Toric Geometry and Equivariant Bifurcations

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To the memory of John David Crawford

Abstract

Many problems in equivariant bifurcation theory involve the computation of invariant functions and equivariant mappings for the action of a torus group. We discuss general methods for finding these based on some elementary considerations related to toric geometry, a powerful technique in algebraic geometry. This approach leads to interesting combinatorial questions about cones in lattices, which lead to explicit calculations of minimal generating sets of invariants, from which the equivariants are easily deduced. We also describe the computation of Hilbert series for torus invariants and equivariants within the same combinatorial framework. As an example, we apply these methods to the interaction of two linear modes of a Euclidean-invariant PDE on a rectangular domain with periodic boundary conditions.

1 Introduction

Equivariant bifurcation theory is the study of parametrised families of ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \tag{1}$$

when the vector field f is equivariant under a group action. More specifically, we shall work in the following context: $x \in \mathbf{R}^m$, $\lambda \in \mathbf{R}^p$ is a bifurcation parameter, and f is a smooth vector field on $\mathbf{R}^m$. Let Γ be a compact Lie group acting linearly on $V = \mathbf{R}^m$. We say that a vector field h on V is Γ-*equivariant* if

$$h(\gamma x) = \gamma h(x) \;\; \forall \gamma \in \Gamma$$

and a function $g : V \to \mathbf{R}$ is Γ-*invariant* if

$$g(\gamma x) = g(x) \ \ \forall \gamma \in \Gamma$$

In equivariant bifurcation theory we usually assume that Γ acts trivially on the bifurcation parameter λ and that f is equivariant in x, so that $f(\gamma x, \lambda) = \gamma f(x, \lambda)$.

The analysis of (1) depends upon obtaining fairly detailed information about equivariant maps f. For the following discussion we temporarily suppress the parameter λ, to simplify notation. The space $\vec{\mathcal{E}}_V(\Gamma)$ of smooth equivariants is a module over the ring $\mathcal{E}_V(\Gamma)$ of smooth invariants. Moreover, the structure of $\vec{\mathcal{E}}_V(\Gamma)$ as a module over $\mathcal{E}_V(\Gamma)$ can be inferred easily from their analogues $\vec{\mathcal{P}}_V(\Gamma)$ and $\mathcal{P}_V(\Gamma)$ for polynomial mappings, by virtue of well known theorems of Poénaru and Schwartz; see Golubitsky *et al.* [17]. These results place the problem within the ambit of classical invariant theory, so that in essence it becomes an algebraic (or algebro-geometric) question.

It is usually fairly difficult to determine $\mathcal{P}_V(\Gamma)$ and $\vec{\mathcal{P}}_V(\Gamma)$, however, except in simple cases. For example, when $\Gamma = \mathbf{SO}(3)$ and $\mathbf{R}^m$ is the 7-dimensional space of spherical harmonics of degree 3, this calculation is already quite complicated and technical (see for example Lauterbach [30]).

From the point of view of invariant theory, one of the best-behaved classes of groups comprises the *torus groups*

$$\mathbf{T}^k \cong \underbrace{\mathbf{S}^1 \times \cdots \times \mathbf{S}^1}_{k}$$

where $\mathbf{S}^1 = \mathbf{R}/2\pi\mathbf{Z}$ is the circle group. Any torus action can be diagonalized over the complex numbers. Each invariant polynomial (under $\mathbf{T}^k$) is a linear combination of invariant monomials. In particular, there are minimal generating sets for the ring of invariants that contain only monomials. Moreover, the invariant monomials are in one-to-one correspondence with the elements of a semigroup S. See for example Wehlau [41] and Kempf [29]. This semigroup was studied by Gordan [22] (see also Danilov [12]), where the ring of $\mathbf{T}^k$-invariant polynomials is proved to be a finitely generated algebra by showing that S is finitely generated as a semigroup. This fact permits the introduction of combinatorial methods when determining invariants and equivariants. Moreover, the equivariants can easily be deduced from the invariants, Gomes and Stewart [20]. The invariant theory of torus actions falls within the area of algebraic geometry known as *toric geometry*, the study of *toric varieties*, surveyed in Danilov [12].

Torus actions are natural in a variety of equivariant bifurcation problems arising from applications. Typically, the appropriate symmetry group is not a torus as such, but a finite extension of a torus. That is, the connected component Γ° of the identity of Γ is a torus, and $Q = \Gamma/\Gamma^\circ$ is finite (as it must be for compact Γ). The invariants and equivariants for Γ can therefore be found by first restricting to Γ°, and then symmetrizing over Q. Circumstances in which such groups arise include:

- Bifurcating waves on a crystallographic lattice. The torus is generated by translations modulo the dual lattice, and Q is the holohedry.

- Problems with circular or cylindrical geometry. Here there is a rotational $\mathbf{T}^1$ symmetry, and in the cylindrical case with periodic boundary conditions, translations modulo the period convert this to $\mathbf{T}^2$. See for example the discussion of Couette-Taylor flow in Golubitsky and Stewart [16], Golubitsky *et al.* [17].

- Say that a PDE posed on a domain $\Omega \subseteq \mathbf{R}^k$ is *Euclidean-invariant* if the image of any solution under a Euclidean transformation of $\mathbf{R}^k$ (that is, a rigid motion) is again a solution, see for example Melbourne [33]. Euclidean-invariant PDEs on multidimensional rectangular domains with periodic boundary conditions have torus symmetry. This fact has been extensively exploited in connection with 'hidden symmetries' in analogous problems with Neumann or Dirichlet boundary conditions. References include Crawford [5, 6, 7, 8], Crawford *et al.* [9, 10], Armbruster and Dangelmayr [1], Dangelmayr and Armbruster [11], Ashwin [2], Ashwin *et al.* [3], Castro [4], Gomes [18, 19], Gomes and Stewart [20, 21], Healey and Kielhöfer [23, 24, 25], Impey [26], Impey *et al.* [27, 28], Manoel [31], Manoel and Stewart [32], and Riley and Winters [36].

In this paper we make a start on putting together the powerful techniques of toric geometry and the prevalence of torus group symmetry in equivariant bifurcation theory. Most of our discussion takes place within one family of examples: steady-state mode interactions between solutions of a Euclidean-invariant PDE on a rectangle $\Omega \subseteq \mathbf{R}^2$, subject to periodic boundary conditions. In section 2 we introduce some elementary ideas and terminology from toric geometry. In section 3 we discuss invariants and equivariants for a special class of torus actions arising in connection with mode interactions in a rectangular domain. Section 4 develops intuition on an example, the $(3,2)-(1,3)$ mode interaction. Section 5 extends the analysis to a general $(k_1,\ell_1)-(k_2,\ell_2)$ mode interaction. In section 6 we compute the Hilbert series of torus invariants for a $(k_1,\ell_1)-(k_2,\ell_2)$ mode interaction. Section 7 relates volumes of fundamental lattice parallelotopes with the number of candidates for the generators of the $\mathbf{T}^2$-invariants. In section 8 we compute the Hilbert series of torus equivariants for a $(k_1,\ell_1)-(k_2,\ell_2)$ mode interaction. Finally section 9 sets up an analogous discussion of general torus actions.

2 Toric Geometry

In the algebro-geometric treatment of the topic, it is usual to work over the field $\mathbf{Q}$ of rational numbers. Bifurcation theorists, on the other hand, will prefer to work over $\mathbf{R} \supseteq \mathbf{Q}$. The elementary parts of the theory (which are all we need here) work equally well in both contexts, so bifurcation theorists should read $\mathbf{Q}$ but think $\mathbf{R}$, especially when visualising the geometry.

Let V be a finite-dimensional vector space over $\mathbf{Q}$. Let $L : V \to \mathbf{Q}$ be a $\mathbf{Q}$-linear map, so that (for non-zero L) the zero-set

$$L^0 = \{v \in V : L(v) = 0\}$$

is a hyperplane. The complement of L^0 decomposes into positive and negative *open linear half-spaces*

$$L^+ = \{v \in V : L(v) > 0\}$$
$$L^- = \{v \in V : L(v) < 0\}$$

and of course $(-L)^{\pm} = L^{\mp}$. Our main interest is in the *closed linear half-space*

$$L^{+0} = \{v \in V : L(v) \geq 0\} = L^+ \cup L^0$$

A *cone* $\sigma \subseteq V$ is the intersection of finitely many closed linear half-spaces:

$$\sigma = \bigcap_{i=1}^{k} L_i^{+0}$$

A *face* of σ is a subset of the form $\sigma \cap L_i^0$. A face of a cone is a cone, and the intersection of finitely many faces is a face. A *fan* in V is a collection Σ of cones such that: (a) every cone of Σ has a vertex; (b) if τ is a face of a cone $\sigma \in \Sigma$, then $\tau \in \Sigma$; (c) if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face both of σ and σ'.

For any set of vectors $\{v_1, ..., v_\ell\} \subseteq V$ we let $\langle v_1, ..., v_\ell \rangle$ be the smallest cone containing $\{v_1, ..., v_\ell\}$. Every cone can be written in this form. If the v_j are linearly independent, then the cone $\langle v_1, ..., v_\ell \rangle$ is said to be *simplicial.* The *dimension* of a cone σ is the dimension over $\mathbf{Q}$ of the $\mathbf{Q}$-vector space spanned by σ. Similarly, a fan is said to be *simplicial* if it consists of simplicial cones.

A *lattice* $\mathcal{M} \subseteq V$ is a free Abelian group of finite rank: its rank is called its *dimension.* A *cone in* $\mathcal{M}$ is a cone in the $\mathbf{Q}$-vector space spanned by $\mathcal{M}$. If σ is a simplicial cone in $\mathcal{M}$ and $\sigma = \langle v_1, ..., v_k \rangle$ where the v_j belong to $\mathcal{M}$ (and are linearly independent), call $P_\sigma^{\mathcal{M}} = \{\sum_{i=1}^k \mu_i v_i : 0 \leq \mu_i \leq 1\}$ its *fundamental parallelotope.* If σ is a cone in $\mathcal{M}$ then $\sigma \cap \mathcal{M}$ is a (commutative) subsemigroup. The following lemma goes back to Gordan, see Danilov [12] or Fulton [14]:

Lemma 2.1 (Gordan) *The semigroup $\sigma \cap \mathcal{M}$ is finitely generated.*

Proof

The proof is straightforward, but for completeness we give it here. By breaking σ up into a union of simplicial cones, we may without loss of generality assume that σ is simplicial, so that $\sigma = \langle v_1, ..., v_k \rangle$ where the v_j belong to $\mathcal{M}$ and are linearly independent. Form the fundamental parallelotope $P_\sigma^{\mathcal{M}} = \{\sum_{i=1}^k \mu_i v_i : 0 \leq \mu_i \leq 1\}$. Since $P_\sigma^{\mathcal{M}}$ is compact and $\mathcal{M}$ is discrete, the intersection $P_\sigma^{\mathcal{M}} \cap \mathcal{M}$ is finite. Clearly any element v of $\sigma \cap \mathcal{M}$ can be represented (uniquely) in the form $\sum_{i=1}^k r_i v_i$ with $r_i \geq 0$, so $r_i = m_i + t_i$ with m_i a nonnegative integer and $0 \leq t_i \leq 1$. Then $v = \sum m_i v_i + v'$, with each v_i and $v' = \sum t_i v_i$ in $P_\sigma^{\mathcal{M}} \cap \mathcal{M}$. Therefore the finite set $P_\sigma^{\mathcal{M}} \cap \mathcal{M}$ generates $\sigma \cap \mathcal{M}$ as a semigroup. ∎

	z_1	z_2	z_3	z_4
θ_1	$e^{ik_1\theta_1}z_1$	$e^{ik_1\theta_1}z_2$	$e^{ik_2\theta_1}z_3$	$e^{ik_2\theta_1}z_4$
θ_2	$e^{i\ell_1\theta_2}z_1$	$e^{-i\ell_1\theta_2}z_2$	$e^{i\ell_2\theta_2}z_3$	$e^{-i\ell_2\theta_2}z_4$
ρ_1	$\overline{z}_2$	$\overline{z}_1$	$\overline{z}_4$	$\overline{z}_3$
ρ_2	z_2	z_1	z_4	z_3

Table 1: Group action for the $(k_1,\ell_1)-(k_2,\ell_2)$ mode interaction.

3 Mode Interactions in a Rectangle

Let $\Omega \subseteq \mathbf{R}^2$ be a rectangular domain in the plane, say $\Omega = [0,A]\times[0,B] (A \neq B)$, and consider a parametrised family of Euclidean-invariant PDEs $\frac{\partial u}{\partial t}+\mathcal{P}(u,\lambda)=0$ subject to periodic boundary conditions (PBC). Steady-state solutions correspond to zeros of the partial differential operator $\mathcal{P}$. Because of Euclidean invariance and PBC, this problem is invariant under a torus group $\mathbf{T}^2$ consisting of translations in $\mathbf{R}^2$ modulo the lattice generated by $(A,0)$ and $(0,B)$. It is also invariant under reflections in the two coordinate directions, leading to a symmetry group $\mathbf{O}(2)\times\mathbf{O}(2)$. Since $A \neq B$ there are no further obvious domain symmetries (although in some cases there may be 'hidden rotations', see Crawford [7]). This symmetry implies that linearized solutions are superpositions of eigenfunctions

$$\sin\left(\frac{2k\pi x}{A}\right)\sin\left(\frac{2\ell\pi y}{B}\right) \qquad \sin\left(\frac{2k\pi x}{A}\right)\cos\left(\frac{2\ell\pi y}{B}\right)$$
$$\cos\left(\frac{2k\pi x}{A}\right)\sin\left(\frac{2\ell\pi y}{B}\right) \qquad \cos\left(\frac{2k\pi x}{A}\right)\cos\left(\frac{2\ell\pi y}{B}\right)$$

where $k,\ell \in \mathbf{N}$ are known as *mode numbers.*

If the parameter λ can be chosen so that modes (k_1,ℓ_1) and (k_2,ℓ_2) bifurcate simultaneously, then the equation undergoes a *mode interaction* at such a value: see Gomes and Stewart [20] for details. At such a mode interaction the bifurcation can be reduced to a finite-dimensional problem, the so-called *(Liapunov-Schmidt) reduced bifurcation equation*, see Golubitsky *et al.* [17]. The reduced bifurcation equation inherits the symmetries of the original problem (provided the reduction procedure is carried out using group-invariant subspaces, which is always possible) and hence is equivariant under a torus group

$$\mathbf{T}^2 = \{(\theta_1,\theta_2) : \theta_1,\theta_2 \in \mathbf{S}^1\}$$

and reflections ρ_1,ρ_2. The group action of $\mathbf{O}(2)\times\mathbf{O}(2)$ for this mode interaction is shown in Table 1. The amplitudes z_1,z_2 belong to the (k_1,ℓ_1) mode and z_3,z_4 to the (k_2,ℓ_2) mode.

We focus on the subgroup $\mathbf{T}^2$. Because the $\mathbf{T}^2$-action is diagonal, all $\mathbf{T}^2$-invariants are generated by monomials

$$z_1^{\alpha_1}\overline{z}_1^{\beta_1}z_2^{\alpha_2}\overline{z}_2^{\beta_2}z_3^{\alpha_3}\overline{z}_3^{\beta_3}z_4^{\alpha_4}\overline{z}_4^{\beta_4} \quad \alpha_j,\beta_j \in \mathbf{N}$$

In order to be invariant, these monomials must satisfy the conditions

$$k_1(\gamma_1+\gamma_2)+k_2(\gamma_3+\gamma_4)=0 \tag{2}$$

$$\ell_1(\gamma_1-\gamma_2)+\ell_2(\gamma_3-\gamma_4)=0 \tag{3}$$

where

$$\gamma_j=\alpha_j-\beta_j \quad (j=1,2,3,4.)$$

Equations (2,3) define a 2-dimensional lattice $\mathcal{L}\subseteq \mathbf{Z}^4$. We study this lattice further in section 5.

Lemma 3.1 *The* $\mathbf{T}^2$*-invariants on* $\mathbf{C}^4$ *are (non-minimally) generated by* $|z_j|^2$, $(j=1,2,3,4)$ *and all monomials of the form* $w_1^{|\gamma_1|}w_2^{|\gamma_2|}w_3^{|\gamma_3|}w_4^{|\gamma_4|}$*, where* $w_j=z_j$ *if* $\gamma_j\geq 0$, $w_j=\overline{z}_j$ *if* $\gamma_j<0$*, and the* γ_j *satisfy (2,3).*

Proof

Clearly $|z_j|^2=z_j\overline{z}_j$ is invariant. Let $m=z_1^{\alpha_1}\overline{z}_1^{\beta_1}z_2^{\alpha_2}\overline{z}_2^{\beta_2}z_3^{\alpha_3}\overline{z}_3^{\beta_3}z_4^{\alpha_4}\overline{z}_4^{\beta_4}$ be an invariant monomial. Let

$$\delta_j=\min(\alpha_j,\beta_j),\ \ j=1,...,4$$

and write m in the form

$$m=\left(|z_1|^2\right)^{\delta_1}\left(|z_2|^2\right)^{\delta_2}\left(|z_3|^2\right)^{\delta_3}\left(|z_4|^2\right)^{\delta_4}m'$$

where $m'=z_1^{\alpha_1'}\overline{z}_1^{\beta_1'}z_2^{\alpha_2'}\overline{z}_2^{\beta_2'}z_3^{\alpha_3'}\overline{z}_3^{\beta_3'}z_4^{\alpha_4'}\overline{z}_4^{\beta_4'}$. Then m' is also invariant, and for each $j=1,...,4$ either $\alpha_j'=0$ or $\beta_j'=0$. Moreover, $\gamma_j'=\alpha_j'-\beta_j'=\gamma_j$, so $m'=w_1^{|\gamma_1|}w_2^{|\gamma_2|}w_3^{|\gamma_3|}w_4^{|\gamma_4|}$. ∎

The following theorem, presumably well known, reduces the computation of equivariants to that for invariants:

Theorem 3.2 *Let* $\mathbf{T}^k$ *act diagonally on* $\mathbf{C}^k$ *with coordinates* $z_j,\overline{z}_j$ *for* $j=1,...,k$.
(a) If $I_1,...,I_s$ *generate the* $\mathbf{C}$*-valued invariants, then the equivariants are generated over the invariants by the mappings:*

$$\text{row}\, j\rightarrow \begin{pmatrix}0\\ \vdots\\ 0\\ \frac{\partial I_g}{\partial \overline{z}_j}\\ 0\\ \vdots\\ 0\end{pmatrix} \tag{4}$$

for $1\leq j\leq k$ *and* $1\leq g\leq s$.
(b) If $I_1,\ldots,I_p$ *is a set of invariant monomials that spans the space of invariants of degree* $d+1$*, then the maps (4), for* $1\leq j\leq k$ *and* $1\leq i\leq p$*, span the equivariants of degree* d.

Proof

Part (a) is proved in Gomes and Stewart [20]. Exactly the same calculations prove part (b). ∎

A monomial of the form $w_1^{|\epsilon_1|}w_2^{|\epsilon_2|}w_3^{|\epsilon_3|}w_4^{|\epsilon_4|}$, where w_j is either equal to z_j or $\overline{z}_j$, is said to be *reduced*. Equivalently, a reduced monomial is one that is not divisible by any $z_j\overline{z}_j$. So Lemma 3.1 states that the invariants are generated by the $|z_j|^2$ together with certain reduced monomials. The form of $w_1^{|\gamma_1|}w_2^{|\gamma_2|}w_3^{|\gamma_3|}w_4^{|\gamma_4|}$ as a polynomial in the z_j and $\overline{z}_j$ depends on the signs of the γ_j. It is this dependence that introduces cones into the analysis, as we see below.

We next describe $\mathcal{L}$ explicitly, making the following simplifying assumptions:

$$\mathrm{hcf}(k_1,k_2)=1 \quad \mathrm{hcf}(\ell_1,\ell_2)=1 \tag{5}$$

If these assumptions do not hold we can factor out the kernel of the group action and thereby ensure that they do hold. This procedure does not change the invariants or equivariants: all it does is replace k_j by $k_j/\mathrm{hcf}(k_1,k_2)$ and ℓ_j by $\ell_j/\mathrm{hcf}(\ell_1,\ell_2)$.

Lemma 3.3 *With the assumptions (5), we have that $\mathcal{L}$ is generated (as a lattice) by the following vectors:*

Type A*:*

$$\begin{pmatrix} k_2 \\ k_2 \\ -k_1 \\ -k_1 \end{pmatrix} \begin{pmatrix} \ell_2 \\ -\ell_2 \\ -\ell_1 \\ \ell_1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(k_2+\ell_2) \\ \frac{1}{2}(k_2-\ell_2) \\ \frac{1}{2}(-k_1-\ell_1) \\ \frac{1}{2}(-k_1+\ell_1) \end{pmatrix}$$

if one pair of mode numbers is of type (odd, odd) and the other pair is either (odd, odd) or (even, even). Here either the first or second generator can be omitted (the third generator is their mean).

Type B*:*

$$\begin{pmatrix} k_2 \\ k_2 \\ -k_1 \\ -k_1 \end{pmatrix} \begin{pmatrix} \ell_2 \\ -\ell_2 \\ -\ell_1 \\ \ell_1 \end{pmatrix}$$

if at least one pair of mode numbers has distinct parities.

Proof

We can solve (2,3). By (2), k_2 divides $\gamma_1+\gamma_2$ and k_1 divides $\gamma_3+\gamma_4$. Hence there exists $a \in \mathbf{Z}$ such that

$$\gamma_1+\gamma_2 = ak_2, \quad \gamma_3+\gamma_4 = -ak_1 \tag{6}$$

Similarly, there exists $b \in \mathbf{Z}$ such that

$$\gamma_1-\gamma_2 = b\ell_2, \quad \gamma_3-\gamma_4 = -b\ell_1 \tag{7}$$

Solving (6,7) we get

$$\begin{aligned}\gamma_1 &= \tfrac{1}{2}(ak_2 + b\ell_2)\\ \gamma_2 &= \tfrac{1}{2}(ak_2 - b\ell_2)\\ \gamma_3 &= \tfrac{1}{2}(-ak_1 - b\ell_1)\\ \gamma_4 &= \tfrac{1}{2}(-ak_1 + b\ell_1)\end{aligned} \tag{8}$$

In addition, we require $\gamma_j \in \mathbf{Z}$, which leads to the two distinct cases as we now show. From (8) we have the following constraints on the parities of a and b: see Table 2. A similar table holds for (k_1, l_1). Note that both tables must be satisfied simultaneously by the same a and b.

k_2	l_2	a	b
odd	odd	same	parity
even	odd	any	even
odd	even	even	any
even	even	any	any

Table 2: Parities of a and b according to the parities of k_2 and l_2.

Interchanging (k_1, l_1) with (k_2, l_2) if necessary we have 10 possibilities for the parities of k_1, l_1, k_2, l_2, five of each violate assumptions (5) and so we do not consider. Using Table 2 we get information on the parities of a and b for the remaining cases. See Table 3. Finally, it follows that for the cases 1 and 4 we can take generators for $\mathcal{L}$ the vectors of Type A. For the cases 2, 3, 6 we have the generators for $\mathcal{L}$ of the Type B. ∎

Case	k_1	l_1	k_2	l_2	a and b
1	odd	odd	odd	odd	same parity
2	odd	odd	even	odd	both even
3	odd	odd	odd	even	both even
4	odd	odd	even	even	same parity
5	even	odd	even	odd	violates (5)
6	even	odd	odd	even	both even
7	even	odd	even	even	violates (5)
8	odd	even	odd	even	violates (5)
9	odd	even	even	even	violates (5)
10	even	even	even	even	violates (5)

Table 3: Parities of k_1, l_1, k_2, l_2 versus parities of a and b.

4 An Example

To motivate the algebra, we consider a typical example, the $(3,2)-(1,3)$ mode interaction. That is,

$$k_1 = 3 \quad \ell_1 = 2 \quad k_2 = 1 \quad \ell_2 = 3$$

This is Type B of Lemma 3.3. Equations (2,3) become:

$$3(\gamma_1+\gamma_2)+(\gamma_3+\gamma_4)=0$$

$$2(\gamma_1-\gamma_2)+3(\gamma_3-\gamma_4)=0$$

which imply that

$$\gamma_3 = -\frac{1}{6}(11\gamma_1+7\gamma_2) \qquad \gamma_4 = -\frac{1}{6}(7\gamma_1+11\gamma_2) \tag{9}$$

Thus we can parametrise solutions by pairs (γ_1,γ_2) in the lattice $\mathcal{M} \subseteq \mathbf{Z}^2$ defined to be the projection of $\mathcal{L}$ onto the first two coordinates of $\mathbf{Z}^4$.

The γ_j are integers if and only if $\gamma_1 \equiv \gamma_2 \pmod 6$. The interpretation of a solution $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ changes when the sign of any γ_j changes; that is, on crossing any of the four lines $\gamma_j = 0$ in $\mathcal{M}$. These lines have equations

$$\gamma_1 = 0 \qquad \gamma_2 = 0 \qquad \gamma_1 = -\frac{7}{11}\gamma_2 \qquad \gamma_1 = -\frac{11}{7}\gamma_2$$

respectively.

Figure 1 shows the half-space $\gamma_1 \geq 0$ in the (γ_1,γ_2)-plane, the lattice $\mathcal{M}$, and the lines $\gamma_j = 0$. Points in $\mathcal{M}$ are shown as black or open dots. The half-space $\gamma_1 \leq 0$ decomposes in the same manner, but rotated by 180°, which has the effect of complex conjugation on the corresponding reduced monomials. The figure divides into four cones (eight including conjugates). The set of these cones and their faces form a fan. Within each cone the choice $w_j = z_j$ or $\overline{z}_j$ is fixed, so the product of two reduced monomials in the same cone is again reduced. We may therefore obtain a (minimal) generating set of reduced monomials, under multiplication, by finding a (minimal) generating set for $\sigma \cap \mathcal{M}$ for each cone σ, under semigroup addition, and taking the union of these sets.

Each cone σ has a *fundamental parallelotope* $P_\sigma^{\mathcal{M}}$, as defined in the proof of Lemma 2.1, whose sides are determined by the vectors $0, v_1, v_2$, where v_1 and v_2 are the non-zero elements of $\mathcal{M}$ of minimal length subject to lying on the faces of the cone. The fundamental parallelotopes are shown shaded in Figure 1. As in the proof of Gordan's Lemma, any minimal generating set lies inside the fundamental parallelotope.

Say that a non-zero element of $\sigma \cap \mathcal{M}$ is σ-*irreducible* if it is not the sum of two non-zero elements of $\sigma \cap \mathcal{M}$. Clearly the σ-*irreducible* elements lie inside the fundamental parallelotope $P_\sigma^{\mathcal{M}}$. It is easy to see that each $\sigma \cap \mathcal{M}$ has a *unique* minimal generating set, which consists of the σ-*irreducible* elements of the finite set

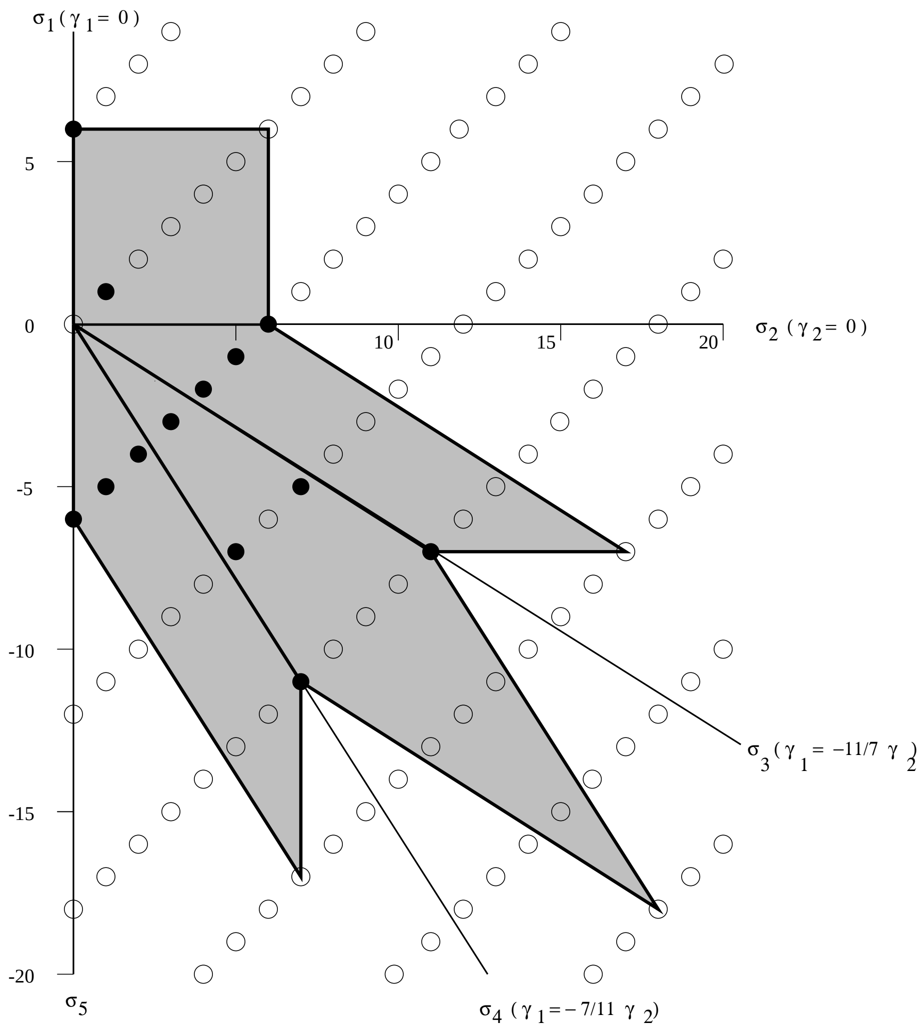

Figure 1: Lattice geometry for the $(3, 2) - (1, 3)$ mode interaction.

$P_\sigma^{\mathcal{M}} \cap \mathcal{M}$. These elements can be found by inspection: in Figure 1 they are marked with a black dot. Reducible elements are marked with an open dot.

In summary, the torus invariants for the $(3, 2) - (1, 3)$ mode interaction are minimally generated by the monomials listed in Table 4, together with the complex conjugates of all but the first four entries (which equal their complex conjugates). The table also lists the degrees of these generating monomials. There are 28 generators, whose degrees range from 2 to 30.

One implication of this table for equivariant bifurcation theory is worth emphasising. The table lists the four obvious invariants $z_j\overline{z}_j$. Apart from these, the lowest degree term that appears has degree 8. Taking Theorem 3.2 into account, we see that the lowest degree equivariant that is not generated from the $z_j\overline{z}_j$ is one of degree 7. The equivariants generated from the $z_j\overline{z}_j$ are equivariant under any of the torus actions in Table 1, independently of the mode numbers. Indeed, they are equivariant under the orthogonal group $\mathbf{O}(8)$ acting on $\mathbf{R}^8 = \mathbf{C}^4$, which has dimension 28,

monomial	degree	monomial	degree
$z_1\overline{z}_1$	2	$z_1^4\overline{z}_2^2\overline{z}_3^5\overline{z}_4$	12
$z_2\overline{z}_2$	2	$z_1^3\overline{z}_2^3\overline{z}_3^2 z_4^2$	10
$z_3\overline{z}_3$	2	$z_1^2\overline{z}_2^4 z_3 z_4^5$	12
$z_4\overline{z}_4$	2	$z_1\overline{z}_2^5 z_3^4 z_4^8$	18
$z_2^6\overline{z}_3^7\overline{z}_4^{11}$	24	$z_1^7\overline{z}_2^5\overline{z}_3^7 z_4$	20
$z_1^6\overline{z}_3^{11}\overline{z}_4^7$	24	$z_1^5\overline{z}_2^7\overline{z}_3 z_4^7$	20
$z_1 z_2\overline{z}_3^3\overline{z}_4^3$	8	$z_1^{11}\overline{z}_2^7\overline{z}_3^{12}$	30
$z_1^5\overline{z}_2\overline{z}_3^8\overline{z}_4^4$	18	$z_1^7\overline{z}_2^{11} z_4^{12}$	30

Table 4: Minimal generators for torus invariants for the $(3,2)-(1,3)$ mode interaction.

whereas the dimension of $\mathbf{T}^2$ is 2. Thus any analysis that does not take degree 7 terms into account will find a profusion of spurious $\mathbf{O}(8)$ group-orbits of solutions. Indeed, any special qualitative features of the $(3,2)-(1,3)$ mode interaction, such as the detailed geometry of regions in parameter space for which solution branches exist, cannot be detected by truncating Taylor series at a degree less than 7. So it is necessary to retain terms of relatively high degree in order to obtain qualitatively accurate bifurcation diagrams. This phenomenon is quite subtle and deserves further study; the most appropriate framework is singularity theory, as in Golubitsky and Schaeffer [15].

In saying this we acknowledge that because torus groups are Abelian, the complex structure of Table 1 implies that there are no axial subgroups, so the Equivariant Branching Lemma [17] cannot be applied here. For torus-equivariant problems, we expect Hopf bifurcations to travelling waves. When the torus action is extended by some finite group, especially one containing reflections, axial subgroups can sometimes occur, and the above remarks remain valid. We have chosen the action of Table 1 because it is convenient to illustrate the general ideas of this paper.

5 The General Mode Interaction

The analysis of the general $(k_1,\ell_1)-(k_2,\ell_2)$ mode interactions follows similar lines. We solve equations (2,3) for γ_3,γ_4 in terms of γ_1,γ_2, leading to:

$$\gamma_3 = \frac{-1}{2k_2\ell_2}\left[(k_1\ell_2+k_2\ell_1)\gamma_1+(k_1\ell_2-k_2\ell_1)\gamma_2\right] \tag{10}$$

$$\gamma_4 = \frac{-1}{2k_2\ell_2}\left[(k_1\ell_2-k_2\ell_1)\,\gamma_1+(k_1\ell_2+k_2\ell_1)\,\gamma_2\right]. \tag{11}$$

Define

$$\Delta = \frac{k_2\ell_1+k_1\ell_2}{k_2\ell_1-k_1\ell_2}.$$

Then $\gamma_3=0$ when $\gamma_2=\Delta\gamma_1$ and $\gamma_4=0$ when $\gamma_2=\frac{1}{\Delta}\gamma_1$.

When we project $\mathbf{Z}^4$ with coordinates $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ onto $\mathbf{Z}^2$ with coordinates (γ_1, γ_2), the hyperplanes in $\mathbf{Z}^4$ defined by (10,11) project to the lines in $\mathbf{Z}^2$ defined by the same equations, since those equations do not depend explicitly on γ_3, γ_4. Therefore we can decompose $\mathcal{L} \subseteq \mathbf{Z}^4$, and simultaneously $\mathcal{M} \subseteq \mathbf{Z}^2$, into cones on which the signs of the γ_j remain constant. So addition in each cone corresponds to multiplication of monomials. On the boundaries of those cones, some $\gamma_j = 0$. Moreover, we can draw the picture in the (γ_1, γ_2)-plane.

The analysis now proceeds as in section 4. For each cone σ we define a fundamental parallelotope $P_\sigma^{\mathcal{M}}$, and σ-irreducible elements, in exactly the same way as for the example. The key computational result is:

Lemma 5.1 *Each lattice cone $\sigma \cap \mathcal{M}$ has a unique minimal generating set, which consists of the σ-irreducible elements of the finite set $P_\sigma^{\mathcal{M}} \cap \mathcal{M}$.* ∎

Again these elements can be found by inspection.

Theorem 5.2 *The ring of the $\mathbf{T}^2$-invariants on $\mathbf{C}^4$ has a unique minimal basis consisting of monomials. This basis is formed by $|z_j|^2$ $(j = 1, 2, 3, 4)$ and the union of the sets of the $\mathbf{T}^2$-invariant reduced monomials corresponding to the σ-irreducible generators in $P_\sigma^{\mathcal{M}} \cap \mathcal{M}$ of $\sigma \cap \mathcal{M}$, for each cone σ.*

Proof

Clearly, by Lemma 3.1, the union $\mathcal{U}$ of the $|z_j|^2$ together with the $\mathbf{T}^2$-invariant reduced monomials corresponding to the σ-irreducible generators in $P_\sigma^{\mathcal{M}} \cap \mathcal{M}$ of $\sigma \cap \mathcal{M}$ for each cone σ generate the ring of the $\mathbf{T}^2$-invariant polynomials.

Suppose that $\mathcal{U}$ is not minimal. Then there is a reduced monomial m in $\mathcal{U}$ that is the product of at least two other reduced monomials, say $m_1,\ m_2$, that are in $\mathcal{U}$. By Lemma 5.1 the monomials $m_1,\ m_2$ have to correspond to two irreducible lattice points of $\mathcal{M}$ that lie in two distinct cones.

The reduced monomials are of the type (a) $w_1^{|\epsilon_1|} w_2^{|\epsilon_2|} w_3^{|\epsilon_3|} w_4^{|\epsilon_4|}$ or (b) $w_{i_1}^{|\epsilon_{i_1}|} w_{i_2}^{|\epsilon_{i_2}|} w_{i_3}^{|\epsilon_{i_3}|}$, where w_j is either equal to z_j or $\overline{z}_j$. We have three possibilities for $m_1,\ m_2$. (Case 1) both are of type (a). (Case 2) both are of type (b). (Case 3) one is of type (a) and the other of type (b).

(Case 1) Since $m_1,\ m_2$ are in distinct cones, then at least one of the w_j is z_j in one cone and $\overline{z}_j$ in the other cone. Thus a power of $|z_j|^2$ appears in $m = m_1 m_2$, which contradicts the hypothesis of m being a reduced monomial.

(Case 2) For this case, m_1 and m_2 correspond to lattice points of $\mathcal{M}$ that lie in two faces of two distinct cones. Therefore at least two of the w_i of m_1 are distinct from two of the w_i of m_2. Again, for this case, a power of some $|z_j|^2$ appears in the product $m_1 m_2$ and m is not reduced. For example, in Figure 1 an element in face σ_1 is of type $z_2^{\gamma_2} \overline{z}_3^{\gamma_3} \overline{z}_4^{\gamma_4}$ and in face σ_3 is of type $z_1^{\gamma_1'} \overline{z}_2^{\gamma_2'} \overline{z}_3^{\gamma_3'}$. The product of two elements lying in faces σ_1 and σ_3 contains a power of $|z_2|^2$.

(Case 3) is similar to (Case 1). ∎

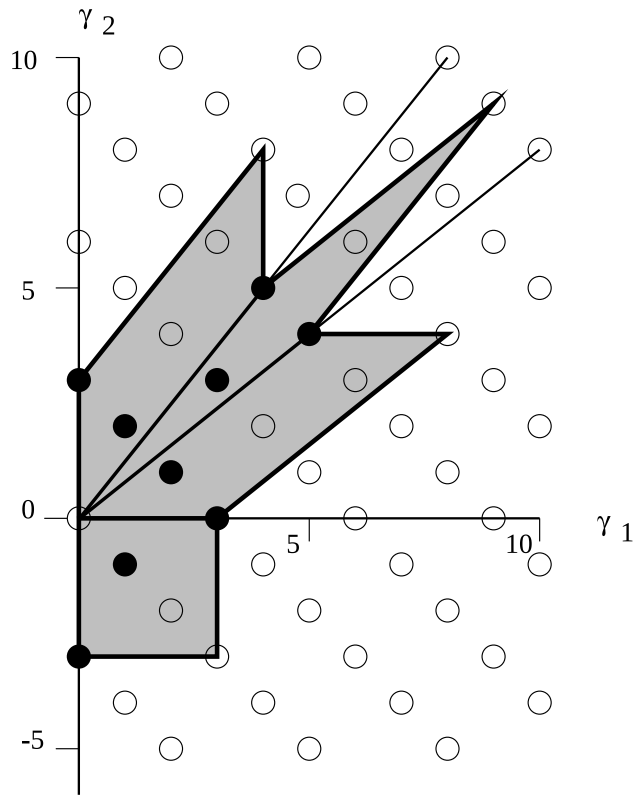

Figure 2: Lattice geometry for the $(1,3)-(3,1)$ mode interaction.

Example 5.3 We consider the $(1,3)-(3,1)$ mode interaction, which is of Type A. Here

$$k_1 = 1 \quad \ell_1 = 3 \quad k_2 = 3 \quad \ell_2 = 1$$

The lattice $\mathcal{L}$ is generated by

$$\begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

The cone boundaries are the γ_1-axis, the γ_2-axis, and the lines $\gamma_3 = 0, \gamma_4 = 0$, whose equations are respectively $\gamma_2 = \frac{5}{4}\gamma_1$ and $\gamma_2 = \frac{4}{5}\gamma_1$, since $\Delta = \frac{5}{4}$. Furthermore,

$$\gamma_3 = \frac{-5\gamma_1 + 4\gamma_2}{3} \qquad \gamma_4 = \frac{4\gamma_1 - 5\gamma_2}{3}$$

and the γ_j are integers if and only if $\gamma_1 + \gamma_2 \equiv 0 \pmod 3$.

The appropriate picture is shown in Figure 2. We can read off a minimal generating system of reduced monomials (omitting complex conjugates), see Table 5. This time there are 20 monomials in a minimal generating set, with degrees ranging from 2 to 12.

monomial	degree	monomial	degree
$z_1\overline{z}_1$	2	$z_1^2 z_2 \overline{z}_3^2 z_4$	6
$z_2\overline{z}_2$	2	$z_1^5 z_2^4 \overline{z}_3^3$	12
$z_3\overline{z}_3$	2	$z_1^3 z_2^3 \overline{z}_3 \overline{z}_4$	8
$z_4\overline{z}_4$	2	$z_1 z_2^2 z_3 \overline{z}_4^2$	6
$z_1\overline{z}_2\overline{z}_3^3 z_4^3$	8	$z_1^4 z_2^5 \overline{z}_4^3$	12
$z_1^3 \overline{z}_3^5 z_4^4$	12	$z_2^3 z_3^4 \overline{z}_4^5$	12

Table 5: Minimal generators for torus invariants for the $(1,3)-(3,1)$ mode interaction.

6 Hilbert Series

The Hilbert series for a group Γ acting on a vector space V is the generating function for the dimensions of the spaces of invariants of given degree. We compute now Hilbert series for torus actions corresponding to $(k_1, l_1) - (k_2, l_2)$ mode interactions using the elementary part of toric geometry considered in this paper. Hilbert series for general torus actions are discussed by Stanley [38] and Renner [35], and the relation between their methods and our results deserves further investigation (but not in this paper).

Let $\mathcal{P}_V^d(\Gamma)$ be the space of homogeneous polynomial invariants of degree d for the action of Γ on V. Then the *Hilbert series* for this action is the formal power series

$$\Phi_\Gamma(t) = \sum_{d=0}^{\infty} \dim(\mathcal{P}_V^d(\Gamma))t^d$$

in the indeterminate t. For a compact Lie group action there is an explicit integral formula, *Molien's Theorem*:

$$\Phi_\Gamma(t) = \int_\Gamma \frac{1}{\det(\mathbf{1} - \gamma t)} \mathrm{d}\mu_\Gamma$$

where μ_Γ is normalised Haar measure on Γ and $\gamma \in \Gamma$. See Molien [34] (or Sturmfels [40]) for the original proof of the finite case, and Sattinger [37] for the extension to a compact group. It is difficult (though not impossible) to use this formula to compute the Hilbert series, but it is often better to proceed by other means — as is the case here.

Let $\Gamma = \mathbf{T}^2$ in the action of section 3, the $(k_1, l_1) - (k_2, l_2)$ mode interaction, for $k_1, l_1, k_2, l_2 \in \mathbf{N}$. Let $m = w_1^{|\gamma_1|} w_2^{|\gamma_2|} w_3^{|\gamma_3|} w_4^{|\gamma_4|}$ be a reduced invariant monomial. Then the integers $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ satisfy the equations

$$\begin{aligned} k_1(\gamma_1 + \gamma_2) + k_2(\gamma_3 + \gamma_4) &= 0 \\ \ell_1(\gamma_1 - \gamma_2) + \ell_2(\gamma_3 - \gamma_4) &= 0 \end{aligned} \tag{12}$$

that define a 2-dimensional lattice $\mathcal{L} \subseteq \mathbf{Z}^4$ (Lemma 3.3). These solutions can be parametrised by pairs (γ_1, γ_2) in a lattice $\mathcal{M} \subseteq \mathbf{Z}^2$ defined to be the projection of $\mathcal{L}$

onto the first two coordinates γ_1, γ_2 of $\mathbf{Z}^4$:

$$\gamma_3 = \frac{-1}{2k_2\ell_2}\left[(k_1\ell_2 + k_2\ell_1)\,\gamma_1 + (k_1\ell_2 - k_2\ell_1)\,\gamma_2\right]$$
$$\gamma_4 = \frac{-1}{2k_2\ell_2}\left[(k_1\ell_2 - k_2\ell_1)\gamma_1 + (k_1\ell_2 + k_2\ell_1)\gamma_2\right] \tag{13}$$

Moreover, for the description of the reduced invariant monomials we just need to consider the half-space $\gamma_1 \geq 0$ in the (γ_1, γ_2)-plane. The lines $\gamma_1 = 0,\ \gamma_2 = 0,\ \gamma_2 = \Delta\gamma_1$ (when $\gamma_3 = 0$) and $\gamma_2 = \frac{1}{\Delta}\gamma_1$ (when $\gamma_4 = 0$), with

$$\Delta = \frac{k_2\ell_1 + k_1\ell_2}{k_2\ell_1 - k_1\ell_2},$$

divide this half-plane into four cones, say A, B, C, D, with faces $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, defined by these lines. We consider here the case when

$$k_2\ell_1 - k_1\ell_2 \neq 0. \tag{14}$$

(When $k_2\ell_1 = k_1\ell_2$, in terms of Hilbert series, we just need to consider for example the cone corresponding to the quadrant $\gamma_1,\ \gamma_2 \geq 0$. The theory developed in this section also applies to this simplified case.) Within each cone the choice $w_j = z_j$ or $\overline{z}_j$ is fixed ($j = 1, \ldots, 4$). On crossing any of the four lines $\gamma_j = 0$ in $\mathcal{M}$, some γ_j changes sign. From the half-space $\gamma_1 \leq 0$, we obtain the conjugates of the reduced invariant monomials of the half-space $\gamma_1 \geq 0$. Thus in terms of the Hilbert series for the action of $\mathbf{T}^2$ on $\mathbf{C}^4$ considered here, the study of the half-space $\gamma_1 \geq 0$ will be enough. Note that since k_1, l_1, k_2, l_2 are natural numbers,

$$|\Delta| > 1.$$

We have two kinds of pictures for the disposition of the half-space $\gamma_1 \geq 0$ into cones, according to whether $\Delta > 1$ (picture (a)) or $\Delta < -1$ (picture (b)). We show these two cases in Figure 3 to fix notation for the rest of this section.

The degree of a reduced invariant monomial $m = w_1^{|\gamma_1|}w_2^{|\gamma_2|}w_3^{|\gamma_3|}w_4^{|\gamma_4|}$ is $\partial m = |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|$. On each cone A, B, C, D, the formula for ∂m is linear in (γ_1, γ_2). Figures 5 and 6 of section 6.1 show the lines $\partial m = c$ in the four cones, for the $(3,2)-(1,3)$ and $(1,3)-(3,1)$ mode interactions respectively. For the first case the number c takes values that are integer multiples of 8, 6, 10, 6. For example, on cone B in Figure 5 we have $\gamma_1 \geq 0, \gamma_2 \leq 0, \gamma_3 < 0, \gamma_4 \leq 0$, so

$$\partial m = \gamma_1 - \gamma_2 - \gamma_3 - \gamma_4 = 4\gamma_1 + 2\gamma_2$$

using (9). For the second case c is multiple of 4, 6, 4, 6 respectively.

Let $\mathcal{Q}^d$ be the space of *reduced* invariant monomials of degree d, and define the *reduced Hilbert series*

$$\Phi_\Gamma^{\mathcal{L}}(t) = \sum_{d=0}^{\infty} \dim(\mathcal{Q}^d)t^d$$

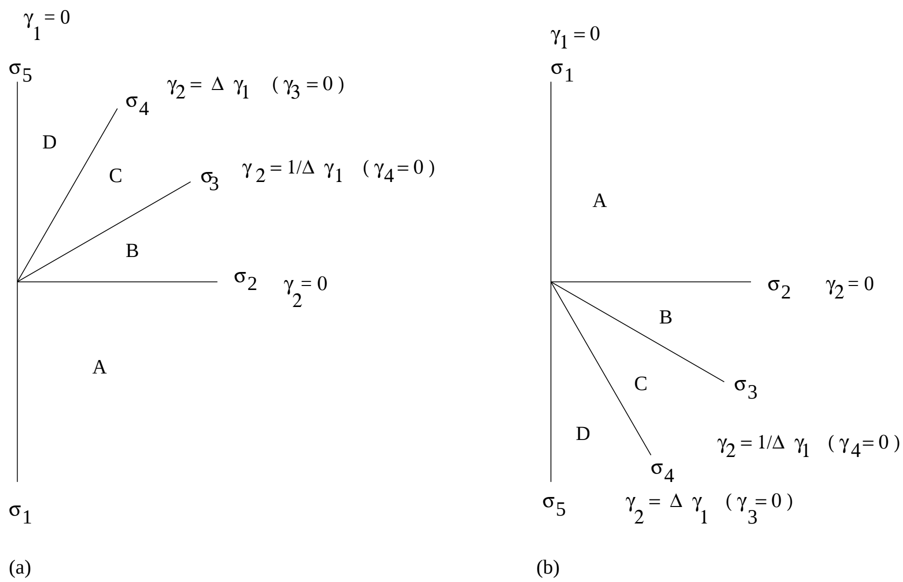

Figure 3: Pictures: (a) $\Delta > 1$. (b) $\Delta < -1$.

which is a Hilbert series for the lattice $\mathcal{L}$ graded by degree in each cone. Lemma 3.1 lets us express $\Phi_\Gamma(t)$ in terms of $\Phi_\Gamma^{\mathcal{L}}(t)$. In fact, since the $|z_j|^2$ $(j = 1, ..., 4)$ are algebraically independent over the reduced monomials, we have:

$$\Phi_\Gamma(t) = \frac{1}{(1-t^2)^4}\Phi_\Gamma^{\mathcal{L}}(t)$$

For any cone σ we define

$$\Phi_\Gamma^\sigma(t) = \sum_{d=0}^{\infty} \dim(\mathcal{Q}_\sigma^d) t^d$$

where $\mathcal{Q}_\sigma^d$ is the subspace of $\mathcal{Q}^d$ spanned by monomials corresponding to lattice points in σ.

Applying the inclusion-exclusion principle (and recalling that from the region $\gamma_1 \leq 0$ we get the complex conjugates of the invariant monomials obtained in the region $\gamma_1 \geq 0$), using notation of Figure 3, we get:

Lemma 6.1

$$\begin{aligned} \Phi_\Gamma^{\mathcal{L}}(t) &= 2\left[\Phi_\Gamma^A(t) + \Phi_\Gamma^B(t) + \Phi_\Gamma^C(t) + \Phi_\Gamma^D(t)\right] \\ &\quad -2\left[\Phi_\Gamma^{\sigma_1}(t) + \Phi_\Gamma^{\sigma_2}(t) + \Phi_\Gamma^{\sigma_3}(t) + \Phi_\Gamma^{\sigma_4}(t)\right] + 1 \end{aligned} \tag{15}$$

where σ_j is either of the two half-rays along which $\gamma_j = 0$, origin included.

Formula (15) converts the computation of $\Phi_\Gamma^{\mathcal{L}}$ to a series of analogous computations within cones (the regions A, B, C, D and their common boundaries, the faces

σ_j, for $j = 1, \ldots, 4$). We next show that the cones B and D (and corresponding faces) have the same Hilbert series, as the apparent symmetry in the figure suggests.

Lemma 6.2

$$\Phi_\Gamma^B(t) = \Phi_\Gamma^D(t)$$

and

$$\Phi_\Gamma^{\sigma_1}(t) = \Phi_\Gamma^{\sigma_2}(t), \quad \Phi_\Gamma^{\sigma_3}(t) = \Phi_\Gamma^{\sigma_4}(t).$$

Proof

Let $m = w_1^{|\gamma_1|} w_2^{|\gamma_2|} w_3^{|\gamma_3|} w_4^{|\gamma_4|}$ be a reduced invariant monomial determined by $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Consider first the case $\Delta > 1$, corresponding to picture (a) of Figure 3. Then γ belongs to cone B if and only if it satisfies equations (12) and $\gamma_1 \geq 0,\ \gamma_2 \geq 0,\ \gamma_3 < 0,\ \gamma_4 \geq 0$. Similarly γ belongs to cone D if and only if it satisfies (12) and $\gamma_1 \geq 0,\ \gamma_2 \geq 0,\ \gamma_3 \geq 0,\ \gamma_4 < 0$. It follows that γ belongs to cone B if and only if $\gamma' = (\gamma_2, \gamma_1, \gamma_4, \gamma_3)$ belongs to cone D. Moreover, $|\gamma| = |\gamma'|$.

Similarly for the case $\Delta < -1$ (picture (b) of Figure 3), γ satisfying (12) belongs to cone B if and only if $\gamma_1 \geq 0,\ \gamma_2 \leq 0,\ \gamma_3 < 0,\ \gamma_4 \leq 0$, and it belongs to cone D if and only if $\gamma_1 \geq 0,\ \gamma_2 \leq 0,\ \gamma_3 \geq 0,\ \gamma_4 > 0$. Now γ belongs to cone B if and only if $\gamma' = (-\gamma_2, -\gamma_1, -\gamma_4, -\gamma_3)$ belongs to cone D, and again $|\gamma| = |\gamma'|$. Thus $\Phi_\Gamma^B(t) = \Phi_\Gamma^D(t)$.

Restricting to the boundaries of the cones B and D, it follows that $\Phi_\Gamma^{\sigma_1}(t) = \Phi_\Gamma^{\sigma_2}(t)$ and $\Phi_\Gamma^{\sigma_3}(t) = \Phi_\Gamma^{\sigma_4}(t)$. ∎

Thus using Lemma 6.2, expression (15) now becomes

$$\Phi_\Gamma^{\mathcal{L}}(t) = 2[\Phi_\Gamma^A(t) + 2\Phi_\Gamma^B(t) + \Phi_\Gamma^C(t)] - 4[\Phi_\Gamma^{\sigma_1}(t) + \Phi_\Gamma^{\sigma_3}(t)] + 1 \tag{16}$$

Definition 6.3 Let σ be a simplicial cone of dimension two in the lattice $\mathcal{M} \subseteq \mathbf{Z}^2$ (projection of $\mathcal{L} \subseteq \mathbf{Z}^4$ on (γ_1, γ_2)). Denote the faces of σ by $\sigma_1,\ \sigma_2$. Let $v_1,\ v_2$ be the two non-zero elements of $\mathcal{M}$ of minimal length subject to lying on the faces $\sigma_1,\ \sigma_2$ of the cone, and let $u_1,\ u_2$ be the corresponding elements in $\mathcal{L} \subseteq \mathbf{Z}^4$ (and so satisfying equations (12)). Define the *fundamental parallelotope* $P_\sigma^{\mathcal{M}}$ of the lattice $\mathcal{M}$ relative to the cone σ to be the polytope with vertices $0,\ v_1,\ v_2,\ v_1 + v_2 \in \mathcal{M} \subseteq \mathbf{Z}^2$. (It is actually a parallelogram in this case, but it is useful to set up terminology for a general setting.) Define the *fundamental parallelotope polynomial* $\mathcal{P}_\sigma^{\mathcal{L}}$ to be

$$\mathcal{P}_\sigma^{\mathcal{L}}(t) = 1 + \sum_{\gamma \in I_\sigma^{\mathcal{L}}} t^{|\gamma|} \tag{17}$$

where

$$I_\sigma^{\mathcal{L}} = \{(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \mathcal{L} \ :\ (\gamma_1, \gamma_2) \in \mathrm{int}(P_\sigma^{\mathcal{M}} \cap \mathcal{M})\}$$

Here $|\gamma| = |\gamma_1| + \cdots + |\gamma_4|$.

We prove now that $\mathcal{P}_{\sigma}^{\mathcal{L}}$ and the degrees of the reduced invariant monomials with exponents $u_1,\ u_2$ determine Φ_{Γ}^{σ} and $\Phi_{\Gamma}^{\sigma_i}$.

Lemma 6.4 *Let σ be any of the simplicial cones A, B, C or D (of dimension two) in the lattice $\mathcal{M} \subseteq \mathbf{Z}^2$ (projection of $\mathcal{L} \subseteq \mathbf{Z}^4$ on (γ_1, γ_2)), and let the faces of σ be $\sigma_1,\ \sigma_2$. Let $v_1,\ v_2$ be the two non-zero elements of $\mathcal{M}$ of minimal length subject to lying on the faces, and let $u_1,\ u_2$ be the corresponding elements in $\mathcal{L} \subseteq \mathbf{Z}^4$. Then*

$$\Phi_{\Gamma}^{\sigma}(t) = \frac{\mathcal{P}_{\sigma}^{\mathcal{L}}(t)}{(1-t^{|u_1|})(1-t^{|u_2|})} \tag{18}$$

and

$$\Phi_{\Gamma}^{\sigma_i}(t) = \frac{1}{(1-t^{|u_i|})} \quad (i = 1, 2).$$

Proof

We have $\sigma =< v_1, v_2 >$ since v_1, v_2 lie in the two distinct faces of the cone. For $v \in \sigma \cap \mathcal{M}$ define

$$\sigma_v = \left\{ v + v^{'} \ : \ v^{'} \in \sigma \right\}.$$

We can think of σ_v as a cone equal to σ but with apex at v (instead of the origin).

Consider $u_1,\ u_2 \in \mathcal{L} \subseteq \mathbf{Z}^4$ the corresponding elements to $v_1,\ v_2 \in \mathcal{M} \subseteq \mathbf{Z}^2$. Let m be a reduced invariant monomial determined by $\gamma^{'} = (\gamma_1^{'}, \gamma_2^{'}, \gamma_3^{'}, \gamma_4^{'}) \in \mathcal{L}$ such that $(\gamma_1^{'}, \gamma_2^{'}) \in \sigma_{v_1} \cap \mathcal{M}$. Then $(\gamma_1^{'}, \gamma_2^{'}) = v_1 + (\gamma_1, \gamma_2)$ for some $(\gamma_1, \gamma_2) \in \sigma \cap \mathcal{M}$. Since v_1 and (γ_1, γ_2) are in $\sigma \cap \mathcal{M}$, let m be the reduced invariant monomial corresponding to $\gamma^{'} = u_1 + \gamma$, where $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, is the lattice point of $\mathcal{L}$ corresponding to the lattice point (γ_1, γ_2) of $\sigma \cap \mathcal{M}$. Then the degree of m is given by

$$\partial m = |\gamma^{'}| = |u_1| + |\gamma|$$

since u_1 and γ have components with same signs. Therefore, if we define an analogous $\Phi_{\Gamma}^{\sigma_{v_1}}$, then

$$\Phi_{\Gamma}^{\sigma_{v_1}}(t) = t^{|u_1|}\Phi_{\Gamma}^{\sigma}(t)$$

Similarly

$$\Phi_{\Gamma}^{\sigma_{v_2}}(t) = t^{|u_2|}\Phi_{\Gamma}^{\sigma}(t)$$

and

$$\Phi_{\Gamma}^{\sigma_{v_1+v_2}}(t) = t^{|u_2|+|u_2|}\Phi_{\Gamma}^{\sigma}(t).$$

Note also that

$$\sigma_{v_1+v_2} = \sigma_{v_1} \cap \sigma_{v_2}$$

(see Figure 4).

Thus

$$\Phi_{\Gamma}^{\sigma}(t) = \mathcal{P}_{\sigma}^{\mathcal{L}}(t) + \left(t^{|u_1|} + t^{|u_2|} - t^{|u_1|+|u_2|}\right)\Phi_{\Gamma}^{\sigma}(t)$$

and so

$$\Phi_{\Gamma}^{\sigma}(t) = \frac{\mathcal{P}_{\sigma}^{\mathcal{L}}(t)}{\left(1-t^{|u_1|}\right)\left(1-t^{|u_2|}\right)}.$$

∎

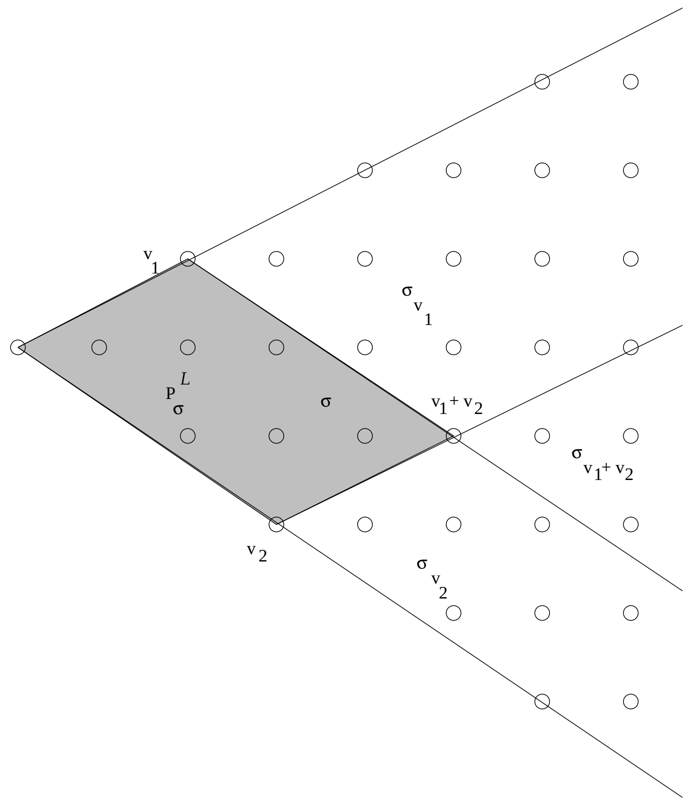

Figure 4: Geometry for the cones σ, σ_{v_1}, σ_{v_2} and $\sigma_{v_1+v_2}$.

Recalling again the notation of Figure 3, it is now straightforward to obtain a formula for $\Phi_\Gamma^{\mathcal{L}}$ (and Φ_Γ) depending on the fundamental parallelotope polynomials for each of the cones A, B, C.

Theorem 6.5 *The Hilbert series for the action of* $\Gamma = \mathbf{T}^2$ *on* $\mathbf{C}^4$ *of section 3 is:*

$$\Phi_\Gamma(t) = \frac{1}{(1-t^2)^4}\Phi_\Gamma^{\mathcal{L}}(t)$$

where

$$\begin{aligned}\Phi_\Gamma^{\mathcal{L}}(t) &= 2\frac{\mathcal{P}_A^{\mathcal{L}}(t)}{\left(1-t^{|u_1|}\right)^2} + 4\frac{\mathcal{P}_B^{\mathcal{L}}(t)}{\left(1-t^{|u_1|}\right)\left(1-t^{|u_3|}\right)} + 2\frac{\mathcal{P}_C^{\mathcal{L}}(t)}{\left(1-t^{|u_3|}\right)^2} \\ &\quad -4\frac{1}{\left(1-t^{|u_1|}\right)} - 4\frac{1}{\left(1-t^{|u_3|}\right)} + 1\end{aligned} \tag{19}$$

Proof

Use Lemmas 6.1, 6.2 and 6.4. ∎

Remark 6.6 For the particular case

$$k_1 l_2 - k_2 l_1 = 0$$

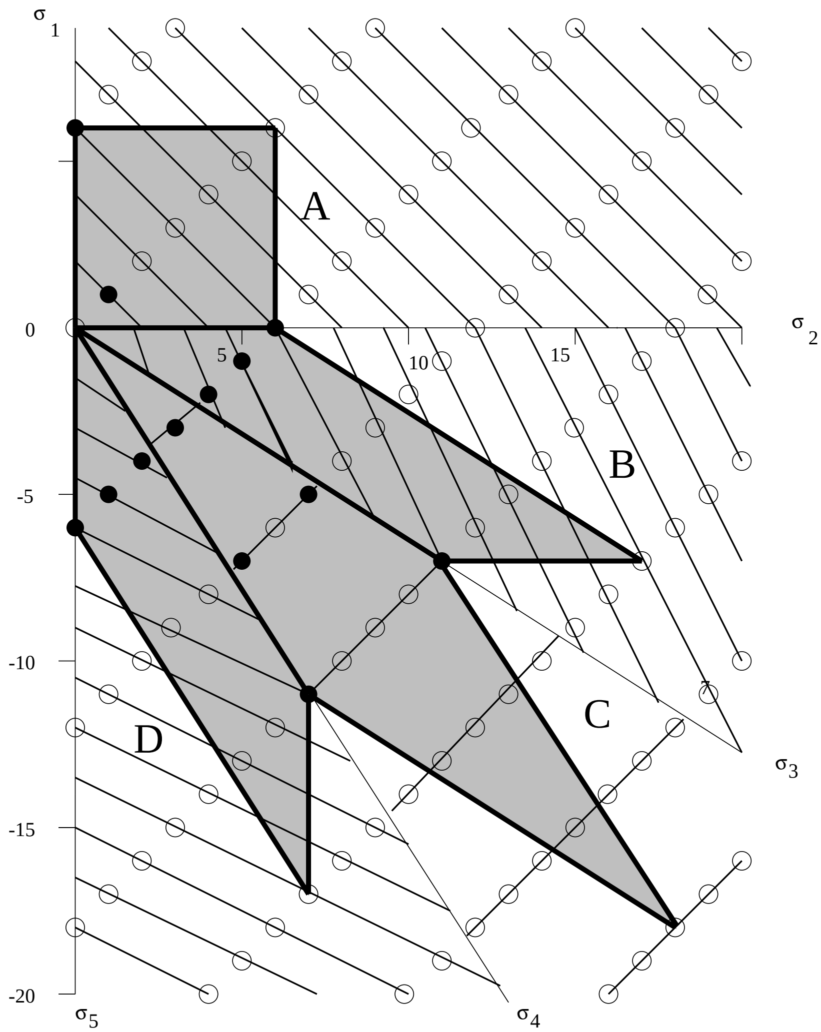

Figure 5: Hilbert series geometry for the $(3,2)-(1,3)$ mode interaction.

it follows from (13) that $\gamma_3,\ \gamma_4$ are integers if and only if $\gamma_1 \equiv 0 \pmod{l_2}$ and $\gamma_2 \equiv 0 \pmod{l_2}$. If we consider the cone A defined by $\gamma_1,\ \gamma_2 \geq 0$, then

$$\begin{aligned}\Phi_\Gamma(t) &= \frac{1}{(1-t^2)^4}\left(4\Phi_\Gamma^A(t) - 4\Phi_\Gamma^{\sigma_1}(t)\right)\\ &= \frac{1}{(1-t^2)^4}\left(\frac{4}{(1-t^{l_1+l_2})^2} - \frac{4}{(1-t^{l_1+l_2})}\right).\end{aligned}$$

6.1 Examples

For illustration, we consider the Hilbert series of the $\mathbf{T}^2$-invariants for the $(3,2)-(1,3)$ mode interaction of section 4, and for the $(3,1)-(1,3)$ mode interaction of section 5.

Example 6.7 Let $\Gamma = \mathbf{T}^2$ in the action of section 4, the $(3,2)-(1,3)$ mode interaction. Figure 5 shows the lines $\partial m = c$ in the four cones, for various values of c. In cones A, B, C, D, the number c takes values that are integer multiples of 8, 6, 10, 6 respectively. The figure also shows the fundamental parallelotope of the lattice $\mathcal{M}$ for each of the cones. Applying Lemma 6.4 to each cone, we have:

$$\begin{aligned}
\Phi_\Gamma^A(t) &= \frac{1+t^8+t^{16}+t^{24}+t^{32}+t^{40}}{(1-t^{24})^2} \\
\Phi_\Gamma^B(t) &= \Phi_\Gamma^D(t) = \frac{1+t^{12}+t^{18}+t^{24}+t^{30}+t^{36}+t^{42}}{(1-t^{24})(1-t^{30})} \\
\Phi_\Gamma^C(t) &= \frac{1+t^{10}+3t^{20}+3t^{30}+3t^{40}+t^{50}}{(1-t^{30})^2} \\
\Phi_\Gamma^{\sigma_1}(t) &= \Phi_\Gamma^{\sigma_2}(t) = \frac{1}{1-t^{24}} \\
\Phi_\Gamma^{\sigma_3}(t) &= \Phi_\Gamma^{\sigma_4}(t) = \frac{1}{1-t^{30}}
\end{aligned}$$

By Theorem 6.5, we obtain an explicit formula for Φ_Γ:

$$\begin{aligned}
\Phi_\Gamma(t) &= \frac{F(t)}{(1-t^2)^4(1-t^{24})^2(1-t^{30})^2} \\
&= 1+4t^2+10t^4+20t^6+37t^8+66t^{10} \\
&\quad + 116t^{12}+196t^{14}+317t^{16}+494t^{18}+\cdots
\end{aligned}$$

Here

$$\begin{aligned}
F(t) = {} & 1+2t^8+2t^{10}+4t^{12}+2t^{16}+4t^{18}+6t^{20}+8t^{24}+12t^{30}+2t^{32}-4t^{34}-4t^{38} \\
& +8t^{40}-4t^{42}-12t^{44}-4t^{46}-9t^{48}+2t^{50}-32t^{54}+2t^{58}-9t^{60}-4t^{62}-12t^{64} \\
& -4t^{66}+8t^{68}-4t^{70}-4t^{74}+2t^{76}+12t^{78}+8t^{84}+6t^{88}+4t^{90}+2t^{92}+4t^{96} \\
& +2t^{98}+2t^{100}+t^{108}
\end{aligned}$$

Example 6.8 Let $\Gamma = \mathbf{T}^2$ in the action of section 5, the $(1,3)-(3,1)$ mode interaction. Figure 6 shows the lines $\partial m = c$ in the four cones, for various values of c. In cones A, B, C, D, the number c takes values that are integer multiples of 4, 6, 4, 6 respectively. Applying Lemma 6.4 to each cone, we compute:

$$\Phi_\Gamma^A(t) = \Phi_\Gamma^C(t) = \frac{1+t^8+t^{16}}{(1-t^{12})^2}$$

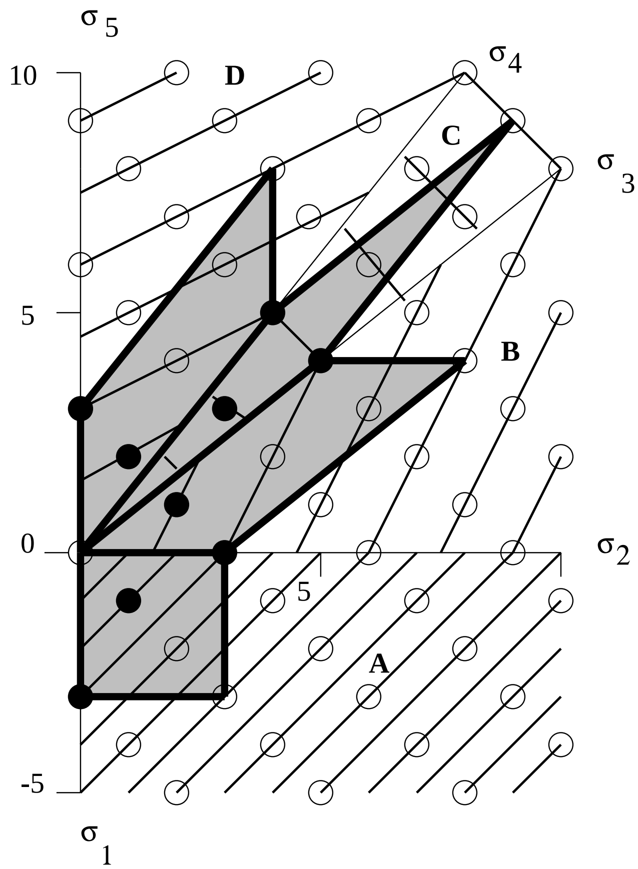

Figure 6: Hilbert series geometry for the $(1,3)-(3,1)$ mode interaction.

$$\Phi^B_\Gamma(t) \;=\; \Phi^D_\Gamma(t) \;=\; \frac{1+t^6+t^{12}+t^{18}}{(1-t^{12})^2}$$

$$\Phi^{\sigma_i}_\Gamma(t) \;=\; \frac{1}{1-t^{12}} \quad (i=1,2,3,4)$$

By combining the above equations (using Theorem 6.5), an explicit formula for Φ_Γ is:

$$\begin{aligned}\Phi_\Gamma(t) &= \frac{1+4t^6+4t^8+10t^{12}+4t^{16}+4t^{18}+t^{24}}{(1-t^2)^4(1-t^{12})^2} \\ &= 1+4t^2+10t^4+24t^6+55t^8+112t^{10} \\ &\quad +216t^{12}+388t^{14}+653t^{16}+1048t^{18}+\cdots\end{aligned}$$

7 Volume, Lattice Points and Generators

As before let $\Gamma = \mathbf{T}^2$ in the action of section 3, the $(k_1, l_1) - (k_2, l_2)$ mode interaction, for $k_1, l_1, k_2, l_2 \in \mathbf{N}$. In this section we calculate the number of lattice points $NI_\sigma^{\mathcal{L}}$ of the lattice $\mathcal{M}$ that lie in the interior of each fundamental parallelotope $P_\sigma^{\mathcal{M}}$, where σ is any of the cones A, B, C or D. This is derived from the volume of the fundamental parallelotope normalized by the volume of one unit cell of $\mathcal{M}$. Recalling Theorem 5.2 of section 5 this number estimates the maximum number of points that we need to check for finding the generators of the reduced monomials for each cone. Moreover, $NI_\sigma^{\mathcal{L}} + 1 = \mathcal{P}_\sigma^{\mathcal{L}}(1)$ where $\mathcal{P}_\sigma^{\mathcal{L}}$ is the fundamental parallelotope polynomial of the lattice $\mathcal{L}$ relative to the cone σ. Since the coefficients of $\mathcal{P}_\sigma^{\mathcal{L}}$ are nonnegative integers, $NI_\sigma^{\mathcal{L}} + 1$ provides a measure of the 'complexity' of $\mathcal{P}_\sigma^{\mathcal{L}}$, namely, the sum of its coefficients. It is therefore of some interest to compute $NI_\sigma^{\mathcal{L}}$ explicitly.

We also calculate the maximum degree of the generators of the ring of the $\mathbf{T}^2$-invariants (generators given by monomials), which gives another way of quantifying how complicated the invariants are.

It is well known that if $\mathcal{L}$ is a full sublattice of $\mathbf{Z}^k$ and D is a fundamental domain for $\mathcal{L}$, then the number of points of $\mathbf{Z}^k$ that lie inside D is equal to the volume $\mathrm{vol}(\mathcal{L}) \equiv \mu(D)$, where μ is k-dimensional Lebesgue measure. (Sketch proof: the quotient map $\phi : \mathbf{R}^k \to \mathbf{R}^k/\mathbf{Z}^k = \mathbf{T}^k$ is volume preserving. The images under ϕ of translates of a fundamental domain for $\mathbf{Z}^k$ by elements of $\mathbf{Z}^k \cap D$ cover $\mathbf{T}^k$ disjointly. Each such translate has measure 1.)

It follows easily that if $\mathcal{L}_1 \subseteq \mathcal{L} \subseteq \mathbf{Z}^k$ are full lattices, and D is a fundamental domain of $\mathcal{L}_1$, then

$$\#\{x : \; x \in \mathcal{L} \cap D\} = \frac{\mathrm{vol}(\mathcal{L}_1)}{\mathrm{vol}(\mathcal{L})}. \tag{20}$$

(Sketch proof: map $\mathcal{L}$ linearly onto $\mathbf{Z}^k$ and note that volumes are scaled according to the determinant of $\mathcal{L}$.)

Let $P_\sigma^{\mathcal{M}}$ be a fundamental parallelotope of the lattice $\mathcal{L}$ relative to the cone σ as defined in Definition 6.3 (so that σ is one of the cones A, B, C or D). Let $NI_\sigma^{\mathcal{L}}$ be the number of interior points of $P_\sigma^{\mathcal{M}} \cap \mathcal{M}$. By Lemma 6.2

$$NI_B^{\mathcal{L}} = NI_D^{\mathcal{L}}$$

Note that if $\mathcal{P}_\sigma^{\mathcal{L}}$ is the fundamental parallelotope polynomial as in Definition 6.3, then

$$\mathcal{P}_\sigma^{\mathcal{L}}(1) = 1 + NI_\sigma^{\mathcal{L}}$$

By Lemma 3.3, the volume of one lattice cell (of $\mathcal{M}$) is $k_2 l_2$ if the mode interaction is of *Type A*, and $2k_2 l_2$ if the mode interaction is of *Type B*. Note that $\mathcal{M}$ (projection of $\mathcal{L}$ onto the first two coordinates γ_1, γ_2) is generated by the vectors (k_2, k_2) and $(1/2(k_2 + l_2), 1/2(k_2 - l_2))$ in the first case, and by (k_2, k_2) and $(l_2, -l_2)$ in the second case.

If $v_1,\ v_2$ are the two non-zero elements of $\mathcal{M}$ of minimal length subject to lying on the faces $\sigma_1,\ \sigma_2$ of the cone σ, then

$$\mathrm{vol}(P_\sigma^{\mathcal{M}}) = \left| \det \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right|$$

and so by (20) we have:

Lemma 7.1

$$\mathcal{P}_\sigma^{\mathcal{L}}(1) = \frac{\left| \det \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right|}{k_2 l_2}$$

if the mode interaction is of Type A*, and*

$$\mathcal{P}_\sigma^{\mathcal{L}}(1) = \frac{\left| \det \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right|}{2k_2 l_2}$$

if the mode interaction is of Type B.

We calculate now the elements v_i for each of the cones A, B, C and D. By Lemma 6.2 from v_1 (or v_5) and v_3 on the faces σ_1 (or σ_5) and σ_3 (recall Figure 3), we obtain the other v_i. Specifically, if $v_1 = (0, a)$ then $v_2 = (-\mathrm{sgn}(\Delta)a, 0)$, and if $v_3 = (b, c)$ then $v_4 = (\mathrm{sgn}(\Delta)c, \mathrm{sgn}(\Delta)b)$.

Lemma 7.2 *Let v_1 be the nonzero element of $\sigma_1 \cap \mathcal{M}$ with minimal length (recall Figure 3). Then*

$$v_1 = \left(0, -\mathrm{sgn}(\Delta)\gamma_2^{'}\right)$$

where

$$\gamma_2^{'} = \begin{cases} \mathrm{lcm}^{'}(k_2, l_2) & \textit{if } \text{Type A } \textit{mode interaction} \\ 2\mathrm{lcm}(k_2, l_2) & \textit{if } \text{Type B } \textit{mode interaction} \end{cases}$$

Here $\mathrm{lcm}(a, b)$ *denotes the lowest (positive) common multiple of a and b, and* $\mathrm{lcm}^{'}(a, b)$ *the lowest common multiple of a and b from the common (positive) multiples of a and b, say m, such that m/a and m/b have the same parity.*

Proof

Elements in $\sigma_1 \cap \mathcal{M}$ satisfy $\gamma_1 = 0$ and so from (13)

$$\begin{aligned} (k_1\ell_2 - k_2\ell_1)\,\gamma_2 &\equiv 0 \pmod{2k_2\ell_2} \\ (k_1\ell_2 + k_2\ell_1)\,\gamma_2 &\equiv 0 \pmod{2k_2\ell_2} \end{aligned} \tag{21}$$

Thus

$$2k_2\ell_1\gamma_2 \equiv 0 \pmod{2k_2\ell_2} \tag{22}$$

$$(k_1\ell_2 - k_2\ell_1)\,\gamma_2 \equiv 0 \pmod{2k_2\ell_2} \tag{23}$$

From (22), it follows that $\gamma_2 \equiv 0 \pmod{\ell_2}$ since $\mathrm{hcf}(\ell_1, \ell_2) = 1$. From (23), it follows that $\gamma_2 \equiv 0 \pmod{k_2}$ since $\mathrm{hcf}(k_1, k_2) = 1$. Thus there are $n_1, n_2 \in \mathbf{Z}$ such that $\gamma_2 = n_1 k_2 = n_2 \ell_2$. Substituting in (23) it follows that $k_1 n_1 - \ell_1 n_2 \equiv 0 \pmod 2$. Now if $n_1,\ n_2$ are integers such that $\gamma_2 = n_1 k_2 = n_2 \ell_2$ and $k_1 n_1 - \ell_1 n_2 \equiv 0 \pmod 2$, then equations (21) are satisfied.

Note that if k_1, ℓ_1 have the same parity and also k_2, ℓ_2 have the same parity, then n_1, n_2 also have to have the same parity. If k_1, ℓ_1 or k_2, ℓ_2 have different parity, then n_1, n_2 have to be both even. ∎

Lemma 7.3 *Let v_3 be the nonzero element of $\sigma_3 \cap \mathcal{M}$ with minimal length (recall Figure 3). Then*

$$v_3 = \left(\gamma_1^{'}, \frac{1}{\Delta}\gamma_1^{'}\right)$$

where

$$\gamma_1^{'} = m^{'} \frac{k_2\ell_1 + k_1\ell_2}{2k_1\ell_1}$$

$$m^{'} = \min_{m \in \mathbf{Z}^+} \left\{ m \ : \ m\frac{k_2\ell_1 + k_1\ell_2}{2k_1\ell_1} \in \mathbf{Z}^+ \right\}$$

$$\Delta = \frac{k_2\ell_1 + k_1\ell_2}{k_2\ell_1 - k_1\ell_2}$$

Proof

Elements in $\sigma_3 \ \cap \ \mathcal{M}$ are of type $(\gamma_1, 1/\Delta\gamma_1)$ where from (13) the integer γ_1 satisfies

$$(k_1\ell_2 + k_2\ell_1)\,\gamma_1 + (k_1\ell_2 - k_2\ell_1)\,\frac{1}{\Delta}\gamma_1 \equiv 0 \pmod{2k_2\ell_2},$$

that is

$$\frac{2k_2\ell_1 2k_1\ell_2}{k_2\ell_1 + k_1\ell_2}\gamma_1 \equiv 0 \pmod{2k_2\ell_2},$$

and so

$$\frac{2k_1\ell_1}{k_2\ell_1 + k_1\ell_2}\gamma_1 \equiv 0 \pmod{1}$$

Note that $\|(\gamma_1, 1/\Delta\gamma_1)\|^2 = \gamma_1^2(1 + |1/\Delta|^2)$ and so $\|(\gamma_1, 1/\Delta\gamma_1)\|^2$ is minimum if γ_1^2 is minimum. ∎

Proposition 7.4 *With the notation of Lemmas 7.2 and 7.3, and Figure 3*

$$\mathcal{P}_A^{\mathcal{L}}(1) = \begin{cases} \dfrac{\mathrm{lcm}^{'}(k_2, l_2)^2}{k_2\ell_2} & \textit{if } \text{Type A } \textit{mode interaction} \\[2ex] \dfrac{2\mathrm{lcm}(k_2, l_2)^2}{k_2\ell_2} & \textit{if } \text{Type B } \textit{mode interaction} \end{cases}$$

$$\mathcal{P}^{\mathcal{L}}_{B}(1) = \mathcal{P}^{\mathcal{L}}_{D}(1) = \begin{cases} \dfrac{m^{'}\mathrm{lcm}^{'}(k_2,l_2)|k_2\ell_1 - k_1\ell_2|}{2k_1\ell_1k_2\ell_2} & \textit{if } \text{Type A } \textit{mode interaction} \\ \\ \dfrac{m^{'}\mathrm{lcm}(k_2,l_2)|k_2\ell_1 - k_1\ell_2|}{2k_1\ell_1k_2\ell_2} & \textit{if } \text{Type B } \textit{mode interaction} \end{cases}$$

$$\mathcal{P}^{\mathcal{L}}_{C}(1) = \begin{cases} \dfrac{m^{'2}}{k_1\ell_1} & \textit{if } \text{Type A } \textit{mode interaction} \\ \\ \dfrac{m^{'2}}{2k_1\ell_1} & \textit{if } \text{Type B } \textit{mode interaction} \end{cases}$$

Proof

Use Lemmas 7.1, 7.2 and 7.3. ∎

Let m be a reduced monomial $w_1^{|\gamma_1|}w_2^{|\gamma_2|}w_3^{|\gamma_3|}w_4^{|\gamma_4|}$, where $w_i = z_i$ if $\gamma_i \geq 0$ and $w_i = \overline{z}_i$ if $\gamma_i < 0$, and where $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \mathcal{L}$. The degree of m is given by

$$\partial m = |\gamma| = |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|$$

Using Lemmas 7.2 and 7.3, we calculate now the degrees of the reduced monomials determined by the $u_i \in \mathcal{L}$ corresponding the lattice points $v_i \in \sigma_i \cap \mathcal{M}$.

Proposition 7.5

$$|u_1| = |u_2| = \begin{cases} \gamma_2^{'}\left(1 + \dfrac{\ell_1}{\ell_2}\right) & \textit{if } \Delta > 0 \\ \\ \gamma_2^{'}\left(1 + \dfrac{k_1}{k_2}\right) & \textit{if } \Delta < 0 \end{cases}$$

where

$$\gamma_2^{'} = \begin{cases} \mathrm{lcm}^{'}(k_2,l_2) & \textit{if } \text{Type A } \textit{mode interaction} \\ 2\mathrm{lcm}(k_2,l_2) & \textit{if } \text{Type B } \textit{mode interaction} \end{cases}$$

$$|u_3| = |u_4| = \begin{cases} m^{'}\dfrac{k_1 + k_2}{k_1} & \textit{if } \Delta > 0 \\ \\ m^{'}\dfrac{\ell_1 + \ell_2}{\ell_1} & \textit{if } \Delta < 0 \end{cases}$$

where

$$m^{'} = \min_{m \in \mathbf{Z}^+} \left\{ m \ : \ m\frac{k_2\ell_1 + k_1\ell_2}{2k_1\ell_1} \in \mathbf{Z}^+ \right\}$$

8 Equivariants for the General Mode Interaction

In section 6 we computed the Hilbert series for the rings of invariants under the torus actions corresponding to $(k_1, l_1) - (k_2, l_2)$ mode interactions. In this section we calculate the Hilbert series for the modules of equivariants for the same torus actions.

Consider a group Γ acting on a vector space V. Let $\vec{\mathcal{P}}_V(\Gamma)$ be the space of equivariants with polynomial components for the action of Γ on V. This is a graded module over the ring $\mathcal{P}_V(\Gamma)$ of the invariants and its *Hilbert series* is the generating function

$$\Psi_\Gamma(t) = \sum_{d=0}^{\infty} \dim(\vec{\mathcal{P}}_V^d(\Gamma))\ t^d$$

where $\vec{\mathcal{P}}_V^d(\Gamma)$ is the space of Γ-equivariants with polynomial components that are homogeneous of degree d. See Worfolk [42], Dias and Stewart [13] and Stewart and Dias [39] for further details. For a compact Lie group action there is an explicit integral formula, that generalizes the Molien Theorem for the equivariants, the *Equivariant Molien Theorem*:

$$\Psi_\Gamma(t) = \int_\Gamma \frac{\mathrm{trace}(\gamma^{-1})}{\det(\mathbf{1} - \gamma t)} \mathrm{d}\mu_\Gamma. \tag{24}$$

where μ_Γ is again the normalised Haar measure on Γ and $\gamma \in \Gamma$. For the proof see Sattinger [37], Worfolk [42].

Using the notation of section 6, we prove:

Theorem 8.1 *The Hilbert series for the equivariants $\vec{\mathcal{P}}_V(\Gamma)$ for the action of $\Gamma = \mathbf{T}^2$ on $\mathbf{C}^4$ of section 3 is:*

$$\Psi_\Gamma(t) = \frac{2}{t(1-t^2)^4} F_\Gamma(t) \tag{25}$$

where

$$F_\Gamma(t) = 4(1+t^2)\left(\Phi_\Gamma^A(t) + \Phi_\Gamma^C(t)\right) + 8(1+t^2)\Phi_\Gamma^B(t) - (10+6t^2)\left(\Phi_\Gamma^{\sigma_1}(t) + \Phi_\Gamma^{\sigma_3}(t)\right) + 4$$

Proof

By Theorem 3.2 (part (b)) every $\mathbf{T}^2$-equivariant (with polynomial components) of degree d can be written as a real linear combination of equivariants of the type (4) for $1 \leq j \leq 4$ and where I_g is a $\mathbf{T}^2$-invariant monomial of degree $d+1$. Thus the number of distinct equivariants of degree d of type (4) (for $j = 1, \ldots, 4$) is equal to the number of distinct invariants I_g of degree $d+1$ such that $\overline{z}_j$ divides I_g (for $j = 1, \ldots, 4$), up to a real constant.

By Lemma 3.1 any such I_g can be written uniquely as

$$I_g = kr \tag{26}$$

where k is a product of terms $z_i \overline{z}_i$, and r is an invariant reduced monomial.

Suppose that

$$\frac{\partial m_1}{\partial \overline{z}_j} = \frac{\partial m_2}{\partial \overline{z}_j}$$

(up to a constant multiple) for invariant monomials m_1, m_2. Then $m_1 = m_2$. Since there is a unique representation

$$m_1 = k_1 r_1, \quad m_2 = k_2 r_2$$

where k_j is a product of terms $z_i \overline{z}_i$, and r_j is an invariant reduced monomial, $k_1 = k_2$ and $r_1 = r_2$. Note that r_1, r_2 correspond to points of the lattice $\mathcal{M}$. Thus different r_j give rise to different m_j (since the representation (26) is unique). Each r_j corresponds to a unique point of the lattice. Therefore, the calculation of the number of distinct equivariants of degree d reduces to the calculation of the number of distinct invariant monomials kr of degree $d+1$, such that $\overline{z}_j$ divides kr, and where r belongs to one of the cones or one of the faces, for $j = 1, \ldots, 4$.

We start with invariants kr where r is a reduced invariant monomial corresponding to a lattice point in the interior of a cone. For easy of explanation, consider the interior of cone A and $\Delta > 1$. The corresponding reduced monomials are of type $z_1^{\gamma_1} \overline{z}_2^{\gamma_2} \overline{z}_3^{\gamma_3} z_4^{\gamma_4}$. Thus invariants $m = k(z) z_1^{\gamma_1} \overline{z}_2^{\gamma_2} \overline{z}_3^{\gamma_3} z_4^{\gamma_4}$ are divided by $\overline{z}_2$ and $\overline{z}_3$, and if $k(z) = |z_1|^2 k^{'}(z)$ where again $k^{'}(z)$ is a product of terms $z_i \overline{z}_i$, then it is divided by $\overline{z}_1$. Similarly, if $k(z) = |z_4|^2 k^{''}(z)$, then m is divided by $\overline{z}_4$. Now the conjugates $k(z) \overline{z}_1^{\gamma_1} z_2^{\gamma_2} z_3^{\gamma_3} \overline{z}_4^{\gamma_4}$ are also invariant and are divided by $\overline{z}_1$ and $\overline{z}_4$, and also by $\overline{z}_2$ or $\overline{z}_3$ in case $k(z)$ contains the factor $|z_2|^2$ or $|z_3|^2$. Thus from equivariants of type (4), where $I_g = k(z) r(z)$ and r is a reduced monomial corresponding to a lattice point in the interior of cone A (or conjugate to it), the contribution to Ψ_Γ is given by

$$2 \frac{1}{t(1-t^2)^4} 4 \left(1 + t^2\right) \left(\Phi_\Gamma^A(t) - \Phi_\Gamma^{\sigma_1}(t) - \Phi_\Gamma^{\sigma_2}(t) + 1\right)$$

For the interior of the other cones (and for $\Delta < -1$) the same formula is obtained (with the corresponding faces).

Consider now the faces, for example σ_1, and again suppose that $\Delta > 1$. Reduced monomials are of the type $\overline{z}_2^{\gamma_2} \overline{z}_3^{\gamma_3} z_4^{\gamma_4}$ (and conjugates $z_2^{\gamma_2} z_3^{\gamma_3} \overline{z}_4^{\gamma_4}$). Thus invariants $k(z) \overline{z}_2^{\gamma_2} \overline{z}_3^{\gamma_3} z_4^{\gamma_4}$ are divided by $\overline{z}_2, \overline{z}_3$, and by $\overline{z}_1, \overline{z}_4$ if $k(z)$ contains a factor $|z_1|^2$ and $|z_4|^2$ respectively. For the invariants $k(z) z_2^{\gamma_2} z_3^{\gamma_3} \overline{z}_4^{\gamma_4}$, these are divided by $\overline{z}_4$, and by $\overline{z}_1, \overline{z}_2, \overline{z}_3$ if $|z_1|^2, |z_2|^2, |z_3|^2$ are factors of $k(z)$. From these, the contribution to Ψ_Γ is given by:

$$2 \frac{1}{t(1-t^2)^4} \left(3 + 5t^2\right) \left(\Phi_\Gamma^{\sigma_1}(t) - 1\right)$$

Repeat the same reasoning for the other cones and faces and use Lemma 6.2.

Finally, invariant monomials of type $k(z)$ give rise to equivariants of the type

$$k_1(z)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ z_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $k_1(z)$ is again a product of $z_i\overline{z}_i$'s. These equivariants contribute to Ψ_Γ with

$$2\frac{4t}{(1-t^2)^4}$$

Summing we get (25). ∎

8.1 Examples

For illustration, we consider the Hilbert series for the $\mathbf{T}^2$-equivariants again for the $(3,2)-(1,3)$ mode interaction and the $(3,1)-(1,3)$ mode interaction as in the examples of section 6.1.

Example 8.2 Let $\Gamma = \mathbf{T}^2$ in the action of section 4, the $(3,2)-(1,3)$ mode interaction. Using Theorem 8.1 and recalling Example 6.7,

$$\Psi_\Gamma(t) = 8t + 32t^3 + 80t^5 + 168t^7 + 328t^9 + 616t^{11} + 1104t^{13} + 1872t^{15} + 3024t^{17} + \cdots$$

Example 8.3 Let $\Gamma = \mathbf{T}^2$ in the action of section 5, the $(1,3)-(3,1)$ mode interaction. Using Theorem 8.1 and recalling Example 6.8,

$$\Psi_\Gamma(t) = 8t + 32t^3 + 96t^5 + 256t^7 + 584t^9 + 1192t^{11} + 2248t^{13} + 3936t^{15} + 6504t^{17} + \cdots$$

For both examples we have also computed Ψ_Γ to degree 11 using Maple and the Molien formula (24), as a check, and we obtain the coefficients stated above to that degree.

9 General Torus Actions

We end by describing the first few steps in extending the above analysis to an arbitrary torus action. Let $(\theta_1, ..., \theta_k)$ be coordinates on $\mathbf{T}^k$, where $\theta_j \in \mathbf{R}/2\pi\mathbf{Z} \cong \mathbf{S}^1$. Let $(z_1, ..., z_r)$ be coordinates on $\mathbf{C}^r$. The general torus action takes the form

$$\theta_i z_j = e^{n_{ij}\theta_i} z_j$$

$$\theta_i \overline{z}_j = e^{-n_{ij}\theta_i} \overline{z}_j$$

where $N = [n_{ij}]_{i=1,\ldots,k}^{j=1,\ldots,r}$ is an integer matrix. Let

$$m = \prod_{j=1}^{r} z_j^{\alpha_j} \overline{z}_j^{\beta_j}$$

be a monomial. Then m is invariant under the action of θ_i if and only if $\sum_{j=1}^{r} n_{ij}\gamma_j = 0 \quad (i = 1, ..., k)$ for integers γ_j, where $\gamma_j = \alpha_j - \beta_j$. Solutions depend only on the differences $\alpha_j - \beta_j$, reflecting the obvious fact that each $|z_j|^2$ is invariant. We may therefore write m uniquely in the form $m = (|z_1|^2)^{\tau_1} \cdots (|z_r|^2)^{\tau_r} m'$, where m' is reduced.

The above equations define a lattice $\mathcal{L}$ whose dimension d equals the rank of N. The possible m' are in one-to-one correspondence with elements of $\mathcal{L}$, as before. Decompose $\mathcal{L}$ into cones on which the γ_j have constant sign.

Each cone σ has a fundamental parallelotope $P_\sigma^{\mathcal{L}}$, and the σ-irreducible elements (which lie inside $P_\sigma^{\mathcal{L}}$) form a finite generating set for the semigroup $\sigma \cap \mathcal{L}$.

Suppose σ is simplicial and it has dimension d. Then it is generated by $v_1, \ldots, v_d$ for some $v_i \in \mathcal{L}$. Consider the rays consisting of positive multiples of each v_i. Take $u_1, \ldots, u_d$ the elements in $\mathcal{L}$ of minimal length subject to lying on these rays. Then formula (18) of Lemma 6.4 generalizes to

$$\Phi_\Gamma^\sigma(t) = \frac{\mathcal{P}_\sigma^{\mathcal{L}}(t)}{(1 - t^{|u_1|}) \cdots (1 - t^{|u_d|})}$$

where $\mathcal{P}_\sigma^{\mathcal{L}}$ is the fundamental parallelotope polynomial as in (17) taking now $P_\sigma^{\mathcal{L}}$, the fundamental parallelotope of the lattice $\mathcal{L}$ relative to the cone σ, the polytope with vertices 0, u_1, …, u_d and all sums of distinct u_j. Similarly Theorem 8.1 generalizes, though we do not attempt to state the general formula here since it involves tedious definitions.

Thus the basic set-up generalises to an arbitrary torus action — which is not surprising, given the original source of toric geometry. It seems plausible that more detailed analysis of torus-equivariant bifurcation problems will be able to exploit deeper features of toric geometry, for example to organise singularity-theoretic classifications (as in Golubitsky and Schaeffer [15], Manoel [31], Manoel and Stewart [32]). Potentially, toric geometry is a rich area for equivariant bifurcation theory.

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