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Bifurcation of Dynamical Systems with Symmetry

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To my father, my first professor of Mathematics and

to my mother, with whom I learned what courage is about.

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Resumo

Nesta tese estudamos bifurcações secundárias de ponto de equilíbrio e bifurcação de Hopf em sistemas com simetria $\mathbf{S}_N$.

Estudamos um sistema geral de equações diferenciais ordinárias que comuta com a acção de permutação do grupo simétrico $\mathbf{S}_{3n}$ em $\mathbf{R}^{3n}$. Usando resultados de teoria de singularidades, econtramos condições suficientes nos coeficientes da truncagem de grau cinco de um campo de vectores $\mathbf{C}^\infty$ geral $\mathbf{S}_{3n}$-equivariante para a existência de um ramo secundário de equilíbrio próximo da origem com simetria $\mathbf{S}_n \times \mathbf{S}_n \times \mathbf{S}_n$ do sistema. Provamos também que sob estas condições as soluções são (genericamente) globalmente instáveis excepto nos casos em que duas bifurcações terciárias ocorrem ao longo do ramo secundário. Nestes casos, o resultado sobre a instabilidade mantém-se apenas para o equilíbrio próximo dos pontos de bifurcação secundária.

Estudamos bifurcação de Hopf com simetria $\mathbf{S}_N$ para a acção standard absolutamente irredutível de $\mathbf{S}_N$ obtida da acção de $\mathbf{S}_N$ por permutação de N coordenadas. Stewart (Symmetry methods in collisionless many-body problems, *J. Nonlinear Sci.* **6** (1996) 543-563) obteve um teorema de classificação para os subgrupos $\mathbf{C}$-axiais de $\mathbf{S}_N \times \mathbf{S}^1$. Usamos esta classificação para provar a existência de ramos de soluções periódicas com simetria $\mathbf{C}$-axial em sistemas de equações diferenciais ordinárias com simetria $\mathbf{S}_N$ postas numa soma directa de duas representações $\mathbf{S}_N$-absolutamente irredutíveis, como resultado de uma bifurcação de Hopf que ocorre quando um parâmetro real é variado. Assumimos que os termos de grau cinco na expansão em série de Taylor do campo de vectores estão na forma normal de Birkhoff e como tal comutam com a acção de $\mathbf{S}_N \times \mathbf{S}^1$. Derivamos, para $N \geq 4$ a função geral $\mathbf{S}_N \times \mathbf{S}^1$-equivariante com componentes polinomiais até grau cinco. Usamos o Teorema de Hopf Equivariante para provar a existência de tais ramos de soluções periódicas (com simetria $\mathbf{C}$-axial). Determinamos ainda as condições (genéricas) nos coeficientes do campo de vectores $\mathbf{S}_N \times \mathbf{S}^1$-equivariante de grau cinco que descrevem a estabilidade e a criticalidade desses ramos de soluções. Encontramos que para alguns grupos $\mathbf{C}$-axiais, em algumas direcções, a truncagem de grau cinco do campo de vectores é necessária para determinar a estabilidade das soluções. Além disso, em alguns casos, mesmo a truncagem de grau cinco é demasiado degenerada (origina um valor próprio nulo que não é forçado pela simetria do problema). Incluímos, então, dois capítulos sobre, respectivamente, bifurcação de Hopf com simetria $\mathbf{S}_4$ e $\mathbf{S}_5$. Estudamos estes dois casos pelas seguintes razões. Quando $N = 4$ temos que a truncagem de grau três do campo de vectores é suficiente para determinar a estabilidade das soluções periódicas garantidas pelo Teorema de Hopf Equivariante. Classificamos todos os diagramas de bifurcação possíveis para bifurcação de Hopf com simetria $\mathbf{S}_4$ e procuramos possíveis ramos de soluções periódicas que podem

bifurcar com isotropia submaximal. Terminamos esta tese com o estudo de bifurcação de Hopf com simetria $\mathbf{S}_5$. Este é o primeiro caso em que necessitamos da truncagem de grau cinco do campo de vectores em algumas direções de forma a determinar a estabilidade das soluções periódicas com simetria $\mathbf{C}$-axial.

Abstract

In this thesis we study secondary steady-state bifurcations and Hopf bifurcation in systems with $\mathbf{S}_N$-symmetry.

We study a general system of ordinary differential equations commuting with the permutation action of the symmetric group $\mathbf{S}_{3n}$ on $\mathbf{R}^{3n}$. Using singularity theory results, we find sufficient conditions on the coefficients of the fifth order truncation of the general smooth $\mathbf{S}_{3n}$-equivariant vector field for the existence of a secondary branch of equilibria near the origin with $\mathbf{S}_n \times \mathbf{S}_n \times \mathbf{S}_n$ symmetry of such system. Moreover, we prove that under such conditions the solutions are (generically) globally unstable except in the cases where two tertiary bifurcations occur along the secondary branch. In these cases, the instability result holds only for the equilibria near the secondary bifurcation points.

We study Hopf bifurcation with $\mathbf{S}_N$-symmetry for the standard absolutely irreducible action of $\mathbf{S}_N$ obtained from the action of $\mathbf{S}_N$ by permutation of N coordinates. Stewart (Symmetry methods in collisionless many-body problems, *J. Nonlinear Sci.* **6** (1996) 543-563) obtains a classification theorem for the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. We use this classification to prove the existence of branches of periodic solutions with $\mathbf{C}$-axial symmetry in systems of ordinary differential equations with $\mathbf{S}_N$-symmetry posed on a direct sum of two such $\mathbf{S}_N$-absolutely irreducible representations, as a result of a Hopf bifurcation occurring as a real parameter is varied. We assume that the degree five terms in the Taylor series expansion of the vector field are in Birkhoff normal form and so commute with the action of $\mathbf{S}_N \times \mathbf{S}^1$. We derive, for $N \geq 4$, the general $\mathbf{S}_N \times \mathbf{S}^1$-equivariant map with polynomial components up to degree five. We use the Equivariant Hopf Theorem to prove the existence of such branches of periodic solutions (with $\mathbf{C}$-axial symmetry). Moreover, we determine the (generic) conditions on the coefficients of the fifth order $\mathbf{S}_N \times \mathbf{S}^1$-equivariant vector field that describe the stability and criticality of those solution branches. We find that for some $\mathbf{C}$-axial groups in some directions the fifth degree truncation of the vector field is needed to determine the stability of the solutions. Furthermore, in some cases, even the fifth degree truncation is too degenerate (it originates a null eigenvalue which is not forced by the symmetry of the problem). We then include two chapters with respectively Hopf bifurcation with $\mathbf{S}_4$ and $\mathbf{S}_5$ symmetry. We include these two cases for the following reasons. When $N = 4$, we have that the third degree truncation of the vector field is enough to determine the stability of the periodic solutions guaranteed by the Equivariant Hopt Theorem. Moreover, we classify all possible bifurcation diagrams for Hopf bifurcation with $\mathbf{S}_4$ symmetry and we look for possible branches of periodic solutions that can bifurcate with submaximal isotropy. We finish this thesis with the study of Hopf bifurcation with $\mathbf{S}_5$ symmetry. This is the first case where we need the fifth degree truncation

of the vector field in some directions in order to determine the stability of the periodic solutions with **C**-axial symmetry.

Résumé

Dans cette thèse nous étudions bifurcations secondaires du point d'équilibre et bifurcation de Hopf dans systèmes avec symétrie $\mathbf{S}_N$.

Nous étudions un système général d'équations différentielles ordinaires qui commutent avec l'action de permutation du groupe symétrique $\mathbf{S}_{3n}$ sur $\mathbf{R}^{3n}$. Employant les résultats de la théorie de singularités, nous trouvons des conditions suffisantes sur les coefficients de la troncation de degré cinq du champ de vecteurs $\mathbf{C}^\infty$ général $\mathbf{S}_{3n}$-equivariant pour l'existence d'une branche secondaire des équilibres près de l'origine avec symétrie $\mathbf{S}_n \times \mathbf{S}_n \times \mathbf{S}_n$ d'un tel système. D'ailleurs, nous montrons que dans de telles conditions les solutions sont (génériquement) globalement instables sauf dans les cas où deux bifurcations tertiaires se produisent le long de la branche secondaire. Dans ces cas, le résultat d'instabilité se mantient seulement pour les équilibres près des points secondaires de bifurcation.

Nous étudions bifurcation de Hopf avec symétrie $\mathbf{S}_N$ pour l'action absolument irréductible standard de $\mathbf{S}_N$ obtenu à partir de l'action de $\mathbf{S}_N$ par la permutation de N coordonnées. Stewart (Symmetry methods in collisionless many-body problems, *J. Nonlinear Sci.* **6** (1996) 543-563) a obtenue un théorème de classification pour les sous-groupes $\mathbf{C}$-axial de $\mathbf{S}_N \times \mathbf{S}^1$. Nous employons cette classification pour prouver l'existence des branches des solutions périodiques avec symétrie $\mathbf{C}$-axial dans les systèmes des équations différentielles ordinaires avec symétrie $\mathbf{S}_N$ posées sur une somme directe de deux telles représentations absolument irréductibles de $\mathbf{S}_N$, comme résultat d'une bifurcation de Hopf qui se produit pendant qu'un paramètre réel est changé. Nous supposons que les termes de degré cinq dans l'expansion de la série de Taylor du champ de vecteurs sont sous la forme normale de Birkhoff et ainsi commute avec l'action de $\mathbf{S}_N \times \mathbf{S}^1$. Nous dérivons, pour $N \geq 4$, la fonction générale $\mathbf{S}_N \times \mathbf{S}^1$-equivariant avec les composants polynômes jusqu'au degré cinq. Nous employons le théorème d' Hopf Equivariant pour prouver l'existence de telles branches de solutions périodiques (avec symétrie $\mathbf{C}$-axial). D'ailleurs, nous déterminons les conditions (génériques) sur les coefficients du cinquième ordre du champ de vecteurs $\mathbf{S}_N \times \mathbf{S}^1$-equivariant qui décrivent la stabilité et la criticalité de ces branches de solutions. Nous constatons que pour certains groupes $\mathbf{C}$-axial dans certaines directions la cinquième troncation du champ de vecteurs est nécessaire pour déterminer la stabilité des solutions. En outre, dans certains cas, même la troncation de degré cinq est trop dégénérée (elle lance une valeur propre nulle qui n'est pas forcée par la symétrie du problème). Nous incluons alors deux chapitres avec respectivement la bifurcation de Hopf avec symétrie $\mathbf{S}_4$ et $\mathbf{S}_5$. Nous incluons ces deux points pour les raisons suivantes. Quand $N = 4$, la troncation de degré trois du champ de vecteurs est assez suffisante pour déterminer la stabilité des solutions périodiques garanties par le théorème d'Hopf Equivariant. D'ailleurs, nous classons

tous les diagrammes possibles de bifurcation pour la bifurcation de Hopf avec symétrie $\mathbf{S}_4$ et nous recherchons les branches possibles des solutions périodiques qui peuvent bifurquer avec l'isotropie submaximale. Nous finissons cette thèse avec l'étude de la bifurcation de Hopf avec symétrie $\mathbf{S}_5$. C'est le premier cas où nous avons besoin de la troncation de degré cinq du champ de vecteurs dans certaines directions afin de déterminer la stabilité des solutions périodiques avec symétrie $\mathbf{C}$-axial.

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Chapter 1

Introduction

"Ao destino agradam as repetições, as variantes, as simetrias."
Jorge Luis Borges, *o Fazedor*

In the sentence of Borges, the argentine writer, *to the destiny please the repetitions, the variants, the symmetries.* This sentence is by itself a motivation to this work. In this thesis we study bifurcation of dynamical systems with symmetry.

In the general theory of symmetric dynamical systems [23] we study a system of ordinary differential equations (ODEs)

$$\frac{dx}{dt} = f(x, \lambda), \tag{1.1}$$

with $x \in V, \lambda \in \mathbf{R}$, where V is a finite-dimensional vector space, λ is the bifurcation parameter and f is a symmetric function.

We say that γ (an invertible $n \times n$ matrix) is a symmetry of (1.1) if $f(\gamma x, \lambda) = \gamma f(x, \lambda)$ for all $x \in V, \lambda \in \mathbf{R}$. A consequence of this is that if $x(t)$ is a solution to (1.1), then so it is $\gamma x(t)$. We say that γ is a symmetry of the solution $x(t)$.

There is a similar consequence for periodic solutions: if $x(t)$ is a T-periodic solution of (1.1), then so is $\gamma x(t)$. Uniqueness of solutions to the initial problem for (1.1) implies that the trajectory of $x(t)$ and $\gamma x(t)$ are either disjoint, in which case we have a *new* periodic solution, or identical, in which case $x(t)$ and $\gamma x(t)$ differ only by a phase shift, that is

$$x(t) = \gamma x(t - t_0)$$

for some t_0. In this case we say that the pair (γ, t_0) is a symmetry of the periodic solution $x(t)$. Thus, symmetries of periodic solutions have both a spatial component γ and a temporal component t_0.

Bifurcation Theory describes how solutions to differential equations can branch as a parameter is varied. It turns out that the symmetry of f imposes restrictions on the bifurcations that can occur. Generically there are two types of bifurcation:

(a) Steady-state bifurcation, when an eigenvalue of $(df)_{0,\lambda}$ passes through 0 (without loss of generality at $\lambda = 0$).

(b) Hopf bifurcation, when a pair of conjugate complex eigenvalues of $(df)_{0,\lambda}$ crosses the imaginary axis with nonzero speed at $\pm\omega i, \omega \neq 0$.

In this work we study both kinds of bifurcation, stated in (a) and (b). We study steady-state secondary bifurcations and Hopf bifurcation in systems with $\mathbf{S}_N$-symmetry.

This work is organized as follows. In Chapter 2 we present the Background needed for the present work.

In Chapter 3 we study secondary bifurcations in systems with all-to-all coupling. The original motivation for the work carried out in this chapter came from evolutionary biology. Cohen and Stewart [7] introduced a system of $\mathbf{S}_N$-equivariant ordinary differential equations (ODEs) that models sympatric speciation as a form of spontaneous symmetry-breaking in a system with $\mathbf{S}_N$-symmetry. Elmhirst [17, 15, 16] studied the stability of the primary branches in such a model and also linked it to a biological specific model of speciation. Stewart *et al.* [40] made numerical studies of relatively concrete models. Here the population is aggregated into N discrete 'cells', with a vector x_j representing values of some phenotypic observable - the phenotype - the organisms form and behavior. If the initial population is monomorphic (single-species) then the system of ODEs representing the time-evolution of the phenotypes should be equivariant under the action of the symmetric group $\mathbf{S}_N$; that is, the model is an example of an *all-to-all coupled system.* Symmetry-breaking bifurcations of the system correspond to the splitting of the population into two or more distinct morphs (species).

Dias and Stewart [11] continue the study of the general cubic truncation of a center manifold reduction of a system of that type, which takes the form

$$\frac{dx_i}{dt} = \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 \tag{1.2}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, D \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i, the coordinates satisfy $x_1 + \cdots + x_N \equiv 0$ and $\pi_j = x_1^j + \cdots + x_N^j$ for $j = 2, 3$. They study the existence, branching geometry and stability of secondary branches of equilibria in such systems. Their study was motivated by numerical simulations showing jump bifurcations between primary branches. These jumps correspond to the loss of stability of the primary branches, see Stewart *et al.* [40]. Primary branches in such systems correspond to partitions of N into two parts p, q with $p + q = N$. Secondary branches correspond to partitions of N into three parts a, b, c with $a + b + c = N$. They remarked that the cubic-order system (1.2) is too degenerate to provide secondary branches if $a = b = c$. We focus our work in this case. We begin by observing why this case is special. When looking for steady-state solutions with symmetry $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$, we restrict the original $\mathbf{S}_N$-equivariant vector field, where $N = 3a$, to the fixed-point subspace of Σ. These equations are now equivariant under the normalizer of Σ inside $\mathbf{S}_N$. Moreover, the group of symmetries acting nontrivially on that fixed-point subspace is the quotient of that normalizer over Σ and it is isomorphic to $\mathbf{D}_3$, the dihedral group of order six. Solutions with Σ-symmetry of the original system correspond to solutions with trivial symmetry for the $\mathbf{D}_3$-symmetric restricted problem. Using singularity results for $\mathbf{D}_3$-equivariant bifurcation problems, see Golubitsky *et al.* [23], we find solutions of that type, by *local analysis near the origin*, assuming nondegeneracy conditions on the coefficients of the fifth order truncation of the system.

In this chapter, we consider a general smooth $\mathbf{S}_N$-equivariant system of ODEs posed on the $\mathbf{S}_N$-absolutely irreducible space, $V_1 = \{x \in \mathbf{R}^n : x_1 + \cdots + x_N = 0\}$, which takes the form

$$\frac{dx}{dt} = G(x, \lambda) \tag{1.3}$$

where

$$\begin{aligned} G_i(x,\lambda) \quad = \quad & \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 + \\ & E(Nx_i^4 - \pi_4) + F(Nx_i^2\pi_2 - \pi_2^2) + Gx_i\pi_3 + \\ & + H(Nx_i^5 - \pi_5) + I(Nx_i^3\pi_2 - \pi_3\pi_2) + \\ & J(Nx_i^2\pi_3 - \pi_3\pi_2) + Lx_i\pi_4 + Mx_i\pi_2^2 + \\ & \text{terms of degree } \geq 6 \end{aligned} \tag{1.4}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, \ldots, M \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i (and the coordinates satisfy $x_1 + \cdots + x_N = 0$). Also $\pi_j = x_1^j + \cdots + x_N^j$ for $j = 2, \ldots, 5$.

In Sections 3.1 and 3.2 we obtain, respectively, the isotropy subgroups for the natural representation of the symmetric group and the general fifth order truncation of (1.3) of any smooth $\mathbf{S}_N$-equivariant vector field posed on the $\mathbf{S}_N$-absolutely irreducible space V_1. In Section 3.3 we present a brief description of the singularity theory of $\mathbf{D}_3$-equivariant bifurcation problems.

In Section 3.4 we suppose $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$. We look for secondary branches of steady-state solutions for the system (1.3) that are Σ-symmetric obtained by bifurcation from a primary branch of solutions with isotropy group (conjugate to) $\mathbf{S}_a \times \mathbf{S}_{2a}$. The restriction of (1.3) to the fixed-point subspace of Σ is $\mathbf{D}_3$-equivariant. $\mathbf{D}_3$-singularity results imply that the existence and stability (in $\mathrm{Fix}(\Sigma)$) of such a secondary branch of solutions near the origin depends only on certain non-degeneracy conditions on the coefficients of the fifth order truncation of the vector field G. Theorem 3.3 describes sufficient conditions on the coefficients of the vector field for the existence of a secondary branch of solutions of (1.3) with that symmetry. Corollary 3.4 describes the parameter regions of stability of those solutions (in $\mathrm{Fix}(\Sigma)$). Finally, in Section 3.5 we discuss the full stability of such a secondary branch. In Theorem 3.8 we obtain the expressions of the eigenvalues that determine the full stability of those solutions. We prove in Theorem 3.9 that these solutions are (generically) globally unstable except in the cases where two tertiary bifurcations occur along the secondary branch. In these cases, the instability result holds only for the equilibria near the secondary bifurcation points. We conclude with an example where two tertiary bifurcations occur along the secondary branch and the solutions along the branch between those tertiary bifurcation points are stable (Example 3.10).

In Chapter 4 we study Hopf bifurcation with $\mathbf{S}_N$-symmetry. This part of the work was motivated by two different branches of physics. Our first motivation for studying this particular group of symmetry came from the ongoing work on series of Josephson junctions arrays over the past years ([20], [42] - [45]). Josephson junctions are superconducting electronic devices capable of generating extraordinarily high frequency voltage oscillations, up to 10^{11} or more. In such devices, it is particularly desirable that the elements oscillate perfectly in phase, in order that the power output reaches practically useful levels. In [42], Tsang *et al.* present a very useful discussion of the symmetries of such systems, noticing that the ODEs governing the dynamics are in fact symmetric under any permutation of the N indices.

But, in fact, the greatest motivation for this work was newtonian mechanics. At this point we want just to conjecture that the results we got in Hopf bifurcation with $\mathbf{S}_N$-symmetry can be used in order to find periodic solutions of symmetric models of celestial dynamics. We will discuss this physical motivation with detail in Section 4.1.

The theory of Hopf Bifurcation with symmetry was developed by Golubitsky and Stewart [25] and by Golubitsky, Stewart, and Schaeffer [23]. Golubitsky and Stewart [24] applied the theory of Hopf bifurcation with symmetry to systems of ordinary differential equations having the symmetries of a regular polygon (this is, with $\mathbf{D}_n$-symmetry). They studied the existence and stability of symmetry-breaking branches of periodic solutions. Finally, they applied their results to a general system of n nonlinear oscillators, coupled symmetrically in a ring, and describe the generic oscillation patterns. Since the development of the theory, some examples were studied with detail, we list some:

- Swift [39] studied Hopf bifurcation with the symmetry of the square (this is, with $\mathbf{D}_4$-symmetry). He found that invariant tori (quasiperiodic solutions with two frequencies) and periodic solutions with "minimal" symmetry bifurcate from the origin for *open regions* of the parameter space of cubic coefficients.

- Iooss and Rossi [29] studied Hopf bifurcation with spherical symmetry ($\mathbf{SO(3)}$-symmetry). In this particular bifurcation the imaginary eigenspace is a direct sum of two copies of the 5-dimensional irreducible representation of the group $\mathbf{SO(3)}$ on the space of the spherical harmonics of order 2. They obtain five different types of bifurcating periodic solutions and the stability conditions and direction of bifurcation are proved for all these solutions. They also show that a family of quasiperiodic solutions may bifurcate directly from an invariant fixed point together with the periodic solutions. Later, Haaf, Roberts and Stewart [26] showed that their results could be obtained in a simpler manner by realizing the space of the spherical harmonics of order 2 as the set of symmetric traceless 3×3 matrices. They prove the generic existence of five types of symmetry-breaking oscillation: two rotating waves and three standing waves and analyse the stabilities of the bifurcating branches, describing the restrictions of the dynamics to various fixed-point spaces of subgroups of $\mathbf{SO(3)}$, and discussing possible degeneracies in the stability conditions.

- Gils and Golubitsky [19] proved that in general, degeneracies arising from Hopf bifurcation in the presence of symmetry, in situations where the normal form equations decouple into phase/amplitude equations lead to secondary torus bifurcations. They apply this result to the case of degenerate Hopf bifurcation with triangular ($\mathbf{D}_3$) symmetry, proving that in codimension two there exist regions of the parameter space where two branches of asymptotic stable 2-tori coexist but where no stable periodic solutions are present.

- Silber and Knobloch [38] studied Hopf bifurcation on a square lattice ($\mathbf{D}_4 \ltimes T^2$-symmetry) and Dias and Stewart [12] studied Hopf bifurcation on a primitive cubic lattice.

- Dias and Paiva ([8], [9]) proved the nonexistence of branches of periodic solutions with submaximal symmetry in Hopf bifurcation problems with dihedral group symmetry ($\mathbf{D}_n$) when $n \neq 4$.

- Abreu and Dias [2] studied Hopf bifurcation on Hemispheres. They considered Hopf bifurcations for reaction-diffusion equations defined on the hemisphere with Newmann boundary conditions on the equator. They showed the effect of hidden symmetries on spherical domains for the type of Hopf bifurcations that can occur. They obtain periodic solutions for the hemisphere problem by extending the problem to the sphere and finding then periodic solutions with spherical spatial symmetries containing the reflection across the equator. The equations on the hemisphere have $\mathbf{O}(2)$-symmetry and the equations on the sphere have spherical symmetry.

We consider the natural action of $\mathbf{S}_N$ on $\mathbf{C}^N$ given by

$$\sigma(z_1, \ldots, z_N) \;=\; \left(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(N)}\right)$$

for $\sigma \in \mathbf{S}_N$ and $(z_1, \ldots, z_N) \in \mathbf{C}^N$. The decomposition of $\mathbf{C}^N$ into invariant subspaces for this action of $\mathbf{S}_N$ is

$$\mathbf{C}^N \;\cong\; \mathbf{C}^{N,0} \oplus V_1$$

where

$$\mathbf{C}^{N,0} \;=\; \{(z_1, \ldots, z_N) \in \mathbf{C}^N : z_1 + \cdots + z_N = 0\}$$

and

$$V_1 \;=\; \{(z, \ldots, z) : \; z \in \mathbf{C}\} \;\cong\; \mathbf{C}.$$

The action of $\mathbf{S}_N$ on V_1 is trivial. The space $\mathbf{C}^{N,0}$ is $\mathbf{S}_N$-simple:

$$\mathbf{C}^{N,0} \;\cong\; \mathbf{R}^{N,0} \oplus \mathbf{R}^{N,0}$$

where $\mathbf{S}_N$ acts absolutely irreducibly on

$$\mathbf{R}^{N,0} \;=\; \{(x_1, \ldots, x_N) \in \mathbf{R}^N : x_1 + \cdots + x_N = 0\} \;\cong\; \mathbf{R}^{N-1}.$$

Suppose now that we have a system of ordinary differential equations (ODEs)

$$\frac{dx}{dt} = f(x, \lambda), \tag{1.5}$$

where $x \in \mathbf{C}^{N,0}$, $\lambda \in \mathbf{R}$ is the bifurcation parameter and $f : \mathbf{C}^{N,0} \times \mathbf{R} \to \mathbf{C}^{N,0}$ is a smooth mapping commuting with the action of $\mathbf{S}_N$ as defined above. Note that $\mathrm{Fix}_{\mathbf{C}^{N,0}}(\mathbf{S}_N) = \{0\}$ and so $f(0, \lambda) \equiv 0$.

We suppose that $(df)_{0,0}$ has eigenvalues $\pm i$. Our aim is to study the generic existence of branches of periodic solutions of (1.5) near the bifurcation point $(x, \lambda) = (0, 0)$. We assume that f is in Birkhoff normal form and so f also commutes with $\mathbf{S}^1$, where the action of $\mathbf{S}^1$ on $\mathbf{C}^{N,0}$ is given by

$$\theta z \;=\; e^{i\theta} z \quad \left(\theta \in \mathbf{S}^1, z \in \mathbf{C}^{N,0}\right).$$

In Section 4.1 we give an overview of the physical motivation for this work.

In Section 4.2 we recall the classification of the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ given by Stewart [41]. There are two types of isotropy subgroups, $\Sigma^I_{q,p}$ and Σ^{II}_q.

In Section 4.3 we calculate the cubic and the fifth order truncation of f in (1.5) for the action of $\mathbf{S}_N \times \mathbf{S}^1$ extended naturally to $\mathbf{C}^N$. We obtain the cubic and the fifth order truncation of f in (1.5) on $\mathbf{C}^{N,0}$ by restricting and projecting onto $\mathbf{C}^{N,0}$.

After describing the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$, we use in Section 4.4 the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions with these symmetries of (1.5) by Hopf bifurcation from the trivial equilibrium at $\lambda = 0$ for a bifurcation problem with symmetry $\Gamma = \mathbf{S}_N$. The main result of this chapter is Theorem 4.13, where we determine the directions of branching and the stability of periodic solutions guaranteed by the Equivariant Hopf Theorem. For solutions with symmetry Σ^{II}_q the terms of the degree three truncation of the vector field determines the criticality of the branches and also the stability of these solutions (near the origin). However, for solutions with symmetry $\Sigma^I_{p,q}$, although the criticality of the branches is determined by the terms of degree three, the stability of solutions in some directions is not. Moreover, in one particular direction, even the degree five truncation is too degenerate (it originates a null eigenvalue which is not forced by the symmetry of the problem).

The remaining two chapters in this thesis are devoted to the study of Hopf bifurcation with $\mathbf{S}_4$ and with $\mathbf{S}_5$ symmetry. From Theorem 4.13 we have that for one of the isotropy subgroups, namely $\Sigma^I_{p,q}$, the stability of solutions in some directions is determined by the fifth degree truncation of the vector field. Furthermore, in one particular direction, even the fifth degree truncation of the vector field is too degenerate. We include these two cases (Hopf bifurcation with $\mathbf{S}_4$ and with $\mathbf{S}_5$ symmetry) with details for the following reasons. When $N = 4$, the directions in which we need the fifth degree truncation of the vector field do not appear in the isotypic decomposition for the action of each isotropy subgroup on $\mathbf{C}^{4,0}$. This means this is the case (in fact the only one) we only need the third degree truncation of the vector field to compute the stability in all directions. We obtain conditions depending on the coefficients of the third order truncation of the vector field that determine the stability and criticality of the branches of periodic solutions guaranteed by the Equivariant Hopf Theorem. This allow us to classify the possible bifurcation diagrams. The case when $N = 5$ is slightly different. In this case, the directions in which we need the degree five truncation of the vector field are present in the isotypic decomposition for some of the isotropy subgroups. Although for the other values of N, the fifth degree truncation of the vector field is still degenerate to determine the stability of the periodic solutions, in the case $N = 5$, the degree five truncation of the vector field determines the stability of all the periodic solutions guaranteed by the Equivariant Hopf Theorem.

In Chapter 5 we study Hopf Bifurcation with $\mathbf{S}_4$-symmetry. We start this Chapter with the study of the branching and stability of periodic solutions with maximal isotropy. The conjugacy classes of isotropy subgroups for the action of $\mathbf{S}_4 \times \mathbf{S}^1$ and the equivariant vector field are derived from the general theorems presented in Chapter 4. In Section 5.1 we look for branches of periodic solutions that can bifurcate with maximal isotropy ($\mathbf{C}$-axial solutions) and we determine the directions of branching and the stability of periodic solutions guaranteed by the Equivariant Hopf Theorem. Although this example has been studied in [4], in this thesis we obtain explicit expressions for the stability which allows us to classify the possible bifurcation diagrams. We do this in Section 5.2, moreover, we give two examples, assigning specific values for the parameters. We finish this chapter by looking for possible branches of periodic solutions that can bifurcate with submaximal isotropy.

In Chapter 6 we study Hopf Bifurcation with $\mathbf{S}_5$-symmetry. This is the first case where the fifth degree truncation of the vector field is necessary in order to determine the branching equations and the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem. Again, we determine the directions of branching and the stability of periodic solutions guaranteed by the Equivariant Hopf Theorem.

Chapter 2

Background

When we study a one-parameter family of systems of ordinary differential equations (ODEs)

$$\frac{dx}{dt} = f(x, \lambda), \tag{2.1}$$

with $x \in V, \lambda \in \mathbf{R}$, where V is a finite-dimensional real vector space, λ is the bifurcation parameter and f commutes with the action of a compact Lie group Γ on V, it turns out that the symmetry of the problem imposes restrictions on the type and the way that solutions can bifurcate from an invariant steady-state equilibrium.

We begin this chapter presenting a few results concerning representation of compact Lie groups.

If $\mathrm{Fix}(\Gamma) = \{0\}$, it follows that $f(0, \lambda) \equiv 0$ and $x = 0$ is an equilibrium of (2.1) for all parameter values of λ. Moreover, if the action of Γ on V is absolutely irreducible, as the Jacobian of f at $(0, \lambda)$, $(df)_{(0,\lambda)}$, commutes with Γ, it follows that $(df)_{(0,\lambda)}$ is a scalar multiple of the identity. Thus $(df)_{(0,\lambda)} = c(\lambda)\mathrm{Id}_V$ where $c: \mathbf{R} \to \mathbf{R}$ is smooth. Suppose that $(df)_{(0,\lambda)}$ is singular, say, at $\lambda = 0$. Then we have that $c(0) = 0$ and $(df)_{(0,0)} = 0$. In Section 2.2 we introduce the Equivariant Branching Lemma [23, Theorem XIII 3.3], which states that if $c^{'}(0) \neq 0$, then for each axial subgroup of Γ there exists a unique branch of equilibria of (2.1) bifurcating from the trivial equilibrium at $\lambda = 0$ with that symmetry.

When the derivative $(df)_{0,0}$ has purely imaginary eigenvalues, then under an additional hypothesis of nondegeneracy, the Equivariant Hopf Theorem [23, Theorem XVI 4.1] states that we can expect bifurcating branches of periodic solutions. These correspond to solutions of (2.1) restricted to two-dimensional fixed-point subspaces of groups that are related now with symmetries that involve the original symmetry group of the problem and an extra group of symmetries called phase-shift symmetries. These extra symmetries can be related to the circle group $\mathbf{S}^1$. See Section 2.3.

2.1 Group Theory

Let $\mathbf{GL}(n)$ denote the group of all invertible linear transformations of the vector space $\mathbf{R}^n$ into itself, or equivalently, the group of nonsingular $n \times n$ matrices over $\mathbf{R}$. As in Golubitsky *et al.* [23, Chapter XII], we define a Lie group to be a closed subgroup Γ of

$\mathbf{GL}(n)$. A Lie group Γ is compact or connected if it is compact or connected as a subset of $\mathbf{R}^{n^2}$. Equivalently, Γ is compact if and only if the entries in the matrices defining Γ are bounded.

Let Γ be a Lie group and let V be a finite-dimensional real vector space. We say that Γ acts (linearly) on V if there is a continuous mapping (the action), from $\Gamma \times V$ to V such that to each (γ, v) makes correspond $\gamma \cdot v$ satisfying:
(a) for each $\gamma \in \Gamma$ the mapping $\rho_\gamma : V \to V$ defined by $\rho_\gamma(v) = \gamma \cdot v$ is linear;
(b) if $\gamma_1, \gamma_2 \in \Gamma$ then $\gamma_1 \cdot (\gamma_2 \cdot v) = (\gamma_1\gamma_2) \cdot v$ for all $v \in V$.

The mapping $\rho : \Gamma \to \mathbf{GL}(V)$ such that $\rho(\gamma) = \rho_\gamma$ is called a *representation* of Γ on V. Here $\mathbf{GL}(V)$ is the group of invertible linear transformations $V \to V$.

Let N be a positive integer. The set of all permutations of $1, 2, \ldots, N$, under the product operation of composition, is a group. It is called the symmetric group, and we write $\mathbf{S}_N$. The number of elements in a group Γ is called the order of Γ, so, the order of $\mathbf{S}_N$ is $N!$.

Remark 2.1 It is easy to see that $\Gamma = \mathbf{S}_N$ acting on $V = \mathbf{R}^N$ by permutation of coordinates

$$\sigma x = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(N)}) \; (\sigma \in \mathbf{S}_N,\ x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N)$$

is an action:

$$\rho(\sigma x) = \rho\left(\sigma(x_1, x_2, \ldots, x_N)\right) = \rho(\underbrace{x_{\sigma^{-1}(1)}}_{y_1}, \ldots, \underbrace{x_{\sigma^{-1}(N)}}_{y_2})$$

with

$$\begin{aligned} \rho(y_1, \ldots, y_N) &= (y_{\rho^{-1}(1)}, \ldots, y_{\rho^{-1}(N)}) &&= (x_{\sigma^{-1}\rho^{-1}(1)}, \ldots, x_{\sigma^{-1}\rho^{-1}(N)}) \\ &= (x_{(\rho\sigma)^{-1}(1)}, \ldots, x_{(\rho\sigma)^{-1}(N)}) &&= (\rho\sigma)x. \end{aligned}$$

$\diamond$

Let Γ be a Lie group acting on the vector space V. A subspace $W \subset V$ is called Γ-*invariant* if $\gamma w \in W$ for all $w \in W, \gamma \in \Gamma$. A representation or action of Γ on V is *irreducible* if the only Γ-invariant subspaces of V are $\{0\}$ and V. A subspace $W \subseteq V$ is said to be Γ-*irreducible* if W is Γ-invariant and the action of Γ on W is irreducible.

The study of a representation of a compact Lie group is often made easier by observing that it decomposes into a direct sum of simpler representations, which are said to be irreducible.

Proposition 2.2 *Let Γ be a compact Lie group acting on V. Let $W \subseteq V$ be a Γ-invariant subspace. Then there exists a Γ-invariant complementary subspace $Z \subseteq V$ such that*

$$V = W \oplus Z.$$

Proof: See [23, Proposition XII 2.1]. $\square$

It follows from this that every representation of a compact Lie group may be written as a direct sum of irreducible subspaces.

Corollary 2.3 (Theorem of Complete Reducibility). *Let Γ be a compact Lie group acting on V. Then there exist Γ-irreducible subspaces $V_1, \ldots, V_s$ of V such that*

$$V = V_1 \oplus \cdots \oplus V_s. \tag{2.2}$$

Proof: See [23, Corollary XII 2.2]. □

In general, the decomposition of V in (2.2) is not unique. The reason for this nonuniqueness in the decomposition of Corollary 2.3 is the occurrence in V of two isomorphic irreducible representations.

Theorem 2.4 *Let Γ be a compact Lie group acting on V.*
(a) Up to Γ isomorphism there are a finite number of distinct Γ-irreducible subspaces of V. Call these $U_1, \ldots, U_t$.
(b) Define W_k to be the sum of all Γ-irreducible subspaces W of V such that W is Γ-isomorphic to U_k. Then

$$V = W_1 \oplus \cdots \oplus W_t.$$

Proof: See [23, Theorem XII 2.5]. □

The subspaces W_k, for $k = 1, \ldots, t$, are called the *isotypic components* of V, of type U_k, for the action of Γ.

We say that a mapping $g : V \to V$ is $\Gamma-$*equivariant* or *commutes* with Γ if

$$g(\gamma \cdot v) = \gamma \cdot g(v)$$

for all $\gamma \in \Gamma$ and $v \in V$.

A special kind of commuting mappings are the linear ones:

Definition 2.5 A *representation* of a group Γ on a vector space V is *absolutely irreducible*, or the *space* V is said to be *absolutely irreducible*, if the only linear mappings on V that commute with Γ are the scalar multiples of the identity. ◇

Remark 2.6 When working with complex representations of compact Lie groups then Schur's Lemma implies that the complex versions of irreducibility and absolute irreducibility are equivalent concepts; however, this is not true for real representations. ◇

Lemma 2.7 *Let Γ be a compact Lie group acting on V. If the action of Γ is absolutely irreducible then it is irreducible.*

Proof: See [23, Lemma XII 3.3]. □

We present now some results about linear maps that commute with the action of a compact Lie group.

Lemma 2.8 *Let Γ be a compact Lie group acting on V, let $A : V \to V$ be a linear mapping that commutes with Γ, and let $W \subset V$ be a Γ-irreducible subspace. Then $A(W)$ is Γ-invariant, and either $A(W) = 0$ or the representation of Γ on W and $A(W)$ are isomorphic.*

Proof: See [23, Lemma XII 3.4]. □

Lemma 2.8 implies:

Theorem 2.9 *Let Γ be a compact Lie group acting on the vector space V. Decompose V into isotypic components*

$$V = W_1 \oplus \cdots \oplus W_s.$$

Let $A : V \to V$ be a linear mapping commuting with Γ. Then

$$A(W_k) \subseteq W_k \tag{2.3}$$

for $k = 1, \ldots, s$.

Proof: See [23, Lemma XII 3.5]. □

Let V and W be n-dimensional real vector spaces and assume that the Lie group acts both on V and W. The actions are said to be *isomorphic*, or the spaces V and W are $\Gamma-$ *isomorphic*, if there exists a (linear) isomorphism $A : V \to W$ such that $A(\gamma \cdot v) = \gamma \cdot A(v)$, for all $v \in V$ and $\gamma \in \Gamma$; that is, we get the same group of matrices if we identify the spaces V and W (via the linear isomorphism).

The symmetry of a mapping imposes restrictions on its form. There are results that permit the description of the $\mathbf{C}^\infty$ functions that are equivariant by Γ. We now describe nonlinear mappings that commute with a group action.

Let Γ be a (compact) Lie group acting on a vector space V. We say that a real-valued function $f : V \to \mathbf{R}$ is *invariant* under Γ if

$$f(\gamma x) = f(x) \tag{2.4}$$

for all $\gamma \in \Gamma,\ x \in V$. An invariant polynomial is defined in the obvious way by taking f to be polynomial. Note that it suffices to verify (2.4) for a set of generators of Γ.

Denote by $\mathcal{P}(\Gamma)\,(\varepsilon(\Gamma))$ the ring of polynomials (C^∞ germs) from V to $\mathbf{R}$ invariant under Γ. Note that $\mathcal{P}(\Gamma)$ is a ring since sums and products of Γ-invariant polynomials are again Γ-invariant.

When there is a finite subset of invariant polynomials $u_1, \ldots, u_s$ such that *every* invariant polynomial may be written as a polynomial function of $u_1, \ldots, u_s$, this set is said to *generate*, or to form a *Hilbert basis* of $\mathcal{P}(\Gamma)$.

Next theorem gives a theoretical foundation for describing invariant polynomials:

Theorem 2.10 (Hilbert-Weyl Theorem) *Let Γ be a compact Lie group acting on V. Then there exists a finite Hilbert basis for the ring $\mathcal{P}(\Gamma)$.*

Proof: See [23, Theorem XII 4.2]. □

A similar result to Theorem 2.10 holds for real analytic functions, moreover, this result remains true for C^∞ germs.

Theorem 2.11 (Schwarz [37]) *Let Γ be a compact Lie group acting on V. Let $u_1, \ldots, u_s$ be a Hilbert basis for the Γ-invariant polynomials $\mathcal{P}(\Gamma)$. Let $f \in \varepsilon(\Gamma)$. Then there exists a smooth germ $h \in \varepsilon_s$ such that*

$$f(x) = h\left(u_1(x), \ldots, u_s(x)\right). \tag{2.5}$$

Here ε_s is the ring of C^∞ germs $\boldsymbol{R}^s \to \boldsymbol{R}$.

Proof: See [23, Theorem XII 4.3]. □

Note that when $\mathcal{P}(\Gamma)$ is a polynomial ring in the Hilbert basis $u_1, \ldots, u_s$, then every invariant polynomial f has *uniquely* the form (2.5). However, even when $\mathcal{P}(\Gamma)$ is a polynomial ring, uniqueness need not hold in (2.4) for $\mathbf{C}^\infty$ germs.

We now describe the restrictions placed on nonlinear mappings. Next lemma states that the product of an equivariant mapping and an invariant function is another equivariant mapping:

Lemma 2.12 *Let $f : V \to \boldsymbol{R}$ be a Γ-invariant function and let $g : V \to V$ be a Γ-equivariant mapping. Then $fg : V \to V$ is Γ-equivariant.*

Proof: See [23, Lemma XII 5.1]. □

Denote now by $\overrightarrow{\mathcal{P}}(\Gamma)$ the space of Γ-equivariant polynomial mappings from V into V and $\overrightarrow{\varepsilon}(\Gamma)$ the space of Γ-equivariant germs (at the origin) $\mathbf{C}^\infty$ from V into V.

The space $\overrightarrow{\mathcal{P}}(\Gamma)$ is a module over the ring $\mathcal{P}(\Gamma)$. Similarly, the space $\overrightarrow{\varepsilon}(\Gamma)$ is a module over the ring $\varepsilon(\Gamma)$.

Let $g_1, \ldots, g_r$ be Γ-equivariant polynomial mappings from V to V such that every $g \in \overrightarrow{\mathcal{P}}(\Gamma)(\overrightarrow{\varepsilon}(\Gamma))$ can be written as

$$g = f_1 g_1 + \cdots + f_r g_r$$

for $f_j \in \mathcal{P}(\Gamma)(\varepsilon(\Gamma))$. Then $g_1, \ldots, g_r$ are said to generate $\overrightarrow{\mathcal{P}}(\Gamma)(\overrightarrow{\varepsilon}(\Gamma))$ over $\mathcal{P}(\Gamma)(\varepsilon(\Gamma))$. If the relation

$$f_1 g_1 + \cdots + f_r g_r \equiv 0$$

where $f_j \in \varepsilon(\Gamma)$ implies that $f_1 \equiv \cdots \equiv f_r \equiv 0$, then we say that $g_1, \ldots, g_r$ *freely generate* $\overrightarrow{\varepsilon}(\Gamma)$ over $\varepsilon(\Gamma)$ and $\overrightarrow{\varepsilon}(\Gamma)$ is called a free module over $\varepsilon(\Gamma)$.

Theorem 2.13 *Let Γ be a compact Lie group acting on V. Then there exists a finite set of Γ-equivariant polynomial mappings $g_1, \ldots, g_r$ that generates the module $\overrightarrow{\mathcal{P}}(\Gamma)$ over the ring $\mathcal{P}(\Gamma)$.*

Proof: See [23, Theorem XII 5.2]. $\square$

Next theorem gives a Γ-equivariant version of Schwarz's theorem:

Theorem 2.14 (Poénaru [36]) *Let Γ be a compact Lie group acting on V and let $g_1, \ldots, g_r$ generate the module $\overrightarrow{\mathcal{P}}(\Gamma)$ over the ring $\mathcal{P}(\Gamma)$. Then $g_1, \ldots, g_r$ generate the module $\overrightarrow{\varepsilon}(\Gamma)$ over the ring $\varepsilon(\Gamma)$.*

Proof: See [23, Theorem XII 5.3]. $\square$

2.2 Symmetry-Breaking in Steady-State Bifurcation

Consider the system of ODEs (2.1) where $f : V \times \mathbf{R} \to V$ is smooth, *commutes* with the action of a compact Lie group Γ on V and $\lambda \in \mathbf{R}$ is a bifurcation parameter. A steady-state solution x for some value of λ satisfies

$$f(x, \lambda) = 0,$$

and since f *commutes* with Γ, if x is a solution, then $\gamma \cdot x$ is also a solution, for $\gamma \in \Gamma$.

Recall that f *commutes* with the action of Γ (or is Γ-*equivariant*) if

$$f(\gamma x, \lambda) = \gamma f(x, \lambda)$$

for all $\gamma \in \Gamma$ and $x \in V$.

We define

$$\Gamma x = \{\gamma \cdot x : \gamma \in \Gamma\}$$

the *orbit* of x under Γ, and

$$\Sigma_x = \{\gamma \in \Gamma \ : \ \gamma x = x\} \subseteq \Gamma$$

the *isotropy subgroup* of $x \in V$ in Γ.

Recall that points in V that are in the same Γ-orbit have conjugate isotropy subgroups.

The *fixed-point space* of a subgroup $\Sigma \subseteq \Gamma$ is the subspace of V defined by

$$\mathrm{Fix}(\Sigma) = \{x \in V \ : \ \gamma x = x, \ \forall \ \gamma \in \Sigma\}$$

For any Γ-equivariant mapping f and any subgroup $\Sigma \subseteq \Gamma$ we have

$$f(\mathrm{Fix}(\Sigma) \times \mathbf{R}) \subseteq \mathrm{Fix}(\Sigma) \times \mathbf{R}$$

This follows from the fact that if $\sigma \in \Sigma$ and $x \in \mathrm{Fix}(\Sigma)$, then $f(x, \lambda) = f(\sigma x, \lambda)$; and as f commutes with Σ, then $f(\sigma x, \lambda) = \sigma f(x, \lambda)$ and so $f(x, \lambda)$ is also fixed by Σ. Note that this result holds even when f is nonlinear. One consequence is that when we look for solutions with a specific isotropy subgroup Σ, we can restrict f to $\mathrm{Fix}(\Sigma)$ $\times\mathbf{R}$ and then solve the equation on this space. Another consequence is the existence of trivial zeros of Γ-equivariant mappings f. Suppose that $\mathrm{Fix}(\Gamma)= \{0\}$. Then it follows that $f(0, \lambda) = 0$ for all $\lambda \in \mathbf{R}$.

The larger is the orbit Γx for $x \in V$, the smaller is the isotropy subgroup Σ_x:

Proposition 2.15 *Let Γ be a compact Lie group acting on V. Then*

(a) If $|\Gamma| < \infty$, then $|\Gamma| = |\Sigma_x||\Gamma x|$

(b) $\dim\Gamma = \dim\Sigma_x + \dim\Gamma x$

Proof: See [23, Proposition XIII 1.2]. $\square$

An important class of isotropy subgroups are called maximal.

Definition 2.16 Let Γ be a Lie group acting on V. An isotropy subgroup $\Sigma \subseteq \Gamma$ is *maximal* if there does not exist an isotropy subgroup Δ of Γ satisfying $\Sigma \subset \Delta \subset \Gamma$. $\diamond$

An isotropy subgroup of Γ is *axial* if it has a 1-dimensional fixed-point space. An equilibrium with axial isotropy is called an *axial equilibrium*, and a branch of axial equilibria is an *axial branch*. Axial subgroups are important because (generically) they lead to solutions for bifurcation problems with symmetry Γ. See Theorem 2.19 below.

Definition 2.17 Let Γ be a Lie group acting on a vector space V. A *steady-state bifurcation problem* with symmetry group Γ is a germ $f \in \overrightarrow{\varepsilon}_{x,\lambda}(\Gamma)$ satisfying $f(0,0) = 0$ and $(df)_{0,0} = 0$. $\diamond$

Here $(df)_{0,0}$ denotes the $n \times n$ Jacobian matrix of derivatives of f with respect to the variables x_j evaluated at $(x, \lambda) = (0, 0)$ assuming V is an n-dimensional real vector space.

Proposition 2.18 *Let $f : \boldsymbol{R}^N \times \boldsymbol{R} \to \boldsymbol{R}^N$ be a one-parameter family of Γ-equivariant mappings with $f(0,0) = 0$. Let $V = \ker(df)_{0,0}$. Then generically the action of Γ on V is absolutely irreducible.*

Proof: See [23, Proposition XIII 3.2]. $\square$

We use the assumption of absolute irreducibility as follows: apply the chain rule to the identity $f(\gamma x, \lambda) = \gamma f(x, \lambda)$ to obtain

$$(df)_{0,\lambda}\gamma = \gamma(df)_{0,\lambda}. \tag{2.6}$$

Absolute irreducibility states that the only matrices commuting with all $\gamma \in \Gamma$ are scalar multiples of the identity. Therefore

$$(df)_{0,\lambda} = c(\lambda)Id. \tag{2.7}$$

Since $(df)_{0,0} = 0$ by the definition of a bifurcation problem with symmetry group Γ we have $c(0) = 0$. We now assume the hypothesis

$$c\prime(0) \neq 0 \tag{2.8}$$

which is valid generically.

Theorem 2.19 (Equivariant Branching Lemma) *Let Γ be a Lie group acting absolutely irreducibly on V and let $f \in \overrightarrow{\varepsilon}_{x,\lambda}(\Gamma)$ be a Γ-equivariant bifurcation problem satisfying (2.8) where $(df)_{0,\lambda}$ is given by (2.7). Let Σ be an isotropy subgroup of Γ satisfying*

$$\dim \mathrm{Fix}(\Sigma) = 1$$

Then there exists a unique smooth solution branch to $f = 0$ such that the isotropy subgroup of each solution is Σ.

Proof: See [23, Theorem XIII 3.3]. □

We now discuss the stability properties of equilibria for (2.1) when the mapping f commutes with the action of a Lie group Γ.

If the action of Γ on V is nontrivial and absolutely irreducible then $\mathrm{Fix}_V(\Gamma) = \{0\}$. Moreover, if f commutes with Γ it follows then that $f(0, \lambda) = 0$ and so $x = 0$ is an equilibrium for all $\lambda \in \mathbf{R}$. In the conditions of the Equivariant Branching Lemma, since it is assumed that $c'(0) \neq 0$, there is an exchange of stability of this trivial equilibrium (for λ near 0). We say that the bifurcating solution branch is *subcritical* if the branch occurs for parameter values of λ where the trivial equilibrium is stable and *supercritical* otherwise. We assume that $c'(0) > 0$ and so $x = 0$ is stable for $\lambda < 0$ and so subcritical branches occur for $\lambda < 0$ and supercritical branches for $\lambda > 0$.

Suppose now that x_0 is an equilibrium solution of (2.1) where f commutes with Γ and let $\Sigma = \Sigma_{x_0}$ be the isotropy subgroup of x_0. The solution x_0 is *asymptotically stable* if every trajectory $x(t)$ of (2.1) which begins near x_0 stays near x_0 for all $t > 0$, and also $\lim_{t\to\infty} x(t) = x_0$. The equilibrium is *neutrally stable* if every trajectory $x(t)$ of (2.1) which begins near x_0 stays near x_0 for all $t > 0$.

Note that

$$T_{x_0}\Gamma x_0 \subseteq \ker(df)_{x_0}$$

where $T_{x_0}\Gamma x_0$ denotes the tangent space of Γx_0 at x_0. To see this, let $y(t) = \gamma(t) \cdot x_0$ be a smooth curve in the orbit Γx_0 with $\gamma(t)$ a smooth curve in Γ such that $\gamma(0) = 1$. Then $\frac{d\gamma}{dt}(0) \cdot x_0$ is an eigenvector of $(df)_{x_0}$ with eigenvalue zero and we have a method for calculating null vectors of $(df)_{x_0}$.

The equilibrium x_0 is *orbitally stable* if x_0 is neutrally stable and if whenever $x(t)$ is a trajectory beginning near x_0, then $\lim_{t\to\infty} x(t)$ exists and lies in Γx_0.

There is a well-known condition for the asymptotic stability known as linear stability: the eigenvalues of $(df)_{x_0}$ all have negative real part. Moreover, if some eigenvalue of $(df)_{x_0}$ has positive real part, then x_0 is unstable.

However, if the isotropy subgroup of x_0 has dimension less than that of Γ, then neither linear stability nor asymptotic stability is possible: in this case the orbit Γx_0 has positive dimension forcing $(df)_{x_0}$ to have zero eigenvalues. So, in presence of symmetry the concepts of linear and asymptotic stability are replaced by linear orbital stability and asymptotic orbital stability as follows:

Definition 2.20 Let x_0 be an equilibrium of (2.1), where f commutes with the action of Γ. The steady state x_0 is *linearly orbitally stable* if the eigenvalues of $(df)_{x_0}$ (other then those arising from $T_{x_0}\Gamma x_0$) have negative real part. ◇

Theorem 2.21 *Linear orbital stability implies (asymptotic) orbital stability.*

Proof: See [23, Theorem XIII 4.3]. □

We now discuss the isotropy restrictions on the Jacobian of f. Let $\Sigma \subseteq \Gamma$ be the isotropy subgroup of x. Then for all $\sigma \in \Sigma$ we have

$$(df)_x \sigma = \sigma (df)_x \tag{2.9}$$

that is, $(df)_x$ commutes with the isotropy subgroup Σ of x.

The commutative relation (2.9) restricts the form of $(df)_x$ as follows. Given Σ we can decompose V into isotypic components

$$V = W_1 \oplus \cdots \oplus W_k$$

as in Theorem 2.4. By Theorem 2.9 we have that

$$(df)_x(W_j) \subseteq W_j. \tag{2.10}$$

We can always take W_1 =Fix(Σ) since Fix(Σ) is the sum of all subspaces of V on which Σ acts trivially.

We conclude that the group Γ affects the form of $(df)_x$ in two ways:
(a) Γ/Σ forces null vectors of $(df)_x$, that is, dim ker$(df)_x \geq$dimΓ/Σ.
(b)$(df)_x$ has invariant subspaces as in (2.10).

The restriction of $(df)_x$ to W_j is often subject to extra conditions. For example, suppose that Σ acts absolutely irreducible on W_j. Then $(df)_x|W_j$ is a scalar multiple of the identity. Even when the action of Γ on W_j is not absolutely irreducible, the form of $(df)_x|W_j$ may be constrained by the symmetry.

In [23, Chapter XIII, Section 4(c)] the authors prove that generically, for certain group actions, the solutions obtained from the Equivariant Branching Lemma are all unstable.

2.3 Symmetry-Breaking in Hopf Bifurcation

Consider a system of ODEs

$$\frac{dx}{dt} = f(x, \lambda), \qquad f(0,0) = 0, \tag{2.11}$$

where $x \in V$, $\lambda \in \mathbf{R}$ is the bifurcation parameter, $f : V \times \mathbf{R} \to V$ is a smooth ($\mathcal{C}^\infty$) mapping and $f(0,\lambda) \equiv 0$ for all $\lambda \in \mathbf{R}$. We say that (2.11) undergoes a *Hopf Bifurcation* at $\lambda = 0$ if $(df)_{0,0}$ has a pair of purely imaginary eigenvalues.

When f commutes with a symmetry group Γ, this symmetry imposes restrictions on the imaginary eigenspace.

Definition 2.22 A representation V of Γ is Γ-simple if either
(a) $V \cong W \oplus W$, where W is absolutely irreducible for Γ, or
(b) V is irreducible, but not absolutely irreducible for Γ. ◇

Proposition 2.23 *Suppose (2.11) where $V = \boldsymbol{R}^n$ and $f : \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n$ commutes with the linear action of a compact Lie group Γ on $\boldsymbol{R}^n$. Suppose that $(df)_{0,0}$ has purely imaginary eigenvalues $\pm i\omega$. Let $G_{i\omega}$ be the corresponding real generalized eigenspace of $(df)_{0,0}$. Then generically $G_{i\omega}$ is Γ-simple. Moreover, $G_{i\omega} = E_{i\omega}$.*

Proof: See [23, Proposition XVI 1.4]. □

Under the conditions of the previous proposition and supposing that $\mathbf{R}^n$ is Γ-simple, after an equivariant change of coordinates and a rescaling of time if necessary, we can assume that $(df)_{0,0}$ has the form

$$(df)_{0,0} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} = J$$

where I_m is the $m \times m$ identity matrix and $m = n/2$. This comes from the following lemma:

Lemma 2.24 *Assume that $\boldsymbol{R}^n$ is Γ-simple, the mapping f is Γ-equivariant and $(df)_{0,0}$ has i as an eigenvalue. Then:*

(a) The eigenvalues of $(df)_{0,\lambda}$ consist of a complex conjugate pair $\sigma(\lambda) \pm i\rho(\lambda)$, each with multiplicity m. Moreover, σ and ρ are smooth functions of λ.

(b) There is an invertible linear map $S : \boldsymbol{R}^n \to \boldsymbol{R}^n$, commuting with Γ, such that

$$(df)_{0,0} = SJS^{-1}.$$

Proof: See [23, Lemma XVI 1.5]. □

Identify the circle $\mathbf{S}^1$ with $\mathbf{R}/2\pi\mathbf{Z}$ and suppose that $x(t)$ is a periodic solution of (2.11) in t of period 2π.

A symmetry of $x(t)$ is an element $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ such that

$$\gamma x(t) = x(t - \theta).$$

The set of all symmetries of $x(t)$ forms a subgroup

$$\Sigma_{x(t)} = \{(\gamma, \theta) \in \Gamma \times \mathbf{S}^1 : \gamma x(t) = x(t - \theta)\}.$$

There is a natural action of $\Gamma \times \mathbf{S}^1$ on the space $\mathcal{C}_{2\pi}$ of 2π-periodic functions from $\mathbf{R}$ into V, defined by

$$(\gamma, \theta) \cdot x = \gamma \cdot x(t + \theta).$$

This is, the action of Γ on $\mathcal{C}_{2\pi}$ is induced through its spatial action on v and $\mathbf{S}^1$ acts by phase shift.

This way, the initial definition of symmetry of the periodic solution $x(t)$ may be rewritten as

$$(\gamma, \theta)x(t) = x(t)$$

and with respect to this action, $\Sigma_{x(t)}$ is the isotropy subgroup of $x(t)$.

So if we assume (2.11) where f commutes with Γ and $(df)_{0,0} = L$ has purely imaginary eigenvalues, we can apply a Liapunov-Schmidt reduction preserving symmetries that will induce a different action of $\mathbf{S}^1$ on a finite-dimensional space, which can be identified with the exponential of $L|_{E_i}$ acting on the imaginary eigenspace E_i of L. The reduced function of f will commute with $\Gamma \times \mathbf{S}^1$. See [23, Section XVI 3].

Basically, the Equivariant Hopf Theorem states that for each isotropy subgroup of $\Gamma \times \mathbf{S}^1$ with two-dimensional fixed-point subspace there exists a unique branch of periodic solutions of (2.11) with that symmetry (with a nondegeneracy crossing condition of the eigenvalues):

Theorem 2.25 (Equivariant Hopf Theorem). *Consider the system of ODEs (2.11), where $f : \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n$ is smooth and commutes with a compact Lie group Γ.*

Assume the generic hypothesis that $\boldsymbol{R}^n$ is Γ-simple and $(df)_{0,0}$ has i as eigenvalue. Thus, after a change of coordinates, we can assume that $(df)_{0,0} = J$, where $m = n/2$. By Lemma 2.24 the eigenvalues of $(df)_{0,\lambda}$ are $\sigma(\lambda) \pm i\rho(\lambda)$ each with multiplicity m. Therefore $\sigma(0) = 0$ and $\rho(0) = 1$.

Assume now that

$$\sigma'(0) \neq 0,$$

that is, the eigenvalues of $(df)_{0,\lambda}$ cross the imaginary axis with nonzero speed.

Let $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ be an isotropy subgroup such that

$$\dim \mathrm{Fix}(\Sigma) = 2.$$

Then there exists a unique branch of small-amplitude periodic solutions to (2.11) with period near 2π, having Σ as their group of symmetries.

Proof: See [23, Theorem XVI 4.1]. □

The basic idea in the Equivariant Hopf Theorem is that small amplitude periodic solutions of (2.11) of period near 2π correspond to zeros of a reduced equation $\phi(x, \lambda, \tau) = 0$ where τ is the period-perturbing parameter. To find periodic solutions of (2.11) with symmetries Σ is equivalent to find zeros of the reduced equation with isotropy Σ and they correspond to the zeros of the reduced equation restricted to $\mathrm{Fix}(\Sigma)$.

The main tool for calculating the stabilities of the periodic solutions (including those guaranteed by the Equivariant Hopf Theorem) is to use a Birkhoff normal form of f: by a suitable coordinate change, up to any given order k, the vector field f can be made to commute with Γ and $\mathbf{S}^1$ (in the Hopf case). This result is the equivariant version of the Poincaré-Birkhoff normal form Theorem. Let $\overrightarrow{\mathcal{P}_k}(\Gamma)$ be the space of the Γ-equivariant homogeneous polynomial mappings of degree k on $\mathbf{R}^n$ and define the linear map $\mathrm{ad}_L : \overrightarrow{\mathcal{P}_k}(\Gamma) \to \overrightarrow{\mathcal{P}_k}(\Gamma)$ by

$$\mathrm{ad}_L\left(\mathrm{P}_k\right)(y) = L\mathrm{P}_k(y) - (d\mathrm{P}_k)_y Ly.$$

Theorem 2.26 *Let f be Γ-equivariant and $L = (df)_0$. Choose a value of k. Then there exists a Γ-equivariant change of coordinates of degree k such that in the new coordinates the system (2.11) has the form*

$$\dot{y} = Ly + f_2(y) + \cdots + f_k(y) + h,$$

where $f_j \in \mathfrak{F}_j, h$ is of order $k+1$ and

$$\overrightarrow{\mathcal{P}}_j(\Gamma) = \mathfrak{F}_j \oplus \mathrm{ad}_L(\overrightarrow{\mathcal{P}}_j(\Gamma)).$$

Proof: See [23, Theorem XVI 5.8]. □

In [14] it is proved that there exists a canonical choice for the complement $\mathfrak{F}_j$ in which the elements of $\mathfrak{F}_j$ commute with a one-parameter group S of mappings defined in terms of the linear part L of f. In the Hopf case, where the derivative $L = J$, the action of S may be interpreted as the symmetries induced by phase-shift $\mathbf{S}^1$. That is, it is possible to choose a complement to $\mathrm{ad}_L(\overrightarrow{\mathcal{P}}_j(\Gamma))$ where the elements commute with $\mathbf{S}^1$ (besides Γ):

$$\overrightarrow{\mathcal{P}}_j(\Gamma) = \overrightarrow{\mathcal{P}}_j(\Gamma \times \mathbf{S}^1) \oplus \mathrm{ad}_L(\overrightarrow{\mathcal{P}}_j(\Gamma))$$

(see [23, Theorem XVI 5.9]). Therefore, when we suppose f in (2.11) is in Birkhoff normal form we suppose that it was made this choice in the complements $\mathrm{ad}_L(\overrightarrow{\mathcal{P}}_j(\Gamma))$ and so f commutes with $\Gamma \times \mathbf{S}^1$. This hypothesis will be very important when calculating the stability of the periodic solutions.

Theorem 2.27 *Suppose that the vector field f in (2.11) is in Birkhoff normal form. Then it is possible to perform a Liapunov-Schmidt reduction on (2.11) such that the reduced equation ϕ has the form*

$$\phi(x, \lambda, \tau) = f(x, \lambda) - (1+\tau)Jx,$$

where τ is the period-scaling parameter.

Proof: See [23, Theorem XVI 10.1]. □

Corollary 2.28 *Suppose that the vector field f in (2.11) is in Birkhoff normal form and that $\phi(x, \lambda, \tau)$ is the mapping obtained by using the Liapunov-Schmidt procedure. Let (x_0, λ_0, τ_0) be a solution to $\phi = 0$, and let $x(t)$ be the corresponding solution of (2.11). Then $x(t)$ is orbitally stable if the $n - d_\Sigma$ (where $d_\Sigma = \mathrm{dim}\Gamma + 1 - \mathrm{dim}\Sigma$) eigenvalues of $(d\phi)_{x_0,\lambda_0,\tau_0}$ which are not forced to zero by the group action have negative real parts.*

Proof: See [23, Corollary XVI 10.2]. □

Thus the assumptions of Birkhoff normal form implies that we can apply the standard Hopf Theorem to $\dot{x} = f(x, \lambda)$ restricted to $\mathrm{Fix}(\Sigma) \times \mathbf{R}$. In this case, exchange of stability happens, so that if the trivial steady-state solution is stable subcritically, then a subcritical branch of periodic solutions with isotropy subgroup Σ is unstable. Supercritical branches may be stable or unstable depending on the signs of the real part of the eigenvalues on the complement of $\mathrm{Fix}(\Sigma)$.

Call the system

$$\dot{y} = Ly + g_2(y) + \cdots + g_k(y)$$

the (kth order) *truncated Birkhoff normal form.*

The dynamics of the truncated Birkhoff normal form are related to, but not identical with, the local dynamics of the system (2.11) around the equilibrium $x = 0$.

On the other hand, in general it is not possible to find a single change of coordinates that puts f into normal form for all orders. And if it is, then there is the problem of the first 'tail'.

The results of Theorem 2.27 and Corollary 2.28 hold when f is in Birkhoff normal form. So, when discussing the stability of the solutions found using the Equivariant Hopf Theorem we suppose that the kth order truncation of f commutes also with $\mathbf{S}^1$ and we use these results. Thus we are ignoring terms of higher order that do not commute necessarily with $\mathbf{S}^1$ and that can change the dynamics (and so the stability of these periodic solutions that exist even for the nontruncated system by the Equivariant Hopf Theorem).

However, in some cases, the stability results for the periodic solutions can hold even when f is of the form

$$\tilde{f}(x, \lambda) + o(\|x\|^k),$$

where $\tilde{f}$ commutes with $\Gamma \times \mathbf{S}^1$ but $o(\|x\|^k)$ commutes only with Γ, provided k is large enough. We use $h(x) = o(\|x\|^k)$ to mean that $h(x)/\|x\|^k \to 0$ as $\|x\| \to 0$.

Suppose that $\dim \mathrm{Fix}(\Sigma) = 2$. Following [23, Definition XVI 11.1] Σ has *p-determined stability* if all eigenvalues of $(d\tilde{f})_{(x_0,\lambda_0)} - (1 + \tau_0)J$, other than those forced to zero by Σ, have the form

$$\mu_j = \alpha_j a^{m_j} + o(a^{m_j})$$

on a periodic solution $x(s)$ of

$$\dot{x} = \tilde{f}(x, \lambda) \tag{2.12}$$

such that $\|x(s)\| = a$, where α_j is a $\mathbf{C}$-valued function of the Taylor coefficients of terms of degree lower or equal p in $\tilde{f}$. We expect that the real parts of the α_j to be generically nonzero: these are the nondegeneracy conditions on the Taylor coefficients of $\tilde{f}$ at the origin that are obtained when computing stabilities along the branches. In this case, we say that $\tilde{f}$ is *nondegenerate* for Σ.

Theorem 2.29 *Suppose that the hypotheses of Theorem 2.25 hold, and the isotropy subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ has p-determined stability. Let $k \geq p$ and assume that $f(x, \lambda) = \tilde{f}(x, \lambda) + o(\|x\|^k)$ where $\tilde{f}$ commutes with $\Gamma \times \mathbf{S}^1$ and is nondegenerate for Σ. Then for λ sufficiently near 0, the stabilities of a periodic solution of $\dot{x} = f(x, \lambda)$ with isotropy Σ are given by the same expressions in the coefficients of f as those that define the stability of a solution of the truncated Birkhoff normal form $\dot{x} = \tilde{f}(x, \lambda)$ with isotropy subgroup Σ.*

Proof: See [23, Theorem XVI 11.2]. $\square$

By Theorem 2.26 there always exists a polynomial change putting f in the form $\tilde{f}(x, \lambda) + o(\|x\|^k)$. Thus, if the p-determined stability condition holds, Theorem 2.29 completes the stability analysis for f.

Chapter 3

Secondary Bifurcations in Systems with All-to-All Coupling

A paper with the contents of this chapter has been published [10].

In this chapter we consider a general system of ordinary differential equations commuting with the permutation action of the symmetric group $\mathbf{S}_{3n}$ on $\mathbf{R}^{3n}$. Using singularity theory results, we find sufficient conditions on the coefficients of the fifth order truncation of the general smooth $\mathbf{S}_{3n}$-equivariant vector field for the existence of a secondary branch of equilibria near the origin with $\mathbf{S}_n \times \mathbf{S}_n \times \mathbf{S}_n$ symmetry of such system. Moreover, we prove that under such conditions the solutions are (generically) globally unstable except in the cases where two tertiary bifurcations occur along the secondary branch. In these cases, the instability result holds only for the equilibria near the secondary bifurcation points.

Let the symmetric group $\Gamma = \mathbf{S}_N$ act on $V = \mathbf{R}^N$ by permutation of coordinates

$$\rho(x_1, \ldots, x_N) = \left(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(N)}\right), \quad \rho \in \mathbf{S}_N, \ (x_1, \ldots, x_N) \in \mathbf{R}^N$$

and consider the restriction of this action onto the standard irreducible

$$V_1 = \{(x_1, x_2, \ldots, x_N) \in V : x_1 + x_2 + \cdots + x_N = 0\} \cong \mathbf{R}^{N-1}.$$

Note that the action of $\mathbf{S}_N$ on V_1 is *absolutely irreducible.* Thus the only matrices commuting with the action of Γ on V_1 are the scalar multiples of the identity. Moreover,

$$V = \{(x_1, x_1, \ldots, x_1) : x_1 \in \mathbf{R}\} \oplus V_1$$

where the action of $\mathbf{S}_N$ on $\{(x_1, x_1, \ldots, x_1) : \ x_1 \in \mathbf{R}\}$ is trivial.

Consider a system of ODEs

$$\frac{dx}{dt} = G(x, \lambda), \tag{3.1}$$

where $x \in V_1$, the vector field $G : V_1 \times \mathbf{R} \to V_1$ is smooth, and $\lambda \in \mathbf{R}$ is a bifurcation parameter. Suppose that G commutes with the action of Γ on V_1. As $\mathrm{Fix}(\Gamma) = \{0\}$, it follows that $G(0, \lambda) \equiv 0$. Thus $x = 0$ is an equilibrium of (3.1) for all parameter values of λ. Moreover, as the action of Γ on V_1 is absolutely irreducible and the Jacobian of G at $(0,\lambda)$, $(dG)_{(0,\lambda)}$, commutes with Γ, it follows that $(dG)_{(0,\lambda)}$ is a scalar multiple of the identity. Thus $(dG)_{(0,\lambda)} = c(\lambda)\mathrm{Id}_{V_1}$ where $c:\ \mathbf{R}\to\mathbf{R}$ is smooth. Suppose that $(dG)_{(0,\lambda)}$ is singular, say at $\lambda=0$. Then we have that $c(0)=0$ and $(dG)_{(0,0)}=0$. By the Equivariant Branching Lemma [23, Theorem XIII 3.3], if $c^{'}(0)\neq 0$, then for each axial subgroup of Γ there exists a unique branch of equilibria of (3.1) bifurcating from the trivial equilibrium at $\lambda=0$ with that symmetry. Any such branch is called a *primary branch*.

In Sections 3.1 and 3.2 we obtain, respectively, the isotropy subgroups for the natural representation of the symmetric group and the general fifth order truncation of (3.1) of any smooth $\mathbf{S}_N$-equivariant vector field posed on the $\mathbf{S}_N$-absolutely irreducible space V_1. In Section 3.3 we present a brief description of the singularity theory of $\mathbf{D}_3$-equivariant bifurcation problems.

In Section 3.4 we suppose $N=3a$ and $\Sigma=\mathbf{S}_a\times\mathbf{S}_a\times\mathbf{S}_a$. We look for secondary branches of steady-state solutions for the system (3.1) that are Σ-symmetric obtained by bifurcation from a primary branch of solutions with isotropy group (conjugate to) $\mathbf{S}_a\times\mathbf{S}_{2a}$. After describing sufficient conditions on the coefficients of the vector field for the existence of a secondary branch of solutions of (3.1) with that symmetry, we describe the parameter regions of stability of those solutions (in $\mathrm{Fix}(\Sigma)$). Finally, in Section 3.5 we discuss the full stability of such a secondary branch, we obtain the expressions of the eigenvalues that determine the full stability of those solutions and we prove that these solutions are (generically) globally unstable except in the cases where two tertiary bifurcations occur along the secondary branch. In these cases, the instability result holds only for the equilibria near the secondary bifurcation points. We conclude this chapter with an example where stability between tertiary bifurcation points on the secondary branch occurs (Example 3.10).

3.1 Isotropy Subgroups of the Symmetric Group for the Natural Representation

The isotropy subgroups of $\mathbf{S}_N$ for the action on V_1 are the same isotropy subgroups of $\mathbf{S}_N$ for the action on V, but the the dimension of every fixed-point subspace is reduced by one. In order to compute isotropy subgroups Σ_x of $\mathbf{S}_N$ acting on V, we partition $\{1,\ldots,N\}$ into disjoint blocks $B_1,\ldots,B_k$ with the property that $x_i=x_j$ if and only if i,j belong to the same block. Let $b_l=|B_l|$. Then

$$\Sigma_x=\mathbf{S}_{b_1}\times\cdots\times\mathbf{S}_{b_k}$$

where $\mathbf{S}_{b_l}$ is the symmetric group on the block B_l. Up to conjugacy, we may assume that

$$\begin{array}{l} B_1=\{1,\ldots,b_1\},\\ B_2=\{b_1+1,\ldots,b_1+b_2\},\\ \ldots,\\ B_k=\{b_1+b_2+\cdots+b_{k-1}+1,\ldots,N\}\end{array}$$

where $b_1\leq b_2\leq\cdots\leq b_{k-1}$. Therefore, conjugacy classes of isotropy subgroups of $\mathbf{S}_N$ are in one-to-one correspondence with partitions of N into non-zero natural numbers arranged in ascending order. If Σ corresponds to a partition of N into k blocks, then the fixed-point subspace in V of Σ has dimension k, and so in V_1 has dimension $k-1$. In particular, the axial subgroups of $\mathbf{S}_N$ are the groups $\mathbf{S}_p \times \mathbf{S}_q$ where $p+q=N$.

3.2 General $\mathbf{S}_N$-Equivariant Mappings

The ring of the smooth Γ-invariants on V is generated by

$$\pi_k = x_1^k + x_2^k + \cdots + x_N^k$$

where $k = 1, \ldots, N$ (see Golubitsky and Stewart [21, Chapter 1, Section 5]). Denote by

$$[x_1^k] = [x_1^k, x_2^k, \ldots, x_N^k]^t,$$

for $k = 0, \ldots, N-1$. Then the module of the Γ-equivariant smooth mappings from V to V is generated over the ring of the smooth Γ-invariants by $[x_1^k]$ for $k = 0, \ldots, N-1$. For a detailed discussion see Golubitsky and Stewart [21, Chapter 2, Section 6].

It follows then that if $G: \ V \to V$ is smooth and commutes with Γ then it has the following form:

$$G(x) = \sum_{k=0}^{N-1} p_k(\pi_1, \ldots, \pi_N)[x_1^k] \tag{3.2}$$

where each $p_k: \ \mathbf{R}^N \to \mathbf{R}$, for $k = 0, \ldots, N-1$ is a smooth function.

From (3.2) we obtain the fifth order truncation of the Taylor expansion of G on V. By imposing the relation $\pi_1 = 0$ and then projecting the result onto V_1 we obtain

$$\begin{array}{rcl} G_i(x,\lambda) & = & \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 + \\ & & E(Nx_i^4 - \pi_4) + F(Nx_i^2\pi_2 - \pi_2^2) + Gx_i\pi_3 + \\ & & +\ H(Nx_i^5 - \pi_5) + I(Nx_i^3\pi_2 - \pi_3\pi_2) + \\ & & J(Nx_i^2\pi_3 - \pi_3\pi_2) + Lx_i\pi_4 + Mx_i\pi_2^2 + \\ & & \text{terms of degree } \geq 6 \end{array} \tag{3.3}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, \ldots, M \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i (and the coordinates satisfy $x_1 + \cdots + x_N = 0$). Also we are taking G such that $(dG)_{(0,\lambda)} = \lambda \mathrm{Id}_{V_1}$. Recall that the Γ-equivariance of G implies that $(dG)_{(0,\lambda)}$ commutes with Γ and so it has the form $c(\lambda)\mathrm{Id}_{V_1}$ where $c: \mathbf{R} \to \mathbf{R}$ is smooth. We are taking the approximation $c(\lambda) \sim \lambda$ since we are assuming that the trivial equilibrium of (3.1) is stable for $\lambda < 0$ and unstable for $\lambda > 0$ and the study done in this work is by local analysis, for parameter values of λ near zero. We show in Section 3.4 that this fifth order truncation captures the presence of a secondary branch of equilibria of (3.1) with symmetry $\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ when $N = 3a$ and its stability by bifurcation from primary branches with axial symmetry.

3.3 $\mathbf{D}_3$-Equivariant Bifurcation Problem

We briefly describe the characterization of $\mathbf{D}_3$-equivariant bifurcation problems obtained by Golubitsky *et al.* [23, Sections XIII 5, XIV 4, XV 3].

Consider the standard action of $\mathbf{D}_3$ on $\mathbf{C} \equiv \mathbf{R}^2$ generated by

$$kz = \overline{z}, \qquad \xi z = e^{2\pi i/3} z \tag{3.4}$$

where $\xi = 2\pi/3$, $\mathbf{D}_3 = \langle k, \xi \rangle$ and $z \in \mathbf{C}$. Up to conjugacy, the only isotropy subgroup of $\mathbf{D}_3$ with one-dimensional fixed-point subspace is $\mathbf{Z}_2(k) = \{1, k\}$.

If $g : \mathbf{C} \times \mathbf{R} \to \mathbf{C}$ is smooth and commutes with this action of $\mathbf{D}_3$ on $\mathbf{C}$ then

$$g(z, \lambda) = p(u, v, \lambda) z + q(u, v, \lambda) \overline{z}^2 \tag{3.5}$$

where $u = z\overline{z}$, $v = z^3 + \overline{z}^3$ and $p, q : \mathbf{R}^3 \to \mathbf{R}$ are smooth functions. Suppose $p(0,0,0) = 0$ and so the linearization of (3.5) at $(z, \lambda) = (0,0)$ is zero. Assume the genericity hypothesis of the Equivariant Branching Lemma $p_\lambda(0,0,0) \neq 0$ and the second nondegeneracy hypothesis $q(0,0,0) \neq 0$. We have then that the only (local) solution branches to $g = 0$ obtained by bifurcation from $(z, \lambda) = (0,0)$ are those obtained using the Equivariant Branching Lemma. That is, those that have $\mathbf{Z}_2(k)$-symmetry or conjugate. Since there is a nontrivial $\mathbf{D}_3$-equivariant quadratic $\overline{z}^2$, by [23, Theorem XIII 4.4], generically, the branch of $\mathbf{Z}_2(k)$ solutions is unstable. Thus in order to find asymptotically stable solutions to a $\mathbf{D}_3$-equivariant bifurcation problem by a *local analysis*, we must consider the degeneracy hypothesis $q(0,0,0) = 0$ and apply unfolding theory.

We state a normal form for the degenerate $\mathbf{D}_3$-equivariant bifurcation problem for which $q(0,0,0) = 0$. We begin by specifying the lower order terms in p and q as follows:

$$\begin{aligned} p(u, v, \lambda) &= \tilde{A}u + \tilde{B}v + \tilde{\alpha}\lambda + \cdots \\ q(u, v, \lambda) &= \tilde{C}u + \tilde{D}v + \tilde{\beta}\lambda + \cdots \end{aligned} \tag{3.6}$$

A $\mathbf{D}_3$-equivariant bifurcation problem g satisfying $p(0,0,0) = 0 = q(0,0,0)$ is called *non-degenerate* if

$$\tilde{\alpha} \neq 0, \quad \tilde{A} \neq 0, \quad \tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A} \neq 0, \quad \tilde{A}\tilde{D} - \tilde{B}\tilde{C} \neq 0. \tag{3.7}$$

Theorem 3.1 *Let g be a $\mathbf{D}_3$-equivariant bifurcation problem. Assume that $p(0,0,0) = 0 = q(0,0,0)$ and g is nondegenerate. Then g is $\mathbf{D}_3$-equivalent to the normal form*

$$h(z, \lambda) = (\epsilon u + \delta\lambda) z + (\sigma u + m v) \overline{z}^2 \tag{3.8}$$

where $\epsilon = \operatorname{sgn} \tilde{A}$, $\delta = \operatorname{sgn} \tilde{\alpha}$, $\sigma = \operatorname{sgn} (\tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A}) \operatorname{sgn} \tilde{\alpha}$, and $m = \operatorname{sgn} (\tilde{A})(\tilde{A}\tilde{D} - \tilde{B}\tilde{C})\tilde{\alpha}^2/(\tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A})^2$.

Proof: See [23, Theorem XIV 4.4]. □

We consider now the bifurcation diagram of bifurcation problems of the type $\dot{z} + h(z, \lambda) = 0$ where h is given by (3.8). The Equivariant Branching Lemma guarantees that there is a unique branch of solutions with $\mathbf{Z}_2(\kappa)$-symmetry that bifurcate from the trivial equilibrium at $\lambda = 0$. Setting $\delta = -1$ and $\epsilon = 1$ in (3.8) so that the trivial solution is asymptotically stable for $\lambda < 0$ and the $\mathbf{Z}_2(\kappa)$-symmetric solutions bifurcate supercritically, we obtain Figure 3.1 (a). Note that the branch of $\mathbf{Z}_2(\kappa)$-solutions splits into two orbits of solutions. The sign of $\sigma = \pm 1$ determines which is stable.

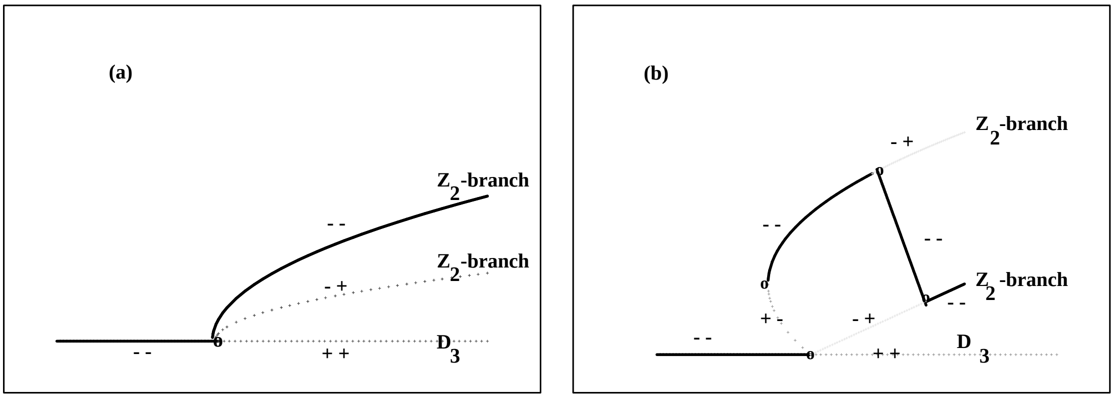

Figure 3.1: (a) Unperturbed $\mathbf{D}_3$-symmetric bifurcation diagram for $\dot{z}+h(z,\lambda)=0$, where h is the normal form $h(z,\lambda)=(u-\lambda)z+(\sigma u+mv)\overline{z}^2$, $\sigma=\pm1$ and $m\neq 0$. [23, Figure XV 4.1 (b)]. (b) Bifurcation diagram for $\dot{z}+H(z,\lambda,\mu,\alpha)=0$, where H is defined by $H(z,\lambda,\mu,\alpha)=(u-\lambda)z+(\sigma u+\mu v+\alpha)\overline{z}^2$, $\sigma=1$, $\alpha<0$ (or $\sigma=-1$, $\alpha>0$) and $\mu>0$ [23, Figure XV 4.2 (c)].

The next theorem states a universal $\mathbf{D}_3$-unfolding for the $\mathbf{D}_3$-normal form of Theorem 3.1.

Theorem 3.2 *The* $\mathbf{D}_3$*-normal form* $h(z,\lambda)=(\epsilon u+\delta\lambda)z+(\sigma u+mv)\overline{z}^2$ *where* $\epsilon,\delta,\sigma=\pm1$ *and* $m\neq 0$*, obtained in Theorem 3.1, has* $\mathbf{D}_3$*-codimension* 2 *and modality* 1*. A universal unfolding of* h *is*

$$H(z,\lambda,\mu,\alpha)=(\epsilon u+\delta\lambda)z+(\sigma u+\mu v+\alpha)\overline{z}^2 \tag{3.9}$$

where (μ,α) *varies near* $(m,0)$*.*

Proof: See [23, Theorem XV 3.3 (b)]. □

We show in Figure 3.1 (b) the bifurcation diagram for $\dot{z}+H(z,\lambda,\mu,\alpha)=0$ where $\delta=-1$, $\epsilon=1$, $\sigma\alpha<0$ and $\mu>0$ in (3.9). Observe the change of stability of the $\mathbf{Z}_2(\kappa)$-symmetric solutions along the branch and the appearance (when $\sigma\alpha<0$) of a secondary branch of solutions with trivial symmetry which are asymptotically stable if $\mu>0$.

Figures 3.1 (a) and (b) appear in [23, Figures XV 4.1 (b), XV 4.2 (c)] with opposite signs for the eigenvalues since the authors consider the eigenvalues of $(dh)_{(z,\lambda)}$ and $(dH)_{(z,\lambda)}$, while we show in Figure 3.1 the signs of the eigenvalues of $-(dh)_{(z,\lambda)}$ and $-(dH)_{(z,\lambda)}$.

3.4 Existence of Secondary Branches

Consider (3.1) where G is defined by (3.3) and suppose $N=3a$ where a is a positive integer. Let $\Sigma=\mathbf{S}_a\times\mathbf{S}_a\times\mathbf{S}_a$ and observe that

$$\text{Fix}(\Sigma) = \{(\underbrace{-x-y,\ldots}_{a};\underbrace{y,\ldots}_{a};\underbrace{x,\ldots}_{a}) : x, y \in \mathbf{R}\}$$

which is two-dimensional. We look for secondary branches of equilibria of (3.1) with symmetry Σ by bifurcation from primary branches with axial isotropy $\mathbf{S}_p \times \mathbf{S}_q$ where $p+q=N$ and $\Sigma \subset \mathbf{S}_p \times \mathbf{S}_q$. We do that by local analysis near the origin using the singularity results stated in Section 3.3. Any such secondary branch must live in the fixed-point subspace $\text{Fix}(\Sigma)$. Moreover, the axial subgroups of $\mathbf{S}_N$ where $N=3a$ containing Σ are

$$\begin{aligned}
\Sigma_1 &= \mathbf{S}_{\{1,\ldots,a\}} \times \mathbf{S}_{\{a+1\ldots,N\}}, \\
\Sigma_2 &= \mathbf{S}_{\{1,\ldots,a,2a+1,\ldots,N\}} \times \mathbf{S}_{\{a+1,\ldots,2a\}}, \\
\Sigma_3 &= \mathbf{S}_{\{1,\ldots,2a\}} \times \mathbf{S}_{\{2a+1,\ldots,N\}}
\end{aligned}$$

and the corresponding one-dimensional fixed-point subspaces are

$$\begin{aligned}
\text{Fix}(\Sigma_1) &= \{(\underbrace{-2x,\ldots}_{a};\underbrace{x,\ldots;x,\ldots}_{2a}) : x \in \mathbf{R}\}, \\
\text{Fix}(\Sigma_2) &= \{(\underbrace{x,\ldots}_{a};\underbrace{-2x,\ldots}_{a};\underbrace{x,\ldots}_{a}) : x \in \mathbf{R}\}, \\
\text{Fix}(\Sigma_3) &= \{(\underbrace{-\frac{1}{2}x,\ldots,-\frac{1}{2}x}_{2a};\underbrace{x,\ldots,x}_{a}) : x \in \mathbf{R}\}.
\end{aligned}$$

Equations (3.1) where G is defined by (3.3) restricted to $\text{Fix}(\Sigma)$ are

$$\begin{aligned}
\frac{dx}{dt} &= \lambda x + B(Nx^2 - \pi_2) + C(Nx^3 - \pi_3) + Dx\pi_2 \\
&\quad + E(Nx^4 - \pi_4) + F(Nx^2\pi_2 - \pi_2^2) + \ Gx\pi_3 \\
&\quad + H(Nx^5 - \pi_5) + I(Nx^3\pi_2 - \pi_3\pi_2) + \\
&\quad J(Nx^2\pi_3 - \pi_3\pi_2) + Lx\pi_4 + Mx\pi_2^2 \\
&\quad + \text{ terms of degree } \geq 6, \\
\frac{dy}{dt} &= \lambda y + B(Ny^2 - \pi_2) + C(Ny^3 - \pi_3) + Dy\pi_2 \\
&\quad + E(Ny^4 - \pi_4) + F(Ny^2\pi_2 - \pi_2^2) + \ Gy\pi_3 \\
&\quad + H(Ny^5 - \pi_5) + I(Ny^3\pi_2 - \pi_3\pi_2) + \\
&\quad J(Ny^2\pi_3 - \pi_3\pi_2) + Ly\pi_4 + My\pi_2^2 \\
&\quad + \text{ terms of degree } \geq 6,
\end{aligned} \tag{3.10}$$

where $\pi_i = N[(-x-y)^i + y^i + x^i)]/3$ for $i = 2,3,4,5$.

Since $\Sigma_1, \Sigma_2, \Sigma_3$ are axial subgroups of $\mathbf{S}_N$ containing Σ, by the Equivariant Branching Lemma, generically there exist branches of equilibria of (3.10) (and so of (3.1)) with isotropy subgroups $\Sigma_1, \Sigma_2, \Sigma_3$. The solutions of equations (3.10) with Σ_1-symmetry satisfy

$y = x$; those with Σ_2-symmetry satisfy $y = -2x$, and finally those with Σ_3-symmetry satisfy $y = -x/2$.

Observe that equations (3.10) correspond to the equations (3.1) where G is defined by (3.3) restricted to $\mathrm{Fix}(\Sigma)$ in coordinates x, y corresponding to the basis $B = (B_1, B_2)$ of the fixed-point subspace $\mathrm{Fix}(\Sigma)$, where $B_1 = (-1, \ldots, -1; 0, \ldots, 0; 1, \ldots, 1)$ and $B_2 = (-1, \ldots, -1; 1, \ldots, 1; 0, \ldots, 0)$. Moreover, those equations are equivariant under the quotient group $N(\Sigma)/\Sigma$ where $N(\Sigma)$ is the normalizer of Σ in $\mathbf{S}_N$. Thus $N(\Sigma)/\Sigma \cong \mathbf{D}_3$ where $\mathbf{D}_3$ is the dihedral group of order 6.

We consider now the basis

$$b = \left(-\frac{2\sqrt{3}}{3} B_1 + \frac{\sqrt{3}}{3} B_2, B_2 \right)$$

of $\mathrm{Fix}(\Sigma)$ and denote the corresponding coordinates by X, Y. Thus $X = \left(-\sqrt{3}x\right)/2$, $Y = x/2 + y$. Identifying $z = X + iY$, we have then that the action of $N(\Sigma)/\Sigma \cong \mathbf{D}_3$ on z is given by (3.4). Moreover, equations (3.10) yield the following equation in z:

$$\frac{dz}{dt} + g(z, \lambda) = 0, \tag{3.11}$$

where

$$\begin{aligned} g(z,\lambda) &= p(u,v,\lambda)z + q(u,v,\lambda)\overline{z}^2, \\ p(u,v,\lambda) &= -\lambda - \frac{N}{3}(3C+2D)u + \frac{\sqrt{3}}{9}N\left(E+G\right)v \\ &\quad -\frac{N}{9}(9H + 6NI + 6L + 4NM)u^2 \\ &\quad + \text{terms of degree } \geq 5, \\ q(u,v,\lambda) &= \frac{\sqrt{3}}{3}NB + \frac{\sqrt{3}}{9}N(3E+2NF)u - \frac{N}{9}(H+NJ)v \\ &\quad + \text{terms of degree } \geq 4, \end{aligned} \tag{3.12}$$

$u = z\overline{z}$ and $v = z^3 + \overline{z}^3$.

Theorem 3.3 *Suppose that $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ and consider (3.1) where G is defined by (3.3). Assume the following conditions on the coefficients of the terms of degree lower or equal to five of G:*

$$3C + 2D < 0, \ (3C+2D)(H+NJ) - (E+G)(3E+2NF) \neq 0 \tag{3.13}$$

and

$$B(3E + 2NF) < 0. \tag{3.14}$$

Then for sufficiently small values of $B \neq 0$, equations (3.10) (and so (3.1)) have a secondary branch of equilibria with symmetry Σ bifurcating from the primary branches with symmetry Σ_i. This is described by:

$$
\begin{aligned}
\lambda \quad & + \frac{N}{3}(3C+2D)(x^2+y^2+xy) - N(E+G)(xy^2+x^2y) \\
& + \frac{N}{9}(9H+6NI+6L+4NM)(x^2+y^2+xy)^2, \\
& + \text{terms of degree } \geq 5 = 0, \\
B \quad & + \frac{1}{3}(3E+2NF)(x^2+y^2+xy) - (H+NJ)(x^2y+xy^2), \\
& + \text{terms of degree } \geq 4 = 0.
\end{aligned}
\tag{3.15}
$$

Proof: The equivariance of equations (3.10) under the group $N(\Sigma)/\Sigma \cong \mathbf{D}_3$ enables us the choice of coordinates X, Y in $\mathrm{Fix}(\Sigma)$ such that the action of $N(\Sigma)/\Sigma$ on $z \equiv X+iY$ is given by (3.4) and equations (3.10) correspond to one equation in z given by (3.11) where g is defined by (3.12). Thus we obtain $\dot{z}+g(z,\lambda)=0$ where $g(z,\lambda)=p(u,v,\lambda)z+q(u,v,\lambda)\overline{z}^2$ and

$$
\begin{aligned}
p(u,v,\lambda) &= -\lambda+\beta_1 u+\beta_2 v+\beta_3 u^2+\text{terms of degree } \geq 5, \\
q(u,v,\lambda) &= \beta_4+\beta_5 u+\beta_6 v+\text{terms of degree } \geq 4,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
& \beta_1 = -\frac{N}{3}(3C+2D),\ \beta_2 = \frac{\sqrt{3}}{9}N(E+G), \\
& \beta_3 = -\tfrac{N}{9}(9H+6NI+6L+4NM),\ \beta_4 = \frac{\sqrt{3}}{3}NB, \\
& \beta_5 = \tfrac{\sqrt{3}}{9}N(3E+2NF),\ \beta_6 = -\tfrac{N}{9}(H+NJ).
\end{aligned}
\tag{3.17}
$$

Note that $p(0,0,0)=0$ and $p_\lambda(0,0,0)\neq 0$. Thus by the Equivariant Branching Lemma there are three branches of steady-state solutions with symmetry $\mathbf{Z}_2(k)$ or conjugate of equation (3.11) obtained by bifurcation from the trivial equilibrium $z=0$ at $\lambda=0$. These correspond to the primary branches with Σ_i-symmetry, for $i=1,2,3$, of equations (3.10) (and so of (3.1)). Observe that solutions of (3.1) with Σ-symmetry correspond to solutions of the $\mathbf{D}_3$-symmetric equation (3.11) with trivial symmetry. Also, note that

$$
q(0,0,0) = \beta_4 = \frac{\sqrt{3}}{3}NB
$$

and so $q(0,0,0)=0$ if and only if $B=0$.

We prove the existence of a secondary branch of solutions with trivial symmetry bifurcating from the primary branches with $\mathbf{Z}_2(k)$-symmetry of (3.11) by showing that g as defined by (3.11) and (3.12) is one of the perturbations contained in the universal unfolding H in Theorem 3.2, where a secondary branch of trivial solutions exist bifurcating from the primary branches with $\mathbf{Z}_2(k)$-symmetry. We do that by considering g with $B = 0$ and finding conditions on the corresponding coefficients such that it is $\mathbf{D}_3$-equivalent to the normal form h of Theorem 3.1.

Comparing (3.6) with (3.16) where β_4 is set to zero (thus $B = 0$), we obtain

$$\tilde{\alpha} = -1, \qquad \tilde{A} = \beta_1, \qquad \tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A} = -\beta_5, \qquad \tilde{A}\tilde{D} - \tilde{B}\tilde{C} = \beta_1\beta_6 - \beta_2\beta_5.$$

Thus g with $B = 0$ is nondegenerate if

$$\beta_1 \neq 0, \quad \beta_5 \neq 0, \quad \beta_1\beta_6 - \beta_2\beta_5 \neq 0$$

and in that case, by Theorem 3.1, it is $\mathbf{D}_3$-equivalent to

$$h(z, \lambda) = (u - \lambda)z + (\sigma u + mv)\overline{z}^2, \tag{3.18}$$

where

$$\sigma = \operatorname{sgn} \beta_5, \ m = \frac{\beta_1\beta_6 - \beta_2\beta_5}{\beta_5^2}.$$

Note that the condition $3C + 2D < 0$ implies that $\epsilon = 1 = \operatorname{sgn} \beta_1$ in the equation (3.8).

By Theorem 3.2, the function g for $\beta_4 \sim 0$ (thus $B \sim 0$), corresponds to a perturbation of (3.18) of the type

$$H(z, \lambda, \mu, \alpha) = (u - \lambda)z + (\sigma u + \mu v + \alpha)\overline{z}^2 \tag{3.19}$$

where (μ, α) varies near $(m, 0)$. Moreover, if condition (3.14) is satisfied and so $\beta_4\beta_5 < 0$, then g corresponds to a perturbation of the type as above where $\alpha\sigma < 0$ and so there is a secondary branch of solutions of trivial symmetry for $dz/dt + H(z, \lambda, \mu, \alpha) = 0$ varying λ and bifurcating from the $\mathbf{Z}_2(k)$-branch of solutions. Observe that solutions of $H(z, \lambda, \mu, \alpha) = 0$ with trivial symmetry satisfy $\mathrm{Re}(z^3) \neq 0$ and so solving $H(z, \lambda, \mu, \alpha) = 0$ is equivalent to solving $u - \lambda = 0$, $\sigma u + \mu v + \alpha = 0$. Now for small enough values of $\alpha \neq 0$ the solutions of $\sigma u + \mu v + \alpha = 0$ (near the origin) form a circlelike curve in the XY-plane of radius approximately $\sqrt{|\alpha/\sigma|}$. It follows that in the (X, Y, λ)-space this curve intersects the $Y = 0$ plane at two points (X^-, λ^-) and (X^+, λ^+) where $X^- < 0 < X^+$ that correspond to the intersection points of the branch with trivial isotropy (for the $\mathbf{D}_3$-problem) and solutions with isotropy $\mathbf{Z}_2(k)$.

The branch of steady-state solutions with trivial symmetry for the $\mathbf{D}_3$-symmetric bifurcation problem $\dot{z} + g(z, \lambda) = 0$ is then given by the equations

$$\begin{aligned} p(u, v, \lambda) &= -\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2 + \text{terms of degree } \geq 5 = 0, && (3.20)\\ q(u, v, \lambda) &= \beta_4 + \beta_5 u + \beta_6 v + \text{terms of degree } \geq 4 = 0. && (3.21) \end{aligned}$$

Now recalling that $z = X + iY$ where $X = \left(-\sqrt{3}x\right)/2$, $Y = x/2 + y$, equations (3.20) and (3.21) in the x, y coordinates are given by (3.15).

$\square$

Corollary 3.4 *Suppose the conditions of Theorem 3.3 and assume that*

$$(3C+2D)(H+JN)-(E+G)(3E+2FN)>0. \tag{3.22}$$

Then the secondary branch of solutions with Σ-symmetry of (3.1) where G is defined by (3.3) and guaranteed by Theorem 3.3 is stable in $\mathrm{Fix}(\Sigma)$.

Proof: We recall equations (3.11), (3.12) and the notation of (3.16), (3.17) in the proof of Theorem 3.3 corresponding to the equations (3.1) restricted to $\mathrm{Fix}(\Sigma)$. Equations (3.20) and (3.21) describe the secondary branch in the $z = X+iY$ coordinate. The stability of these solutions is determined by

$$\begin{aligned}
\mathrm{tr}\ \left((dg)_{(z,\lambda)}\right) &= 2\left[up_u + \tfrac{v}{2}(3p_v+q_u)+3u^2q_v\right] \\
&= 2\left[\beta_1 u + (3\beta_2+\beta_5)\tfrac{v}{2} + (2\beta_3+3\beta_6)u^2\right] \\
&\quad + \text{terms of degree} \geq 5,
\end{aligned}$$

$$\begin{aligned}
\det\ \left((dg)_{(z,\lambda)}\right) &= 3(p_vq_u - p_uq_v)(z^3-\overline{z}^3)^2 \\
&= 12(\beta_1\beta_6-\beta_2\beta_5+2\beta_3\beta_6 u)\left(\mathrm{Im}\ (z^3)\right)^2 \\
&\quad + \text{terms of degree} \geq 10
\end{aligned}$$

and so the solutions (near the origin) are stable if $\beta_1 > 0$ and $\beta_1\beta_6 - \beta_2\beta_5 > 0$, that is, if conditions (3.13) and (3.22) are satisfied.

The same conclusion can be derived from the fact that $\mathbf{D}_3$-equivalence preserves the asymptotic stability of the solutions with trivial symmetry [23, Section XV 4]. Note that (3.12) corresponds to a perturbation of (3.18) of the type (3.19) where $\alpha\sigma < 0$ (by (3.14)). Thus the secondary branch is stable if $\mu > 0$. As μ varies near m and $\mathrm{sgn}(m) = \mathrm{sgn}(\beta_1\beta_6-\beta_2\beta_5)$, if condition (3.22) is satisfied then $\beta_1\beta_6-\beta_2\beta_5 > 0$ and so $m > 0$. Thus the local bifurcation diagram of equation (3.11) corresponds to the bifurcation diagram of $dz/dt + H(z,\lambda,\mu,\alpha) = 0$, where H is defined by (3.19), that appear in Figure 3.1 (b). Therefore the secondary branch of steady-state solutions with trivial symmetry bifurcating from the branch of steady-state solutions with $\mathbf{Z}_2(k)$-symmetry is stable. □

Observe that Theorem 3.3 guarantees the existence of the secondary branch if $q(0,0,0)$ is sufficiently small. We finish this section by considering (3.1) truncated to the fifth order. We specify in the next corollary a sufficient condition on the coefficients of the truncated vector field that guarantees $q(0,0,0)$ to be sufficiently small.

Corollary 3.5 *Suppose that $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ and consider*

$$\frac{dx}{dt} = G(x, \lambda) \tag{3.23}$$

where G is defined by

$$\begin{array}{rcl} G_i(x,\lambda) & = & \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 \\ & & +\ E(Nx_i^4 - \pi_4) + F(Nx_i^2\pi_2 - \pi_2^2) + Gx_i\pi_3 \\ & & +\ H(Nx_i^5 - \pi_5) + I(Nx_i^3\pi_2 - \pi_3\pi_2) \\ & & +J(Nx_i^2\pi_3 - \pi_3\pi_2) + Lx_i\pi_4 + Mx_i\pi_2^2 \end{array} \tag{3.24}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, \ldots, M \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i (and the coordinates satisfy $x_1 + \cdots + x_N = 0$). Also $\pi_j = x_1^j + \cdots + x_N^j$ for $j = 2, \ldots, 5$. Assume the following conditions on the coefficients of G:

$$\begin{array}{l} 3C + 2D < 0, \\ (3C + 2D)(H + NJ) - (E + G)(3E + 2NF) \neq 0, \\ B(3E + 2NF) < 0, \end{array} \tag{3.25}$$

and

$$H + NJ \neq 0. \tag{3.26}$$

Then for small values of $B \neq 0$ such that

$$\frac{3B}{3E + 2NF} + \frac{(3E + 2NF)^2}{9(H + NJ)^2} > 0 \tag{3.27}$$

equation (3.23) has a secondary branch of equilibria with symmetry Σ bifurcating from the primary branches with symmetry Σ_i. This is described by:

$$\begin{array}{rl} \lambda & +\ \frac{N}{3}(3C + 2D)(x^2 + y^2 + xy) - N(E + G)(xy^2 + x^2y) \\ & +\ \frac{N}{9}(9H + 6NI + 6L + 4NM)(x^2 + y^2 + xy)^2 = 0, \\ & \\ B & +\ \frac{1}{3}(3E + 2NF)(x^2 + y^2 + xy) - (H + NJ)(x^2y + xy^2) = 0. \end{array} \tag{3.28}$$

Proof: As before, we take coordinates X, Y in $\mathrm{Fix}(\Sigma)$ so that equations (3.23) restricted to $\mathrm{Fix}(\Sigma)$ yield one equation in $z \equiv X + iY$. This is given by $dz/dt + g(z, \lambda) = 0$, where $g(z, \lambda) = p(u, v, \lambda)z + q(u, v, \lambda)\overline{z}^2$, $p(u, v, \lambda) = -\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2$, $q(u, v, \lambda) = \beta_4 + \beta_5 u + \beta_6 v$ and $\beta_1, \ldots, \beta_6$ are defined by (3.17). By Theorem 3.3, if the conditions (3.25) are satisfied, provided $B \neq 0$ is sufficiently small, (3.23) has a secondary branch of solutions that correspond to the solutions of the $\mathbf{D}_3$-equivariant problem $dz/dt + g(z, \lambda) = 0$ with trivial symmetry. Moreover, the branch is described by the following two equations in z:

$$-\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2 = 0, \tag{3.29}$$

$$\beta_4 + \beta_5 u + \beta_6 v = 0 \tag{3.30}$$

and recall that solutions with $\mathbf{Z}_2(\kappa)$-symmetry satisfy $Y = 0$.

Set

$$r(X) = \frac{\beta_4}{\beta_5} + X^2 + 2\frac{\beta_6}{\beta_5}X^3$$

and recall that $\beta_4\beta_5 < 0$ by (3.25). We describe now generic conditions on the β_i's such that $r(X)$ has three real zeros. We have that

$$r^{'}(X) = 2X\left(1 + \frac{3\beta_6}{\beta_5}X\right)$$

Assume (3.26) and so $\beta_6 \neq 0$. As $r(0) = \beta_4/\beta_5 < 0$ by (3.25), if $r(-\beta_5/(3\beta_6)) > 0$ we have that r has three real solutions, X^-, X^+, X^*, where $X^- < 0 < X^+$ and $X^* < -\beta_5/(3\beta_6) < X^-$ if $\beta_6/\beta_5 > 0$, or $X^* > -\beta_5/(3\beta_6) > X^+$ if $\beta_6/\beta_5 < 0$. Thus if $r(-\beta_5/(3\beta_6)) > 0$ then in the (X, Y, λ)-space the curve given by (3.30) intersects the $Y = 0$ plane at two points (X^-, λ^-) and (X^+, λ^+) where $X^- < 0 < X^+$ that correspond to the intersection points of the branch with trivial isotropy. Condition $r(-\beta_5/(3\beta_6)) > 0$ is equivalent to (3.27). $\square$

3.5 Secondary Branches: Full Stability

In this section we study the stability of the solutions of the secondary branch obtained in Theorem 3.3 in the transversal directions to $\mathrm{Fix}(\Sigma)$. As before we assume that $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$.

Given an equilibrium X_0 of (3.1) in the Σ-branch obtained in Theorem 3.3, in order to analyze the stability of this solution, we need to compute the eigenvalues of the Jacobian $(dG)_{X_0}$. We use now the decomposition of V_1 into isotypic components for the action of Σ to block-diagonalize the Jacobian on V_1. We have

$$V_1 = \mathrm{Fix}(\Sigma) \oplus (U_1 \oplus U_2 \oplus U_3)$$

where

$$\begin{aligned}
U_1 &= \{(x_1, \ldots, x_a; 0, \ldots, 0; 0, \ldots, 0) \in V_1 : x_1 + \cdots + x_a = 0\},\\
U_2 &= \{(0, \ldots, 0; x_{a+1}, \ldots, x_{2a}; 0, \ldots, 0) \in V_1 : x_{a+1} + \cdots + x_{2a} = 0\},\\
U_3 &= \{(0, \ldots, 0; 0, \ldots, 0; x_{2a+1}, \ldots, x_{3a}) \in V_1 : x_{2a+1} + \cdots + x_{3a} = 0\}.
\end{aligned}$$

The action of Σ is absolutely irreducible on each U_i, for $i = 1, 2, 3$ and trivial on $\mathrm{Fix}(\Sigma)$. Moreover, $\dim U_i = a - 1$. Since $(dG)_{X_0}$ commutes with Σ,

$$(dG)_{X_0} = \begin{pmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & C_9 \end{pmatrix} \tag{3.31}$$

where C_1, C_5, C_9 commute with S_a.

Suppose M is a square matrix of order a with rows $l_1, \ldots, l_a$ and commuting with S_a. It follows then that $M = (l_1, (12)\cdot l_1, \cdots, (1a)\cdot l_1)^t$, where if $l_1 = (m_1, \ldots, m_a)$ then

$(1i) \cdot l_1 = (m_i, m_2, \ldots, m_{i-1}, m_1, m_{i+1}, \ldots, m_a)$. Moreover, l_1 is invariant under S_{a-1} in the last $a-1$ entries and so it has the following form: $(m_1, m_2, \ldots, m_2)$. Applying this to C_1, C_5, C_9 we get

$$C_i = \begin{pmatrix} a_i & & & \\ & \ddots & b_i & \\ & b_i & \ddots & \\ & & & a_i \end{pmatrix} \tag{3.32}$$

for $i = 1, 5, 9$, where

$$a_1 = (\partial G_1/\partial x_1)_{X_0}, \quad a_5 = (\partial G_{a+1}/\partial x_{a+1})_{X_0}, \quad a_9 = (\partial G_{2a+1}/\partial x_{2a+1})_{X_0},$$

$$b_1 = (\partial G_1/\partial x_2)_{X_0}, \quad b_5 = (\partial G_{a+1}/\partial x_{a+2})_{X_0}, \quad b_9 = (\partial G_{2a+1}/\partial x_{2a+2})_{X_0}.$$

The other symmetry restrictions on the C_i, for $i \neq 1, 5, 9$, imply that the rest of the matrices each have one identical entry. From this we obtain a basis for each U_i composed by eigenvectors of $(dG)_{X_0}$: $U_1 = \{\nu_1, \ldots, \nu_{a-1}\}$, $U_2 = \{\psi_1, \ldots, \psi_{a-1}\}$, $U_3 = \{\phi_1, \ldots, \phi_{a-1}\}$ where

$$\begin{aligned}
\nu_1 &= (1, -1, 0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0), \\
\nu_2 &= (0, 1, -1, 0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0), \cdots, \\
\nu_{a-1} &= (0, \ldots, 0, 1, -1; 0, \ldots, 0; 0, \ldots, 0), \\
\psi_1 &= (0, \ldots, 0; 1, -1, 0, \ldots, 0; 0, \ldots, 0), \\
\psi_2 &= (0, \ldots, 0; 0, 1, -1, \ldots, 0; 0, \ldots, 0), \cdots, \\
\psi_{a-1} &= (0, \ldots, 0; 0, \ldots, 0, 1, -1; 0, \ldots, 0), \\
\phi_1 &= (0, \ldots, 0; 0, \ldots, 0; 1, -1, 0, \ldots, 0), \\
\phi_2 &= (0, \ldots, 0; 0, \ldots, 0; 0, 1, -1, 0, \ldots, 0), \cdots, \\
\phi_{a-1} &= (0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0, 1, -1).
\end{aligned}$$

Moreover the eigenvalue associated with ν_i is

$$\lambda_1 = a_1 - b_1 = (\partial G_1/\partial x_1)_{X_0} - (\partial G_1/\partial x_2)_{X_0},$$

the one associated with ψ_i is

$$\lambda_2 = a_5 - b_5 = (\partial G_{a+1}/\partial x_{a+1})_{X_0} - (\partial G_{a+1}/\partial x_{a+2})_{X_0}$$

and the one associated with ϕ_i is

$$\lambda_3 = a_9 - b_9 = (\partial G_{2a+1}/\partial x_{2a+1})_{X_0} - (\partial G_{2a+1}/\partial x_{2a+2})_{X_0}.$$

The branching conditions for Σ of Theorem 3.3 and the symmetry of G yield:

Lemma 3.6 *Let X_0 be an equilibrium of (3.10) in the Σ-branch obtained in Theorem 3.3. Then the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $(dG)_{X_0}$ are*

$$\begin{aligned}
\lambda_1 &= N(x + 2y)(2x + y)S_2(x, -x - y), \\
\lambda_2 &= N(x + 2y)(y - x)S_2(x, y), \\
\lambda_3 &= N(x - y)(2x + y)S_2(y, x),
\end{aligned} \tag{3.33}$$

where

$$S_2(x,y) = \begin{array}{l} C + Ey + \left(\frac{2}{3}NI + H\right)(x^2+y^2+xy) + Hy^2 \\ +\text{terms of degree} \geq 3, \end{array} \tag{3.34}$$

and x and y are as in the second equation of (3.15):

$$\begin{array}{l} B + \dfrac{1}{3}(3E+2NF)(x^2+y^2+xy) - (H+NJ)(x^2y+xy^2) \\ +\text{terms of degree } \geq 4 = 0. \end{array} \tag{3.35}$$

Remark 3.7 Suppose X_0 corresponds to a solution of the primary branch with Σ_1-symmetry. Note that the isotypic decomposition of V_1 for the action of Σ_1 is

$$V_1 = W_0 \oplus W_1 \oplus W_2$$

where

$$\begin{array}{l} W_0 = \mathrm{Fix}(\Sigma_1) = \{(-2x,\ldots;x,\ldots;x,\ldots) : x \in \mathbf{R}\}, \\ W_1 = \{(x_1,\ldots,x_a;0,\ldots,0) \in V_1 : x_1 + \cdots + x_a = 0\}, \\ W_2 = \{(0,\ldots,0;x_{a+1},\ldots,x_{3a}) \in V_1 : x_{a+1} + \cdots + x_{3a} = 0\}. \end{array}$$

The action of Σ_1 is absolutely irreducible on each W_1, W_2 and trivial on W_0. It follows then that the Jacobian $(dG)_{X_0}$ has (at most) three distinct real eigenvalues, μ_j, one for each W_j, with multiplicity dim W_j.

The stability in $\mathrm{Fix}(\Sigma)$ for the solution with Σ_1-symmetry is determined by the eigenvalue μ_0 associated with $W_0 = \mathrm{Fix}(\Sigma_1)$ and μ_2 since $\mathrm{Fix}(\Sigma) \bigcap W_2 \neq \{0\}$.

Suppose now that X_0 corresponds to a solution of the Σ-branch and of the Σ_1-branch. Then the eigenvalue μ_2 is zero and it is associated with the eigenspace W_2. Moreover, $U_2 \subseteq W_2$ and $U_3 \subseteq W_2$. Therefore X_0 is a zero of λ_2 and λ_3, and we have the factor $y - x$ in the expressions for λ_2 and λ_3 that appear in (3.33). Similarly, we justify the factors $x + 2y$ and $2x + y$ in those expressions. $\diamond$

Lemma 3.6 and (the proof of) Corollary 3.4 lead to the following result:

Theorem 3.8 *Assume the conditions of Theorem 3.3. Let X_0 be an equilibrium of (3.10) (and so of (3.1)) in the Σ-branch obtained in Theorem 3.3. Then the eigenvalues λ_i for $i = 1,\ldots,5$ of $(dG)_{X_0}$ determining the stability of X_0 are λ_i for $i = 1,\ldots,5$ where*

$$
\begin{aligned}
\lambda_1 &= N(x+2y)(2x+y)S_2(x,-x-y),\\
\lambda_2 &= N(x+2y)(y-x)S_2(x,y),\\
\lambda_3 &= N(x-y)(2x+y)S_2(y,x),\\
\lambda_4\lambda_5 &= \frac{N^2}{9}\left[(3C+2D)(H+NJ)-(E+G)(3E+2NF)\right]\\
&\quad\times(x-y)^2(x+2y)^2(y+2x)^2\\
&\quad+\frac{2}{27}N^2(9H+6NI+6L+4NM)(H+NJ)(x^2+y^2+xy)(x-y)^2\\
&\quad\times(x+2y)^2(y+2x)^2\\
&\quad+\text{ terms of degree }\geq 10,\\
\lambda_4+\lambda_5 &= \frac{2}{3}N(3C+2D)(x^2+y^2+xy)-N(6E+2NF+3G)(x^2y+xy^2)\\
&\quad+\frac{2}{9}N(21H+12NI+3NJ+12L+8NM)(x^2+y^2+xy)^2\\
&\quad+\text{ terms of degree }\geq 5,
\end{aligned}
$$

where $S_2(x,y)$ is as in (3.34) and x,y satisfy (3.35).

We discuss now the stability of the equilibria in the secondary branch of steady-state solutions of (3.1) with symmetry Σ obtained in Theorem 3.3 for small values of $B \neq 0$.

Locally, near the origin, equation (3.35) in the x,y-plane is approximately an ellipse. Tertiary bifurcation points in the secondary branch occur if and only if the curve $S_2(x,y)=0$ intersects the curve (3.35). Generically, the curve $S_2(x,y)=0$ near the origin is approximately an ellipsis or an hyperbola. The distinction between these two cases depends on the sign of the product $(2NI+3H)(2NI+7H)$. It follows then that, generically, only three distinct situations can occur: the number of intersections between the curve $S_2(x,y)=0$ and the Σ-branch in the xy-plane is zero, two or four. See Figure 3.2. Identifying points in the same $\mathbf{D}_3$-orbit, these correspond to zero, one and two tertiary bifurcations along the secondary branch, respectively.

We show below that the solutions of the Σ-branch are generically unstable in the first two cases. In the third case, we prove the instability of the equilibria of the secondary branch only near the secondary bifurcation points.

Theorem 3.9 *Assume the conditions of Theorem 3.3 and let X_0 be an equilibrium of the secondary branch of steady-state solutions of (3.1) with symmetry Σ obtained in Theorem 3.3 for sufficiently small values of $B \neq 0$. Then the solutions of the secondary branch near the secondary bifurcation points are generically unstable.*

Proof: Under the conditions of Theorem 3.3 there is a secondary branch of equilibria of (3.10) near the origin obtained by bifurcation from the primary branches with Σ_i-symmetry for $i=1,2,3$. Denote by $(x_i^-,y_i^-),(x_i^+,y_i^+)$ where $x_i^- < x_i^+$ the projections at the xy-plane of the intersections between the Σ-branch and the Σ_i-branch. Here x,y denote coordinates on $\mathrm{Fix}(\Sigma)$ (recall the beginning of Section 3.4).

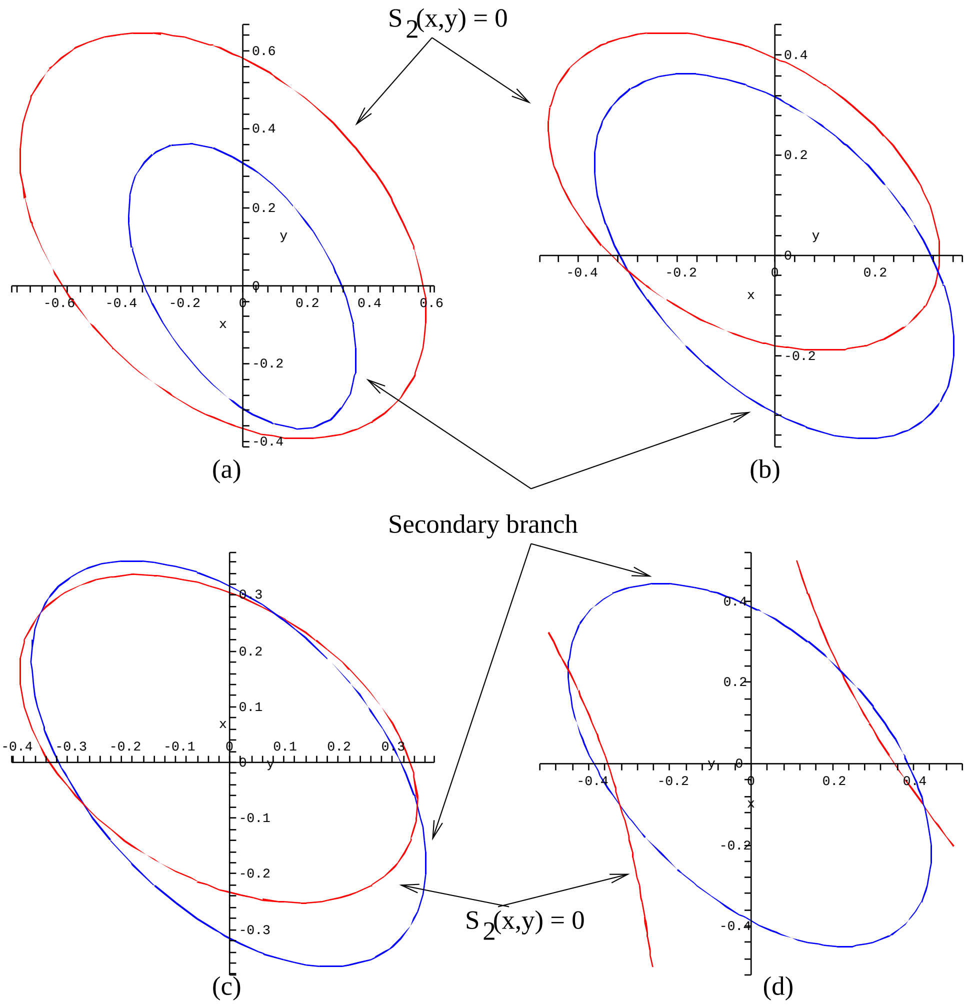

Figure 3.2: Intersections in the xy-plane between the Σ-branch and the curve $S_2(x, y) = 0$. (a) Zero intersections. (b) Two intersections. (c-d) Four intersections.

Let X_0 be an equilibrium of (3.10) in the Σ-branch not corresponding to one of the intersections between the Σ-branch and the Σ_i-branches and consider the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $(dG)_{X_0}$ as in Lemma 3.6 (defining the stability of X_0 at the isotypic components U_1, U_2, U_3 for the action of Σ).

We divide the proof in two cases. First, we suppose that $S_2(x, y) \neq 0$ along the secondary branch. Note that

$$\lambda_1\lambda_2\lambda_3 = -N^3(x + 2y)^2(2x + y)^2(y - x)^2 S_2(x, -x - y)S_2(x, y)S_2(y, x)$$

where sgn $(S_2(x, y))$ = sgn $(S_2(y, x))$ = sgn $(S_2(x, -x - y))$ since $S_2(x, y)$ does not change sign along the Σ-branch. Therefore, in order for X_0 to be (linearly) stable we need sgn $(S_2(x,y)) > 0$ and $\lambda_1\lambda_2 > 0$, $\lambda_1\lambda_3 > 0$, $\lambda_2\lambda_3 > 0$. Now, the signs of these products depend on $(2x+y)(y-x)$, $(x+2y)(x-y)$, $(-1)(x+2y)(y+2x)$ and so there are no values of x, y such that these three products are positive. Thus X_0 is unstable.

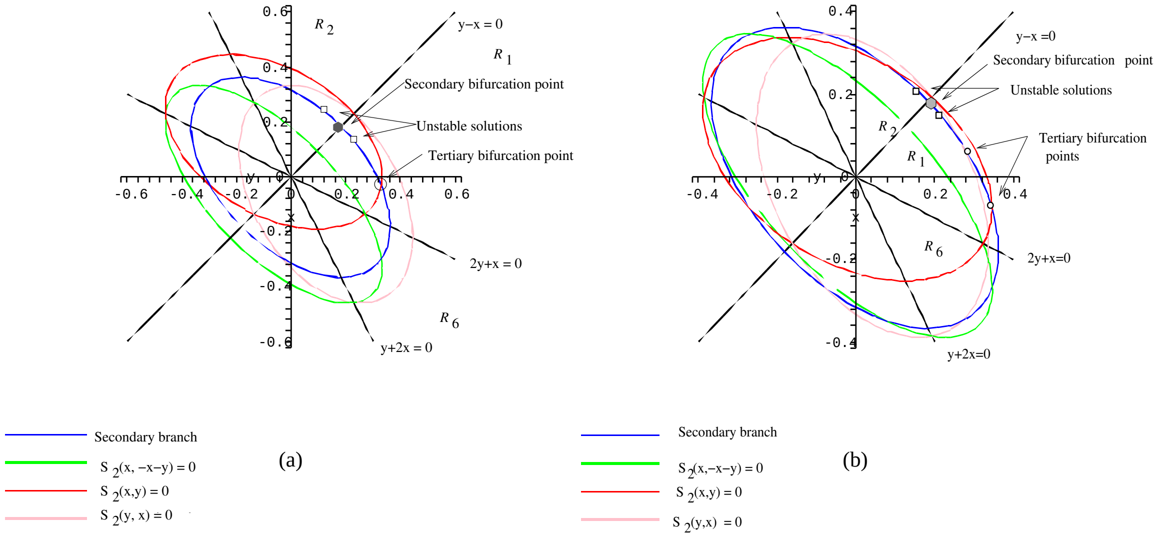

Figure 3.3: Examples where the curve $S_2(x,y) = 0$ intersects the secondary branch and one of the intersections belongs to the region $\mathcal{R}_1$. In each example the two unstable points in the Σ-branch marked with a square are in the same $\mathbf{D}_3$-orbit. (a) There are two intersection points. (b) There are four intersection points.

Suppose now that there is an equilibrium X_0 of the secondary branch with symmetry Σ such that

$$S_2(x_0, y_0) = 0$$

where (x_0, y_0) is the projection of X_0 at the xy-plane. Generically, we can assume that X_0 is not an intersection point between the Σ-branch and one of the Σ_i-branches, for $i = 1, 2, 3$. We have then a tertiary bifurcation at $\lambda = \lambda_0$ from the secondary branch which implies the sign change of one of the eigenvalues determining the stability of the steady-state solutions of the Σ-branch near X_0. By the above discussion, generically, we have two cases: the curve $S_2(x,y) = 0$ intersects the curve (3.35) in two or four points. We have then that the three curves $S_2(y,x) = 0$, $S_2(x,y) = 0$, $S_2(x,-x-y) = 0$ intersect the curve (3.35) in six points (one $\mathbf{D}_3$-orbit) or twelve points (two $\mathbf{D}_3$-orbits), respectively. Recall situations (b)-(d) of Figure 3.2.

Denote by

$$\begin{aligned}
\mathcal{R}_1 &= \{(x,y) \in \mathbf{R}^2 : y - x \leq 0,\ 2y + x \geq 0\},\\
\mathcal{R}_2 &= \{(x,y) \in \mathbf{R}^2 : y + 2x \geq 0,\ y - x \geq 0\},\\
\mathcal{R}_6 &= \{(x,y) \in \mathbf{R}^2 : y + 2x \geq 0,\ 2y + x \leq 0\}
\end{aligned}$$

and assume $(x_0, y_0) \in \mathcal{R}_1$. (The other cases are addressed in a similar way.) Then the eigenvalue λ_2 determining the stability of the equilibrium points in the Σ-branch that belong to the region $\mathcal{R}_1$ changes sign.

Observe that if $(x, y) \in \mathcal{R}_1$ is a steady-state solution in the Σ-branch then $(y, x) \in \mathcal{R}_2$ is also a solution in the Σ-branch and so with the same stability. Note that as (x, y) represents a vector

$$X = (\underbrace{-x-y, \ldots}_{a}; \underbrace{y, \ldots}_{a}; \underbrace{x, \ldots}_{a})$$

in $\mathrm{Fix}(\Sigma)$ then (y, x) corresponds to σX where σ is any permutation that fixes the first set of a coordinates and exchanges the second block of a coordinates with the third block of a coordinates. That is,

$$\sigma X = (\underbrace{-x-y, \ldots}_{a}; \underbrace{x, \ldots}_{a}; \underbrace{y, \ldots}_{a}) \in \mathrm{Fix}(\Sigma).$$

Any such σ belongs to $N(\Sigma)$.

We consider now an open set $\mathcal{O} \subset \mathcal{R}_1 \cup \mathcal{R}_2 \subset \mathbf{R}^2$ containing (x_1^+, y_1^+) such that:
(i) $\mathcal{R}_2 \cap \mathcal{O} = \sigma(\mathcal{R}_1 \cap \mathcal{O})$;
(ii) $S_2(x, y)$ does not change sign in the Σ-branch along $\mathcal{O}$.
We have then that the sign of the eigenvalue λ_2 for an equilibrium X of the secondary branch in $\mathcal{R}_1 \cap \mathcal{O}$ is opposite of the sign of λ_2 for $\sigma X \in \mathcal{R}_2 \cap \mathcal{O}$. Moreover, X and σX have the same stability. Thus, X has eigenvalues with opposite signs and so it is unstable. In Figure 3.3 (a) we show an example where the curve $S_2(x, y) = 0$ intersects at two points the secondary branch in the xy-plane and one of the intersections belongs to the region $\mathcal{R}_1$. Up to symmetry, there is one tertiary bifurcation along the Σ-branch. In the example of Figure 3.3 (b) the curve $S_2(x, y) = 0$ intersects at four points the secondary branch in the xy-plane (and one of the intersections belongs to the region $\mathcal{R}_1$). Up to symmetry, there are two tertiary bifurcations along the Σ-branch.

Similarly, taking steady-state solutions of the Σ-branch close to the point (x_3^+, y_3^+) in the region $\mathcal{R}_1$ where $S_2(x, y)$ does not vary the sign and their orbits by $\mathbf{D}_3$ in the region $\mathcal{R}_6$ we conclude the instability of the steady-state solutions of the Σ-branch close to the point (x_3^+, y_3^+) in the region $\mathcal{R}_1$.

In the example of Figure 3.3 (a) we have instability of equilibria in the Σ-branch. In the case of Figure 3.3 (b) the solutions of the secondary branch near the secondary bifurcation points are unstable. $\square$

We show now an example illustrating the situation where solutions of the secondary branch between tertiary bifurcation points (in the region $\mathcal{R}_1$) are stable.

Example 3.10 We consider (3.23), that is, (3.1) where G is truncated to the fifth order, $N = 6$ and we assume the following parameter values:

$$\begin{aligned}
&B = -0.15,\ C = -1,\ D = 1, \\
&E = 0.9,\ F = 0.025,\ G = -1.9, \\
&H = -8,\ I = 4.25,\ J = 1.35\,.
\end{aligned}$$

The conditions of Corollary 3.5 are satisfied. Therefore, the system (3.23) has a branch of equilibria with symmetry Σ bifurcating from the primary branches with Σ_i-symmetry,

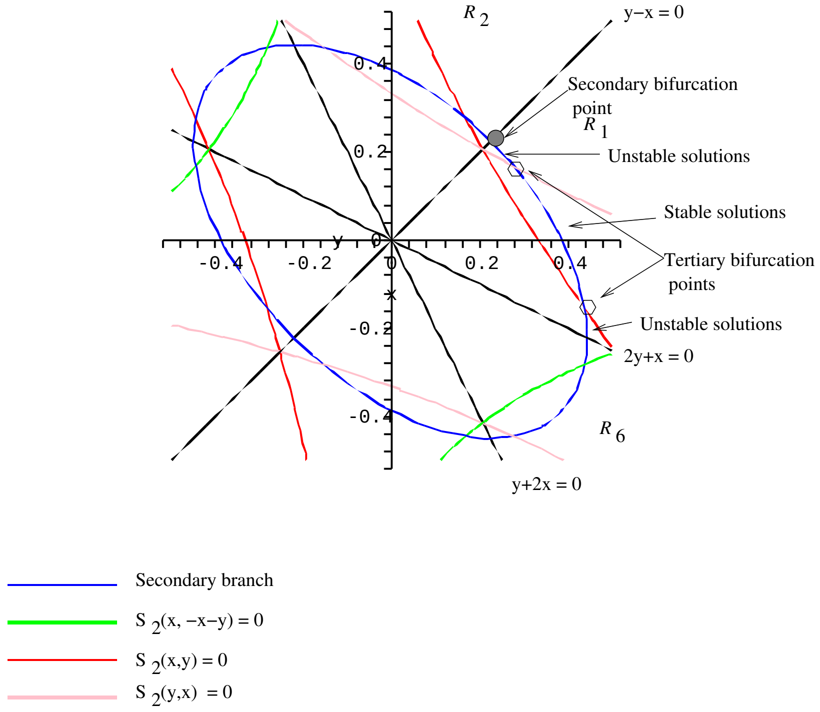

Figure 3.4: Example where solutions of the secondary branch between the tertiary bifurcation points (in region $\mathcal{R}_1$) are stable.

for $i = 1, 2, 3$, which is described by (3.28). In particular, x, y satisfy

$$-0.15 + x^2 + y^2 + xy - 0.1(x^2 y + xy^2) = 0 \tag{3.36}$$

and the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ defined by (3.33) depend on

$$S_2(x, y) = -1 + 0.9y + 9(x^2 + y^2 + xy) - 8y^2.$$

In Figure 3.4 we show the curves $S_2(x, y) = 0$ and (3.36) near the origin. Observe that $S_2(x, y) = 0$ is an hyperbola intersecting (3.36) at four points. Moreover, equilibria in the Σ-branch between the tertiary bifurcation points for example in region $\mathcal{R}_1$ (following the notation of the above proof) are stable: it is clear from Figure 3.4 that $\lambda_1, \lambda_2, \lambda_3 < 0$; for the above parameter values $\lambda_4, \lambda_5 < 0$ using Corollary 3.4 or the expressions of Theorem 3.8. These statements are independent of the values of the parameters L, M. See Figure 3.5 for a schematic representation of the bifurcation diagram showing the amplitude and stability change of the Σ-branch with the primary bifurcation parameter. $\diamond$

We state now sufficient conditions on the coefficients of G in (3.1) that imply the instability of the all Σ-branch of solutions obtained in Theorem 3.3.

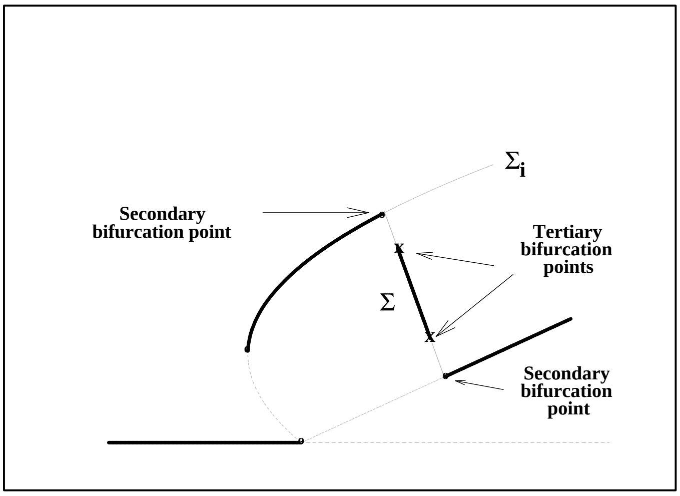

Figure 3.5: Bifurcation diagram showing the amplitude and stability change of the Σ-branch with the primary bifurcation parameter for $N = 6$ and the parameter values of Example 3.10. The Σ-branch solutions near the secondary bifurcation points (dashed lines) are unstable (in the transverse directions to Fix(Σ)) and between the tertiary bifurcation points (solid lines) are stable (in Fix(Σ) and in the transverse directions to Fix(Σ)).

Corollary 3.11 *Suppose the conditions of Theorem 3.3 and assume $H \neq 0$. Let*

$$\Delta = E^2 - 4H\left[C - \frac{B(2NI + 3H)}{3E + 2NF}\right]$$

and if $\Delta > 0$ define

$$y_{\pm} = \frac{-E \pm \sqrt{\Delta}}{2H}, \quad \Delta^*_{\pm} = -3y^2_{\pm} - \frac{12B}{3E + 2NF}.$$

For parameter values such that

$$\begin{array}{l} (i)\ \Delta < 0, \quad or \\ (ii)\ \Delta > 0,\ \Delta^*_{+} < 0,\ \Delta^*_{-} < 0, \quad or \\ (iii)\ \Delta > 0,\ \Delta^*_{+}\Delta^*_{-} < 0, \end{array} \tag{3.37}$$

the solutions of the Σ-branch guaranteed by Theorem 3.3 (that do not correspond to secondary and tertiary bifurcation points) are unstable.

Proof: The instability of the solutions of the Σ-branch guaranteed by Theorem 3.3 follows directly from the proof of Theorem 3.9 if the curves $S_2(x, y) = 0$ where S_2 appears in (3.34) and (3.35), near the origin, intersect at zero or two points only. We find sufficient conditions on the coefficients of G in (3.1) that imply the above situations.

Near the origin we have

$$S_2(x, y) = 0 \Leftrightarrow C + Ey + \left(\frac{2}{3}NI + H\right)(x^2 + y^2 + xy) + Hy^2 + \text{terms of degree} \geq 3 = 0$$

and the equation of the Σ-branch is

$$B+\ \frac{1}{3}(3E+2NF)(x^2+y^2+xy)+\text{terms of degree }\ \geq 3=0.$$

We start by solving

$$\begin{cases} C+Ey+\left(\frac{2}{3}NI+H\right)(x^2+y^2+xy)+Hy^2=0, \\ B+\ \frac{1}{3}(3E+2NF)(x^2+y^2+xy). \end{cases}$$

Trivial calculations show that if conditions (3.37) are satisfied then this system has zero or two real solutions. Now recall that singularity theory methods were used in Theorem 3.3 to prove the existence of the Σ-branch near the origin for sufficiently small values of the parameter B. Higher order terms will not change the geometric properties of the curves $S_2(x,y)=0$ and (3.35) from the point of view of their intersections near the origin as long as the conditions of Theorem 3.3 are satisfied.

$\square$

Chapter 4

Hopf Bifurcation with $\mathbf{S}_N$-Symmetry

In this chapter we study Hopf bifurcation with $\mathbf{S}_N$-Symmetry. The basic existence theorem for Hopf bifurcation in the symmetric case is the Equivariant Hopf Theorem, which involves $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ (in this case). Stewart [41] obtains a classification theorem for $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. We use this classification to prove the existence of branches of periodic solutions in systems of ordinary differential equations with $\mathbf{S}_N$-symmetry taking the restriction of the standard action of $\mathbf{S}_N$ on $\mathbf{C}^N$ onto a $\mathbf{S}_N$-simple space. We derive, for $N \geq 4$, the general $\mathbf{S}_N \times \mathbf{S}^1$-equivariant smooth map up to degree five. We use the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions and we determine conditions on the parameters that describe the stability of the different types of bifurcating periodic solutions.

Consider the natural action of $\mathbf{S}_N$ on $\mathbf{C}^N$ where $\sigma \in \mathbf{S}_N$ acts by permutation of coordinates:

$$\sigma(z_1, \ldots, z_N) = \left(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(N)}\right) \tag{4.1}$$

where $(z_1, \ldots, z_N) \in \mathbf{C}^N$. Observe the following decomposition of $\mathbf{C}^N$ into invariant subspaces for this action:

$$\mathbf{C}^N \cong \mathbf{C}^{N,0} \oplus V_1$$

where

$$\mathbf{C}^{N,0} = \{(z_1, \ldots, z_N) \in \mathbf{C}^N : z_1 + \cdots + z_N = 0\}$$

and

$$V_1 = \{(z, \ldots, z) : z \in \mathbf{C}\} \cong \mathbf{C}.$$

The action of $\mathbf{S}_N$ on V_1 is trivial and the space $\mathbf{C}^{N,0}$ is $\mathbf{S}_N$-simple:

$$\mathbf{C}^{N,0} \cong \mathbf{R}^{N,0} \oplus \mathbf{R}^{N,0}$$

where $\mathbf{S}_N$ acts absolutely irreducibly on

$$\mathbf{R}^{N,0} = \{(x_1, \ldots, x_N) \in \mathbf{R}^N : x_1 + \cdots + x_N = 0\} \cong \mathbf{R}^{N-1}.$$

By Proposition 2.23, if we have a local Γ-equivariant Hopf bifurcation problem, generically the centre subspace at the Hopf bifurcation point is Γ-simple. We make that assumption here. Thus we consider a general $\mathbf{S}_N$-equivariant system of ODEs

$$\frac{dz}{dt} = f(z, \lambda), \tag{4.2}$$

where $z \in \mathbf{C}^{N,0}, \lambda \in \mathbf{R}$ is the bifurcation parameter and $f : \mathbf{C}^{N,0} \times \mathbf{R} \to \mathbf{C}^{N,0}$ is smooth and commutes with the restriction of the natural action (4.1) of $\mathbf{S}_N$ on $\mathbf{C}^N$ to the $\mathbf{S}_N$-simple space $\mathbf{C}^{N,0}$. Observe that $f(0, \lambda) \equiv 0$ since $\mathrm{Fix}_{\mathbf{C}^{N,0}}(\mathbf{S}_N) = \{0\}$.

We study Hopf bifurcation of (4.2) from the trivial equilibrium, say, at $\lambda = 0$, and so we assume that $(df)_{0,0}$ has purely imaginary eigenvalues $\pm i$ (after rescaling time if necessary). Thus if we denote the eigenvalues of $(df)_{0,\lambda}$ by $\sigma(\lambda) \pm i\rho(\lambda)$ (recall Lemma 2.24) then $\sigma(0) = 0, \rho(0) = 1$ and we make the standard hypothesis of the Equivariant Hopf Theorem:

$$\sigma'(0) \neq 0.$$

Under the above hypothesis, we can assume that the action of $\mathbf{S}^1$ on the centre space $\mathbf{C}^{N,0}$ of $(df)_{0,0}$ (that can be identified with the exponential of $(df)_{0,0}$) is given by multiplication by $e^{i\theta}$:

$$\theta(z_1, \ldots, z_N) \ = \ e^{i\theta} (z_1, \ldots, z_N) \tag{4.3}$$

for $\theta \in \mathbf{S}^1$, $(z_1, \ldots, z_N) \in \mathbf{C}^{N,0}$.

In Section 4.1 we give an overview of the physical motivation for this work.

In Section 4.2 we recall the classification of the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ given by Stewart [41].

In Section 4.3 we calculate the cubic and the fifth order truncation of f in (4.2) for the action of $\mathbf{S}_N \times \mathbf{S}^1$ extended to $\mathbf{C}^N$ defined by (4.1) and (4.3). We obtain the cubic and the fifth order truncation of f in (4.2) on $\mathbf{C}^{N,0}$ by restricting and projecting onto $\mathbf{C}^{N,0}$.

We describe the two types of $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$: $\Sigma^I_{q,p}$ and Σ^{II}_q (Theorem 4.1). We use in Section 4.4 the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions with these symmetries of (4.2) by Hopf bifurcation from the trivial equilibrium at $\lambda = 0$ for a bifurcation problem with symmetry $\Gamma = \mathbf{S}_N$. The main result of this chapter is Theorem 4.13. In this theorem we determine the directions of branching and the stability of periodic solutions guaranteed by the Equivariant Hopf Theorem. For solutions with symmetry Σ^{II}_q the terms of degree three determines the criticality of the branches and also the stability of these solutions (near the origin). However, for solutions with symmetry $\Sigma^I_{p,q}$, although the criticality of the branches is determined by the terms of degree three, the stability of solutions in some directions is not. Moreover, in one particular direction, even the degree five truncation is too degenerate (it originates a null eigenvalue which is not forced by the symmetry of the problem).

4.1 Physical Motivation

An appropriate symmetry context for studying periodic solutions to the equal-mass many-body problem in the plane was formulated by Stewart [41].

Consider a system of N equal point particles in $\mathbf{R}^2$. Let $q_1, \ldots, q_N$ be the position coordinates and $p_1, \ldots, p_N$ the momentum coordinates. Then the motion is governed by a smooth Hamiltonian $H : \mathbf{R}^{2N} \times \mathbf{R}^{2N} \to \mathbf{R}$ with $2N$ degrees of freedom. The explicit form of the Hamiltonian for the planar N-Body problem is given by

$$H(p_i, q_i) = \sum_{i=1}^{N} \frac{\|p_i\|^2}{2m_i} - \sum_{1 \leq i \leq j \leq N} \frac{m_i m_j}{\|q_i - q_j\|} \tag{4.4}$$

where $q_i \in \mathbf{R}^2$ is the position, $p_i \in \mathbf{R}^2$ is the momentum and $m_i \in \mathbf{R}_+$ is the mass of the ith particle. Furthermore, we have the Hamilton's equations:

$$\begin{cases} \dot{q}_i = \partial H / \partial p_i \\ \dot{p}_i = -\partial H / \partial q_i. \end{cases} \tag{4.5}$$

See for example [1] and [28].

There is a reduction of the problem that simplifies the analysis. We may choose a system of coordinates whose origin is at the center of mass, this is, we may restrict the dynamics to the subspace

$$q_1 + \cdots + q_N = 0 \ , \ p_1 + \cdots + p_N = 0.$$

The equations of motion of many physical systems are invariant under the action of a group due to the fact that the physical systems possess certain symmetries [32].

In [31] van der Meer studied Hamiltonian Hopf bifurcation in the presence of a compact symmetry group G. He classified the expected actions of G and showed that near four-dimensional fixed-point subspaces of subgroups of $G \times \mathbf{S}^1$ the bifurcation of periodic solutions is diffeomorphic to the standard Hamiltonian Hopf bifurcation in two degrees of freedom. Furthermore, he presented examples with $\mathbf{O}(2), \mathbf{SO}(2)$ and $\mathbf{SU}(2)$ symmetry.

In [6] Chossat *et al.* studied the appearance of branches of relative periodic orbits in Hamiltonian Hopf bifurcation processes in the presence of compact symmetry groups that do not generically exist in the dissipative framework. Examples are given with $\mathbf{O}(2)$ and $\mathbf{SO}(3)$ symmetry.

Our goal is to ask if there is any possible correspondence between this work and the general questions arising from the N-Body problems in hamiltonian dynamics. Montgomery [35] suggested us two ideas as an application of the work we carried out in this chapter to the N-Body problem. The first suggestion follows from the figure of eight, a solution to the three-body problem and consists in applying our work to the choreographes for N-body systems which have $\mathbf{S}_N$ as a symmetry group; the second idea would be applying our results to Saturn's rings. In what follows we try to explore these ideas with more detail.

Chenciner and Montgomery [5] presented a periodic orbit for the newtonian problem of three equal masses in the plane in which the three bodies chase each other around a fixed eight-shaped curve. This solutions has symmetry $\mathbf{S}_3$.

Suppose now that we have N equal Newtonian masses *dancing* around a fixed curve. The eight is such a solution. For each N we can obtain such a solution where the curve is a circle by placing the N points at the vertices of a regular N-gon inscribed in the circle and then rotating this N-gon at the proper frequency. Until the moment, except for the eight solution, there are no rigorous proof of any equal mass planar N-body solutions (see the expository article by Montgomery [34]).

To make N equal bodies perform a desired dance, begin with the circle $\mathbf{S}^1 = \mathbf{R}/T\mathbf{Z}$ of circumference T. The cyclic group $\mathbf{Z}_N$ of order N acts on this circle with its generator ω acting by $\omega(t) = t + T/N$, which is to say by rotation by $2\pi/N$. It acts on $\mathbf{C}^N$ by $\omega(x_1, x_2, \ldots, x_N) = (x_N, x_1, \ldots, x_{N-1})$. Then $\mathbf{Z}_N$ acts on the space of all loops $x : \mathbf{S}^1 \to \mathbf{C}^N$ by $(\omega x)(t) = \omega(x\omega^{-1}(t))$. A fixed point of this action on loops is a map $x : \mathbf{S}^1 \to \mathbf{C}^N$ satisfying $x_{j+1}(t) = x_1(t - jT/N)$. Such a map is called a *choreography.* In a choreography all N masses travel along the same closed planar curve $q(t) = x_1(t)$, staggered in phase from each other by T/N. This action of $\mathbf{Z}_N$ on loops x leaves the N-body action $A(x)$ invariant when the masses are all equal. If we restrict A to the fixed points of the action - the choreographes - and find a collision-free choreography which is a critical point for this restricted A, then this choreography will be a solution to the N-body problem. Excluding collisions breaks up the set of choreographies into countable many different components, which are called *choreography classes.* With an argument of Poincaré, if the potential is strong-force, then there is a solution realizing each choreography class. In the case of N-bodies, all the choreographies are possible, which have $\mathbf{S}_N$ as a symmetry group. For further detail see [34].

Another interesting problem for applying Hopf bifurcation with $\mathbf{S}_N$ symmetry would be the rings of Saturn, the brightest and best known planetary ring system. In [33] Meyer and Schmidt gave a simple mathematical model for braided rings of a planet based on Maxwell's model for the rings of Saturn. Their rings models are Hamiltonian systems of two degrees of freedom which have an equilibrium point which corresponds to a central configuration.

4.2 C-Axial Subgroups of $\mathbf{S}_N \times \mathbf{S}^1$

In order to apply the Equivariant Hopf Theorem we require information on the $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. Such subgroups are of the type $H^\theta = \{(h, \theta(h)) : h \in H\}$ where $H \subseteq \mathbf{S}_N$ and $\theta : H \to \mathbf{S}^1$ is a group homomorphism (see [23, Definition XVI 7.1, Proposition XVI 7.2]). Also they are maximal with respect to fixing a complex line $\mathbf{C}z = \{\mu z : \mu \in \mathbf{C}\}$, where $\mu \neq 0$. A vector z such that the isotropy subgroup Σ_z in $\mathbf{S}_N \times \mathbf{S}^1$ fixes only $\mathbf{C}z$ is called an *axis.*

Theorem 4.1 (Stewart [41]) *Suppose that $N \geq 2$. Then the axes of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ have orbit representatives as follows:*
Type I
Let $N = qk + p$ where $2 \leq k \leq N$, $q \geq 1$, $p \geq 0$. Let $\xi = e^{2\pi i/k}$ and set

$$z = \left(\underbrace{1, \ldots, 1}_{q}; \underbrace{\xi, \ldots, \xi}_{q}; \underbrace{\xi^2, \ldots, \xi^2}_{q}; \ldots; \underbrace{\xi^{k-1}, \ldots, \xi^{k-1}}_{q}; \underbrace{0, \ldots, 0}_{p}\right). \tag{4.6}$$

Type II
Let $N = q + p$, $1 \leq q < N/2$ *and set*

$$z = \left(\underbrace{1, \ldots, 1}_{q}; \underbrace{a, \ldots, a}_{p} \right) \tag{4.7}$$

where $a = -q/p$.

Proof: See Stewart [41, Theorem 7]. □

Next we consider the corresponding isotropy subgroups as in [41]. For type I we have **C**-axial subgroups $H^\theta = \Sigma_z$ where

$$\Sigma_z = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p \stackrel{\text{def}}{=} \Sigma^I_{q,p}. \tag{4.8}$$

Here $\wr$ denotes the wreath product (see Hall [27, p. 81]) and the tilde indicates that $\mathbf{Z}_k$ is twisted into $\mathbf{S}^1$. Let

$$K = \ker(\theta) = \mathbf{S}^1_q \times \cdots \times \mathbf{S}^k_q \times \mathbf{S}_p, \tag{4.9}$$

where $\mathbf{S}^j_q$ is the symmetric group on $B_j = \{(j-1)q+1, \ldots, jq\}$ and $\mathbf{S}_p$ is the symmetric group on $B_0 = \{kq+1, \ldots, , n\}$. Observe that if $\mathbf{S}_r$ acts by permutating $\{1, \ldots, r\}$ then it is generated by $(1\ 2), (1\ 3), \ldots, (1\ r)$.

Let $\alpha = (1\ q+1\ 2q+1\ \ldots\ (k-1)q+1)$ and $\xi = 2\pi/k$. Then $\Sigma^I_{q,p}$ is generated by (α, ξ) and K.

For the type II, the isotropy subgroup is

$$\Sigma_z = S_q \times S_p \stackrel{\text{def}}{=} \Sigma^{II}_q \tag{4.10}$$

where the respective factors are the symmetric groups on $\{1, \ldots, q\}$ and $\{q+1, \ldots, N\}$. Thus we have the generators

$$(1\ 2), \ldots, (1\ q), (q+1\ q+2), \ldots, (q+1\ N).$$

Table 4.1 lists the generators for the isotropy subgroups $\Sigma^I_{q,p}$ and Σ^{II}_q.

4.3 Equivariant Vector Field

In order to determine the direction of branching and the stability of the bifurcating branches of periodic solutions of (4.2), we must compute the general form of a $\mathbf{S}_N \times \mathbf{S}^1$-equivariant bifurcation problem. We start by calculating the cubic and the fifth order truncation of f in (4.2) for the action of $\mathbf{S}_N \times \mathbf{S}^1$ extended to $\mathbf{C}^N$ defined by (4.1) and (4.3). We obtain then the cubic and the fifth order truncation of f in (4.2) on $\mathbf{C}^{N,0}$ by restricting and projecting onto $\mathbf{C}^{N,0}$.

Isotropy Subgroup	Generators
$\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$	$(1\ 2), \ldots, (1\ q), \ldots, (kq+1\ kq+2), \ldots, (kq+1\ N)$ (α, ξ) where $\alpha = (1\ q+1\ 2q+1\ \ldots\ (k-1)q+1)$ and $\xi = 2\pi/k$
$\Sigma^{II}_{q,p} = \mathbf{S}_q \times \mathbf{S}_p$	$(1\ 2), \ldots, (1\ q), (q+1\ q+2), \ldots, (q+1\ N)$

Table 4.1: Generators for the isotropy subgroups $\Sigma^I_{q,p}$ and Σ^{II}_q.

Theorem 4.2 *Suppose $N \geq 4$. Let $f : \mathbf{C}^N \to \mathbf{C}^N$ be $\mathbf{S}_N \times \mathbf{S}^1$-equivariant with respect to the action defined by (4.1) and (4.3) with polynomial components of degree lower or equal than 3. Then $f = (f_1, f_2, \ldots, f_N)$ where*

$$\begin{array}{rcl} f_1(z_1, z_2, \ldots, z_N) & = & \displaystyle\sum_{i=-1}^{11} a_i h_i(z_1, z_2, \ldots, z_N) \\ f_2(z_1, z_2, \ldots, z_N) & = & f_1(z_2, z_1, \ldots, z_N) \\ & \vdots & \\ f_N(z_1, z_2, \ldots, z_N) & = & f_1(z_N, z_2, \ldots, z_1) \end{array} \tag{4.11}$$

and

$$\begin{array}{rclrcl} h_{-1}(z) & = & z_1 + \cdots + z_N, & h_0(z) & = & z_1 \\ h_1(z) & = & |z_1|^2 z_1 & & & \\ h_2(z) & = & z_1^2 \displaystyle\sum_{j=1}^N \overline{z}_j, & h_3(z) & = & |z_1|^2 \displaystyle\sum_{i=1}^N z_i \\ h_4(z) & = & z_1 \displaystyle\sum_{k=1}^N |z_k|^2, & h_5(z) & = & z_1 \displaystyle\sum_{i=1}^N z_i \sum_{k=1}^N \overline{z}_k, \\ h_6(z) & = & \overline{z}_1 \displaystyle\sum_{j=1}^N z_j^2, & h_7(z) & = & \overline{z}_1 \displaystyle\sum_{i=1}^N z_i \sum_{j=1}^N z_j, \\ h_8(z) & = & \displaystyle\sum_{j=1}^N |z_j|^2 z_j, & h_9(z) & = & \displaystyle\sum_{i=1}^N z_i^2 \sum_{k=1}^N \overline{z}_k, \\ h_{10}(z) & = & \displaystyle\sum_{i=1}^N z_i \sum_{k=1}^N |z_k|^2, & h_{11}(z) & = & \displaystyle\sum_{i=1}^N z_i \sum_{j=1}^N z_j \sum_{k=1}^N \overline{z}_k, \end{array} \tag{4.12}$$

for constants $a_j \in \mathbf{C}$. Also we denote $|z_j|^2 = z_j \overline{z}_j$ for $j = 1, \ldots, N$.

Proof: Let $f : \mathbf{C}^N \to \mathbf{C}^N$ be $\mathbf{S}_N \times \mathbf{S}^1$-equivariant with polynomial components of degree lower or equal than 3. For $z = (z_1, \ldots, z_N)$ define $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_N)$. Using multi-indices we have that $f = (f_1, \dots, f_N)$ where each f_j can be written as

$$f_j(z) \;=\; \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta.$$

Here each $a_{\alpha\beta} \in \mathbf{C}$ and $\alpha, \beta \in \left(\mathbf{Z}_0^+\right)^N$.

Now f is $\mathbf{S}_N \times \mathbf{S}^1$-equivariant if and only if f is $\mathbf{S}_N$-equivariant and $\mathbf{S}^1$-equivariant. The $\mathbf{S}^1$-equivariance is equivalent as saying that each f_j satisfies

$$f_j(e^{i\theta} z) = e^{i\theta} f_j(z) \tag{4.13}$$

for all $\theta \in \mathbf{S}^1$ and $z \in \mathbf{C}^N$. Note that condition (4.13) implies that

$$\sum a_{\alpha\beta} z^\alpha \overline{z}^\beta = \sum a_{\alpha\beta} e^{i\theta(\alpha-\beta-1)} z^\alpha \overline{z}^\beta.$$

Thus each $a_{\alpha\beta} = 0$ unless

$$|\alpha| = |\beta| + 1.$$

In particular, it follows that each f_j is odd as a polynomial in $z, \overline{z}$. Since f has degree lower or equal to three, it follows that each component f_j can only contain degree 1 or degree 3 monomials.

The equivariance of f under $\mathbf{S}_N$ is equivalent to the invariance say of the first component f_1 under $\mathbf{S}_{N-1}$ in the last $N-1$-coordinates $z_2, \dots, z_N$, and then

$$f(z) \;=\; \left(f_1(z_1, z_2, \dots, z_{N-1}, z_N), f_1(z_2, z_1, \dots, z_{N-1}, z_N), \dots, f_1(z_N, z_2, \dots, z_{N-1}, z_1)\right).$$

This follows from

$$f\left((1i)(z_1, z_2, \dots, z_N)\right) \;=\; (1i) f\left(z_1, z_2, \dots, z_N\right)$$

for $i = 2, 3, \dots, N$.

The rest of the proof consists in characterizing the first component f_1. That is, we describe the polynomials of degree lower or equal to 3 that are $\mathbf{S}_{N-1}$-invariant in the last $N-1$-coordinates $z_2, \dots, z_N$ and satisfy (4.13).

For the degree one polynomials we have that f_1 has to be a linear combination of the monomials z_1 and $z_2 + \cdots + z_N$. That is, any linear polynomial in $z, \overline{z}$ that is $\mathbf{S}_{N-1}$-invariant in the last $N-1$ coordinates and satisfies (4.13), is a linear combination of z_1 and $z_2 + \cdots + z_N$, or alternatively, of z_1 and $z_1 + z_2 + \cdots + z_N$. We obtain the equivariants where the first component is h_{-1} and h_0.

For the degree three polynomials, f_1 is a linear combination of monomials of the following types:

(a) $z_1^2 \overline{z}_1 = z_1 |z_1|^2$ and we obtain h_1;

(b) $z_1^2 \sum_{j=2}^N \overline{z}_j$, and we obtain h_2 using h_1;

(c) $z_1 \overline{z}_1 \sum_{j=2}^N z_j$, and so we get h_3 using h_1;

(d) $z_1 p(z_2, \dots, z_N)$ where $p(z_2, \dots, z_N)$ has degree two in $z, \overline{z}$ and it is $S_{N-1} \times \mathbf{S}^1$-invariant. After some manipulations we obtain h_4 and h_5.

(e) $\overline{z}_1 p(z_2, \ldots, z_N)$ where $p(z_2, \ldots, z_N)$ has degree two in $z_2, \ldots, z_N$, it is S_{N-1}-invariant and does not depend on the $\overline{z}_j$. We obtain h_6 and h_7.

(f) $p(z_2, \ldots, z_N)$ where p is S_{N-1}-invariant and satisfies (4.13). We get h_8, h_9, h_{10} and h_{11}.

□

Remark 4.3 From Theorem 4.2 we have that the number of linearly independent cubic $\mathbf{S}_N \times \mathbf{S}^1$-equivariants on $\mathbf{C}^N$ (over $\mathbf{C}$) is 11 for $N \geq 4$. This result is in agreement with [3, Proposition 6.4]. ◇

Remark 4.4 There are two (real) valued invariants of degree two for the action of $\mathbf{S}_N \times \mathbf{S}^1$ on $\mathbf{C}^N$:

$$I_1 = \sum_{j=1}^{N} |z_j|^2, \qquad I_2 = \sum_{i=1}^{N} z_i \sum_{j=1}^{N} \overline{z}_j. \tag{4.14}$$

To see this note that a polynomial function $p: \mathbf{C}^N \to \mathbf{R}$ can be written as

$$p(z) = \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta$$

where $z \in \mathbf{C}^N$. Now f is $\mathbf{S}_N \times \mathbf{S}^1$-invariant if and only if it is invariant by $\mathbf{S}^1$ and $\mathbf{S}_N$. The $\mathbf{S}^1$-invariance is equivalent as saying that $p(z) = p(e^{i\theta} z)$ for all $\theta \in \mathbf{S}^1$ and $z \in \mathbf{C}^N$, this condition implies that

$$\sum a_{\alpha\beta} z^\alpha \overline{z}^\beta = \sum a_{\alpha\beta} e^{i\theta(\alpha-\beta)} z^\alpha \overline{z}^\beta.$$

Thus, $p(z)$ is $\mathbf{S}^1$-invariant if and only if $|\alpha| = |\beta|$ for the coefficients $a_{\alpha,\beta} \neq 0$. In particular this implies that the $\mathbf{S}_N \times \mathbf{S}^1$-invariants have even degree in $z, \overline{z}$. ◇

Theorem 4.5 *Suppose $N \geq 4$. Consider the action of $\mathbf{S}_N \times \mathbf{S}^1$ on $\mathbf{C}^N$ defined by (4.1) and (4.3). Let $f : \mathbf{C}^N \to \mathbf{C}^N$ be $\mathbf{S}_N \times \mathbf{S}^1$-equivariant with homogeneous polynomial components of degree 5. Then $f = (f_1, f_2, \ldots, f_N)$ where*

$$\begin{aligned} f_1(z_1, z_2, \ldots, z_N) &= \sum_{i=1}^{52} a_i h_i(z_1, z_2, \ldots, z_N), \\ f_2(z_1, z_2, \ldots, z_N) &= f_1(z_2, z_1, \ldots, z_N), \\ &\vdots \\ f_N(z_1, z_2, \ldots, z_N) &= f_1(z_N, z_2, \ldots, z_1), \end{aligned}$$

and

$$h_1(z) = |z_1|^4 z_1, \qquad h_2(z) = z_1 \sum_{j=1}^{N} |z_j|^4,$$

$$h_3(z) = z_1 \sum_{i=1}^{N} z_i^2 \sum_{k=1}^{N} \overline{z}_k^2, \qquad h_4(z) = z_1 \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} |z_j|^2,$$

$$h_5(z) = z_1 \sum_{k=1}^{N} |z_k|^2 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} \overline{z}_j, \qquad h_6(z) = z_1 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \overline{z}_k \sum_{l=1}^{N} \overline{z}_l,$$

$$h_7(z) = z_1 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \overline{z}_k^2, \qquad h_8(z) = z_1 \sum_{i=1}^{N} z_i^2 \sum_{j=1}^{N} \overline{z}_j \sum_{k=1}^{N} \overline{z}_k,$$

$$h_9(z) = z_1 \sum_{i=1}^{N} |z_i|^2 z_i \sum_{j=1}^{N} \overline{z}_j, \qquad h_{10}(z) = z_1 \sum_{i=1}^{N} |z_i|^2 \overline{z}_i \sum_{j=1}^{N} z_j,$$

$$h_{11}(z) = z_1^2 \sum_{k=1}^{N} z_k \sum_{i=1}^{N} \overline{z}_i^2, \qquad h_{12}(z) = z_1^2 \sum_{k=1}^{N} |z_k|^2 \overline{z}_k,$$

$$h_{13}(z) = z_1^2 \sum_{k=1}^{N} |z_k|^2 \sum_{j=1}^{N} \overline{z}_j, \qquad h_{14}(z) = z_1^2 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} \overline{z}_j \sum_{k=1}^{N} \overline{z}_k,$$

$$h_{15}(z) = z_1^3 \sum_{j=1}^{N} \overline{z}_j{}^2, \qquad h_{16}(z) = z_1^3 \sum_{i=1}^{N} \overline{z}_i \sum_{j=1}^{N} \overline{z}_j,$$

$$h_{17}(z) = \overline{z}_1 \sum_{i=1}^{N} z_i^3 \sum_{j=1}^{N} \overline{z}_j, \qquad h_{18}(z) = \overline{z}_1 \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} z_j^2,$$

$$h_{19}(z) = \overline{z}_1 \sum_{i=1}^{N} |z_i|^2 z_i^2, \qquad h_{20}(z) = \overline{z}_1 \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} z_j \sum_{k=1}^{N} z_k,$$

$$h_{21}(z) = \overline{z}_1 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j \sum_{k=1}^{N} z_k \sum_{j=1}^{N} \overline{z}_l, \qquad h_{22}(z) = \overline{z}_1 \sum_{i=1}^{N} z_i^2 \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \overline{z}_k,$$

$$h_{23}(z) = \overline{z}_1 \sum_{i=1}^{N} |z_i|^2 z_i \sum_{j=1}^{N} z_j, \qquad h_{24}(z) = \overline{z}_1^2 \sum_{k=1}^{N} z_k^3,$$

$$h_{25}(z) = \overline{z}_1^2 \sum_{k=1}^{N} z_k^2 \sum_{i=1}^{N} z_i, \qquad h_{26}(z) = \overline{z}_1^2 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j \sum_{k=1}^{N} z_k,$$

$$h_{27}(z) = |z_1|^2 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \overline{z}_k, \qquad h_{28}(z) = |z_1|^2 \sum_{i=1}^{N} z_i^2 \sum_{k=1}^{N} \overline{z}_k,$$

$$h_{29}(z) = |z_1|^2 \sum_{k=1}^{N} |z_k|^2 z_k, \qquad h_{30}(z) = |z_1|^2 \sum_{k=1}^{N} |z_k|^2 \sum_{j=1}^{N} z_j,$$

$$h_{31}(z) = z_1 |z_1|^2 \sum_{j=1}^{N} |z_j|^2, \qquad h_{32}(z) = z_1 |z_1|^2 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} \overline{z}_j,$$

$$
\begin{array}{rclcrcl}
h_{33}(z) & = & z_1^2|z_1|^2 \sum_{k=1}^N \overline{z}_k, & & h_{34}(z) & = & \overline{z}_1|z_1|^2 \sum_{i=1}^N z_i^2, \\
h_{35}(z) & = & \overline{z}_1|z_1|^2 \sum_{i=1}^N z_i \sum_{j=1}^N z_j, & & h_{36}(z) & = & |z_1|^4 \sum_{i=1}^N z_i, \\
h_{37}(z) & = & \sum_{i=1}^N z_i^3 \sum_{l=1}^N \overline{z}_l \sum_{m=1}^N \overline{z}_m, & & h_{38}(z) & = & \sum_{i=1}^N z_i \sum_{j=1}^N z_j \sum_{k=1}^N z_k \sum_{l=1}^N \overline{z}_l \sum_{m=1}^N \overline{z}_m, \\
h_{39}(z) & = & \sum_{i=1}^N z_i^3 \sum_{k=1}^N \overline{z}_k^2, & & h_{40}(z) & = & \sum_{i=1}^N z_i^2 \sum_{j=1}^N z_j \sum_{k=1}^N \overline{z}_k \sum_{l=1}^N \overline{z}_l, \\
h_{41}(z) & = & \sum_{i=1}^N z_i^2 \sum_{j=1}^N z_j \sum_{k=1}^N \overline{z}_k^2, & & h_{42}(z) & = & \sum_{i=1}^N z_i \sum_{j=1}^N z_j \sum_{l=1}^N z_l \sum_{k=1}^N \overline{z}_k^2, \\
h_{43}(z) & = & \sum_{i=1}^N |z_i|^2 \sum_{j=1}^N |z_j|^2 \sum_{k=1}^N z_k, & & h_{44}(z) & = & \sum_{i=1}^N |z_i|^4 z_i, \\
h_{45}(z) & = & \sum_{i=1}^N |z_i|^4 \sum_{k=1}^N z_k, & & h_{46}(z) & = & \sum_{i=1}^N |z_i|^2 \sum_{k=1}^N z_k^2 \sum_{l=1}^N \overline{z}_l, \\
h_{47}(z) & = & \sum_{i=1}^N |z_i|^2 \sum_{j=1}^N z_j \sum_{k=1}^N z_k \sum_{l=1}^N \overline{z}_l, & & h_{48}(z) & = & \sum_{i=1}^N |z_i|^2 z_i^2 \sum_{l=1}^N \overline{z}_l, \\
h_{49}(z) & = & \sum_{i=1}^N |z_i|^2 z_i \sum_{l=1}^N |z_l|^2, & & h_{50}(z) & = & \sum_{i=1}^N |z_i|^2 \overline{z}_i \sum_{l=1}^N z_l^2, \\
h_{51}(z) & = & \sum_{i=1}^N |z_i|^2 \overline{z}_i \sum_{l=1}^N z_l \sum_{j=1}^N z_j, & & h_{52}(z) & = & \sum_{i=1}^N |z_i|^2 z_i \sum_{l=1}^N z_l \sum_{j=1}^N \overline{z}_j,
\end{array}
$$

for constants $a_j \in \mathbf{C}$. Also we denote $|z_j|^2 = z_j\overline{z}_j$ and $|z_j|^4 = z_j^2\overline{z}_j^2$ for $j = 1, \ldots, N$.

Proof: We will describe in cases (a) to (k) the $\mathbf{S}_N \times \mathbf{S}^1$-equivariants where the first component can be written as

$$z_1^a \overline{z_1}^b p(z_2, \ldots, z_N)$$

where $a, b \in \mathbf{Z}_0^+$ and $a + b > 0$. Case (l) considers the situation in which $a + b = 0$. Recall the proof of Theorem 4.2. For the degree five homogeneous polynomials, f_1 is a linear combination of monomials of the following types:

(a) $z_1^3\overline{z}_1^2 = z_1|z_1|^4$ and we obtain h_1;

(b) We describe the polynomials of degree five in the form $h_i(z) = z_1 p(z_2, \ldots, z_N)$ which are $\mathbf{S}_{N-1}$-invariant and satisfy (4.13). Note that if we write

$$h_i(z) = \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta$$

then condition (4.13) is equivalent to the condition

$$|\alpha| = |\beta| + 1 \tag{4.15}$$

for $a_{\alpha\beta} \neq 0$.

We describe $p(z_2, \ldots, z_N)$ which are invariant by permutation of coordinates and have degree four. Moreover, from (4.15) we have that $p(z_2, \ldots, z_N)$ must have degree two in $z_2, \ldots, z_N$ and degree two in $\overline{z}_2, \ldots, \overline{z}_N$.

The only polynomials that satisfy those conditions are

$$p_6(z) = \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \overline{z}_k \sum_{l=1}^{N} \overline{z}_l,$$

$$\begin{aligned}
p_2(z) &= \sum_{j=1}^{N} |z_j|^4, & p_3(z) &= \sum_{i=1}^{N} z_i^2 \sum_{k=1}^{N} \overline{z}_k^2, \\
p_4(z) &= \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} |z_j|^2, & p_5(z) &= \sum_{k=1}^{N} |z_k|^2 \sum_{i=1}^{N} z_i \sum_{j=1}^{N} \overline{z}_j, \\
p_7(z) &= \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j \sum_{k=1}^{N} \overline{z}_k^2, & p_8(z) &= \sum_{i=1}^{N} z_i^2 \sum_{j=1}^{N} \overline{z}_j \sum_{k=1}^{N} \overline{z}_k, \\
p_9(z) &= \sum_{i=1}^{N} |z_i|^2 z_i \sum_{j=1}^{N} \overline{z}_j, & p_{10}(z) &= \sum_{i=1}^{N} |z_i|^2 \overline{z}_i \sum_{j=1}^{N} z_j.
\end{aligned}$$

In such a way we obtain $h_2, h_3, \ldots, h_{10}$.

(c) We describe the polynomials of degree five in the form $h_i(z) = z_1^2 p(z_2, \ldots, z_N)$. Using the same argument as above, we must now describe the polynomials $p(z_2, \ldots, z_N)$ of degree three which have degree one in $z_2, \ldots, z_N$ and degree two in $\overline{z}_2, \ldots, \overline{z}_N$.

We get the polynomials $p_{14}(z) = \sum_{i=1}^{N} z_i \sum_{j=1}^{N} \overline{z}_j \sum_{k=1}^{N} \overline{z}_k$ and

$$p_{11}(z) = \sum_{k=1}^{N} z_k \sum_{i=1}^{N} \overline{z}_i^2, \qquad p_{12}(z) = \sum_{k=1}^{N} |z_k|^2 \overline{z}_k, \qquad p_{13}(z) = \sum_{k=1}^{N} |z_k|^2 \sum_{j=1}^{N} \overline{z}_j.$$

In such a way we obtain h_{11}, h_{12}, h_{13} and h_{14}.

(d) We describe the polynomials in the form $z_1^3 p(z_2, \ldots, z_N)$ where $p(z_2, \ldots, z_N)$ has degree two in $\overline{z}_2, \ldots, \overline{z}_N$. We have $p_{16}(z) = \sum_{i=1}^{N} \overline{z}_i \sum_{j=1}^{N} \overline{z}_j$ and $p_{15}(z) = \sum_{j=1}^{N} \overline{z}_j{}^2$. We obtain h_{15} and h_{16}.

(e) We describe the polynomials of the form $h_i(z) = \overline{z}_1 p(z_2, \dots, z_N)$ where $p(z_2, \dots, z_N)$ has degree three in $z_2, \dots, z_N$, degree one in $\overline{z}_2, \dots, \overline{z}_N$ and is S_{N-1}-invariant.

We have

$$p_{21}(z) = \sum_{i=1}^N z_i \sum_{j=1}^N z_j \sum_{k=1}^N z_k \sum_{j=1}^N \overline{z}_l,$$

$$\begin{aligned} p_{17}(z) &= \sum_{i=1}^N z_i^3 \sum_{j=1}^N \overline{z}_j, & p_{18}(z) &= \sum_{i=1}^N |z_i|^2 \sum_{j=1}^N z_j^2, \\ p_{19}(z) &= \sum_{i=1}^N |z_i|^2 z_i^2, & p_{20}(z) &= \sum_{i=1}^N |z_i|^2 \sum_{j=1}^N z_j \sum_{k=1}^N z_k, \\ p_{22}(z) &= \sum_{i=1}^N z_i^2 \sum_{j=1}^N z_j \sum_{k=1}^N \overline{z}_k, & p_{23}(z) &= \sum_{i=1}^N |z_i|^2 z_i \sum_{j=1}^N z_j \end{aligned}$$

and we obtain $h_{17}, h_{18}, h_{19}, h_{20}, h_{21}, h_{22}$ and h_{23}.

(f) We describe the polynomials $h_i(z) = \overline{z}_1^2 p(z_2, \dots, z_N)$ where $p(z_2, \dots, z_N)$ is of degree three in $z_2, \dots, z_N$ and is S_{N-1}-invariant. We have $p_{26}(z) = \sum_{i=1}^N z_i \sum_{j=1}^N z_j \sum_{k=1}^N z_k$ and

$$p_{24}(z) = \sum_{k=1}^N z_k^3, \qquad p_{25}(z) = \sum_{k=1}^N z_k^2 \sum_{i=1}^N z_i$$

and we obtain h_{24}, h_{25} and h_{26}.

(g) We describe the polynomials of the form $h_i(z) = |z_1|^2 p(z_2, \dots, z_N)$ where $p(z_2, \dots, z_N)$ has degree two in $z_2, \dots, z_N$, degree one in $\overline{z}_2, \dots, \overline{z}_N$ and is S_{N-1}-invariant. We have $p_{27}(z) = \sum_{i=1}^N z_i \sum_{j=1}^N z_j \sum_{k=1}^N \overline{z}_k$ and

$$p_{28}(z) = \sum_{i=1}^N z_i^2 \sum_{k=1}^N \overline{z}_k, \qquad p_{29}(z) = \sum_{k=1}^N |z_k|^2 z_k, \qquad p_{30}(z) = \sum_{k=1}^N |z_k|^2 \sum_{j=1}^N z_j.$$

We obtain h_{27}, h_{28}, h_{29} and h_{30}.

(h) We describe the polynomials of the form $h_i(z) = |z_1|^2 z_1 p(z_2, \dots, z_N)$ where $p(z_2, \dots, z_N)$ has degree two in $z, \overline{z}$. We list now the $\mathbf{S}_{N-1}$-invariant polynomials which have degree one in z and $\overline{z}$:

$$p_{31}(z) = \sum_{j=1}^N |z_j|^2, \qquad p_{32}(z) = \sum_{i=1}^N z_i \sum_{j=1}^N \overline{z}_j.$$

We obtain h_{31} and h_{32}.

(i) We describe the polynomials of the form $h_i(z) = |z_1|^2 z_1^2 p(z_2, \ldots, z_N)$ where $p(z_2, \ldots, z_N)$ has degree one in $\overline{z}_2, \ldots, \overline{z}_N$. We get $p_{33}(z) = \sum_{k=1}^{N} \overline{z}_k$ and we obtain h_{33}.

(j) We describe the polynomials of the form $h_i(z) = |z_1|^2 \overline{z}_1 p(z_2, \ldots, z_N)$ where $p(z_2, \ldots, z_N)$ has degree two in $z_2, \ldots, z_N$. We have $p_{34}(z) = \sum_{i=1}^{N} z_i^2$ and $p_{35}(z) = \sum_{i=1}^{N} z_i \sum_{j=1}^{N} z_j$ and so we get h_{34} and h_{35}.

(k) We describe the polynomials of the form $h_i(z) = |z_1|^4 p(z_2, \ldots, z_N)$ where $p(z_2, \ldots, z_N)$ has degree one in $z_2, \ldots, z_N$. We have the only polynomial $p_{36}(z) = \sum_{i=1}^{N} z_i$ and we get h_{36}.

(l) We describe the polynomials h_i of degree five in $z, \overline{z}$ which are S_{N-1}-invariant in the last $N-1$ coordinates and satisfy (4.13) and we get $h_{37}, \ldots, h_{52}$.

$\square$

Theorem 4.6 *Suppose $N \geq 4$. Consider the action of $\mathbf{S}_N \times \mathbf{S}^1$ on $\mathbf{C}^{N,0}$ defined by (4.1) and (4.3). Let $f : \mathbf{C}^{N,0} \to \mathbf{C}^{N,0}$ be $\mathbf{S}_N \times \mathbf{S}^1$-equivariant with polynomial components of degree less or equal than 3. Then*

$$f(z) = aF_1(z) + bF_2(z) + cF_3(z) + dF_4(z) \tag{4.16}$$

where $a, b, c, d \in \mathbf{C}$,

$$F_1(z_1, z_2, \ldots, z_N) = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix},$$

$$F_2(z_1, z_2, \ldots, z_N) = \left[\begin{pmatrix} |z_1|^2 z_1 \\ |z_2|^2 z_2 \\ \vdots \\ |z_N|^2 z_N \end{pmatrix} - \frac{1}{N} \sum_{k=1}^{N} |z_k|^2 z_k \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right],$$

$$F_3(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N} z_i^2 \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_N \end{pmatrix},$$

$$F_4(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N} |z_i|^2 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$$

and $z_N = -z_1 - \cdots - z_{N-1}$.

Proof: We restrict to $\mathbf{C}^{N,0}$ and project onto $\mathbf{C}^{N,0}$ the $\mathbf{S}_N \times \mathbf{S}^1$-equivariant functions from $\mathbf{C}^N$ to $\mathbf{C}^N$ obtained in Theorem 4.2. Note that if $z \in \mathbf{C}^{N,0}$ then $z_1 + \cdots + z_N = 0$ and $\overline{z}_1 + \cdots + \overline{z}_N = 0$. The nonzero functions obtained in this way are the ones corresponding to the first component being h_0, h_1, h_4 and h_6.

$\square$

Remark 4.7 From Theorem 4.6 we have that the number of linearly independent cubic $\mathbf{S}_N \times \mathbf{S}^1$-equivariants on $\mathbf{C}^{N,0}$ (over $\mathbf{C}$) is 3 for $N \geq 4$. This result is in agreement with [3, Theorem 6.5]. $\diamond$

Remark 4.8 Any polynomial function $p : \mathbf{C}^{N,0} \to \mathbf{R}$ invariant under $\mathbf{S}_N \times \mathbf{S}^1$ of degree 2 is a scalar multiple of

$$I_1 = \sum_{j=1}^{N} |z_j|^2 \tag{4.17}$$

where $z_N = -z_1 - z_2 - \cdots - z_{N-1}$. This follows from the restriction of I_1 and I_2 of Remark 4.4 to $\mathbf{C}^{N,0}$. $\diamond$

Remark 4.9 Observe that for $N = 3$ we have $F_3(z_1, z_2, z_3) = 6F_2(z_1, z_2, z_3) - 2F_4(z_1, z_2, z_3)$ for $(z_1, z_2, z_3) \in \mathbf{C}^{3,0}$ and so we obtain (over the complex field) only two linearly independent cubic $\mathbf{S}_3 \times \mathbf{S}^1$-equivariants, as it is known. See for example Dias and Paiva [9]. $\diamond$

Theorem 4.10 *Suppose $N \geq 4$. Consider the action of $\mathbf{S}_N \times \mathbf{S}^1$ on $\mathbf{C}^{N,0}$ defined by (4.1) and (4.3). Let $f : \mathbf{C}^{N,0} \to \mathbf{C}^{N,0}$ be $\mathbf{S}_N \times \mathbf{S}^1$-equivariant with homogeneous polynomial components of degree* 5*. Then f is a linear combination of $F_5, \ldots, F_{16}$ where*

$$F_5(z_1, z_2, \ldots, z_N) = \left[\begin{pmatrix} |z_1|^4 z_1 \\ |z_2|^4 z_2 \\ \vdots \\ |z_N|^4 z_N \end{pmatrix} - \frac{1}{N} \sum_{k=1}^{N} |z_k|^4 z_k \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right],$$

$$F_6(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N} |z_i|^4 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix},$$

$$F_7(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N} z_i^2 \sum_{j=1}^{N} \overline{z}_j^2 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix},$$

$$F_8(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} |z_j|^2 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix},$$

$$F_9(z_1, z_2, \ldots, z_N) = \left[\sum_{j=1}^{N} |z_j|^2 \overline{z}_j \begin{pmatrix} z_1^2 \\ z_2^2 \\ \vdots \\ z_N^2 \end{pmatrix} - \frac{1}{N} \sum_{k=1}^{N} z_i^2 \sum_{j=1}^{N} |z_j|^2 \overline{z}_j \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right],$$

$$F_{10}(z_1, z_2, \ldots, z_N) = \left[\sum_{j=1}^{N} \overline{z}_j^2 \begin{pmatrix} z_1^3 \\ z_2^3 \\ \vdots \\ z_N^3 \end{pmatrix} - \frac{1}{N} \sum_{k=1}^{N} z_i^3 \sum_{j=1}^{N} \overline{z}_j^2 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right],$$

$$F_{11}(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N} |z_i^2| \sum_{j=1}^{N} z_j^2 \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_N \end{pmatrix},$$

$$F_{12}(z_1, z_2, \ldots, z_N) = \sum_{i=1}^{N} |z_i^2| z_i^2 \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_N \end{pmatrix},$$

$$F_{13}(z_1, z_2, \ldots, z_N) = \left[\sum_{j=1}^{N} z_j^3 \begin{pmatrix} \overline{z}_1^2 \\ \overline{z}_2^2 \\ \vdots \\ \overline{z}_N^2 \end{pmatrix} - \frac{1}{N} \sum_{k=1}^{N} \overline{z}_i^2 \sum_{j=1}^{N} z_j^3 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right],$$

$$F_{14}(z_1, z_2, \ldots, z_N) = \left[\sum_{k=1}^{N} |z_k|^2 z_k \begin{pmatrix} |z_1|^2 \\ |z_2|^2 \\ \vdots \\ |z_N|^2 \end{pmatrix} - \frac{1}{N} \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} |z_j|^2 z_j \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right],$$

$$F_{15}(z_1, z_2, \ldots, z_N) = \left[\sum_{k=1}^{N} |z_k|^2 \begin{pmatrix} |z_1|^2 z_1 \\ |z_2|^2 z_2 \\ \vdots \\ |z_N|^2 z_N \end{pmatrix} - \frac{1}{N} \sum_{i=1}^{N} |z_i|^2 z_i \sum_{j=1}^{N} |z_j|^2 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right],$$

$$F_{16}(z_1, z_2, \ldots, z_N) = \left[\sum_{k=1}^{N} z_k^2 \begin{pmatrix} |z_1|^2 \overline{z}_1 \\ |z_2|^2 \overline{z}_2 \\ \vdots \\ |z_N|^2 \overline{z}_N \end{pmatrix} - \frac{1}{N} \sum_{i=1}^{N} |z_i|^2 \overline{z}_i \sum_{j=1}^{N} z_j^2 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right]$$

and $z_N = -z_1 - \cdots - z_{N-1}$.

Proof: We restrict to $\mathbf{C}^{N,0}$ and project onto $\mathbf{C}^{N,0}$ the $\mathbf{S}_N \times \mathbf{S}^1$-equivariant functions from $\mathbf{C}^N$ to $\mathbf{C}^N$ obtained in Theorem 4.5. Note that if $z \in \mathbf{C}^{N,0}$ then $z_1 + \cdots + z_N = 0$ and $\overline{z}_1 + \cdots + \overline{z}_N = 0$. The nonzero functions obtained in this way are the ones corresponding to the first component being h_1, h_2, h_3, h_4, h_{12}, h_{15}, h_{18}, h_{19}, h_{24}, h_{29}, h_{31} and h_{34}.

□

Remark 4.11 Any polynomial function $p : \mathbf{C}^{N,0} \to \mathbf{R}$ invariant under $\mathbf{S}_N \times \mathbf{S}^1$ of degree four is a linear combination of the following $\mathbf{S}_N \times \mathbf{S}^1$-invariants:

$$I_3 = \sum_{i=1}^{N} z_i^2 \sum_{k=1}^{N} \overline{z}_k^2, \quad I_5 = \sum_{j=1}^{N} |z_j|^4, \quad I_6 = \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} |z_j|^2, \tag{4.18}$$

where $z_N = -z_1 - z_2 - \ldots - z_{N-1}$. ◇

Remark 4.12 For $N = 5$ we have

$$\begin{aligned} F_{13}(z) = \; & 30F_5(z) - \tfrac{9}{2}F_6(z) + \tfrac{3}{4}F_7(z) + \tfrac{3}{2}F_8(z) - 3F_9(z) - \tfrac{3}{2}F_{10}(z) + \\ & \tfrac{3}{2}F_{11}(z) - 3F_{12}(z) - 6F_{14}(z) - 9F_{15}(z) - \tfrac{9}{2}F_{16}(z) \end{aligned}$$

where $z = (z_1, z_2, z_3, z_4, z_5)$ and so we obtain (over the complex field) only eleven linearly independent $\mathbf{S}_5 \times \mathbf{S}^1$-equivariants with homogeneous polynomial components of degree five. ◇

4.4 Periodic Solutions with Maximal Isotropy

Consider the system of ODEs

$$\frac{dz}{dt} = f(z, \lambda), \tag{4.19}$$

where $f : \mathbf{C}^{N,0} \times \mathbf{R} \to \mathbf{C}^{N,0}$ is smooth, commutes with $\Gamma = \mathbf{S}_N$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0, \rho(0) = 1$ and $\sigma'(0) \neq 0$.

If we suppose that the Taylor series of degree five of f around $z = 0$ commutes also with $\mathbf{S}^1$, then by Theorems 4.6 and 4.10 we can write $f = (f_1, f_2, \ldots, f_N)$, where

$$\begin{aligned}
&f_1(z_1, \ldots, z_N, \lambda) = \mu(\lambda) z_1 + f_1^{(3)}(z_1, \ldots, z_N, \lambda) + f_1^{(5)}(z_1, \ldots, z_N, \lambda) + \cdots \\
&f_2(z_1, \ldots, z_N, \lambda) = f_1(z_2, z_1, \ldots, z_N, \lambda) \\
&\cdots \\
&f_N(z_1, \ldots, z_N, \lambda) = f_1(z_N, z_2, \ldots, z_1, \lambda)
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
f_1^{(3)}(z_1, \ldots, z_N, \lambda) \;=\;& A_1 \left[|z_1|^2 z_1 - \tfrac{1}{N} \textstyle\sum_{k=1}^N |z_k|^2 z_k \right] \; + \\
& A_2 \, \overline{z}_1 \textstyle\sum_{k=1}^N z_k^2 \; + \; A_3 \, z_1 \sum_{k=1}^N |z_k|^2 \\[1em]
f_1^{(5)}(z_1, \ldots, z_N, \lambda) \;=\;& A_4 \left[|z_1|^4 z_1 - \tfrac{1}{N} \textstyle\sum_{k=1}^N |z_k|^4 z_k \right] + A_5 z_1 \textstyle\sum_{i=1}^N |z_i|^4 + \\
& A_6 z_1 \textstyle\sum_{i=1}^N z_i^2 \sum_{j=1}^N \overline{z}_j^2 + A_7 z_1 \sum_{i=1}^N |z_i|^2 \sum_{j=1}^N |z_j|^2 + \\
& A_8 \left[z_1^2 \textstyle\sum_{j=1}^N |z_j|^2 \overline{z}_j - \tfrac{1}{N} \sum_{k=1}^N z_i^2 \sum_{j=1}^N |z_j|^2 \overline{z}_j \right] + \\
& A_9 \left[z_1^3 \textstyle\sum_{j=1}^N \overline{z}_j^2 - \tfrac{1}{N} \sum_{k=1}^N z_i^3 \sum_{j=1}^N \overline{z}_j^2 \right] + \\
& A_{10} \left[\overline{z}_1 \textstyle\sum_{i=1}^N |z_i^2| \sum_{j=1}^N z_j^2 \right] \; + \; A_{11} \overline{z}_1 \textstyle\sum_{i=1}^N |z_i^2| z_i^2 + \\
& A_{12} \left[\overline{z}_1^2 \textstyle\sum_{j=1}^N z_j^3 - \tfrac{1}{N} \sum_{k=1}^N \overline{z}_i^2 \sum_{j=1}^N z_j^3 \right] + \\
& A_{13} \left[|z_1|^2 \textstyle\sum_{k=1}^N |z_k|^2 z_k - \tfrac{1}{N} \sum_{i=1}^N |z_i|^2 \sum_{j=1}^N |z_j|^2 z_j \right] + \\
& A_{14} \left[|z_1|^2 z_1 \textstyle\sum_{k=1}^N |z_k|^2 - \tfrac{1}{N} \sum_{i=1}^N |z_i|^2 z_i \sum_{j=1}^N |z_j|^2 \right] + \\
& A_{15} \left[|z_1|^2 \overline{z}_1 \textstyle\sum_{k=1}^N z_k^2 - \tfrac{1}{N} \sum_{i=1}^N |z_i|^2 \overline{z}_i \sum_{j=1}^N z_j^2 \right]
\end{aligned}$$

with $z_N = -z_1 - \cdots - z_{N-1}$. The coefficients A_i, for $i = 1, \ldots, 15$ are complex smooth functions of λ, $\mu(0) = i$ and $\mathrm{Re}(\mu'(0)) \neq 0$. Suppose that $\mathrm{Re}(\mu'(0)) > 0$. Rescaling λ if necessary we can suppose that

$$\mathrm{Re}(\mu(\lambda)) = \lambda + \cdots$$

where $+\cdots$ stands for higher order terms in λ. Thus the trivial solution of (4.19) is stable for λ negative and unstable for λ positive (near zero).

Throughout, subscripts r and i on the coefficients $A_1, \ldots, A_{15}$ refer to real and imaginary parts.

Isotropy Subgroup	Fixed-Point Subspace
$\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ $N = kq + p,\ 2 \le k \le N,$ $q \ge 1, p \ge 0$	$\left\{ \left(\underbrace{z_1, \ldots}_{q}; \underbrace{\xi z_1, \ldots}_{q}; \ldots; \underbrace{\xi^{k-1} z_1, \ldots}_{q}; \underbrace{0, \ldots, 0}_{p} \right) : z_1 \in \mathbf{C} \right\}$
$\Sigma^{II}_q = \mathbf{S}_q \times \mathbf{S}_p$ $N = q + p,\ 1 \le q < \frac{N}{2}$	$\left\{ \left(\underbrace{z_1, \ldots, z_1}_{q}; \underbrace{-\frac{q}{p} z_1, \ldots, -\frac{q}{p} z_1}_{p} \right) : z_1 \in \mathbf{C} \right\}$

Table 4.2: $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ and fixed-point subspaces. Here $\xi = e^{2\pi i/k}$.

Isotropy Subgroup	Branching Equations
$\Sigma^I_{q,p},\ 2 < k \le N$ $N = kq + p,$ $q \ge 1, p \ge 0$	$\nu(\lambda) + (A_1 + kqA_3)\lvert z\rvert^2 + \cdots = 0$
$\Sigma^I_{q,p}\ ,\ k = 2$ $N = 2q + p,$ $q \ge 1, p \ge 0$	$\nu(\lambda) + [A_1 + 2q(A_2 + A_3)]\lvert z\rvert^2 + \cdots = 0$
Σ^{II}_q $N = q + p,$ $1 \le q < \frac{N}{2}$	$\nu(\lambda) + A_1 \left[1 - \frac{q}{N}\left(1 - \frac{q^2}{p^2}\right)\right] \lvert z\rvert^2 + (A_2 + A_3) q \left(1 + \frac{q}{p}\right) \lvert z\rvert^2 + \cdots = 0$

Table 4.3: Branching equations for $\mathbf{S}_N \times \mathbf{S}^1$ Hopf bifurcation. Here $\nu(\lambda) = \mu(\lambda) - (1+\tau)i$ and $+\cdots$ stands for higher order terms.

Isotropy Subgroup	Branching Equations
$\Sigma^I_{q,p}$, $2 < k \leq N$ $N = kq + p$, $q \geq 1, p \geq 0$	$\lambda \;=\; -\,(A_{1r} + kqA_{3r})\lvert z\rvert^2 \;+\; \cdots$
$\Sigma^I_{q,p}$, $k = 2$ $N = 2q + p$, $q \geq 1, p \geq 0$	$\lambda \;=\; -\,[A_{1r} + 2q(A_{2r} + A_{3r})]\lvert z\rvert^2 \;+\; \cdots$
Σ^{II}_q $N = q + p$, $1 \leq q < \frac{N}{2}$	$\lambda \;=\; -\,A_{1r}\left[1 - \frac{q}{N}\left(1 - \frac{q^2}{p^2}\right)\right]\lvert z\rvert^2 \;-\; (A_{2r} + A_{3r})q\left(1 + \frac{q}{p}\right)\lvert z\rvert^2 + \cdots$

Table 4.4: Branching equations for $\mathbf{S}_N$ Hopf bifurcation. Subscript r on the coefficients refer to the real part and $+\cdots$ stands for higher order terms.

Theorem 4.13 *Consider the system (4.19) where f is as in (4.20). Assume that* $\mathrm{Re}(\mu'(0)) > 0$*, such that the trivial equilibrium is stable if $\lambda < 0$ and it is unstable if $\lambda > 0$ (near the origin). For each type of the isotropy subgroups of the form $\Sigma^I_{q,p}$ and Σ^{II}_q listed in Table 4.2, let $\Delta_0, \ldots, \Delta_r$ be the functions of $A_1, \ldots, A_{15}$ listed in Tables 4.5, 4.6 and 4.7 evaluated at $\lambda = 0$. Then:*

(1) For each Σ_i the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$. Tables 4.3 and 4.4 list the branching equations.

(2) For each Σ_i, if $\Delta_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all j, then the branch of periodic solutions is stable near $\lambda = 0$ and $z = 0$.

Proof: Our aim is to study periodic solutions of (4.19) obtained by Hopf bifurcation from the trivial equilibrium. Note that we are assuming that f satisfies the conditions of the Equivariant Hopf Theorem.

From Proposition 5.1 we have (up to conjugacy) the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. See Table 4.2. Therefore, we can use the Equivariant Hopf Theorem to prove the existence of periodic solutions with these symmetries for a bifurcation problem with symmetry $\Gamma = \mathbf{S}_N$.

Periodic solutions of (4.19) of period $2\pi/(1+\tau)$ are in one-to-one correspondence with the zeros of $g(z, \lambda, \tau)$, the reduced function obtained by the Lyapunov-Schmidt procedure where τ is the period-perturbing parameter. Moreover, by Theorem 2.27, assuming that f commutes with $\Gamma \times \mathbf{S}^1$, $g(z, \lambda, \tau)$ has the explicit form

$$g(z, \lambda, \tau) \;=\; f(z, \lambda) - (1 + \tau)iz. \tag{4.21}$$

Isotropy Subgroup	Δ_0
$\Sigma^I_{q,p},\ 2 < k \leq N$ $N = kq + p,\ q \geq 1, p \geq 0$	$A_{1r} + kqA_{3r}$
$\Sigma^I_{q,N-2q},\ k = 2$ $N = 2q + p,\ q \geq 1, p \geq 0$	$A_{1r} + 2q(A_{2r} + A_{3r})$
Σ^{II}_q $N = q + p,\ 1 \leq q < \frac{N}{2}$	$A_{1r}\left[1 - \frac{q}{N}\left(1 - \frac{q^2}{p^2}\right)\right] + (A_{2r} + A_{3r})q\left(1 + \frac{q}{p}\right)$

Table 4.5: Stability for $\mathbf{S}_N$ Hopf bifurcation in the direction of $W_0 = \mathrm{Fix}(\Sigma)$.

Throughout denote by $\nu(\lambda) = \mu(\lambda) - (1+\tau)i$. By Corollary 2.28, if $z(t)$ is a periodic solution of (4.19) with $\lambda = \lambda_0$ and $\tau = \tau_0$, and (z_0, λ_0, τ_0) is the corresponding solution of (4.21), then there is a correspondence between the Floquet multipliers of $z(t)$ and the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ such that a multiplier lies inside (respectively outside) the unit circle if and only if the corresponding eigenvalue has negative (respectively positive) real part. So, we determine the stability of each type of bifurcating periodic orbit by calculating the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ (to the lowest order in z).

Recall Table 4.2. As g commutes with $\Gamma \times \mathbf{S}^1$, it maps $\mathrm{Fix}(\Sigma)$ into itself (where Σ is either of type $\Sigma^I_{q,p}$ or Σ^{II}_q). By the Equivariant Hopf Theorem, for each of the conjugacy classes $\Sigma^I_{q,p}$ and Σ^{II}_q in Table 4.1, we have a distinct branch of periodic solutions of (4.19) that are in correspondence with the zeros of g with isotropy $\Sigma^I_{q,p}$ and Σ^{II}_q. These zeros are found by solving $g|_{\mathrm{Fix}(\Sigma^I_{q,p})} = 0$ and $g|_{\mathrm{Fix}(\Sigma^{II}_q)} = 0$ (and $\mathrm{Fix}(\Sigma^I_{q,p}), \mathrm{Fix}(\Sigma^{II}_q)$ are two-dimensional). Note that to find the zeros of g, it suffices to look at representative points on $\Gamma \times \mathbf{S}^1$ orbits. See Tables 4.3 and 4.4.

Let $\Sigma_{z_0} \subset \Gamma$ be the isotropy subgroup of z_0. Then, for $\sigma \in \Sigma_{z_0}$ we have

$$(dg)_{z_0}\sigma = \sigma(dg)_{z_0}. \tag{4.22}$$

That is, $(dg)_{z_0}$ commutes with the isotropy subgroup Σ of z_0.

For the two types of isotropy subgroups $\Sigma^I_{q,p}$ and Σ^{II}_q in Table 4.1, it is possible to put the Jacobian matrix $(dg)_{z_0}$ into block diagonal form. We do this by decomposing $\mathbf{C}^{N,0}$ into subspaces, each of which is invariant under a different representation of the corresponding isotropy subgroup. The isotypic components for the action of $\Sigma^I_{q,p}$ and Σ^{II}_q on $\mathbf{C}^{N,0}$ are listed in Table 4.8.

Specifically, for $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ we form the isotypic decomposition

$$\mathbf{C}^{N,0} = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus \sum_{j=2}^{k-1} P_j \tag{4.23}$$

Isotropy Subgroup	$\Delta_1, \ldots, \Delta_r$
$\Sigma^I_{p,q}, k=2$ $N = 2q+p$ $q \geq 1, p \geq 0$	$\left(1-\frac{4q}{N}\right) A_{1r} - 2qA_{2r},\ \text{if } p \geq 1$ $-\left\vert\left(1-\frac{4q}{N}\right) A_1 - 2qA_2\right\vert^2 + \left\vert\left(1-\frac{2q}{N}\right) A_1 + 2qA_2\right\vert^2,\ \text{if } p \geq 1$ $-A_{1r} - 2qA_{2r},\ \text{if } p > 1$ $-\left(\vert A_1 + 2qA_2\vert^2 - \vert 2qA_2\vert^2\right),\ \text{if } p > 1$ $A_{1r} - 2qA_{2r},\ \text{if } q \geq 2$ $-\left(\vert A_1 - 2qA_2\vert^2 - \vert A_1 + 2qA_2\vert^2\right),\ \text{if } q \geq 2$
$\Sigma^I_{p,q},\ k=3$ $N = 3q+p,$ $q \geq 1, p \geq 0$	$\left(1-\frac{6q}{N}\right) A_{1r},\ \text{if } p \geq 1$ $-\vert\left(1-\frac{6q}{N}\right) A_1\vert^2 + \vert A_1\vert^2,\ \text{if } p \geq 1$ $A_{1r},\ \text{if } p > 1$ $-\vert A_1\vert^2,\ \text{if } p > 1$ $-\left(-3q+\frac{6q}{N}\right) \text{Re}(A_1\overline{A}_{12}),\ \text{if } q \geq 2$ $A_{1r} + 6A_{2r}$ $-\left(\vert A_1 + 6A_2\vert^2 - \vert\left(1-\frac{3}{N}\right) A_1\vert^2\right)$
Σ^{II}_q $N = q+p,\ 1 \leq q \leq \frac{N}{2}$	$\left(1+\frac{q}{N}-\frac{q^3}{Np^2}\right) A_{1r} - q\left(1+\frac{q}{p}\right) A_{2r}$ $-\left(\left\vert\left(1+\frac{q}{N}-\frac{q^3}{Np^2}\right) A_1 - q\left(1+\frac{q}{p}\right) A_2\right\vert^2 - \left\vert A_1 + q\left(1+\frac{q}{p}\right) A_2\right\vert^2\right)$ $\left(-1+\frac{q}{N}-\frac{q^3}{Np^2}+\frac{2q^2}{p^2}\right) A_{1r} - q\left(1+\frac{1}{p}\right) A_{2r}$ $-\left(\left\vert\left(-1+\frac{q}{N}-\frac{q^3}{Np^2}+\frac{2q^2}{p^2}\right) A_1 - q\left(1+\frac{1}{p}\right) A_2\right\vert^2 - \left\vert\frac{q^2}{p^2} A_1 + q\left(1+\frac{q}{p}\right) A_2\right\vert^2\right)$

Table 4.6: Stability for $\mathbf{S}_N$ Hopf bifurcation.

Isotropy Subgroup	$\Delta_1, \ldots, \Delta_r$
	$\left(1 - \frac{2kq}{N}\right) A_{1r}$, if $p \geq 1$
$\Sigma^I_{p,q},\ 3 < k \leq N$	$-\lvert \left(1 - \frac{2kq}{N}\right) A_1\rvert^2 + \lvert A_1\rvert^2$, if $p \geq 1$
$N = kq + p$	A_{1r}, if $p > 1$
$q \geq 1,\ p \geq 0$	$-\lvert A_1\rvert^2$, if $p > 1$
	If $k > 3, q \geq 2$ then the fifth degree truncation is too degenerate to determine the stability in the directions in W_3
	A_{1r}
	$-\left(\lvert A_1\rvert^2 - \lvert \left(1 - \frac{kq}{N}\right) A_1\rvert^2\right)$
	$A_{1r} + 2kqA_{2r}$, if $k \geq 4$
	$-(\lvert A_1 + 2kqA_2\rvert^2 - \lvert A_1\rvert^2)$, if $k \geq 4$
	$-\mathrm{Re}(A_1\overline{\xi}_1) + \mathrm{Re}(2A_1\overline{A}_4 + kqA_1\overline{A}_{14})$, if $k \geq 5$
	$-\mathrm{Re}(A_1\overline{\xi}_2) + \mathrm{Re}(2A_1\overline{A}_4 + kqA_1\overline{A}_{14})$, if $k \geq 6$

Table 4.7: Stability for $\mathbf{S}_N$ Hopf bifurcation. Here $\xi_1 = 2A_4 + 3kqA_{12} + q(kq-1)\left(2 - \frac{2kq}{N}\right) A_{13} + kqA_{14} + q(kq-1)\left(1 - \frac{2kq}{N}\right) A_{14} + 2q(kq-1)A_{15}$ and $\xi_2 = \xi_1 - 3kqA_{12} - kqA_{14}$.

where $W_0 = \mathrm{Fix}(\Sigma^I_{q,p}), W_1$ and the $k-2$ subspaces $P_j, j = 2, \cdots, k-1$ are complex one-dimensional subspaces, invariant under $\Sigma^I_{q,p}$. Moreover, W_2 and W_3 are complex invariant subspaces of dimension respectively $p-1$ and $k(q-1)$ that are the sum of two isomorphic real absolutely irreducible representations of dimension respectively $p-1$ and $k(q-1)$ of $\Sigma^I_{q,p}$.

Note that if $p = 0$ we have $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k}$ and then W_1 does not occur in the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$. Moreover, we only have the occurrence of W_2 in the isotypic decomposition if $p \geq 2$. Furthermore, we only have the isotypic component W_3 if $q \geq 2$ and P_j if $k > 2$.

For $\Sigma^{II}_q = \mathbf{S}_q \times \mathbf{S}_p$ we form the isotypic decomposition

$$\mathbf{C}^{N,0} = W_0 \oplus W_1 \oplus W_2 \tag{4.24}$$

where $W_0 = \mathrm{Fix}(\Sigma^{II}_q)$ and W_1, W_2 are complex invariant subspaces of dimension respectively $q-1$ and $p-1$ that are the sum of two isomorphic real absolutely irreducible representations of dimension respectively $q-1$ and $p-1$ of Σ^{II}_q.

Note that as the group action forces some of the Floquet multipliers to be equal to one, it also forces the corresponding eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to be equal to zero. (Recall [23, Theorem XVI 6.2].) The eigenvectors associated with these eigenvalues are the tangent vectors to the orbit of $\mathbf{S}_N \times \mathbf{S}^1$ through z_0. If the solution z_0 has symmetry Σ_{z_0}, then the group orbit has the dimension of $\left(\Gamma \times \mathbf{S}^1\right)/\Sigma_{z_0}$ and so the number of zero eigenvalues of

Type of Isotropy Subgroup	Isotypic components
$\Sigma^I_{q,p}$	$W_0 = \{(\underbrace{z_1,\dots,z_1}_{q};\underbrace{\xi z_1,\dots,\xi z_1}_{q};\dots;\underbrace{\xi^{k-1}z_1,\dots,\xi^{k-1}z_1}_{q};\underbrace{0,\dots,0}_{p}) : z_1\in\mathbf{C}\}$
$N=kq+p$, $2\le k\le N$	$W_1 = \{(\underbrace{z_1,\dots,z_1}_{kq};\underbrace{-\frac{kq}{p}z_1,\dots,-\frac{kq}{p}z_1}_{p}) : z_1\in\mathbf{C}\}$ if $p\ge 1$
$q\ge 1,\ p\ge 0$	$W_2 = \{(0,\dots,0;\underbrace{z_1,\dots,z_{p-1},-z_1-\cdots-z_{p-1}}_{p}) : z_1,\dots,z_{p-1}\in\mathbf{C}\}$ if $p\ge 2$
	$W_3 = \{(\underbrace{z_1,\dots,z_{q-1},z_q}_{q};\dots;\underbrace{z_{q(k-1)+1},\dots,z_{kq-1},z_{kq}}_{q};\underbrace{0,\dots,0}_{p})\}$ if $q\ge 2$
	$P_j = \{(\underbrace{z_1,\dots}_{q};\underbrace{\xi^j z_1,\dots}_{q};\dots;\underbrace{\xi^{j(k-1)}z_1,\dots}_{q};\underbrace{0,\dots,0}_{p}) : z_1\in\mathbf{C}\}$ and $j=2,\dots,k-1$
Σ^{II}_q	$W_0 = \mathrm{Fix}(\Sigma^{II}_q) = \left\{\left(\underbrace{z_1,\dots,z_1}_{q};\underbrace{-\frac{q}{p}z_1,\dots,-\frac{q}{p}z_1}_{p}\right) : z_1\in\mathbf{C}\right\}$
$N=q+p$, $1\le q<\frac{N}{2}$	$W_1 = \{(z_1,\dots,z_{q-1},-z_1-\cdots-z_{q-1},0,\dots,0) : z_1,\dots,z_{q-1}\in\mathbf{C}\}$ if $q>1$
	$W_2 = \{(0,\dots,0,z_{q+1},\dots,z_{N-1},-z_{q+1}-\cdots-z_{N-1}) : z_{q+1},\dots,z_{N-1}\in\mathbf{C}\}$ if $p>1$

Table 4.8: Isotypic components of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$ and Σ^{II}_q. Here, in W_3 we have $z_q = -z_1-\cdots-z_{q-1},\dots,z_{kq} = -z_{q(k-1)+1}-\cdots-z_{kq-1}$ and $z_1,\dots,z_{q-1},\dots,z_{q(k-1)+1},\dots,z_{kq-1}\in\mathbf{C}$.

$(dg)_{(z_0,\lambda_0,\tau_0)}$ forced by the group action is

$$d_{\Sigma_{z_0}} = 1-\dim(\Sigma_{z_0})$$

since $\dim\left(\mathbf{S}_N\times\mathbf{S}^1\right)=1$. Now, since in our case, the groups $\Sigma_{z_0}=\Sigma^I_{q,p}$ and $\Sigma_{z_0}=\Sigma^{II}_q$ are discrete, then there is one eigenvalue forced by the symmetry to be zero (this is, we get $d_{\Sigma_{z_0}}=1$).

To compute the eigenvalues it is convenient to use the complex coordinates. We take co-ordinate functions on $\mathbf{C}^N$

$$z_1, \overline{z}_1, z_2, \overline{z}_2, \ldots, z_N, \overline{z}_N$$

These correspond to a basis B for $\mathbf{C}^N$ with elements denoted by

$$b_1, \overline{b}_1, b_2, \overline{b}_2, \ldots, b_N, \overline{b}_N. \tag{4.25}$$

Recall that an $\mathbf{R}$-linear mapping on $\mathbf{C} \equiv \mathbf{R}^2$ has the form

$$\omega \mapsto \alpha\omega + \beta\overline{\omega} \tag{4.26}$$

where $\alpha, \beta \in \mathbf{C}$. The matrix of this mapping in these coordinates

$$M = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{4.27}$$

has

$$\mathrm{tr}(M) \;=\; 2\mathrm{Re}(\alpha) \qquad \det(M) \;=\; |\alpha|^2 - |\beta|^2 \tag{4.28}$$

The eigenvalues of this matrix are

$$\frac{\mathrm{tr}(M)}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(M)}{2}\right)^2 - \det(M)} \tag{4.29}$$

If one eigenvalue is zero, then $\det(M) = 0$ and the sign of the other eigenvalue (if it is not zero) is given by the sign of the real part of α. If M has no zero eigenvalues, then the eigenvalues have negative real part if and only if the determinant is positive and the trace is negative.

$\left(\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p, \text{where } N = qk + p,\ 2 \le k \le N,\ q \ge 1,\ p \ge 0\right)$

The fixed-point subspace of $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ is

$$\mathrm{Fix}\left(\Sigma^I_{q,p}\right) \;=\; \{(\underbrace{z, \ldots, z}_{q}; \underbrace{\xi z, \ldots, \xi z}_{q}; \ldots; \underbrace{\xi^{k-1} z, \ldots, \xi^{k-1} z}_{q}; \underbrace{0, \ldots, 0}_{p}) : z \in \mathbf{C}\}$$

where $\xi = e^{2\pi i/k}$. Using the equation (4.21) where f is as in (4.20), after dividing by z we have if $k \neq 2$

$$\nu(\lambda) + (A_1 + kqA_3)|z|^2 \;+\; \cdots \;=\; 0 \tag{4.30}$$

where $+\ \cdots$ denotes terms of higher order in z and $\overline{z}$, and taking the real part of this equation, we obtain,

$$\lambda = -(A_{1r} + kqA_{3r})|z|^2 \;+\; \cdots \tag{4.31}$$

It follows that if $A_1 + kqA_3 < 0$, then the branch bifurcates supercritically.

In the particular case $k = 2$ we have

$$\nu(\lambda) + [A_1 + 2q(A_2 + A_3)]|z|^2 \ + \ \cdots \ = \ 0 \tag{4.32}$$

and taking the real part of this equation,

$$\lambda = -[A_{1r} + 2q(A_{2r} + A_{3r})]|z|^2 + \cdots \tag{4.33}$$

where the functions A_{ir} for $i = 1, 2, 3$ are evaluated at $\lambda = 0$. It follows in this case that if $A_{1r} + 2q(A_{2r} + A_{3r}) < 0$, then the branch bifurcates supercritically.

Throughout we denote by (z_0, λ_0, τ_0) a zero of $g(z, \lambda, \tau) = 0$ with $z_0 \in \mathrm{Fix}(\Sigma)$. Specifically, we wish to calculate $(dg)_{(z_0,\lambda_0,\tau_0)}$.

Recall the generators for $\Sigma^I_{q,p}$ given in Table 4.1. With respect to the basis B, any "real" matrix commuting with $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} M_1 & M_3 & M_4 & \dots & M_{k+1} & M_{k+2} \\ M^{\xi^2}_{k+1} & M^{\xi^2}_1 & M^{\xi^2}_3 & \dots & M^{\xi^2}_k & M^{\xi^2}_{k+2} \\ \vdots & \ddots & & \vdots & & \\ M^{\xi^{2(k-1)}}_3 & & & \dots & M^{\xi^{2(k-1)}}_1 & M^{\xi^{2(k-1)}}_{k+2} \\ M_{k+3} & M^{\xi^2}_{k+3} & M^{\xi^4}_{k+3} & \dots & M^{\xi^{2(k-1)}}_{k+3} & M_{k+4} \end{pmatrix} \tag{4.34}$$

where M_1 commutes with $\mathbf{S}_q$, M_{k+4} commutes with $\mathbf{S}_p$ and the other matrices are defined below.

Suppose M is a square matrix of order a with rows $l_1, \dots, l_a$ and commuting with S_a. It follows then that $M = (l_1, (12) \cdot l_1, \cdots, (1a) \cdot l_1)^t$, where if $l_1 = (m_1, \dots, m_a)$ then $(1i) \cdot l_1 = (m_i, m_2, \dots, m_{i-1}, m_1, m_{i+1}, \dots, m_a)$. Moreover, l_1 is invariant under S_{a-1} in the last $a - 1$ entries and so it has the following form: $(m_1, m_2, \dots, m_2)$. Applying this to M_1 and M_{k+4} we get

$$M_1 \ = \ \begin{pmatrix} C_1 & C_2 & \dots & C_2 \\ C_2 & C_1 & \dots & C_2 \\ \vdots & & \ddots & \vdots \\ C_2 & C_2 & \dots & C_1 \end{pmatrix}, \quad M_{k+4} \ = \ \begin{pmatrix} C_{k+4} & C_{k+5} & \dots & C_{k+5} \\ C_{k+5} & C_{k+4} & \dots & C_{k+5} \\ \vdots & & \ddots & \vdots \\ C_{k+5} & C_{k+5} & \dots & C_{k+4} \end{pmatrix},$$

where M_1 is a $2q \times 2q$ matrix and M_{k+4} is a $2p \times 2p$ matrix.

The other symmetry restrictions on the M_i, for $i = 3, \dots, k+3$, imply that each have one identical entry,

$$M_i \ = \ \begin{pmatrix} C_i & \dots & C_i \\ & \ddots & \\ C_i & \dots & C_i \end{pmatrix}.$$

Note that each M_i for $i = 1, \dots, k+1$ is a $2q \times 2q$ matrix and M_{k+2}, M_{k+3} are, respectively, $2q \times 2p$ and $2p \times 2q$ matrices. Furthermore, we have

$$M_1^{\xi^j} = \begin{pmatrix} C_1^{\xi^j} & C_2^{\xi^j} & \dots & C_2^{\xi^j} \\ C_2^{\xi^j} & C_1^{\xi^j} & \dots & C_2^{\xi^j} \\ \vdots & & \ddots & \vdots \\ C_2^{\xi^j} & C_2^{\xi^j} & \dots & C_1^{\xi^j} \end{pmatrix}$$

for $j = 2, \dots, 2(k-1)$ and

$$M_l^{\xi^j} = \begin{pmatrix} C_l^{\xi^j} & C_l^{\xi^j} & \dots & C_l^{\xi^j} \\ C_l^{\xi^j} & C_l^{\xi^j} & \dots & C_l^{\xi^j} \\ \vdots & & \ddots & \vdots \\ C_l^{\xi^j} & C_l^{\xi^j} & \dots & C_l^{\xi^j} \end{pmatrix}$$

for $l = 3, \dots, k+3$ and $j = 2, \dots, 2(k-1)$.

Now, each C_i is of the type

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c_i'} & \overline{c_i} \end{pmatrix}, \quad C_i^{\xi^j} = \begin{pmatrix} c_i & \xi^j c_i' \\ \overline{\xi}^j \overline{c_i'} & \overline{c_i} \end{pmatrix}$$

for $i = 1, \dots, k+3, j = 2, \dots, 2(k-1)$ and

$$C_{k+2} = \begin{pmatrix} c_{k+2} & c_{k+2}' \\ \overline{c_{k+2}'} & \overline{c_{k+2}} \end{pmatrix}, \quad C_{k+4} = \begin{pmatrix} c_{k+4} & c_{k+4}' \\ \overline{c_{k+4}'} & \overline{c_{k+4}} \end{pmatrix}, \quad C_{k+5} = \begin{pmatrix} c_{k+5} & c_{k+5}' \\ \overline{c_{k+5}'} & \overline{c_{k+5}} \end{pmatrix},$$

where

$$\begin{array}{llll}
c_1 = \frac{\partial g_1}{\partial z_1}, & c_1' = \frac{\partial g_1}{\partial \overline{z}_1}, & c_2 = \frac{\partial g_1}{\partial z_2}, & c_2' = \frac{\partial g_1}{\partial \overline{z}_2}, \\
c_3 = \frac{\partial g_1}{\partial z_{q+1}}, & c_3' = \frac{\partial g_1}{\partial \overline{z}_{q+1}}, & \dots & c_{k+1} = \frac{\partial g_1}{\partial z_{q(k-1)+1}}, \quad c_{k+1}' = \frac{\partial g_1}{\partial \overline{z}_{q(k-1)+1}}, \\
c_{k+2} = \frac{\partial g_1}{\partial z_{kq+1}}, & c_{k+2}' = \frac{\partial g_1}{\partial \overline{z}_{kq+1}}, & c_{k+3} = \frac{\partial g_{kq+1}}{\partial z_1}, & c_{k+3}' = \frac{\partial g_{kq+1}}{\partial \overline{z}_1}, \\
c_{k+4} = \frac{\partial g_N}{\partial z_N}, & c_{k+4}' = \frac{\partial g_N}{\partial \overline{z}_N}, & c_{k+5} = \frac{\partial g_N}{\partial z_{N-1}}, & c_{k+5}' = \frac{\partial g_N}{\partial \overline{z}_{N-1}},
\end{array}$$

calculated at (z_0, λ_0, τ_0).

Throughout we denote by $(dg)_{(z_0,\lambda_0,\tau_0)}|W_k$ the restriction of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to the subspace W_k.

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ where

$$W_0 = \{(\underbrace{z_1, \dots, z_1}_{q}; \underbrace{\xi z_1, \dots, \xi z_1}_{q}; \dots; \underbrace{\xi^{k-1} z_1, \dots, \xi^{k-1} z_1}_{q}; \underbrace{0, \dots, 0}_{p}) : z_1 \in \mathbf{C}\}.$$

The tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector

$$(\underbrace{iz,\ldots,iz}_{q};\underbrace{i\xi z,\ldots,i\xi z}_{q};\cdots;\underbrace{i\xi^{k-1}z,\ldots,i\xi^{k-1}z}_{q};\underbrace{0,\ldots,0}_{p}).$$

Note that

$$\frac{d}{dt}(e^{it}z,\ldots,e^{it}z,\ldots,e^{it}\xi^{k-1}z,\ldots,e^{it}\xi^{k-1}z)\big|_{t=0}=(iz,\ldots,iz,\ldots,i\xi^{k-1}z,\ldots,i\xi^{k-1}z).$$

Now since $g(\mathrm{Fix}(\Sigma^I_{q,p}))\subseteq\mathrm{Fix}(\Sigma^I_{q,p})$ we have that $g(\mathrm{Fix}(\Sigma^I_{q,p}))$ is two-dimensional. Thus, $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ is as in (4.26) and the matrix of this mapping has the form (4.27). The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is given by

$$2\mathrm{Re}(\alpha)=\quad 2\mathrm{Re}(A_1+kqA_3)|z|^2+\cdots$$

if $k\geq 3$, whose sign is determined by $A_{1r}+kqA_{3r}$ if it is assumed nonzero (where $A_{1r}+kqA_{3r}$ is calculated at zero). In the particular case $k=2$, the nonzero eigenvalue is given by

$$2\mathrm{Re}(\alpha)=\quad 2\mathrm{Re}[A_1+2q(A_2+A_3)]|z|^2+\cdots$$

whose sign is determined by $A_{1r}+2q(A_{2r}+A_{3r})$ if it is assumed nonzero (where $A_{1r}+2q(A_{2r}+A_{3r})$ is calculated at zero).

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ where

$$W_1\;=\;\{(\underbrace{z_1,\ldots,z_1}_{kq};\underbrace{-\frac{kq}{p}z_1,\ldots,-\frac{kq}{p}z_1}_{p}):z_1\in\mathbf{C}\}.$$

We have $\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right)z\rightarrow\alpha z+\beta\overline{z}$ where

$$\begin{array}{l}\alpha=c_1+(q-1)c_2+qc_3-2qc_4,\\ \beta=c'_1+(q-1)c'_2+qc'_3-2qc'_4,\end{array}$$

for $k=2$. Recall that this case is special case since the branching equation is different than the one we obtain for $k\geq 3$. Thus, we study this case separately. Since

$$\begin{array}{rcl}c_1&=&\left[\left(1-\frac{2}{N}\right)A_1+(2-2q)A_2+A_3\right]|z|^2+\cdots,\\ c_2&=&\left(-\frac{2}{N}A_1+2A_2+A_3\right)|z|^2+\cdots,\\ c_3&=&\left(-\frac{2}{N}A_1-2A_2-A_3\right)|z|^2+\cdots,\\ c_4&=&0,\\ &&\\ c'_1&=&\left[\left(1-\frac{1}{N}\right)A_1+2qA_2+A_3\right]z^2+\cdots,\\ c'_2&=&\left(-\frac{1}{N}A_1+A_3\right)z^2+\cdots,\\ c'_3&=&\left(-\frac{1}{N}A_1-A_3\right)z^2+\cdots,\\ c'_4&=&0,\end{array}\tag{4.35}$$

it follows, for $k=2$ that

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = 2\mathrm{Re}\left[\left(1-\tfrac{4q}{N}\right)A_1 - 2qA_2\right]|z|^2 + \cdots ,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = \left(\left|\left(1-\tfrac{4q}{N}\right)A_1 - 2qA_2\right|^2 - \left|\left(1-\tfrac{2q}{N}\right)A_1 + 2qA_2\right|^2\right)|z|^4 + \cdots . \tag{4.36}$$

If $k \geq 3$ we have

$$\begin{aligned} \alpha &= c_1 + (q-1)c_2 + qc_3 + \cdots + qc_{k+1} - kqc_{k+2}, \\ \beta &= c_1' + (q-1)c_2' + qc_3' + \cdots + qc_{k+1}' - kqc_{k+2}'. \end{aligned}$$

Since

$$\begin{array}{lcl} c_1 &=& \left[\left(1-\frac{2}{N}\right)A_1 + 2A_2 + A_3\right]|z|^2 + \cdots , \\ c_2 &=& \left(-\frac{2}{N}A_1 + 2A_2 + A_3\right)|z|^2 + \cdots , \\ c_3 &=& \left(-\frac{2}{N}A_1 + 2\xi A_2 + \overline{\xi}A_3\right)|z|^2 + \cdots , \\ \cdots && \\ c_{k+1} &=& \left(-\frac{2}{N}A_1 + 2\xi^{k-1}A_2 + \overline{\xi^{k-1}}A_3\right)|z|^2 + \cdots , \\ c_{k+2} &=& 0, \\ && \\ c_1' &=& \left[\left(1-\frac{1}{N}\right)A_1 + A_3\right]z^2 + \cdots , \\ c_2' &=& \left(-\frac{1}{N}A_1 + A_3\right)z^2 + \cdots , \\ c_3' &=& \left(-\frac{1}{N}\xi^2 A_1 + \xi A_3\right)z^2 + \cdots , \\ \cdots && \\ c_{k+1}' &=& \left(-\frac{1}{N}\xi^{2(k-1)}A_1 + \xi^{k-1}A_3\right)z^2 + \cdots , \\ c_{k+2}' &=& 0, \end{array} \tag{4.37}$$

it follows that

$$\begin{aligned} \mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) &= 2\mathrm{Re}\left[\left(1-\tfrac{2kq}{N}\right)A_1\right]|z|^2 + \cdots , \\ \det((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) &= \left(\left|\left(1-\tfrac{2kq}{N}\right)A_1\right|^2 - |A_1|^2\right)|z|^4 + \cdots . \end{aligned} \tag{4.38}$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ where

$$W_2 \;=\; \{(0,\ldots,0;\underbrace{z_1,\ldots,z_{p-1},-z_1-\cdots-z_{p-1}}_{p}) : z_1,\ldots,z_{p-1} \in \mathbf{C}\}.$$

Recall that we only have this isotypic component in the decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$ when $p > 1$. Recall (4.9) of Section 4.2. The action of $K \subset \Sigma^I_{q,p}$ on W_2 decomposes in the following way:

$$W_2 = W_2^1 \oplus W_2^2$$

where

$$
\begin{aligned}
W_2^1 &= \{(0,\ldots,0;\underbrace{x_1,\ldots,x_{p-1},-x_1-\cdots-x_{p-1}}_{p}) : x_1,\ldots,x_{p-1}\in\mathbf{R}\},\\
W_2^2 &= \{(0,\ldots,0;\underbrace{ix_1,\ldots,ix_{p-1},-ix_1-\cdots-ix_{p-1}}_{p}) : x_1,\ldots,x_{p-1}\in\mathbf{R}\}.
\end{aligned}
$$

Moreover, the actions of K on W_2^1 and on W_2^2 are K-isomorphic and are K-absolutely irreducible. Thus, it is possible to choose a basis of W_2 such that $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the new coordinates has the form

$$
\begin{pmatrix} a\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} & b\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} \\ c\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} & d\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} \end{pmatrix} \tag{4.39}
$$

where $\mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)}$ is the $(p-1)\times(p-1)$ identity matrix. Furthermore, the eigenvalues of (4.39) are the eigenvalues of $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ each with multiplicity $p-1$.

With respect to the basis B' of W_2 given by

$$
b_{kq+1}-b_N, \bar{b}_{kq+1}-\bar{b}_N, b_{kq+2}-b_N, \bar{b}_{kq+2}-\bar{b}_N, \ldots, b_{N-1}-b_N, \bar{b}_{N-1}-\bar{b}_N
$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the following block diagonal form

$$
(dg)_{(z_0,\lambda_0,\tau_0)}|W_2 = \mathrm{diag}(C_{k+4}-C_{k+5},\ldots,C_{k+4}-C_{k+5}).
$$

The eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ are the eigenvalues of $C_{k+4}-C_{k+5}$, each with multiplicity $p-1$. The eigenvalues of $C_{k+4}-C_{k+5}$ have negative real part if and only if

$$
\mathrm{tr}(C_{k+4}-C_{k+5}) \;<\; 0 \;\wedge\; \det(C_{k+4}-C_{k+5}) \;>\; 0.
$$

If $k=2$ then

$$
\begin{aligned}
c_{k+4} &= -(A_1+2qA_2)\,|z|^2+\cdots, & c'_{k+4} &= 2qA_2z^2+\cdots,\\
c_{k+5} &= 0, & c'_{k+5} &= 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = 2\mathrm{Re}\,(-A_1-2qA_2)\,|z|^2+\cdots,\\
&\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \Big(|A_1+2qA_2|^2-|2qA_2|^2\Big)\,|z|^4+\cdots.
\end{aligned} \tag{4.40}
$$

If $k\geq 3$ then

$$
\begin{aligned}
c_{k+4} &= -A_1|z|^2+\cdots, & c'_{k+4} &= 0,\\
c_{k+5} &= 0, & c'_{k+5} &= 0,
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
&\operatorname{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots\,, \\
&\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = |A_1|^2\,|z|^4 + \cdots\,.
\end{aligned}
\tag{4.41}
$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_3$ where

$$
W_3 \;=\; \{(\underbrace{z_1,\ldots,z_{q-1},z_q}_{q};\ldots;\underbrace{z_{q(k-1)+1},\ldots,z_{kq-1},z_{kq}}_{q};\underbrace{0,\ldots,0}_{p}) : z_1,\ldots,z_{kq}\in\mathbf{C}\}
$$

with $z_q = -z_1 - \cdots - z_{q-1}, \ldots, z_{kq} = -z_{q(k-1)+1} - \cdots - z_{kq-1}$.

Recall that we only have this isotypic component in the decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$ when $q \geq 2$.

With respect to the basis B' of W_3 given by

$$
\begin{aligned}
& b_1 - b_q, \overline{b}_1 - \overline{b}_q, \ldots, b_{q-1} - b_q, \overline{b}_{q-1} - \overline{b}_q, \\
& b_{q+1} - b_{2q}, \overline{b}_{q+1} - \overline{b}_{2q}, \ldots, b_{2q-1} - b_{2q}, \overline{b}_{2q-1} - \overline{b}_{2q}, \\
& \cdots, \\
& b_{q(k-1)+1} - b_{kq}, \overline{b}_{q(k-1)+1} - \overline{b}_{kq}, \ldots, b_{kq-1} - b_{kq}, \overline{b}_{kq-1} - \overline{b}_{kq},
\end{aligned}
$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_3$ in the following block diagonal form:

$$
(dg)_{(z_0,\lambda_0,\tau_0)}|W_3 = \mathrm{diag}(\underbrace{C_1 - C_2,\ldots}_{q-1};\underbrace{C_1^{\xi^2} - C_2^{\xi^2},\ldots}_{q-1};\ldots;\underbrace{C_1^{\xi^{2(k-1)}} - C_2^{\xi^{2(k-1)}},\ldots}_{q-1}).
$$

Note that we have

$$
\begin{aligned}
&\operatorname{tr}(C_1^{\xi^j} - C_2^{\xi^j}) = \operatorname{tr}(C_1 - C_2), \\
&\det(C_1^{\xi^j} - C_2^{\xi^j}) = \det(C_1 - C_2).
\end{aligned}
$$

Recall (4.35), it follows that for $k = 2$

$$
\begin{aligned}
&\operatorname{tr}(C_1 - C_2) = 2\mathrm{Re}\,(A_1 - 2qA_2)\,|z|^2 + \cdots\,, \\
&\det(C_1 - C_2) = \left(|A_1 - 2qA_2|^2 - |A_1 + 2qA_2|^2\right)|z|^4 + \cdots\,.
\end{aligned}
\tag{4.42}
$$

Recall now (4.37). It follows that for $k \geq 3$ we have

$$
\det(C_1 - C_2) = 0.
$$

Thus, the degree three truncation is too degenerate (it originates a null eigenvalue which is not forced by the symmetry of the problem). We consider now the degree five truncation and we get that $k = 3$ is a particular case. Note that the fifth degree truncation of the branching equations are different in the cases $k = 3$ and $k > 3$, thus, we get different expressions for the derivatives. We study the case $k = 3$ first. We have

$$
\begin{aligned}
c_1 \;=\;& \left[\left(1-\tfrac{2}{N}\right)A_1 + 2A_2 + A_3\right]|z|^2 + \\
& \left[\left(2-\tfrac{3}{N}\right)A_4 + 2A_5 + 6qA_7 + A_8\right]|z|^4 + \\
& \left[6qA_{10} + 3A_{11} + (3-3q)A_{12}\right]|z|^4 + \\
& \left[\left(2-\tfrac{6q}{N}\right)A_{13} + \left(3q+1-\tfrac{6q}{N}\right)A_{14} + 2A_{15}\right]|z|^4 + \cdots, \\
c_2 \;=\;& \left(-\tfrac{2}{N}A_1 + 2A_2 + A_3\right)|z|^2 + \\
& \left(-\tfrac{3}{N}A_4 + 2A_5 + 6qA_7 + A_8 + 6qA_{10} + 3A_{11}\right)|z|^4 + \\
& \left[3A_{12} + A_{13}\left(2-\tfrac{6q}{N}\right) + A_{14}\left(1-\tfrac{6q}{N}\right) + 2A_{15}\right]|z|^4 + \cdots, \\
c_1' \;=\;& \left[\left(1-\tfrac{1}{N}\right)A_1 + A_3\right]z^2 + \\
& \left[\left(2-\tfrac{2}{N}\right)A_4 + 2A_5 + 6qA_7 + 2A_8\right]|z|^2z^2 + \\
& \left[\left(2-\tfrac{6q}{N}\right)A_9 + A_{11}\right]|z|^2z^2 + \\
& \left[\left(1-\tfrac{3q}{N}\right)A_{13} + \left(3q+1-\tfrac{3q}{N}\right)A_{14}\right]|z|^2z^2 + \cdots, \\
c_2' \;=\;& \left(-\tfrac{1}{N}A_1 + A_3\right)z^2 + \\
& \left(-\tfrac{2}{N}A_4 + 2A_5 + 6qA_7\right)|z|^2z^2 + \\
& \left[2A_8 + A_9\left(2-\tfrac{6q}{N}\right) + A_{11} - \tfrac{6q}{N}A_{12}\right]|z|^2z^2 + \\
& \left[A_{13}\left(1-\tfrac{3q}{N}\right) + A_{14}\left(1-\tfrac{3q}{N}\right)\right]|z|^2z^2 + \cdots,
\end{aligned}
\tag{4.43}
$$

it follows that

$$
\begin{aligned}
\mathrm{tr}(C_1 - C_2) =\;& 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots, \\
\det(C_1 - C_2) =\;& \left|A_1 + (2A_4 - 3qA_{12} + 3qA_{14})|z|^2\right|^2 |z|^4 - \\
& \left|A_1 + (2A_4 - \tfrac{6q}{N}A_{12} + 3qA_{14})|z|^2\right|^2 |z|^4 + \cdots = \\
& \left(-3q + \tfrac{6q}{N}\right) 2\mathrm{Re}\left(A_1\overline{A_{12}}\right)|z|^6 + \cdots.
\end{aligned}
\tag{4.44}
$$

Now, for $k > 3$ since

$$
\begin{aligned}
c_1 \;=\;& \left[\left(1-\tfrac{2}{N}\right)A_1 + 2A_2 + A_3\right]|z|^2 + \\
& \left[\left(2-\tfrac{3}{N}\right)A_4 + 2A_5 + 2kqA_7 + A_8\right]|z|^4 + \\
& \left[2kqA_{10} + 3A_{11} + 3A_{12}\right]|z|^4 + \\
& \left[\left(2-\tfrac{2kq}{N}\right)A_{13} + \left(kq+1-\tfrac{2kq}{N}\right)A_{14} + 2A_{15}\right]|z|^4 + \cdots, \\
c_2 \;=\;& \left(-\tfrac{2}{N}A_1 + 2A_2 + A_3\right)|z|^2 \\
& \left(-\tfrac{3}{N}A_4 + 2A_5 + 2kqA_7 + A_8\right)|z|^4 + \\
& \left[2kqA_{10} + 3A_{11} + 3A_{12} + A_{13}\left(2-\tfrac{2kq}{N}\right) + A_{14}\left(1-\tfrac{2kq}{N}\right) + 2A_{15}\right]|z|^4 + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
c_1' \;=\;\; & \left[\left(1-\tfrac{1}{N}\right)A_1+A_3\right]z^2+\\
& \left[\left(2-\tfrac{2}{N}\right)A_4+2A_5+2kqA_7+2A_8\right]|z_1|^2z^2+\\
& \left[2A_9+A_{11}+\left(1-\tfrac{kq}{N}\right)A_{13}+\left(kq+1-\tfrac{kq}{N}\right)A_{14}\right]|z|^2z^2+\cdots,
\end{aligned}
$$

$$
\begin{aligned}
c_2' \;=\;\; & \left(-\tfrac{1}{N}A_1+A_3\right)z^2\\
& \left(-\tfrac{2}{N}A_4+2A_5+2kqA_7\right)|z|^2z^2+\\
& \left(2A_8+2A_9+A_{11}\right)|z|^2z^2+\\
& \left[A_{13}\left(1-\tfrac{kq}{N}\right)+A_{14}\left(1-\tfrac{kq}{N}\right)\right]|z|^2z^2+\cdots,
\end{aligned}
$$

it follows that $\det(C_1-C_2)=0$. In this case, when $k>3$ and when this component appears in the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$, the five degree truncation is too degenerate in order to determine the stability of the system.

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|P_j$ where

$$
P_j \;=\; \{(\underbrace{z_1,\ldots,z_1}_{q};\underbrace{\xi^j z_1,\ldots,\xi^j z_1}_{q};\ldots;\underbrace{\ldots,\xi^{j(k-1)}z_1}_{q};\underbrace{0,\ldots,0}_{p}):z_1\in\mathbf{C}\}
$$

and $2\le j\le k-1$. We have $\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_j\right)z\to\alpha z+\beta\overline{z}$ where

$$
\begin{aligned}
\alpha &= c_1+(q-1)c_2+q\xi^j c_3+\cdots+q\xi^{(k-1)j}c_{k+1},\\
\beta &= c_1'+(q-1)c_2'+q\overline{\xi^j}c_3'+\cdots+q\overline{\xi^{(k-1)j}}c_{k+1}'.
\end{aligned}
\tag{4.45}
$$

When we substitute the expressions for the derivatives given by (4.37) in (4.45), we get that the case $k=3$ is a particular case. If $j=2$ and $k\ge 4$ we have

$$
\begin{aligned}
&\operatorname{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right)=2\mathrm{Re}\,(A_1)\,|z|^2+\cdots,\\
&\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right)=\left(|A_1|^2-\left|\left(1-\tfrac{kq}{N}\right)A_1\right|^2\right)|z|^4+\cdots,
\end{aligned}
\tag{4.46}
$$

but it the particular case $k=3$ it follows that

$$
\begin{aligned}
&\operatorname{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right)=2\mathrm{Re}\,(A_1+6A_2)\,|z|^2+\cdots,\\
&\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right)=\left(|A_1+6A_2|^2-\left|\left(1-\tfrac{3}{N}\right)A_1\right|^2\right)|z|^4+\cdots.
\end{aligned}
\tag{4.47}
$$

Consider now $j=k-1$. It follows that

$$
\begin{aligned}
&\operatorname{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-1}\right)=2\mathrm{Re}\,(A_1+2kqA_2)\,|z|^2+\cdots,\\
&\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-1}\right)=\left(|A_1+2kqA_2|^2-|A_1|^2\right)|z|^4+\cdots.
\end{aligned}
\tag{4.48}
$$

Moreover, if we consider $2<j\le k-2$, then we obtain that $\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_j\right)=0$. Thus, we need to consider the five degree truncation of (4.20). Since

$$
\begin{aligned}
c_1 \;=\;& \left[\left(1-\tfrac{2}{N}\right)A_1+2A_2+A_3\right]|z|^2\;+\\
& \left[\left(2-\tfrac{3}{N}\right)A_4+2A_5+2kqA_7+A_8+2kqA_{10}+3A_{11}\right]|z|^4+\\
& \left[3A_{12}+\left(2-\tfrac{2kq}{N}\right)A_{13}+\left(kq+1-\tfrac{2kq}{N}\right)A_{14}+2A_{15}\right]|z|^4+\cdots,\\[1ex]
c_2 \;=\;& \left(-\tfrac{2}{N}A_1+2A_2+A_3\right)|z|^2+\\
& \left[-\tfrac{3}{N}A_4+2A_5+2kqA_7+A_8+2kqA_{10}+3A_{11}\right]|z|^4\;+\\
& \left[3A_{12}+\left(2-\tfrac{2kq}{N}\right)A_{13}+\left(1-\tfrac{2kq}{N}\right)A_{14}+2A_{15}\right]|z|^4+\cdots,\\[1ex]
c_3 \;=\;& \left(-\tfrac{2}{N}A_1+2\xi A_2+\overline{\xi}A_3\right)|z|^2+\\
& \left[-\tfrac{3}{N}A_4+2\overline{\xi}A_5+2kq\overline{\xi}A_7+\overline{\xi^2}A_8+2kq\xi A_{10}+3\xi A_{11}\right]|z|^4\;+\\
& \left[3\xi^2A_{12}+\left(2-\tfrac{2kq}{N}\right)A_{13}+\left(1-\tfrac{2kq}{N}\right)A_{14}+2\xi A_{15}\right]|z|^4+\cdots,\\
\cdots&\\
c_{k+1} \;=\;& \left(-\tfrac{2}{N}A_1+2\xi^{k-1}A_2+\overline{\xi^{k-1}}A_3\right)|z|^2+\\
& \left[-\tfrac{3}{N}A_4+2\overline{\xi^{k-1}}A_5+2kq\overline{\xi^{k-1}}A_7+\overline{\xi^{2(k-1)}}A_8+2kq\xi^{k-1}A_{10}+3\xi^{k-1}A_{11}\right]|z|^4\;+\\
& \left[3\xi^{2(k-1)}A_{12}+\left(2-\tfrac{2kq}{N}\right)A_{13}+\left(1-\tfrac{2kq}{N}\right)A_{14}+2\xi^{k-1}A_{15}\right]|z|^4+\cdots,\\[1ex]
c_1' \;=\;& \left[\left(1-\tfrac{1}{N}\right)A_1+A_3\right]z^2\;+\\
& \left[\left(2-\tfrac{2}{N}\right)A_4+2A_5+2kqA_7+2A_8\right]|z|^2z^2+\\
& \left[2A_9+A_{11}+\left(1-\tfrac{kq}{N}\right)A_{13}+\left(kq+1-\tfrac{kq}{N}\right)A_{14}\right]|z|^2z^2+\cdots,\\[1ex]
c_2' \;=\;& \left(-\tfrac{1}{N}A_1+A_3\right)z^2\\
& \left[-\tfrac{2}{N}A_4+2A_5+2kqA_7+2A_8\right]|z|^2z^2+\\
& \left[2A_9+A_{11}\right]|z|^2z^2+\\
& \left[\left(1-\tfrac{kq}{N}\right)A_{13}+\left(1-\tfrac{kq}{N}\right)A_{14}\right]|z|^2z^2+\cdots,\\[1ex]
c_3' \;=\;& \left(-\tfrac{1}{N}\xi^2A_1+\xi A_3\right)z^2\;+\\
& \left[-\tfrac{2}{N}\xi^2A_4+2\xi A_5+2kq\xi A_7+2A_8\right]|z|^2z^2+\\
& \left[2\overline{\xi}A_9+\xi^3A_{11}\right]+\\
& \left[\left(1-\tfrac{kq}{N}\right)\xi^2A_{13}+\xi\left(1-\tfrac{kq}{N}\xi\right)A_{14}\right]|z|^2z^2+\cdots\\
\ldots,&\\
c_{k+1}' \;=\;& \left(-\tfrac{1}{N}\xi^{2(k-1)}A_1+\xi^{k-1}A_3\right)z^2\;+\\
& \left[-\tfrac{2}{N}\xi^{2(k-1)}A_4+2\xi^{k-1}A_5+2kq\xi^{k-1}A_7+2A_8\right]|z|^2z^2+\\
& \left[2\overline{\xi}A_9+\xi^{3(k-1)}A_{11}\right]+\\
& \left[\left(1-\tfrac{kq}{N}\right)\xi^{2(k-1)}A_{13}+\xi^{k-1}\left(1-\tfrac{kq}{N}\xi^{k-1}\right)A_{14}\right]|z|^2z^2+\cdots,
\end{aligned}
$$

it follows for $j=k-2$ (note that we only have this isotypic component when $k\geq 5$) that

$$\begin{array}{rcl}\mathrm{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-2}\right) & = & 2\mathrm{Re}\,(A_1)\,|z|^2+\cdots\,,\\ \det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-2}\right) & = & \left(|A_1+\xi_1|z|^2|^2-|A_1+(2A_4+kqA_{14})|z|^2|^2\right)|z|^4+\cdots\\ & = & [2\mathrm{Re}(A_1\overline{\xi}_1)-2\mathrm{Re}(2A_1\overline{A}_4+kqA_1\overline{A}_{14}]|z|^6+\cdots\,,\end{array} \tag{4.49}$$

where

$$\begin{array}{rl}\xi_1 = & 2A_4+3kqA_{12}+q(kq-1)\left(2-\frac{2kq}{N}\right)A_{13}+kqA_{14}+\\ & +q(kq-1)\left(1-\frac{2kq}{N}\right)A_{14}+2q(kq-1)A_{15}.\end{array}$$

Furthermore, for $3\leq j\leq k-3$ (note that we only have this isotypic component when $k\geq 6$) we get

$$\begin{array}{rcl}\mathrm{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_j\right) & = & 2\mathrm{Re}\,(A_1)\,|z|^2+\cdots\,,\\ \det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_j\right) & = & \left(|A_1+\xi_2|z|^2|^2-|A_1+(2A_4+kqA_{14})|z|^2|^2\right)|z|^4+\cdots\\ & = & [2\mathrm{Re}(A_1\overline{\xi}_2)-2\mathrm{Re}(2A_1\overline{A}_4+kqA_1\overline{A}_{14}]|z|^6+\cdots\,,\end{array} \tag{4.50}$$

with

$$\xi_2=\xi_1-3kqA_{12}-kqA_{14}.$$

$\left(\Sigma_q^{II}=\mathbf{S}_q\times\mathbf{S}_p,\ \text{where } N=q+p,\ 1\leq q<\frac{N}{2}\right)$

The fixed-point subspace of $\Sigma_q^{II}=\mathbf{S}_q\times\mathbf{S}_p$ is

$$\mathrm{Fix}\left(\Sigma^I_{q,p}\right) \;=\; \left\{\left(\underbrace{z,\ldots,z}_{q};\ \underbrace{-\frac{q}{p}z,\ldots,-\frac{q}{p}z}_{p}\right) : z\in\mathbf{C}\right\}.$$

Using the equation (4.21) where f is as in (4.20), after dividing by z we have

$$\nu(\lambda)\;+\;A_1\left[1-\frac{q}{N}\left(1-\frac{q^2}{p^2}\right)\right]|z|^2+(A_2+A_3)q\left(1+\frac{q}{p}\right)|z|^2+\cdots\;=\;0 \tag{4.51}$$

where $+\ \cdots$ denotes terms of higher order in z and $\overline{z}$, and taking the real part of this equation, we obtain,

$$\lambda\;=\;-\;A_{1r}\left[1-\frac{q}{N}\left(1-\frac{q^2}{p^2}\right)\right]|z|^2-(A_{2r}+A_{3r})q\left(1+\frac{q}{p}\right)|z|^2+\cdots\,. \tag{4.52}$$

It follows that if $A_{1r}\left[1-\frac{q}{N}\left(1-\frac{q^2}{p^2}\right)\right]|z|^2+(A_{2r}+A_{3r})q\left(1+\frac{q}{p}\right)<0$, then the branch bifurcates supercritically.

Let $\Sigma_q^{II}=\mathbf{S}_q\times\mathbf{S}_p$ be the isotropy subgroup of $z_0 = \left(z,\dots,z;\,-\frac{q}{p}z,\dots,-\frac{q}{p}z\right)$. Recall the generators for Σ_q^{II} given in Table 4.1.

Suppose M is a square $(q+p)\times(q+p)$ matrix with rows $l_1,\dots,l_q,l_{q+1},\dots,l_{q+p}$ and commuting with $S_q\times S_p$. Then

$$M=(l_1,(12)\cdot l_1,\dots,(1q)\cdot l_1;l_{q+1},(q+1\ q+2)\cdot l_{q+1},\dots,(q+1\ q+p)\cdot l_{q+1})$$

where if $l_1=(m_1,\dots,m_{q+p})$ then

$$(1i)\cdot l_1=(m_i,m_2,\dots,m_{i-1},m_1,m_{i+1},\dots,m_{q+p}).$$

Moreover, l_1 is $S_{q-1}\times S_p$-invariant and l_{q+1} is $S_q\times S_{p-1}$-invariant. Applying this to $(dg)_{(z_0,\lambda_0,\tau_0)}$ we have

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & & C_6 & C_2 & & C_2\\ & \ddots & & & \ddots & \\ C_6 & & C_1 & C_2 & & C_2\\ C_3 & & C_3 & C_4 & & C_5\\ & \ddots & & & \ddots & \\ C_3 & & C_3 & C_5 & & C_4 \end{pmatrix} \tag{4.53}$$

where C_i for $i=1,\dots,5$ are the 2×2 matrices

$$C_i=\begin{pmatrix} c_i & c_i'\\ \overline{c}_i' & \overline{c}_i\end{pmatrix}$$

and

$$c_1=\tfrac{\partial g_1}{\partial z_1},\quad c_1'=\tfrac{\partial g_1}{\partial \overline{z}_1},\quad c_6=\tfrac{\partial g_1}{\partial z_2},\quad c_6'=\tfrac{\partial g_1}{\partial \overline{z}_2},\quad c_2=\tfrac{\partial g_1}{\partial z_{q+1}},\quad c_2'=\tfrac{\partial g_1}{\partial \overline{z}_{q+1}},$$

$$c_3=\tfrac{\partial g_{q+1}}{\partial z_1},\quad c_3'=\tfrac{\partial g_{q+1}}{\partial \overline{z}_1},\quad c_4=\tfrac{\partial g_{q+1}}{\partial z_{q+1}},\quad c_4'=\tfrac{\partial g_{q+1}}{\partial \overline{z}_{q+1}},\quad c_5=\tfrac{\partial g_{q+1}}{\partial z_{q+2}},\quad c_5'=\tfrac{\partial g_{q+1}}{\partial \overline{z}_{q+2}},$$

calculated at (z_0,λ_0,τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z,\overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z=\alpha z+\beta\overline{z}$ where

$$\begin{aligned}\alpha&=c_1+(q-1)c_6-[q(N-q)/p]c_2,\\ \beta&=c_1'+(q-1)c_6'-[q(N-q)/p]c_2'.\end{aligned}$$

The tangent vector to the orbit of $\Gamma\times\mathbf{S}^1$ through z_0 is the eigenvector

$$\left(iz,\dots,iz,-i\frac{q}{p}z,\dots,-i\frac{q}{p}z\right).$$

Note that

$$\frac{d}{dt}\left(e^{it}z,\ldots,e^{it}z,-e^{it}\frac{q}{p}z,\ldots,-e^{it}\frac{q}{p}z\right)|_{t=0}=\left(iz,\ldots,iz,-i\frac{q}{p}z,\ldots,-i\frac{q}{p}z\right).$$

The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha)=2\mathrm{Re}\left[A_1\left(1-\frac{q}{N}+\frac{q^3}{Np^2}\right)+(A_2+A_3)q\left(1+\frac{q}{p}\right)\right]|z|^2+\cdots$$

whose sign is determined by

$$A_{1r}\left[1-\frac{q}{N}\left(1-\frac{q^2}{p^2}\right)\right]+(A_{2r}+A_{3r})q\left(1+\frac{q}{p}\right)$$

if it is assumed nonzero (where A_{1r},A_{2r},A_{3r} are calculated at zero).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ where

$$W_1 = \left\{\left(z_1,\ldots,z_{q-1},-z_1-\cdots-z_{q-1};\underbrace{0,\ldots,0}_{p}\right):z_1,\ldots,z_{q-1}\in\mathbf{C}\right\}.$$

The action of Σ_q^{II} on W_1 decomposes in the following way

$$W_1=W_1^1\oplus W_1^2$$

where

$$W_1^1 = \left\{\left(x_1,\ldots,x_{q-1},-x_1-\cdots-x_{q-1};\underbrace{0,\ldots,0}_{p}\right):x_1,\ldots,x_{q-1}\in\mathbf{R}\right\},$$

$$W_1^2 = \left\{\left(ix_1,\ldots,ix_{q-1},-ix_1-\cdots-ix_{q-1};\underbrace{0,\ldots,0}_{p}\right)x_1,\ldots,x_{q-1}\in\mathbf{R}\right\}.$$

Moreover, the actions of Σ_q^{II} on W_1^1 and on W_1^2 are Σ_q^{II}-isomorphic and are Σ_q^{II}-absolutely irreducible. Thus, it is possible to choose a basis of W_1 such that $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ in the new coordinates has the form

$$\begin{pmatrix} a\ \mathrm{Id}_{(\mathrm{q}-1)\times(\mathrm{q}-1)} & b\ \mathrm{Id}_{(\mathrm{q}-1)\times(\mathrm{q}-1)}\\ c\ \mathrm{Id}_{(\mathrm{q}-1)\times(\mathrm{q}-1)} & d\ \mathrm{Id}_{(\mathrm{q}-1)\times(\mathrm{q}-1)}\end{pmatrix} \tag{4.54}$$

where $\mathrm{Id}_{(\mathrm{q}-1)\times(\mathrm{q}-1)}$ is the $(q-1)\times(q-1)$ identity matrix. Furthermore, the eigenvalues of (4.54) are the eigenvalues of $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ each with multiplicity $q-1$.

With respect to the basis B' of W_1 given by

$$b_1 - b_q, \overline{b}_1 - \overline{b}_q, b_2 - b_q, \overline{b}_2 - \overline{b}_q, \ldots, b_{q-1} - b_q, \overline{b}_{q-1} - \overline{b}_q$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ in the following block diagonal form

$$(dg)_{(z_0,\lambda_0,\tau_0)}|W_1 = \mathrm{diag}(C_1 - C_6, C_1 - C_6, \ldots, C_1 - C_6).$$

The eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ are the eigenvalues of $C_1 - C_6$, each with multiplicity $q-1$. The eigenvalues of $C_1 - C_6$ have negative real part if and only if

$$\mathrm{tr}(C_1 - C_6) < 0 \ \wedge \ \det(C_1 - C_6) > 0.$$

Since

$$\begin{aligned}
c_1 &= \left(1 - \frac{2}{N} + \frac{q}{N} - \frac{q^3}{Np^2}\right) A_1|z|^2 + \left(2 - q - \frac{q^2}{p}\right) A_2|z|^2 + A_3|z|^2 + \cdots, \\
c_1' &= \left(1 - \frac{1}{N}\right) A_1 z^2 + q\left(1 + \frac{q}{p}\right) A_2 z^2 + A_3 z^2 + \cdots, \\
c_6 &= -\frac{2}{N} A_1|z|^2 + 2A_2|z|^2 + A_3|z|^2 + \cdots, \\
c_6' &= -\frac{1}{N} A_1 z^2 + A_3 z^2 + \cdots,
\end{aligned}$$

it follows that

$$\begin{aligned}
\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = \quad & 2\mathrm{Re}\left[\left(1 + \frac{q}{N} - \frac{q^3}{Np^2}\right) A_1 - q\left(1 + \frac{q}{p}\right) A_2\right] |z|^2 + \cdots, \\
\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = \quad & \left|\left(1 + \frac{q}{N} - \frac{q^3}{Np^2}\right) A_1 - q\left(1 + \frac{q}{p}\right) A_2\right|^2 |z|^4 - \\
& \left|A_1 + q\left(1 + \frac{q}{p}\right) A_2\right|^2 |z|^4 + \cdots.
\end{aligned} \tag{4.55}$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ where

$$W_2 \ = \ \{(0, \ldots, 0, z_{q+1}, \ldots, z_{N-1}, -z_{q+1} - \cdots - z_{N-1}) : z_{q+1}, \ldots, z_{N-1} \in \mathbf{C}\}.$$

The action of Σ_q^{II} on W_2 decomposes in the following way

$$W_2 = W_2^1 \oplus W_2^2$$

where

$$W_2^1 = \left\{ \left(\underbrace{0, \dots, 0}_{q}, x_{q+1}, \dots, x_{N-1}, -x_{q+1} - \cdots - x_{N-1}, \right) : x_{q+1}, \dots, x_{N-1} \in \mathbf{R} \right\},$$

$$W_2^2 = \left\{ \left(\underbrace{0, \dots, 0}_{q}, ix_{q+1}, \dots, ix_{N-1}, -ix_{q+1} - \cdots - ix_{N-1}, \right) : x_{q+1}, \dots, x_{N-1} \in \mathbf{R} \right\}.$$

Moreover, the actions of Σ_q^{II} on W_2^1 and on W_2^2 are Σ_q^{II}-isomorphic and are Σ_q^{II}-absolutely irreducible. Thus, it is possible to choose a basis of W_2 such that $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the new coordinates has the form

$$\begin{pmatrix} a \ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} & b \ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} \\ c \ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} & d \ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} \end{pmatrix} \tag{4.56}$$

where $\mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})}$ is the $(N-q-1)\times(N-q-1)$ identity matrix. Furthermore, the eigenvalues of (4.56) are the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ each with multiplicity $N-q-1$.

With respect to the basis B' of W_2 given by

$$b_{q+1} - b_N, \bar{b}_{q+1} - \bar{b}_N, b_{q+2} - b_N, \bar{b}_{q+2} - \bar{b}_N, \dots, b_{N-1} - b_N, \bar{b}_{N-1} - \bar{b}_N$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the following block diagonal form

$$(df)_{(z_0,\lambda_0,\tau_0)}|W_2 = \mathrm{diag}(C_4 - C_5, C_4 - C_5, \dots, C_4 - C_5).$$

The eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ are the eigenvalues of $C_4 - C_5$, each with multiplicity $N-q-1$. The eigenvalues of $C_4 - C_5$ have negative real part if and only if

$$\mathrm{tr}(C_4 - C_5) < 0 \ \wedge \ \det(C_4 - C_5) > 0.$$

Since

$$c_4 = \left[-1 + \frac{q}{N}\left(1 - \frac{q^2}{p^2}\right) + 2\left(1 - \frac{1}{N}\right)\frac{q^2}{p^2} \right] A_1|z|^2 + \left(-q - \frac{q}{p} + \frac{2q^2}{p^2} \right) A_2|z|^2 + \frac{q^2}{p^2} A_3|z|^2 + \cdots,$$

$$c_4' = \frac{q^2}{p^2}\left(1 - \frac{1}{N}\right) A_1 z^2 + q\left(1 + \frac{q}{p}\right) A_2 z^2 + \frac{q^2}{p^2} A_3 z^2 + \cdots,$$

$$c_5 = -\frac{2}{N}\frac{q^2}{p^2} A_1|z|^2 + \frac{2q^2}{p^2} A_2|z|^2 + \frac{q^2}{p^2} A_3|z|^2 + \cdots,$$

$$c_5' = -\frac{1}{N}\frac{q^2}{p^2} A_1 z^2 + \frac{q^2}{p^2} A_3 z^2 + \cdots,$$

it follows that

$$
\begin{aligned}
\operatorname{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \quad & 2\mathrm{Re}\left[\left(-1+\frac{q}{N}-\frac{q^3}{Np^2}+\frac{2q^2}{p^2}\right)A_1 - q\left(1+\frac{1}{p}\right)A_2\right]|z|^2+\cdots\,, \\
\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \quad & \left|\left(-1+\frac{q}{N}-\frac{q^3}{Np^2}+\frac{2q^2}{p^2}\right)A_1 - q\left(1+\frac{1}{p}\right)A_2\right|^2|z|^4 - \\
& \left|\frac{q^2}{p^2}A_1 + q\left(1+\frac{q}{p}\right)A_2\right|^2|z|^4+\cdots\,.
\end{aligned}
\tag{4.57}
$$

□

Chapter 5

Hopf Bifurcation with S_4-symmetry

In this chapter we consider Hopf bifurcation with $\mathbf{S}_N$-symmetry for the special case $N = 4$. This is the only case where the degree three truncation of a general $\mathbf{S}_N$-equivariant vector field determines the stability and the criticality of the branches of periodic solutions guaranteed by the Equivariant Hopf Theorem. In particular, we obtain the possible bifurcation diagrams according the conditions depending on the coefficients of the third degree truncation of the vector field.

Following Chapter 4, we consider the action of $\mathbf{S}_4 \times \mathbf{S}^1$ on $\mathbf{C}^{4,0}$ given by

$$(\sigma, \theta)(z_1, z_2, z_3, z_4) \;=\; e^{i\theta}\left(z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}, z_{\sigma^{-1}(4)}\right) \tag{5.1}$$

for $(z_1, z_2, z_3, z_4) \in \mathbf{C}^{4,0} \;=\; \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 : z_1 + z_2 + z_3 + z_4 = 0\}$, $\sigma \in \mathbf{S}_4$ and $\theta \in \mathbf{S}^1$.

We study Hopf bifurcation with $\mathbf{S}_4$-symmetry and so we take

$$\frac{dz}{dt} = f(z, \lambda), \tag{5.2}$$

where $f : \mathbf{C}^{4,0} \times \mathbf{R} \to \mathbf{C}^{4,0}$ is smooth, commutes with $\mathbf{S}_4$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0, \rho(0) = 1$ and $\sigma'(0) \neq 0$.

As in Chapter 4, the main steps are: describe the $\mathbf{C}$-axial subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathbf{C}^{4,0}$; use the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions with these symmetries of (5.2) by Hopf bifurcation from the trivial equilibrium at $\lambda = 0$. In Theorem 5.2 we determine (generically) the directions of branching and the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem. We prove that it is enough to consider the degree three truncation of f. Although we may obtain the results of Theorem 5.2 from Theorem 4.11, we present the essential results in the proof. In Section 5.2 we classify the possible bifurcation diagrams for the nondegenerate Hopf bifurcation with $\mathbf{S}_4$-symmetry and we give two examples, assigning specific values for the parameters. We finish this chapter by looking for possible branches of periodic solutions that can bifurcate with submaximal isotropy. We prove that the only isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ with fixed-point subspace of dimension 2 are $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_2$.

Isotropy Subgroup	Generators	Orbit Representative	Fixed-Point Subspace
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$	$((1423), \pi)$, $((13)(24), \pi)$	$(1, 1, -1, -1)$	$\{(z_1, z_1, -z_1, -z_1) :\ z_1 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$	(34), $((12), \pi)$	$(1, -1, 0, 0)$	$\{(z_1, -z_1, 0, 0) :\ z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3$	$\left((123), \frac{2\pi}{3}\right)$	$(1, \xi, \xi^2, 0)$	$\left\{(z_1, \xi z_1, \xi^2 z_1, 0) :\ z_1 \in \mathbf{C}\right\}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$	$\left((1234), \frac{\pi}{2}\right)$	$(1, i, -1, -i)$	$\{(z_1, iz_1, -z_1, -iz_1) :\ z_1 \in \mathbf{C}\}$
$\Sigma_5 = \mathbf{S}_3$	(23), (24)	$(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$	$\left\{(z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1) :\ z_1 \in \mathbf{C}\right\}$

Table 5.1: $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathbf{C}^{4,0}$, generators, orbit representatives and fixed-point subspaces. Here $\xi = e^{2\pi i/3}$.

5.1 Periodic solutions with maximal isotropy

From Theorem 4.1 we obtain a description of the $\mathbf{C}$-axial subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathbf{C}^{4,0}$:

Proposition 5.1 *There are five conjugacy classes of $\mathbf{C}$-axial subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ for the action on $\mathbf{C}^{4,0}$ given by (5.1). They are listed, together with their generators, orbit representatives and fixed-point subspaces in Table 5.1.*

Let f be as in (5.2). If we suppose that the Taylor series of degree three of f around $z = 0$ commutes also with $\mathbf{S}^1$, then by Theorem 4.6 and taking $N = 4$, we can write $f = (f_1, f_2, f_3, f_4)$, where

$$
\begin{array}{rcl}
f_1(z_1, z_2, z_3, z_4, \lambda) & = & \mu(\lambda) z_1 \ + \ \frac{1}{4} A_1 (3|z_1|^2 z_1 - |z_2|^2 z_2 - |z_3|^2 z_3 - |z_4|^2 z_4) \ + \\
 & & A_2 \overline{z}_1 \left(z_1^2 + z_2^2 + z_3^2 + z_4^2\right) + A_3 z_1 \left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\right) + \\
 & & \text{terms of degree} \geq 5 \\
f_2(z_1, z_2, z_3, z_4, \lambda) & = & f_1(z_2, z_1, z_3, z_4, \lambda) \\
f_3(z_1, z_2, z_3, z_4, \lambda) & = & f_1(z_3, z_2, z_1, z_4, \lambda) \\
f_4(z_1, z_2, z_3, z_4, \lambda) & = & f_1(z_4, z_2, z_3, z_1, \lambda)
\end{array}
\tag{5.3}
$$

with $z_4 = -z_1 - z_2 - z_3$. The coefficients A_1, A_2 and A_3 are complex smooth functions of λ, $\mu(0) = i$ and $\mathrm{Re}(\mu'(0)) \neq 0$. Throughout, subscripts r and i on the coefficients A_1, A_2 and A_3 refer to the real and imaginary parts.

Next Theorem follows from Theorem 4.13. Recall Table 4.2. Each of the five $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathbf{C}^{4,0}$ listed in Table 5.1 are of the form $\Sigma^I_{q,p}$ or

Isotropy Subgroup	Branching Equations
Σ_1	$\nu + (A_1 + 4A_2 + 4A_3)\|z\|^2 + \cdots \;=\; 0$
Σ_2	$\nu + (A_1 + 2A_2 + 2A_3)\|z\|^2 + \cdots \;=\; 0$
Σ_3	$\nu + (A_1 + 3A_3)\|z\|^2 + \cdots \;=\; 0$
Σ_4	$\nu + (A_1 + 4A_3)\|z\|^2 + \cdots \;=\; 0$
Σ_5	$\nu + \frac{1}{3}(\frac{7}{3}A_1 + 4A_2 + 4A_3)\|z\|^2 + \cdots \;=\; 0$

Table 5.2: Branching equations for $\mathbf{S}_4 \times \mathbf{S}^1$ Hopf bifurcation. Here $\nu(\lambda) = \mu(\lambda) - (1+\tau)i$ and $+\cdots$ stands for higher order terms.

Σ_q^{II}. Specifically, we have that $\Sigma_i, i = 1, \ldots, 4$ are of the form $\Sigma_{q,p}^{I}$ and Σ_5 is of the form Σ_q^{II}.

Theorem 5.2 *Consider the system (5.2) where f is as in (5.3). Assume that* $\mathrm{Re}(\mu'(0)) > 0$ *(such that the trivial equilibrium is stable if $\lambda < 0$ and unstable if $\lambda > 0$, for λ near zero). For each isotropy subgroup Σ_i, for $i = 1, \ldots, 5$ listed in Table 5.1, let $\Delta_0, \ldots, \Delta_r$ be the functions of A_1, A_2 and A_3 listed in Table 5.4 evaluated at $\lambda = 0$. Then:*

(1) For each Σ_i the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$. Tables 5.2 and 5.3 list the branching equations.

(2) For each Σ_i, if $\Delta_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all j, then the branch of periodic solutions is stable near $\lambda = 0$ and $z = 0$.

Remark 5.3 Observe that periodic solutions with symmetry Σ_3 guaranteed by Theorem 5.2 are always unstable since generically $\Delta_2 = |A_1|^2 > 0$. If $A_{1r} > 0$, then solutions with symmetry Σ_4 are unstable and if $A_{2r} < 0$, then solutions with symmetry Σ_2 are also unstable. $\diamond$

Proof: Our aim is to study periodic solutions of (5.2) obtained by Hopf bifurcation from the trivial equilibrium. Note that we are assuming that f satisfies the conditions of the Equivariant Hopf Theorem.

From Proposition 5.1 we have (up to conjugacy) the $\mathbf{C}$-axial subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$. Therefore, we can use the Equivariant Hopf Theorem to prove the existence of periodic solutions with these symmetries for a bifurcation problem with symmetry $\Gamma = \mathbf{S}_4$.

Isotropy Subgroup	Branching Equations
Σ_1	$\lambda = -\left(A_{1r} + 4A_{2r} + 4A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_2	$\lambda = -\left(A_{1r} + 2A_{2r} + 2A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_3	$\lambda = -\left(A_{1r} + 3A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_4	$\lambda = -\left(A_{1r} + 4A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_5	$\lambda = -\frac{1}{3}\left(\frac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r}\right)\lvert z\rvert^2 + \cdots$

Table 5.3: Branching equations for $\mathbf{S}_4$ Hopf bifurcation. Subscript r on the coefficients refer to the real part and $+\cdots$ stands for higher order terms.

Periodic solutions of (5.2) of period near $2\pi/(1+\tau)$ are in one-to-one correspondence with the zeros of a function $g(z,\lambda,\tau)$, with explicit form given by (4.21) if f commutes with $\mathbf{S}_4 \times \mathbf{S}^1$.

Recall the isotypic decomposition for each type of isotropy subgroups $\Sigma^I_{q,p}$ and Σ^{II}_q given by (4.23) and (4.24). For the three isotropy subgroups Σ_i, for $i = 2, 3, 4$, in Table 5.1, the isotypic decomposition takes, respectively, the form

$$\begin{array}{l} \mathbf{C}^{4,0} = W_0 \oplus W_1 \oplus W_2 \\ \mathbf{C}^{4,0} = W_0 \oplus W_1 \oplus P_2 \\ \mathbf{C}^{4,0} = W_0 \oplus P_2 \oplus P_3 \end{array} \tag{5.4}$$

where $W_0 = \mathrm{Fix}(\Sigma_i), W_1, W_2, P_2$ and P_3 are the complex one-dimensional isotypic components for the action of Σ_i on $\mathbf{C}^{4,0}$. It follows then that $(dg)_{z_0}(W_j) \subseteq W_j$ for $j = 0, 1, 2$ and $(dg)_{z_0}(P_j) \subseteq P_j$ for $j = 2, 3$ since $(dg)_{z_0}$ commutes with Σ_i. For Σ_1 and Σ_5 we obtain that

$$\mathbf{C}^{4,0} = W_0 \oplus W_3$$

and

$$\mathbf{C}^{4,0} = W_0 \oplus W_2$$

where W_3, W_2 are complex two-dimensional invariant subspaces that are the sum of two isomorphic real absolutely irreducible representations of dimension 2 of Σ_1 and Σ_5, respectively. Again we have $(dg)_{z_0}(W_j) \subseteq W_j$ for $j = 0, 2, 3$.

Table 5.5 gives the isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of each of the isotropy subgroups Σ_i listed in Table 5.1.

Throughout we denote by (z_0, λ_0, τ_0) a zero of $g(z,\lambda,\tau) = 0$ with $z_0 \in \mathrm{Fix}(\Sigma_i)$. Specifically, for $i = 1, \ldots, 5$, we calculate now $(dg)_{(z_0,\lambda_0,\tau_0)}$.

Isotropy Subgroup	Δ_0	$\Delta_1, \ldots, \Delta_r$
Σ_1	$A_{1r} + 4A_{2r} + 4A_{3r}$	$A_{1r} - 4A_{2r}$ $- \left(\|A_1 - 4A_2\|^2 - \|A_1 + 4A_2\|^2\right)$
Σ_2	$A_{1r} + 2A_{2r} + 2A_{3r}$	$-A_{1r} - 2A_{2r}$ $- \left(\|A_1 + 2A_2\|^2 - \|2A_2\|^2\right)$ $-A_{2r}$ $- \left(4\|A_2\|^2 - \|\frac{1}{2}A_1 + 2A_2\|^2\right)$
Σ_3	$A_{1r} + 3A_{3r}$	$-A_{1r}$ $\|A_1\|^2$ $A_{1r} + 6A_{2r}$ $- \left(\|A_1 + 6A_2\|^2 - \|\frac{1}{4}A_1\|^2\right)$
Σ_4	$A_{1r} + 4A_{3r}$	A_{1r} $-\|A_1\|^2$ $A_{1r} + 8A_{2r}$ $- \left(\|A_1 + 8A_2\|^2 - \|A_1\|^2\right)$
Σ_5	$\frac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r}$	$-5A_{1r} - 12A_{2r}$ $- \left(\|5A_1 + 12A_2\|^2 - \|A_1 + 12A_2\|^2\right)$

Table 5.4: Stability for $\mathbf{S}_4$ Hopf bifurcation.

Isotropy subgroup and Orbit Representative	Isotypic components of $\mathbf{C}^{4,0}$
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$ $z = (z_1, z_1, -z_1, -z_1)$	$W_0 = \mathrm{Fix}(\Sigma_1) = \{(z_1, z_1, -z_1, -z_1) : z_1 \in \mathbf{C}\}$ $W_3 = \{(z_1, -z_1, z_2, -z_2) : z_1, z_2 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$ $z = (z_1, -z_1, 0, 0)$	$W_0 = \mathrm{Fix}(\Sigma_2) = \{(z_1, -z_1, 0, 0) : z_1 \in \mathbf{C}\}$ $W_1 = \{(z_1, z_1, -z_1, -z_1) : z_1 \in \mathbf{C}\}$ $W_2 = \{(0, 0, z_1, -z_1) : z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3$ $z = (z_1, \xi z_1, \xi^2 z_1, 0)$	$W_0 = \mathrm{Fix}(\Sigma_3) = \{(z_1, \xi z_1, \xi^2 z_1, 0) : z_1 \in \mathbf{C}\}$ $W_1 = \{(z_1, z_1, z_1, -3z_1) : z_1 \in \mathbf{C}\}$ $P_2 = \{(z_1, \xi^2 z_1, \xi z_1, 0) : z_1 \in \mathbf{C}\}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$ $z = (z_1, iz_1, -z_1, -iz_1)$	$W_0 = \mathrm{Fix}(\Sigma_4) = \{(z_1, iz_1, -z_1, -iz_1) : z_1 \in \mathbf{C}\}$ $P_2 = \{(z_1, -z_1, z_1, -z_1) : z_1 \in \mathbf{C}\}$ $P_3 = \{(z_1, -iz_1, -z_1, iz_1) : z_1 \in \mathbf{C}\}$
$\Sigma_5 = \mathbf{S}_3$ $z = \left(z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1\right)$	$W_0 = \mathrm{Fix}(\Sigma_5) = \left\{\left(z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1\right) : z_1 \in \mathbf{C}\right\}$ $W_2 = \{(0, z_2, z_3, -z_2 - z_3) : z_2, z_3, \in \mathbf{C}\}$

Table 5.5: Isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of each of the isotropy subgroups listed in Table 5.1. Here $\xi = e^{2\pi i/3}$.

To compute the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ we use complex coordinates $z_1, \overline{z}_1, \ldots, z_4, \overline{z}_4$ corresponding to a basis B for $\mathbf{C}^4$ with elements denoted by $b_1, \overline{b}_1, b_2, \overline{b}_2, b_3, \overline{b}_3, b_4, \overline{b}_4$.

(Σ_1)

The fixed-point subspace of Σ_1 is $z_3 = z_4 = -z$ and $z_1 = z_2 = z$. The isotropy subgroup $\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$ is of the type $\Sigma^I_{q,p}$ with $k = q = 2$ and $p = 0$. Using this in (4.32) and (4.33) we get the branching equations for Σ_1 listed in Tables 5.2 and 5.3. It follows that if $A_{1r} + 4A_{2r} + 4A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 5.5. The isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of Σ_1 is $\mathbf{C}^{4,0} \cong W_0 \oplus W_3$ where

$$\begin{array}{rcl} W_0 & = & \mathrm{Fix}(\Sigma_1) = \{(z_1, z_1, -z_1, -z_1) : \; z_1 \in \mathbf{C}\}\,, \\ W_3 & = & \{(z_1, -z_1, z_2, -z_2) : \; z_1, z_2 \in \mathbf{C}\}\,. \end{array}$$

Moreover, Σ_1 is isomorphic to $\mathbf{D}_4$, the dihedral group of order 8. Recall (4.34). With respect to the basis B, any "real" matrix commuting with Σ_1 has the form (note that $\xi^2 = 1$ if $\xi = e^{i2\pi/2}$)

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_2 & C_3 & C_3 \\ C_2 & C_1 & C_3 & C_3 \\ C_3 & C_3 & C_1 & C_2 \\ C_3 & C_3 & C_2 & C_1 \end{pmatrix}$$

where C_1, C_2, C_3 are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c'_i \\ \overline{c}'_i & \overline{c}_i \end{pmatrix}$$

and

$$c_1 = \frac{\partial g_1}{\partial z_1}, \quad c'_1 = \frac{\partial g_1}{\partial \overline{z}_1}, \quad c_2 = \frac{\partial g_1}{\partial z_2}, \quad c'_2 = \frac{\partial g_1}{\partial \overline{z}_2}, \quad c_3 = \frac{\partial g_1}{\partial z_3}, \quad c'_3 = \frac{\partial g_1}{\partial \overline{z}_3}$$

calculated at (z_0, λ_0, τ_0) (note that $\xi^2 = 1$).

Throughout we denote by $(dg)_{(z_0,\lambda_0,\tau_0)}|W_j$ the restriction of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to the subspace W_j.

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z \mapsto \alpha z + \beta \overline{z}$ where

$$\alpha = c_1 + c_2 - 2c_3,$$
$$\beta = c'_1 + c'_2 - 2c'_3.$$

Note that $\frac{d}{dt}(e^{it}z, e^{it}z, -e^{it}z, -e^{it}z)\big|_{t=0} = (iz, iz, -iz, -iz)$ and so a tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz, iz, -iz, -iz)$.

The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = 2\mathrm{Re}\left(A_1 + 4A_2 + 4A_3\right)|z|^2 + \cdots$$

whose sign is determined by $A_{1r} + 4A_{2r} + 4A_{3r}$ if it is assumed nonzero (where $A_{1r} + 4A_{2r} + 4A_{3r}$ is calculated at zero).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_3$. From (4.42) with $q = 2$ it follows that

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = 2\mathrm{Re}(A_1 - 4A_2)|z|^2 + \cdots,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \left(|A_1 - 4A_2|^2 - |A_1 + 4A_2|^2\right)|z|^4 + \cdots.$$

(Σ_2)

The fixed-point subspace of Σ_2 is $z_4 = z_3 = 0$ and $z_2 = -z_1 = -z$. The isotropy subgroup $\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$ is of the type $\Sigma^I_{q,p}$ with $k = 2, q = 1$ and $p = 2$. Using this in (4.32) and (4.33) we get the branching equations for Σ_2 listed in Tables 5.2 and 5.3. It follows that if $A_{1r} + 2A_{2r} + 2A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 5.5. The isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of Σ_2 is $\mathbf{C}^{4,0} \cong W_0 \oplus W_1 \oplus W_2$ where

$$\begin{aligned} W_0 &= \mathrm{Fix}(\Sigma_2) = \{(z_1, -z_1, 0, 0) : z_1 \in \mathbf{C}\}, \\ W_1 &= \{(z_1, z_1, -z_1, -z_1) : \ z_1 \in \mathbf{C}\}, \\ W_2 &= \{(0, 0, z_1, -z_1) : \ z_1 \in \mathbf{C}\}. \end{aligned}$$

Recall (4.34). With respect to the basis B, any "real" matrix commuting with Σ_2 has the form (note that $\xi^2 = 1$ if $\xi = e^{i2\pi/2}$)

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_3 & C_4 & C_4 \\ C_3 & C_1 & C_4 & C_4 \\ C_5 & C_5 & C_6 & C_7 \\ C_5 & C_5 & C_7 & C_6 \end{pmatrix}$$

where C_i, for $i = 1, \ldots, 7$ are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c'_i \\ \overline{c}'_i & \overline{c}_i \end{pmatrix}$$

and

$$c_1 = \tfrac{\partial g_1}{\partial z_1}, \quad c'_1 = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad c_3 = \tfrac{\partial g_1}{\partial z_2}, \quad c'_3 = \tfrac{\partial g_1}{\partial \overline{z}_2}, \quad c_4 = \tfrac{\partial g_1}{\partial z_3}, \quad c'_4 = \tfrac{\partial g_1}{\partial \overline{z}_3}$$

$$c_5 = \tfrac{\partial g_3}{\partial z_1}, \quad c'_5 = \tfrac{\partial g_3}{\partial \overline{z}_1}, \quad c_6 = \tfrac{\partial g_3}{\partial z_3}, \quad c'_6 = \tfrac{\partial g_3}{\partial \overline{z}_3}, \quad c_7 = \tfrac{\partial g_3}{\partial z_4}, \quad c'_7 = \tfrac{\partial g_3}{\partial \overline{z}_4}$$

calculated at (z_0, λ_0, τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$, we have $\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_0\right) z \mapsto \alpha z + \beta \overline{z}$ where

$$\begin{aligned} \alpha &= c_1 - c_3, \\ \beta &= c'_1 - c'_3. \end{aligned}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz, -iz, 0, 0)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = 2\mathrm{Re}\left(A_1 + 2A_2 + 2A_3\right)|z|^2 + \cdots$$

whose sign is determined by $A_{1r} + 2A_{2r} + 2A_{3r}$ if it is assumed nonzero (where $A_{1r} + 2A_{2r} + 2A_{3r}$ is calculated at zero).

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ and $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$. Respectively, from (4.36) with $N = 4, q = 1$ and from (4.40) with $q = 1$ it follows that

$$\operatorname{trace}\big((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\big) = 2\mathrm{Re}(-2A_2)|z|^2 + \cdots ,$$

$$\det\big((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\big) = \big(|-2A_2|^2 - |\tfrac{1}{2}A_1 + 2A_2|^2\big)\,|z|^4 + \cdots ,$$

$$\operatorname{trace}\big((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\big) = 2\mathrm{Re}(-A_1 - 2A_2)|z|^2 + \cdots ,$$

$$\det\big((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\big) = \big(|-A_1 - 2A_2|^2 - |2A_2|^2\big)\,|z|^4 + \cdots .$$

(Σ_3)

The fixed-point subspace of Σ_3 is $z_4 = 0, z_3 = \xi^2 z, z_2 = \xi z$ and $z_1 = z$ with $\xi = e^{2\pi i/3}$. The isotropy subgroup $\Sigma_3 = \widetilde{\mathbf{Z}}_3$ is of the type $\Sigma^I_{q,p}$ with $k = 3, q = 1$ and $p = 1$. Using this in (4.30) and (4.31) we get the branching equations for Σ_3 listed in Tables 5.2 and 5.3. It follows that if $A_{1r} + 3A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 5.5. The isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of Σ_3 is $\mathbf{C}^{4,0} \cong W_0 \oplus W_1 \oplus P_2$ where

$$\begin{array}{rcl} W_0 &=& \mathrm{Fix}(\Sigma_3) \;=\; \big\{(z_1, \xi z_1, \xi^2 z_1, 0):\; z_1 \in \mathbf{C}\big\}, \\ W_1 &=& \{(z_1, z_1, z_1, -3z_1):\; z_1 \in \mathbf{C}\}, \\ P_2 &=& \big\{(z_1, \xi^2 z_1, \xi z_1, 0):\; z_1 \in \mathbf{C}\big\}. \end{array}$$

Recall (4.34). With respect to the basis B, any "real" matrix commuting with Σ_3 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_3 & C_4 & C_5 \\ C_4^{\xi^2} & C_1^{\xi^2} & C_3^{\xi^2} & C_5 \\ C_3^{\xi^4} & C_4^{\xi^4} & C_1^{\xi^4} & C_5 \\ C_6 & C_6^{\xi^2} & C_6^{\xi^4} & C_7 \end{pmatrix}$$

where $C_i, C_i^{\xi^j}$, for $i = 1,3,4,6, j = 2,4$ are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix} \quad C_i^{\xi^j} = \begin{pmatrix} c_i & \xi^j c_i' \\ \overline{\xi}^j \overline{c}_i' & \overline{c}_i \end{pmatrix} \quad C_5 = \begin{pmatrix} c_5 & c_5' \\ \overline{c}_5' & \overline{c}_5 \end{pmatrix} \quad C_7 = \begin{pmatrix} c_7 & c_7' \\ \overline{c}_7' & \overline{c}_7 \end{pmatrix},$$

$\xi = e^{i2\pi/3}$ and

$$c_1 = \tfrac{\partial g_1}{\partial z_1}, \quad c_1' = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad c_3 = \tfrac{\partial g_1}{\partial z_2}, \quad c_3' = \tfrac{\partial g_1}{\partial \overline{z}_2}, \quad c_4 = \tfrac{\partial g_1}{\partial z_3}, \quad c_4' = \tfrac{\partial g_1}{\partial \overline{z}_3},$$

$$c_5 = \tfrac{\partial g_1}{\partial z_4}, \quad c_5' = \tfrac{\partial g_1}{\partial \overline{z}_4}, \quad c_6 = \tfrac{\partial g_4}{\partial z_1}, \quad c_6' = \tfrac{\partial g_4}{\partial \overline{z}_1}, \quad c_7 = \tfrac{\partial g_4}{\partial z_4}, \quad c_7' = \tfrac{\partial g_4}{\partial \overline{z}_4}.$$

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$, we have

$\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_0\right)$ $z \to \alpha z + \beta\overline{z}$ where

$$\begin{aligned}\alpha &= c_1 + \xi c_3 + \xi^2 c_4,\\ \beta &= c_1' + \overline{\xi} c_3' + \overline{\xi^2} c_4'.\end{aligned}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz_1, i\xi z_1, i\xi^2 z_1, 0)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = 2\mathrm{Re}(A_1 + 3A_3)|z|^2 + \cdots$$

whose sign is determined by $A_{1r} + 3A_{3r}$ if it is assumed nonzero (where $A_{1r} + 3A_{3r}$ is calculated at zero).

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$. From (4.38) with $N = 4, k = 3, q = 1$ it follows that

$$\mathrm{trace}\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = 2\mathrm{Re}(-\tfrac{1}{2}A_1)|z|^2 + \cdots,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = \left(\left|-\tfrac{1}{2}A_1\right|^2 - |A_1|^2\right)|z|^4 + \cdots.$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|P_2$. From (4.47) with $N = 4$ it follows that

$$\mathrm{trace}\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = 2\mathrm{Re}\left(A_1 + 6A_2\right)|z|^2 + \cdots,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = \left(|A_1 + 6A_2|^2 - |\tfrac{1}{4}A_1|^2\right)|z|^4 + \cdots.$$

(Σ_4)

The fixed-point subspace of Σ_4 is $z_4 = -iz, z_3 = -z, z_2 = iz$ and $z_1 = z$. The isotropy subgroup $\Sigma_4 = \widetilde{\mathbf{Z}}_4$ is of the type $\Sigma^I_{q,p}$ with $k = 4, q = 1$ and $p = 0$. Using this in (4.30) and (4.31) we get the branching equations for Σ_4 listed in Tables 5.2 and 5.3. It follows that if $A_{1r} + 4A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 5.5. The isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of Σ_4 is $\mathbf{C}^{4,0} \cong W_0 \oplus P_2 \oplus P_3$ where

$$\begin{aligned}W_0 &= \mathrm{Fix}(\Sigma_4) = \{(z_1, iz_1, -z_1, -iz_1) : z_1 \in \mathbf{C}\},\\ P_2 &= \{(z_1, -z_1, z_1, -z_1) : z_1 \in \mathbf{C}\},\\ P_3 &= \{(z_1, -iz_1, -z_1, iz_1) : z_1 \in \mathbf{C}\}.\end{aligned}$$

Recall (4.34). With respect to the basis B any "real" matrix commuting with Σ_4 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_3 & C_4 & C_5 \\ C_5^{\xi^2} & C_1^{\xi^2} & C_3^{\xi^2} & C_4^{\xi^2} \\ C_4^{\xi^4} & C_5^{\xi^4} & C_1^{\xi^4} & C_3^{\xi^4} \\ C_3^{\xi^6} & C_4^{\xi^6} & C_5^{\xi^6} & C_1^{\xi^6} \end{pmatrix}$$

where

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix} \quad C_i^{\xi^j} = \begin{pmatrix} c_i & \xi^j c_i' \\ \overline{\xi}^j \overline{c}_i' & \overline{c}_i \end{pmatrix}$$

for $i = 1,3,4,5,\ j = 2,4,6,\ \xi = e^{i2\pi/4} = i$ and

$$c_1 = \frac{\partial g_1}{\partial z_1},\quad c_1' = \frac{\partial g_1}{\partial \overline{z}_1},\quad c_3 = \frac{\partial g_1}{\partial z_2},\quad c_3' = \frac{\partial g_1}{\partial \overline{z}_2},\quad c_4 = \frac{\partial g_1}{\partial z_3},\quad c_4' = \frac{\partial g_1}{\partial \overline{z}_3},\quad c_5 = \frac{\partial g_1}{\partial z_4},\quad c_5' = \frac{\partial g_1}{\partial \overline{z}_4}.$$

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$, we have $\big((dg)_{(z_0,\lambda_0,\tau_0)}|W_0\big)\, z \to \alpha z + \beta \overline{z}$ where

$$\begin{array}{l} \alpha = c_1 + ic_3 - c_4 - ic_5, \\ \beta = c_1' - ic_3' - c_4' + ic_5'. \end{array}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz_1, -z_1, -iz_1, z_1)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = 2\mathrm{Re}(A_1 + 4A_3)|z|^2 + \cdots$$

whose sign is determined by $A_{1r} + 4A_{3r}$ if it is assumed nonzero (where $A_{1r} + 4A_{3r}$ is calculated at zero).

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|P_2$. From (4.46) with $N = 4, k = 4, q = 1$ it follows that

$$\mathrm{trace}\,\big((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\big) = 2\mathrm{Re}(A_1)|z|^2 + \cdots ,$$

$$\det\big((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\big) = |A_1|^2|z|^4 + \cdots .$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|P_3$. From (4.48) with $j = k-1 = 3, q = 1$ it follows that

$$\mathrm{tr}\,\big((dg)_{(z_0,\lambda_0,\tau_0)}|P_3\big) = 2\mathrm{Re}(A_1 + 8A_2)|z|^2 + \cdots ,$$

$$\det\big((dg)_{(z_0,\lambda_0,\tau_0)}|P_3\big) = \big(|A_1 + 8A_2|^2 - |A_1|^2\big)\,|z|^4 + \cdots .$$

(Σ_5)

The fixed-point subspace of Σ_5 is $z_2 = z_3 = z_4 = -\frac{1}{3}z$ and $z_1 = z$. The isotropy subgroup $\Sigma_5 = \mathbf{S}_3$ is of the type Σ_q^{II} with $q = 1$ and $p = 3$. Using this in (4.51) and (4.52) we get the branching equations for Σ_5 listed in Tables 5.2 and 5.3. It follows that if $\frac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 5.5. The isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of Σ_5 is $\mathbf{C}^{4,0} \cong W_0 \oplus W_2$ where

$$\begin{array}{rcl} W_0 &=& \mathrm{Fix}(\Sigma_5) \;=\; \big\{\big(z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1\big):\ z_1 \in \mathbf{C}\big\}, \\ W_2 &=& \{(0, z_2, z_3, -z_2 - z_3):\ z_2, z_3, \in \mathbf{C}\}. \end{array}$$

Recall (4.53). With respect to the basis B, any "real" matrix commuting with Σ_5 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} \;=\; \begin{pmatrix} C_1 & C_2 & C_2 & C_2 \\ C_3 & C_4 & C_5 & C_5 \\ C_3 & C_5 & C_4 & C_5 \\ C_3 & C_5 & C_5 & C_4 \end{pmatrix}$$

where C_i for $i = 1, \ldots, 5$ are 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix}$$

and

$$c_1 = \tfrac{\partial g_1}{\partial z_1}, \quad c_1' = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad c_2 = \tfrac{\partial g_1}{\partial z_2}, \quad c_2' = \tfrac{\partial g_1}{\partial \overline{z}_2}, \quad c_3 = \tfrac{\partial g_2}{\partial z_1}, \quad c_3' = \tfrac{\partial g_2}{\partial \overline{z}_1},$$

$$c_4 = \tfrac{\partial g_2}{\partial z_2}, \quad c_4' = \tfrac{\partial g_2}{\partial \overline{z}_2}, \quad c_5 = \tfrac{\partial g_2}{\partial z_3}, \quad c_5' = \tfrac{\partial g_2}{\partial \overline{z}_3},$$

calculated at (z_0, λ_0, τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z = \alpha z + \beta \overline{z}$ where

$$\begin{aligned} \alpha &= c_1 - c_2, \\ \beta &= c_1' - c_2'. \end{aligned}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $\left(iz, -\frac{1}{3}iz, -\frac{1}{3}iz, -\frac{1}{3}iz\right)$ and the matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = \frac{2}{3}\mathrm{Re}\left(\frac{7}{3}A_1 + 4A_2 + 4A_3\right)|z|^2 + \cdots$$

whose sign is determined by $\frac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r}$ if it is assumed nonzero (where $\frac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r}$ is calculated at zero).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$. From (4.57) with $N = 4, q = 1, p = 3$ it follows that

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \tfrac{2}{9}\mathrm{Re}(-5A_1 - 12A_2)|z|^2 + \cdots,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \left(\left|\tfrac{1}{9}(-5A_1 - 12A_2)\right|^2 - \left|\tfrac{1}{3}(\tfrac{1}{3}A_1 + 4A_2)\right|^2\right)|z|^4 + \cdots.$$

$\square$

5.2 Bifurcation diagrams

The previous section determined that the solution stabilities depend on the following coefficients

$$A_1, A_2, A_{3r} \tag{5.5}$$

of the degree three truncation of f.

In this section we classify the possible bifurcation diagrams as a function of these coefficients.

Recall the stability results for these solutions summarized in Table 5.4. From this we obtain the following non-degeneracy conditions:

$$\begin{array}{l}
\text{(a) } A_{1r} + 4A_{2r} + 4A_{3r} \neq 0 \\
\text{(b) } A_{1r} - 4A_{2r} \neq 0 \\
\text{(c) } |A_1 - 4A_2|^2 - |A_1 + 4A_2|^2 \neq 0 \\
\text{(d) } A_{1r} + 2A_{2r} + 2A_{3r} \neq 0 \\
\text{(e) } A_{1r} + 2A_{2r} \neq 0 \\
\text{(f) } |A_1 + 2A_2|^2 - 4|A_2|^2 \neq 0 \\
\text{(g) } A_{2r} \neq 0 \\
\text{(h) } \left(4|A_2|^2 - |\tfrac{1}{2}A_1 + 2A_2|^2\right) \neq 0 \\
\text{(i) } A_{1r} + 3A_{3r} \neq 0 \\
\text{(j) } A_{1r} + 4A_{3r} \neq 0 \\
\text{(k) } A_{1r} + 6A_{2r} \neq 0 \\
\text{(l) } A_{1r} \neq 0 \\
\text{(n) } A_{1r} + 8A_{2r} \neq 0 \\
\text{(o) } |A_1 + 8A_2|^2 - |A_1|^2 \neq 0 \\
\text{(p) } \tfrac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r} \neq 0 \\
\text{(q) } 5A_{1r} + 12A_{2r} \neq 0 \\
\text{(r) } |5A_1 + 12A_2|^2 - |A_1 + 12A_2|^2 \neq 0 \\
\text{(s) } |A_1 + 6A_2|^2 - |\tfrac{1}{4}A_1|^2 \neq 0
\end{array} \tag{5.6}$$

The inequalities (5.6) divide the parameter space (5.5) into regions characterized by (possibly) distinct bifurcation diagrams. In Figures 5.1 and 5.2 we assume, respectively, $A_{1r} < 0$ and $A_{1r} > 0$ and we consider the various regions of the (A_{2r}, A_{3r})-parameter space defined by (5.6).

Figures 5.3-5.5 show the bifurcation diagrams corresponding to the regions of parameter space of Figure 5.1. An asterisk on solution indicates that it is possible for the solution to be unstable, depending on the sign of

$$\begin{array}{ll}
(*) & |A_1 - 4A_2|^2 - |A_1 + 4A_2|^2 \\
(**) & |A_1 + 2A_2|^2 - 4|A_2|^2 \text{ and } 4|A_2|^2 - |\tfrac{1}{2}A_1 + 2A_2|^2 \\
(***) & |A_1 + 8A_2|^2 - |A_1|^2 \\
(****) & |5A_1 + 12A_2|^2 - |A_1 + 12A_2|^2
\end{array} \tag{5.7}$$

Furthermore, note that the Σ_3 solution is never stable.

On Figures 5.6-5.8 we show the bifurcation diagrams concerning regions of the parameter space of Figure 5.2.

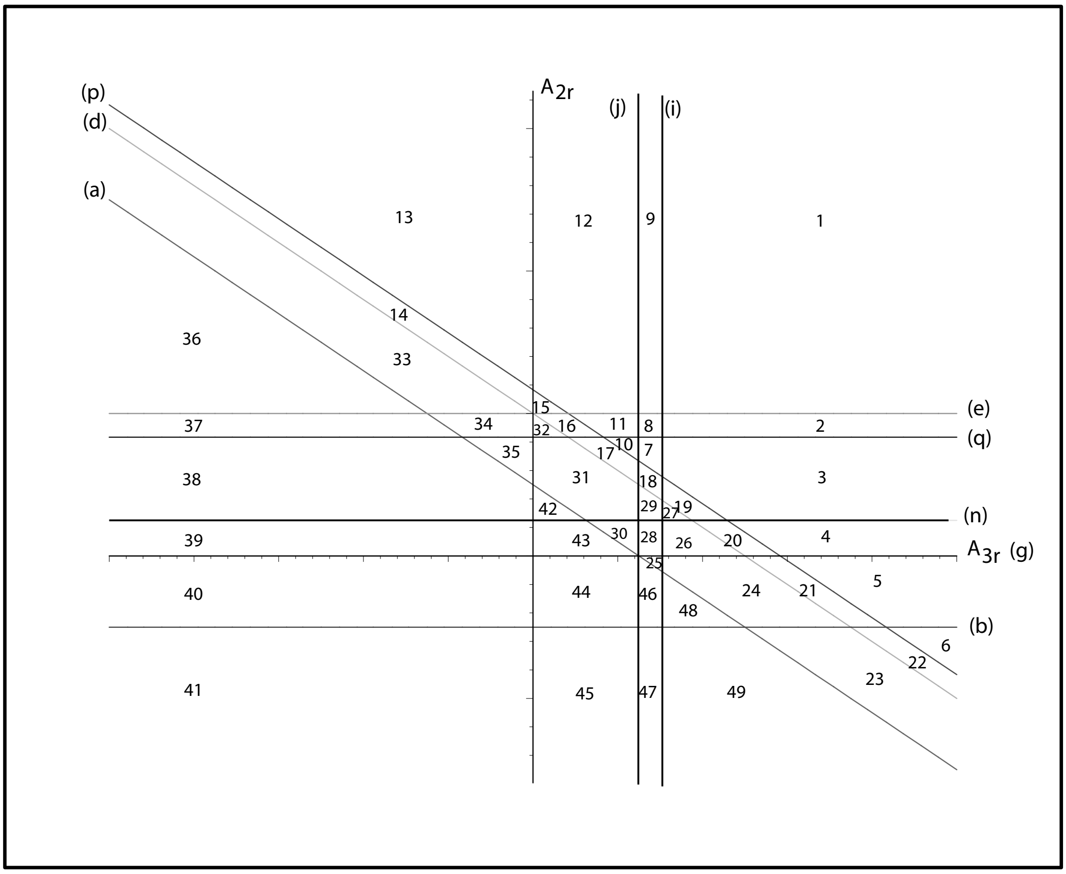

Figure 5.1: Regions of the (A_{3r}, A_{2r})-parameter space defined by the lines corresponding to the equations (5.6). Here we assume $A_{1r} < 0$. Lines are labelled according to which of the corresponding expressions on (5.6) vanishes on them.

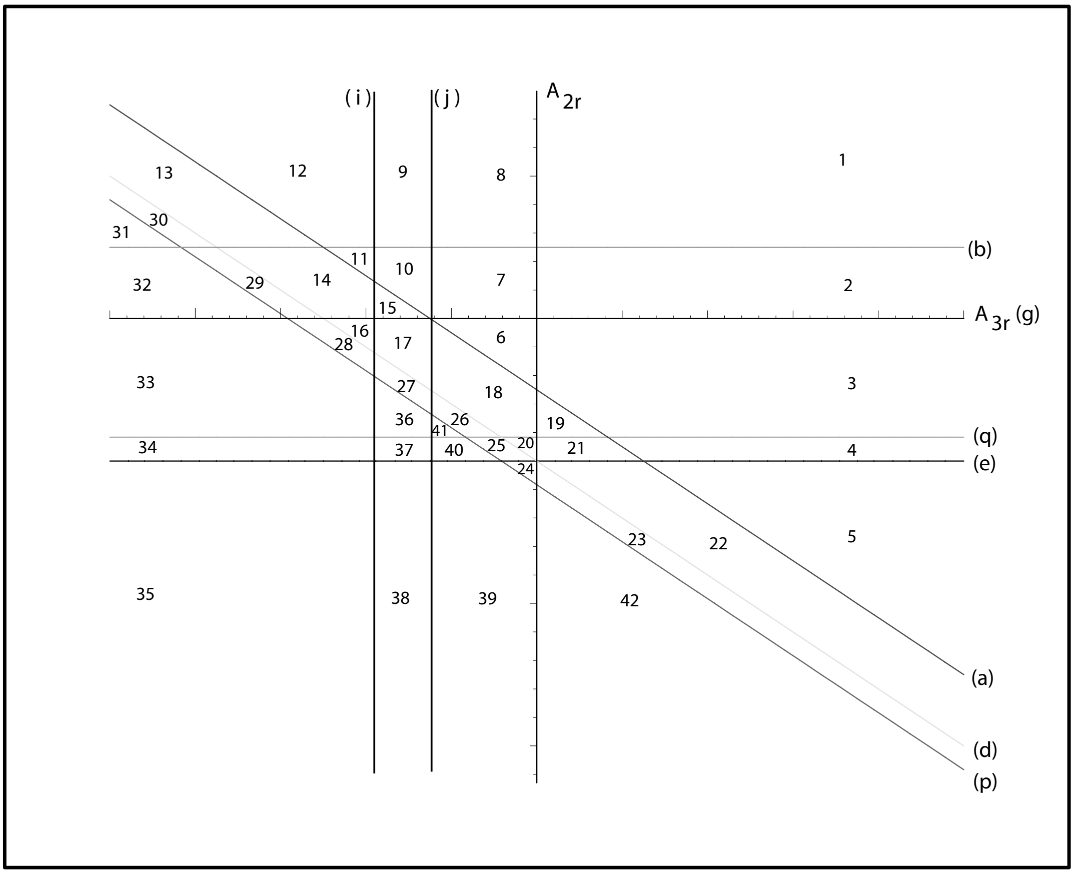

Figure 5.2: Regions of the (A_{3r}, A_{2r})-parameter space defined by the lines corresponding to the equations (5.6). Here we assume $A_{1r} > 0$. Lines are labelled according to which of the corresponding expressions on (5.6) vanishes on them.

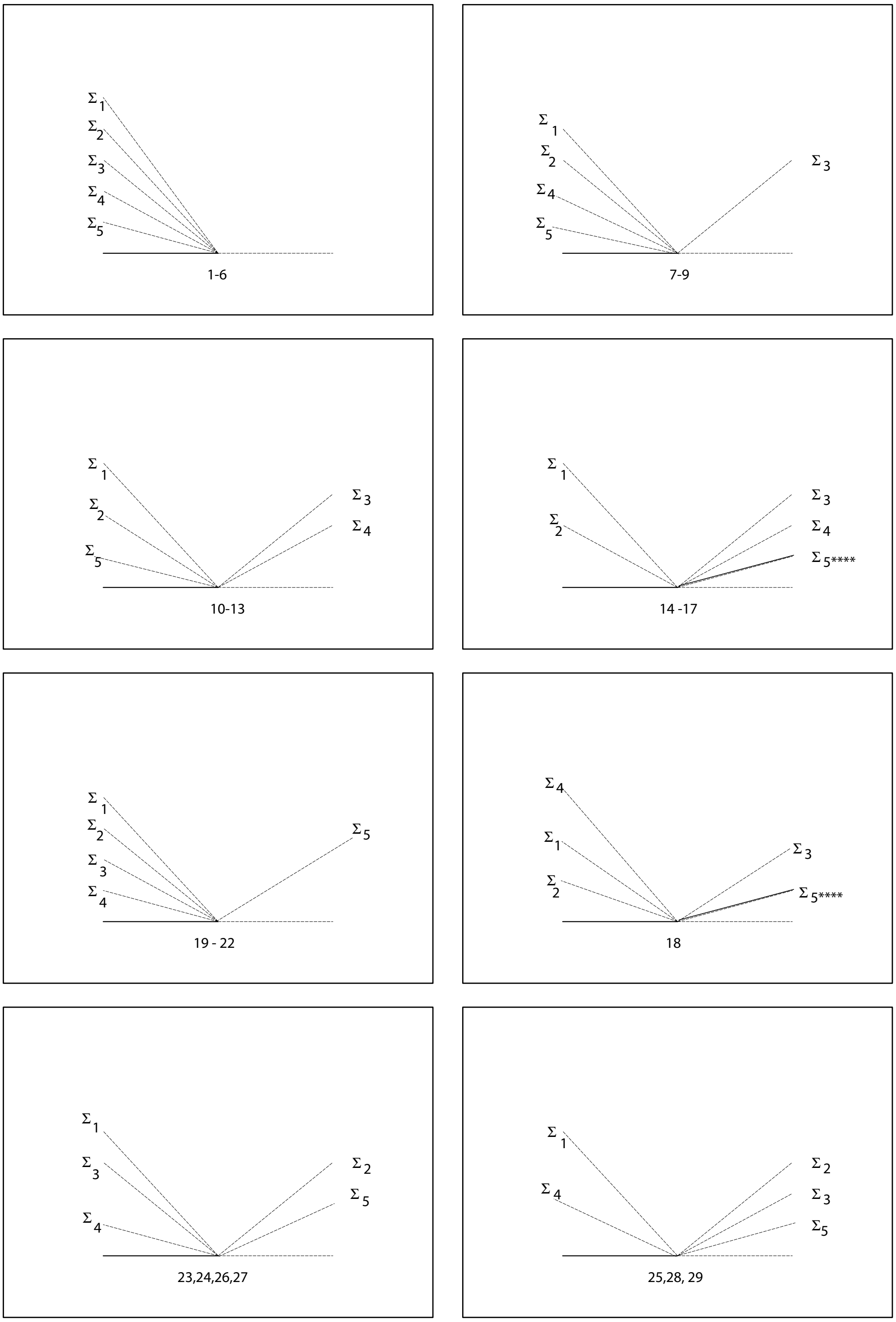

Figure 5.3: Bifurcation diagrams for the nondegenerate Hopf bifurcation with $\mathbf{S}_4$ symmetry. Broken (unbroken) bifurcation curves indicate unstable (stable) solutions. An asterisk on solution indicates that it is possible for the solution to be unstable, depending on the sign of (5.7). The diagrams are plotted for$A_{1r} < 0$.

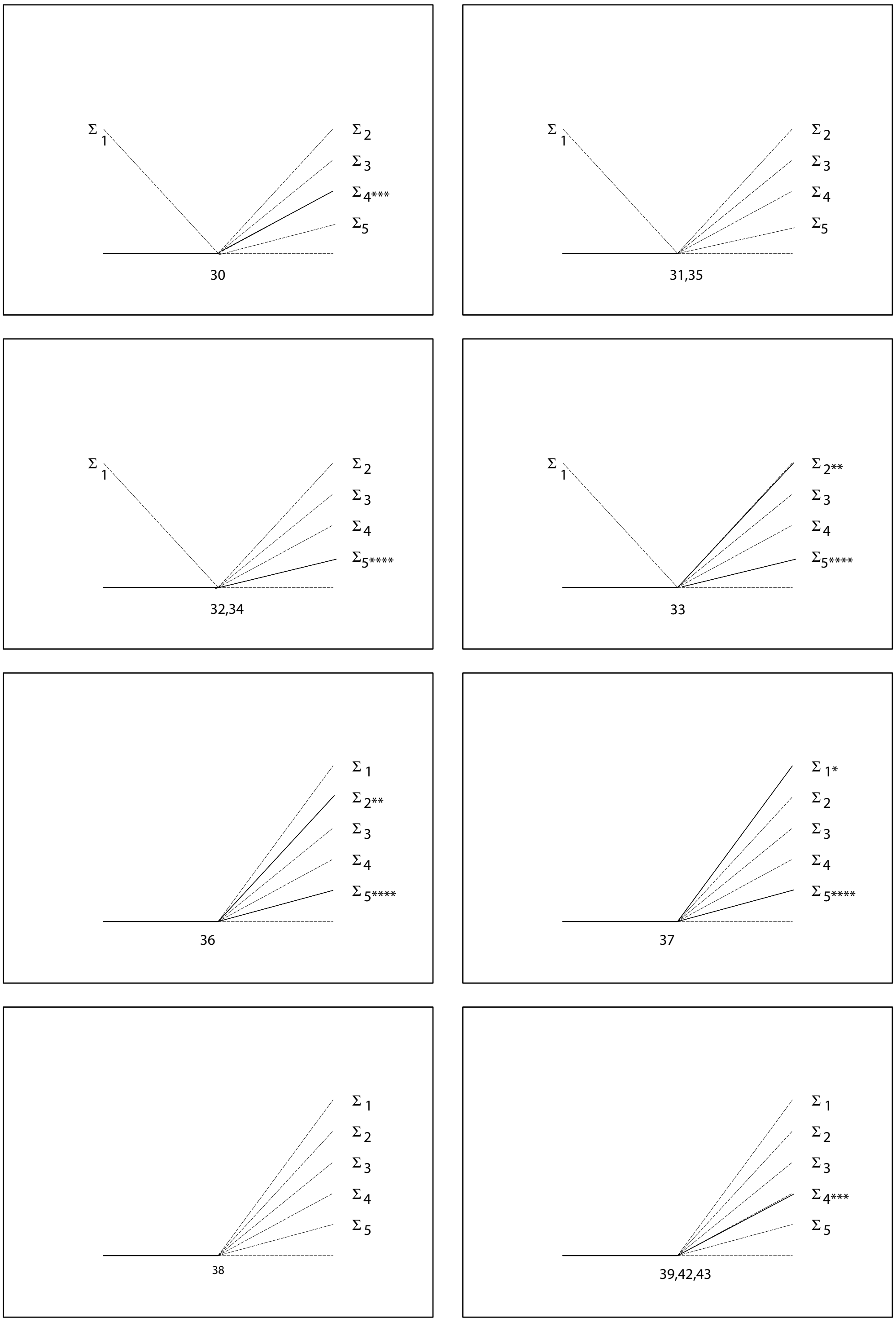

Figure 5.4: Continuation of Figure 5.3.

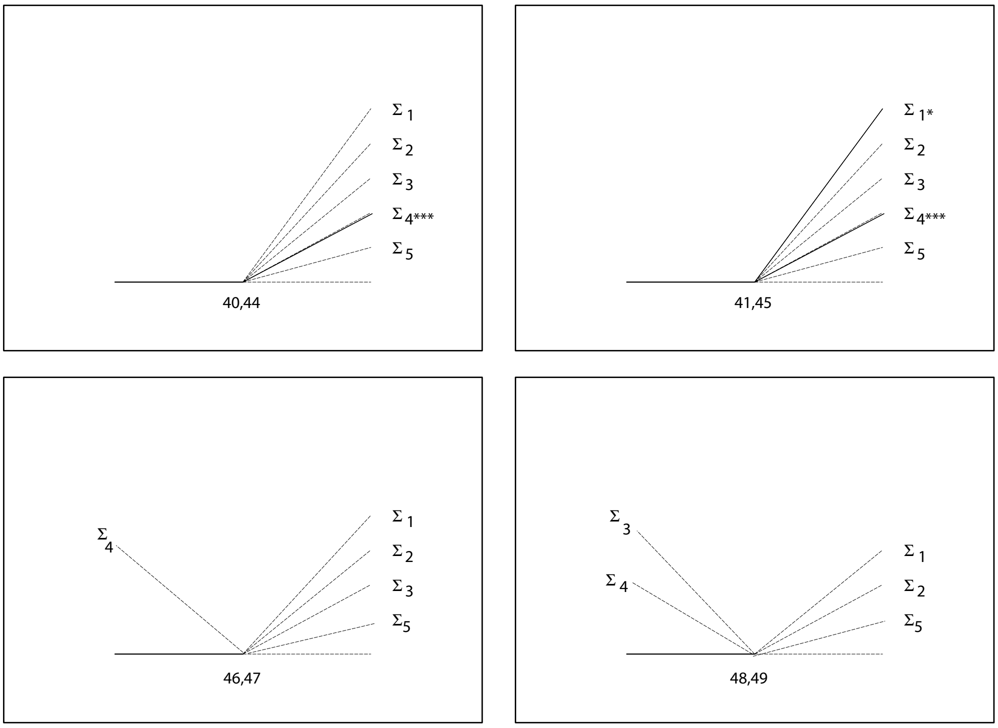

Figure 5.5: Continuation of Figure 5.3.

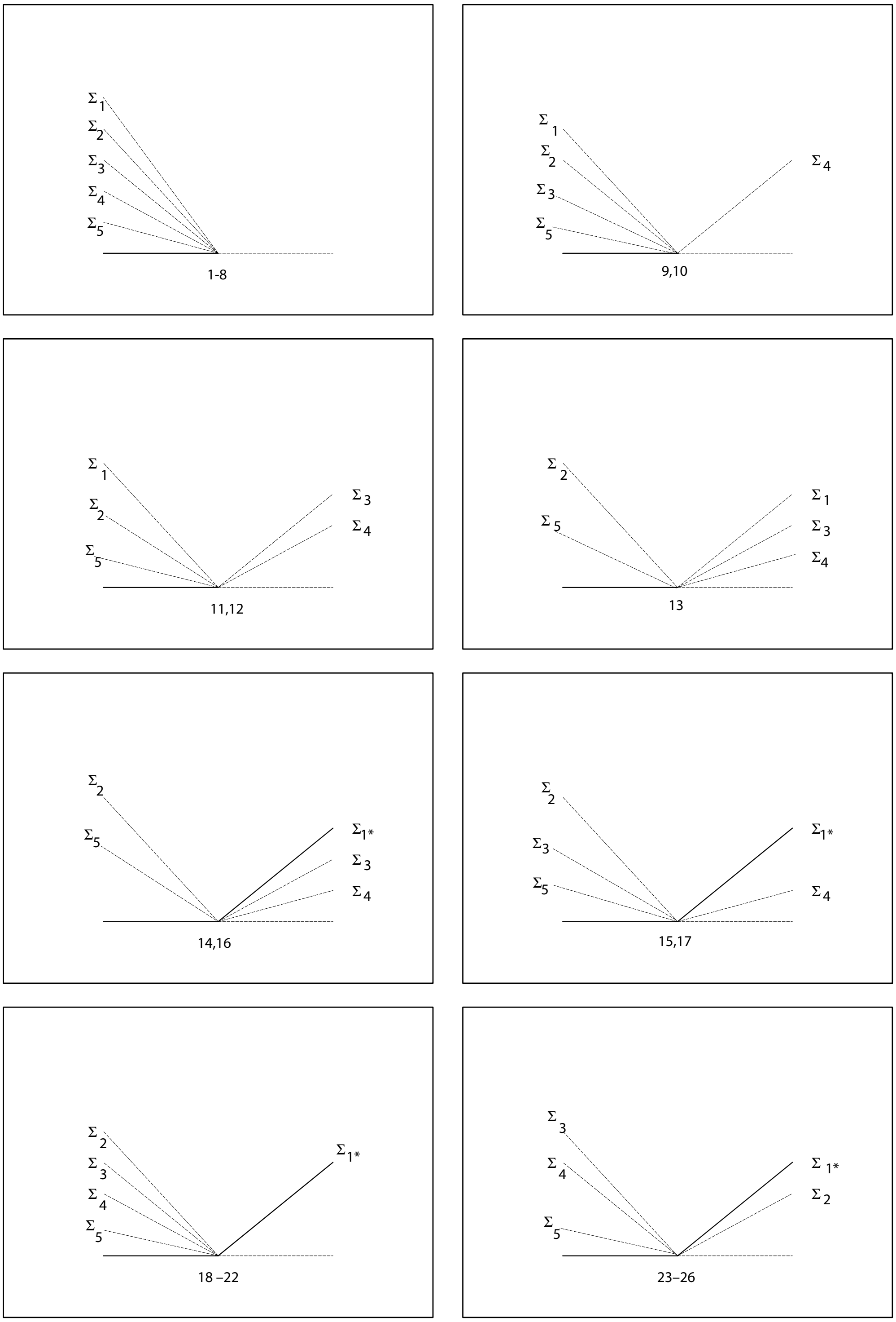

Figure 5.6: Bifurcation diagrams for the nondegenerate Hopf bifurcation with $\mathbf{S}_4$ symmetry. Broken (unbroken) bifurcation curves indicate unstable (stable) solutions. An asterisk on solution indicates that it is possible for the solution to be unstable, depending on the sign of (5.7). The diagrams are plotted for $A_{1r} > 0$.

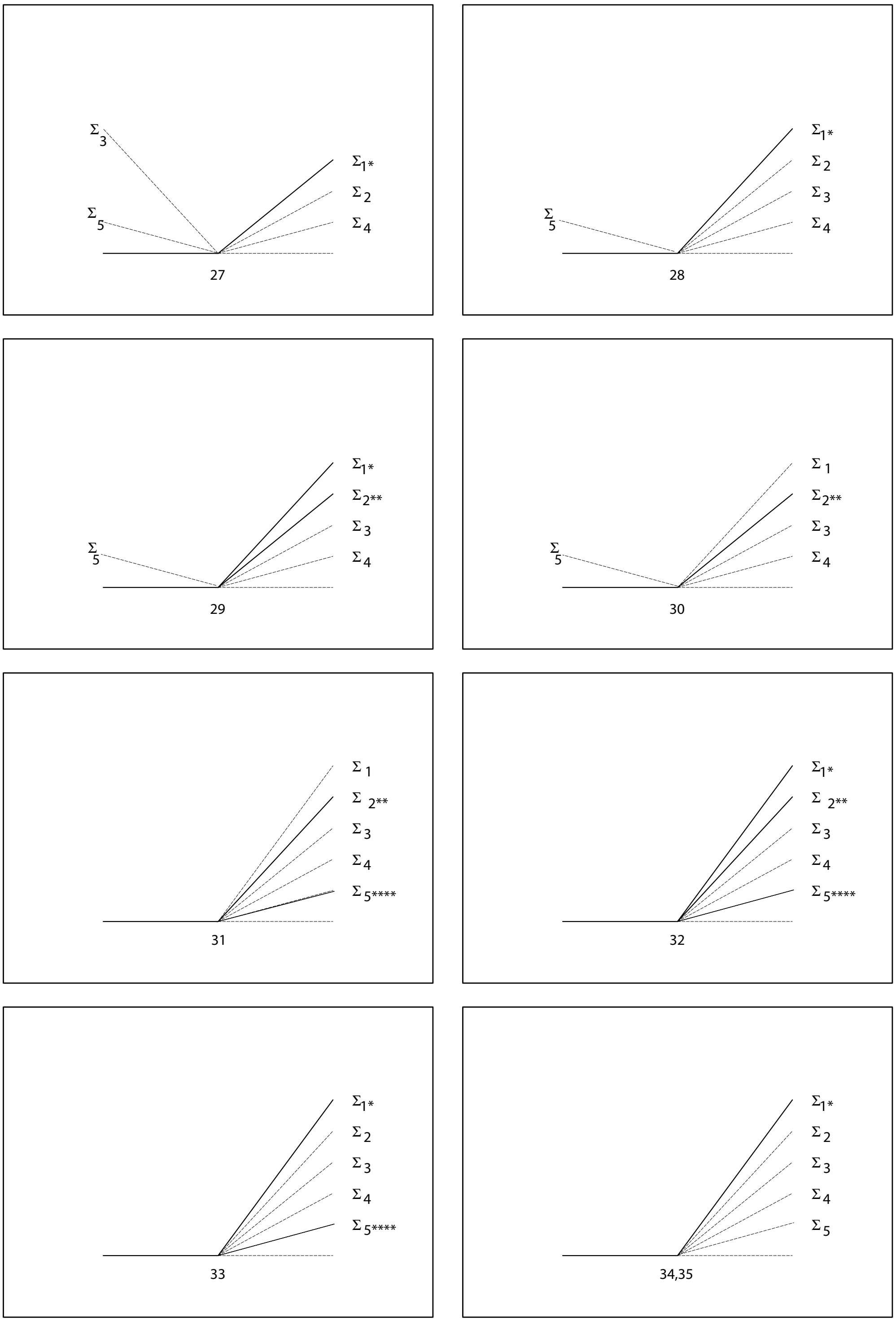

Figure 5.7: Continuation of Figure 5.6

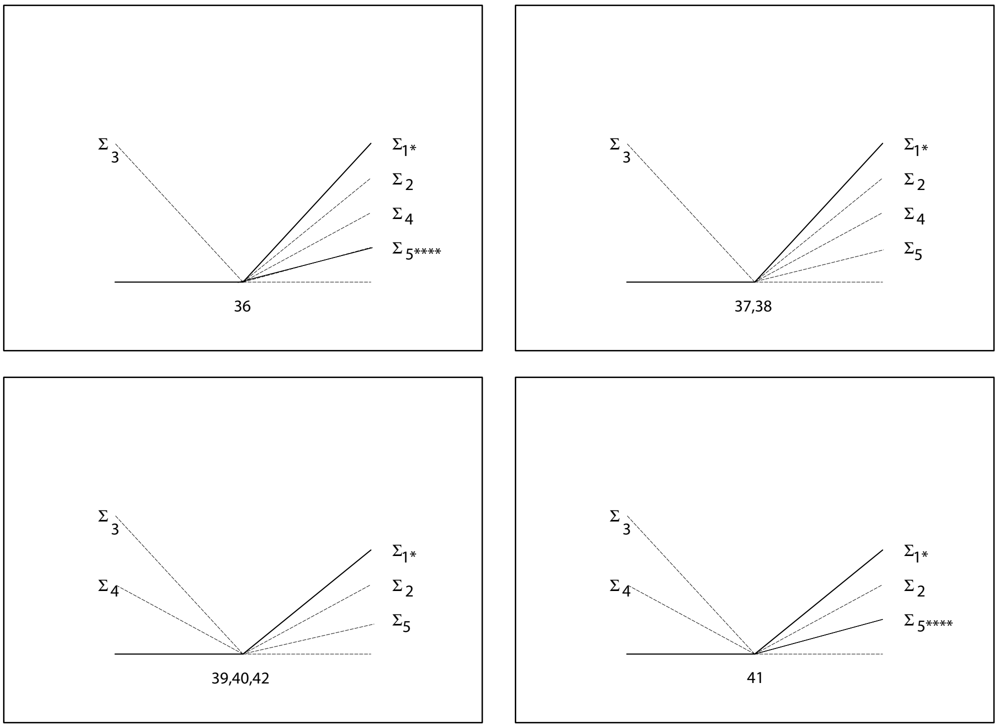

Figure 5.8: Continuation of Figure 5.6

5.3 Examples

We consider the system of ODEs (5.2) where f is as in (5.3). We assume the following parameter values:

$$A_{1r} = 1,\ A_{1i} = 1,\ A_{2r} = 0.3,\ A_{2i} = -0.7,\ A_{3r} = -5.$$

From Theorem 5.2 and Table 5.4 we obtain that for the isotropy subgroups Σ_1, Σ_2 and Σ_5, the corresponding branches of periodic solutions are supercritical and the solutions are stable (near the origin). Furthermore, we obtain that for the isotropy subgroups Σ_3 and Σ_4, the corresponding branches of periodic solutions are supercritical and unstable. This situation corresponds to the bifurcation diagram for region 32 in Figure 5.7.

We assume now the following parameter values:

$$A_{1r} = -1,\ A_{1i} = -1,\ A_{2r} = 1,\ A_{2i} = 2,\ A_{3r} = -4.$$

We get that the branches of periodic solutions with Σ_i-symmetry for $i = 1, \ldots, 5$ bifurcate supercritically. Moreover, the solutions with Σ_1, Σ_5-symmetry are stable and the solutions with $\Sigma_2, \Sigma_3, \Sigma_4$-symmetry are unstable (near the origin). This situation corresponds to the bifurcation diagram for region 37 in Figure 5.4.

5.4 Periodic solutions with submaximal isotropy

In the previous section we considered the possible branches of periodic solutions with maximal isotropy that could generically bifurcate for the system (5.2). We look now for possible branches of periodic solutions that can bifurcate with submaximal isotropy.

We have that $\widetilde{\mathbf{Z}}_2 = \langle((13)(24), \pi)\rangle$ and $\mathbf{S}_2 = \langle(23)\rangle$ are submaximal isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$. See Table 5.6. We will study $g|_{\mathrm{Fix}(\Delta)}$ where Δ is either $\widetilde{\mathbf{Z}}_2$ or $\mathbf{S}_2$. By Proposition 5.4 below, we have that the normalizer of Δ in $\mathbf{S}_4 \times \mathbf{S}^1$, where $\Delta = \widetilde{\mathbf{Z}}_2$ or $\mathbf{S}_2$, is the largest subgroup of $\mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathrm{Fix}(\Delta)$. We start by computing these normalizers.

Let Σ be a subgroup of Γ. We define the *normalizer* of Σ in Γ as:

$$N_\Gamma(\Sigma) = \{\gamma \in \Gamma : \gamma\Sigma\gamma^{-1} = \Sigma\}.$$

Proposition 5.4 *Let Σ be an isotropy subgroup of Γ. Then $N_\Gamma(\Sigma)$ is the largest subgroup of Γ that leaves* $\mathrm{Fix}(\Sigma)$ *invariant.*

Proof: See for example [18, Proposition 5.2.2]. □

The following lemma will be extremely useful. Recall that by [23, Definition XVI 7.1] the isotropy subgroups $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ are always of the form $G^\theta = \{(g, \theta(g)) \in \Gamma \times \mathbf{S}^1 : g \in G\}$ where $G \subseteq \Gamma$ and $\theta : G \to \mathbf{S}^1$ is a group homomorphism. Denote by $K = Ker(\theta)$.

Lemma 5.5 *Let $G^\theta \subseteq \Gamma \times \mathbf{S}^1$. Then $N_{\Gamma \times \mathbf{S}^1}(G^\theta) = C(G, K) \times \mathbf{S}^1$ where $C(G, K) = \{\gamma \in \Gamma : \gamma g \gamma^{-1} g^{-1} \in K, \forall g \in G\}$.*

Proof: See [22, Lemma 2.5]. $\square$

Remark 5.6 Let $G = \mathbf{D}_n = \langle a, b : a^n = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$, and write the $2n$ elements of G in the form $a^i b^j$ with $0 \le i \le n-1,\ 0 \le j \le 1$. Let H be any group, and suppose that H contains elements x and y which satisfy

$$x^n = y^2 = 1,\ y^{-1}xy = x^{-1}.$$

The function $h : G \to H$ defined by

$$h : a^i b^j \to x^i y^j \ (0 \le i \le n-1,\ 0 \le j \le 1)$$

is an isomorphism. For a proof of this result see for example [30, Chapter 1].

$\diamond$

We now prove the following result:

Lemma 5.7 *Let* $\widetilde{\mathbf{Z}}_2 = \langle ((13)(24), \pi) \rangle$ *and* $\mathbf{S}_2 = \langle (23) \rangle$. *Then:*
(a) If $\Sigma = \widetilde{\mathbf{Z}}_2$ *then* $N_{\mathbf{S}_4 \times \mathbf{S}^1}(\Sigma) \cong \mathbf{D}_4 \times \mathbf{S}^1$.
(b) If $\Sigma = \mathbf{S}_2$ *then* $N_{\mathbf{S}_4 \times \mathbf{S}^1}(\Sigma) \cong \mathbf{D}_2 \times \mathbf{S}^1$.

Proof: We start by proving (a). Let $\Sigma = \widetilde{\mathbf{Z}}_2 = \langle ((13)(24), \pi) \rangle \subseteq \mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathbf{C}^{4,0}$. Then $\mathrm{Fix}(\Sigma) = \{(z_1, z_2, -z_1, -z_2) :\ z_1 \in \mathbf{C}\}$. We have $G^\theta = \widetilde{\mathbf{Z}}_2 = \{Id, ((13)(24), \pi)\} \subseteq \mathbf{S}_4 \times \mathbf{S}^1$, where G is the projection of G^θ into $\mathbf{S}_4$, that is,

$$G = \Pi_{\mathbf{S}_4}(\widetilde{\mathbf{Z}}_2) = \{Id, (13)(24)\} \tag{5.8}$$

and θ is the homomorphism

$$\begin{aligned} \theta : \qquad G &\to \mathbf{S}^1 \\ Id &\mapsto 0 \\ (13)(24) &\mapsto \pi \end{aligned}$$

with kernel given by

$$K = \mathrm{Ker}(\theta) = \{g \in G : \theta(g) = 0\} = \{Id\}. \tag{5.9}$$

The centralizer $C(G, K)$ is given by

$$\begin{aligned} C(G,K) &= \{\gamma \in \mathbf{S}_4 : \gamma g \gamma^{-1} g^{-1} \in K, \forall g \in G\} \\ &= \{\sigma \in \mathbf{S}_4 : \sigma g \sigma^{-1} g^{-1} = Id, \forall g \in \{Id, (12)(34)\}\} \\ &= \{\sigma \in \mathbf{S}_4 : \sigma(13)(24) = (13)(24)\sigma\} \\ &= \{Id, (24), (12)(34), (1432), (13)(24), (1234), (14)(23), (13)\}. \end{aligned} \tag{5.10}$$

We apply now Lemma 5.5. We have

$$N_{\mathbf{S}_4 \times \mathbf{S}^1}(\widetilde{\mathbf{Z}}_2) = C(G, K) \times \mathbf{S}^1$$

Isotropy Subgroup	Generators	Fixed-Point Subspace
$\Delta_1 = \widetilde{\mathbf{Z}}_2$	$((13)(24), \pi)$	$\{(z_1, z_2, -z_1, -z_2):\ z_1 \in \mathbf{C}\}$
$\Delta_2 = \mathbf{S}_2$	(23)	$\{(z_1, z_2, z_2, -z_1 - 2z_2):\ z_1 \in \mathbf{C}\}$

Table 5.6: Generators and fixed-point subspaces corresponding to the isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ with fixed-point subspaces of complex dimension two.

where $C(G, K)$ is given by (5.10).

Recall Remark 5.6. Since

$$\mathbf{D}_4 = \langle a, b : a^4 = b^2 = Id, b^{-1}ab = a^{-1} \rangle$$

and taking $x = (1432),\ y = (24)$ we have $C(G, K) \cong \mathbf{D}_4$. Thus we have proved that

$$N_{\Gamma \times \mathbf{S}^1}(\widetilde{\mathbf{Z}}_2) \cong \mathbf{D}_4 \times \mathbf{S}^1.$$

We now prove (b). Let $\Sigma = \mathbf{S}_2 = \langle (23) \rangle \subseteq \mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathbf{C}^{4,0}$. Then $\mathrm{Fix}(\Sigma) = \{(z_1, z_2, z_2, -z_1 - 2z_2):\ z_1 \in \mathbf{C}\}$. Recall Proposition 5.4. Clearly, $(2\,3)$, $(1\,4)$ and $(2\,3)(1\,4)$ are the only permutations from the twenty four elements of $\mathbf{S}_4$ which leaves $\mathrm{Fix}(\Sigma)$ invariant. Moreover, every $\theta \in \mathbf{S}^1$ leaves $\mathrm{Fix}(\Sigma)$ invariant. Thus, we have

$$N_\Gamma(\Sigma) = \{Id, (2\ 3), (1\ 4), (2\ 3)(1\ 4)\} \times \mathbf{S}^1.$$

Set $H = \{Id, (2\ 3), (1\ 4), (2\ 3)(1\ 4)\}$. Recall Remark 5.6. Since

$$\mathbf{D}_2 = \langle a, b : a^2 = b^2 = Id, b^{-1}ab = a^{-1} \rangle$$

take $x = (2\ 3),\ y = (1\ 4) \in H$. Then H and $\mathbf{D}_2$ are isomorphic. Thus, we have

$$N_\Gamma(\Sigma) \cong \mathbf{D}_2 \times \mathbf{S}^1.$$

□

In [4], Ashwin and Podvigina considered Hopf bifurcation with the group $\mathbf{O}$ of rotational symmetries of the cube. The group $\mathbf{O}$ is isomorphic to $\mathbf{S}_4$ and it has two non-isomorphic real irreducible representations of dimension three. In [4] they consider the irreducible representation of $\mathbf{O}$ corresponding to rotational symmetries of a cube in $\mathbf{R}^3 = W$. When studying Hopf bifurcation, they take two copies of this irreducible representation. Specifically, they consider the action of $\mathbf{O} \times \mathbf{S}^1$ on $W \oplus W$ generated by:

$$\begin{array}{l} \rho_{111}(z_1, z_2, z_3) = (z_2, z_3, z_1) \\ \rho_{001}(z_1, z_2, z_3) = (z_2, -z_1, z_3) \\ \gamma_\theta(z_1, z_2, z_3) = e^{i\theta}(z_1, z_2, z_3)\ (\theta \in \mathbf{S}^1). \end{array} \tag{5.11}$$

Although the permutation group $\mathbf{S}_4$ is isomorphic to the group of rotations of a cube, the action of $\mathbf{O}$ on W and the natural action of $\mathbf{S}_4$ on $\mathbf{R}^{4,0}$ are not isomorphic. Recall that $\mathbf{R}^{4,0} = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$. However, the action of $\mathbf{O} \times \mathbf{S}^1$ on $W \oplus W$ and the action of $\mathbf{S}_4 \times \mathbf{S}^1$ on $\mathbf{C}^{4,0}$ are isomorphic (see [4]).

In [4], the isotropy lattice for the Γ-simple action of $\mathbf{O} \times \mathbf{S}^1$ on $\mathbf{C}^3$ is obtained and the isotropy subgroups with fixed-point subspaces of complex dimension two have normalizers given, respectively, by $\mathbf{D}_4 \times \mathbf{S}^1$ and $\mathbf{D}_2 \times \mathbf{S}^1$. These are in correspondence with the normalizers of $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_2$ as we obtained above. We list in Table 5.6 the submaximal isotropy subgroups $\mathbf{S}_4 \times \mathbf{S}^1$ with fixed-point subspaces of complex dimension two and their respective generators.

As was stated, when f is supposed to commute also with $\mathbf{S}^1$, then the problem of finding periodic solutions of $\dot{z} = f(z, \lambda)$ can be transformed to the problem of finding the zeros of $\dot{z} = g(z, \lambda, \tau)$ where $g = f - (1 + \tau)iz$. However, for the branches of periodic solutions with submaximal isotropy that are found here, we can no longer guarantee that they exist for (5.2) if f commutes only with $\mathbf{S}_4$ (even with the third order Taylor series commuting with $\mathbf{S}^1$). These solutions branches are guaranteed only for the third order truncation with which we work from now on. Consider the truncation of f as in (5.3) of degree three and the respective reduced vector field $g = f - (1 + \tau)iz$ of the same degree.

Recall Table 5.6. When we restrict g to $\mathrm{Fix}(\widetilde{\mathbf{Z}}_2) = \{(z_1, z_2, -z_1, -z_2) : \; z_1 \in \mathbf{C}\}$, we obtain the following system:

$$
\begin{array}{l}
\dot{z}_+ = z_+ \left(\lambda + i\omega + A(|z_+|^2 + |z_-|^2) + B|z_+|^2\right) + C\overline{z}_+ z_-^2 \\
\dot{z}_- = z_- \left(\lambda + i\omega + A(|z_+|^2 + |z_-|^2) + B|z_-|^2\right) + C\overline{z}_- z_+^2
\end{array}
\qquad (5.12)
$$

where $(z_+, z_-) \in \mathbf{C}^2$, $A = 2A_3, B = A_1 + 2A_2$ and $C = 2A_2$. This is the normal form for the generic Hopf bifurcation problem with symmetry $\mathbf{D}_4$ studied, for example, by Swift [39].

The nontrivial solutions in the space $\mathrm{Fix}(\widetilde{\mathbf{Z}}_2) = \{(z_1, z_2, -z_1, -z_2) : \; z_1 \in \mathbf{C}\}$ with maximal isotropy are the solutions with symmetry $\widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}, \widetilde{\mathbf{Z}}_4, \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$, , corresponding, respectively, to zeros of type $z_+ = z_-$, $z_+ = iz_-$ and $z_+ = 0$ (note that for solutions corresponding to the isotropy subgroup $\widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$ we have that $(z_1, 0, -z_1, 0)$ is conjugated to $(z_1, -z_1, 0, 0)$). Their stability properties are studied in [23], [24] and [39].

By [39], in addition to these periodic solutions, there can be a fourth branch of periodic solutions to (5.12) with $z_+ \neq z_-$ and $z_+ z_- \neq 0$. Thus, these correspond to solutions of (5.2) where f is as in (5.3) truncated to the third order with $\widetilde{\mathbf{Z}}_2$-symmetry. Moreover, this solution branch exists if

$$
\left|\mathrm{Re}[2(A_1 + 2A_2)\overline{A}_2]\right| < |2A_2|^2 < |A_1 + 2A_2|^2
$$

and the solutions are generically unstable.

Chapter 6

Hopf Bifurcation with $\mathbf{S}_5$-symmetry

In this chapter we consider Hopf bifurcation with $\mathbf{S}_N$-symmetry for the special case $N = 5$.

For general N, from Theorem 4.13 we know that the stability of some of the periodic solutions guaranteed by the Equivariant Hopf Theorem in some directions is determined by the fifth degree truncation of the vector field. Furthermore, in one particular direction, even the fifth degree truncation of the vector field is too degenerate to determine their stability. When $N = 5$ the directions in which we need the degree five truncation of the vector field are present in the isotypic decomposition for some of the $\mathbf{C}$-axial isotropy subgroups.

Recall Theorem 4.1 and Section 4.2. We have two types of $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$: $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ and $\Sigma^{II}_q = \mathbf{S}_q \times \mathbf{S}_p$. From Theorem 4.13, we have that if $k > 3$ and $q \geq 2$ in $\Sigma^I_{q,p}$, then the fifth degree truncation of the vector field is too degenerate to determine the stability of solutions with those symmetries in some particular directions. In the case $N = 5$, the isotropy subgroups that we find are all of the form $q < 2$ except one of them (see Σ_1 in Table 6.1), but for this one we have $k = 2$. Thus, this is the first case where the fifth degree truncation of the vector field is necessary to determine the stability of such solutions. Moreover, the degree five truncation of a general $\mathbf{S}_5$-equivariant vector field determines the stability and the criticality of the branches of periodic solutions guaranteed by the Equivariant Hopf Theorem. We consider so this special case and we give the explicit conditions on the coefficients of the general degree 5 vector field equivariant under $\mathbf{S}_5 \times \mathbf{S}^1$ determining the stability and the criticality of those solutions.

Recall Chapter 4. Consider the action of $\mathbf{S}_5 \times \mathbf{S}^1$ on $\mathbf{C}^{5,0}$ given by

$$(\sigma, \theta)(z_1, z_2, z_3, z_4, z_5) \;=\; e^{i\theta}\left(z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}, z_{\sigma^{-1}(4)}, z_{\sigma^{-1}(5)}\right) \tag{6.1}$$

for $\sigma \in S_5, \theta \in \mathbf{S}^1$ and $(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^{5,0}$, with

$$\mathbf{C}^{5,0} = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : z_1 + z_2 + z_3 + z_4 + z_5 = 0\}.$$

In this chapter we study Hopf bifurcation with $\mathbf{S}_5$-symmetry. We consider the system of ODEs

$$\frac{dz}{dt} = f(z, \lambda), \tag{6.2}$$

Isotropy Subgroup	Generators	Fixed-Point Subspace
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$	(12), (34), $((13)(24),\pi)$	$\{(z_1, z_1, -z_1, -z_1, 0) : \ z_1 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_3$	(34), (35), $((12),\pi)$	$\{(z_1, -z_1, 0, 0, 0) : \ z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3 \times \mathbf{S}_2$	(45), $\left((123), \frac{2\pi}{3}\right)$	$\left\{(z_1, \xi z_1, \xi^2 z_1, 0, 0) : \ z_1 \in \mathbf{C}\right\}, \ \xi = e^{2\pi i/3}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$	$\left((1234), \frac{\pi}{2}\right)$	$\{(z_1, iz_1, -z_1, -iz_1, 0) : \ z_1 \in \mathbf{C}\}$
$\Sigma_5 = \widetilde{\mathbf{Z}}_5$	$\left((12345), \frac{2\pi}{5}\right)$	$\left\{(z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1) : \ z_1 \in \mathbf{C}\right\}, \ \xi = e^{2\pi i/5}$
$\Sigma_6 = \mathbf{S}_2 \times \mathbf{S}_3$	(12), $(34), (35)$	$\left\{(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1) : \ z_1 \in \mathbf{C}\right\}$
$\Sigma_7 = \mathbf{S}_4$	(23), (24), (25)	$\left\{(z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1) : \ z_1 \in \mathbf{C}\right\}$

Table 6.1: $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$ acting on $\mathbf{C}^{5,0}$, generators and fixed-point subspaces.

where $f : \mathbf{C}^{5,0} \times \mathbf{R} \to \mathbf{C}^{5,0}$ is smooth, commutes with $\mathbf{S}_5$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0, \rho(0) = 1$ and $\sigma'(0) \neq 0$.

After recalling the $\mathbf{C}$-axial subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$ acting on $\mathbf{C}^{5,0}$, we use the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions with these symmetries of (6.2) by Hopf bifurcation from the trivial equilibrium at $\lambda = 0$. In Theorem 6.2 we determine (generically) the directions of branching and the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem. For that we need, in this case, the degree five truncation of the Taylor expansion around the bifurcation point.

From Theorem 4.1 we obtain a description of the $\mathbf{C}$-axial subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$ acting on $\mathbf{C}^{5,0}$:

Proposition 6.1 *There are seven conjugacy classes of $\mathbf{C}$-axial subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$ for the action on $\mathbf{C}^{5,0}$. They are listed, together with their generators and fixed-point subspaces in Table 6.1.*

Let f be as in (6.2). If we suppose that the Taylor series of degree five of f around $z = 0$ commutes also with $\mathbf{S}^1$, then by Theorems 4.6 and 4.10, taking $N = 5$, we can write $f = (f_1, f_2, f_3, f_4, f_5)$, where

Isotropy Subgroup	Branching Equations
Σ_1	$\nu + (A_1 + 4A_2 + 4A_3)\lvert z\rvert^2 + \cdots = 0$
Σ_2	$\nu + (A_1 + 2A_2 + 2A_3)\lvert z\rvert^2 + \cdots = 0$
Σ_3	$\nu + (A_1 + 3A_3)\lvert z\rvert^2 + \cdots = 0$
Σ_4	$\nu + (A_1 + 4A_3)\lvert z\rvert^2 + \cdots = 0$
Σ_5	$\nu + (A_1 + 5A_3)\lvert z\rvert^2 + \cdots = 0$
Σ_6	$\nu + \frac{1}{3}\left(\frac{7}{3}A_1 + 10A_2 + 10A_3\right)\lvert z\rvert^2 + \cdots = 0$
Σ_7	$\nu + \frac{1}{4}\left(\frac{13}{4}A_1 + 5A_2 + 5A_3\right)\lvert z\rvert^2 + \cdots = 0$

Table 6.2: Branching equations for $\mathbf{S}_5 \times \mathbf{S}^1$ Hopf bifurcation. Here $\nu(\lambda) = \mu(\lambda) - (1 + \tau i)$ and $+\cdots$ stands for higher order terms.

Isotropy Subgroup	Branching Equations
Σ_1	$\lambda = -(A_{1r} + 4A_{2r} + 4A_{3r})\lvert z\rvert^2 + \cdots$
Σ_2	$\lambda = -(A_{1r} + 2A_{2r} + 2A_{3r})\lvert z\rvert^2 + \cdots$
Σ_3	$\lambda = -(A_{1r} + 3A_{3r})\lvert z\rvert^2 + \cdots$
Σ_4	$\lambda = -(A_{1r} + 4A_{3r})\lvert z\rvert^2 + \cdots$
Σ_5	$\lambda = -(A_{1r} + 5A_{3r})\lvert z\rvert^2 + \cdots$
Σ_6	$\lambda = -\frac{1}{3}\left(\frac{7}{3}A_{1r} + 10A_{2r} + 10A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_7	$\lambda = -\frac{1}{4}\left(\frac{13}{4}A_{1r} + 5A_{2r} + 5A_{3r}\right)\lvert z\rvert^2 + \cdots$

Table 6.3: Branching equations for $\mathbf{S}_5$ Hopf bifurcation. Subscript r on the coefficients refer to the real part and $+\cdots$ stands for higher order terms.

Isotropy Subgroup	Δ_0	$\Delta_1, \ldots, \Delta_r$
Σ_1	$A_{1r} + 4A_{2r} + 4A_{3r}$	$-\frac{3}{5}A_{1r} - 4A_{2r}$ $-\left(\lvert -\frac{3}{5}A_1 - 4A_2\rvert^2 - \lvert\frac{1}{5}A_1 + 4A_2\rvert^2\right)$ $A_{1r} - 4A_{2r}$ $-\left(\lvert A_1 - 4A_2\rvert^2 - \lvert A_1 + 4A_2\rvert^2\right)$
Σ_2	$A_{1r} + 2A_{2r} + 2A_{3r}$	$\frac{1}{5}A_{1r} - 2A_{2r}$ $-\left(\lvert\frac{1}{5}A_1 - 2A_2\rvert^2 - \lvert\frac{3}{5}A_1 + 2A_2\rvert^2\right)$ $-A_{1r} - 2A_{2r}$ $-\left(\lvert A_1 + 2A_2\rvert^2 - \lvert 2A_2\rvert^2\right)$
Σ_3	$A_{1r} + 3A_{3r}$	$-A_{1r}$ $-\left(\lvert A_1 + 6A_2\rvert^2 - \lvert\frac{2}{5}A_1\rvert^2\right)$ $\lvert A_1\rvert^2$ $A_{1r} + 6A_{2r}$ A_{1r} $-\lvert A_1\rvert^2$
Σ_4	$A_{1r} + 4A_{3r}$	$-\lvert A_1\rvert^2$ $-A_{1r}$ A_{1r} $-\left(\lvert A_1 + 8A_2\rvert^2 - \lvert A_1\rvert^2\right)$ $A_{1r} + 8A_{2r}$
Σ_5	$A_{1r} + 5A_{3r}$	$-\lvert A_1\rvert^2$ A_{1r} $-\mathrm{Re}[A_1(\overline{\xi}_1 - \overline{\xi}_2)]$ $A_{1r} + 10A_{2r}$ $-\left(\lvert A_1 + 10A_2\rvert^2 - \lvert A_1\rvert^2\right)$
Σ_6	$\frac{7}{3}A_{1r} + 10A_{2r} + 10A_{3r}$	$\frac{11}{3}A_{1r} - 10A_{2r}$ $-\left(\lvert\frac{1}{3}\left(\frac{11}{3}A_1 - 10A_2\right)\rvert^2 - \lvert A_1 + \frac{10}{3}A_2\rvert^2\right)$ $\frac{1}{3}A_{1r} - 8A_{2r}$ $-\left(\lvert\frac{1}{3}\left(\frac{1}{3}A_1 - 8A_2\right)\rvert^2 - \lvert\frac{1}{3}\left(\frac{4}{3}A_1 + 10A_2\right)\rvert^2\right)$
Σ_7	$\frac{13}{4}A_{1r} + 5A_{2r} + 5A_{3r}$	$\mathrm{Re}(-\frac{55}{80}A_1 - \frac{5}{4}A_2)$ $-\left(\lvert -\frac{55}{80}A_1 - \frac{5}{4}A_2\rvert^2 - \lvert\frac{1}{16}A_1 + \frac{5}{4}A_2\rvert^2\right)$

Table 6.4: Stability for $\mathbf{S}_5$ Hopf bifurcation. Here $\xi_1 = 2A_4 + 10A_{14}$ and $\xi_2 = 2A_4 + 5A_{11} + 5A_{14}$. Note that solutions with Σ_3 and Σ_4 symmetry are always unstable.

Isotropy subgroup and Orbit Representative	Isotypic components of $\mathbf{C}^{5,0}$
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$ $z = (z_1, z_1, -z_1, -z_1, 0)$	$W_0 = \mathrm{Fix}(\Sigma_1) = \{(z_1, z_1, -z_1, -z_1, 0) : z_1 \in \mathbf{C}\}$ $W_1 = \{(z_1, z_1, z_1, z_1, -4z_1) : z_1 \in \mathbf{C}\}$ $W_3 = \{(z_1, -z_1, z_2, -z_2, 0) : z_1, z_2 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_3$ $z = (z_1, -z_1, 0, 0, 0)$	$W_0 = \mathrm{Fix}(\Sigma_2) = \{(z_1, -z_1, 0, 0, 0) : z_1 \in \mathbf{C}\}$ $W_1 = \left\{(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1) : z_1 \in \mathbf{C}\right\}$ $W_2 = \{(0, 0, z_1, z_2, -z_1 - z_2) : z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3 \times \mathbf{S}_2$ $z = (z_1, \xi z_1, \xi^2 z_1, 0, 0)$ $\xi = e^{2\pi i/3}$	$W_0 = \mathrm{Fix}(\Sigma_3) = \left\{(z_1, \xi z_1, \xi^2 z_1, 0, 0) : z_1 \in \mathbf{C}\right\}$ $W_1 = \left\{(z_1, z_1, z_1, -\frac{3}{2}z_1, -\frac{3}{2}z_1) : z_1 \in \mathbf{C}\right\}$ $W_2 = \{(0, 0, 0, z_1, -z_1) : z_1 \in \mathbf{C}\}$ $P_2 = \left\{(z_1, \xi^2 z_1, \xi^4 z_1, 0, 0) : z_1 \in \mathbf{C}\right\}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$ $z = (z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, 0)$ $\xi = i$	$W_0 = \mathrm{Fix}(\Sigma_4) = \left\{(z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, 0) : z_1 \in \mathbf{C}\right\}$ $W_1 = \{(z_1, z_1, z_1, z_1, -4z_1) : z_1 \in \mathbf{C}\}$ $P_2 = \left\{(z_1, \xi^2 z_1, \xi^4 z_1, \xi^6 z_1, 0) : z_1 \in \mathbf{C}\right\}$ $P_3 = \left\{(z_1, \xi^3 z_1, \xi^6 z_1, \xi^9 z_1, 0) : z_1 \in \mathbf{C}\right\}$
$\Sigma_5 = \widetilde{\mathbf{Z}}_5$ $z = (z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1)$ $\xi = e^{2\pi i/5}$	$W_0 = \mathrm{Fix}(\Sigma_5) = \left\{(z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1) : z_1 \in \mathbf{C}\right\}$ $P_2 = \left\{(z_1, \xi^2 z_1, \xi^4 z_1, \xi^6 z_1, \xi^8 z_1) : z_1 \in \mathbf{C}\right\}$ $P_3 = \left\{(z_1, \xi^3 z_1, \xi^6 z_1, \xi^9 z_1, \xi^{12} z_1) : z_1 \in \mathbf{C}\right\}$ $P_4 = \left\{(z_1, \xi^4 z_1, \xi^8 z_1, \xi^{12} z_1, \xi^{16} z_1) : z_1 \in \mathbf{C}\right\}$
$\Sigma_6 = \mathbf{S}_2 \times \mathbf{S}_3$ $z = \left(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1\right)$	$W_0 = \mathrm{Fix}(\Sigma_6) = \left\{\left(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1\right) : z_1 \in \mathbf{C}\right\}$ $W_1 = \{(z_1, -z_1, 0, 0, 0) : z_1, z_2, \in \mathbf{C}\}$ $W_2 = \{(0, 0, z_1, z_2, -z_1 - z_2) : z_1, z_2, \in \mathbf{C}\}$
$\Sigma_7 = \mathbf{S}_4$ $z = \left(z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1\right)$	$W_0 = \mathrm{Fix}(\Sigma_7) = \left\{\left(z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1\right) : z_1 \in \mathbf{C}\right\}$ $W_1 = \{(0, z_2, z_3, z_4, -z_2 - z_3 - z_4) : z_2, z_3, z_4, \in \mathbf{C}\}$

Table 6.5: Isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of each of the isotropy subgroups listed in Table 6.1.

$$
\begin{aligned}
&f_1(z,\lambda) = \mu(\lambda) z_1 \ + \ f_1^{(3)}(z) \ + \ f_1^{(5)}(z) \ + \ \cdots \\
&f_2(z,\lambda) = f_1(z_2, z_1, z_3, z_4, z_5, \lambda) \\
&\cdots \\
&f_5(z,\lambda) = f_1(z_5, z_2, z_3, z_4, z_1, \lambda)
\end{aligned}
\tag{6.3}
$$

where

$$
f_1^{(3)}(z) = \textstyle\sum_{i=1}^{3} A_j f_{1,j}(z),
$$

$$
f_1^{(5)}(z) = \textstyle\sum_{i=4, i\neq 12}^{15} A_j f_{1,j}(z),
$$

with $z_5 = -z_1 - z_2 - z_3 - z_4$. The coefficients $A_i,\ i = 1, \ldots, 15,\ i \neq 12$ are complex smooth functions of λ, $\mu(0) = i$ and $\mathrm{Re}(\mu'(0)) \neq 0$. Note that by Remark 4.12 we don't have the term $f_{1,12}$ and the other $f_{1,i}$ are given by

$$
\begin{array}{lcl}
f_{1,1}(z) &=& \left[\frac{4}{5}|z_1|^2 z_1 - \frac{1}{5}\left(|z_2|^2 z_2 + |z_3|^2 z_3 + |z_4|^2 z_4 + |z_5|^2 z_5\right)\right] \\
f_{1,2}(z) &=& \overline{z}_1 \left(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2\right) \\
f_{1,3}(z) &=& z_1 \left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2\right) \\
\\
f_{1,4}(z) &=& \left[\frac{4}{5}|z_1|^4 z_1 - \frac{1}{5}\left(|z_2|^4 z_2 + |z_3|^4 z_3 + |z_4|^4 z_4 + |z_5|^4 z_5\right)\right] \\
f_{1,5}(z) &=& z_1 \left(|z_1|^4 + |z_2|^4 + |z_3|^4 + |z_4|^4 + |z_5|^4\right) \\
f_{1,6}(z) &=& z_1 \left(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2\right)\left(\overline{z}_1^2 + \overline{z}_2^2 + \overline{z}_3^2 + \overline{z}_4^2 + \overline{z}_5^2\right) \\
f_{1,7}(z) &=& z_1 \left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2\right)^2 \\
f_{1,8}(z) &=& z_1^2 \left(|z_1|^2 \overline{z}_1 + |z_2|^2 \overline{z}_2 + |z_3|^2 \overline{z}_3 + |z_4|^2 \overline{z}_4 + |z_5|^2 \overline{z}_5\right) - \\
\multicolumn{3}{l}{\frac{1}{5}(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2)\left(|z_1|^2 \overline{z}_1 + |z_2|^2 \overline{z}_2 + |z_3|^2 \overline{z}_3 + |z_4|^2 \overline{z}_4 + |z_5|^2 \overline{z}_5\right)} \\
f_{1,9}(z) &=& z_1^3 \left(\overline{z}_1^2 + \overline{z}_2^2 + \overline{z}_3^2 + \overline{z}_4^2 + \overline{z}_5^2\right) - \\
\multicolumn{3}{l}{\frac{1}{5}(z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3)\left(\overline{z}_1^2 + \overline{z}_2^2 + \overline{z}_3^2 + \overline{z}_4^2 + \overline{z}_5^2\right)} \\
f_{1,10}(z) &=& \overline{z}_1 \left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2\right)(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2) \\
f_{1,11}(z) &=& \overline{z}_1 \left(|z_1|^2 z_1^2 + |z_2|^2 z_2^2 + |z_3|^2 z_3^2 + |z_4|^2 z_4^2 + |z_5|^2 z_5^2\right) \\
f_{1,13}(z) &=& |z_1|^2 \left(|z_1|^2 z_1^2 + |z_2|^2 z_2 + |z_3|^2 z_3 + |z_4|^2 z_4 + |z_5|^2 z_5\right) - \\
\multicolumn{3}{l}{\frac{1}{5}\left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2\right)\left(|z_1|^2 z_1 + |z_2|^2 z_2 + |z_3|^2 z_3 + |z_4|^2 z_4 + |z_5|^2 z_5\right)} \\
f_{1,14}(z) &=& |z_1|^2 z_1 \left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2\right) - \\
\multicolumn{3}{l}{\frac{1}{5}\left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2\right)\left(|z_1|^2 z_1 + |z_2|^2 z_2 + |z_3|^2 z_3 + |z_4|^2 z_4 + |z_5|^2 z_5\right)} \\
f_{1,15}(z) &=& |z_1|^2 \overline{z}_1 (z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2) - \\
\multicolumn{3}{l}{-\frac{1}{5}(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2)\left(|z_1|^2 z_1 + |z_2|^2 z_2 + |z_3|^2 z_3 + |z_4|^2 z_4 + |z_5|^2 z_5\right)}
\end{array}
$$

Next Theorem follows from Theorem 4.13. Recall Table 4.2. Each of the seven $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$ acting on $\mathbf{C}^{5,0}$ listed in Table 6.1 are of the form $\Sigma^I_{q,p}$ or Σ^{II}_q. Specifically, we have that $\Sigma_i, i = 1, \ldots, 5$ are of the form $\Sigma^I_{q,p}$ and Σ_6, Σ_7 are of the form Σ^{II}_q.

Theorem 6.2 *Consider the system (6.2) where f is as in (6.3). Assume that* $\mathrm{Re}(\mu'(0)) >$ 0 *(such that the trivial equilibrium is stable if $\lambda < 0$ and unstable if $\lambda > 0$ for λ near zero). For each isotropy subgroup Σ_i, for $i = 1, \ldots, 7$ listed in Table 6.1, let $\Delta_0, \ldots, \Delta_r$ be the functions of $A_1, \ldots, A_{15}$ listed in Table 6.4 evaluated at $\lambda = 0$. Then:*

(1) For each Σ_i the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$. Tables 6.2 and 6.3 list the branching equations.

(2) For each Σ_i, if $\Delta_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all j, then the branch of periodic solutions is stable near $\lambda = 0$ and $z = 0$.

Proof: We include the relevant data of the proof of Theorem 4.13 specialized to the case $N = 5$ for completeness. Our aim is to study periodic solutions of (6.2) obtained by Hopf bifurcation from the trivial equilibrium where f satisfies the conditions of the Equivariant Hopf Theorem.

From Proposition 6.1 we have (up to conjugacy) the $\mathbf{C}$-axial subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$. Therefore, we can use the Equivariant Hopf Theorem to prove the existence of periodic solutions with these symmetries for a bifurcation problem with symmetry $\Gamma = \mathbf{S}_5$.

Periodic solutions of (6.2) of period near $2\pi/(1+\tau)$ are in one-to-one correspondence with the zeros of a function $g(z, \lambda, \tau)$ with explicit form given by (4.21).

Recall the isotypic decomposition for each type of the isotropy subgroups $\Sigma^I_{q,p}$ and Σ^{II}_q given by (4.23) and (4.24). For the seven isotropy subgroups Σ_i, for $i = 1, \ldots, 7$, in Table 6.1, it is possible to put the Jacobian matrix $(dg)_{z_0}$ into block diagonal form. We do this by decomposing $\mathbf{C}^{5,0}$ into isotypic components for the action of each isotropy subgroup Σ_i. Specifically, for $i = 3, 4, 5$ we form, respectively, the isotypic decomposition

$$\begin{array}{l} \mathbf{C}^{5,0} = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \\ \mathbf{C}^{5,0} = W_0 \oplus W_1 \oplus P_2 \oplus P_3 \\ \mathbf{C}^{5,0} = W_0 \oplus P_2 \oplus P_3 \oplus P_4 \end{array} \tag{6.4}$$

where $W_0 = \mathrm{Fix}(\Sigma_i), W_1, W_2, P_2, P_3$ and P_4 are complex one-dimensional subspaces, invariant under Σ_i. It follows then that $(dg)_{z_0}(W_j) \subset W_j$ for $j = 0, 1, 2$ and $(dg)_{z_0}(P_j) \subset P_j$ for $j = 2, 3, 4$ since $(dg)_{z_0}$ commutes with Σ_i (recall (4.24)).

Furthermore, for Σ_3 and Σ_5 we have that W_1, W_2, P_2, P_3 and P_4 are irreducible representations of complex type.

For Σ_4 we have that W_1, P_2 are irreducible representations of complex type. Moreover, we have that $P_3 = P_{3,R} \oplus P_{3,I}$, with $P_{3,R} \cong P_{3,I}$ and $P_{3,R}, P_{3,I}$ are absolutely irreducible.

For Σ_1, Σ_2 and Σ_6 we obtain that $\mathbf{C}^{5,0} = W_0 \oplus W_1 \oplus W_2$, where W_1 is a complex one-dimensional subspace, invariant under Σ_i and W_2, W_3 are a complex two-dimensional invariant subspaces that are the sum of two isomorphic real absolutely irreducible representations of dimension 2. Again we have $(dg)_{z_0}(W_j) \subseteq W_j$ for $j = 0, 1, 2, 3$. Note that in these cases we have $W_1 = W_{1,R} \oplus W_{1,I}$ with $W_{1,R} \cong W_{1,I}$ and the actions of $\Sigma_1, \Sigma_2, \Sigma_6$ on $W_{1,R}, W_{1,I}$ are (absolutely) irreducible.

For Σ_7 we obtain that $\mathbf{C}^{5,0} = W_0 \oplus W_1$, where W_1 is a complex three-dimensional invariant subspace that is the sum of two isomorphic real absolutely irreducible representations of dimension 2 of Σ_7 and we have $(dg)_{z_0}(W_j) \subseteq W_j$ for $j = 0, 1$.

Table 6.5 gives the isotypic decomposition of $\mathbf{C}^{5,0}$ for each of the isotropy subgroups Σ_i listed in Table 6.1.

Throughout we denote by (z_0, λ_0, τ_0) a zero of $g(z, \lambda, \tau) = 0$ with $z_0 \in \mathrm{Fix}(\Sigma_i)$. Specifically, for $i = 1, \ldots, 7$, we wish to calculate $(dg)_{(z_0, \lambda_0, \tau_0)}$.

To compute the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ we use complex coordinates $z_1, \overline{z}_1, \ldots, z_5, \overline{z}_5$, corresponding to a basis B for $\mathbf{C}^5$ with elements denoted by $b_1, \overline{b}_1, \ldots, b_5, \overline{b}_5$.

(Σ_1)

The fixed-point subspace of Σ_1 is $\mathrm{Fix}(\Sigma_1) = \{(z_1, z_1, -z_1, -z_1, 0) : z_1 \in \mathbf{C}\}$. The isotropy subgroup $\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$ is of the type $\Sigma^I_{q,p}$ with $k = q = 2$ and $p = 1$. Using this in (4.32) and (4.33) we get the branching equations for Σ_1 listed in Tables 6.2 and 6.3. It follows that if $A_{1r} + 4A_{2r} + 4A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 6.5 for the isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of Σ_1.

By (4.34), we have that with respect to the basis B, any "real" matrix commuting with Σ_1 has the form (note that $\xi^2 = 1$ where $\xi = e^{i2\pi/2}$)

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_2 & C_3 & C_3 & C_4 \\ C_2 & C_1 & C_3 & C_3 & C_4 \\ C_3 & C_3 & C_1 & C_2 & C_4 \\ C_3 & C_3 & C_2 & C_1 & C_4 \\ C_5 & C_5 & C_5 & C_5 & C_6 \end{pmatrix}$$

where

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix}$$

for $i = 1, \ldots, 6$ and

$$c_1 = \frac{\partial g_1}{\partial z_1}, \quad c_1' = \frac{\partial g_1}{\partial \overline{z}_1}, \quad c_2 = \frac{\partial g_1}{\partial z_2}, \quad c_2' = \frac{\partial g_1}{\partial \overline{z}_2}, \quad c_3 = \frac{\partial g_1}{\partial z_3}, \quad c_3' = \frac{\partial g_1}{\partial \overline{z}_3},$$

$$c_4 = \frac{\partial g_1}{\partial z_5}, \quad c_4' = \frac{\partial g_1}{\partial \overline{z}_5}, \quad c_5 = \frac{\partial g_5}{\partial z_1}, \quad c_5' = \frac{\partial g_5}{\partial \overline{z}_1}, \quad c_6 = \frac{\partial g_5}{\partial z_5}, \quad c_6' = \frac{\partial g_5}{\partial \overline{z}_5},$$

calculated at (z_0, λ_0, τ_0).

As before $(dg)_{(z_0,\lambda_0,\tau_0)}|W_j$ denotes the restriction of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to the subspace W_j.

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z \mapsto \alpha z + \beta \overline{z}$ where

$$\begin{array}{l} \alpha = c_1 + c_2 - 2c_3, \\ \beta = c_1' + c_2' - 2c_3'. \end{array}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz, iz, -iz, -iz, 0)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = 2\mathrm{Re}(A_1 + 4A_2 + 4A_3)|z|^2 \ + \ \cdots$$

whose sign is determined by $A_{1r} + 4A_{2r} + 4A_{3r}$ if it is assumed nonzero.

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$. From (4.36) with $N = 5, k = 2, q = 2$ it follows that

$$\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = 2\mathrm{Re}\left(-\tfrac{3}{5}A_1 - 4A_2\right)|z|^2 + \cdots$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = \left(|-\tfrac{3}{5}A_1 - 4A_2|^2 - |\tfrac{1}{5}A_1 + 4A_2|^2\right)|z|^4 + \cdots$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_3$. From (4.42) with $q = 2$ it follows that

$$\begin{array}{l} \mathrm{tr}\,(C_1 - C_2) = 2\mathrm{Re}\,(A_1 - 4A_2)\,|z|^2 + \cdots \\ \det\,(C_1 - C_2) = \left(|A_1 - 4A_2|^2 - |A_1 + 4A_2|^2\right)|z|^4 + \cdots \end{array}$$

(Σ_2)

The fixed-point subspace of Σ_2 is $\mathrm{Fix}(\Sigma_2) = \{(z_1, -z_1, 0, 0, 0) : z_1 \in \mathbf{C}\}$. The isotropy subgroup $\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_3$ is of the type $\Sigma^I_{q,p}$ with $k = 2, p = 3$ and $q = 1$. Using this in (4.32) and (4.33) we get the branching equations for Σ_2 listed in Tables 6.2 and 6.3. It follows that if $A_{1r} + 2A_{2r} + 2A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 6.5 for the isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of Σ_2. Recall (4.34). With respect to the basis B, any "real" matrix commuting with Σ_2 has the form (note that $\xi^2 = 1$ since $\xi = e^{i2\pi/2}$)

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_3 & C_4 & C_4 & C_4 \\ C_3 & C_1 & C_4 & C_4 & C_4 \\ C_5 & C_5 & C_6 & C_7 & C_7 \\ C_5 & C_5 & C_7 & C_6 & C_7 \\ C_5 & C_5 & C_7 & C_7 & C_6 \end{pmatrix}$$

where C_i for $i = 1, \ldots, 7$ are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c'_i \\ \overline{c}'_i & \overline{c}_i \end{pmatrix}$$

and

$$c_1 = \tfrac{\partial g_1}{\partial z_1}, \quad c'_1 = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad c_3 = \tfrac{\partial g_1}{\partial z_2}, \quad c'_3 = \tfrac{\partial g_1}{\partial \overline{z}_2}, \quad c_4 = \tfrac{\partial g_1}{\partial z_3}, \quad c'_4 = \tfrac{\partial g_1}{\partial \overline{z}_3}$$

$$c_5 = \tfrac{\partial g_3}{\partial z_1}, \quad c'_5 = \tfrac{\partial g_3}{\partial \overline{z}_1}, \quad c_6 = \tfrac{\partial g_3}{\partial z_3}, \quad c'_6 = \tfrac{\partial g_3}{\partial \overline{z}_3}, \quad c_7 = \tfrac{\partial g_3}{\partial z_4}, \quad c'_7 = \tfrac{\partial g_3}{\partial \overline{z}_4}$$

calculated at (z_0, λ_0, τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z \mapsto \alpha z + \beta\overline{z}$ where

$$\begin{array}{l} \alpha = c_1 - c_3 \\ \beta = c'_1 - c'_3 \end{array}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz, -iz, 0, 0, 0)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = \quad 2\mathrm{Re}(A_1 + 2A_2 + 2A_3)|z|^2 \ + \ \cdots$$

whose sign is determined by $A_{1r} + 2A_{2r} + 2A_{3r}$ if it is assumed nonzero.

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$. From (4.36) with $N = 5, q = 1$ it follows that

$$\mathrm{trace}\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = 2\mathrm{Re}\left(\tfrac{1}{5}A_1 - 2A_2\right)|z|^2 + \cdots$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = \left(\left|\tfrac{1}{5}A_1 - 2A_2\right|^2 - \left|\tfrac{3}{5}A_1 + 2A_2\right|^2\right)|z|^4 + \cdots$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$. From (4.40) with $q = 1$ it follows that

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = -2\mathrm{Re}\left(A_1 + 2A_2\right)|z|^2 + \cdots$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \left(|A_1 + 2A_2|^2 - |2A_2|^2\right)|z|^4 + \cdots$$

(Σ_3)

The fixed-point subspace of Σ_3 is $\mathrm{Fix}(\Sigma_3) = \left\{(z_1, \xi z_1, \xi^2 z_1, 0, 0) : z_1 \in \mathbf{C}\right\}$. The isotropy subgroup $\Sigma_3 = \widetilde{\mathbf{Z}}_3 \times \mathbf{S}_2$ is of the type $\Sigma_{q,p}^I$ with $k = 3, p = 2$ and $q = 1$. Using this in (4.30) and (4.31) we get the branching equations for Σ_3 listed in Tables 6.2 and 6.3. It follows that if $A_{1r} + 3A_{3r} < 0$ then the branch bifurcates supercritically.

Recall Table 6.5 for the isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of Σ_3.

Recall (4.34). With respect to the basis B, any "real" matrix commuting with Σ_3 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_3 & C_4 & C_5 & C_5 \\ C_4^{\xi^2} & C_1^{\xi^2} & C_3^{\xi^2} & C_5 & C_5 \\ C_3^{\xi^4} & C_4^{\xi^2} & C_1^{\xi^4} & C_5 & C_5 \\ C_6 & C_6^{\xi^2} & C_6^{\xi^4} & C_7 & C_8 \\ C_6 & C_6^{\xi^2} & C_6^{\xi^4} & C_8 & C_7 \end{pmatrix}$$

where $C_i, C_i^{\xi^j}$ for $i = 1, 3, 4, 6, \ j = 2, 3$ are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \bar{c}_i' & \bar{c}_i \end{pmatrix}, \qquad C_i^{\xi^j} = \begin{pmatrix} c_i & \xi^j c_i' \\ \bar{\xi}^j \bar{c}_i' & \bar{c}_i \end{pmatrix},$$

the matrices $C_l, l = 5, 7, 8$ are given by

$$C_l = \begin{pmatrix} c_l & c_l' \\ \bar{c}_l' & \bar{c}_l \end{pmatrix}$$

$\xi = e^{i2\pi/3}$ and

$$c_1=\frac{\partial g_1}{\partial z_1},\quad c_1'=\frac{\partial g_1}{\partial \overline{z}_1},\quad c_3=\frac{\partial g_1}{\partial z_2},\quad c_3'=\frac{\partial g_1}{\partial \overline{z}_2},\quad c_4=\frac{\partial g_1}{\partial z_3},\quad c_4'=\frac{\partial g_1}{\partial \overline{z}_3}$$

$$c_5=\frac{\partial g_1}{\partial z_4},\quad c_5'=\frac{\partial g_1}{\partial \overline{z}_4},\quad c_6=\frac{\partial g_4}{\partial z_1},\quad c_6'=\frac{\partial g_4}{\partial \overline{z}_1},\quad c_7=\frac{\partial g_4}{\partial z_4},\quad c_7'=\frac{\partial g_4}{\partial \overline{z}_4},\quad c_8=\frac{\partial g_4}{\partial z_5},\quad c_8'=\frac{\partial g_4}{\partial \overline{z}_5}$$

calculated at (z_0,λ_0,τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z,\overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z\mapsto \alpha z+\beta\overline{z}$ where

$$\begin{aligned}\alpha&=c_1+\xi c_3+\xi^2 c_4\\ \beta&=c_1'+\overline{\xi}c_3'+\overline{\xi^2}c_4'\end{aligned}$$

A tangent vector to the orbit of $\Gamma\times\mathbf{S}^1$ through z_0 is the eigenvector $(iz,i\xi z,i\xi^2 z,0,0)$ and the matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha)=\ \ 2\mathrm{Re}(A_1+3A_3)|z|^2\ +\ \cdots$$

whose sign is determined by $A_{1r}+3A_{3r}$ if it is assumed nonzero.

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$. From (4.38) with $N=5,k=3,q=1$ it follows that

$$\mathrm{trace}\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right)=2\mathrm{Re}\left(-\tfrac{1}{5}A_1\right)|z|^2\ +\ \cdots,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right)=-\tfrac{24}{25}\left|A_1\right|^2|z|^4+\cdots.$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ (corresponds to W_2 in the general case). From (4.41) it follows that

$$\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\right)=2\mathrm{Re}\left(A_1\right)|z|^2+\cdots,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\right)=\left|A_1\right|^2|z|^4+\cdots.$$

Finally, we compute $(dg)_{(z_0,\lambda_0,\tau_0)}|P_2$. This component corresponds to P_2 in the general case. From (4.47) with $N=5$ it follows that

$$\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right)=2\mathrm{Re}\left(A_1+6A_2\right)|z|^2+\cdots,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right)=\left(|A_1+6A_2|^2-\left|\tfrac{2}{5}A_1\right|^2\right)|z|^4+\cdots.$$

(Σ_4)

The fixed-point subspace of Σ_4 is $\mathrm{Fix}(\Sigma_4)=\left\{(z_1,\xi z_1,\xi^2 z_1,\xi^3 z_1,0): z_1\in\mathbf{C}\right\}$. The isotropy subgroup $\Sigma_4=\widetilde{\mathbf{Z}}_4$ is of the type $\Sigma^I_{q,p}$ with $k=4,p=1$ and $q=1$. Using this in (4.30) and (4.31) we get the branching equations for Σ_4 listed in Tables 6.2 and 6.3. It follows that if $A_{1r}+4A_{3r}\ <\ 0$ then the branch bifurcates supercritically.

Recall Table 6.5 for the isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of Σ_4.

Recall (4.34). With respect to the basis B, any "real" matrix commuting with Σ_4 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_3 & C_4 & C_5 & C_6 \\ C_5^{\xi^2} & C_1^{\xi^2} & C_3^{\xi^2} & C_4^{\xi^2} & C_6 \\ C_4^{\xi^4} & C_5^{\xi^4} & C_1^{\xi^4} & C_3^{\xi^4} & C_6 \\ C_3^{\xi^6} & C_4^{\xi^6} & C_5^{\xi^6} & C_1^{\xi^6} & C_6 \\ C_7 & C_7^{\xi^2} & C_7^{\xi^4} & C_7^{\xi^6} & C_8 \end{pmatrix}$$

where $C_i, C_i^{\xi^j}$ for $i = 1,3,4,5,7,\ j = 2,4,6$ are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix}, \qquad C_i^{\xi^j} = \begin{pmatrix} c_i & \xi^j c_i' \\ \overline{\xi}^j \overline{c}_i' & \overline{c}_i \end{pmatrix},$$

$\xi = e^{i2\pi/4}$, the matrices C_6 and C_8 are given by

$$C_6 = \begin{pmatrix} c_6 & c_6' \\ \overline{c}_6' & \overline{c}_6 \end{pmatrix}, \qquad C_8 = \begin{pmatrix} c_8 & c_8' \\ \overline{c}_8' & \overline{c}_8 \end{pmatrix},$$

and

$$c_1 = \tfrac{\partial g_1}{\partial z_1},\quad c_1' = \tfrac{\partial g_1}{\partial \overline{z}_1},\quad c_3 = \tfrac{\partial g_1}{\partial z_2},\quad c_3' = \tfrac{\partial g_1}{\partial \overline{z}_2},\quad c_4 = \tfrac{\partial g_1}{\partial z_3},\quad c_4' = \tfrac{\partial g_1}{\partial \overline{z}_3}$$

$$c_5 = \tfrac{\partial g_1}{\partial z_4},\quad c_5' = \tfrac{\partial g_1}{\partial \overline{z}_4},\quad c_6 = \tfrac{\partial g_1}{\partial z_5},\quad c_6' = \tfrac{\partial g_1}{\partial \overline{z}_5},\quad c_7 = \tfrac{\partial g_5}{\partial z_1},\quad c_7' = \tfrac{\partial g_5}{\partial \overline{z}_1},\quad c_8 = \tfrac{\partial g_5}{\partial z_5},\quad c_8' = \tfrac{\partial g_5}{\partial \overline{z}_5}$$

calculated at (z_0, λ_0, τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z \mapsto \alpha z + \beta\overline{z}$ where

$$\begin{aligned} \alpha &= c_1 + \xi c_3 + \xi^2 c_4 + \xi^3 c_5 \\ \beta &= c_1' + \overline{\xi} c_3' + \overline{\xi^2} c_4' + \overline{\xi^3} c_5' \end{aligned}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz, i\xi z, i\xi^2 z, i\xi^3 z, 0)$ and the matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = 2\mathrm{Re}(A_1 + 4A_3)|z|^2 \ + \ \cdots$$

whose sign is determined by $A_{1r} + 4A_{3r}$ if it is assumed nonzero.

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$. From (4.38) with $N = 5, k = 4, q = 1$ it follows that

$$\mathrm{trace}\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = 2\mathrm{Re}\left(-\tfrac{3}{5}A_1\right)|z|^2 + \cdots,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = \left|-\tfrac{4}{5}A_1\right|^2 |z|^4 + \cdots.$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|P_2$. From (4.46) with $N = 5, k = 4, q = 1$ it follows that

$$\mathrm{trace}\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots ,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = \tfrac{24}{25}\,|A_1|^2\,|z|^4 + \cdots .$$

Finally, we compute $(dg)_{(z_0,\lambda_0,\tau_0)}|P_3$. From (4.48) with $k=4, q=1$ it follows that

$$\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_3\right) = 2\mathrm{Re}\,(A_1+8A_2)\,|z|^2 + \cdots ,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_3\right) = \left(|A_1+8A_2|^2 \;-\; |A_1|^2\right)|z|^4 + \cdots .$$

(Σ_5)

The fixed-point subspace of Σ_5 is $\mathrm{Fix}(\Sigma_5) = \left\{(z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1) : z_1 \in \mathbf{C}\right\}$. The isotropy subgroup $\Sigma_5 = \widetilde{\mathbf{Z}}_5$ is of the type $\Sigma^I_{q,p}$ with $k=5, p=0$ and $q=1$. Using this in (4.30) and (4.31) we get the branching equations for Σ_5 listed in Tables 6.2 and 6.3. It follows that if $A_{1r}+5A_{3r}<0$ then the branch bifurcates supercritically.

Recall Table 6.5 for the isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of Σ_5.

Recall (4.34). With respect to the basis B, any "real" matrix commuting with Σ_5 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_3 & C_4 & C_5 & C_6 \\ C_6^{\xi^2} & C_1^{\xi^2} & C_3^{\xi^2} & C_4^{\xi^2} & C_5^{\xi^2} \\ C_5^{\xi^4} & C_6^{\xi^4} & C_1^{\xi^4} & C_3^{\xi^4} & C_4^{\xi^4} \\ C_4^{\xi^6} & C_5^{\xi^6} & C_6^{\xi^6} & C_1^{\xi^6} & C_3^{\xi^6} \\ C_3^{\xi^8} & C_4^{\xi^8} & C_5^{\xi^8} & C_6^{\xi^8} & C_1^{\xi^8} \end{pmatrix}$$

where $C_i, C_i^{\xi^j}$ for $i=1,3,4,5,6,\ j=2,4,6,8$ are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix}, \qquad C_i^{\xi^j} = \begin{pmatrix} c_i & \xi^j c_i' \\ \overline{\xi}^j \overline{c}_i' & \overline{c}_i \end{pmatrix},$$

$\xi = e^{i2\pi/5}$ and

$$c_1 = \tfrac{\partial g_1}{\partial z_1},\quad c_1' = \tfrac{\partial g_1}{\partial \overline{z}_1},\quad c_3 = \tfrac{\partial g_1}{\partial z_2},\quad c_3' = \tfrac{\partial g_1}{\partial \overline{z}_2},\quad c_4 = \tfrac{\partial g_1}{\partial z_3},\quad c_4' = \tfrac{\partial g_1}{\partial \overline{z}_3}$$

$$c_5 = \tfrac{\partial g_1}{\partial z_4},\quad c_5' = \tfrac{\partial g_1}{\partial \overline{z}_4},\quad c_6 = \tfrac{\partial g_1}{\partial z_5},\quad c_6' = \tfrac{\partial g_1}{\partial \overline{z}_5}$$

calculated at (z_0,λ_0,τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z,\overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z \mapsto \alpha z + \beta\overline{z}$ where

$$\begin{aligned} \alpha &= c_1 + \xi c_3 + \xi^2 c_4 + \xi^3 c_5 + \xi^4 c_6 \\ \beta &= c_1' + \overline{\xi} c_3' + \overline{\xi^2} c_4' + \overline{\xi^3} c_5' + \overline{\xi^4} c_6' \end{aligned}$$

A tangent vector to the orbit of $\Gamma\times\mathbf{S}^1$ through z_0 is the eigenvector $(iz, i\xi z, i\xi^2 z, i\xi^3 z, i\xi^4 z)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = \ 2\mathrm{Re}(A_1 + 5A_3)|z|^2 \ + \ \cdots$$

whose sign is determined by $A_{1r} + 5A_{3r}$ if it is assumed nonzero.

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|P_2$. From (4.46) with $N = 5, k = 5, q = 1$ it follows that

$$\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots\,,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = |A_1|^2\,|z|^4 + \cdots\,.$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|P_3$. From (4.49) with $N = 5, k = 5, q = 1$ it follows that

$$\begin{aligned}
\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_3\right) &= 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots\,,\\
\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_3\right) &= \left(\left|A_1 + \xi_1|z|^2\right|^2 \ - \ \left|A_1 + \xi_2|z|^2\right|^2\right)|z|^4\\
&= 2\mathrm{Re}\left[A_1\left(\bar{\xi}_1 - \bar{\xi}_2\right)\right]|z|^6 + \cdots\,,
\end{aligned}$$

where

$$\begin{aligned}
\xi_1 &= \ 2A_4 + 10A_{14},\\
\xi_2 &= \ 2A_4 + 5A_{11} + 5A_{14}.
\end{aligned}$$

Finally, we compute $(dg)_{(z_0,\lambda_0,\tau_0)}|P_4$. From (4.48) with $N = 5, k = 5, q = 1$ it follows that

$$\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_4\right) = 2\mathrm{Re}\,(A_1 + 10A_2)\,|z|^2 + \cdots\,,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_4\right) = \left(|A_1 + 10A_2|^2 \ - \ |A_1|^2\right)|z|^4.$$

(Σ_6)

The fixed-point subspace of Σ_6 is $z_3 = z_4 = z_5 = -\frac{2}{3}z$ and $z_1 = z_2 = z$. The isotropy subgroup $\Sigma_6 = \mathbf{S}_2\times\mathbf{S}_3$ is of the type Σ_q^{II} with $q = 2$ and $p = 3$. Using this in (4.51) and (4.52) we get the branching equations for Σ_6 listed in Tables 6.2 and 6.3. It follows that if $\frac{7}{3}A_{1r} + 10A_{2r} + 10A_{3r} < 0$ then the branch bifurcates supercritically.

Let $\Sigma_6 = \mathbf{S}_2\times\mathbf{S}_3$ be the isotropy subgroup of $z_0 = \left(z, z, -\frac{2}{3}z, -\frac{2}{3}z, -\frac{2}{3}z\right)$. Recall Table 6.5 for the isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of Σ_6.

Recall (4.53). With respect to the basis B, any "real" matrix commuting with Σ_6 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} \ = \ \begin{pmatrix} C_1 & C_6 & C_2 & C_2 & C_2\\ C_6 & C_1 & C_2 & C_2 & C_2\\ C_3 & C_3 & C_4 & C_5 & C_5\\ C_3 & C_3 & C_5 & C_4 & C_5\\ C_3 & C_3 & C_5 & C_5 & C_4 \end{pmatrix}$$

where C_i for $i = 1, \ldots, 6$ are 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix}$$

and

$$c_1 = \frac{\partial g_1}{\partial z_1}, \quad c_1' = \frac{\partial g_1}{\partial \overline{z}_1}, \quad c_2 = \frac{\partial g_1}{\partial z_3}, \quad c_2' = \frac{\partial g_1}{\partial \overline{z}_3}, \quad c_3 = \frac{\partial g_3}{\partial z_1}, \quad c_3' = \frac{\partial g_3}{\partial \overline{z}_1}$$

$$c_4 = \frac{\partial g_3}{\partial z_3}, \quad c_4' = \frac{\partial g_3}{\partial \overline{z}_3}, \quad c_5 = \frac{\partial g_3}{\partial z_4}, \quad c_5' = \frac{\partial g_3}{\partial \overline{z}_4} \quad c_6 = \frac{\partial g_1}{\partial z_2}, \quad c_6' = \frac{\partial g_1}{\partial \overline{z}_2}$$

calculated at (z_0, λ_0, τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z = \alpha z + \beta \overline{z}$ where

$$\alpha = c_1 + c_6 - 2c_2$$
$$\beta = c_1' + c_6' - 2c_2'$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $\left(iz, iz, -\frac{2}{3}iz, -\frac{2}{3}iz, -\frac{2}{3}iz\right)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = \frac{2}{3}\mathrm{Re}\left(\frac{7}{3}A_1 + 10A_2 + 10A_3\right)|z|^2 + \cdots$$

whose sign is determined by $7/3A_{1r} + 10A_{2r} + 10A_{3r}$ if it is assumed nonzero (where $\frac{7}{3}A_{1r} + 10A_{2r} + 10A_{3r}$ is calculated at zero).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$. From (4.55) with $N = 5, q = 2, p = 3$ it follows that

$$\mathrm{tr}\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = \tfrac{2}{3}\mathrm{Re}\left(\tfrac{11}{3}A_1 - 10A_2\right)|z|^2 + \cdots,$$

$$\det\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) = \left(\left|\tfrac{1}{3}(\tfrac{11}{3}A_1 - 10A_2)\right|^2 - \left|A_1 + \tfrac{10}{3}A_2\right|^2\right)|z|^4 + \cdots.$$

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$. From (4.57) with $N = 5, q = 2, p = 3$ it follows that

$$\mathrm{tr}(C_4 - C_5) = \tfrac{2}{3}\mathrm{Re}\left(\tfrac{1}{3}A_1 - 8A_2\right)|z|^2 + \cdots,$$

$$\det(C_4 - C_5) = \left(\left|\tfrac{1}{3}(\tfrac{1}{3}A_1 - 8A_2)\right|^2 - \left|\tfrac{1}{3}(\tfrac{4}{3}A_1 + 10A_2)\right|^2\right)|z|^4 + \cdots.$$

(Σ_7)

The fixed-point subspace of Σ_7 is $z_2 = z_3 = z_4 = z_5 = -\frac{1}{4}z$ and $z_1 = z$. The isotropy subgroup $\Sigma_7 = \mathbf{S}_4$ is of the type Σ_q^{II} with $q = 1$ and $p = 4$. Using this in (4.51) and

(4.52) we get the branching equations for Σ_7 listed in Tables 6.2 and 6.3. It follows that if $\frac{13}{4}A_{1r} + 5A_{2r} + 5A_{3r} < 0$ then the branch bifurcates supercritically.

Let $\Sigma_7 = \mathbf{S}_4$ be the isotropy subgroup of $z_0 = \left(z, -\frac{1}{4}z, -\frac{1}{4}z, -\frac{1}{4}z, -\frac{1}{4}z\right)$. Recall Table 6.5 for the isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of Σ_7. Recall (4.53). With respect to the basis B, any "real" matrix commuting with Σ_7 has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & C_2 & C_2 & C_2 & C_2 \\ C_3 & C_4 & C_5 & C_5 & C_5 \\ C_3 & C_5 & C_4 & C_5 & C_5 \\ C_3 & C_5 & C_5 & C_4 & C_5 \\ C_3 & C_5 & C_5 & C_5 & C_4 \end{pmatrix}$$

where C_i for $i = 1, \ldots, 6$ are 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix}$$

and

$$c_1 = \tfrac{\partial g_1}{\partial z_1}, \quad c_1' = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad c_2 = \tfrac{\partial g_1}{\partial z_2}, \quad c_2' = \tfrac{\partial g_1}{\partial \overline{z}_2}, \quad c_3 = \tfrac{\partial g_2}{\partial z_1}, \quad c_3' = \tfrac{\partial g_2}{\partial \overline{z}_1}$$

$$c_4 = \tfrac{\partial g_2}{\partial z_2}, \quad c_4' = \tfrac{\partial g_2}{\partial \overline{z}_2}, \quad c_5 = \tfrac{\partial g_2}{\partial z_3}, \quad c_5' = \tfrac{\partial g_2}{\partial \overline{z}_3}$$

calculated at (z_0, λ_0, τ_0).

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z = \alpha z + \beta \overline{z}$ where

$$\begin{aligned} \alpha &= c_1 - c_2 \\ \beta &= c_1' - c_2' \end{aligned}$$

A tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $\left(iz, -\frac{1}{4}iz, -\frac{1}{4}iz, -\frac{1}{4}iz, -\frac{1}{4}iz\right)$. The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = \frac{1}{2}\mathrm{Re}\left(\frac{13}{4}A_{1r} + 5A_{2r} + 5A_{3r}\right)|z|^2 + \cdots$$

whose sign is determined by $\frac{13}{4}A_{1r} + 5A_{2r} + 5A_{3r}$ if it is assumed nonzero (where $\frac{13}{4}A_{1r} + 5A_{2r} + 5A_{3r}$ is calculated at zero).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$. From (4.57) with $N = 5, q = 1, p = 4$ it follows that

$$\mathrm{tr}(C_4 - C_5) = 2\mathrm{Re}\left(-\tfrac{55}{80}A_1 - \tfrac{5}{4}A_2\right)|z|^2 + \cdots,$$

$$\det(C_4 - C_5) = \left(\left|-\tfrac{55}{80}A_1 - \tfrac{5}{4}A_2\right|^2 - \left|\tfrac{1}{16}A_1 + \tfrac{5}{4}A_2\right|^2\right)|z|^4 + \cdots.$$

□

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