Heteroclinic network dynamics on joining coupled cell networks

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Abstract

We present a method of combining coupled cell systems to get dynamics supporting robust simple heteroclinic networks given by the product of robust simple heteroclinic networks (cycles). We consider coupled cell networks, with no assumption on symmetry, and combine them via the join operation. Assuming that the dynamics of the component networks supports robust simple heteroclinic cycles or networks, we show that the join dynamics realizes a more complex heteroclinic network given by the product of those cycles or networks. Moreover, the equilibria in the product heteroclinic network correspond to partially synchronous states. Assuming no symmetry for the component coupled cell networks, one of the key points for the existence and robustness of the heteroclinic dynamics are the flow invariant subspaces forced by the network structure - the synchrony subspaces. The other key point is that the (linear) stability of equilibria in the join dynamics is determined by the (linear) stability of equilibria in the component dynamics. The first point depends only at the network structure of the component networks. The second one depends both at the components network structures and the convenient choice of the join coupling. The proposed method is general and can be applied to the join of symmetric or asymmetric networks. Here, we illustrate it through the join of two asymmetric coupled cell networks where robust simple heteroclinic cycles between fully synchronous equilibria occur. We obtain robust simple heteroclinic networks for the join dynamics between partially synchronised equilibria for the associated join network.

AMS classification scheme numbers: 37C29 34C15 37C80

1 Introduction

A wide range of applications in many diverse areas, such as biology, economics, physics and ecology, are modelled through coupled cell systems. See, for example, Boccaletti *et al.* [13] and references therein. The network structure associated to a coupled cell system can have a nontrivial impact on its dynamical behaviour. One such example is the occurrence of robust heteroclinic cycles and networks.

In terms of applications, particularly in computational neuroscience, heteroclinic phenomena have been deserving a growing interest. See, for example, Ashwin *et al.* [8, 11], Neves *et al.* [22, 23], Rabinovich *et al.* [24, 25].

Heteroclinic cycles and networks do not occur in a robust way for general systems, but it is well known they do in the presence of flow invariant subspaces. Such flow invariant subspaces can arise, for example, as a consequence of symmetry, that is, due to the equivariance with respect to a symmetry group, see for example Homburg and Knobloch [19]. Another setting where flow invariant subspaces appear and thus robust heteroclinic phenomena can occur is in replicator dynamics and bimatrix games, see Aguiar [1].

In the context of coupled cell systems there are flow-invariant subspaces that are not related to the symmetry or other specific features of the particular systems but instead are forced by the associated network structure. These are the network *synchrony subspaces*. More specifically, the *synchrony subspaces* of a network are polydiagonals, spaces described by equalities of groups of cell coordinates, that are flow-invariant for every coupled cell system that have structure consistent with the network. Moreover, the synchrony subspaces of a network are in one-to-one correspondence with the polydiagonal subspaces that are left invariant under the network adjacency matrix (or adjacency matrices in case different types of interactions occur), see [26, 17, 4]. Observe that, if the coupled cell network is symmetric under the action of a nontrivial group Γ of permutations of the nodes (the cells), then the fixed-point subspaces of subgroups of Γ, for the induced (permutation) action of Γ on the total phase space of the associated coupled cell systems are flow-invariant. In general, for a fixed network structure, symmetric or asymmetric, the set of the network synchrony subspaces contains properly the set of fixed-point subspaces. See Antoneli and Stewart [7].

Robust heteroclinic phenomena in symmetric systems have been studied extensively in recent years as becomes patent from the review article by Homburg and Knobloch [19]. Recently, interest has been given to the study of the existence of heteroclinic cycles and networks in coupled cell systems, induced by the associated set of network synchrony subspaces. See, for example, Aguiar *et al.* [2], Ashwin *et al.* [12] and Field [15]. Note that, in general, the network has no symmetries and so these flow-invariant subspaces are not fixed-point subspaces. See also the work of Chossat *et al.* [14] in the context of Hopfield networks. The existence of heteroclinic networks can lead to the occurrence of complex dynamics, see for example, Aguiar *et al.* [3, 5], Homburg *et al.* [18], Labouriau *et al.* [21], Kirk *et al.* [20], Homburg *et al.* [19] and references therein. For coupled cell systems displaying unexpected heteroclinic behaviour see, for example, the review in Ashwin *et al.* [9] and Ashwin *et al.* [10].

In this paper, we show how the join operation on coupled cell networks can be used to construct dynamics supporting robust simple heteroclinic networks given by the product of robust simple heteroclinic cycles (or networks). Briefly, the key points in our proposal method of constructing heteroclinic behaviour for join dynamics, from heteroclinic dynamics of the component network dynamics, are: the existence of synchrony subspaces for the join obtained from the synchrony subspaces of the component networks, see Aguiar and Ruan [6]; the linear stability of equilibria at the heteroclinic network for the join dynamics is determined by the linear stability of equilibria for the component networks. Specifically, assume, for example, that the component networks $\mathcal{N}_1$ and $\mathcal{N}_2$ support robust heteroclinic connections between the equilibria $\mathbf{p}$, $\mathbf{q}$ for $\mathcal{N}_1$ and $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$ for $\mathcal{N}_2$. The coupling for the join coupled cell systems can be chosen such that $(\mathbf{p}, \overline{\mathbf{p}})$, $(\mathbf{p}, \overline{\mathbf{q}})$, $(\mathbf{q}, \overline{\mathbf{p}})$, $(\mathbf{q}, \overline{\mathbf{q}})$ are equilibria for the join $\mathcal{N}_1 * \mathcal{N}_2$. Moreover, the stability of these equilibria depend mildly on the coupling function and strongly on the stability of the equilibria $\mathbf{p}$, $\mathbf{q}$ for $\mathcal{N}_1$ and $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$ for $\mathcal{N}_2$. In particular, if $\mathbf{p}$, $\mathbf{q}$, $\overline{\mathbf{p}}$, $\overline{\mathbf{q}}$ have full synchrony, then the new equilibria $(\mathbf{p}, \overline{\mathbf{p}})$, $(\mathbf{p}, \overline{\mathbf{q}})$, $(\mathbf{q}, \overline{\mathbf{p}})$, $(\mathbf{q}, \overline{\mathbf{q}})$ have partial synchrony. These apply to a specific asymmetric network structure example where it has been proved recently the existence of associated coupled cell systems that support, in a robust way, simple heteroclinic cycles between two fully synchronised equilibria, see Aguiar *et al.* [2]. For this case, we prove the existence of coupled cell dynamics for the join network supporting a robust simple heteroclinic network with four partially synchronous equilibria. The method followed here can be used iteratively and can be generalised to other coupled cell networks with dynamics realising heteroclinic networks as, for example, in Ashwin *et al.* [12] and Field [15].

The paper is organized in the following way. In Section 2 we introduce the basic facts concerning coupled cell networks, heteroclinic networks/cycles, and the join operation on networks. In Section 3 we propose a general method, via the join operation on networks, to construct dynamics supporting robust simple heteroclinic networks given by the product of robust simple heteroclinic cycles (or networks). We apply the method to an example in Section 4.

2 Coupled cell networks and the join of networks

In this section we review the main points concerning coupled cell networks and the join operation on networks. We also give a brief review on heteroclinic cycles and networks.

2.1 Coupled cell networks

Following [26, 17, 16], a *coupled cell network* $\mathcal{N}$ is a directed graph whose nodes represent the cells and the directed arrows the couplings. Nodes are represented by the same symbol if they correspond to the same individual dynamics. Similarly, identical edges correspond to couplings of the same type. For cell i of $\mathcal{N}$, let $I_i = \{i_1, \ldots, i_{k_i}\}$ be the multiset of the cells with edges directed to cell i – the *input set of cell* i. Two cells i and j are said to be *input isomorphic* if there is a bijection between I_i and I_j preserving the edge types. A network is called *homogeneous* if all the cells are input isomorphic. For each edge type of an n-cell network, we can consider the $n \times n$ *adjacency matrix* where the ij entry is the number of edges of that type from cell j to cell i. Thus, a network $\mathcal{N}$ with p edge types has p adjacency matrices.

Example 2.1 Consider the homogeneous network structure in Figure 1 appearing in [2]. There are two edge types corresponding to the adjacency matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that $I_1 = \{2, 3\}$, $I_2 = \{1, 3\}$ and $I_3 = \{1, 2\}$. Moreover, all the cells receive one edge of each type. It follows then that all the cells are input isomorphic. For example, the function $\gamma : I_1 \to I_2$ such that $\gamma(2) = 1$, $\gamma(3) = 3$ is a bijection between the two input sets preserving the edges types. $\diamond$

A coupled cell system compatible with a coupled cell network $\mathcal{N}$ is a system of ordinary differential equations $\dot{\mathbf{X}} = F(\mathbf{X})$ with structure consistent with $\mathcal{N}$ in the following way. If P_i is the cell phase space of cell i then the total phase space is given by the product of the cell phase spaces. In this paper we assume that $P_i = \mathbf{R}$; if $\mathcal{N}$ has n cells then $P = \mathbf{R}^n$. Denote by $X = (x_1, \ldots, x_n) \in P$ and let $F = (f_1, \ldots, f_n)$. We have then that

$$\dot{x}_i = f_i \left(x_i; x_{i_1}, \ldots, x_{i_{k_i}} \right)$$

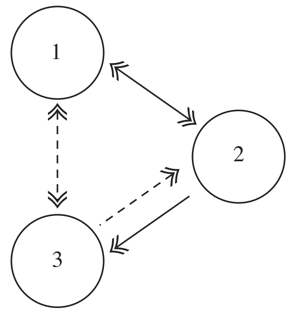

Figure 1: A three-cell homogeneous network with two edge types: every cell receives two directed edges, one of each type.

where $f_i : \mathbf{R} \times \mathbf{R}^{k_i} \to \mathbf{R}$ is smooth. It will also hold that if $i_l, i_s \in I_i$ correspond to edges directed to cell i of the same type, then f_i is invariant under the permutation of the variables x_{i_l}, x_{i_s}. Moreover, if i and j are input isomorphic, then f_i and f_j differ only by permutation of the corresponding variables associated with I_i, I_j.

Example 2.2 Consider the homogeneous network structure in Figure 1. The admissible coupled cell systems have the form

$$\begin{array}{l} \dot{x}_1 = f(x_1; x_2, x_3) \\ \dot{x}_2 = f(x_2; x_1, x_3) \\ \dot{x}_3 = f(x_3; x_2, x_1) \end{array} , \tag{2.1}$$

with $f : \mathbf{R}^3 \to \mathbf{R}$ a smooth function. $\diamond$

A subspace Δ of $\mathbf{R}^n$ given by equalities of certain cell coordinates is a *synchrony subspace* for the coupled cell network $\mathcal{N}$ when it is flow-invariant for any coupled cell system with structure consistent with $\mathcal{N}$. From the results of Stewart and Golubitsky *et al.* [26, 17] it follows that a subspace given by equalities of certain cell coordinates is a synchrony subspace for $\mathcal{N}$ if and only if it is left invariant under the adjacency matrices of $\mathcal{N}$. Moreover, the coupled cell systems consistent with a network $\mathcal{N}$ restricted to a synchrony subspace S are coupled cell systems with structure consistent with a smaller network, called the *quotient network* of $\mathcal{N}$ by S. See Aguiar and Dias [4] for a characterisation and computation algorithm of the set of synchrony subspaces of a network which form a lattice under the subset inclusion relation.

Example 2.3 Consider the two-cell network $\mathcal{Q}$ in Figure 2. The admissible coupled cell systems for $\mathcal{Q}$ have the form

$$\begin{array}{l} \dot{y}_1 = \overline{f}(y_1; y_2, y_1) \\ \dot{y}_2 = \overline{f}(y_2; y_1, y_1) \end{array} \tag{2.2}$$

where $\overline{f} : \mathbf{R}^3 \to \mathbf{R}$ is a smooth function. For the homogeneous network in Figure 1 there are four synchrony subspaces. See Table 1. Taking for example the synchrony subspace $S^3 = \{(y_1, y_2, y_1) \in \mathbf{R}^3\}$, the coupled cell systems equations (2.1) restricted to S^3 are equations (2.2), taking $\overline{f} = f$. We say that $\mathcal{Q}$ is the quotient network of the network in Figure 1 by S^3. $\diamond$

Observe that if $\mathcal{N}$ is an homogeneous coupled cell network with n cells, then the diagonal subspace $\Delta = \{\mathbf{x} : x_1 = \cdots = x_n\}$ is a synchrony subspace. Denoting the adjacency matrices of $\mathcal{N}$ by A_i, then

$P = \mathbf{R}^3$	$S^2 = \{\mathbf{x} :\ x_1 = x_2\}$	$S^3 = \{\mathbf{x} :\ x_1 = x_3\}$	$\Delta = \{\mathbf{x} :\ x_1 = x_2 = x_3\}$

Table 1: The synchrony subspaces for the network in Figure 1.

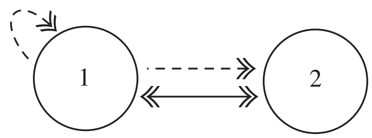

Figure 2: A two-cell quotient network $\mathcal{Q}$ of the network in Figure 1 by the synchrony subspace $S^3 = \{\mathbf{x} :\ x_1 = x_3\}$.

if $\mathbf{p} = (p, \ldots, p)$ and we have a coupled cell systems $\dot{\mathbf{X}} = F(\mathbf{X})$ consistent with $\mathcal{N}$, it follows that the linearization of the vector field at $\mathbf{p}$ has the following form:

$$(\partial F)_{\mathbf{p}} = \alpha \mathrm{Id}_n + \sum_{i \in I_1} \beta_i A_i$$

where $\alpha = (\partial f / \partial x_1)_{\mathbf{p}}$ and $\beta_i = (\partial f / \partial x_i)_{\mathbf{p}}$.

Example 2.4 For the homogeneous network in Figure 1 let $\mathbf{p} = (p, p, p)$. Recall the network adjacency matrices A_1 and A_2 listed in Example 2.1. The linearization at $\mathbf{p}$ of the vector field determined by equations (2.1) is:

$$J(\mathbf{p}) = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{bmatrix} = \alpha \mathrm{Id}_3 + \beta A_1 + \gamma A_2,$$

where $\alpha = \frac{\partial f}{\partial x}(\mathbf{p})$, $\beta = \frac{\partial f}{\partial y}(\mathbf{p})$ and $\gamma = \frac{\partial f}{\partial z}(\mathbf{p})$.

Note that $J(\mathbf{p})$ has eigenvalues $\mu_1 = \alpha + \beta + \gamma, \mu_2 = \alpha - \beta, \mu_3 = \alpha - \gamma$ associated with the following eigenlines:

$$\begin{array}{l} E_1 :\ x_1 = x_2 = x_3; \\ E_2 :\ x_2 = -(1 + \gamma\beta^{-1})x_1,\ x_1 = x_3; \\ E_3 :\ x_3 = -(1 + \beta\gamma^{-1})x_1,\ x_1 = x_2\,. \end{array}$$

Using the notation of Table 1, note that E_1, E_2 are contained in the synchrony subspace S^3; also, E_1, E_3 are contained in the synchrony subspace S^2. $\diamond$

2.2 Heteroclinic cycles and networks

There is an *heteroclinic cycle* connecting a sequence of k saddle equilibria $\mathbf{p}_0, \ldots, \mathbf{p}_{k-1}$ if the unstable manifold of $\mathbf{p}_i$ intersects nontrivially the stable manifold of $\mathbf{p}_{i+1}$, for each $i = 1, \ldots, k \pmod{k}$. By a *robust simple heteroclinic cycle* it is meant one in which the heteroclinic connections are one dimensional and lying on a two-dimensional invariant subspace. In the context of this work, the invariant subspaces are network synchrony subspaces.

A connected assembly of heteroclinic cycles forms a *heteroclinic network*. Thus, in a heteroclinic network, every equilibria has at least one incoming and one outgoing connection; and given any two equilibria in the network, there is a sequence of connections taking one to the other. In case the heteroclinic cycles are robust and simple the heteroclinic network is a *robust simple heteroclinic network*.

We say that a heteroclinic network (cycle) $\mathcal{H}$ is *attracting in a flow-invariant manifold M containing* $\mathcal{H}$ if there exists a neighbourhood V of $\mathcal{H}$ such that any trajectory with initial condition in $V \cap M$ is attracted to $\mathcal{H}$.

In [2] it is proved the existence of admissible vector fields for the network structure in Figure 1 supporting robust attracting simple heteroclinic cycles involving two fully synchronous equilibria $\mathbf{p}$ and $\mathbf{q}$. For example, following the notation of Example 2.4 above, assuming that $W^u(\mathbf{p}) \subset \{\mathbf{x} : \ x_1 = x_3\}$, thus $\mu_2 = \alpha - \beta > 0$, and assuming that $W^s(\mathbf{p}) \subset \{\mathbf{x} : \ x_1 = x_2\}$, thus $\mu_3 = \alpha - \gamma < 0$, and the opposite for the invariant manifolds $W^u(\mathbf{q})$, $W^s(\mathbf{q})$ and corresponding eigenvalues, then there are admissible vector fields f supporting the existence of the heteroclinic cycle depicted in Figure 3.

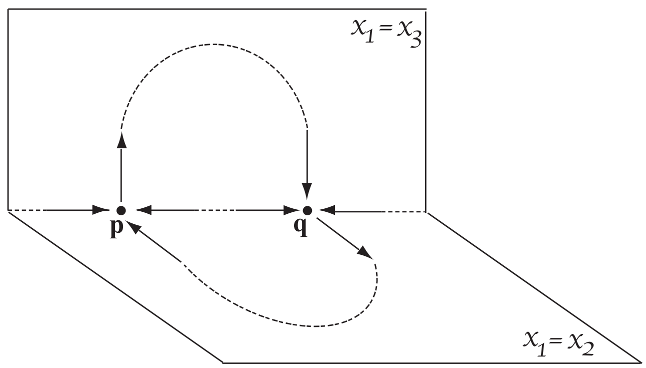

Figure 3: In [2], it is proved that the network structure in Figure 1 supports a robust attracting simple heteroclinic cycle, as the one depicted here, involving two fully synchronous equilibria $\mathbf{p}$ and $\mathbf{q}$, with connections contained in the synchrony subspaces $S^2 = \{\mathbf{x} : \ x_1 = x_2\}$ and $S^3 = \{\mathbf{x} : \ x_1 = x_3\}$.

A heteroclinic network (cycle) can be schematized by a directed graph where the saddle equilibria and corresponding orbits (in forward time) linking them are represented by the nodes and directed edges, respectively. In that sense, we say that a heteroclinic network $\mathcal{H}$ is given by the *product of two heteroclic networks (cycles)*, say $\mathcal{H}_1$ and $\mathcal{H}_2$, when it can be represented by the cartesian product of the two directed graphs associated with the heteroclinic networks (cycles). That is, the nodes of the product $\mathcal{H}_1 \times \mathcal{H}_2$ are the cartesian product of the two sets of nodes of $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, if we denote by ij the node of the product where i corresponds to a node in $\mathcal{H}_1$ and j in $\mathcal{H}_2$, then there is an arrow from the node ij to the node kl if and only if $i = k$ and there is a directed edge from j to l in $\mathcal{H}_2$, or $j = l$ and there is a direcetd edge from i to k in $\mathcal{H}_1$. We call $\mathcal{H}$ a *product heteroclinic network*.

2.3 The join of two networks

The usual definition of *join* of graphs is given by the disjoint union of all graphs together with additional arrows added between every two cells from distinct graphs. In Aguiar and Ruan [6], the join of two coupled cell networks $\mathcal{N}_1$ and $\mathcal{N}_2$ is the network $\mathcal{N}_1 * \mathcal{N}_2$ defined in the following way: the set of cells of $\mathcal{N}_1 * \mathcal{N}_2$ is the union of the sets of cells of $\mathcal{N}_1$ and $\mathcal{N}_2$; the set of edges is the union of the sets of edges of the two networks plus bidirected edges connecting every cell of $\mathcal{N}_1$ to every cell of $\mathcal{N}_2$. Here, we consider a more general definition of join of two networks given by the disjoint union of the two networks together with additional directed arrows added between every two cells from the two networks, but where directed arrows from one group to the other are not necessarily of the same type. Moreover, we allow the absence of arrows from the cells of one network to the cells of the other one. In particular, the disjoint union of the two networks is a join special case.

For $i = 1, 2$, consider the network $\mathcal{N}_i$, with set of cells (nodes) $\mathcal{C}_i$ and set of arrows $\mathcal{E}_i$. Denote the adjacency matrices of $\mathcal{N}_1$ by A_i, for $i = 1, \ldots, p_1$ and the adjacency matrices of $\mathcal{N}_2$ by B_j, for $j = 1, \ldots, p_2$. Assume $r_1 = \#\mathcal{C}_1$ and $r_2 = \#\mathcal{C}_2$. The *join* of $\mathcal{N}_1$ and $\mathcal{N}_2$, denoted by $\mathcal{N}_1 * \mathcal{N}_2$, is the network with set of cells $\mathcal{C}_1 \cup \mathcal{C}_2$ and adjacency matrices

$$\begin{pmatrix} A_i & 0_{r_1,r_2} \\ 0_{r_2,r_1} & 0_{r_2,r_2} \end{pmatrix}, \ i = 1, \ldots, p_1; \quad \begin{pmatrix} 0_{r_1,r_1} & 0_{r_1,r_2} \\ 0_{r_2,r_1} & B_j \end{pmatrix}, \ j = 1, \ldots, p_2; \quad \begin{pmatrix} 0_{r_1,r_1} & C_{r_1,r_2} \\ 0_{r_2,r_1} & 0_{r_2,r_2} \end{pmatrix}; \quad \begin{pmatrix} 0_{r_1,r_1} & 0_{r_1,r_2} \\ D_{r_2,r_1} & 0_{r_2,r_2} \end{pmatrix},$$

where each C or D can be the zero matrix $0_{k,l}$ or the $1_{k,l}$ matrix, of the corresponding orders. Here $n_{k,l}$ denotes the $k \times l$ matrix with entries all equal to n.

Example 2.5 The network in Figure 4 is the six-cell join of two networks $\mathcal{N}_1$ and $\mathcal{N}_2$ both with the network structure given in Figure 1. Recall the adjacency matrices A_1, A_2, in Example 2.1, of the network in Figure 1. The network $\mathcal{N}_1 * \mathcal{N}_2$ has adjacency matrices

$$\begin{pmatrix} A_i & 0_{3,3} \\ 0_{3,3} & 0_{3,3} \end{pmatrix}, \begin{pmatrix} 0_{3,3} & 0_{3,3} \\ 0_{3,3} & A_i \end{pmatrix}, \begin{pmatrix} 0_{3,3} & 1_{3,3} \\ 1_{3,3} & 0_{3,3} \end{pmatrix},$$

if we consider only one join coupling type. Otherwise, instead of the adjacency matrix at the right, we have the following two adjacency matrices:

$$\begin{pmatrix} 0_{3,3} & 1_{3,3} \\ 0_{3,3} & 0_{3,3} \end{pmatrix}, \begin{pmatrix} 0_{3,3} & 0_{3,3} \\ 1_{3,3} & 0_{3,3} \end{pmatrix}.$$

Here $i = 1, 2$. $\diamond$

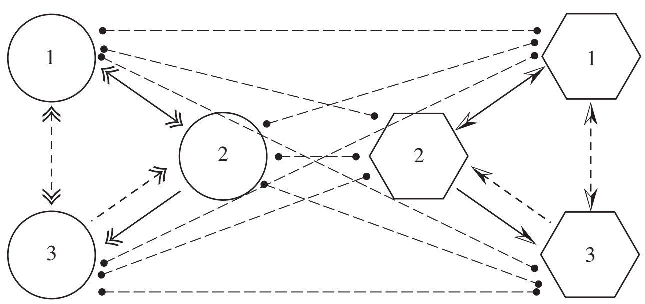

Figure 4: The join network $\mathcal{N}_1 * \mathcal{N}_2$ for networks $\mathcal{N}_1$ and $\mathcal{N}_2$ both with network structure given in Figure 1. Here, the arrow type from cells in $\mathcal{N}_1$ to cells in $\mathcal{N}_2$ and from cells in $\mathcal{N}_2$ to cells in $\mathcal{N}_1$ are drawn equally in order to make easier the visualization.

Synchrony subspaces for the join of networks

For $i = 1, 2$, let S_i be a synchrony subspace for $\mathcal{N}_i$. Then $S_1 \times S_2$ is a synchrony subspace for $\mathcal{N}_1 * \mathcal{N}_2$. If $\mathcal{N}_1, \mathcal{N}_2$ are homogeneous networks it follows then that each admits the full diagonal space as synchrony space, say Δ_i for $\mathcal{N}_i$. It follows then that $\mathcal{N}_1 * \mathcal{N}_2$ has at least the following nontrivial synchrony subspaces: $\Delta_1 \times \Delta_2$, $\Delta_1 \times P_2$ and $P_1 \times \Delta_2$. For the description of all the synchrony subspaces for the join network that are obtained from the synchrony subspaces of the component networks see Aguiar and Ruan [6].

Example 2.6 The synchrony subspaces for the join network $\mathcal{N}_1 * \mathcal{N}_2$, in Figure 4, are listed in Table 2. $\diamond$

$P_1 \times P_2$
$P_1 \times S_2^2 = \{(\mathbf{x},\mathbf{y}):\ y_1 = y_2\}$
$P_1 \times S_2^3 = \{(\mathbf{x},\mathbf{y}):\ y_1 = y_3\}$
$P_1 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}):\ y_1 = y_2 = y_3\}$
$S_1^2 \times P_2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2\}$
$S_1^2 \times S_2^2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2,\ y_1 = y_2\}$
$S_1^2 \times S_2^3 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2,\ y_1 = y_3\}$
$S_1^2 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2,\ y_1 = y_2 = y_3\}$
$S_1^3 \times P_2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_3\}$
$S_1^3 \times S_2^2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_3,\ y_1 = y_2\}$
$S_1^3 \times S_2^3 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_3,\ y_1 = y_3\}$
$S_1^3 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_3,\ y_1 = y_2 = y_3\}$
$\Delta_1 \times P_2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2 = x_3\}$
$\Delta_1 \times S_2^2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2 = x_3,\ y_1 = y_2\}$
$\Delta_1 \times S_2^3 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2 = x_3,\ y_1 = y_3\}$
$\Delta_1 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}):\ x_1 = x_2 = x_3,\ y_1 = y_2 = y_3\}$

Table 2: The synchrony subspaces for $\mathcal{N}_1 * \mathcal{N}_2$ in Figure 4. We use the notation of Table 1.

Admissible coupled cell systems for the join of networks

Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are two (connected) homogeneous networks where $\mathcal{N}_1$ has r_1 cells and $\mathcal{N}_2$ has r_2 cells. Consider the join network $\mathcal{N}_1 * \mathcal{N}_2$ which has $r = r_1 + r_2$ cells. Assuming that the internal dynamics of cells in $\mathcal{N}_1 * \mathcal{N}_2$ is one-dimensional, the state space of $\mathcal{N}_1 * \mathcal{N}_2$ is the cartesian product of the phase spaces of the two networks: $\mathbf{R}^{r_1+r_2} \cong \mathbf{R}^{r_1} \times \mathbf{R}^{r_2}$.

Let $\dot{\mathbf{x}} = F(\mathbf{x})$, for $\mathbf{x} \in \mathbf{R}^{r_1+r_2}$, be a coupled cell system where the vector field F is admissible for the join network $\mathcal{N}_1 * \mathcal{N}_2$. Denote by $x_i \in \mathbf{R}$ and $y_j \in \mathbf{R}$ the coordinates associated to the cell i of $\mathcal{N}_1$ and cell j of $\mathcal{N}_j$, respectively. Let I_i be the set of cells in the network $\mathcal{N}_1$ that are coupled to cell i on $\mathcal{N}_1$ and I_j be the set of cells in network $\mathcal{N}_2$ that are coupled to cell j of $\mathcal{N}_2$. It follows then that the input set of cell i in the join network $\mathcal{N}_1 * \mathcal{N}_2$ is the union of I_i with the set of cells $\mathcal{C}_2$. Similarly, the input set of cell j is the union of I_j and $\mathcal{C}_1$. A vector field consistent with the structure of the join network has then the i and j components of the form

$$\begin{array}{l} \dot{x}_i = f\left(x_i; x_{I_i}; \overline{y}_{\mathcal{C}_2}\right) \\ \dot{y}_j = g\left(y_j; y_{I_j}; \overline{x}_{\mathcal{C}_1}\right) \end{array} . \tag{2.3}$$

Here, the invariance of f (resp. g) under some of the variables in x_{I_i} (resp. y_{I_j}) depends on the structure of $\mathcal{N}_1$ (resp. $\mathcal{N}_2$). Note that the functions f, g are independent of $i \in \mathcal{C}_1$ and $j \in \mathcal{C}_2$, as the join network maintains homogeneity for cells in $\mathcal{C}_1$ and cells in $\mathcal{C}_2$. Also, we use a bar in f to denote that f is invariant under the variables y_k for $k \in \mathcal{C}_2$, as all edges from cells in $\mathcal{C}_2$ to cells in $\mathcal{C}_1$ are of the same edge type. Similarly, for g.

Example 2.7 For the join network $\mathcal{N}_1 * \mathcal{N}_2$ in Figure 4, assuming the cells are one-dimensional, the admissible coupled cell systems have the form

$$\begin{array}{l} \dot{x}_1 = f(x_1; x_2, x_3, \overline{y_1, y_2, y_3}) \\ \dot{x}_2 = f(x_2; x_1, x_3, \overline{y_1, y_2, y_3}) \\ \dot{x}_3 = f(x_3; x_2, x_1, \overline{y_1, y_2, y_3}) \\ \dot{y}_1 = g(y_1; y_2, y_3, \overline{x_1, x_2, x_3}) \\ \dot{y}_2 = g(y_2; y_1, y_3, \overline{x_1, x_2, x_3}) \\ \dot{y}_3 = g(y_3; y_2, y_1, \overline{x_1, x_2, x_3}) \end{array} , \tag{2.4}$$

with $f, g : \mathbf{R}^6 \to \mathbf{R}$ smooth functions.

$\diamond$

Linear admissible vector fields for the join of networks

Let $\mathbf{p} = (p, \ldots, p) \in \mathbf{R}^{r_1}$ and $\overline{\mathbf{p}} = (\overline{p}, \ldots, \overline{p}) \in \mathbf{R}^{r_2}$. That is, $(\mathbf{p}, \overline{\mathbf{p}}) \in \Delta_1 \times \Delta_2$. Suppose that $\dot{\mathbf{x}} = F(\mathbf{x})$, for $\mathbf{x} \in \mathbf{R}^{r_1+r_2}$, is a coupled cell system where the vector field F is admissible for the join network $\mathcal{N}_1 * \mathcal{N}_2$. Using the notation of equations (2.3), take for example cell $1 \in \mathcal{C}_1$ and the input set I_1 formed by the cells in $\mathcal{C}_1$ that have directed edges to cell 1 and choose one cell $i_l \in I_1$ for an edge of type A_i. Similarly, take for example cell $1 \in \mathcal{C}_2$ and the input set I_1 formed by the cells in $\mathcal{C}_2$ that have directed edges to cell 1 and choose one cell $j_m \in I_1$ for an edge of type B_j. Consider the following partial derivatives:

$$\alpha_1 = (\partial f / \partial x_1)_{(\mathbf{p}, \overline{\mathbf{p}})}, \qquad \alpha_2 = (\partial g / \partial y_1)_{(\mathbf{p}, \overline{\mathbf{p}})},$$

$$\beta_i = (\partial f / \partial x_{i_l})_{(\mathbf{p}, \overline{\mathbf{p}})}, \qquad \tau_j = (\partial g / \partial y_{j_m})_{(\mathbf{p}, \overline{\mathbf{p}})},$$

$$\gamma_1 = (\partial f / \partial y_1)_{(\mathbf{p}, \overline{\mathbf{p}})}, \qquad \gamma_2 = (\partial g / \partial x_1)_{(\mathbf{p}, \overline{\mathbf{p}})} .$$

Recall that A_i denotes an adjacency matrix of one type of couplings for the network $\mathcal{N}_1$ and B_j an adjacency matrix of one type of couplings for the network $\mathcal{N}_2$. Thus, α_1, α_2 are the linearized internal dynamics, β_i, τ_j are the linearized couplings strengths for the couplings in the networks $\mathcal{N}_1$, $\mathcal{N}_2$, respectively, and γ_1, γ_2 are the linearized coupling strength for the couplings between the two networks. We have then that

$$(\partial F)_{(\mathbf{p}, \overline{\mathbf{p}})} = \alpha_1 \begin{pmatrix} \mathrm{Id}_{r_1} & 0_{r_1,r_2} \\ 0_{r_2,r_1} & 0_{r_2,r_2} \end{pmatrix} + \sum_{i=1}^{p_1} \beta_i \begin{pmatrix} A_i & 0_{r_1,r_2} \\ 0_{r_2,r_1} & 0_{r_2,r_2} \end{pmatrix} +$$

$$\alpha_2 \begin{pmatrix} 0_{r_1,r_1} & 0_{r_1,r_2} \\ 0_{r_2,r_1} & \mathrm{Id}_{r_2} \end{pmatrix} + \sum_{j=1}^{p_2} \tau_j \begin{pmatrix} 0_{r_1,r_1} & 0_{r_1,r_2} \\ 0_{r_2,r_1} & B_j \end{pmatrix} +$$

$$\gamma_1 \begin{pmatrix} 0_{r_1,r_1} & 1_{r_1,r_2} \\ 0_{r_2,r_1} & 0_{r_2,r_2} \end{pmatrix} + \gamma_2 \begin{pmatrix} 0_{r_1,r_1} & 0_{r_1,r_2} \\ 1_{r_2,r_1} & 0_{r_2,r_2} \end{pmatrix} .$$

Example 2.8 For the join network $\mathcal{N}_1 * \mathcal{N}_2$ in Figure 4, the linearization of the vector field determined by equations (2.4) at $(\mathbf{p},\overline{\mathbf{p}}) \in \{(x_1,x_1,x_1,y_1,y_1,y_1) : x_1,y_1 \in \mathbf{R}\}$ is:

$$
\begin{array}{rcl}
(\partial F)_{(\mathbf{p},\overline{\mathbf{p}})} & = & \left(\begin{array}{cc} \alpha_1 \mathrm{Id}_3 + \beta_1 A_1 + \beta_2 A_2 & 0_{3,3} \\ 0_{3,3} & 0_{3,3} \end{array} \right) \\
& + & \left(\begin{array}{cc} 0_{3,3} & 0_{3,3} \\ 0_{3,3} & \alpha_2 \mathrm{Id}_2 + \tau_1 A_1 + \tau_2 A_2 \end{array} \right) \\
& + & \gamma_1 \left(\begin{array}{cc} 0_{3,3} & 1_{3,3} \\ 0_{3,3} & 0_{3,3} \end{array} \right) + \gamma_2 \left(\begin{array}{cc} 0_{3,3} & 0_{3,3} \\ 1_{3,3} & 0_{3,3} \end{array} \right) .
\end{array}
$$

Considering $f(x,y,z,t,w,l)$, $g(x,y,z,t,w,l)$ in equations (2.4), we have that

$$
\alpha_1 = (\partial f/\partial x)_{(\mathbf{p},\overline{\mathbf{p}})}, \quad \beta_1 = (\partial f/\partial y)_{(\mathbf{p},\overline{\mathbf{p}})}, \quad \beta_2 = (\partial f/\partial z)_{(\mathbf{p},\overline{\mathbf{p}})}, \quad \gamma_1 = (\partial f/\partial t)_{(\mathbf{p},\overline{\mathbf{p}})},
$$

$$
\alpha_2 = (\partial g/\partial x)_{(\mathbf{p},\overline{\mathbf{p}})}, \quad \tau_1 = (\partial g/\partial y)_{(\mathbf{p},\overline{\mathbf{p}})}, \quad \tau_2 = (\partial g/\partial z)_{(\mathbf{p},\overline{\mathbf{p}})}, \quad \gamma_2 = (\partial g/\partial t)_{(\mathbf{p},\overline{\mathbf{p}})} .
$$

$\diamond$

3 Product heteroclinic networks given by the join of networks

As before, we take two homogeneous networks, $\mathcal{N}_1$ and $\mathcal{N}_2$, where $\mathcal{N}_1$ has r_1 cells and $\mathcal{N}_2$ has r_2 cells and consider the join network $\mathcal{N}_1 * \mathcal{N}_2$ with $r = r_1 + r_2$ cells. Let us assume that the internal dynamics of cells in $\mathcal{N}_1$ and $\mathcal{N}_2$ is one-dimensional and so the state space of $\mathcal{N}_1 * \mathcal{N}_2$ is the cartesian product of the phase spaces of the two networks: $\mathbf{R}^{r_1} \times \mathbf{R}^{r_2}$.

Our aim is to construct admissible coupled cell systems for $\mathcal{N}_1 * \mathcal{N}_2$ supporting robust simple product heteroclinic dynamics from the hypothesis that the admissible equations for $\mathcal{N}_1$ and $\mathcal{N}_2$ support robust simple heteroclinic networks (cycles).

We impose two conditions:

(A) If $\mathbf{p} \in \Delta_1$ and $\overline{\mathbf{p}} \in \Delta_2$ are equilibria involved in such heteroclinic networks (cycles) for the admissible equations for $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, then consider the hyperplanes contained in the synchrony subspaces $P_1 \times \Delta_2$ and $\Delta_1 \times P_2$, respectively,

$$
H_{\overline{\mathbf{p}}} = \{(\mathbf{x},\mathbf{y}) : \ \mathbf{y} = \overline{\mathbf{p}}\} \quad \text{and} \quad H_{\mathbf{p}} = \{(\mathbf{x},\mathbf{y}) : \ \mathbf{x} = \mathbf{p}\},
$$

and assume that these hyperplanes are flow-invariant for the join equations (2.3).

(B) The dynamics of equations (2.3) restricted to $H_{\overline{\mathbf{p}}}$ ($H_{\mathbf{p}}$) is conjugated to the dynamics associated with the network $\mathcal{N}_1$ ($\mathcal{N}_2$).

Note that condition (A) implies that $(\mathbf{p},\overline{\mathbf{p}}) \in \Delta_1 \times \Delta_2$ is an equilibrium for the join equations (2.3).

Theorem 3.1 *Consider two homogeneous networks $\mathcal{N}_1$ and $\mathcal{N}_2$ with dynamics supporting a robust simple heteroclinic network (cycle) involving the hyperbolic saddle equilibria $\mathbf{p}_i \in \Delta_1$, for $i = 1,\ldots,n_1$, and $\overline{\mathbf{p}}_j \in \Delta_2$, for $j = 1,\ldots,n_2$, respectively. Assume conditions (A) and (B).*

Then, in the restriction to the invariant manifold $\Delta_1 \times P_2 \cup P_1 \times \Delta_2$, the dynamics of $\mathcal{N}_1 * \mathcal{N}_2$ supports a robust simple heteroclinic network

$$\mathcal{H} = \left(\bigcup_{i=1}^{n_1} \overline{\mathcal{H}}_i\right) \bigcup \left(\bigcup_{j=1}^{n_2} \mathcal{H}_j\right)$$

involving the equilibria $(\mathbf{p}_i, \overline{\mathbf{p}}_j)$. Here, for each $i = 1, \ldots, n_1$, $\overline{\mathcal{H}}_i$ denotes the heteroclinic network (cycle), contained in the hyperplane $H_{\mathbf{p}_i}$, involving the equilibria $(\mathbf{p}_i, \overline{\mathbf{p}}_j)$ for $j = 1, \ldots, n_2$. Also, for each $j = 1, \ldots, n_2$, $\mathcal{H}_j$ denotes the heteroclinic network (cycle), contained in the hyperplane $H_{\overline{\mathbf{p}}_j}$, involving the equilibria $(\mathbf{p}_i, \overline{\mathbf{p}}_j)$ for $i = 1, \ldots, n_1$.

Proof The existence of the robust simple heteroclinic networks (cycles) $\overline{\mathcal{H}}_i$ in $H_{\mathbf{p}_i}$ and $\mathcal{H}_j$ in $H_{\overline{\mathbf{p}}_j}$ follows from conditions (A) and (B). Note that the union of these networks (cycles) is a connected invariant set as every intersection of $\overline{\mathcal{H}}_i$ with $\mathcal{H}_j$ is nonempty (consists of a unique equilibrium) and so $\mathcal{H}$ is a robust simple heteroclinic network.

Note that the equilibria and heteroclinic connections of $\mathcal{H}$ are contained in two-dimensional and three-dimensional synchrony subspaces, respectively. More precisely, the equilibria and heteroclinic connections of $\mathcal{H}$ are contained in products of an hyperbolic equilibria by an one-dimensional and two-dimensional synchrony subspaces, respectively. Since small perturbations of equations (2.3) that preserve the join network structure preserve the synchrony subspaces, it follows the robustness of the heteroclinic network $\mathcal{H}$. □

Consider now equations (2.3), where f, g take the following form:

$$\begin{aligned} f\left(x_i; x_{I_i}; \overline{y}_{\mathcal{C}_2}\right) &= f_1(x_i; x_{I_i})\left(1 + h_1\left(\overline{y}_{\mathcal{C}_2}\right)\right), \\ g\left(y_j; y_{I_j}; \overline{x}_{\mathcal{C}_1}\right) &= f_2(y_j; y_{I_j})\left(1 + h_2\left(\overline{x}_{\mathcal{C}_1}\right)\right), \end{aligned} \tag{3.5}$$

with f_1, f_2, h_1, h_2 smooth functions. Thus equations (2.3) take the form

$$\begin{aligned} \dot{x}_i &= f_1(x_i; x_{I_i})\left(1 + h_1\left(\overline{y}_{\mathcal{C}_2}\right)\right) \\ \dot{y}_j &= f_2(y_j; y_{I_j})\left(1 + h_2\left(\overline{x}_{\mathcal{C}_1}\right)\right) \end{aligned} . \tag{3.6}$$

Assuming that $f_1(\mathbf{p}) = 0$ and $f_2(\overline{\mathbf{p}}) = 0$, it follows that $(\mathbf{p}, \overline{\mathbf{p}}) \in \Delta_1 \times \Delta_2$ is an equilibrium of equations (3.6). Moreover, taking J_1 to be the Jacobian at $\mathbf{p}$ of equations for $\mathcal{N}_1$ with f_1 and J_2 the Jacobian at $\overline{\mathbf{p}}$ of equations for $\mathcal{N}_2$ with f_2, the Jacobian for the system (3.6) at $(\mathbf{p}, \overline{\mathbf{p}})$ has the following form:

$$\operatorname{diag}\left((1 + h_1(\overline{\mathbf{p}}))J_1, (1 + h_2(\mathbf{p}))J_2\right) .$$

Lemma 3.2 *Consider the join coupled cell system (3.6). Assume $f_1(\mathbf{p}) = 0$ and $f_2(\overline{\mathbf{p}}) = 0$, where $\mathbf{p} \in \Delta_1$ and $\overline{\mathbf{p}} \in \Delta_2$. Take J_1 to be the Jacobian at $\mathbf{p}$ of equations for $\mathcal{N}_1$ with f_1 and J_2 the Jacobian at $\overline{\mathbf{p}}$ of equations for $\mathcal{N}_2$ with f_2. Assume that*

$$1 + h_1(\overline{\mathbf{p}}) > 0 \text{ and } 1 + h_2(\mathbf{p}) > 0 . \tag{3.7}$$

Then:
(i) The equilibrium $(\mathbf{p}, \overline{\mathbf{p}}) \in \Delta_1 \times \Delta_2$ of (3.6) has stability determined by (the signs of the eigenvalues of) J_1 and J_2.
(ii) The hyperplanes $H_{\overline{\mathbf{p}}}$ and $H_{\mathbf{p}}$ are flow-invariant and contained in the synchrony subspaces $P_1 \times \Delta_2$ and $\Delta_1 \times P_2$, respectively.
(iii) The dynamics of equations (3.6) restricted to $H_{\overline{\mathbf{p}}}$ ($H_{\mathbf{p}}$) is conjugated to the dynamics associated with the network $\mathcal{N}_1$ ($\mathcal{N}_2$).

Proof Statements (i-ii) follow trivially from the discussion above. For (iii), note that the restriction of (3.6) to $P_1 \times \Delta_2$ is a coupled cell system associated with a quotient network with $r_1 + 1$ cells . More precisely, this quotient network admitts $\mathcal{N}_1$ as a subnetwork where all cells of $\mathcal{N}_1$ receive the same input from the cell representing the cells of $\mathcal{N}_2$ in that quotient. This cell can be seen as a controller. In the case that the controller cell is at the equilibrium state $\overline{\mathbf{p}}$ then the dynamics of the restricted system,

$$\begin{array}{l} \dot{x}_i = f_1(x_i; x_{I_i})\,(1 + h_1(\overline{\mathbf{p}})) \\ \dot{y}_j = 0 \end{array}, \tag{3.8}$$

is conjugated to the dynamics associated with the network $\mathcal{N}_1$, as we are assuming $1 + h_1(\overline{\mathbf{p}}) > 0$. The same holds considering $\Delta_1 \times P_2$, the network $\mathcal{N}_2$, the equilibrium $\mathbf{p}$ and the hyperplane $H_{\mathbf{p}}$, assuming the condition $1 + h_2(\mathbf{p}) > 0$.

Corollary 3.3 *Under the conditions of Lemma 3.2, assuming the dynamics of $\mathcal{N}_1$ and $\mathcal{N}_2$ supports a robust simple heteroclinic network (cycle) involving the hyperbolic saddle equilibria $\mathbf{p}_i \in \Delta_1$, for $i = 1, \ldots, n_1$, and $\overline{\mathbf{p}}_j \in \Delta_2$, for $j = 1, \ldots, n_2$, respectively, the result in Theorem 3.1 applies to equations (3.6). That is, in the restriction to the invariant manifold $\Delta_1 \times P_2 \cup P_1 \times \Delta_2$, the dynamics of $\mathcal{N}_1 * \mathcal{N}_2$ supports a robust simple product heteroclinic network as described in Theorem 3.1.*

4 Example

In this section, we consider the join network $\mathcal{N}_1 * \mathcal{N}_2$ in Figure 4 and the associated admissible coupled cell systems in equations (4.9) below. We consider individual cell dynamics f_1 and f_2 associated to $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, such that the corresponding admissible vector fields support the existence of robust simple heteroclinic cycles. Note that we are assuming no symmetry for f_1 and f_2 and that the flow invariant subspaces guaranteeing the existence and robustness of the heteroclinic cycles are the synchrony subspaces determined only by the corresponding associated network structures.

Join equations

Considering the equations (2.1) for the coupled cell systems admissible by the network structure of both network $\mathcal{N}_1$ and $\mathcal{N}_2$, we take admissible equations for $\mathcal{N}_1 * \mathcal{N}_2$ in the following way:

$$\begin{array}{l} \dot{x}_1 = f_1(x_1; x_2, x_3)\,[1 + h_1(\overline{y_1, y_2, y_3})] \\ \dot{x}_2 = f_1(x_2; x_1, x_3)\,[1 + h_1(\overline{y_1, y_2, y_3})] \\ \dot{x}_3 = f_1(x_3; x_2, x_1)\,[1 + h_1(\overline{y_1, y_2, y_3})] \\ \dot{y}_1 = f_2(y_1; y_2, y_3)\,[1 + h_2(\overline{x_1, x_2, x_3})] \\ \dot{y}_2 = f_2(y_2; y_1, y_3)\,[1 + h_2(\overline{x_1, x_2, x_3})] \\ \dot{y}_3 = f_2(y_3; y_2, y_1)\,[1 + h_2(\overline{x_1, x_2, x_3})] \end{array} \tag{4.9}$$

with $h_1, h_2:\ \mathbf{R}^3 \to \mathbf{R}$ smooth functions and f_1, f_2 admissible functions for the networks $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively.

Assumptions on equilibria and stability

For $p, q, \overline{p}, \overline{q} \in \mathbf{R}$ denote by $\mathbf{p} = (p,p,p) \in \mathbf{R}^3$, $\mathbf{q} = (q,q,q) \in \mathbf{R}^3$, $\overline{\mathbf{p}} = (\overline{p},\overline{p},\overline{p}) \in \mathbf{R}^3$ and $\overline{\mathbf{q}} = (\overline{q},\overline{q},\overline{q}) \in \mathbf{R}^3$. We will assume that:
(i) The points $\mathbf{p}$ and $\mathbf{q}$ are two non-zero distinct hyperbolic saddle equilibria with 1-dimensional unstable manifolds for the equations (2.1) with $f = f_1$.
(ii)The points $\overline{\mathbf{p}}$ and $\overline{\mathbf{q}}$, distinct from $\mathbf{p}$ and $\mathbf{q}$, are two non-zero distinct hyperbolic saddle equilibria with 1-dimensional unstable manifolds for the equations (2.1) with $f = f_2$.
In fact, we have the following stability assumptions:

$$f_1(\mathbf{p}) = f_1(\mathbf{q}) = 0, \qquad W^u(\mathbf{p}) \subset \{\mathbf{x}:\ x_1 = x_3\}, \quad W^u(\mathbf{q}) \subset \{\mathbf{x}:\ x_1 = x_2\},$$

$$f_2(\overline{\mathbf{p}}) = f_2(\overline{\mathbf{q}}) = 0, \qquad W^u(\overline{\mathbf{p}}) \subset \{\mathbf{y}:\ y_1 = y_3\}, \quad W^u(\overline{\mathbf{q}}) \subset \{\mathbf{y}:\ y_1 = y_2\}.$$

Using the notation of Example 2.4, for equilibria of equations (2.1) with $f = f_1$, considering $f_1(x,y,z)$, denote:

$$\alpha_1 = \tfrac{\partial f_1}{\partial x}(\mathbf{p}), \quad \beta_1 = \tfrac{\partial f_1}{\partial y}(\mathbf{p}), \quad \gamma_1 = \tfrac{\partial f_1}{\partial z}(\mathbf{p}),$$

$$\alpha_2 = \tfrac{\partial f_1}{\partial x}(\mathbf{q}), \quad \beta_2 = \tfrac{\partial f_1}{\partial y}(\mathbf{q}), \quad \gamma_2 = \tfrac{\partial f_1}{\partial z}(\mathbf{q}).$$

Similarly, using the notation of Example 2.4, for equilibria of equations (2.1) with $f = f_2$, considering $f_2(x,y,z)$, denote:

$$\alpha_3 = \tfrac{\partial f_2}{\partial x}(\overline{\mathbf{p}}), \quad \beta_3 = \tfrac{\partial f_2}{\partial y}(\overline{\mathbf{p}}), \quad \gamma_3 = \tfrac{\partial f_2}{\partial z}(\overline{\mathbf{p}}),$$

$$\alpha_4 = \tfrac{\partial f_2}{\partial x}(\overline{\mathbf{q}}), \quad \beta_4 = \tfrac{\partial f_2}{\partial y}(\overline{\mathbf{q}}), \quad \gamma_4 = \tfrac{\partial f_2}{\partial z}(\overline{\mathbf{q}}).$$

Recalling Example 2.4, we have that the eigenvalues of $J(\mathbf{p})$ for (2.1) with $f = f_1$ are

$$\mu_1^1 = \alpha_1 + \beta_1 + \gamma_1, \quad \mu_2^1 = \alpha_1 - \beta_1, \quad \mu_3^1 = \alpha_1 - \gamma_1$$

and are associated with the following eigenlines

$$E_1:\ x_1 = x_2 = x_3, \quad E_3:\ x_2 = -(1+\gamma\beta^{-1})x_1,\ x_1 = x_3, \quad \text{and} \quad E_2:\ x_3 = -(1+\beta\gamma^{-1})x_1,\ x_1 = x_2.$$

We are assuming that $W^u(\mathbf{p}) \subset \{\mathbf{x}:\ x_1 = x_3\}$, thus we suppose that $\mu_2^1 = \alpha_1 - \beta_1 > 0$. Similarly, as we are assuming that $W^u(\mathbf{q}) \subset \{\mathbf{x}:\ x_1 = x_2\}$, we suppose that $\mu_3^2 = \alpha_2 - \gamma_2 > 0$. Using the notation of Table 1, note that E_1, E_2 are contained in the synchrony subspace S_1^2 and E_1, E_3 are contained in the synchrony subspace S_1^3.

Assume so that:
(i) Eigenvalues of $J(\mathbf{p})$ for (2.1) with $f = f_1$:

$$\mu_1^1 = \alpha_1 + \beta_1 + \gamma_1 < 0, \quad \mu_2^1 = \alpha_1 - \beta_1 > 0, \quad \mu_3^1 = \alpha_1 - \gamma_1 < 0;$$

(ii) Eigenvalues of $J(\mathbf{q})$ for (2.1) with $f = f_1$:

$\mu_1^2 = \alpha_2 + \beta_2 + \gamma_2 < 0, \quad \mu_2^2 = \alpha_2 - \beta_2 < 0, \quad \mu_3^2 = \alpha_2 - \gamma_2 > 0.$

Moreover, assume that:
(iii) Eigenvalues of $J(\overline{\mathbf{p}})$ for (2.1) with $f = f_2$:

$\mu_1^3 = \alpha_3 + \beta_3 + \gamma_3 < 0, \quad \mu_2^3 = \alpha_3 - \beta_3 > 0, \quad \mu_3^3 = \alpha_3 - \gamma_3 < 0;$

(iv) Eigenvalues of $J(\overline{\mathbf{q}})$ for (2.1)with $f = f_2$:

$\mu_1^4 = \alpha_4 + \beta_4 + \gamma_4 < 0, \quad \mu_2^4 = \alpha_4 - \beta_4 < 0, \quad \mu_3^4 = \alpha_4 - \gamma_4 > 0.$

Heteroclinic cycles for the $\mathcal{N}_1$ and $\mathcal{N}_2$ admissible equations

As shown in [2], there is cell dynamics f_1 for the cells in $\mathcal{N}_1$ such that there are admissible vector fields supporting a robust attracting simple heteroclinic cycle involving $\mathbf{p}$ and $\mathbf{q}$. Similarly, there are admissible vector fields for $\mathcal{N}_2$ that support a robust attracting simple heteroclinic cycle involving $\overline{\mathbf{p}}$ and $\overline{\mathbf{q}}$. Recall the discussion at Section 2.2.

Consider the three-dimensional invariant hyperplanes

$$H_{\overline{\mathbf{p}}} = \{(\mathbf{x}, \mathbf{y}) : \ y_1 = y_2 = y_3 = \overline{p}\} \quad \text{and} \quad H_{\overline{\mathbf{q}}} = \{(\mathbf{x}, \mathbf{y}) : \ y_1 = y_2 = y_3 = \overline{q}\}$$

contained in $P_1 \times \Delta_2$,

$$H_{\mathbf{p}} = \{(\mathbf{x}, \mathbf{y}) : \ x_1 = x_2 = x_3 = p\} \quad \text{and} \quad H_{\mathbf{q}} = \{(\mathbf{x}, \mathbf{y}) : \ x_1 = x_2 = x_3 = q\}$$

contained in $\Delta_1 \times P_2$, and the following conditions:

$$1 + h_1(\overline{\mathbf{p}}) > 0, \quad 1 + h_1(\overline{\mathbf{q}}) > 0, \quad 1 + h_2(\mathbf{p}) > 0, \quad 1 + h_2(\mathbf{q}) > 0\,. \tag{4.10}$$

Lemma 4.1 *Assume conditions (4.10). There are admissible equations (4.9) such that in the restriction to the union of the synchrony subspaces $P_1 \times \Delta_2$ and $\Delta_1 \times P_2$, the network $\mathcal{N}_1 * \mathcal{N}_2$ supports the existence of the following robust attracting simple heteroclinic cycles:*

(a) $\mathcal{H}_1$, contained in $H_{\overline{\mathbf{p}}}$, involving the equilibria $(\mathbf{p}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{p}})$;

(b) $\mathcal{H}_2$, contained in $H_{\overline{\mathbf{q}}}$, involving the equilibria $(\mathbf{p}, \overline{\mathbf{q}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$;

(c) $\mathcal{H}_3$, contained in $H_{\mathbf{p}}$, involving the equilibria $(\mathbf{p}, \overline{\mathbf{p}})$ and $(\mathbf{p}, \overline{\mathbf{q}})$;

(d) $\mathcal{H}_4$, contained in $H_{\mathbf{q}}$, involving the equilibria $(\mathbf{q}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$.

The heteroclinic cycles $\mathcal{H}_1, \mathcal{H}_2$ are attracting in $P_1 \times \Delta_2$ and the heteroclinic cycles $\mathcal{H}_3, \mathcal{H}_4$ are attracting in $\Delta_1 \times P_2$.

Proof We argue the existence of the heteroclinic cycle $\mathcal{H}_1$. The existence of the other heteroclinic cycles can be shown analogously.

The restriction of the join equations (4.9) to the four-dimensional synchrony subspace $P_1 \times \Delta_2$ is given by:

$$\begin{array}{l} \dot{x}_1 = f_1(x_1; x_2, x_3)\,[1 + h_1(\overline{y_1; y_1, y_1})] \\ \dot{x}_2 = f_1(x_2; x_1, x_3)\,[1 + h_1(\overline{y_1; y_1, y_1})] \\ \dot{x}_3 = f_1(x_3; x_2, x_1)\,[1 + h_1(\overline{y_1; y_1, y_1})] \\ \dot{y}_1 = f_2(y_1; y_1, y_1)\,[1 + h_2(\overline{x_1, x_2, x_3})] \end{array} . \tag{4.11}$$

The associated quotient network is pictured in Figure 5. We recall Lemma 3.2 (iii) that, in the particular situation where the individual dynamics of the controller cell is at the equilibrium state $\overline{\mathbf{p}}$, then the dynamics of the restricted systems is conjugated to the dynamics associated with the network $\mathcal{N}_1$, and we can conclude the existence of a robust simple heteroclinic cycle. In fact, the restriction of equations (4.11) to the three-dimensional invariant hyperplane $H_{\overline{\mathbf{p}}}$ take the form

$$\begin{array}{l} \dot{x}_1 = f_1(x_1; x_2, x_3)\,\left[1 + h_1(\overline{\overline{p}, \overline{p}, \overline{p}})\right] \\ \dot{x}_2 = f_1(x_2; x_1, x_3)\,\left[1 + h_1(\overline{\overline{p}, \overline{p}, \overline{p}})\right] \\ \dot{x}_3 = f_1(x_3; x_2, x_1)\,\left[1 + h_1(\overline{\overline{p}, \overline{p}, \overline{p}})\right] \\ \dot{y}_1 = 0 \end{array} .$$

Assuming conditions (4.10) and taking into account Lemma 3.2 (i), we have that the stability of the equilibria $(\mathbf{p}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{p}})$, in the restriction to the hyperplane $H_{\overline{\mathbf{p}}}$ is determined by the stability of $\mathbf{p}$ and $\mathbf{q}$, respectively. Using the same arguments for the local and global construction of heteroclinic orbits presented in Sections 5.2 and 5.3 of [2], that allow to conclude the support by the dynamics of $\mathcal{N}_1$ of a robust attracting simple heteroclinic cycle involving the two equilibria $\mathbf{p}$ and $\mathbf{q}$ in Δ_1 with heteroclinic connections in S_1^2 and S_1^3, we can conclude the existence of cell dynamics such that there exists a robust attracting simple heteroclinic cycle $\mathcal{H}_1$ in the invariant hyperplane $H_{\overline{\mathbf{p}}}$ involving the two equilibria $(\mathbf{p}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{p}})$ in $\Delta_1 \times \{\overline{\mathbf{p}}\}$ and with heteroclinic connections in $S_1^2 \times \{\overline{\mathbf{p}}\}$ and $S_1^3 \times \{\overline{\mathbf{p}}\}$.

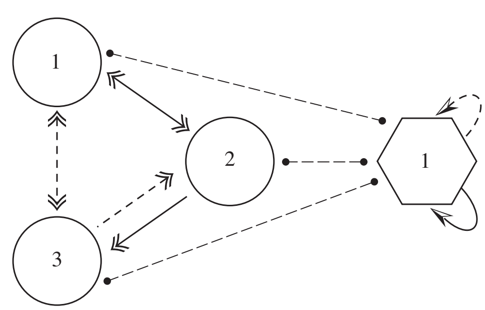

Figure 5: Quotient network of the join network $\mathcal{N}_1 * \mathcal{N}_2$ by the four-dimensional synchrony subspace $P_1 \times \Delta_2$. Here, P_1 is the total phase space for $\mathcal{N}_1$ and Δ_2 is the full synchrony subspace for $\mathcal{N}_2$.

□

Lemma 4.2 *Assuming conditions*

$$1 + h_1(\mathbf{y}) > 0, \qquad 1 + h_2(\mathbf{x}) > 0, \quad \forall (\mathbf{x}, \mathbf{y}) \in P_1 \times P_2, \tag{4.12}$$

in the restriction of the join equations (4.9) to the synchrony subspace $P_1 \times \Delta_2$ ($\Delta_1 \times P_2$) the invariant hyperplanes $H_{\overline{\mathbf{p}}}$ and $H_{\overline{\mathbf{q}}}$ ($H_{\mathbf{p}}$ and $H_{\mathbf{q}}$) are attractors.

Proof We present the proof for the hyperplane $H_{\overline{\mathbf{p}}}$. The proof for the other hyperplanes is analogous. Consider equations (4.11) that correspond to the restriction of the join equations (4.9) to the synchrony subspace $P_1 \times \Delta_2$. The result follows from the fact that for every point in $H_{\overline{\mathbf{p}}}$ the eigenvalue of the Jacobian matrix of equations (4.11) in the direction transversal to $H_{\overline{\mathbf{p}}}$ is negative. In fact, that eigenvalue is given by $(\alpha_3 + \beta_3 + \gamma_3)(1 + h_2(\mathbf{x}))$ which is negative taking the assumptions. Note that the last row of the Jacobian matrix of equations (4.11) at a point of the form $(\mathbf{x}, \overline{\mathbf{p}})$ has all the entries equal to zero with the exception of the entry in the last column which is $(\alpha_3 + \beta_3 + \gamma_3)(1 + h_2(\mathbf{x}))$.
□

From the conclusions above we get then the following result.

Theorem 4.3 *Assuming conditions (4.10), in the restriction to the invariant manifold $\Delta_1 \times P_2 \cup P_1 \times \Delta_2$, there are admissible join equations (4.9) for $\mathcal{N}_1 * \mathcal{N}_2$ supporting a robust simple heteroclinic network $\mathcal{H} = \cup_{i=1}^4 \mathcal{H}_i$ involving the equilibria $(\mathbf{p}, \overline{\mathbf{p}})$, $(\mathbf{p}, \overline{\mathbf{q}})$, $(\mathbf{q}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$. The heteroclinic network $\mathcal{H}$ is robust for small perturbations that preserve the join network structure. Moreover, assuming conditions (4.12), there are admissible equations of the form (4.9) such that the heteroclinic network $\mathcal{H}$ is attracting in $\Delta_1 \times P_2 \cup P_1 \times \Delta_2$. See Figure 6 for a schematic representation of the heteroclinic network $\mathcal{H}$.*

Proof The existence of the robust simple heteroclinic network $\mathcal{H}$ follows from Corollary 3.3 and the analysis above of the dynamics associated to the join network $\mathcal{N}_1 * \mathcal{N}_2$ in the restriction to the synchrony subspaces $P_1 \times \Delta_2$ and $\Delta_1 \times P_2$.

From Lemma 4.1, we have that there are admissible vector fields f_1 such that the heteroclinic cycles $\mathcal{H}_1, \mathcal{H}_2$ are attracting in $H_{\overline{\mathbf{p}}}$ and $H_{\overline{\mathbf{q}}}$, respectively, and there are admissible vector fields f_2 such that the heteroclinic cycles $\mathcal{H}_3, \mathcal{H}_4$ are attracting in $H_{\mathbf{p}}$ and $H_{\mathbf{q}}$, respectively. From Lemma 4.2, we have that the invariant hyperplanes $H_{\overline{\mathbf{p}}}$ and $H_{\overline{\mathbf{q}}}$ are attractors in $P_1 \times \Delta_2$ and the invariant hyperplanes $H_{\mathbf{p}}$ and $H_{\mathbf{q}}$ are attractors in $\Delta_1 \times P_2$. We can conclude then that, there are admissible equations of the form (4.9) such that the heteroclinic network $\mathcal{H}$ is attracting in $\Delta_1 \times P_2 \cup P_1 \times \Delta_2$.
□

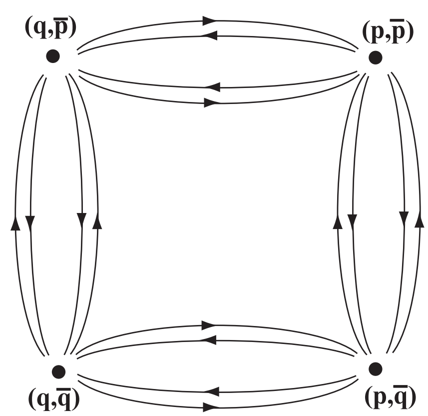

Figure 6: Schematic representation of the robust simple heteroclinic network $\mathcal{H}$ supported by the join network $\mathcal{N}_1 * \mathcal{N}_2$ equations (4.9).

Remark 4.4 (i) The equilibria in the heteroclinic network $\mathcal{H}$ are partially synchronous equilibria. (ii) The manifold $\Delta_1 \times P_2 \cup P_1 \times \Delta_2$ is not a (synchrony) subspace. (iii) Since the unstable manifolds of all the four equilibria in the heteroclinic network $\mathcal{H}$ are two-dimensional it is expected that $\mathcal{H}$ is a subnetwork of a bigger network that also contains heteroclinic connections in $P_1 \times P_2 \backslash (P_1 \times \Delta_2 \cup \Delta_1 \times P_2)$. For example, on the assumptions that $1 + h_1(\mathbf{y}) \neq 0$, $1 + h_2(\mathbf{x}) \neq 0$ for $(\mathbf{x}, \mathbf{y}) \in P_1 \times P_2$, there are no more equilibria beyond those obtained by the product of the equilibria in $\mathcal{N}_1$ and $\mathcal{N}_2$. It is expected that there are heteroclinic connections from $(\mathbf{p}, \overline{\mathbf{p}})$ to $(\mathbf{q}, \overline{\mathbf{q}})$ in $S_1^3 \times S_2^3$, from $(\mathbf{p}, \overline{\mathbf{q}})$ to $(\mathbf{q}, \overline{\mathbf{p}})$ in $S_1^3 \times S_2^2$, from $(\mathbf{q}, \overline{\mathbf{p}})$ to $(\mathbf{p}, \overline{\mathbf{q}})$ in $S_1^2 \times S_2^3$ and from $(\mathbf{q}, \overline{\mathbf{q}})$ to $(\mathbf{p}, \overline{\mathbf{p}})$ in $S_1^2 \times S_2^2$. Thus, in the total phase space $P_1 \times P_2$, the network $\mathcal{H}$ is not an attractor. $\diamond$

5 Conclusions

In this work we have described a method to combine, via the join operation, two networks $\mathcal{N}_1$ and $\mathcal{N}_2$, with dynamics realising a robust simple heteroclinic network (cycle), in order to get a coupled cell network with dynamics supporting a robust simple heteroclinic network $\mathcal{H}$ given by the product of the two networks (cycles).

An observation that can be inferred from our construction is that the existence of the heteroclinic network $\mathcal{H}$ occurs even for the particular cases where one, or both, coupling functions is the zero function. That is, taking the join special cases, either by using the join of two networks just as its disjoint union or by taking the join of two networks where there are only join arrows from the cells of one network to the cells of the other network. The same key points imply the aforementioned existence of product heteroclinic networks for the join.

We have assumed no symmetry for the component networks of the join as we aimed to construct heteroclinic networks not forced by symmetry but only by the network structure. It easily follows that it is also possible to get a product heteroclinic network by doing the join of two symmetric coupled cell systems with heteroclinic cycles (or networks) contained in the union of fixed point subspaces, instead of the union of synchrony subspaces.

The procedure presented here can be applied iteratively. For example, if we consider a network $\mathcal{N}_3$ with the same network structure as $\mathcal{N}_1$ and $\mathcal{N}_2$, and do the join of $\mathcal{N}_1 * \mathcal{N}_2$ with $\mathcal{N}_3$ then there are coupled cell systems associated to the join network $(\mathcal{N}_1 * \mathcal{N}_2) * \mathcal{N}_3$ supporting a robust simple heteroclinic network involving eight partially synchronous equilibria, given by the product of the heteroclinic network $\mathcal{H}$ with the heteroclinic cycle supported by the dynamics associated to $\mathcal{N}_3$. Moreover, the method used generalizes to other coupled cell network structures realising heteroclinic cycles or networks.

A natural extension of this work is to consider the construction of complex heteroclinic networks based on simpler heteroclinic networks (cycles) using other network graph operations as, for example, the product operation.

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