ENGEL ELEMENTS IN GROUPS AND DYNAMICAL SYSTEMS DEFINING NILPOTENCY IN FINITE GROUPS

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ABSTRACT. A relationship between two natural dynamical systems on groups is established thereby giving a new characterization of Engel elements. Using this connection, various closure properties for the sets of left and right Engel elements are established. Some other dynamical systems and their relationship with Engel elements and nilpotency for finite groups are also considered.

1. INTRODUCTION

Engel groups and left and right Engel elements in groups have been extensively studied since the 1950's and even justify a subsection in the AMS classification scheme. Investigations have explored connections between Engel properties and nilpotency properties but also structural properties of groups in which every element has an Engel property of small index. See [10, 11] for the earlier theory and Subsection 20F45 of review journals for more recent work.

Recall the standard notation in group theory for iterated commutators: $[x,{}_1y] = [x,y] = x^{-1}y^{-1}xy$ and $[x,{}_{n+1}y] = [[x,{}_ny],y]$. For a group G, consider the set $R(G)$ of all elements x of G such that, for every $y \in G$, there exists $r \geq 1$ such that $[x,{}_ry] = 1$. The elements of $R(G)$ are known as *right Engel elements* of G [11, Section 12.3]. Also let $L(G)$ denote the set of all *left Engel elements* of G consisting of those $x \in G$ such that, for every $y \in G$, there exists $r \geq 1$ such that $[y,{}_rx] = 1$. Note that both $L(G)$ and $R(G)$ are stable under automorphisms of G. While it is known that $R(G)^{-1} \subseteq L(G)$, it seems to remain an open problem the precise relationship between these two subsets of G and whether they are subgroups of G. A negative result in this direction, but which falls short of solving any of those problems, is an example that shows that no power of a right r-Engel element (x such that, for every y, $[x,{}_ry] = 1$) need be a left r-Engel element (similarly defined),

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even in a nilpotent group [9]. On the other hand, obviously $G = L(G)$ if and only if $G = R(G)$, in which case G is said to be an *Engel group*.

Denote by $Z(G)$ the center of a group G. Another useful subgroup of G is its *hypercenter* which is defined as the union ζG of the transfinite sequence of characteristic subgroups given recursively as follows: $\zeta_0 G = 1$, $\zeta_{\alpha+1}G/\zeta_\alpha G = Z(G/\zeta_\alpha G)$, and $\zeta_\lambda G = \bigcup_{\alpha<\lambda} \zeta_\alpha G$ for a limit ordinal λ. Once two consecutive terms in this transfinite sequence are equal, the sequence is constant from that point on. Hence the hypercenter of G is $\zeta_\alpha G$, where α is the cardinal of G. In particular, the hypercenter of a finite group G is $\zeta_\omega G$. Recall that a finite group G is nilpotent if and only if $\zeta_\omega G = G$. More generally, a group is said to be *hypercentral* if $\zeta G = G$.

The *Fitting subgroup* of a group G is the subgroup $\mathrm{Fit}(G)$ generated by all normal nilpotent subgroups of G. The *Hirsch-Plotkin radical* of G is the unique maximal locally nilpotent normal subgroup $\mathrm{HP}(G)$ of G. It is easy to show that $\mathrm{HP}(G) \subseteq L(G)$ and $\zeta G \subseteq R(G)$ [11]. By a Theorem of Zorn [14], for a locally finite group G, $R(G)$ is a normal subgroup of G and, therefore, a subgroup of $\mathrm{HP}(G)$, which in turn coincides with $L(G)$ (cf. [11, Section 12.3]). More generally, by a Theorem of Gruenberg [7], in a solvable group G, $L(G) = \mathrm{HP}(G)$. By a Theorem of Baer [5], the same relationship holds for a group satisfying the maximal condition on subgroups in which, moreover, $R(G) = \zeta G$. In case G is finite, the Hirsch-Plotkin radical coincides with $\mathrm{Fit}(G)$.

This note results from work which the author has been developing concerning the dynamics of implicit operators on finite semigroups [3, 2]. The starting point is a close relationship between the iterated commutator, as defined above, and a certain form of iterated conjugation. This leads to various new characterizations of nilpotency in finite groups as well as to alternative characterizations of Engel elements in arbitrary groups. The latter is explored in Section 4 of the paper to establish what appear to be new closure properties for the sets $L(G)$ and $R(G)$, including closure by product by the hypercenter in both cases and by the Fitting subgroup in the case of $L(G)$. In Section 3, we also make some elementary observations between our discrete dynamical systems in a group and the associated topological dynamical systems in the profinite completion of the group.

2. Recursive conjugation

We start by relating two different dynamical systems on a group G. The iterated commutator defines one of the dynamical systems: iteratively apply the operator $x \mapsto [x, y]$ to obtain the successive $[x, {}_n y]$. This can also be seen as the action of the operator $(a, b) \mapsto ([a, b], b)$. The other dynamical system consists in taking a pair (a, b) of elements of G and iteratively applying the operator $(a, b) \mapsto (b^{-1}ab, a)$.

Lemma 1. *Let a and b be two elements of an arbitrary group G. Define recursively the sequence $(u_n)_n$ by*

$$u_{-1} = a,\ u_0 = b^{-1}ab,\ u_{n+1} = u_{n-1}^{-1} u_n u_{n-1} \quad (n \geq 0). \tag{1}$$

Then, for every $n \geq 0$, $u_{n-1} = u_n$ if and only if $[b^{-1}, {}_{n+1} a] = 1$.

Proof. Consider the sequence defined recursively by

$$z_0 = b,\ z_{n+1} = z_n u_{n-1} \quad (n \geq 0). \tag{2}$$

Then an easy induction shows that

$$u_n = z_n^{-1} a z_n \quad (n \geq 0). \tag{3}$$

Hence

$$\begin{aligned} u_{n-1} = u_n \quad &\Leftrightarrow \quad u_{n-1}^{-1} z_n^{-1} a z_n = 1 \\ &\Leftrightarrow \quad z_n z_{n-1}^{-1} a^{-1} z_{n-1} z_n^{-1} a = 1 \\ &\Leftrightarrow \quad a^{-n} z_n z_{n-1}^{-1} a^{n-1} \cdot a^{-1} \cdot a^{1-n} z_{n-1} z_n^{-1} a^n \cdot a = 1 \\ &\Leftrightarrow \quad [a^{1-n} z_{n-1} z_n^{-1} a^n, a] = 1 \end{aligned}$$

Thus to prove the lemma it suffices to show that the following equality holds for every $n \geq 0$:

$$[b^{-1}, {}_n a] = a^{1-n} z_{n-1} z_n^{-1} a^n. \tag{4}$$

For $n = 1$, plugging in the values of $z_0 = b$ and $z_1 = ba$, one easily verifies that both sides of (4) are equal to $ba^{-1}b^{-1}a$. Proceeding by induction, assume that (4) holds for a certain value of n. Then, using (2) and (3), we obtain

$$\begin{aligned} [b^{-1}, {}_{n+1} a] = [[b^{-1}, {}_n a], a] &= a^{-n} z_n z_{n-1}^{-1} a^{n-1} \cdot a^{-1} \cdot a^{1-n} z_{n-1} z_n^{-1} a^n \cdot a \\ &= a^{-n} z_n \cdot z_{n-1}^{-1} a^{-1} z_{n-1} \cdot z_n^{-1} a^{n+1} \\ &= a^{-n} z_n \cdot u_{n-1}^{-1} z_n^{-1} \cdot a^{n+1} \\ &= a^{1-(n+1)} z_n z_{n+1}^{-1} a^{n+1} \end{aligned}$$

which shows that (4) holds for every $n \geq 0$ and proves the lemma. □

Let $\psi(a, b) = (v_1, v_2)$ be an operator which makes sense on a group G. For a group G, denote by $\Delta = \Delta_G$ the diagonal $\{(g, g) : g \in G\}$ of $G \times G$. Suppose that $\psi(\Delta) \subseteq \Delta$ and consider the following two subsets of G:

- $L_\psi(G)$ is the set of all $a \in G$ such that, for every $b \in G$, there exists $r \geq 1$ such that $\psi^r(a, b) \in \Delta$;
- $R_\psi(G)$ is the set of all $b \in G$ such that, for every $a \in G$, there exists $r \geq 1$ such that $\psi^r(a, b) \in \Delta$.

Let $\varphi(a, b) = (b^{-1}ab, a)$. By Lemma 1, $L(G) = L_\varphi(G)$ and $R(G) = R_\varphi(G)^{-1}$. Note that $\varphi^{n+1}(b, c) = \varphi^n(c^{-1}bc, b)$. This implies immediately that $R_\varphi(G) \subseteq L_\varphi(G)$, which gives another proof that $R(G)^{-1} \subseteq L(G)$.

As a direct application of Zorn's Theorem and Lemma 1, we obtain the following result.

Corollary 2. *A finite group G is nilpotent if and only if $L_\varphi(G) = G$.* □

While, by an example of Golod [6], not every Engel group is even locally nilpotent, Wilson and Zel'manov [13] showed that a profinite Engel group must be locally nilpotent and Medvedev [8] further proved that every compact Engel group is locally nilpotent.

3. Other dynamical systems

One may ask which dynamical systems share the property of defining nilpotent finite groups in the above sense. In [2], we consider the dynamical systems $\psi(a, b) = (v_1, v_2)$ whose components are defined by binary implicit operations[1] and we show that, if the equation $v_1 = v_2$ is trivial for finite Abelian groups then, for every finite nilpotent group G, every point of $G \times G$ is eventually transformed by the iterate operator into a diagonal point, that is all periodic points lie in the diagonal $\Delta_G = \{(g, g) : g \in G\}$. For a binary operator ψ acting on $G \times G$, denote by $P_\psi(G)$ the set of periodic points of $G \times G$ under the action of the transformation ψ. Note that the property

$$P_\psi(G) \subseteq \Delta_G \tag{5}$$

of a group G with respect to an operator ψ defined by implicit operations is preserved under taking homomorphic images, subgroups and finite direct products. In particular, the finite groups which satisfy it form a pseudovariety. In fact, the property (5) is expressed precisely by a pseudoidentity.

If π is the set of primes which divide the trace of the frequency matrix of the pair (v_1, v_2) (whose (i, j)-entry is obtained by counting, in the free profinite group on one generator, which is also a profinite ring, the number of times the jth generator occurs in v_i), then one can show that every extension of a finite nilpotent group by a finite solvable π-group satisfies property (5) for the operator $\psi(a, b) = (v_1, v_2)$. Thus, if one seeks operators for which property (5) characterizes finite nilpotent groups, one should consider only those operators for which the trace of the frequency matrix is invertible. Of course, if $v_1 = v_2$ then every finite group will satisfy property (5). So, we should also assume the restriction $v_1 \neq v_2$. Further restrictions are needed since, for instance, the symmetric group S_3 satisfies property (5) for the operator $\psi(a, b) = (b^{-1}ab, bab^{-1})$. For this same operator, the set $L_\psi(G)$ is not necessarily a subgroup of a given finite group G as the calculation of this set in S_4 shows that it has 16 elements.

For a group G, denote by $\hat{G}$ its *profinite completion*. To construct it, one may take all homomorphisms $G \to G/N$, where N is a normal subgroup of finite index, and consider the induced homomorphism $\iota : G \to H$ into the product H of all such quotients. The group H is a profinite group under the product topology and $\hat{G}$ may be defined to be the closure of the subgroup

[1] Briefly, an *implicit operation* is an operation which commutes with homomorphisms. In particular, terms in an algebraic language define implicit operations on the corresponding finite algebras. An important example of a unary implicit operation on finite semigroups associates with each element s of a finite semigroup its unique idempotent power, denoted s^ω. Implicit operations on the algebras of a pseudovariety V also have natural interpretations on pro-V algebras, that is on compact algebras which are residually V as topological algebras. See [1, 4] for an introduction to the theory of implicit operations and its role in finite semigroup theory and its applications. An operator $\psi(a_1, \ldots, a_n) = (v_1, \ldots, v_n)$ on profinite groups G whose components are of the form $v_i = (\pi_i)_G(a_1, \ldots, a_n)$ for an implicit operation π_i is called an *n-ary implicit operator*. By abuse of notation which should not lead to confusion, we often use the same notation π_i and v_i. More formally, we should consider an operator $\psi = (\pi_1, \ldots, \pi_n)$ and write $\psi_G(a_1, \ldots, a_n) = (v_1, \ldots, v_n)$ for its interpretation in a given profinite group G, but this distinction is not relevant for the present paper.

$\iota(G)$. Note that $\iota : G \to \hat{G}$ is an embedding if and only if G is residually finite.

Now, the monoid $\operatorname{End} M$ of continuous endomorphisms of a finitely generated profinite monoid M is itself a profinite monoid under the pointwise convergence topology and the evaluation mapping $(\psi, m) \mapsto \psi(m)$ is continuous [4]. Let F_n be the free profinite group on the free generators $x_1, \dots, x_n$. A continuous endomorphism Ψ of F_n defines an n-ary implicit operator $\psi = \big(\Psi(x_1), \dots, \Psi(x_n)\big)$ on profinite groups and Ψ is in turn completely determined by ψ. Moreover, the correspondence $\Psi \mapsto \psi$ defines a homomorphism from $\operatorname{End} F_n$ onto the monoid of n-ary implicit operators on profinite groups and hence the latter is also a profinite monoid under a suitable topology. It is easy to check that this topology has the following property: if the sequence $(\psi_r)_r$ converges to ψ in the monoid of n-ary implicit operators and the sequence $(\bar{a}_r)_r$ converges to $\bar{a}$ in G^n, for a profinite group G, then $\big(\psi_r(\bar{a}_r)\big)_r$ converges to $\psi(\bar{a})$.

Proposition 3. *Let ψ be a binary operator on groups defined by group words such that $\psi(\Delta) \subseteq \Delta$. A group G is such that, for every $a, b \in \hat{G}$, $\psi^\omega(a, b) \in \Delta_{\hat{G}}$ if and only if every finite quotient of G satisfies property* (5).

Proof. Suppose first that, for every $a, b \in \hat{G}$, $\psi^\omega(a, b) \in \Delta_{\hat{G}}$. In a finite group, $\psi^\omega(a, b) = \psi^{n!}(a, b)$ for all sufficiently large n. Hence, in every finite quotient G/N of G, $\psi^{n!}(a, b) \in \Delta_{G/N}$ for all sufficiently large n. Since ψ maps $\Delta_{G/N}$ into itself, it follows that G/N satisfies property (5). Conversely, assume that every finite quotient of G satisfies property (5) and let $a, b \in \hat{G}$. Given a normal subgroup N of G of finite index, the closure $\overline{N}$ of N in $\hat{G}$ is a normal subgroup such that $\hat{G}/\overline{N} \simeq G/N$. Then $\psi^\omega(a, b)\overline{N} \in \Delta_{\hat{G}/\overline{N}}$ by hypothesis and, since this holds for all such N, it follows that $\psi^\omega(a, b) \in \Delta_{\hat{G}}$. □

The following proposition provides an extension of well-known results in Engel theory.

Proposition 4. *Let $\psi(a, b) = (v_1, v_2)$ be a binary implicit operator such that the pseudoidentity $v_1 = v_2$ is valid in all finite Abelian groups and $v_1(1, b) = 1$. Then, for every finite group G, Fit$(G) \subseteq L_\psi(G)$ and $\zeta G \subseteq R_\psi(G)$.*

Proof. Let $\psi^\omega(x, y) = (w_1, w_2)$, where x, y are free generators of a free profinite group. Then we have already observed that, by a result from [2], the pseudoidentity $w_1 = w_2$ holds in every finite nilpotent group. Moreover, in the given finite group G, from $v_1(1, b) = 1$ it follows that $\psi^n(1, b) \in 1 \times G$ for all $b \in G$. Hence, since the subgroup generated by b is certainly nilpotent, we conclude that $w_2(1, b) = w_1(1, b) = 1$.

Let N be a nilpotent normal subgroup of G and let $a \in N$ and $b \in G$. Then $w_i(a, b)N = w_i(1, b)N = N$ by the above, which shows that $w_i(a, b) \in N$ and so

$$\psi^\omega(a, b) = \psi^\omega\big(\psi^\omega(a, b)\big) = \psi^\omega\big(w_1(a, b), w_2(a, b)\big) \in \Delta_G$$

since N is nilpotent. Hence $N \subseteq L_\psi(G)$ and so also $\mathrm{Fit}(G) \subseteq L_\psi(G)$.

Consider next the hypercenter $\zeta_\omega G$. We prove inductively that the following property holds for every integer $r \geq 0$:

$$\psi^\omega(a, az) \in \Delta \text{ for all } a \in G \text{ and } z \in \zeta_r G. \tag{6}$$

Indeed, (6) is trivial for $r = 0$. Assuming it is true for $r = k$, let $z \in \zeta_{k+1}G$ and let $a \in G$. Then $w_i(a, az)\zeta_k G = w_i(a, a)w_i(1, z)\zeta_k G = w_i(a, a)\zeta_k G$ since $z\zeta_k G$ is central in $G/\zeta_k G$ and $w_i(1, z) = 1$. Hence the element $t = \big(w_1(a, az)\big)^{-1}w_2(a, az)$ belongs to $\zeta_k G$. Applying the induction hypothesis, we deduce that

$$\psi^\omega(a, az) = \psi^\omega\big(\psi^\omega(a, az)\big) = \psi^\omega(a', a't) \in \Delta$$

where $a' = w_1(a, az)$. This shows that (6) holds for every $r \geq 0$.

Suppose next that $b \in \zeta_n G$ and let $a \in G$. Let $N = \zeta_{n-1}G$. Then, since bN is central in G/N, we have $w_i(a, b)N = w_i(a, 1)w_i(1, b)N = w_i(a, 1)N = w_1(a, 1)N$, taking also into account that the pseudoidentity $w_1 = w_2$ is valid in all finite Abelian groups. Hence $z = \big(w_1(a, b)\big)^{-1}w_2(a, b) \in N$ and so

$$\psi^\omega(a, b) = \psi^\omega\big(\psi^\omega(a, b)\big) = \psi^\omega(w_1(a, b), w_1(a, b)z) \in \Delta_G$$

by (6), which completes the proof. □

The above result fails without the assumption $v_1(1, b) = 1$. For example, this is the case for the group S_4 and the (Prouhet-Thuë-Morse) operator (ab, ba), whose trace is 2, but also for the same group and the operator $(ab^{-1}a, b^{-1}a^2)$, whose trace is 1.

Calculations with GAP [12] suggest that $\theta(a, b) = (a\,[b, a], a)$ might be another operator for which, in every finite group G, $L_\theta(G) = \mathrm{Fit}(G)$. But the simple argument of taking (cyclic) conjugates which underlies the proof of Lemma 1 does not work for this operator. Nevertheless, we are able to prove the analog of Theorem 2 for the operator θ by a direct argument which is an adaptation of the proof of Zorn's Theorem presented in [11]. Although the argument is given for a somewhat more general situation, this seems to be again a rather special case. For instance, just to indicate a few examples which were found with the help of GAP, S_3 satisfies property (5) for the operators $(a\,[a, b^2], a)$ and $(a\,[b, a^2], a)$, while A_4 satisfies that property for the operator $(a\,[a, b]^2, a)$. Note that the operator φ considered earlier in this paper is given by $(a\,[a, b], a)$.

We say that a binary implicit operation w *behaves like a commutator* if, in every finite group, the following equivalence holds:

$$w(a, b) = 1 \text{ if and only if } ab = ba. \tag{7}$$

Say that a dynamical system is *aperiodic* if all its periodic points are fixed points.

Theorem 5. *Let w be a binary implicit operation which behaves like a commutator and let θ be the binary operator defined by $\theta(a, b) = (aw, a)$. Then a finite group G is nilpotent if and only if the dynamical system $(G \times G, \theta)$ is aperiodic.*

Proof. Note that the property that all periodic points of the dynamical system are fixed points is equivalent to $P_\theta(G) \subseteq \Delta_G$. Indeed, the first property is equivalent $\theta(a, b) = (a, b)$ whenever (a, b) is in the image $\mathrm{Im}\,\theta^\omega$.

By (7) and the definition of θ, the condition $\theta(a,b) = (a,b)$ is equivalent to $a = b$. Hence all periodic points are fixed points if and only if they all lie in the diagonal, that is $P_\theta(G) \subseteq \Delta_G$.

We already observed that since the equation of the components of θ is trivial on Abelian groups, every finite nilpotent group G satisfies the property $P_\theta(G) \subseteq \Delta_G$. Conversely, suppose there exists some finite non-nilpotent group which satisfies that property and consider such a group G of minimum order. By a theorem of O. J. Schmidt [11, Statement 9.1.9], $G = QP$ for a normal Sylow p-subgroup P and a cyclic Sylow q-subgroup Q, where p and q are distinct primes. On the other hand, by P. Hall's criterion for nilpotency [11, Statement 5.2.10], since P is a nilpotent normal subgroup of G, if G/P' were also nilpotent, where P' denotes the commutator subgroup of P, then G would be nilpotent, in contradiction with the initial assumption. But, by the minimality assumption on G, since G/P' still satisfies property (5) for the operator θ, the minimality of G implies that the subgroup P' must be trivial, that is P must be Abelian.

Let a be a generator of the cyclic group Q. We claim that the centralizer $C_P(a)$ is nontrivial. Since P is Abelian, the subgroup $C_P(a)$ is contained in the center Z of G. Once the claim is established, we obtain $Z \neq 1$ and we may then deduce that G/Z is nilpotent by the minimality of G. But then, by considering the upper central series of the group G and taking into account the minimality of G, we immediately conclude that G is nilpotent, in contradiction with the initial assumption. This shows that it suffices to establish the claim to complete the proof.

Let $b \in P \setminus \{1\}$. Define, recursively, a sequence of elements of G as follows:

$$u_{-1} = b,\ u_0 = a,\ u_{n+1} = u_n\, w(u_{n-1}, u_n).$$

Note that if $u_n = u_{n+1}$ then $u_n = u_m$ for every $m \geq n$. Now, since the equation of the components of θ holds in G by hypothesis, there is some n such that $u_n = u_{n+1}$. Let n be the smallest integer for which $u_n = u_{n+1}$. If $n = -1$ or $n = 0$, then $b \in C_P(a)$ and the claim is verified. Suppose next that $n > 0$ for a given $b \in P \setminus \{1\}$.

From $u_{n+1} = u_n$, we deduce that $[u_{n-1}, u_n] = 1$. Substituting in this equation $u_n = u_{n-1}\, w(u_{n-2}, u_{n-1})$, we obtain

$$[u_{n-1}, w(u_{n-2}, u_{n-1})] = 1. \tag{8}$$

Now, for $k \geq 0$, since $P \trianglelefteq G$, an easy induction argument shows that $u_k \in aP$. If $n = 1$ then (8) says that $w(b,a) \in C_P(a) \setminus \{1\}$ by the minimality of n. If $n > 1$, then $g = a^{-1}u_{n-1}$ and $h = w(u_{n-2}, u_{n-1})$ are both elements of P and (8) gives $g^{-1}a^{-1}h^{-1}agh = 1$ so that $a = h^{-1}aghg^{-1} = h^{-1}ah$ since P is Abelian. Hence $h \in C_P(a) \setminus \{1\}$ again by the minimality of n. □

Say that an implicit operator ψ is *aperiodic* on a profinite group G if, in G, $\psi^\omega = \psi^{\omega+1}$. Note that, if G is finite, this condition is equivalent to the dynamical system (G^n, ψ) being aperiodic, where n is the arity of ψ.

Corollary 6. *Let w be a binary implicit operation which behaves like a commutator and let θ be the binary operator defined by $\theta(a,b) = (aw, a)$. Then a profinite group G is pro-nilpotent if and only if θ is aperiodic in G.* □

4. Back to recursive conjugation

The results of Section 2 and the examples of Section 3 show that the operator φ is in a sense very special. Section 3 also provides an example of another operator which shares the property of defining nilpotency for finite groups through the triviality of the cycles in its orbits. But we do not know which other operators share that property. So it is perhaps worth to further study the operator φ.

Throughout this section, G denotes an arbitrary group. The following technical lemma explores the structure of the iterates of the operator φ in some special cases.

Lemma 7. *Suppose that $a, b \in G$ are such that $\varphi^n(a, b) \in \Delta$ for some $n \geq 0$. If b belongs to a normal subgroup A such that $c \in C_G(A)$, then $\varphi^m(ac, b) \in \Delta$ for some $m \geq 0$.*

Proof. Define two sequences $(u_k)_{k\geq -1}$ and $(v_k)_{k\geq -1}$ recursively as follows:

$$u_{-1} = a,\ u_0 = b^{-1}ab,\ u_{k+2} = u_k^{-1}u_{k+1}u_k,$$
$$v_{-1} = ac,\ v_0 = b^{-1}acb,\ v_{k+2} = v_k^{-1}v_{k+1}v_k.$$

We claim that, for every $n \geq 0$,

$$v_n = u_n c. \tag{9}$$

This is obvious for $n = 0$ and $v_1 = b^{-1}acb = b^{-1}abc = u_1c$ since b and c commute. Suppose that (9) holds for $n \in \{k, k+1\}$. Then $v_{k+2} = c^{-1}u_k^{-1}u_{k+1}cu_kc$ and so (9) holds for $n = k+2$ if we can show that $u_n^{-1}u_{n+1} \in A$ for every $n \geq 0$. For $n = 0$, we have $u_n^{-1}u_{n+1} = a^{-1}b^{-1}a \cdot b \in A$ since A is a normal subgroup of G. Suppose that $u_n^{-1}u_{n+1} \in A$. Then

$$\begin{aligned} u_{n+1}^{-1}u_{n+2} &= u_n^{-1}(u_{n+1}u_n^{-1})^{-1}u_n^{-1}u_{n+1}u_n \\ &= u_n^{-1} \cdot (u_n \cdot u_n^{-1}u_{n+1} \cdot u_n^{-1})^{-1} \cdot u_n^{-1}u_{n+1} \cdot u_n \end{aligned}$$

belongs to A since A is normal subgroup of G. This completes the proof of the claim that (9) holds for every $n \geq 0$.

Using (9), from the hypothesis that $u_n = u_{n+1}$ for some $n \geq -1$ we deduce that $v_n = v_{n+1}$ for some $n \geq 0$, which proves the lemma. □

This leads to the following closure properties for $L(G)$.

Theorem 8. *The set $L(G)$ is closed under multiplication by Fit(G).*

Proof. Since Fit(G) is generated by all nilpotent normal subgroups of G, it suffices to show that $L(G)N \subseteq L(G)$ for every nilpotent normal subgroup N of G. We proceed by induction on the nilpotent class of N. So, assume the result holds whenever we have a group with a normal subgroup which is of nilpotent class smaller than that of N. If $N = 1$, then the desired conclusion is trivial. We assume therefore that $N \neq 1$. Then the center $A = Z(N)$ is a characteristic subgroup of N and so it is a normal subgroup of G. In the quotient group G/A, the normal subgroup N/A is nilpotent of class smaller than that of N. Hence we may apply the induction hypothesis to G/A and the subgroup N/A.

Consider next $c \in N$ and $a \in L(G)$. Given any element $b \in G$, there exists $r \geq 0$ such that $\varphi^r(a, b) \in \Delta_G$. In the quotient group G/A, we have

$aA \in L(G/A)$ and $cA \in N/A$. By the induction hypothesis, there exists $m \geq 0$ such that $\varphi^m(acA, bA) \in \Delta_{G/A}$. Let $(u, v) = \varphi^m(ac, b)$. Then we have $v = ug$ for some $g \in A$. Note that $\varphi^k(u, ug) = \varphi^k(u, g)$ for every $k \geq 1$. Moreover, since u is a conjugate of ac, it is of the form $u = a'c'$ where a' is a conjugate of a and c' is a conjugate of c. Hence $a' \in L(G)$, $c' \in N$, $g \in A$, and c' centralizes A since $A = Z(N)$. By Lemma 7, it follows that $\varphi^n(u, g) = \varphi^n(a'c', g) \in \Delta_G$ for some $n \geq 1$. Hence $\varphi^{m+n}(ac, b) \in \Delta_G$ and, since $b \in G$ is arbitrary, this shows that $ac \in L(G)$. □

The same argument may be used to establish the following result.

Theorem 9. *For a nilpotent normal subgroup N of G which is contained in $R(G)$, we have $R(G)\, N \subseteq R(G)$.* □

An adequate transfinite induction scheme also allows us to show that $L(G)$ and $R(G)$ are unions of cosets of the hypercenter.

Theorem 10. *Each of the subsets $L(G)$ and $R(G)$ is closed by multiplication by the hypercenter.*

Proof. We show, by induction on an ordinal α, that $L(G)\, \zeta_\alpha G \subseteq L(G)$. Since the inclusion is trivial in case $\alpha = 0$, it suffices to consider the case where $\alpha = \beta + 1$ is a successor ordinal. Let $N = \zeta_\beta G$ and consider $a \in L(G)$ and $c \in \zeta_\alpha G$. Given $b \in G$, there exists $r \geq 0$ such that $\varphi^r(a, b) \in \Delta_G$. Since, in the quotient group G/N, the element cN is central, by Lemma 7 there exists $m \geq 0$ such that $\varphi^m(acN, bN) \in \Delta_{G/N}$. Let $(u, v) = \varphi^m(ac, b)$. Then we have $v = ug$ for some $g \in N$. Since $\varphi^k(u, ug) = \varphi^k(u, g)$ for every $k \geq 1$ and $g \in N \subseteq \zeta G \subseteq R(G)^{-1} = R_\varphi(G)$, it follows that there exists $n \geq 1$ such that $\varphi^n(u, g) \in \Delta_G$. Hence $\varphi^{m+n}(ac, b) \in \Delta_G$, which shows that $ac \in L(G)$.

The proof for the case of $R(G)$ is similar but simpler since, for a central element cN of G/N, $\varphi(a, bc)N = \varphi(a, b)N$, thereby replacing the reference to Lemma 7 in this case. □

We can also prove the following technical result.

Proposition 11. *Suppose that $a, b \in G$ are such that $\varphi^n(a, b) \in \Delta$ for some $n \geq 0$. Further let c be an element of an Abelian normal subgroup A of G. If $a \in L(G)$ or $c \in R(G)$, then $\varphi^m(ac, b) \in \Delta$ for some $m \geq 0$.*

Proof. As in the proof of Lemma 1, define two sequences $(u_k)_{k \geq -1}$ and $(z_k)_{k \geq 0}$ recursively as follows:

$$u_{-1} = a,\ u_0 = b^{-1}ab,\ u_{k+2} = u_k^{-1} u_{k+1} u_k,$$
$$z_0 = b,\ z_{k+1} = z_k u_{k-1} \quad (k \geq 0).$$

Define a new sequence $(v_k)_{k \geq -1}$ by

$$v_{-1} = ac,\ v_0 = b^{-1}acb,\ v_{k+2} = v_k^{-1} v_{k+1} v_k.$$

Let $t_k = z_k^{-1} c z_k$ so that $u_{k-1}^{-1} t_k u_{k-1} = t_{k+1}$ and

$$t_{k+1} = t_k\, [t_k, u_{k-1}] \tag{10}$$

Let $(w_k)_{k \geq 0}$ be defined recursively by

$$w_0 = c,\ w_{k+1} = [w_k, u_k]\, t_k \tag{11}$$

and let

$$s_0 = 1,\ \ s_{k+1} = s_k w_k = w_0 w_1 \cdots w_k.$$

Since A is a normal subgroup of G and $c \in A$, each t_k, w_k, s_k belongs to A.

We claim that

$$v_k = s_k^{-1} u_k s_k \cdot t_k \tag{12}$$

for every $k \geq 0$. For $k = 0$, we have $v_0 = b^{-1}acb = b^{-1}abb^{-1}cb = s_0^{-1}u_0 s_0 t_0$. For $k = 1$, we have

$$\begin{aligned} v_1 &= v_{-1}^{-1} v_0 v_{-1} \\ &= c^{-1}u_{-1}^{-1}u_0 u_{-1} c \cdot c^{-1} u_{-1}^{-1} t_0 u_{-1} c \\ &= c^{-1} u_1 c \cdot t_1 \quad \text{since } u_{-1}^{-1} t_0 u_{-1} = t_1 \text{ and } c, t_1 \in A \\ &= s_1^{-1} u_1 s_1 \cdot t_1. \end{aligned}$$

Assume next that the formula (12) holds for $k \in \{r, r+1\}$. Then we have

$$\begin{aligned} v_{r+2} &= v_r^{-1} v_{r+1} v_r \\ &= v_r^{-1} \cdot (w_r^{-1} \cdot s_r^{-1} u_{r+1} s_r \cdot w_r \cdot t_{r+1}) \cdot v_r \end{aligned}$$

by (12) with $k = r+1$ and the definition of s_{r+1}. We split the last conjugate under v_r into four conjugates which we compute separately using (12) with $k = r$ for the value of v_r:

$$\begin{aligned} v_r^{-1} w_r v_r &= t_r^{-1} s_r^{-1} \cdot u_r^{-1} s_r w_r s_r^{-1} u_r \cdot s_r t_r \\ &= u_r^{-1} w_r u_r \quad \text{since } w_r, u_r^{-1} w_r u_r, t_r, s_r \in A \\ &= w_r\,[w_r, u_r], \end{aligned}$$

$$\begin{aligned} v_r^{-1} s_r^{-1} u_{r+1} s_r v_r &= t_r^{-1} s_r^{-1} u_r^{-1} s_r \cdot s_r^{-1} u_{r+1} s_r \cdot s_r^{-1} u_r s_r t_r \\ &= t_r^{-1} s_r^{-1} \cdot u_r^{-1} u_{r+1} s_r s_r^{-1} u_r \cdot s_r t_r \\ &= t_r^{-1} s_r^{-1} u_{r+2} s_r t_r, \end{aligned}$$

$$\begin{aligned} v_r^{-1} t_{r+1} v_r &= t_r^{-1} s_r^{-1} u_r^{-1} s_r \cdot t_{r+1} \cdot s_r^{-1} u_r s_r t_r \\ &= u_r^{-1} t_{r+1} u_r \quad \text{since } u_r^{-1} s_r t_{r+1} s_r^{-1} u_r, s_r, t_r, t_{r+1} \in A \\ &= t_{r+2}. \end{aligned}$$

Putting it all together and once again taking into account that A is Abelian, we obtain

$$\begin{aligned} v_{r+2} &= (w_r\,[w_r, u_r])^{-1} \cdot t_r^{-1} s_r^{-1} u_{r+2} s_r t_r \cdot [w_r, u_r]\, w_r \cdot t_{r+2} \\ &= ([w_r, u_r]\, t_r)^{-1} \cdot w_r^{-1} s_r^{-1} u_{r+2} s_r w_r \cdot [w_r, u_r]\, t_r \cdot t_{r+2} \\ &= w_{r+1}^{-1} \cdot s_{r+1}^{-1} u_{r+2} s_{r+1} \cdot w_{r+1} \cdot t_{r+2} \\ &= s_{r+2}^{-1} u_{r+2} s_{r+2} \cdot t_{r+2}, \end{aligned}$$

which establishes the claim.

Assume that $u_{n+1} = u_n$. Then, $u_m = u_n$ for all $m \geq n$ and so, in view of formula (12), $s_m^{-1} s_{m+1} = w_m$, formula (10), and since $w_m, s_m, t_m, [t_m, u_n]$

belong to the Abelian subgroup A, we have, for $m \geq n$,

$$\begin{aligned} v_{m+1} = v_m \quad &\Leftrightarrow \quad s_{m+1}^{-1} u_n s_{m+1} t_{m+1} = s_m^{-1} u_n s_m t_m \\ &\Leftrightarrow \quad w_m^{-1} u_n w_m [t_m, u_n] = u_n \\ &\Leftrightarrow \quad [w_m, u_n] = [t_m, u_n]. \end{aligned}$$

Hence, to complete the proof of the proposition, it suffices to establish the equality

$$[w_m, u_n] = [t_m, u_n] \tag{13}$$

for some $m \geq n$. In order to prove this equality, we claim next that, for $k \geq 0$,

$$[w_{n+k}, u_n] = [t_{n+k}, u_n] \Leftrightarrow [w_n, {}_k u_n] = [t_n, {}_k u_n]. \tag{14}$$

To prove (14), it suffices to apply inductively the following equalities for integers $k, r \geq 0$:

$$\begin{aligned} [w_{n+k+1}, {}_r u_n] &= \big[[w_{n+k}, u_n] t_{n+k}, {}_r u_n\big] = [w_{n+k}, {}_{r+1} u_n]\,[t_{n+k}, {}_r u_n] \\ [t_{n+k+1}, {}_r u_n] &= \big[[t_{n+k}, u_n] t_{n+k}, {}_r u_n\big] = [t_{n+k}, {}_{r+1} u_n]\,[t_{n+k}, {}_r u_n] \end{aligned}$$

where we use the observation that, since the normal subgroup A is Abelian, the transformation

$$x \in A \mapsto [x, y] \tag{15}$$

is an endomorphism of A for every $y \in G$ (in particular for $y = u_n$) and formulas (10) and (11). This proves the claim that the equivalence (14) holds for all $k \geq 1$.

Finally, we show that

$$[w_n, {}_k u_n] = [t_n, {}_k u_n] = 1 \quad \text{for all sufficiently large } k, \tag{16}$$

from which it follows that (13) holds for all sufficiently large m. Suppose first that $a \in L(G)$. Then $u_n \in L(G)$ since $L(G)$ is closed under automorphisms of G and u_n is a conjugate of a. Hence (16) holds by definition of $L(G)$.

Suppose next that $c \in R(G)$. Since (15) is an endomorphism of A, the product of two elements of A (respectively the inverse) which lie in $R(G)$ also lies in $R(G)$. In particular, since $R(G)$ is closed under conjugation in G, we conclude that $w_n \in R(G)$. Also, $t_n \in R(G)$ since t_n is a conjugate of c. This shows that (16) holds and finishes the proof of the proposition. □

Dually, we obtain the following result.

Proposition 12. *Suppose that $a, b \in G$ are such that $\varphi^n(a, b) \in \Delta$ for some $n \geq 0$. Further let c be an element of an Abelian normal subgroup A of G. If $a \in L(G)$ or $c \in R(G)$, then $\varphi^m(ca, b) \in \Delta$ for some $m \geq 0$.*

Proof. We deduce Proposition 12 from Proposition 11 by a duality argument. Let H be the group which is obtained from G by reversing the multiplication: $x \cdot y = yx$. Inversion defines an isomorphism $\eta : G \to H$. Let $\eta_2 : G \times G \to H \times H$ be the isomorphism which is η component-wise. Given a binary operator ψ defined by group terms, denote by ψ_K its interpretation in a group K. Then we have the equality

$$\eta_2 \circ \psi_G = \psi_H \circ \eta_2. \tag{17}$$

From Lemma 1, we deduce that $L(G) = \eta\big(L(H)\big)$ and $R(G) = \eta\big(R(H)\big)$. Moreover, also using equation (17), the hypothesis that $\varphi_G^n(a,b) \in \Delta$ for some $n \geq 0$ yields $\varphi_H^n(a^{-1}, b^{-1}) \in \Delta$ for the same n. Since we assume that either $a^{-1} \in L(H)$ or $c^{-1} \in R(H)$, Proposition 11 guarantees the existence of some $m \geq 0$ such that $\varphi_H^m(a^{-1} \cdot c^{-1}, b^{-1}) \in \Delta$. Hence, by (17), we also have $\varphi_G^m(ca, b) \in \Delta$. □

We conclude with an application of Proposition 11 and its dual Proposition 12. It is an improvement of Theorem 9 at the expense of replacing nilpotent normal subgroups by Abelian normal subgroups.

Theorem 13. *For an Abelian normal subgroup A of G, the intersection $A \cap R(G) \cap R(G)^{-1}$ is a (normal) subgroup of G.*

Proof. Let $b \in R(G)$ and $c \in A \cap R(G) \cap R(G)^{-1}$. The statement of the theorem is equivalent to showing that $bc \in R(G)$.

Given any $a \in G$, there exists $n \geq 0$ such that $\varphi^n(a, b) \in \Delta$ and so also such that

$$\varphi^n(b^{-1}ab, a) \in \Delta. \tag{18}$$

We need to show that $\varphi^m(a, bc) \in \Delta$ for some $m \geq 0$, for which it suffices to show that there exists some $\ell \geq 0$ such that

$$\varphi^\ell(c^{-1}b^{-1}abc, a) \in \Delta. \tag{19}$$

Since $c \in A \cap R(G)$, from (18), using Proposition 11, it follows that there exists some $m \geq 0$ such that

$$\varphi^m(b^{-1}abc, a) \in \Delta. \tag{20}$$

Similarly, since $c^{-1} \in R(G)$, using Proposition 12, we deduce from (20) that there exists $\ell \geq 0$ such that condition (19) holds. This completes the proof of the theorem. □

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