Spatially Periodic Patterns of Synchrony in Lattice Networks

By Ana Paula S. Dias and Eliana Manuel Pinho† ‡

CMUP and Departamento de Matemática Pura, Universidade do Porto
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

We consider n-dimensional Euclidean lattice networks with nearest neighbour coupling architecture. The associated lattice dynamical systems are infinite systems of ordinary differential equations, the cells, indexed by the points in the lattice. A pattern of synchrony is a finite-dimensional flow-invariant subspace for all lattice dynamical systems with the given network architecture. These subspaces correspond to a classification of the cells into k classes, or colours, and are described by a local colouring rule, named balanced colouring.

Previous results with planar lattices show that patterns of synchrony can exhibit several behaviours such as periodicity. Considering sufficiently extensive couplings, spatial periodicity appears for all the balanced colourings with k colours. However, there is not a direct way of relating the local colouring rule and the colouring of the whole lattice network.

Given an n-dimensional lattice network with nearest neighbour coupling architecture, and a local colouring rule with k colours, we state a necessary and sufficient condition for the existence of a spatially periodic pattern of synchrony. This condition involves finite coupled cell networks, whose couplings are bidirectional and whose cells are coloured according to the given rule. As an intermediate step, we obtain the proportion of the cells for each colour, for the lattice network and any finite bidirectional network with the same balanced colouring. A crucial tool in obtaining our results is a classical theorem of graph theory concerning the factorisation of even degree regular graphs, a class of graphs where lattice networks are included.

Keywords: lattice dynamical systems, coupled cell networks, balanced colourings, patterns of synchrony, spatially periodic patterns

Mathematics Subject Classification: 82B20, 34C15, 37L60, 05C90

1. Introduction

(*a*) *context*

The theory of coupled cell systems, developed in [Stewart *et al.* 2003] and [Golubitsky *et al.* 2005], applies to the case of a *lattice network*, $G_{\mathcal{L}}$, where an infinite number of identical cells are disposed on a lattice $\mathcal{L}$. Each cell is a system of

† E-mail: apdias@fc.up.pt and eopinho@fc.up.pt

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ordinary differential equations with phase space $\mathbf{R}^d$, with $d \in \mathbf{N}$, and is coupled to the same number of cells in its neighbourhood, such that the whole system is invariant by the translations of $\mathcal{L}$.

In this work we consider n-dimensional Euclidean lattices with nearest neighbour coupling architecture. If $\mathcal{J}$ is the set of the nearest neighbours of the origin then a *lattice dynamical system* is defined by the infinite number of ordinary differential equations:

$$\dot{x}_l = f\left(x_l, \overline{x_{l+g_1}, \ldots, x_{l+g_v}}\right) \quad \text{with } l \in \mathcal{L} \tag{1.1}$$

where $v = \#\mathcal{J}$ is the *valence* or *degree*, $g_1, \ldots, g_v \in \mathcal{J}$ and f is invariant under all permutations of the variables under the bar. Lattice networks are constant degree graphs, called *regular graphs*.

This follows the general definition of lattice networks and lattice dynamical systems considered in [Antoneli *et al.* 2005] where it is assumed that the coupling structure is determined by the distances between lattice points.

Patterns of synchrony are finite-dimensional flow-invariant subspaces for all lattice dynamical systems with a given network architecture and are formed by setting equal the coordinates in different cells. The general theory of coupled cell systems [Golubitsky *et al.* 2005, Stewart *et al.* 2003] shows that finding patterns of synchrony is equivalent to finding *balanced colourings* of cells, where the colour of the cell $l \in \mathcal{L}$ defines the number of its neighbours with each colour. In a nearest neighbour coupling architecture, each balanced colouring corresponds to a *colouring matrix* $A = (a_{ij})$. The entry a_{ij} is the number of cells with colour j that are coupled to a cell of colour i, with $i, j \in \mathcal{U} = \{1, \ldots, k\}$. Here $\mathcal{U}$ is the set that indexes the k different colours of the balanced colouring. Our study concerns balanced colourings for a general number of colours k.

Equations (1.1), can be restricted to the flow-invariant subspace defined by identifying the cell coordinates of cells with the same colour, denoted by x_i with $i \in \mathcal{U}$. Thus, the restriction is defined by the finite-dimensional system with k equations:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & f\left(x_1, \overline{y_{11}, \ldots, y_{1v}}\right), \\ & \vdots & \\ \dot{x}_k & = & f\left(x_k, \overline{y_{k1}, \ldots, y_{kv}}\right). \end{array} \right. \tag{1.2}$$

where the arrays $(y_{i1}, \ldots, y_{iv})$ have a_{ij} entries equal to x_j. This system can be described by a *quotient network*, a coupled cell network with k cells, one of each colour, having adjacency matrix A.

See Section (d) for an illustration of these concepts.

There are several results concerning patterns of synchrony in lattice networks. In what follows, we consider that a *family of patterns* is a set of patterns in a lattice network having the same balanced colouring rule, for a given architecture, and that a *spatially periodic pattern of synchrony* corresponds to a balanced colouring of an n-dimensional lattice network that is periodic along n linearly independent directions.

[Golubitsky *et al.* 2004] describes an infinite family of two-colour patterns of synchrony on planar square lattice systems with nearest neighbour coupling and [Wang and Golubitsky 2005] classifies all possible two-colour patterns of synchrony of planar square and hexagonal lattice networks with two different architectures — nearest or both nearest and next nearest neighbour couplings. For nearest neighbour coupling architecture they show that there are two (resp. three) infinite continuum families in the square (resp. hexagonal) lattice and eight (resp. ten) isolated patterns. Moreover, these include spatially periodic and nonperiodic colourings. The technique followed in this enumeration was strongly based at: (i) The enumeration of all possible two-cell networks with one type of edge and where each cell receives four (resp. six) inputs. (ii) Checking which of these two-cell networks was corresponding to a balanced colouring of the fixed planar lattice with nearest neighbour coupling. From the tractable point of view this method, followed for $k = 2$ and $n = 2$, does not generalize for all n or k.

[Antoneli *et al.* 2005] studies k-colour patterns of synchrony in Euclidean lattices proving that, for planar lattices, they are periodic if the couplings are sufficiently extensive.

(*b*) *our results*

Our aim is to relate the local structure, given by the rule of the balanced colouring, with the long-range behaviour of the patterns of synchrony, for a general n-dimensional lattice and a general balanced k-colouring.

Given an n-dimensional Euclidean lattice $\mathcal{L}$, with nearest neighbour coupling architecture, and a $k \times k$ matrix A, our main result, Theorem 5.4, states a necessary and sufficient condition for the existence of a spatially periodic pattern of synchrony in the lattice network, corresponding to the balanced colouring defined by A.

The condition involves the *bidirectional networks* having the colouring matrix A, networks that have all the couplings bidirectional and even valence. These networks admit a factorisation into 2-factors — a decomposition of the couplings which produces networks with the same cells and valence two. This defines a set Σ of permutations of their cells, one permutation for each 2-factor. If there is a homomorphism between the group generated by these permutations and the lattice $\mathcal{L}$ then we say that Σ and $\mathcal{J}$ are *identifiable*. Let the set $\mathcal{D}_A$ be formed by the sets of permutations Σ associated to all the factorisations of all the possible finite bidirectional networks with colouring matrix A. We state our main result:

Theorem 5.4. *Let A be a colouring matrix with even valence v and let $\mathcal{L}$ be an Euclidean lattice with $\#\mathcal{J} = v$. The lattice network with nearest neighbour coupling architecture $G_{\mathcal{L}}$ has a periodic balanced colouring with the colouring matrix A if and only if $\mathcal{D}_A$ has an element identifiable with $\mathcal{J}$.*

We highlight some features of our results and the underlying methods.

One important point is that, given a k-colouring rule and a n-dimensional Euclidean lattice network with nearest neighbour coupling architecture, the existence of a periodic pattern of synchrony is translated into a local problem, involving finite networks with special properties. These finite networks are easily handled and standard methods in matrix theory can be used to treat them. In particular, if we approach the problem by imposing restrictions upon the fundamental domain of possible patterns of synchrony, the set of finite networks to be considered is fixed. These restrictions may concern dimension and geometry, and arise naturally, for example, in problems with periodic boundary conditions. See the *second example* in this section and the proof of Theorem 5.4 (*From lattice to finite bidirectional networks*).

Another aspect is the constructive nature of the method. The determination of an element in $\mathcal{D}_A$ identifiable with $\mathcal{J}$, leads immediately to the construction of a periodic pattern of synchrony and the same is true in the converse direction. We present several examples of these constructions in this section and in Section 6. Moreover, the proof of Theorem 5.4 is itself constructive. A crucial tool in obtaining our results is a classical theorem of graph theory concerning the factorisation of even degree regular graphs, a class of graphs where lattice networks are included, suggesting the possibility of applying other graph theory techniques in related problems.

Finally, the method for establishing the existence of periodic patterns of synchrony stands for lattices of any dimension and k-colourings for all k. Moreover, as described in Section 6, knowing patterns of synchrony for an Euclidean lattice network, gives information about patterns in lattices with different dimensions or different geometry.

(*c*) *article structure*

We continue the introduction with two examples illustrating our results and the methods we have used to prove them. The rest of the paper is organized in the following way. In Section 2 we introduce the notation and background on coupled cell networks and lattice networks used in this work. Section 3 specifies the definition of balanced colouring of the cells in a network to the class of identical-edge homogeneous networks. Moreover, we introduce a matrix formulation of balanced colouring and formalize the notions of symmetry and spatial periodicity of balanced colourings. Sections 4 and 5 are the core of our work. Section 4 is dedicated to the construction and the decomposition of finite bidirectional networks. We state results concerning the ratio of the cells for each colour on a balanced colouring of both finite bidirectional and lattice networks, and ensure the factorisation into 2-factors of the considered finite bidirectional networks. In Section 5 we prove our main result, Theorem 5.4. Under the conditions stated in this theorem, we show how to relate a periodic balanced colouring in a finite bidirectional network to the same balanced colouring in a lattice network, using a projection of the lattice into the finite network. Finally, in Section 6, we remark on several consequences of Theorem 5.4, where some "inclusions" involving balanced colourings with different dimensions are established. Several examples of periodic patterns of synchrony for the standard cubic lattice are presented.

(*d*) *First example*

We consider an infinite number of identical cells disposed on a lattice and ask if it is possible to colour these cells with two different colours, say black and grey, such that the following rule is followed: each black cell has four grey neighbour cells and each grey cell has two black neighbours, where the neighbours of each cell are its closest cells of the lattice.

If each cell in the lattice is a system of ordinary differential equations, coupled to the nearest neighbour cells that are identical systems, this colouring problem corresponds to the definition of a flow invariant subspace of the total phase space, see [Antoneli *et al.* 2005] for a complete description of these systems.

Applying the main result of this paper (Theorem 5.4) we ensure that the answer to that question is affirmative for the planar square, the planar hexagonal and the standard cubic lattices. Moreover, we have a method to construct a periodic two-colouring of each of these lattice networks respecting the given rule.

(i) *the planar square lattice network*

Formally, we begin with a planar square lattice

$$\mathcal{L} = \{(1,0),(0,1)\}_{\mathbf{Z}}$$

and consider the nearest neighbour coupling architecture — each element $l \in \mathcal{L}$ interacts with the elements in $l + \mathcal{J} = \{l \pm l_1, l \pm l_2\}$, where $l_1 = (1,0)$ and $l_2 = (0,1)$, see figure 1.

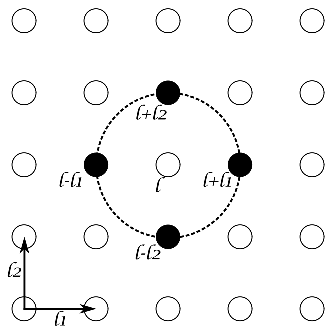

Figure 1. The nearest neighbours of $l \in \mathcal{L}$ are the elements in the black dots.

Given the lattice and the architecture above, the two-dimensional lattice network that we consider is the one represented in figure 2, where the couplings are bidirectional: each cell receives four inputs from its neighbour cells — we say the valence of the graph is 4.

Choosing a phase space $\mathbf{R}^d$ for each point in the lattice, a lattice dynamical system consistent with the given lattice architecture is described by:

$$\dot{x}_l = f\left(x_l, \overline{x_{l+l_1}, x_{l-l_1}, x_{l+l_2}, x_{l-l_2}}\right). \tag{1.3}$$

Here $l \in \mathcal{L}$, $x_l \in \mathbf{R}^d$, and f is invariant under all permutations of the variables under the bar.

(ii) *the balanced colouring*

Consider a set of two different colours, black and grey, indexed by the elements in $\mathcal{U} = \{1,2\}$, and the rule *each black cell receives four inputs from grey cells and each grey cell receives two inputs from black cells.* This is *case 42* considered

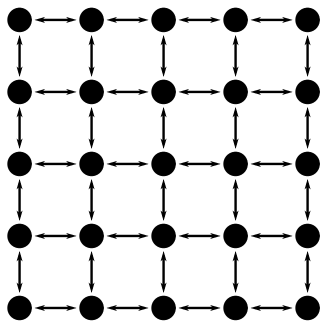

Figure 2. A portion of the two-dimensional lattice network for a square lattice architecture with nearest neighbour coupling.

in [Wang and Golubitsky 2005, page 633] and we will use this terminology throughout.

This rule can be described by the adjacency matrix

$$A = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$$

where the (i, j) entry is the number of inputs that each cell of colour i receives from cells of colour j or, equivalently, by a coupled cell network with 2 cells, one for each colour, as in figure 3.

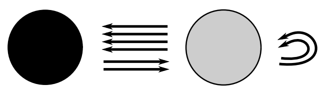

Figure 3. Each black cell receives four inputs from grey cells and each grey cell receives two inputs from grey cells and two inputs from black cells.

We construct now a periodic two-colouring of the lattice network respecting the above rule. We show in Theorem 5.4 that this colouring is related to the decomposition of a finite network whose couplings are bidirectional and whose cells are coloured under the same rule. In Sections (iii) to (vi) we choose a finite network with bidirectional couplings, decompose it and build a periodic colouring of the planar square lattice.

Observe that equations (1.3), when restricted to the flow-invariant subspace defined by identifying the cell coordinates of cells with the same colour, form the finite-dimensional system

$$\left\{ \begin{array}{l} \dot{x}_b = g\left(x_b, \overline{x_g, x_g, x_g, x_g}\right), \\ \dot{x}_g = g\left(x_g, \overline{x_g, x_b, x_g, x_b}\right), \end{array} \right. \tag{1.4}$$

where x_b and x_g denote the coordinates, respectively, of black and grey cells, and belong to $\mathbf{R}^d$, for some $d \in \mathbf{N}$.

(iii) *proportion of cells for each colour*

By Lemmas 4.3 and 4.8 we obtain the proportion p_i of cells of colour $i \in \mathcal{U}$ in this balanced 2-colouring of the square lattice network with nearest neighbour coupling architecture, as well as in the balanced colouring of a finite network with bidirectional couplings and valence 4. Specifically, it is the i^{th} coordinate of the left eigenvector of the matrix A, associated to the eigenvalue 4, whose coordinates sum 1:

$$(p_1, p_2)A = 4(p_1, p_2).$$

The eigenvector such that $p_1 + p_2 = 1$ is

$$\mathbf{p}^T = \left(\frac{1}{3}, \frac{2}{3}\right)$$

meaning that the ratio of black and grey cells in a balanced colouring corresponding to case 42, is $p_1/p_2 = 1/2$.

(iv) *the finite networks with bidirectional couplings*

We construct coupled cell networks, with bidirectional couplings, corresponding to case 42. By Lemma 4.3 the number of cells, for each colour, in these networks follow the proportion given by $\mathbf{p}$. We choose the networks having the smallest number of cells. Since $3\mathbf{p}^T = (1, 2)$, three is that smallest number. Moreover, there are two possible networks with three cells, see figure 4.

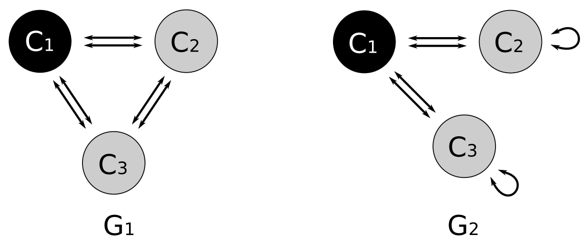

Figure 4. Two coupled cell networks where all the couplings are bidirectional. Each black cell receives four inputs from grey cells and each grey cell receives two inputs from grey cells and two inputs from black cells.

Let $\{c_1, c_2, c_3\}$ be the set of cells, where c_1 is the black cell and c_2 and c_3 are the grey cells. The adjacency matrices B_1 and B_2 of these bidirectional coupled cell networks have in the (i, j)-position the number of inputs that cell c_i receives from cell c_j:

$$B_1 = \left(\begin{array}{c|cc} 0 & 2 & 2 \\ \hline 2 & 0 & 2 \\ 2 & 2 & 0 \end{array}\right) \qquad \text{and} \qquad B_2 = \left(\begin{array}{c|cc} 0 & 2 & 2 \\ \hline 2 & 2 & 0 \\ 2 & 0 & 2 \end{array}\right).$$

(v) *decomposition in cycles*

By Lemma 4.4 each of these matrices can be decomposed into the sum of 2 (half the valence) symmetric matrices whose lines sum 2. For this example we have

$$B_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

corresponding to the decomposition of the couplings shown in figure 5.

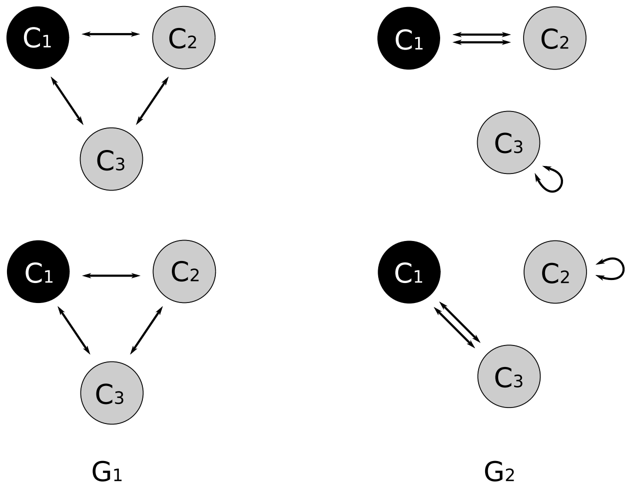

Figure 5. The decomposition of the couplings for the two bidirectional networks of figure 4. Both G_1 and G_2 are decomposed into bidirectional networks with valence 2.

Each one of the matrices obtained can be written as the sum $M_\sigma + M_\sigma^T$ for some permutation matrix M_σ associated to a permutation $\sigma \in \mathbf{S}_3$. Here 3 is the number of cells in the bidirectional coupled cell network that we are considering. For example,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

If M_{σ_1} is the matrix associated to the permutation $\sigma_1 = (c_1c_2c_3)$,

$$M_{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and taking $\sigma_2 = \sigma_1$, we have that B_1 is the sum

$$B_1 = \left(M_{\sigma_1} + M_{\sigma_1}^T\right) + \left(M_{\sigma_2} + M_{\sigma_2}^T\right) = 2\left(M_{\sigma_1} + M_{\sigma_1}^T\right).$$

(vi) *the periodic pattern*

Since σ_1 and σ_2 commute then, by Theorem 5.4, we can construct a periodic balanced two-colouring of the lattice network as shown in figure 6. Beginning with any cell c_i in any position $l \in \mathcal{L}$, the cell in position $l + l_j$ will be $\sigma_j(c_i)$, for $j = 1, 2$.

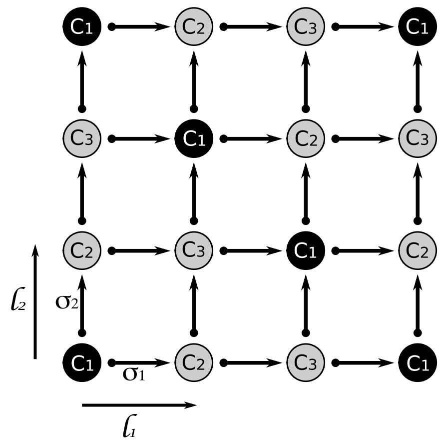

Figure 6. The permutations σ_1 and σ_2 are associated, respectively, to the generators l_1 and l_2 of the lattice.

The resulting coloured pattern is represented in figure 7, together with its noncolinear periods

$$3l_1 \quad \text{and} \quad 2l_1 + l_2.$$

By Theorem 5.4, these periods correspond to the compositions of the permutations $\sigma_1^3 = \epsilon$ and $\sigma_1^2 \circ \sigma_2 = \epsilon$, where ϵ denotes the identity element of $\mathbf{S}_3$.

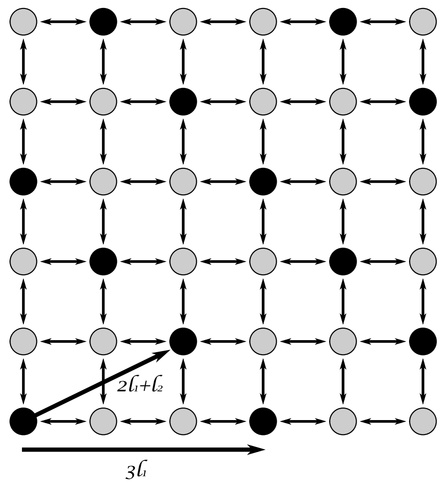

Figure 7. The periodic pattern obtained for case 42 with the two noncolinear periods $3l_1$ and $2l_1 + l_2$.

(vii) *the pattern is always found*

A similar decomposition for the matrix B_2 results into

$$B_2 = \left(M_{\sigma_3} + M_{\sigma_3}^T\right) + \left(M_{\sigma_4} + M_{\sigma_4}^T\right)$$

where $\sigma_3 = (c_1c_2)(c_3)$ and $\sigma_4 = (c_1c_3)(c_2)$. Since these permutations do not commute, the above construction is not possible. However, Theorem 5.4 ensures that, if there is a periodic pattern satisfying the required balanced colouring condition on the lattice, then there is always a finite bidirectional network whose decomposition allows its construction, see the next example in Section (e).

(viii) *the planar hexagonal and the standard cubic lattice networks*

Now we repeat the above construction for two lattice networks with valence 6. Let $\mathcal{L}$ be the standard cubic lattice

$$\mathcal{L} = \{(1,0,0),(0,1,0),(0,0,1)\}_{\mathbf{Z}}$$

with nearest neighbour coupling architecture: $\mathcal{J} = \{\pm l_1, \pm l_2, \pm l_3\}$, where $l_1 = (1,0,0)$, $l_2 = (0,1,0)$ and $l_3 = (0,0,1)$. See figure 8.

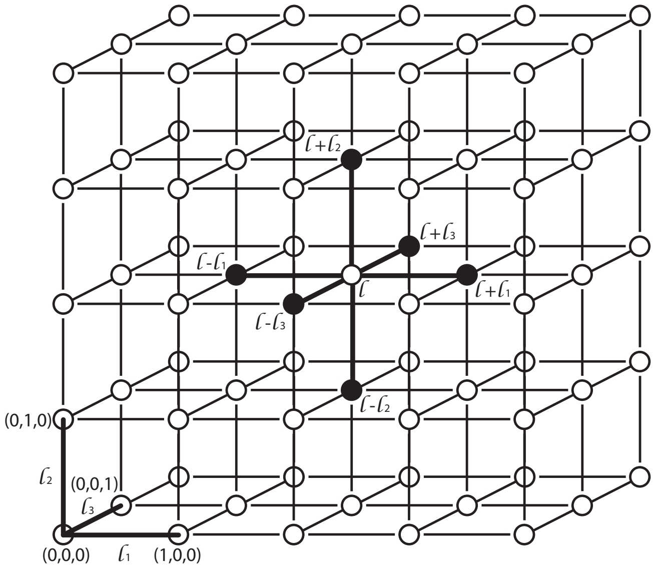

Figure 8. The nearest neighbours of $l \in \mathcal{L}$ are the elements in the black dots.

Consider also the planar hexagonal lattice

$$\mathcal{L} = \left\{ (1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \right\}_{\mathbf{Z}}$$

with nearest neighbour coupling architecture: $\mathcal{J} = \{\pm l_1, \pm l_2, \pm l_3\}$, where $l_1 = (1,0)$, $l_2 = (1/2, \sqrt{3}/2)$ and $l_3 = l_2 - l_1$, see figure 9.

The colouring rule corresponding to *case 42* is now described by the adjacency matrix

$$A = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$$

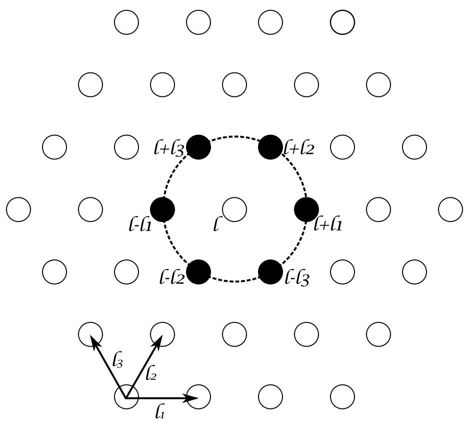

Figure 9. The nearest neighbours of $l \in \mathcal{L}$ are the elements in the black dots.

and the proportion of cells with each colour is

$$\mathbf{p}^T = \left(\frac{1}{3}, \frac{2}{3}\right)$$

as in the previous example with valence 4.

One finite bidirectional network with three cells corresponding to *case 42*, represented in figure 10, has adjacency matrix B:

$$B = \left(\begin{array}{c|cc} 2 & 2 & 2 \\ \hline 2 & 2 & 2 \\ 2 & 2 & 2 \end{array}\right) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

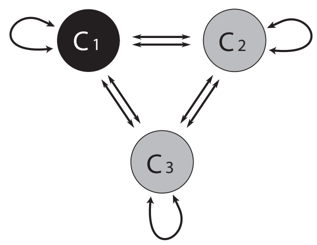

Figure 10. A bidirectional coupled cell network. Each black cell receives four inputs from grey cells and each grey cell receives two inputs from grey cells and two inputs from black cells.

As in the previous case, the first two matrices in the decomposition of B are associated to the permutations σ_1 and σ_2:

$$\sigma_1 = \sigma_2 = (c_1 c_2 c_3).$$

The third matrix, $2Id_3$, is associated to

$$\sigma_3 = \epsilon.$$

These permutations commute and, by Theorem 5.4, this condition ensures that we can build a periodic balanced colouring of the given cubic lattice if we associate each permutation to a generator. See the resulting pattern in figure 11.

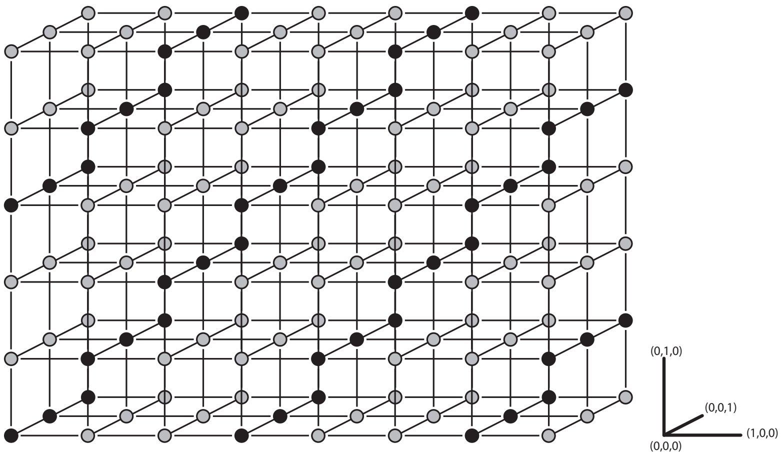

Figure 11. A periodic pattern obtained for case 42.

For the hexagonal lattice, the elements l_1, l_2 and l_3 are not linearly independent, $l_3 = l_2 - l_1$. By Theorem 5.4, we can associate the permutation σ_i to the direction l_i if $\sigma_3 = \sigma_2 \circ \sigma_1^{-1}$. This condition is verified and, thus, we built the periodic pattern in figure 12, one of the two balanced colourings corresponding to *case 42* in the planar hexagonal lattice with nearest neighbour architecture, described in [Wang and Golubitsky 2005, page 636].

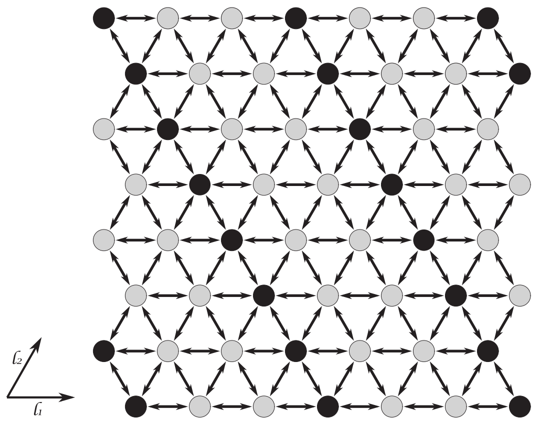

Figure 12. A periodic pattern obtained for case 42.

In Section 6 we present other examples of periodic balanced colourings of the standard cubic lattice with nearest neighbour coupling architecture.

(e) second example

In this section we follow the converse direction of the previous example. We begin with a spatially periodic pattern with a balanced colouring in a planar hexagonal lattice and then construct a finite network with bidirectional couplings and with the same colouring rule as the lattice network.

(i) *the periodic colouring in the lattice*

Let $\mathcal{L}$ be the planar hexagonal lattice described above

$$\mathcal{L} = \left\{(1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}_{\mathbf{Z}}$$

with nearest neighbour coupling architecture: $\mathcal{J} = \{\pm l_1, \pm l_2, \pm l_3\}$, where $l_1 = (1,0)$, $l_2 = (1/2, \sqrt{3}/2)$ and $l_3 = l_2 - l_1$, see figure 9.

Consider a colouring of the cells on this lattice with two colours, black and grey, such that each black cell receives five inputs from grey cells and one input from a black cell, and each grey cell receives two inputs from black cells and four inputs from grey cells. This is *case 52* considered in [Wang and Golubitsky 2005, page 636] and we present in figure 13 the only possible pattern respecting this colouring rule, as proved in [Wang and Golubitsky 2005, Theorem 1.7].

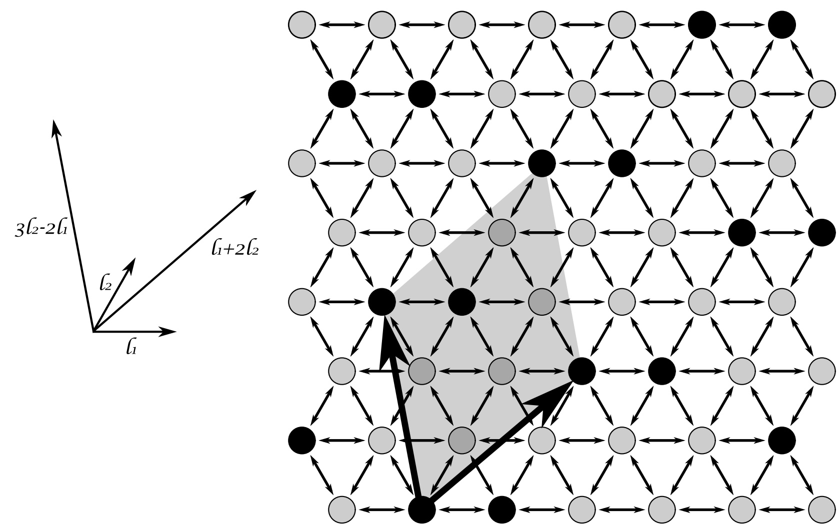

Figure 13. The periodic pattern for case 52 with the two noncolinear periods $l_1 + 2l_2$ and $3l_2 - 2l_1$. The fundamental domain, whose periodic repetition forms the pattern, is in light grey.

We can identify all the cells of the hexagonal lattice that differ by a period of the coloured pattern, obtaining seven different types of cells that compose the fundamental domain — a portion of the pattern that is periodically repeated. Let $\{c_1, \ldots, c_7\}$ be the set of different cells up to periods and see how they form the pattern in figure 15.

(ii) *the finite network*

Now we associate a permutation of the cells $c_1, \ldots, c_7$ to each direction of the couplings in the lattice network. See figure 14.

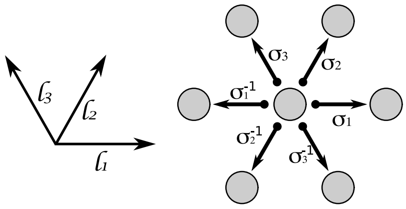

Figure 14. The directions of the nearest neighbour couplings and the associated permutations. Here $l_3 = l_2 - l_1$ where l_1 and l_2 generate the planar hexagonal lattice.

In figure 15 we see that the permutations $\sigma_1 = (c_1c_2c_6c_3c_7c_4c_5)$, $\sigma_2 = (c_1c_3c_5c_6c_4c_2c_7)$ and $\sigma_3 = (c_1c_6c_7c_5c_2c_3c_4)$ describe the sequence of cells along the given directions. These sequences define the couplings of a finite coupled cell network with the set of cells $\{c_1, \ldots, c_7\}$.

The inverse permutations, $\sigma_1^{-1} = (c_1c_5c_4c_7c_3c_6c_2)$, $\sigma_2^{-1} = (c_1c_7c_2c_4c_6c_5c_3)$ and $\sigma_3^{-1} = (c_1c_4c_3c_2c_5c_7c_6)$, correspond to the sequences of cells in the pattern along the opposite directions. If we consider both σ_i and σ_i^{-1}, then the couplings of the finite network are all bidirectional. We obtain the coupled cell network in figure 16, having adjacency matrix

$$B = \left(\begin{array}{cc|ccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

given by

$$B = \left(M_{\sigma_1} + M_{\sigma_1}^T\right) + \left(M_{\sigma_2} + M_{\sigma_2}^T\right) + \left(M_{\sigma_3} + M_{\sigma_3}^T\right).$$

The colouring of this finite coupled cell network follows the rule of the original colouring of the hexagonal lattice. Therefore, if we were looking for a colouring of the hexagonal lattice for case 52, our results imply that the existence of this finite network would give us a decomposition ensuring a periodic pattern on the lattice network with the given balanced colouring rule.

2. Definitions — coupled cell networks

In this section we describe the notation and concepts concerning the coupled cell networks structure.

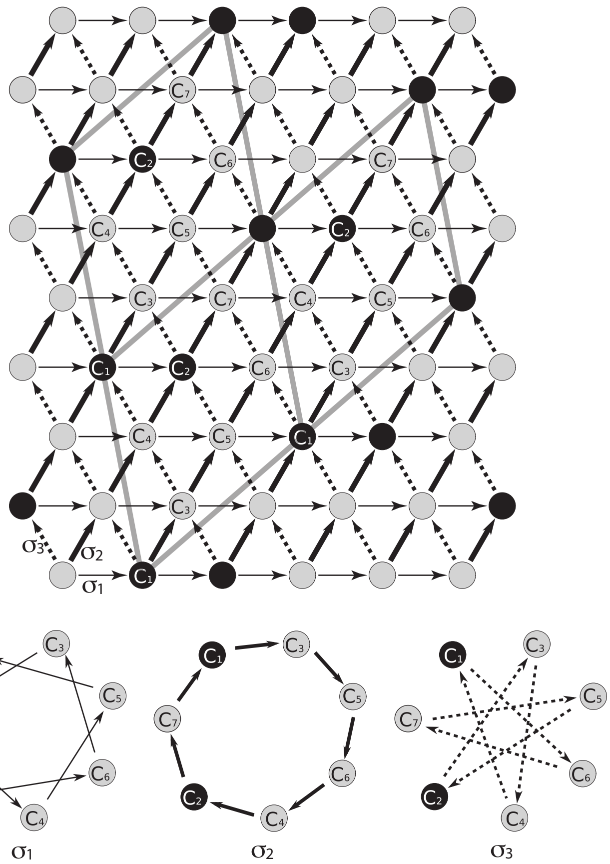

Figure 15. The sequence of the cells $c_1, \ldots, c_7$ associated to the permutations $\sigma_1 = (c_1c_2c_6c_3c_7c_4c_5)$, $\sigma_2 = (c_1c_3c_5c_6c_4c_2c_7)$ and $\sigma_3 = (c_1c_6c_7c_5c_2c_3c_4)$ along each direction of the lattice and for a finite coupled cell network.

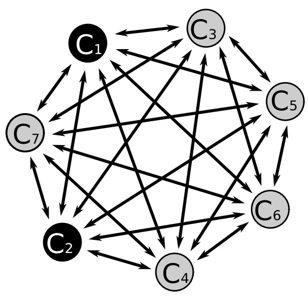

Figure 16. The coupled cell network with bidirectional couplings obtained from case 52 of a hexagonal lattice.

(*a*) *coupled cell network*

We use the definition of coupled cell network with a countable number of cells in [Antoneli *et al.* 2005]:

Definition 2.1. [Antoneli *et al.* 2005, Definition 2.1] A *coupled cell network* G consists of:

1. a countable set $\mathcal{C}$ of cells,
2. an equivalence relation $\sim_C$ on cells in $\mathcal{C}$,
3. a countable set $\mathcal{E}$ of edges or arrows,
4. an equivalence relation $\sim_E$ on edges in $\mathcal{E}$. The *edge type* of edge e is the $\sim_E$-equivalence class of e,
5. (local finiteness) there is a *head map* $\mathcal{H} : \mathcal{E} \to \mathcal{C}$ and a *tail map* $\mathcal{T} : \mathcal{E} \to \mathcal{C}$ such that for every $c \in \mathcal{C}$ the sets $\mathcal{H}^{-1}(c)$ and $\mathcal{T}^{-1}(c)$ are finite.
6. (consistency condition) equivalent arrows have equivalent tails and equivalent heads; that is, if $e_1 \sim_E e_2$ in $\mathcal{E}$, then every $\mathcal{H}(e_1) \sim_C \mathcal{H}(e_2)$ and $\mathcal{T}(e_1) \sim_C \mathcal{T}(e_2)$.

With notation $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. ♢

(*b*) *input set of a cell and homogeneous coupled cell network*

We consider networks that may have multiple edges and self-coupling. Thus, we follow the definition of input set of a cell and of isomorphic input sets in [Golubitsky *et al.* 2005, Definitions 2.2 and 2.3]:

Definition 2.2. Let $c \in \mathcal{C}$. The *input set of* c is $I(c) = \{e \in \mathcal{E} : \mathcal{H}(e) = c\}$ with a fixed ordering. An element of $I(c)$ is called an *input edge* or *input arrow* of c. We say that *c receives m inputs from d* if $m = \#\{e \in I(c) : \mathcal{T}(e) = d\}$. ♢

Definition 2.3. Two input sets $I(c)$ and $I(d)$ are *isomorphic* if there is a bijection $\beta : I(c) \to I(d)$ that preserves the input edge type:

$$i \sim_E \beta(i) \qquad \text{for all } i \in I(c).$$

♢

Observe that if two cells have isomorphic input sets then, in particular, they are in the same $\sim_C$-equivalence class.

Definition 2.4. A coupled cell network is *homogeneous* if all the cells have isomorphic input sets. ♢

From now on we consider *identical-edge homogeneous networks*, that is, homogeneous coupled cell networks with only one edge type. We denote these by $G = (\mathcal{C}, \mathcal{E})$. For this class of networks, the input sets $I(c)$ have only one edge type and a fixed cardinality called *valence*, $v = \#I(c)$, for all $c \in \mathcal{C}$.

(c) lattice and nearest neighbours

An *n-dimensional lattice* $\mathcal{L}$ is a set

$$\mathcal{L} = \{l_1, \ldots, l_n\}_{\mathbf{Z}} \subset \mathbf{R}^n$$

where the n elements $l_1, \ldots, l_n \in \mathbf{R}^n$, the *generators of the lattice*, are linearly independent.

Let $|l|$ denote the Euclidean norm of $l \in \mathbf{R}^n$. The set

$$\{|l| : l \in \mathcal{L}\}$$

is a countable set and the possible Euclidean distances of $l \in \mathcal{L}$ to the origin are

$$r_0, r_1, r_2, \ldots$$

such that $r_0 = 0$ and $r_i < r_{i+1}$ for all $i \in \mathbf{N}$.

Considering the set of cells $\mathcal{J}$ defined by

$$\mathcal{J} = \{l \in \mathcal{L} : |l| = r_1\}$$

then the *nearest neighbours* of $l \in \mathcal{L}$ are the elements on $l + \mathcal{J} = \{l + g : g \in \mathcal{J}\}$.

Definition 2.5. An *n-dimensional Euclidean lattice* is a lattice $\mathcal{L} = \{l_1, \ldots, l_n\}_{\mathbf{Z}}$ such that $|l_i| = r_1$ for all $i \in \{1, \ldots, n\}$. ◊

Observe that if $\mathcal{L} = \{l_1, \ldots, l_n\}_{\mathbf{Z}}$ is an Euclidean lattice then we have $\{l_1, \ldots, l_n\} \subset \mathcal{J}$.

(d) n-dimensional lattice network

We define an n-dimensional lattice network with nearest neighbour coupling architecture, a special case of the definition given by [Antoneli *et al.* 2005, Definition 2.5] for more general architectures.

Definition 2.6. An *n-dimensional lattice network with nearest neighbour coupling architecture* consists of:

1. an n-dimensional lattice $\mathcal{L}$,
2. a homogeneous coupled cell network $G_{\mathcal{L}}$ whose cells are indexed by $\mathcal{L}$,
3. $I(c) = \{e : \mathcal{T}(e) \in c + \mathcal{J}\}$ for all $c \in \mathcal{L}$,
4. all the arrows in $I(0)$ have the same edge type.

◊

The valence of a lattice network $G_{\mathcal{L}}$ is a geometric property of $\mathcal{L}$ and equals $\#\mathcal{J}$ for the nearest neighbour architecture. Since $-\mathcal{L} = \mathcal{L}$ and $|-l| = |l|$, the set $\mathcal{J}$ has even cardinality. If $\mathcal{L} = \{l_1, \ldots, l_n\}_{\mathbf{Z}}$ is an Euclidean lattice then we can choose $l_{n+1}, \ldots, l_{v/2} \in \mathcal{L}$ such that $\mathcal{J} = \{\pm l_1, \ldots, \pm l_n, \pm l_{n+1}, \ldots, \pm l_{v/2}\}$.

All the couplings in the lattice networks are bidirectional.

3. Definitions — balanced colourings

In this section we specify the definitions of balanced equivalence relation on $\mathcal{C}$, with k equivalence classes, and of quotient network given in [Golubitsky *et al.* 2005, Sections 4 and 5] to the class of identical-edge homogeneous networks. Moreover, we introduce a matrix formulation of balanced colouring that will be used in the remaining sections.

(a) *balanced k-colourings*

Definition 3.1. Let $G = (\mathcal{C}, \mathcal{E})$ be a coupled cell network and suppose that we colour the cells in $\mathcal{C}$ with k different colours. Let the colours be indexed by the set $\mathcal{U} = \{1, \ldots, k\}$. The *k-colouring* ξ *of the network* G is the function

$$\begin{array}{rcl} \xi : \mathcal{C} & \longrightarrow & \mathcal{U} \\ c & \longmapsto & \xi(c) \end{array}$$

where $\xi(c)$ is the colour of cell c.

In order to use matrices we define an analogous vector notation

$$\begin{array}{rcl} \vec{\xi} : \mathcal{C} & \longrightarrow & \{\mathbf{e}_i, i \in \mathcal{U}\} \\ c & \longmapsto & \vec{\xi}(c) = \mathbf{e}_{\xi(c)} \end{array}$$

where $\{\mathbf{e}_i, i \in \mathcal{U}\}$ is the canonical basis of $\mathbf{R}^k$. ◊

Unless otherwise stated, all the vectors are column vectors. We will refer to the colouring as ξ or $\vec{\xi}$, without distinction, since they are equivalent definitions by the isomorphism $i \longleftrightarrow \mathbf{e}_i$ between $\mathcal{U}$ and $\{\mathbf{e}_i, i \in \mathcal{U}\}$.

Definition 3.2. Let $G = (\mathcal{C}, \mathcal{E})$ be an identical-edge homogeneous coupled cell network with a k-colouring ξ. For any finite subset $\mathcal{F} \subset \mathcal{E}$ we define the vector $\mathbf{q}(\mathcal{F}, \xi) = (q_1, \ldots, q_k)^T$ whose i^{th} coordinate is the number of edges in $\mathcal{F}$ with tail cell of colour i:

$$q_i = \#\{e \in \mathcal{F} : \xi(\mathcal{T}(e)) = i\}.$$

In vector notation,

$$\mathbf{q}(\mathcal{F}, \xi) = \sum_{e \in \mathcal{F}} \vec{\xi}(\mathcal{T}(e)).$$

◊

Now we express the concepts of balanced colouring and quotient network in terms of the vectors $\mathbf{q}(I(c), \xi)$, $c \in \mathcal{C}$, for a colouring ξ of an identical cell homogeneous network $G = (\mathcal{C}, \mathcal{E})$.

Definition 3.3. Let $G = (\mathcal{C}, \mathcal{E})$ be an identical-edge homogeneous coupled cell network. A k-colouring ξ of G is *balanced* if, for all cells $c, d \in \mathcal{C}$,

$$\xi(c) = \xi(d) \quad \text{implies} \quad \mathbf{q}(I(c), \xi) = \mathbf{q}(I(d), \xi).$$

◊

Thus, in a balanced k-colouring we identify each colour $i \in \mathcal{U}$ with a particular set of colours for the tails $\mathcal{T}(I(c))$, where c is any cell with colour i. This induces the next definition.

(*b*) *quotient networks*

Definition 3.4. Let $G = (\mathcal{C}, \mathcal{E})$ be an identical-edge homogeneous coupled cell network with input sets $I(c)$, for $c \in \mathcal{C}$, and ξ a balanced k-colouring of G with colours in $\mathcal{U} = \{1, \ldots, k\}$.

Let $G_0 = (\mathcal{C}_0, \mathcal{E}_0)$ be the coupled cell network with k cells, $\mathcal{C}_0 = \{c_1, \ldots, c_k\}$, with input sets $I_0(c)$, for $c \in \mathcal{C}_0$, and a k-colouring ξ_0 such that $\xi_0(c_i) = i$ for $i \in \mathcal{U}$. If, for all $i \in \mathcal{U}$, $\mathbf{q}(I_0(c_i), \xi_0) = \mathbf{q}(I(c), \xi)$ for any $c \in \mathcal{C}$ such that $\xi(c) = i$, then G_0 is the *quotient network* of G by ξ. $\diamondsuit$

The above definition implies, in particular, that the valences of G and G_0 are the same.

(*c*) *adjacency matrix*

Definition 3.5. Let $G = (\mathcal{C}, \mathcal{E})$ be an identical-edge homogeneous network with $\mathcal{C} = \{c_1, \ldots, c_m\}$. The *adjacency matrix of* G is the $m \times m$ matrix $B = (b_{ij})$ where b_{ij} is the number of edges from cell c_j to cell c_i. The adjacency matrix characterizes G and thus we write G_B. $\diamondsuit$

The sum of all the entries in a line of the adjacency matrix B of G_B is the valence v.

We refer to the adjacency matrix B of a network $G_B = (\mathcal{C}, \mathcal{E})$ assuming we have fixed an ordering of the set $\mathcal{C}$. Unless otherwise stated, given a k-colouring ξ of G_B, with the colours indexed by $\mathcal{U} = \{1, \ldots, k\}$, we enumerate the cells of $\mathcal{C}$ by

$$\mathcal{C} = \{c_1, \ldots, c_m\}, \text{ such that } \xi(c_i) \leq \xi(c_j) \text{ for } i < j.$$

Consider the k-colouring ξ_0 of a network G_0 with k cells $c_1, \ldots, c_k$ and with adjacency matrix $A = (a_{ij})$. The entry a_{ij} is the number of inputs from the j-colour cell that the i-colour cell receives. Therefore, the i^{th} line of the adjacency matrix of G_0 is the transpose of the vector $\mathbf{q}(I_0(c_i), \xi_0)$. Let $G = (\mathcal{C}, \mathcal{E})$ be a coupled cell network with a balanced k-colouring ξ. If G_0 is the quotient of G then, for any cell $c \in \mathcal{C}$ with colour i, the vector $\mathbf{q}(I(c), \xi)$ is the transpose of the i^{th} line of A. Thus, any cell of colour i receives a_{ij} inputs from cells with colour j.

Definition 3.6. A *k-colouring matrix* is the adjacency matrix A of a network G_0 with k cells, one of each colour. If G is a network having a balanced k-colouring with quotient network G_0 then we say that *G has the k-colouring matrix A.* $\diamondsuit$

(*d*) *matrix formulation of a balanced colouring*

Lemma 3.7. *Let $\vec{\xi}$ be a balanced k-colouring of an identical-edge homogeneous coupled cell network $G = (\mathcal{C}, \mathcal{E})$ having the k-colouring matrix A. Therefore,*

$$A^T \vec{\xi}(c) = \sum_{d \in I(c)} \vec{\xi}(\mathcal{T}(d)) \quad \textit{for all } c \in \mathcal{C}. \tag{3.1}$$

For the particular case of a lattice network $G_{\mathcal{L}}$, with nearest neighbours indexed by $\mathcal{J}$ and k-colouring matrix A, the above equation is

$$A^T \vec{\xi}(l) = \sum_{g \in \mathcal{J}} \vec{\xi}(l + g) \quad \textit{for all } l \in \mathcal{L}. \tag{3.2}$$

Proof. If $\xi(c) = i$ then the transpose of $A^T\vec{\xi}(c)$ is the i^{th} line of the matrix A, *i.e.*, $\mathbf{q}(I(c), \xi)$. By definition this vector equals $\sum_{d\in I(c)} \vec{\xi}(\mathcal{T}(d))$.

For (3.2), notice that

$$\{l + g : g \in \mathcal{J}\} = \{\mathcal{T}(d) : d \in I(l)\} \tag{3.3}$$

for all $l \in \mathcal{L}$. □

(e) permuting the colours

Suppose we have a network $G = (\mathcal{C}, \mathcal{E})$ with a k-colouring matrix A and corresponding to a balanced k-colouring ξ, with colours indexed by $\mathcal{U} = \{1, \ldots, k\}$. Let σ be a permutation of the elements of $\mathcal{U}$ and let $\sigma \cdot \xi$ be the colouring of G such that $(\sigma \cdot \xi)(c) = \sigma(\xi(c))$, for $c \in \mathcal{C}$. Thus, $\sigma \cdot \xi$ is also a balanced k-colouring and has the k-colouring matrix $M_\sigma A M_\sigma^T$, where M_σ is the $k \times k$ permutation matrix corresponding to σ, *i.e.*, $M_\sigma \mathbf{e_i} = \mathbf{e_j}$ if $j = \sigma(i)$, for $i, j \in \mathcal{U}$.

To prove the statement above, let $A = (a_{ij})$ and let A_σ be the adjacency matrix of the colouring after permuting the colours by σ. The entry a_{ij} is the number of inputs from cells with colour j that each cell of colour i receives, for the initial balanced k-colouring. In the new colouring, $\sigma \cdot \xi$, the cells with colour $\sigma(i)$ receive a_{ij} inputs from the cells with colour $\sigma(j)$, *i.e.*, the matrix A_σ has the value a_{ij} on the position $(\sigma(i), \sigma(j))$.

If A commutes with the permutation matrix M_σ then $M_\sigma A M_\sigma^T = A$, since $M_\sigma^{-1} = M_\sigma^T$ for permutation matrices. This permutation $\sigma \in \mathbf{S}_k$ does not change the k-colouring matrix A.

Therefore, if $\mathcal{X}_{G,k}$ is the space of balanced k-colourings of the network G, there is a natural action of the permutation group $\mathbf{S}_k$:

$$\begin{array}{rcl} \mathbf{S}_k \times \mathcal{X}_{G,k} & \longrightarrow & \mathcal{X}_{G,k} \\ (\sigma, \xi) & \longmapsto & \sigma \cdot \xi \end{array}$$

where we denote the identity permutation by ϵ.

Using vector notation, $(\sigma \cdot \vec{\xi})(c) = M_\sigma \vec{\xi}(c)$ for $c \in \mathcal{C}$. Equation (3.1) becomes, for $M_\sigma \vec{\xi}(c)$,

$$A_\sigma^T M_\sigma \vec{\xi}(c) = \sum_{d\in I(c)} M_\sigma \vec{\xi}(\mathcal{T}(d)) \quad \text{for all } c \in \mathcal{C}$$

and is another way of showing that $A_\sigma = M_\sigma A M_\sigma^T$.

Remark 3.8. We describe balanced k-colourings of a network up to these permutations, *i.e.*, up to the $\mathbf{S}_k$-orbits on $\mathcal{X}_{G,k}$. ♢

For colourings of a lattice network $G_\mathcal{L}$ we can define other symmetries, inherited from the symmetries of the lattice $\mathcal{L}$. This is done in the next subsections.

(f) spatial transformation of a colouring for lattice networks

Let $G_\mathcal{L}$ be a lattice network. Since the cells are indexed by $\mathcal{L}$, we define a k-colouring ξ of this lattice network as

$$\begin{array}{rcl} \xi : \mathcal{L} & \longrightarrow & \mathcal{U} \\ l & \longmapsto & \xi(l) \end{array}$$

where $\xi(l)$ is the colour of the cell in the position $l \in \mathcal{L}$, which we call *cell* l.

The *pattern associated to* ξ is the set

$$\Psi_\xi = \{(l, \xi(l)), l \in \mathcal{L}\}.$$

Let $\Gamma = \mathcal{L} \dot{+} \mathbf{H} \subset \mathbf{R}^n \dot{+} \mathbf{O}(n)$ be the symmetry group of the lattice $\mathcal{L}$, where $\mathbf{H}$ is the holohedry of $\mathcal{L}$. We denote the elements in Γ by $\gamma = (t, \delta)$, where t is a translation belonging to $\mathcal{L}$ and δ is an orthogonal matrix, and they transform $x \in \mathbf{R}^n$ into $\gamma x = t + \delta x$. See [Antoneli *et al.* 2007] for some remarks relating the symmetries of the lattice and the symmetries of the equations governing the corresponding lattice dynamical systems.

Notice that the set $\mathcal{J} \subseteq \mathcal{L}$ is $\mathbf{H}$-invariant. This follows from the fact that $\mathcal{J}$ is defined by a distance and the holohedry is the group of orthogonal transformations such that $\mathbf{H}(\mathcal{L}) = \mathcal{L}$.

The usual action of Γ at the spaces of functions $\mathcal{X}_{G,k}$ is the *scalar action*:

$$(\gamma \cdot \xi)(l) = \xi(\gamma^{-1} l) = (\xi \gamma^{-1})(l) \quad \text{for all } \gamma \in \Gamma \text{ and } l \in \mathcal{L}.$$

Suppose that we have a pattern Ψ_{ξ_1}

$$\Psi_{\xi_1} = \{(l, \xi_1(l)), l \in \mathcal{L}\}$$

and we transform the colouring by a symmetry of the lattice $\gamma \in \Gamma$ obtaining $\xi_2 = \gamma \cdot \xi_1$. The new pattern is

$$\Psi_{\xi_2} = \{(l, \xi_2(l)), l \in \mathcal{L}\} = \{(l, \xi_1(\gamma^{-1} l)), l \in \mathcal{L}\}.$$

This set equals $\{(\gamma l, \xi_1(l)), \gamma l \in \gamma\mathcal{L}\}$ and, since $\gamma\mathcal{L} = \mathcal{L}$,

$$\Psi_{\xi_2} = \{(\gamma l, \xi_1(l)), \gamma l \in \mathcal{L}\}.$$

Thus, Ψ_{ξ_2} is the initial pattern after an Euclidean transformation.

Remark 3.9. We describe patterns up to these transformations. ◊

(*g*) *periodicity*

For the nearest neighbour coupling architecture, the structure of the lattice network $G_\mathcal{L}$ is defined by $\mathcal{L}$ and so we use the notation $\mathcal{X}_{\mathcal{L},k}$ for the space of the balanced k-colourings of a lattice network $G_\mathcal{L}$. The group acting on $\mathcal{X}_{\mathcal{L},k}$ is, then, $\Theta = \Gamma \times \mathbf{S}_k$, where, as before, $\Gamma = \mathcal{L} \dot{+} \mathbf{H}$ with the following action

$$\begin{array}{rcl} \Theta \times \mathcal{X}_{\mathcal{L},k} & \longrightarrow & \mathcal{X}_{\mathcal{L},k} \\ ((\gamma, \sigma), \xi) & \longmapsto & (\gamma, \sigma) \cdot \xi \end{array}$$

where $((\gamma, \sigma) \cdot \xi)(l) = \sigma(\xi(\gamma^{-1} l))$ for all $l \in \mathcal{L}$.

We say that a pattern Ψ_ξ has a symmetry $(\gamma, \sigma) \in \Theta$ if the associated colouring is (γ, σ)-invariant:

$$(\gamma, \sigma) \cdot \xi(l) = \xi(l) \quad \text{for all } l \in \mathcal{L}.$$

In particular, we say that Ψ_ξ *has a period* $t \in \mathcal{L}$, $t \neq 0$, if

$$((t, Id_n), \epsilon) \cdot \xi(l) = \xi(l - t) = \xi(l) \quad \text{for all } l \in \mathcal{L}.$$

Definition 3.10. Let ξ be a balanced k-colouring of a n-dimensional lattice network $G_{\mathcal{L}}$. The pattern Ψ_ξ is *periodic* if it has periods along n non-colinear directions. If we denote the set of its periods by $\tilde{\mathcal{L}}$

$$\tilde{\mathcal{L}} = \{\tilde{l}_1, \ldots, \tilde{l}_n\}_{\mathbf{Z}} \subseteq \mathcal{L},$$

for n non-colinear elements $\tilde{l}_1, \ldots, \tilde{l}_n \in \mathbf{R}^n$, then we also say that Ψ_ξ is $\tilde{\mathcal{L}}$*-periodic.* $\diamondsuit$

Let Ψ_ξ be a $\tilde{\mathcal{L}}$-periodic pattern for $\tilde{\mathcal{L}} = \{\tilde{l}_1, \ldots, \tilde{l}_n\}_{\mathbf{Z}}$ and consider the n-dimensional parallelepiped $\{\tilde{l}_1, \ldots, \tilde{l}_n\}_{[0,1)}$. The intersection of this parallelepiped with $\mathcal{L}$ is a set $\mathcal{C}$, isomorphic to the quotient $\mathcal{L}/\tilde{\mathcal{L}}$. The *fundamental domain* of the pattern Ψ_ξ is its restriction to

$$\{(l, \xi(l)) : l \in \mathcal{C}\}.$$

In fact the pattern is a regular repetition of the fundamental domain, along the n non-colinear directions $\tilde{l}_1, \ldots, \tilde{l}_n$.

Given a lattice $\mathcal{L}$ and a k-colouring matrix A, we prove in Theorem 5.4 a necessary and sufficient condition for the existence of a periodic pattern Ψ_ξ associated to a balanced k-colouring ξ with matrix A.

4. Decomposition of finite bidirectional netwoks

Our main result (Theorem 5.4) relates the existence of spatially periodic balanced colourings of a lattice network, for a given colouring matrix A, to the decomposition of finite coupled cell networks with the same colouring matrix A and having all the couplings bidirectional. In this section we study networks with bidirectional couplings in two major steps — stating the proportion of cells for each colour in a balanced colouring (both for finite bidirectional and lattice networks) and ensuring the factorisation of the considered finite bidirectional networks.

Definition 4.1. A *bidirectional network* is a coupled cell network where all the couplings are bidirectional. $\diamondsuit$

In a bidirectional network, given any two distinct cells, there is a direct path between them. Therefore, the adjacency matrix of a bidirectional network is an irreducible matrix (see Section d). Moreover, if ξ is a balanced k-colouring of a bidirectional network with the k-colouring matrix A, then A is also an irreducible matrix.

Bidirectional networks have symmetric adjacency matrices described in the next lemma.

(*a*) *number of cells for bidirectional networks*

Lemma 4.2. *Let ξ be a k-colouring of a finite bidirectional identical-edge homogeneous network $G_B = (\mathcal{C}, \mathcal{E})$ with colours indexed by the set $\mathcal{U} = \{1, \ldots, k\}$. Let $A = (a_{ij})$, with $i, j \in \mathcal{U}$, be the corresponding k-colouring matrix. Let p_i be the* *proportion of cells with colour i, for $i \in \mathcal{U}$, and let $\#\mathcal{C} = m$. Then the adjacency matrix B of the network has the following block structure:*

$$B = \left(\begin{array}{c|c|c|c} B_{11} & B_{12} & \cdots & B_{1k} \\ \hline B_{21} & B_{22} & \cdots & B_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline B_{k1} & B_{k2} & \cdots & B_{kk} \end{array}\right) \tag{4.1}$$

where, for all $i, j \in \mathcal{U}$,

1. *B_{ij} is a $mp_i \times mp_j$ matrix,*
2. *the sum of any line of B_{ij} is a_{ij},*
3. *$B_{ji} = B_{ij}^T$,*
4. *the elements in the diagonals of B_{ii} are even numbers.*

Proof. To prove (1) and (2), note that for all $i, j \in \mathcal{U}$, the entry a_{ij} is the number of inputs from cells with colour j that a cell of colour i receives. Now we have mp_i cells of colour i and mp_j cells of colour j. The inputs that the first ones receive from the second ones are the entries of a $mp_i \times mp_j$ matrix B_{ij} whose lines sum a_{ij} because the colouring is balanced, see [Aguiar *et al.* 2007, Theorem 3.5].

In a bidirectional network, each edge $e \in \mathcal{E}$ with $\mathcal{T}(e) = c$ and $\mathcal{H}(e) = d$, for $c, d \in \mathcal{C}$, implies the existence of another edge $f \in \mathcal{E}$ in the opposite direction, $\mathcal{T}(f) = d$ and $\mathcal{H}(f) = c$. Thus, B is symmetric and $B_{ji} = B_{ij}^T$ for all $i, j \in \mathcal{U}$. In particular, self-couplings arise in pairs and the k square matrices B_{ii} have even numbers in their diagonals. □

Let A be a $k \times k$ matrix with fixed valence v (that is, the sum of the entries of any line is v) and consider the possible identical-edge homogeneous bidirectional networks $G_B = (\mathcal{C}, \mathcal{E})$, with finite $\mathcal{C}$, having A as k-colouring matrix. The proportion of cells for each one of the k colours that G_B must have is stated in Lemma 4.3 below.

Lemma 4.3. *Let $G = (\mathcal{C}, \mathcal{E})$ be a finite bidirectional identical-edge homogeneous network with a k-colouring ξ, with colours in $\mathcal{U} = \{1, \ldots, k\}$, and corresponding to a k-colouring matrix $A = (a_{ij})$, $i, j \in \mathcal{U}$, with valence v. Suppose that $\#\mathcal{C} = m$ and that p_i is the proportion of cells with colour i, for $i \in \mathcal{U} = \{1, \ldots, k\}$. Then the vector*

$$\mathbf{p}^T = (p_1, \ldots, p_k) \in [0, 1]^k$$

is a left eigenvector of A generating the one-dimensional eigenspace that corresponds to the eigenvalue v. Moreover, the entries of A satisfy the condition

$$p_i a_{ij} = p_j a_{ji} \quad \forall i, j \in \mathcal{U}. \tag{4.2}$$

Proof. By Lemma 4.2, the adjacency matrix of G has the block structure (4.1). Thus, the summation of all the entries of B_{ij} is $mp_i a_{ij}$, for all $i, j \in \mathcal{U}$. Since B is symmetric and $B_{ji} = B_{ij}^T$ for all $i, j \in \mathcal{U}$, the sum of all the entries of these two matrices must coincide, implying condition (4.2). Summing the two sides we obtain

$$\begin{aligned} \sum_{i=1}^{k} mp_i a_{ij} &= \sum_{i=1}^{k} mp_j a_{ji} \\ &= mp_j \sum_{i=1}^{k} a_{ji} \\ &= mp_j v \end{aligned}$$

where the last equality follows from the definition of valence. Now the left hand side summation is the j^{th} component of the product $m\mathbf{p}^T A$. Therefore $\mathbf{p}^T A = v\mathbf{p}^T$.

Since A is an irreducible matrix (see Section 4 (d)), then the valence v has multiplicity one, see for example [Brualdi and Ryser 1992, Lemma 5.1.1]. Therefore $\mathbf{p}^T$ generates the eigenspace of A corresponding to the eigenvalue v and it is the unique vector in this space whose coordinates sum 1. □

Observe that Lemma 4.3 defines the proportion of cells for each colour, in a bidirectional network $G = (\mathcal{C}, \mathcal{E})$ that admits a k-colouring corresponding to a fixed k-colouring matrix. It follows, in particular, that there is a minimum number of cells $\#\mathcal{C}$. Moreover, condition (4.2) restricts the structure of the matrices that can be k-colouring matrices for identical-edge homogeneous bidirectional networks.

The results stated in the previous lemma are also valid for lattice networks. We describe that in Section 4 (d), below.

(*b*) *decomposition of B*

A bidirectional identical-edge homogeneous network $G = (\mathcal{C}, \mathcal{E})$ is a graph whose vertex set is $\mathcal{C}$ and whose undirected edges are the bidirectional couplings. For a fixed valence v, it is a *v-regular graph*, *i.e.*, a graph such that all the vertices are incident with exactly v edges. A 2-*factor* of G is a vertex-disjoint union of 2-regular subgraphs of G that covers $\mathcal{C}$.

Since v is even, these graphs have a factorisation into $v/2$ graphs that are 2-regular. This is a major tool in our work and is stated in Lemma 4.4, below.

Lemma 4.4. *[Petersen 1891] Every v-regular graph, with v even, has a factorisation into* 2*-factors.*

Outline of the proof. [Bondy and Murty 1976, pages 72, 75 and 229] Let G be a $2k$-regular graph with the set of vertices $\mathcal{C} = \{v_1, \ldots, v_m\}$; without loss of generality, assume G is connected. Let G_1 be an Euler tour in G. Form a bipartite graph G' with bipartition (X, Y), where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$ by joining x_i to y_j whenever v_i immediately precedes v_j on G_1. Show that G' is 1-factorable using Hall's Theorem, and hence that G is 2-factorable. □

The statement of Lemma 4.4 above, can be formulated in the following way. A bidirectional identical-edge homogeneous network $G_B = (\mathcal{C}, \mathcal{E})$, with even valence v, can be decomposed into $v/2$ bidirectional networks

$$G_{B_1}, \ldots, G_{B_{v/2}}$$

where, for $\mathcal{V} = \{1, \ldots, v/2\}$,

1. $G_{B_i} = (\mathcal{C}, \mathcal{E}_i)$, for $i \in \mathcal{V}$,
2. the disjoint union $\cup_{i \in \mathcal{V}} \mathcal{E}_i$ is $\mathcal{E}$ or, equivalently, $\sum_{i=1}^{v/2} B_i = B$,
3. G_{B_i} is a bidirectional network with valence 2, for $i \in \mathcal{V}$.

Lemma 4.5. *Let $\#\mathcal{C} = m \in \mathbf{N}$ and let $G_B = (\mathcal{C}, \mathcal{E})$ be a bidirectional identical-edge homogeneous network with adjacency matrix B and even valence v. Then, for $\mathcal{V} = \{1, \ldots, v/2\}$, there is a set of permutations*

$$\Sigma = \{\sigma_i, \sigma_i^{-1} : i \in \mathcal{V}\} \subset \mathbf{S}_m$$

such that, for all $c \in \mathcal{C}$,

$$\{\mathcal{T}(e) : e \in I(c)\} = \{\sigma(c) : \sigma \in \Sigma\}. \tag{4.3}$$

Equivalently, B can be written as a sum

$$B = \sum_{i=1}^{v/2} \left(M_{\sigma_i} + M_{\sigma_i}^T \right) \tag{4.4}$$

where M_σ is the $m \times m$ permutation matrix associated to $\sigma \in \Sigma$.

Proof. Consider a factorisation of G into 2-factors, $G_{B_1}, \ldots, G_{B_{v/2}}$. For all $i \in \mathcal{V}$, the network $G_{B_i} = (\mathcal{C}, \mathcal{E}_i)$ is a vertex-disjoint union of cycles that covers $\mathcal{C}$. If we choose one direction for each cycle then we define a digraph associated to a permutation σ_i of the cells in $\mathcal{C}$. The inverse permutation, σ_i^{-1}, corresponds to the converse digraph, obtained by reversing the direction of the cycles. The union of these digraphs is G_{B_i} and, therefore, the set of cells coupled to any cell $c \in \mathcal{C}$ can be described by Σ, as in (4.3).

If $e \in \mathcal{E}_i$ then either $\mathcal{H}(e) = \sigma_i(\mathcal{T}(e))$ or $\mathcal{H}(e) = \sigma_i^{-1}(\mathcal{T}(e))$ and, equivalently, $B_i = M_{\sigma_i} + M_{\sigma_i}^T$, since $M_{\sigma_i^{-1}} = M_{\sigma_i}^T$. Thus (4.4) follows from $\cup_{i \in \mathcal{V}} \mathcal{E}_i = \mathcal{E}$ which is equivalent to $\sum_{i=1}^{v/2} B_i = B$. □

(*c*) *the set* $\mathcal{D}_A$

Lemma 4.5 states that every finite bidirectional identical-edge homogeneous network with even valence, can be associated to sets of permutations Σ. In fact, each Σ defines only one coupled cell network, see the proof of Lemma 4.5. However, if a bidirectional network has various decompositions into 2-factors, then it will have different associated sets Σ. Thus, we can identify each of these permutations sets by $\mathcal{C}$ and by the partition of the couplings: $\Sigma = \Sigma(\mathcal{C}, \mathcal{E}_1, \ldots, \mathcal{E}_{v/2})$.

Definition 4.6. Let A be a $k \times k$ matrix with fixed valence v, with v even, and let $\mathcal{V} = \{1, \ldots, v/2\}$. The set $\mathcal{D}_A$ is

$$\mathcal{D}_A = \left\{ \begin{array}{ccc} \Sigma = \Sigma(\mathcal{C}, \mathcal{E}_1, \ldots, \mathcal{E}_{v/2}) & : & G = (\mathcal{C}, \cup_{i \in \mathcal{V}} \mathcal{E}_i) \text{ is a bidirectional} \\ & & \text{network with colouring matrix } A \end{array} \right\}.$$

◊

Remark 4.7. The elements in the set $\mathcal{D}_A$ represent all the decompositions of all the finite bidirectional networks having k-colouring matrix A.

If A is not irreducible then there are no connected bidirectional networks with k-colouring matrix A and, thus, the set $\mathcal{D}_A$ is empty. Now consider condition (4.2), *i.e.*, $p_i a_{ij} = p_j a_{ji}$ for all $i, j \in \{1, \dots, k\}$, where $(p_1, \dots, p_k)^T$ is any left eigenvector of $A = (a_{ij})$ associated to v . If the entries of A do not satisfy this condition then there are no bidirectional networks having k-colouring matrix A and $\mathcal{D}_A$ is also an empty set. ◊

(d) proportion of colours for lattice networks

Now we present a result concerning the ratio of the cells for each colour on a balanced colouring of a lattice network. Lemma 4.8 below, extends Lemma 4.3 for lattice networks, a class of bidirectional networks with an infinite number of cells.

Consider a random walk through the coloured cells of a lattice network $G_{\mathcal{L}}$ where the possible steps are the bidirectional nearest neighbour couplings between cells. Suppose that $G_{\mathcal{L}}$ has a balanced k-colouring with k-colouring matrix A and that, at each step, we register the colour i of the cell, with $i \in \mathcal{U} = \{1, \dots, k\}$. Thus, indexing the steps by time $t = 0, 1, 2, \dots$, we obtain a sequence of random variables $\{X_t\}$, where $X_t \in \mathcal{U}$ is the state-space for time t.

Being in a cell of colour i, the probability of going to a cell of colour j in the next step is $p_{ij} = a_{ij}/v$, the number of inputs from cells of colour j that a cell of colour i receives, over the valence. This probability does not change with t and does not depend on the previous steps. Therefore this random walk is a Markov chain with transition matrix

$$\frac{1}{v}A = (p_{ij}) \,,$$

where the entry p_{ij} is the conditional probability

$$p_{ij} = \frac{P[\text{a cell has colour } i \text{ and receives an input from a cell of colour } j]}{P[\text{a cell has colour } i]}. \tag{4.5}$$

Given a cell of any colour, it intercommunicates with cells of all the colours in $\mathcal{U}$, via the bidirectional couplings. Therefore, this is an irreducible Markov chain (see [Isaacson and Madsen 1976, Theorem II.1.1]).

Let $\mathbf{p}^T = (p_1, \dots, p_k) \in [0,1]^k$ be the left eigenvector of A:

$$\mathbf{p}^T \frac{1}{v} A = \mathbf{p}^T \Leftrightarrow \mathbf{p}^T A = v \mathbf{p}^T$$

normalised so that $\sum_{i=1}^{k} p_i = 1$, called the invariant probability vector. By [Isaacson and Madsen 1976, Theorems III.2.2. and III.2.4.] this vector exists and is the limit distribution of the colours, independent of the starting point of the random walk. In another interpretation, p_i is the fraction of time that the process is expected to spend in cells of colour i, for a large number of steps, see [Kemeni and Snell 1976, Section 4.2]. Thus, p_i represents the proportion of the cells with color $i \in \mathcal{U}$ in the whole colouring: P[a cell has colour i].

The probability P[a cell has colour i and receives an input from a cell of colour j] equals, by the above considerations, $p_i p_{ij} = p_i a_{ij}/v$, for all $i, j \in \{1, \dots, k\}$. This probability is the proportion of couplings between the cells of colour j and the cells of colour i in the whole set of couplings of the lattice network. Since the couplings are bidirectional, it equals P[a cell has colour j and receives an input from a cell of colour i] $= p_j p_{ji} = p_j a_{ji}/v$. It follows that $p_i a_{ij} = p_j a_{ji}$ for all $i, j \in \{1, \ldots, k\}$.

This proves the next lemma.

Lemma 4.8. *Let $\mathcal{U} = \{1, \ldots, k\}$ and let $G_{\mathcal{L}}$ be a lattice network with nearest neighbour coupling architecture having a balanced k-colouring with k-colouring matrix $A = (a_{ij})$. Let p_i be the proportion of cells having colour i, for $i \in \mathcal{U}$ and v the valence. Then $\mathbf{p}^T = (p_1, \ldots, p_k) \in [0,1]^k$ is the unique vector whose components sum to 1 and that is a left eigenvector of A, corresponding to the eigenvalue v. Moreover, $p_i a_{ij} = p_j a_{ji}$ for all $i, j \in \mathcal{U}$.*

By Lemmas 4.3 and 4.8 the proportion p_i of colour $i \in \mathcal{U}$ is the same as in the balanced k-colouring of a lattice network or in any other finite network that shares the same k-colouring matrix A.

(*e*) *periodic patterns*

Suppose Ψ_ξ is a periodic pattern of a balanced k-colouring on a lattice network. Thus, the fundamental domain of the pattern must have a set of cells whose colours respect the proportion in $\mathbf{p}$ and the number of cells of each colour in the fundamental domain are the components of the vector $m\mathbf{p}$ for some $m \in \mathbf{N}$.

5. Balanced colourings in lattices

In this section we define a function that projects lattice networks into finite bidirectional networks under some conditions. Using this projection function we obtain our main result (Theorem 5.4): we show that any periodic balanced k-colouring of an n-dimensional lattice network with nearest neighbour coupling architecture can be obtained from a finite bidirectional network admitting the same balanced k-colouring rule.

Let A be a k-colouring matrix with fixed valence v, with v even. For $\mathcal{V} = \{1, \ldots, v/2\}$, let $\Sigma = \{\sigma_i, \sigma_i^{-1} : i \in \mathcal{V}\} \in \mathcal{D}_A$. Let $\mathcal{L} = \{l_1, \ldots, l_n\}_{\mathbf{Z}}$ be an Euclidean lattice with nearest neighbours indexed by $\mathcal{J} = \{l_i, -l_i : i \in \mathcal{V}\}$, where $v/2 \geq n$.

Suppose that

1. the permutations commute: $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ for all $i, j \in \mathcal{V}$, where $\circ$ denotes the composition,

and let ϕ_τ be a function

$$\phi_\tau : \mathcal{J} \longrightarrow \Sigma$$

associated to a permutation τ of the elements in $\mathcal{V}$, such that the following conditions are verified:

2. related inverse elements: $\phi_\tau(l_i) = \sigma_{\tau(i)}$ and $\phi_\tau(-l_i) = \sigma^{-1}_{\tau(i)}$ for all $i \in \mathcal{V}$,

3. consistency: for $v/2 > n$, if $l_i = \sum_{j=1}^{v/2} m_j l_j$, with $m_1, \ldots, m_{v/2} \in \mathbf{Z}$, then $\sigma_{\tau(i)} = \sigma^{m_1}_{\tau(1)} \circ \cdots \circ \sigma^{m_{v/2}}_{\tau(v/2)}$.

Remark 5.1. Given $\Sigma \in \mathcal{D}_A$, there always exists ϕ_τ satisfying condition (2) and it it is used only to define ϕ_τ. Condition (3) depends on the set Σ as well as on $\mathcal{J}$ and condition (1) defines a subset of $\mathcal{D}_A$ that can be the empty set, as we show in the next example. Consider the matrix

$$A = \begin{pmatrix} 0 & 4 \\ 3 & 1 \end{pmatrix}.$$

This matrix has left eigenvector $7\mathbf{p}^T = (3, 4)$ associated to $v = 4$. In figure 17 we represent one bidirectional network G with 3 cells of one colour and 4 cells of another colour, that has the two-colouring matrix A. Lemma 4.5 ensures the existence of $\Sigma \in \mathcal{D}_A$, that is, a decomposition of G. Consider, as an example, the factorisation into 2-factors G_1 and G_2, represented in figure 18 and corresponding to $\Sigma = \{\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-2}\}$ with the non-commuting permutations $\sigma_1 = (c_1c_4c_3c_7c_6c_2c_5)$ and $\sigma_2 = (c_1c_6c_3c_5c_4c_2c_7)$.

Suppose that there is a factorisation such that the permutations in Σ commute. By Theorem 5.4, below, this Σ ensures the existence of a periodic balanced colouring, with the two-colouring matrix A, of the planar square lattice network with nearest neighbour coupling architecture, since for the planar square lattice network we have $v/2 = n$ and, thus, (3) is not a restriction. However, [Wang and Golubitsky 2005, Theorem 1.7] proves that there is not such a colouring, and we conclude that the permutations in Σ do not commute. ♢

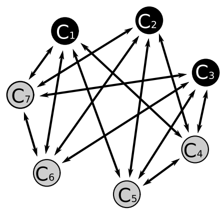

Figure 17. A bidirectional network where each black cell receives four inputs from grey cells and each grey cell receives one input from grey cells and three inputs from black cells.

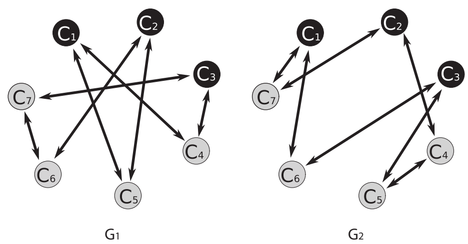

Figure 18. A decomposition of the bidirectional network in figure 17. The 2-factors G_1 and G_2 correspond to the non-commuting permutations $\sigma_1 = (c_1c_4c_3c_7c_6c_2c_5)$ and $\sigma_2 = (c_1c_6c_3c_5c_4c_2c_7)$.

Definition 5.2. Let $\mathcal{J}$ and Σ be such that there exists $\phi \equiv \phi_\tau$ satisfying the conditions (1)-(3) above. Then we say that they are *identifiable* and we use the notation

$$\phi(l) = \sigma_l \quad \text{for } l \in \mathcal{J}$$

knowing that, for all $l, g \in \mathcal{J}$,

- $\sigma_l \circ \sigma_g = \sigma_g \circ \sigma_l$;
- $\sigma_{-l} = \sigma_l^{-1}$;
- $\sigma_{l+g} = \sigma_l \circ \sigma_g$, if $l + g \in \mathcal{J}$.

$\Diamond$

Since the lattice is Euclidean, the set $\mathcal{J}$ generates the abelian group $(\mathcal{L}, +)$ that we denote by $< \mathcal{J} >$. Analogously, the set $\Sigma = \{\sigma_i, \sigma_i^{-1} : i \in \mathcal{V}\}$, with commuting permutations (condition (1) above), generates an abelian group that we denote by $< \Sigma >$.

The function ϕ induces the homomorphism

$$\Phi : < \mathcal{J} > \quad \longrightarrow \quad < \Sigma >$$

such that $\Phi(l) = \phi(l)$ for all $l \in \mathcal{J}$. From the definition of Φ, it follows that Φ is an epimorphism, that is, $\Phi(\mathcal{L}) = < \Sigma >$.

Therefore, saying that $\mathcal{J}$ and Σ are identifiable means that there exists a homomorphism between $< \mathcal{J} > \cong \mathcal{L}$ and $< \Sigma >$.

Theorem 5.3. *Let $\mathcal{L}$ be an n-dimensional Euclidean lattice, A a k-colouring matrix with fixed even valence v and $\Sigma = \Sigma(\mathcal{C}, \mathcal{E}_1, \ldots, \mathcal{E}_{v/2}) \in \mathcal{D}_A$. If $\mathcal{J}$ and Σ are identifiable by ϕ, then there is a function*

$$\Pi : \mathcal{L} \quad \longrightarrow \quad \mathcal{C}$$

such that

$$\Pi(l + g) = \sigma_g(\Pi(l)) \quad \textit{for all } l \in \mathcal{L} \textit{ and for all } g \in < \mathcal{J} > \tag{5.1}$$

where $\sigma_g = \Phi(g)$ is the image of $g \in < \mathcal{J} >$ by Φ.

Moreover, Π has set of periods $\tilde{\mathcal{L}}$, the kernel of Φ,

$$ker(\Phi) = \{l \in \mathcal{L} : \Phi(l) = \epsilon\} = \tilde{\mathcal{L}}$$

with periods along n linearly independent directions.

Proof. Let $\alpha_{\mathcal{L}}$ and $\alpha_{\mathcal{C}}$ be the actions of $< \mathcal{J} >$ on $\mathcal{L}$ and $< \Sigma >$ on $\mathcal{C}$, respectively, defined by:

$$\begin{array}{rcl} \alpha_{\mathcal{L}} : < \mathcal{J} > \times \mathcal{L} & \longrightarrow & \mathcal{L} \\ (g, l) & \longmapsto & g + l \end{array} \qquad \begin{array}{rcl} \alpha_{\mathcal{C}} : < \Sigma > \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ (\sigma, c) & \longmapsto & \sigma(c) \end{array} .$$

If $\mathcal{J}$ and Σ are identifiable by ϕ then we can consider the induced homomorphism $\Phi : < \mathcal{J} > \longrightarrow < \Sigma >$ and define a function $\Pi : \mathcal{L} \longrightarrow \mathcal{C}$ such that the next diagram commutes:

$$\begin{array}{ccccc} & & & \alpha_{\mathcal{L}} & \\ < \mathcal{J} > & \times & \mathcal{L} & \longrightarrow & \mathcal{L} \\ \Phi \downarrow & & \downarrow \Pi & & \downarrow \Pi \\ < \Sigma > & \times & \mathcal{C} & \longrightarrow & \mathcal{C} \\ & & & \alpha_{\mathcal{C}} & \end{array} .$$

Thus, for all $l \in \mathcal{L}$ and $g \in < \mathcal{J} >$,

$$\Pi(\alpha_{\mathcal{L}}(g,l)) = \alpha_{\mathcal{C}}(\sigma_g, \Pi(l)) \Leftrightarrow \Pi(l+g) = \sigma_g(\Pi(l)).$$

If $g \in \tilde{\mathcal{L}}$ then $\sigma_g = \epsilon$ and $\Pi(l+g) = \epsilon(\Pi(l)) = \Pi(l)$ for all $l \in \mathcal{L}$. For each $g \in \mathcal{J}$ there in an integer s_g such that $\sigma_g^{s_g} = \epsilon$ and, therefore, $s_g g \in \tilde{\mathcal{L}}$. Since the lattice is Euclidean, there are elements of $\mathcal{J}$ along n linearly independent directions and the result follows. □

It follows that the function Π, obtained in the above theorem, covers the lattice $\mathcal{L}$ with the cells of $\mathcal{C}$ such that the colouring ξ of $\mathcal{C}$, induces a colouring of $\mathcal{L}$. Moreover, we show in the next theorem that the colouring is balanced for the lattice network $G_{\mathcal{L}}$.

Theorem 5.4. *Let A be a colouring matrix with even valence v and let $\mathcal{L}$ be an Euclidean lattice with $\#\mathcal{J} = v$. The lattice network with nearest neighbour coupling architecture $G_{\mathcal{L}}$ has a periodic balanced colouring with the colouring matrix A if and only if $\mathcal{D}_A$ has an element identifiable with $\mathcal{J}$.*

Proof — From finite bidirectional networks to lattice networks. If $\mathcal{J}$ and Σ are identifiable, with $\Sigma = \Sigma(\mathcal{C}, \mathcal{E}_1, \ldots, \mathcal{E}_{v/2})$ then, by Theorem 5.3 there is a function $\Pi : \mathcal{L} \longrightarrow \mathcal{C}$ such that $\Pi(l+g) = \sigma_g(\Pi(l))$ for all $l \in \mathcal{L}$ and for all $g \in < \mathcal{J} >$. Beginning with any cell $l \in \mathcal{L}$, let $\Pi(l) = c \in \mathcal{C}$. The remaining domain of Π is the orbit of l by group $< \mathcal{J} >$, having images given by the orbit of c under $< \Sigma >$.

Since the cells of $\mathcal{C}$ are coloured by the balanced k-colouring ξ with k-colouring matrix A, this function Π induces the colouring $\xi_{\mathcal{L}}$ of $G_{\mathcal{L}}$:

$$\xi_{\mathcal{L}}(l) = \xi(\Pi(l)) \quad \text{for all } l \in \mathcal{L}.$$

Moreover, $\xi_{\mathcal{L}}$ is also a balanced k-colouring with k-colouring matrix A, as we show now. Recall the matrix formulation (3.1) and (3.2) of balanced colourings of networks and lattice networks. We have

$$\begin{aligned}
A^T \vec{\xi}_{\mathcal{L}}(l) &= A^T \vec{\xi}(\Pi(l)) \\
&= \sum_{d \in I(\Pi(l))} \vec{\xi}(\mathcal{T}(d)), \text{ since } \xi \text{ is balanced,} \\
&= \sum_{g \in \mathcal{J}} \vec{\xi}(\sigma_g(\Pi(l))), \text{ see (3.3),} \\
&= \sum_{g \in \mathcal{J}} \vec{\xi}(\Pi(l+g)), \text{ by (5.1) ,} \\
&= \sum_{g \in \mathcal{J}} \vec{\xi}_{\mathcal{L}}(l+g), \text{ by the definition of } \xi_{\mathcal{L}}.
\end{aligned}$$

Moreover, $\xi_{\mathcal{L}}$ inherits the periods of Π: if $\tilde{l}$ is a period of Π then, for all $l \in \mathcal{L}$,

$$\xi_{\mathcal{L}}(l+\tilde{l}) = \xi(\Pi(l+\tilde{l})) = \xi(\Pi(l)) = \xi_{\mathcal{L}}(l).$$

Therefore, by Theorem 5.3, the colouring $\xi_{\mathcal{L}}$ has set of periods $\tilde{\mathcal{L}}$, with periods along n linearly independent directions.

From lattice networks to finite bidirectional networks. Let $G_{\mathcal{L}}$ have a periodic balanced k-colouring $\xi_{\mathcal{L}}$, with k-colouring matrix A and with set of periods $\tilde{\mathcal{L}} = \{\tilde{l}_1, \ldots, \tilde{l}_n\}_{\mathbf{Z}} \subset \mathcal{L}$, an n-dimensional lattice, and consider the n-dimensional parallelepiped
$\{\tilde{l}_1, \ldots, \tilde{l}_n\}_{[0,1)}$. The intersection of this parallelepiped with $\mathcal{L}$ is a set $\mathcal{C}$, isomorphic to the quotient $\mathcal{L}/\tilde{\mathcal{L}}$.

Let $\Pi : \mathcal{L} \longrightarrow \mathcal{C}$ be the composition of the isomorphism $\mathcal{L}/\tilde{\mathcal{L}} \cong \mathcal{C}$ with the projection $\mathcal{L} \longrightarrow \mathcal{L}/\tilde{\mathcal{L}}$. Notice that Π has the set of periods $\tilde{\mathcal{L}}$. Now consider the permutations of the elements in $\mathcal{C}$ defined by

$$\sigma_g(\Pi(l)) = \Pi(l+g) \quad \text{for } g \in \mathcal{J}. \tag{5.2}$$

By definition, $\sigma_{l+g} = \sigma_l \circ \sigma_g$ for all $l, g \in \mathcal{J}$. Therefore, we have $\sigma_{-l} = \sigma_l^{-1}$ and $\sigma_l \circ \sigma_g = \sigma_g \circ \sigma_l$ for all $l, g \in \mathcal{J}$. It follows that $\mathcal{J}$ and $\Sigma = \{\sigma_g : g \in \mathcal{J}\}$ are identifiable. Let $G = (\mathcal{C}, \mathcal{E})$ be the finite identical-edge homogeneous network such that, for all $c \in \mathcal{C}$,

$$\{\mathcal{T}(e) : e \in I(c)\} = \{\sigma_g(c) : g \in \mathcal{J}\}. \tag{5.3}$$

Since $g \in \mathcal{J}$ implies $-g \in \mathcal{J}$, the network G has all the couplings bidirectional.

Consider the colouring ξ of $\mathcal{C}$ defined by:

$$\xi(\Pi(l)) = \xi_{\mathcal{L}}(l) \quad \text{for all } l \in \mathcal{L}.$$

We have

$$\begin{aligned} A^T\vec{\xi}(\Pi(l)) &= A^T\vec{\xi}_{\mathcal{L}}(l) \\ &= \sum_{g\in\mathcal{J}} \vec{\xi}_{\mathcal{L}}(l+g), \text{ since } \xi_{\mathcal{L}} \text{ is balanced}, \\ &= \sum_{g\in\mathcal{J}} \vec{\xi}(\Pi(l+g)), \text{ by the definition of } \xi_{\mathcal{L}}, \\ &= \sum_{g\in\mathcal{J}} \vec{\xi}(\sigma_g(\Pi(l))), \text{ by (5.2) }, \\ &= \sum_{d\in I(\Pi(l))} \vec{\xi}(\mathcal{T}(d)), \text{ see (5.3)}. \end{aligned}$$

Thus ξ is a balanced k-colouring of $G = (\mathcal{C}, \mathcal{E})$, with the k-colouring matrix A. Therefore $\Sigma \in \mathcal{D}_A$ and the proof is complete. □

The proof of this theorem is illustrated by the examples in the first section.

Theorem 5.4 relates the local structure of the colourings, given by the colouring matrix A, to their long-range behaviour in lattice networks.

6. Some consequences of Theorem 5.4

In this section we present some corollaries of Theorem 5.4, as well as some examples, relating balanced colourings of lattices with different dimensions.

(a) corollaries

In what follows we denote by $\mathbf{Z}^m$ the m-dimensional lattice generated by the m vectors in the canonical basis of $\mathbf{R}^m$. Note that $\mathbf{Z}^m$ is an Euclidean lattice. For this lattice network, with nearest neighbour coupling architecture, we have $v/2 = m$ and one element $\Sigma = \{\sigma_i, \sigma_i^{-1} : i \in \mathcal{V}\} \in \mathcal{D}_A$ with $\mathcal{V} = \{1, \ldots, v/2\}$, is identifiable with $\mathcal{J}$ if and only if the permutations commute.

Corollary 6.1. *Let A be a colouring matrix with even valence v and let $\mathcal{L}$ be an n-dimensional Euclidean lattice, with $v/2 > n$. If the lattice network with nearest neighbour coupling architecture $G_{\mathcal{L}}$ has a periodic balanced colouring with the colouring matrix A then there is periodic balanced colouring of a $v/2$-dimensional Euclidean lattice $\mathbf{Z}^{v/2}$ having the same colouring matrix A.*

Proof. Under the hypotheses of the corollary and by Theorem 5.4, there is an element $\Sigma \in \mathcal{D}_A$ identifiable with $\mathcal{J}$, the set that indexes the nearest neighbours in $\mathcal{L}$. In particular, the permutations in Σ commute and so Σ is also identifiable with the set that indexes the v nearest neighbours of $\mathbf{Z}^{v/2}$. Thus, the result follows by Theorem 5.4. □

Corollary 6.2. *Let A be a k-colouring matrix with even valence v and let $\mathcal{L} = \mathbf{Z}^n$. Let $r \in \mathbf{Z}^+$. If the lattice network with nearest neighbour coupling architecture $G_{\mathcal{L}}$ has a periodic balanced colouring with the colouring matrix A then there is a periodic balanced colouring of the $(n+r)$-dimensional Euclidean lattice $\mathbf{Z}^{n+r}$ having the k-colouring matrix $A + 2r\mathrm{Id}_k$.*

Proof. Under the hypotheses of the corollary and by Theorem 5.4, there is an element $\Sigma \in \mathcal{D}_A$ identifiable with $\mathcal{J}$, the set that indexes the nearest neighbours in $\mathcal{L}$. The set $\Sigma \cup \Sigma'$, where $\Sigma' = \{\epsilon, \ldots, \epsilon\}$ with $\#\Sigma' = 2r$, is identifiable with the set that indexes the nearest neighbours of $\mathbf{Z}^{r+n}$ and, thus, corresponds to a balanced colouring with colouring matrix A'. Let $A = (a_{ij})$ and $A' = (a'_{ij})$. In $\mathbf{Z}^{n+r}$, each cell with colour i is connected to a_{ij} cells of colour j, if $i \neq j$, according to Σ. By Σ', beyond these connections, each cell is coupled $2r$ times to cells with the same colour. That is, $a'_{ij} = a_{ij}$, if $i \neq j$, and $a'_{ij} = a_{ij} + 2r$, if $i = j$. □

Remark 6.3. By Corollary 6.2 it follows, in particular, that for a given k, the number of balanced colourings of $\mathbf{Z}^{n+1}$ with nearest neighbour coupling architecture, is larger than or equal to the number of balanced colourings of $\mathbf{Z}^n$ with nearest neighbour coupling architecture.

We think that a similar result may hold for n-dimensional lattices, with $v/2 \neq n$, *i.e.*, given two Euclidean lattices $\mathcal{L}$ and $\mathcal{L}^1$ with dimensions, respectively, n and $n+1$ and such that $\mathcal{L} = \{l_1, \ldots, l_n\}_{\mathbf{Z}}$ and $\mathcal{L}^1 = \{l_1, \ldots, l_n, l_{n+1}\}_{\mathbf{Z}}$, we make the following conjecture. For a given k, the number of balanced colourings of $\mathcal{L}^1$, with nearest neighbour coupling architecture, is larger than or equal to the number of balanced colourings of $\mathcal{L}$ with nearest neighbour coupling architecture. ♢

(b) examples

Consider the standard cubic lattice

$$\mathcal{L} = \{(1,0,0), (0,1,0), (0,0,1)\}_{\mathbf{Z}}$$

with nearest neighbour coupling architecture: $\mathcal{J} = \{\pm l_1, \pm l_2, \pm l_3\}$, where $l_1 = (1,0,0)$, $l_2 = (0,1,0)$ and $l_3 = (0,0,1)$.

Next we present three examples of balanced colourings of $\mathcal{L}$ with two colours, black and grey, indexed by the elements in $\mathcal{U} = \{1,2\}$.

(i) *case 43*

Let A be the matrix

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}.$$

By [Wang and Golubitsky 2005], there are balanced colourings of the planar hexagonal lattice with colouring matrix A. Thus, by Corollary 6.1, the existence of a pattern with this colouring matrix in the standard cubic lattice is ensured.

For the two-colouring matrix A, the proportion of cells with each colour is given by the vector $(3/7, 4/7)^T$. Consider the seven cell bidirectional network with adjacency matrix

$$B = \left(\begin{array}{ccc|cccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

admitting the two-colouring rule defined by A. Note that we have the decomposition

$$B = \left(\begin{array}{ccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) + \left(\begin{array}{ccc|cccc} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right) + \left(\begin{array}{ccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right),$$

that corresponds to the commuting permutations $\sigma_1 = (b_1 b_2 b_3 g_1 g_2 g_3 g_4)$, $\sigma_2 = (b_1 b_3 g_2 g_4 b_2 g_1 g_3)$ and $\sigma_3 = (b_1 g_1 g_4 b_3 g_3 b_2 g_2)$ defining the balanced colouring in figure 19.

Note that the restriction of this pattern to the planes with constant third coordinate, is a two-dimensional pattern (see figure 20) whose proportion of cells with each colour is given by the vector $(3/7, 4/7)^T$. However this is not a balanced colouring. The only colouring matrix with valence 4 that has this left eigenvector corresponding to the eigenvalue 4 is

$$\begin{pmatrix} 0 & 4 \\ 3 & 1 \end{pmatrix}.$$

By [Wang and Golubitsky 2005], there is not a balanced pattern in the square lattice with this colouring matrix and, thus, this restriction could not be balanced.

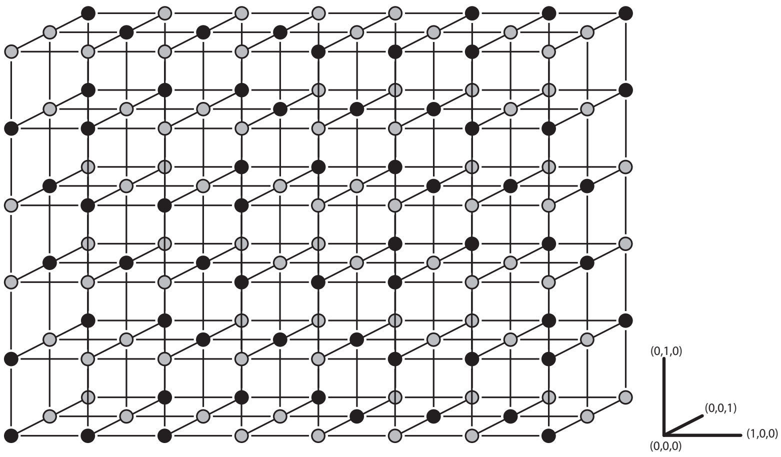

Figure 19. A periodic pattern for the standard cubic lattice with nearest neighbour coupling architecture obtained for the *case 43*.

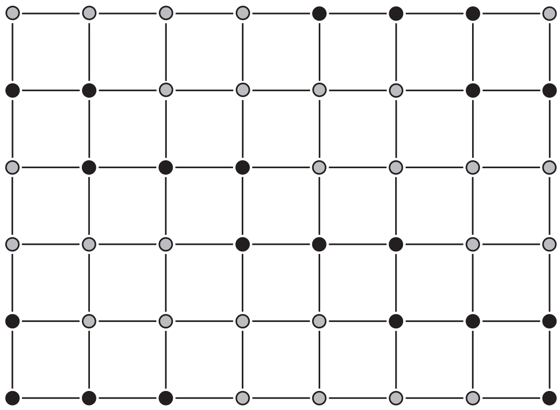

Figure 20. Restriction to $\{(x, y, 0) : x, y \in \mathbf{R}\}$ of the periodic pattern in figure 19.

(ii) *case 41*

Let

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}.$$

The proportion of cells with each colour is given by the vector $(1/5, 4/5)^T$. So we can take the five cell bidirectional network with adjacency matrix

$$B = \left(\begin{array}{c|cccc} 2 & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right) = \left(\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right) + \left(\begin{array}{c|cccc} 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) + 2\mathrm{Id}_5.$$

This decomposition corresponds to the commuting permutations $\sigma_1 = (b_1 g_1 g_2 g_3 g_4)$, $\sigma_2 = (b_1 g_2 g_4 g_1 g_3)$ and $\sigma_3 = \epsilon$ that generate the pattern in figure 21.

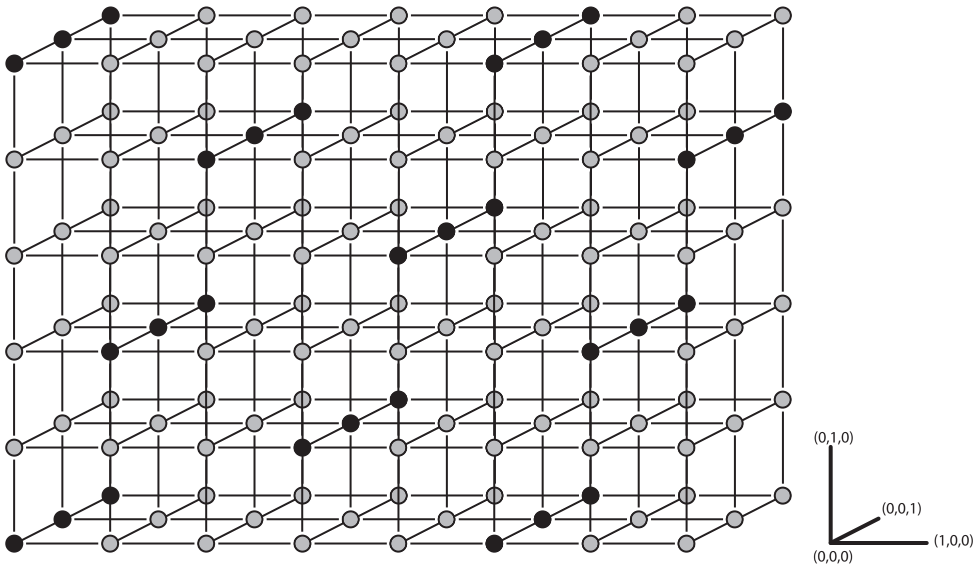

Figure 21. A periodic pattern for the standard cubic lattice with nearest neighbour coupling architecture obtained for the *case 41*.

Remark 6.4. This colouring matrix does not correspond to a balanced colouring of the planar hexagonal lattice, by the enumeration in [Wang and Golubitsky 2005]. Although the permutations commute, they do not verify the condition $\sigma_3 = \sigma_2 \circ \sigma_1^{-1}$ which is necessary for the identification of the permutations and the directions of the nearest neighbours in the planar hexagonal lattice network. Note that, given A and for nearest neighbour coupling architecture, Corollary 6.1 states that the number of periodic patterns with the k-colouring matrix A in $\mathbf{Z}^{v/2}$, is greater or equal to the number of periodic patterns with the same colouring matrix in other n-dimensional Euclidean lattices with valence v, if $v/2 > n$. In the example above these numbers are not equal. ◊

Since

$$A - 2\mathrm{Id}_2 = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$$

and this colouring matrix corresponds to a pattern in the planar square lattice (see [Wang and Golubitsky 2005]), the three-dimensional pattern in $\mathcal{L}$ was ensured by Corollary 6.2.

(iii) *case 33*

Let

$$A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

The proportion of cells with each colour is given by the vector $(1/2, 1/2)^T$. We define a four cell bidirectional network with adjacency matrix

$$B = \left(\begin{array}{cc|cc} 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \\ \hline 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \end{array}\right) = 3\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right).$$

Therefore, we associate each generator of the lattice to the permutation $(b_1 b_2 g_1 g_2)$ and we obtain the pattern in figure 22.

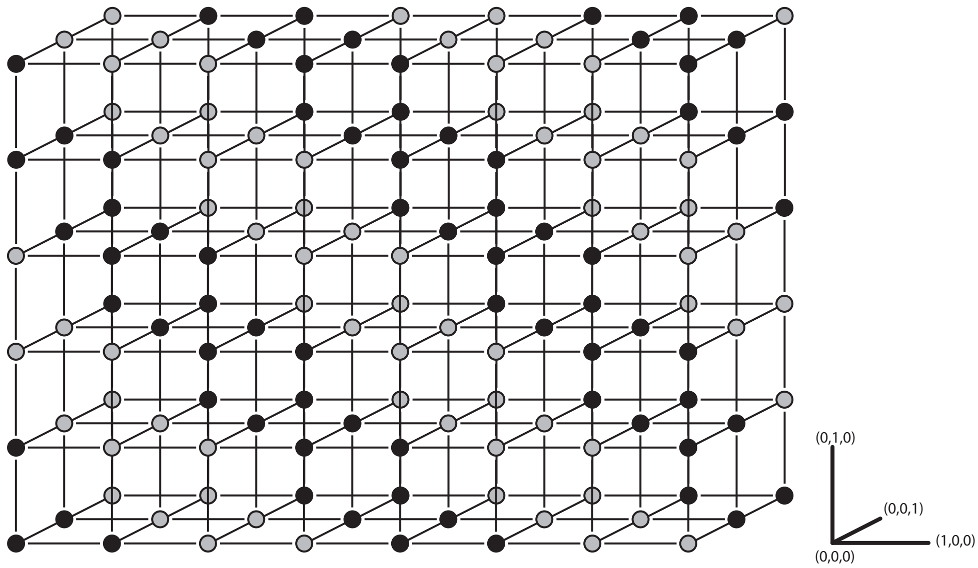

Figure 22. A periodic pattern for the standard cubic lattice with nearest neighbour coupling architecture obtained for the *case 33*.

Remark 6.5. By [Wang and Golubitsky 2005], there is one balanced colouring of the planar hexagonal lattice with colouring matrix

$$A = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

Thus, by Corollary 6.1, the existence of a periodic pattern with this colouring matrix in $\mathcal{L} = \mathbf{Z}^3$ is ensured. However in the standard cubic lattice, A corresponds to an infinite family of patterns as we describe below.

The restriction of the pattern in figure 22 to the planes with constant third coordinate correspond to *case 22* in the planar square lattice, where an infinite family of patterns can be obtained by the diagonal method, interchanging colours along diagonal where black and grey cells are alternate ([Wang and Golubitsky 2005, Section 3.2]). In figure 23 we show one of these diagonal whose cells have alternating colours. In fact, all the diagonals that are parallel to the one represented, are alternating diagonals. Thus, for a given alternating diagonal d_r =

$\{(r+x, x, 0) : x \in \mathbf{R}\}$, with $r \in \mathbf{Z}$, we can define an infinite set of alternating diagonals $F_r = \{d_r + s(0,0,1) : s \in \mathbf{Z}\}$ (figure 24) in the plane $\{(r+x, x, y) : x, y \in \mathbf{R}\}$. The lines $a_s = \{(r+2s, 2s, y) : y \in \mathbf{R}\} \cap \mathcal{L}$ are equally coloured for all $s \in \mathbf{Z}$, see lines "**a**" in figure 25, and the lines $b_s = \{(r+2s+1, 2s+1, y) : y \in \mathbf{R}\} \cap \mathcal{L}$ are equal for all $s \in \mathbf{Z}$ (lines "**b**" in figure 25). Interchanging lines **a** and **b** we obtain a new balanced colouring with the same colouring matrix A. This happens because along the third coordinate everything remains, and in the planes with constant third coordinate we interchange colours along alternating diagonals.

Therefore, this is a three-dimensional version of the diagonal method described in [Golubitsky *et al.* 2004]. ◊

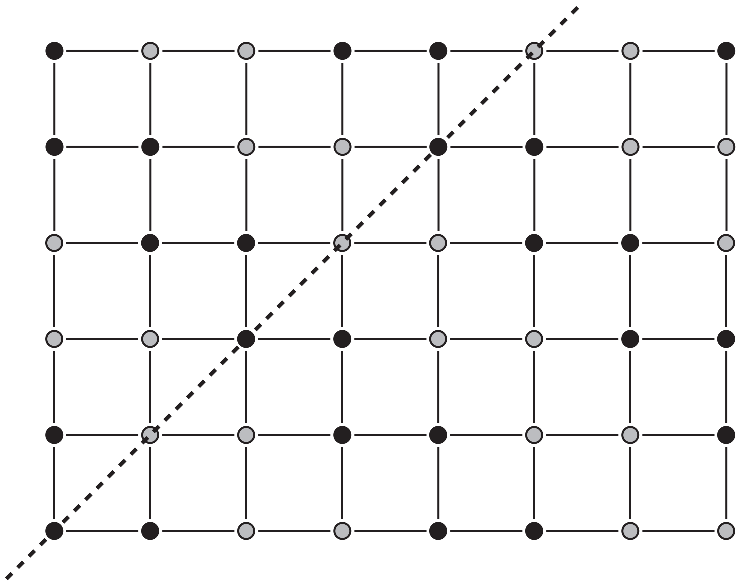

Figure 23. Restriction to $\{(x, y, 0) : x, y \in \mathbf{R}\}$ of the periodic pattern in figure 22. This plane has an alternating diagonal.

7. Acknowledgements

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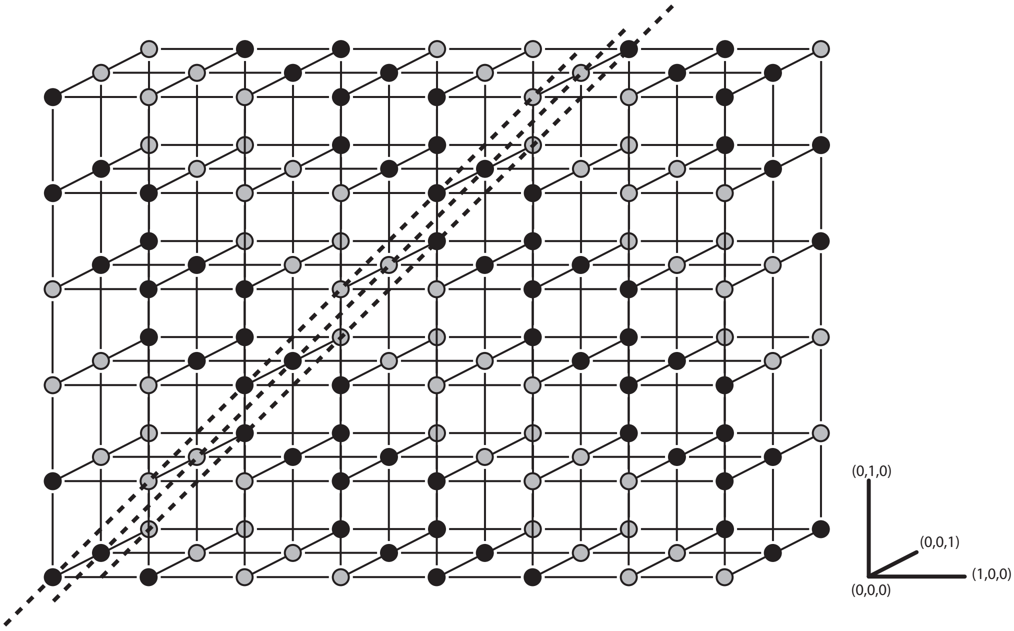

Figure 24. The alternating diagonals of the parallel planes with constant third coordinate in the periodic pattern in figure 22.

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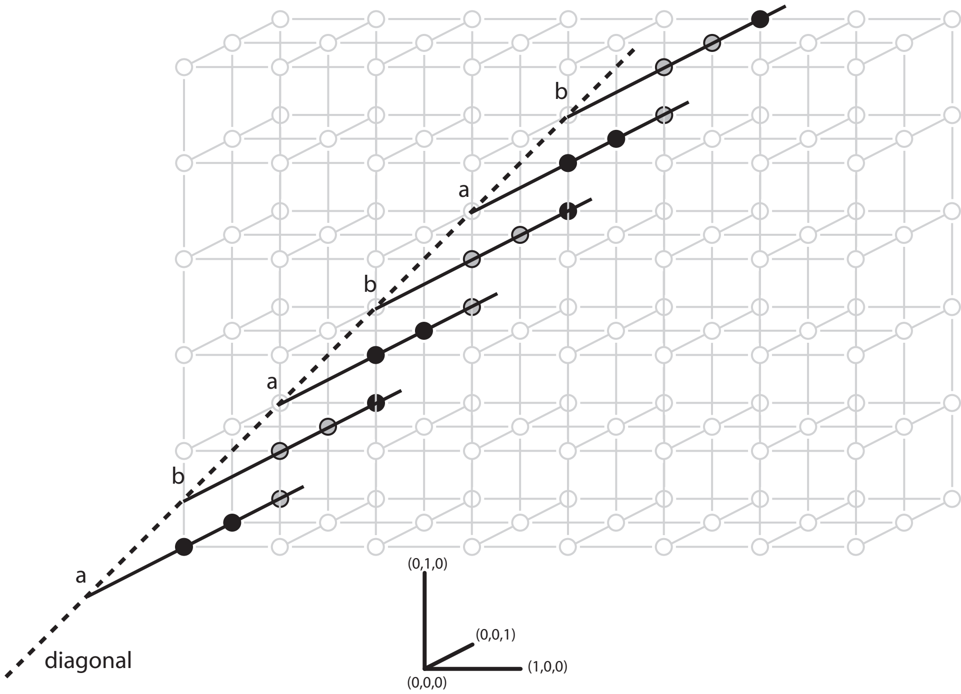

Figure 25. One possible generalisation of the diagonal method for three-dimensional patterns. Interchanging the lines A and B along the "diagonal" corresponds to the diagonal method applied in every plane with constant third coordinate.