Hopf bifurcation in coupled cell networks with abelian symmetry[1]

Ana Paula S. Dias† e *Rui C. Paiva*◇
†CMUP
Departamento de Matemática
Universidade do Porto

◇CMUP
Instituto Politécnico de Leiria

Abstract

We consider symmetric coupled cell networks of differential equations. We show that already at the level of Abelian symmetry, very degenerate codimension-one bifurcations can occur. This degenerate behaviour occurs due to the restrictions that the symmetry group of the network and the network structure impose at the associated coupled cell networks of differential equations.

1 Introduction

Consider a system of ordinary differential equations (ODEs)

$$\dot{x} = f\,(x, \lambda)\,, \quad f\,(0, \lambda) \equiv 0, \tag{1}$$

where $x \in V = \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $f\colon V \times \mathbb{R} \to V$ is smooth. We say that (1) presents *Hopf bifurcation* at $\lambda = 0$ if $(df)_{(0,0)}$ has a pair of purely imaginary eigenvalues. Here $(df)_{(x,\lambda)}$ represents the Jacobian matrix of f relatively to x, evaluated at (x, λ). Suppose now that f *commutes* with the linear action of a compact Lie group Γ on $\mathbb{R}^n$, that is, f is Γ-*equivariant*:

$$f(\gamma x, \lambda) = \gamma f(x, \lambda)$$

for all $\gamma \in \Gamma$, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

In general Γ-equivariant systems multiple eigenvalues of $(df)_{(0,0)}$ often occur. At this situation, a *steady-state mode* occurs when $(df)_{(0,0)}$ has a zero eigenvalue and the corresponding eigenspace is generically Γ-*absolutely irreducible* (Golubitsky *et al.* [3] Proposition XIII 3.2). A *Hopf mode* occurs when $(df)_{(0,0)}$ has a pair of conjugate imaginary eigenvalues and the corresponding eigenspace is generically Γ-*simple* (See Golubitsky et al. [3]

[1] Autor para contacto: Rui C. Paiva (e-mail: `rui.paiva@ipleiria.pt`).

Proposition XVI 1.4). We obtain the existence of branches of equilibria and periodic solutions with symmetry groups satisfying the conditions of the Equivariant Branching Lemma (Golubitsky *et al.* [3] Theorem XIII 3.3) or the Equivariant Hopf Theorem (Golubitsky *et al.* [3] Theorem XVI 4.1.). A *critical mode* occurs when the eigenvalues of $(df)_{(0,0)}$ lie on the imaginary axis. Generically we expect, in a one parameter equivariant system, to have only one critical mode.

The occurrence of multiple critical modes is called *mode interaction*. For example, in systems with more than one parameter, multiple critical modes are expected, see for instance Golubitsky *et al.* [3] Chapter XIX. Since there are two types of critical mode (steady-state and Hopf), there are three types of mode interaction: steady-state/steady-state, Hopf/steady-state and Hopf/Hopf.

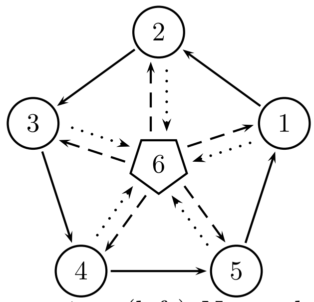
In this work we consider one parameter equivariant systems of ODEs that correspond to coupled cell networks of differential equations. We show that very degenerate behaviour concerning the phenomena of local Hopf bifurcation from a fully symmetric equilibrium can occur.

2 Symmetric Coupled Cell Networks

Standard examples of dynamical systems include networks of coupled *cells*, that is, coupled ODEs. Here, a network is represented by a directed graph whose nodes and edges are classified according to associated labels or *types*. The nodes (or cells) of a network $\mathcal{G}$ represent dynamical systems, and the edges (*arrows*) represent couplings. Cells with the same label indicate that they have 'identical' internal dynamics; arrows with the same label represent 'identical' couplings. We follow the theory developed by Stewart, Golubitsky and Pivato [6] and Golubitsky, Stewart and Török [4]. The *input set* of a cell is the set of edges directed to that cell. Coupled cell systems are dynamical systems compatible with the architecture or topology of a directed graph representing the network. Formally, they are defined in the following way. Each cell c is equipped with a phase space P_c, and the *total phase space* of the network is the cartesian product $P = \prod_c P_c$. A vector field f is called *admissible* (or $\mathcal{G}$-*admissible*) if its component f_c for cell c depends only on variables associated with the cell c and the cells coupled to c. Moreover, the components for cells c, d that have isomorphic input sets are identical up to a suitable permutation of the relevant variables.

In the study of network dynamics there is an important class of networks, namely, networks that possess a group of symmetries. In this context there is a group of permutations of the cells (and arrows) that preserves the network structure (including cell-types and arrow-types) and its action on P is by permutation of cell coordinates. The $\mathcal{G}$-admissible functions are equivariant functions. However, it is not always true that the set of $\mathcal{G}$-admissible functions coincide with the set of equivariant functions. So care must be taken when applying the theory of equivariant dynamical systems to such dynamical systems.

Example 2.1 *The network $\mathcal{G}$ of Figure 1 (left) has $\mathbb{Z}_5$-symmetry. We can write the $\mathcal{G}$-admissible system $\dot{x} = f(x, \lambda)$ as in Figure 1 (right). Here,*



$$\begin{cases} \dot{x}_1 = f_1(x_1, x_5, x_6, \lambda) \\ \dot{x}_2 = f_1(x_2, x_1, x_6, \lambda) \\ \dot{x}_3 = f_1(x_3, x_2, x_6, \lambda) \\ \dot{x}_4 = f_1(x_4, x_3, x_6, \lambda) \\ \dot{x}_5 = f_1(x_5, x_4, x_6, \lambda) \\ \dot{x}_6 = f_2(x_6, \overline{x_1, x_2, x_3, x_4, x_5}, \lambda) \end{cases}$$

Figure 1: (left) Network $\mathcal{G}$ with $\mathbb{Z}_5$-symmetry. (right) ODE system $\mathcal{G}$-admissible.

$f\colon \mathbb{R}^{5k} \times \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}^{5k} \times \mathbb{R}^l$ is $\mathcal{G}$-admissible, $f_1\colon \mathbb{R}^{2k} \times \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}^k$, $f_2\colon \mathbb{R}^l \times \mathbb{R}^{5k} \times \mathbb{R} \to \mathbb{R}^l$ are smooth maps and $\lambda \in \mathbb{R}$ is the bifurcation parameter. The function f_2 is invariant by permutation of the arguments under the bar reflecting the fact that all the edges directed to cell 6 correspond to 'identical'couplings.

3 Local Bifurcation in Coupled Cell Networks with Abelian Symmetry

One of the main questions in the theory of coupled cell networks is the following: in what way the network architecture may affect the kinds of bifurcations that are expected to occur in a coupled cell network? In Dias and Paiva [2] we address this question by focusing on networks with an abelian symmetry group that permutes cells transitively. Observe that distinct networks (with the same number of cells) can have the same symmetry group (see Figure 2). In addressing this question, the first concern is with the spectrum of the linearized vector field (Jacobian matrix) at the equilibrium solution when parameters are varied, in particular with the analysis of how eigenvalues typically cross the imaginary axis. Dias and Lamb [1] address this question in the case of abelian symmetric networks. They obtain a result that states that in an abelian symmetric connected coupled cell network $\mathcal{G}$ where the symmetry group acts transitively by permutation of the cells of the network, if the phase space of cells has dimension greater than one, the codimension one eigenvalue movements across the imaginary axis of the linearization of the $\mathcal{G}$-admissible vector fields at a fully symmetric equilibrium are independent of the network structure and are identical to the corresponding eigenvalue movements in general equivariant vector fields. The next example shows that this result is incomplete when we study codimension one Hopf bifurcation assuming that the phase space of the cells is one-dimensional.

Example 3.1 An example of the difficulties that can arise when we study codimension one Hopf bifurcation on coupled cell networks is given by the ring of twelve cells on the left of the Figure 2 with symmetry $\Gamma = \langle (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12), (1\ 4\ 7\ 10)(2\ 5\ 8\ 11)(3\ 6\ 9\ 12) \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_4$. We assume that the phase space of the cells is one-dimensional and that the network dynamics have a group invariant equilibrium x_0.

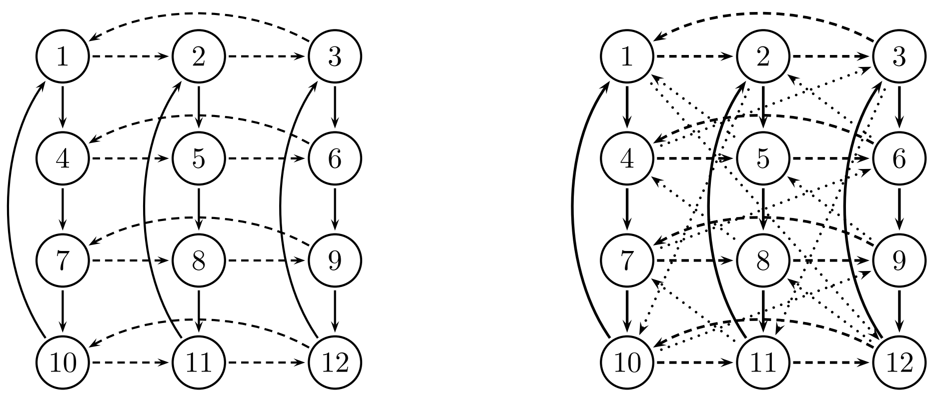


Figure 2: Networks $\mathcal{G}_1$ (left) and $\mathcal{G}_2$ (right) with $\mathbb{Z}_3 \times \mathbb{Z}_4$-symmetry.

Then, a general $\mathcal{G}_2$-admissible linear map (representing the Jacobian matrix at such an equilibrium for a $\mathcal{G}_2$-admissible system of ODEs) has the form

$$L = \begin{bmatrix} A & C & 0 & B \\ B & A & C & 0 \\ 0 & B & A & C \\ C & 0 & B & A \end{bmatrix}$$

where

$$A = \begin{bmatrix} a & 0 & c \\ c & a & 0 \\ 0 & c & a \end{bmatrix}, \ B = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}, \ C = \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & d \\ d & 0 & 0 \end{bmatrix}$$

and a, b, c, d are real-valued smooth functions of λ. The spectrum of L includes the eigenvalues

$$a - \frac{1}{2}c - \frac{\sqrt{3}}{2}d \pm i\left(b + \frac{\sqrt{3}}{2}c + \frac{1}{2}d\right) \ \text{ and } \ a - \frac{1}{2}c + \frac{\sqrt{3}}{2}d \pm i\left(b - \frac{\sqrt{3}}{2}c + \frac{1}{2}d\right).$$

Consider the network $\mathcal{G}_1$ $(d = 0)$ shown in Figure 2 (left). Then the spectrum of L includes

$$a - \frac{1}{2}c \pm i(b + \frac{\sqrt{3}}{2}c) \ \text{ and } \ a - \frac{1}{2}c \pm i\left(b - \frac{\sqrt{3}}{2}c\right).$$

as eigenvalues. Observe that for the network $\mathcal{G}_1$ if $c(0) = 2a(0)$ then we have two distinct pairs of complex eigenavlues crossing the imaginary axis for $\lambda = 0$. These are associated with two distinct $\mathbb{Z}_3 \times \mathbb{Z}_4$-irreducible subspaces. We say we have a codimension one Hopf/Hopf mode interaction given by $\pm i(b(0) + \sqrt{3}a(0))$ and $\pm i(b(0) - \sqrt{3}a(0))$. This situation cannot generically occur for the network $\mathcal{G}_2$. ◇

Generically, in general equivariant linear systems, in case of codimension one eigenvalue crossings with the imaginary axis, the phenomenon occurring in Example 3.1 for the network $\mathcal{G}_1$ is not expected. In Paiva [5] and Dias and Paiva [2] we describe the networks with abelian symmetry groups that can present this phenomenon in terms of the complex *characters* of the abelian groups. Essentially, each cell of the network $\mathcal{G}$ corresponds to a unique element of the abelian symmetry group Γ. With this identification, we associate with $\mathcal{G}$ a set $S \subseteq \Gamma$ that depends on the group elements corresponding to the present couplings. In case $\mathcal{G}$ is connected, S generates Γ. Moreover, the eigenvalues of the linearization of the $\mathcal{G}$-admissible vector fields at a fully symmetric equilibrium depend on the characters of Γ evaluated on the elements of S. We determine, using this fact, generic conditions involving the complex characters of Γ that permit in one parameter coupled cell networks of ODEs the occurrence of Hopf bifurcation associated to the crossings with the imaginary axis of two or more distinct pairs of complex eigenvalues of linearization.

The next step is to study this question for non abelian symmetry groups.

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