoidentity become coterminal *profinite paths* over such a graph. A pseudovariety of semigroupoids has *finite vertex rank* if it admits a basis of pseudoidentities for which there is a bound on the number of vertices of the finite directed graphs on which they are written; otherwise, we say that it has *infinite vertex rank*.

A semigroup S may be viewed as a semigroupoid with a single (virtual) vertex whose edges are the elements of S. The smallest pseudovariety of semigroupoids containing a given pseudovariety V of semigroups is called the *global of* V and is denoted $g\mathsf{V}$. The largest pseudovariety of semigroupoids whose semigroups are the members of a pseudovariety V of semigroups consists of all semigroupoids whose *local semigroups* (that is the semigroups of all loops at some vertex) belong to V; it is called the *local of* V and is denoted $\ell\mathsf{V}$. A pseudovariety V of semigroups is *local* if $g\mathsf{V} = \ell\mathsf{V}$. Note that, if $\mathsf{V} = [\![\Sigma]\!]$ is local, then $g\mathsf{V}$ is also defined by the pseudoidentities from Σ, now viewed as pseudoidentities over one-vertex graphs, because such pseudoidentities clearly define $\ell\mathsf{V}$.

Many globals of pseudovarieties of semigroups have been computed. Here is a sample of them.

- The pseudovarieties A, R, all nontrivial pseudovarieties $\mathsf{V} \subseteq \mathsf{G}$ [59], DS [34], and DA [5], DG (announced by J. Kad'ourek, in 2005) are all local.

- For the *trivial pseudovariety* $\mathsf{I} = [\![x = y]\!]$, we have

$$g\mathsf{I} = [\![x = y;\ \bullet \underset{y}{\overset{x}{\rightrightarrows}} \bullet\ ]\!]$$

so that $g\mathsf{I}$ does not consist just of semigroupoids with only one edge.
- For $\mathsf{Com} = [\![xy = yx]\!]$,

$$g\mathsf{Com} = [\![xyz = zyx;\ \bullet \underset{y}{\overset{x,z}{\rightleftarrows}} \bullet\ ]\!]\ [60].$$

- The known proofs of the following result are considered quite difficult:

$$g\mathsf{J} = [\![(xy)^\omega xt(zt)^\omega = (xy)^\omega (zt)^\omega;\ \bullet \underset{y,t}{\overset{x,z}{\rightleftarrows}} \bullet\ ]\!]\ [36, 37, 3].$$

- For $m \geq 2$ and $k \geq 1$ (or $k = \omega$), $g\ [\![xy = yx, x^{k+m} = x^m]\!]$ has infinite vertex rank [11].
- For $n \geq 1$, $g\big(\mathsf{A} * (\mathsf{G} * \mathsf{A})^n\big)$ has infinite vertex rank [51].

Back to semidirect products of pseudovarieties, for whose calculation categories, semigroupoids, and globals were first introduced in the theory of finite semigroups, we have the following basis theorem. For a directed graph Γ, and an edge $e \in \Gamma$, we denote by $\alpha(e)$ and $\omega(e)$ respectively the start and end vertices of e.

Theorem 4.3 ([20])**.** *If $g\mathsf{V} = [\![\Sigma]\!]$, where Σ is a set of semigroupoid pseudoidentities over finite graphs with a uniform bound on the number of vertices then $\mathsf{V} * \mathsf{W}$ is defined by the pseudoidentities of the form $t\,u(w_1, \ldots, w_n) = t\,v(w_1, \ldots, w_n)$ such that $\big(u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n); \Gamma\big) \in \Sigma$, $(p) \underset{v(x_1,\ldots,x_n)}{\overset{u(x_1,\ldots,x_n)}{\rightrightarrows}} (q)$, $\gamma : \Gamma \to \overline{\Omega}_A\mathsf{S}$ is a continuous graph homomorphism, $\gamma(x_i) = w_i$, $\gamma(p) = t$, and $\mathsf{W} \models \gamma(\alpha(x_i))w_i = \gamma(\omega(x_i))$.*

Just as in the case of Mal'cev products, one may then prove the following result. The system of equations associated with a finite digraph Γ takes the elements (vertices and edges) of Γ as variables and has an equation $xy = z$ for each edge y from the vertex x to the vertex z.

Theorem 4.4 ([6])**.** *If $g\mathsf{V}$ is decidable and of finite vertex rank, and W is hyperdecidable with respect to systems of equations associated with finite graphs, then $\mathsf{V} * \mathsf{W}$ is decidable.*

The need for the finite vertex rank hypothesis in Theorem 4.4 coming from the boundedness hypothesis of Theorem 4.3 is rather unfortunate, although no counter-example is known to Theorem 4.3 with that hypothesis dropped. A recent more general basis theorem for which no counterpart similar to Theorem 4.3 seems to have yet been found is the following.

Theorem 4.5 ([50])**.** *Let V and W be pseudovarieties of semigroups and let $g\mathsf{V} = [\![\Sigma]\!]$. Then $\mathsf{V} * \mathsf{W}$ is defined by all pseudoidentities of the form $tu = tv$ over finite alphabets A such that, for each A-generated semigroup $T \in \mathsf{W}$, there exist $(\pi = \rho; \Gamma) \in \Sigma$, $(p) \underset{\rho}{\overset{\pi}{\rightrightarrows}} (q)$, and a labeling $\gamma : \Gamma \to \overline{\Omega}_A\mathsf{S}$ such that $\gamma(p) = t$, $\hat{\gamma}(\pi) = u$, $\hat{\gamma}(\rho) = v$, and $T \models \gamma(\alpha(e))\gamma(e) = \gamma(\omega(e)$ for every edge $e \in \Gamma$.*

5. Tameness

To obtain actual algorithms for hyperdecidability is usually a very hard task. Steinberg and the author [16, 17] suggested a different approach, namely to prove a stronger property! Indeed, the difficulty in a brute force approach is that there are too many (uncountably many) candidates for solutions modulo V of a given system of equations. Suppose that if there is a solution modulo V of a given finite system of equations in $\overline{\Omega}_A\mathsf{S}$, then there is also one in $\Omega_A^\sigma\mathsf{S}$, where σ is some countable *implicit signature*, meaning a set of implicit operations which includes binary multiplication. Then at least the above difficulty disappears. And indeed one obtains a stronger property than hyperdecidability.

We say that a pseudovariety V is *(σ-)tame* with respect to a class $\mathcal{C}$ of systems of equations if the following conditions hold:

- *(σ-reducibility)* for every system in $\mathcal{C}$, over a finite set X of variables, and associated clopen constraints $K_x \subseteq \overline{\Omega}_A\mathsf{S}$ ($x \in X$), if there is a solution $X \to \overline{\Omega}_A\mathsf{S}$ modulo V, then there is a solution $X \to \Omega_A^\sigma\mathsf{S}$ modulo V;
- V is recursively enumerable, meaning that there is some Turing machine which outputs successively all representatives of the isomorphism classes of members of V and nothing else;
- the word problem for $\Omega_A^\sigma\mathsf{V}$ is algorithmically solvable.

We say that V is *completely (σ-)tame* if V is σ-tame with respect to the class of all finite systems of equations in the signature σ.

Among the properties that compose tameness, reducibility is often the most difficult to prove although sometimes the word problem is also quite complicated. It is out of the scope of this survey to give a precise idea of any nontrivial proof of tameness. We proceed rather to mention some important tameness resullts.

- The pseudovariety G is κ-tame with respect to systems of equations associated with finite digraphs. This is a reformulation [16] of a celebrated theorem of Ash [22] which has already been mentioned several times in this survey. It follows from results of Coulbois and Khélif [27] that G is not completely κ-tame.
- The pseudovariety J is completely κ-tame. This is trivial since J is κ-full (see Section 6) and $\overline{\Omega}_A\mathsf{J} = \Omega_A^\kappa\mathsf{J}$. A proof of hyperdecidability of J which does not depend on this fact and which actually tries to exhibit a reasonable algorithm turns out to be rather complicated [21].
- The pseudovariety G_p is tame but not κ-tame [8]. The author's proof of this result, based on several other works [53, 39, 57], led him to explore connections with dynamical systems, the suitable signature that has been found being a countably infinite signature which is constructed by iteration of implicit operations.
- The pseudovariety LSl of all finite semigroups S whose subsemigroups of the form eSe, with $e \in S$ idempotent, are commutative and consist only of idempotents, is κ-tame [26].
- The pseudovariety R is completely κ-tame [13].

It remains as open problems to find a signature with respect to which G is completely tame, and to determine whether A is completely κ-tame (the word problem for $\Omega_A^\kappa\mathsf{A}$ has been solved by McCammond [41] and κ-tameness for A has been announced by J. Rhodes in the late 1990's but remains unpublished). A more general question is whether V and W both tame (with respect to some suitable signatures and classes of systems of equations) implies that $\mathsf{V} * \mathsf{W}$ is also tame. Note that, since tameness, even with respect to the single equation $x = y$, implies decidability, an affirmative answer to this question would provide a solution to the Krohn-Rhodes complexity problem, provided A and G do have the suitable tameness properties.

6. Computing closures of rational languages

We introduce in this section another related problem.

One can easily show that the induced topology on $\Omega_A^\sigma\mathsf{V}$, as a subspace of $\overline{\Omega}_A\mathsf{V}$, is its *pro-*$\mathsf{V}$ *topology*, that is the smallest topology which renders continuous all homomorphisms of σ-algebras $\Omega_A^\sigma\mathsf{V} \to S$ with $S \in \mathsf{V}$. Given $L \subseteq \Omega_A^\sigma\mathsf{V}$, denote by $\mathrm{cl}_{\sigma,\mathsf{V}}(L)$ its closure in the pro-V topology of $\Omega_A^\sigma\mathsf{V}$.

A general problem which has shown to be rather important in some instances is the following: given a rational subset $L \subseteq \Omega_A^\kappa \mathsf{V}$, compute $\mathrm{cl}_{\sigma,\mathsf{V}}(L)$, or at least decide whether a given $w \in \Omega_A^\sigma \mathsf{V}$ belongs to $\mathrm{cl}_{\sigma,\mathsf{V}}(L)$. Often, $A^+ \hookrightarrow \Omega_A^\kappa \mathsf{V}$ and L is just a rational language of A^+.

The historical source of this question is the following. Given a finite semigroup S and an onto homomorphism $\varphi : A^+ \to S$, consider the set $K_{\mathsf{G}}(S)$ of all $s \in S$ such that, for every homomorphism $\psi : A^+ \to G$ into a finite group G, $s \in \varphi(\psi^{-1}(1))$. Then it is easy to show that $S \in \mathsf{V} \,ⓜ\, \mathsf{G}$ if and only if $K_{\mathsf{G}}(S) \in \mathsf{V}$. An alternative characterization has been noted by Pin [45]:

$$K_{\mathsf{G}}(S) = \{s \in S : 1 \in \mathrm{cl}_{\mathsf{G}}\big(\psi(\varphi^{-1}(s))\big) \subseteq \Omega_A^\kappa \mathsf{G}\}.$$

The Rhodes "type II" conjecture stated that $K_{\mathsf{G}}(S) \cup \{1\}$ is the smallest subsemigroup of S^1 which contains the idempotents and, if it contains s and $aba = a$ or $bab = b$, then it also contains asb. The conjecture was the motivation for Ash's work and in fact it is proved in [22], that is, it is a (simple) consequence (in a sense a particular case) of the tameness of G. Pin and Reutenauer [47] proved that it suffices to establish that the product of finitely many finitely generated subgroups of the free group is closed in the pro-G topology. In fact, it turns out that this property is formally equivalent to the type II conjecture [30]. It was proved directly and independently in this form by Ribes and Zalesskiĭ [52] using profinite group theory.

Consider now the natural projection

$$p_{\mathsf{V}} : \overline{\Omega}_A \mathsf{S} \to \overline{\Omega}_A \mathsf{V}$$
$$a \in A \mapsto a$$

We say that a pseudovariety V is *σ-full* if, for every finite set A and for every rational language $L \subseteq A^+$, the set $p_{\mathsf{V}}(\mathrm{cl}_{\mathsf{S}}(L))$ is closed in $\Omega_A^\sigma \mathsf{V}$. For example, G [28], J (this is an exercise, taking into account knowledge of $\overline{\Omega}_A \mathsf{J}$), R and A (recently proved by the author with J. C. Costa and M. Zeitoun) are all κ-full.

The property of σ-fullness allows us to obtain a simplified formulation of σ-reducibility which we proceed to present. Consider a system Σ of equations over a finite set X of variables with constraints of the form $\overline{L_x}$ $(x \in X)$, where each L_x is a rational language. We say that $\varphi : X \to \overline{\Omega}_A \mathsf{V}$ is a *solution (of the system satisfying the constraints)* if

- $\varphi(x) \in \overline{L_x}$ $(\forall x \in X)$
- $\hat{\varphi}(u) = \hat{\varphi}(v)$ $(\forall (u = v) \in \Sigma)$

If a pseudovariety V is σ-full then it is σ-reducible for a system of equations over a finite set X of variables if and only if, for every choice L_x $(x \in X)$ of rational languages, if the system with the $\overline{L_x}$ as constraints admits a solution $\varphi : X \to \overline{\Omega}_A \mathsf{V}$ then it also admits a solution $\psi : X \to \Omega_A^\sigma \mathsf{V}$. In particular, if V is σ-full and $\overline{\Omega}_A \mathsf{V} = \Omega_A^\sigma \mathsf{V}$, then V is trivially σ-reducible.

It is thus worthwhile to understand the topological closure operation, at least for rational languages, within structures of the form $\Omega_A^\sigma \mathsf{V}$. We proceed to describe a *natural* procedure for computing closures of rational languages.

Suppose that $A^+ \subseteq \Omega_A^\sigma \mathsf{V}$. It is easy to see that, for $L, K \subseteq A^+$:

(1) if L is finite, then $\mathrm{cl}_{\sigma,\mathsf{V}}(L) = L$;
(2) $\mathrm{cl}_{\sigma,\mathsf{V}}(L \cup K) = \mathrm{cl}_{\sigma,\mathsf{V}}(L) \cup \mathrm{cl}_{\sigma,\mathsf{V}}(K)$;
(3) $\mathrm{cl}_{\sigma,\mathsf{V}}(L)\,\mathrm{cl}_{\sigma,\mathsf{V}}(K) \subseteq \mathrm{cl}_{\sigma,\mathsf{V}}(LK)$;
(4) $\langle \mathrm{cl}_{\sigma,\mathsf{V}}(L) \rangle_\sigma \subseteq \mathrm{cl}_{\sigma,\mathsf{V}}(L^+)$,

where $\langle X \rangle_\sigma$ denotes the σ-subalgebra of $\Omega_A^\sigma \mathsf{V}$ generated by X. If both inclusions (3) and (4) turn out to be always equalities, then we have a *natural procedure* to "compute" $\mathrm{cl}_{\sigma,\mathsf{V}}(L)$ in case $L \subseteq A^+$ is rational: we successively commute the topological closure operation with the rational operations using the equality versions of the above formulas.

Here are some examples for which the natural procedure works.

- It follows from [47, 52] that the natural procedure works for G with respect to the signature κ.
- It is easy to see that the natural procedure works for J, κ.
- Together with J. C. Costa and M. Zeitoun, the author has recently shown that the natural procedure works for both R and A over κ.
- Ribes and Zalesskiĭ [53] also gave an algorithm to compute the pro-G_p closure of a rational language in the free group $\Omega_A^\kappa \mathsf{G}_p = \Omega_A^\kappa \mathsf{G}$. See also [39] for complexity issues. These results had a strong influence in the eventual proof that G_p is tame [8].

In general, one may ask for which pseudovarieties does the natural procedure work for the signature κ. This remains an open problem.

7. Connection with Model Theory

We conclude this survey with a connection with a remarkable result in Model Theory, where an equivalent form of Ash's tameness theorem was discovered independently of semigroup theory.

We say that a class $\mathcal{R}$ of relational structures of the same type has the *finite extension property for partial automorphisms* (*FEPPA*) if for every finite $R \in \mathcal{R}$ and every set P of partial automorphisms of R, if there exists an extension $S \in \mathcal{R}$ of R for which every $f \in P$ extends to a total automorphism of S, then there exists such an extension $S \in \mathcal{R}$ which is finite.

By a *homomorphism* of relational structures of the same type we mean a function that preserves the given structures' relations in the forward direction. For a class $\mathcal{R}$ of relational structures, let $\operatorname{Excl}\mathcal{R}$ denote the class of all structures S for which there is no homomorphism $R \to S$ with $R \in \mathcal{R}$.

Theorem 7.1 ([32]). *For every finite set $\mathcal{R}$ of finite structures of a finite relational language, $Excl\,\mathcal{R}$ satisfies the FEPPA.*

Herwig and Lascar also showed that this property is formally equivalent to the following property of free groups. For a subgroup H of a free group F and elements $x, y \in F$, write $x \equiv_H y$ if $xH = yH$.

Theorem 7.2 ([32]). *Consider a finite system of equations of one of the following forms*

$$X \equiv_H Yg \quad \text{and} \quad X \equiv_H g,$$

where X and Y are variables, the H are finitely generated subgroups of the free group F, and the g are elements of F. If the system has no solution in F, then one may replace each subgroup H by a subgroup of F of finite index containing H such that the system remains without solution.

By suitably encoding one problem into the other, Delgado and the author [14, 15] have shown that Theorem 7.2 is formally equivalent to the tameness of G, thus deriving also Theorem 7.1 from the tameness of G.

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