Secondary Bifurcations in Systems with All-to-All Coupling. Part II.

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Abstract

In a recent paper Dias and Stewart (Secondary Bifurcations in Systems with All-to-All Coupling, *Proc. R. Soc. Lond. A* (2003) **459**, 1969-1986.) studied the existence, branching geometry, and stability of secondary branches of equilibria in all-to-all coupled systems of differential equations, that is, equations that are equivariant under the permutation action of the symmetric group $\mathbf{S}_N$. They consider the most general cubic-order system of this type. Primary branches in such systems correspond to partitions of N into two parts p, q with $p+q=N$. Secondary branches correspond to partitions of N into three parts a, b, c with $a+b+c=N$. They prove that except in the case $a=b=c$ secondary branches exist and are (generically) globally unstable in the cubic-order system. In this work they realized that the cubic order system is too degenerate to provide secondary branches if $a=b=c$. In this paper we consider a general system of ordinary differential equations commuting with the permutation action of the symmetric group $\mathbf{S}_{3n}$ on $\mathbf{R}^{3n}$. Using singularity theory results, we find sufficient conditions on the coefficients of the fifth order truncation of the general smooth $\mathbf{S}_{3n}$-equivariant vector field for the existence of a secondary branch of equilibria near the origin with $\mathbf{S}_n \times \mathbf{S}_n \times \mathbf{S}_n$ symmetry of such system. Moreover, we prove that under such conditions the solutions are (generically) globally unstable except in the cases where two tertiary bifurcations occur along the secondary branch. In these cases, the instability result holds only for the equilibria near the secondary bifurcation points. We show an example where stability between tertiary bifurcation points on the secondary branch occurs.

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1 Introduction

The original motivation for this work came from evolutionary biology. Cohen and Stewart [1] introduced a system of $\mathbf{S}_N$-equivariant ordinary differential equations (ODEs) that models sympatric speciation as a form of spontaneous symmetry-breaking in a system with $\mathbf{S}_N$-symmetry. Elmhirst [3–5] studied the stability of the primary branches in such a model and also linked it to a biological specific model of speciation. Stewart *et al.* [8] made numerical studies of relatively concrete models. Here the population is aggregated into N discrete 'cells', with a

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vector x_j representing values of some phenotypic observable - the phenotype - the organisms form and behavior. If the initial population is monomorphic (single-species) then the system of ODEs representing the time-evolution of the phenotypes should be equivariant under the action of the symmetric group $\mathbf{S}_N$; that is, the model is an example of an *all-to-all coupled system*. Symmetry-breaking bifurcations of the system correspond to the splitting of the population into two or more distinct morphs (species).

Dias and Stewart [2] continue the study of the general cubic truncation of a center manifold reduction of a system of that type, which takes the form

$$\frac{dx_i}{dt} = \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 \tag{1.1}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, D \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i, the coordinates satisfy $x_1 + \cdots + x_N \equiv 0$ and $\pi_j = x_1^j + \cdots + x_N^j$ for $j = 2, 3$. Their study was motivated by numerical simulations showing jump bifurcations between primary branches. These jumps correspond to the loss of stability of the primary branches, see Stewart *et al.* [8]. Primary branches in such systems correspond to partitions of N into two parts p, q with $p + q = N$. Secondary branches correspond to partitions of N into three parts a, b, c with $a+b+c = N$. They remarked that the cubic-order system (1.1) is too degenerate to provide secondary branches if $a = b = c$. We focus our work in this case. We begin by observing why this case is special. When looking for steady-state solutions with symmetry $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$, we restrict the original $\mathbf{S}_N$-equivariant vector field, where $N = 3a$, to the fixed-point subspace of Σ. These equations are now equivariant under the normalizer of Σ inside $\mathbf{S}_N$. Moreover, the group of symmetries acting nontrivially on that fixed-point subspace is the quotient of that normalizer over Σ and it is isomorphic to $\mathbf{D}_3$, the dihedral group of order six. Solutions with Σ-symmetry of the original system correspond to solutions with trivial symmetry for the $\mathbf{D}_3$-symmetric restricted problem. Using singularity results for $\mathbf{D}_3$-equivariant bifurcation problems, see Golubitsky *et al.* [7], we find solutions of that type, by *local analysis near the origin*, assuming nondegeneracy conditions on the coefficients of the fifth order truncation of the system.

In this paper we consider a general smooth $\mathbf{S}_N$-equivariant system of ODEs posed on the $\mathbf{S}_N$-absolutely irreducible space, $V_1 = \{x \in \mathbf{R}^n : x_1 + \cdots + x_N = 0\}$, which takes the form

$$\frac{dx}{dt} = G(x, \lambda) \tag{1.2}$$

where

$$\begin{array}{rcl} G_i(x,\lambda) & = & \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 \\ & & +\ E(Nx_i^4 - \pi_4) + F(Nx_i^2\pi_2 - \pi_2^2) + Gx_i\pi_3 \\ & & +\ H(Nx_i^5 - \pi_5) + I(Nx_i^3\pi_2 - \pi_3\pi_2) + J(Nx_i^2\pi_3 - \pi_3\pi_2) + Lx_i\pi_4 + Mx_i\pi_2^2 \\ & & +\ \text{terms of degree } \geq 6 \end{array} \tag{1.3}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, \ldots, M \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i (and the coordinates satisfy $x_1 + \cdots + x_N = 0$). Also $\pi_j = x_1^j + \cdots + x_N^j$ for $j = 2, \ldots, 5$.

The aim of this paper is to study the existence, branching geometry and stability of secondary branches of equilibria with $\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ symmetry of the system (1.2) where G is defined by (1.3) and $N = 3a$. We do that from the point of view of singularity theory. Thus the analysis is local, that is, it is restricted to a neighborhood of the trivial equilibrium and for values of the bifurcation parameter λ near zero.

In Section 2 we review some results related to equivariant bifurcation theory of $\mathbf{S}_N$-symmetric systems. In particular, we obtain the general fifth order truncation of (1.2) of any smooth $\mathbf{S}_N$-

equivariant vector field posed on the $\mathbf{S}_N$-absolutely irreducible space V_1. We finish the section with a brief description of the singularity theory of $\mathbf{D}_3$-equivariant bifurcation problems.

In section 3 we suppose $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$. We look for secondary branches of steady-state solutions for the system (1.2) that are Σ-symmetric obtained by bifurcation from a primary branch of solutions with isotropy group (conjugate to) $\mathbf{S}_a \times \mathbf{S}_{2a}$. As mentioned above the restriction of (1.2) to the fixed-point subspace of Σ is $\mathbf{D}_3$-equivariant. $\mathbf{D}_3$-singularity results imply that the existence and stability (in $\mathrm{Fix}(\Sigma)$) of such a secondary branch of solutions near the origin depends only on certain nondegeneracy conditions on the coefficients of the fifth order truncation of the vector field G. Theorem 3.1 describes sufficient conditions on the coefficients of the vector field for the existence of a secondary branch of solutions of (1.2) with that symmetry. Corollary 3.2 describes the parameter regions of stability of those solutions (in $\mathrm{Fix}(\Sigma)$). Finally, in Section 4 we discuss the full stability of such a secondary branch. In Theorem 4.3 we obtain the expressions of the eigenvalues that determine the full stability of those solutions. We prove in Theorem 4.4 that these solutions are (generically) globally unstable except in the cases where two tertiary bifurcations occur along the secondary branch. In these cases, the instability result holds only for the equilibria near the secondary bifurcation points. We conclude with an example where two tertiary bifurcations occur along the secondary branch and the solutions along the branch between those tertiary bifurcation points are stable (Example 4.5).

2 Background

In this section we review some key points related to equivariant bifurcation theory of $\mathbf{S}_N$-symmetric systems and we give a brief description of the singularity theory results for $\mathbf{D}_3$-equivariant bifurcation problems.

2.1 Equivariant Bifurcation Theory for the Symmetric Group

For the basics of equivariant bifurcation theory see for example Golubitsky *et al.* [7, Chapters XII, XIII].

Let the symmetric group $\Gamma = \mathbf{S}_N$ act on $V = \mathbf{R}^N$ by permutation of coordinates

$$\rho(x_1, \ldots, x_N) = \left(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(N)}\right), \quad \rho \in \mathbf{S}_N,\ (x_1, \ldots, x_N) \in \mathbf{R}^N$$

and consider the restriction of this action onto the standard irreducible

$$V_1 = \{(x_1, x_2, \ldots, x_N) \in V : x_1 + x_2 + \cdots + x_N = 0\} \cong \mathbf{R}^{N-1}.$$

Note that the action of $\mathbf{S}_N$ on V_1 is *absolutely irreducible*. Thus the only matrices commuting with the action of Γ on V_1 are the scalar multiples of the identity. Moreover,

$$V = \{(x_1, x_1, \ldots, x_1) : x_1 \in \mathbf{R}\} \oplus V_1$$

where the action of $\mathbf{S}_N$ on $\{(x_1, x_1, \ldots, x_1) :\ x_1 \in \mathbf{R}\}$ is trivial.

We say that $G :\ V_1 \to V_1$ *commutes* with the action of Γ on V_1, or it is Γ-*equivariant*, if

$$G(\rho x) = \rho G(x)$$

for all $\rho \in \Gamma$ and $x \in V_1$. Also, $p :\ V_1 \to \mathbf{R}$ is Γ-*invariant* if $p(\rho x) = p(x)$ for all $\rho \in \Gamma, x \in V_1$. Given $x \in V_1$, the subgroup of Γ

$$\Sigma_x = \{\gamma \in \Gamma\ :\ \gamma x = x\}$$

is the *isotropy subgroup* of x. The *fixed-point space* of a subgroup $\Sigma \subseteq \Gamma$ is the subspace of V_1 defined by

$$\mathrm{Fix}(\Sigma) = \{x \in V_1 \; : \; \gamma x = x, \; \forall \, \gamma \in \Sigma\}.$$

An isotropy subgroup of Γ is said to be *axial* if it has a one-dimensional fixed-point space. If $G : \; V_1 \to V_1$ is Γ-equivariant and Σ is a subgroup of Γ we have

$$G(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma).$$

Consider a system of ODEs

$$\frac{dx}{dt} = G(x, \lambda) \tag{2.4}$$

where $x \in V_1$, the vector field $G : V_1 \times \mathbf{R} \to V_1$ is smooth, and $\lambda \in \mathbf{R}$ is a bifurcation parameter. Suppose that G commutes with the action of Γ on V_1. As $\mathrm{Fix}(\Gamma) = \{0\}$, it follows that $G(0, \lambda) \equiv 0$. Thus $x = 0$ is an equilibrium of (2.4) for all parameter values of λ. Moreover, as the action of Γ on V_1 is absolutely irreducible and the Jacobian of G at $(0, \lambda)$, $(dG)_{(0,\lambda)}$, commutes with Γ, it follows that $(dG)_{(0,\lambda)}$ is a scalar multiple of the identity. Thus $(dG)_{(0,\lambda)} = c(\lambda)\mathrm{Id}_{V_1}$ where $c : \; \mathbf{R} \to \mathbf{R}$ is smooth. Suppose that $(dG)_{(0,\lambda)}$ is singular, say at $\lambda = 0$. Then we have that $c(0) = 0$ and $(dG)_{(0,0)} = 0$. By the Equivariant Branching Lemma [7, Theorem XIII3.3], if $c^{'}(0) \neq 0$, then for each axial subgroup of Γ there exists a unique branch of equilibria of (2.4) bifurcating from the trivial equilibrium at $\lambda = 0$ with that symmetry. Any such branch is called a *primary branch*.

We end this review describing the isotropy subgroups of Γ and the general form of G.

Isotropy Subgroups of the Symmetric Group for the Natural Representation

The isotropy subgroups of $\mathbf{S}_N$ for the action on V_1 are the same isotropy subgroups of $\mathbf{S}_N$ for the action on V, but the the dimension of every fixed-point subspace is reduced by one. In order to compute the isotropy subgroups Σ_x of $\mathbf{S}_N$ acting on V, we partition $\{1, \ldots, N\}$ into disjoint blocks $B_1, \ldots, B_k$ with the property that $x_i = x_j$ if and only if i, j belong to the same block. Let $b_l = |B_l|$. Then

$$\Sigma_x = \mathbf{S}_{b_1} \times \cdots \times \mathbf{S}_{b_k}$$

where $\mathbf{S}_{b_l}$ is the symmetric group on the block B_l. Up to conjugacy, we may assume that

$$B_1 = \{1, \ldots, b_1\}, \; B_2 = \{b_1 + 1, \ldots, b_1 + b_2\}, \; \ldots, \; B_k = \{b_1 + b_2 + \cdots + b_{k-1} + 1, \ldots, N\}$$

where $b_1 \leq b_2 \leq \cdots \leq b_k$. Therefore, conjugacy classes of isotropy subgroups of $\mathbf{S}_N$ are in one-to-one correspondence with partitions of N into nonzero natural numbers arranged in ascending order. If Σ corresponds to a partition of N into k blocks, then the fixed-point subspace in V of Σ has dimension k, and so in V_1 has dimension $k - 1$. In particular, the axial subgroups of $\mathbf{S}_N$ are the groups $\mathbf{S}_p \times \mathbf{S}_q$ where $p + q = N$.

General $\mathbf{S}_N$-Equivariant Mappings

The ring of the smooth Γ-invariants on V is generated by $\pi_k = x_1^k + x_2^k + \cdots + x_N^k$ where $k = 1, \ldots, N$. Denote by $[x_1^k] = [x_1^k, x_2^k, \ldots, x_N^k]^t$, for $k = 0, \ldots, N-1$. Then the module of the Γ-equivariant smooth mappings from V to V is generated over the ring of the smooth Γ-invariants by $[x_1^k]$ for $k = 0, \ldots, N-1$. For a detailed discussion see Golubitsky and Stewart [6, Chapter

1, Section 5.]. It follows then that if $G:\ V \to V$ is smooth and commutes with Γ then it has the following form:

$$G(x) = \sum_{k=0}^{N-1} p_k(\pi_1, \ldots, \pi_N)[x_1^k] \tag{2.5}$$

where each $p_k:\ \mathbf{R}^N \to \mathbf{R}$, for $k = 0, \ldots, N-1$ is a smooth function.

From (2.5) we obtain the fifth order truncation of the Taylor expansion of G on V. By imposing the relation $\pi_1 = 0$ and then projecting the result onto V_1 we obtain (1.3) where we are taking G such that $(dG)_{(0,\lambda)} = \lambda \mathrm{Id}_{V_1}$. Recall that the Γ-equivariance of G implies that $(dG)_{(0,\lambda)}$ commutes with Γ and so it has the form $c(\lambda)\mathrm{Id}_{V_1}$ where $c : \mathbf{R} \to \mathbf{R}$ is smooth. We are taking the approximation $c(\lambda) \sim \lambda$ since we are assuming that the trivial equilibrium of (1.2) is stable for $\lambda < 0$ and unstable for $\lambda > 0$ and the study done in this paper is by local analysis, for parameter values of λ near zero. We show in Section 3 that this fifth order truncation captures the presence of a secondary branch of equilibria of (1.2) with symmetry $\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ when $N = 3a$ and its stability by bifurcation from primary branches with axial symmetry.

2.2 $\mathbf{D}_3$-Equivariant Bifurcation Problem

We briefly describe the characterization of $\mathbf{D}_3$-equivariant bifurcation problems obtained by Golubitsky *et al.* [7, Sections XIII5, XIV4, XV3].

Consider the standard action of $\mathbf{D}_3$ on $\mathbf{C} \equiv \mathbf{R}^2$ generated by

$$kz = \overline{z}, \qquad \xi z = e^{2\pi i/3} z \tag{2.6}$$

where $\xi = 2\pi/3$, $\mathbf{D}_3 = \langle k, \xi \rangle$ and $z \in \mathbf{C}$. Up to conjugacy, the only isotropy subgroup of $\mathbf{D}_3$ with one-dimensional fixed-point subspace is $\mathbf{Z}_2(k) = \{1, k\}$.

If $g : \mathbf{C} \times \mathbf{R} \to \mathbf{C}$ is smooth and commutes with this action of $\mathbf{D}_3$ on $\mathbf{C}$ then

$$g(z, \lambda) = p(u, v, \lambda)z + q(u, v, \lambda)\overline{z}^2 \tag{2.7}$$

where $u = z\overline{z}$, $v = z^3 + \overline{z}^3$ and $p, q : \mathbf{R}^3 \to \mathbf{R}$ are smooth functions. Suppose $p(0, 0, 0) = 0$ and so the linearization of (2.7) at $(z, \lambda) = (0, 0)$ is zero. Assume the genericity hypothesis of the Equivariant Branching Lemma [7] $p_\lambda(0, 0, 0) \neq 0$ and the second nondegeneracy hypothesis $q(0, 0, 0) \neq 0$. We have then that the only (local) solution branches to $g = 0$ obtained by bifurcation from $(z, \lambda) = (0, 0)$ are those obtained using the Equivariant Branching Lemma. That is, those that have $\mathbf{Z}_2(k)$-symmetry or conjugate. Since there is a nontrivial $\mathbf{D}_3$-equivariant quadratic $\overline{z}^2$, by [7, Theorem XIII4.4], generically, the branch of $\mathbf{Z}_2(k)$ solutions is unstable. Thus in order to find asymptotically stable solutions to a $\mathbf{D}_3$-equivariant bifurcation problem by a *local analysis*, we must consider the degeneracy hypothesis $q(0, 0, 0) = 0$ and apply unfolding theory.

We state a normal form for the degenerate $\mathbf{D}_3$-equivariant bifurcation problem for which $q(0, 0, 0) = 0$. We begin by specifying the lower order terms in p and q as follows:

$$\begin{aligned} p(u, v, \lambda) &= \tilde{A}u + \tilde{B}v + \tilde{\alpha}\lambda + \cdots \\ q(u, v, \lambda) &= \tilde{C}u + \tilde{D}v + \tilde{\beta}\lambda + \cdots \end{aligned} \tag{2.8}$$

A $\mathbf{D}_3$-equivariant bifurcation problem g satisfying $p(0, 0, 0) = 0 = q(0, 0, 0)$ is called *nondegenerate* if

$$\tilde{\alpha} \neq 0, \quad \tilde{A} \neq 0, \quad \tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A} \neq 0, \quad \tilde{A}\tilde{D} - \tilde{B}\tilde{C} \neq 0. \tag{2.9}$$

Theorem 2.1 [7] *Let g be a $\mathbf{D}_3$-equivariant bifurcation problem. Assume that $p(0,0,0) = 0 = q(0,0,0)$ and g is nondegenerate. Then g is $\mathbf{D}_3$-equivalent to the normal form*

$$h(z,\lambda) = (\epsilon u + \delta\lambda)z + (\sigma u + mv)\overline{z}^2 \tag{2.10}$$

where $\epsilon = \text{sgn } \tilde{A}$, $\delta = \text{sgn } \tilde{\alpha}$, $\sigma = \text{sgn } (\tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A})\text{sgn } \tilde{\alpha}$, and $m = \text{sgn } (\tilde{A})(\tilde{A}\tilde{D} - \tilde{B}\tilde{C})\tilde{\alpha}^2/(\tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A})^2$.

Proof: See Golubitsky *et al.* [7, Theorem XIV4.4]. □

We consider now the bifurcation diagram of bifurcation problems of the type $\dot{z} + h(z,\lambda) = 0$ where h is given by (2.10). The Equivariant Branch Lemma guarantees that there is a unique branch of solutions with $\mathbf{Z}_2(\kappa)$-symmetry that bifurcate from the trivial equilibrium at $\lambda = 0$. Setting $\delta = -1$ and $\epsilon = 1$ in (2.10) so that the trivial solution is asymptotically stable for $\lambda < 0$ and the $\mathbf{Z}_2(\kappa)$-symmetric solutions bifurcate supercritically, we obtain Figure 1 (a). Note that the branch of $\mathbf{Z}_2(\kappa)$-solutions splits into two orbits of solutions. The sign of $\sigma = \pm 1$ determines which is stable.

The next theorem states a universal $\mathbf{D}_3$-unfolding for the $\mathbf{D}_3$-normal form of Theorem 2.1.

Theorem 2.2 [7] *The $\mathbf{D}_3$-normal form $h(z,\lambda) = (\epsilon u + \delta\lambda)z + (\sigma u + mv)\overline{z}^2$ where $\epsilon, \delta, \sigma = \pm 1$ and $m \neq 0$, obtained in Theorem 2.1, has $\mathbf{D}_3$-codimension 2 and modality 1. A universal unfolding of h is*

$$H(z,\lambda,\mu,\alpha) = (\epsilon u + \delta\lambda)z + (\sigma u + \mu v + \alpha)\overline{z}^2 \tag{2.11}$$

where (μ,α) varies near $(m,0)$.

Proof: See Golubitsky *et al.* [7, Theorem XV3.3 (b)]. □

We show in Figure 1 (b) the bifurcation diagram for $\dot{z} + H(z,\lambda,\mu,\alpha) = 0$ where $\delta = -1$, $\epsilon = 1$, $\sigma\alpha < 0$ and $\mu > 0$ in (2.11). Observe the change of stability of the $\mathbf{Z}_2(\kappa)$-symmetric solutions along the branch and the appearance (when $\sigma\alpha < 0$) of a secondary branch of solutions with trivial symmetry which are asymptotically stable if $\mu > 0$. Figures 1 (a) and (b) appear in [7, Figures XV4.1 (b), XV4.2 (c)] with opposite signs for the eigenvalues since the authors consider the eigenvalues of $(dh)_{(z,\lambda)}$ and $(dH)_{(z,\lambda)}$, while we show in Figure 1 the signs of the eigenvalues of $-(dh)_{(z,\lambda)}$ and $-(dH)_{(z,\lambda)}$.

3 Existence of Secondary Branches

Consider (1.2) where G is defined by (1.3) and suppose $N = 3a$ where a is a positive integer. Let $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ and observe that

$$\text{Fix}(\Sigma) = \{(\underbrace{-x-y,\ldots}_{a};\underbrace{y,\ldots}_{a};\underbrace{x,\ldots}_{a}) : x, y \in \mathbf{R}\}$$

which is two-dimensional. We look for secondary branches of equilibria of (1.2) with symmetry Σ by bifurcation from primary branches with axial isotropy $\mathbf{S}_p \times \mathbf{S}_q$ where $p + q = N$ and $\Sigma \subset \mathbf{S}_p \times \mathbf{S}_q$. We do that by local analysis near the origin using the singularity results stated in Section 2.2. Any such secondary branch must live in the fixed-point subspace $\text{Fix}(\Sigma)$. Moreover, the axial subgroups of $\mathbf{S}_N$ containing Σ are

$$\Sigma_1 = \mathbf{S}_{\{1,\ldots,a\}} \times \mathbf{S}_{\{a+1\ldots,N\}}, \quad \Sigma_2 = \mathbf{S}_{\{1,\ldots,a,2a+1,\ldots,N\}} \times \mathbf{S}_{\{a+1,\ldots,2a\}}, \quad \Sigma_3 = \mathbf{S}_{\{1,\ldots,2a\}} \times \mathbf{S}_{\{2a+1,\ldots,N\}}$$

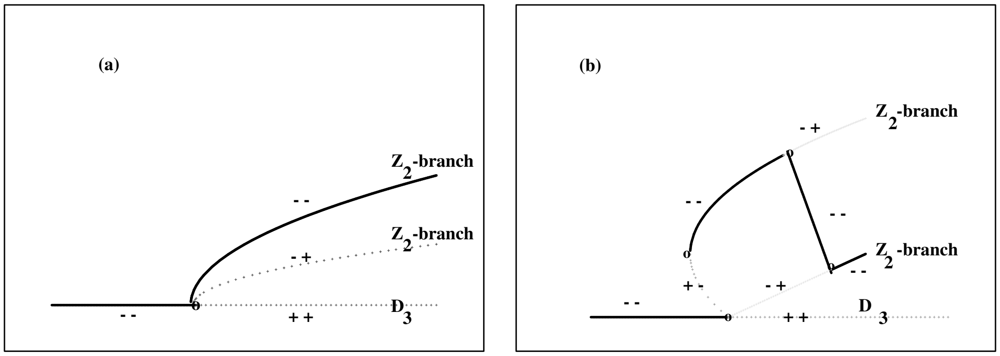

Figure 1: (a) Unperturbed $\mathbf{D}_3$-symmetric bifurcation diagram for $\dot{z} + h(z, \lambda) = 0$, where h is the normal form $h(z, \lambda) = (u - \lambda)z + (\sigma u + mv)\overline{z}^2$, $\sigma = \pm 1$ and $m \neq 0$. [7, Figure XV4.1 (b)]. (b) Bifurcation diagram for $\dot{z} + H(z, \lambda) = 0$, where H is defined by $H(z, \lambda, \mu, \alpha) = (u - \lambda)z + (\sigma u + \mu v + \alpha)\overline{z}^2$, $\sigma = 1$, $\alpha < 0$ (or $\sigma = -1$, $\alpha > 0$) and $\mu > 0$ [7, Figure XV4.2 (c)].

and the corresponding one-dimensional fixed-point subspaces are

$$\mathrm{Fix}(\Sigma_1) = \{(\underbrace{-2x, \ldots}_{a}; \underbrace{x, \ldots; x, \ldots}_{2a}) : x \in \mathbf{R}\}, \ \mathrm{Fix}(\Sigma_2) = \{(\underbrace{x, \ldots}_{a}; \underbrace{-2x, \ldots}_{a}; \underbrace{x, \ldots}_{a}) : x \in \mathbf{R}\},$$

$$\mathrm{Fix}(\Sigma_3) = \{(\underbrace{-\frac{1}{2}x, \ldots, -\frac{1}{2}x}_{2a}; \underbrace{x, \ldots, x}_{a}) : x \in \mathbf{R}\}.$$

Equations (1.2) where G is defined by (1.3) restricted to $\mathrm{Fix}(\Sigma)$ are

$$\begin{aligned}
\frac{dx}{dt} &= \lambda x + B(Nx^2 - \pi_2) + C(Nx^3 - \pi_3) + Dx\pi_2 + E(Nx^4 - \pi_4) + F(Nx^2\pi_2 - \pi_2^2) \\
&\quad + Gx\pi_3 + H(Nx^5 - \pi_5) + I(Nx^3\pi_2 - \pi_3\pi_2) + J(Nx^2\pi_3 - \pi_3\pi_2) + Lx\pi_4 + Mx\pi_2^2 \\
&\quad + \text{terms of degree } \geq 6, \\
\frac{dy}{dt} &= \lambda y + B(Ny^2 - \pi_2) + C(Ny^3 - \pi_3) + Dy\pi_2 + E(Ny^4 - \pi_4) + F(Ny^2\pi_2 - \pi_2^2) \\
&\quad + Gy\pi_3 + H(Ny^5 - \pi_5) + I(Ny^3\pi_2 - \pi_3\pi_2) + J(Ny^2\pi_3 - \pi_3\pi_2) + Ly\pi_4 + My\pi_2^2 \\
&\quad + \text{terms of degree } \geq 6,
\end{aligned} \tag{3.12}$$

where $\pi_i = N[(-x - y)^i + y^i + x^i)]/3$ for $i = 2, 3, 4, 5$.

Since $\Sigma_1, \Sigma_2, \Sigma_3$ are axial subgroups of $\mathbf{S}_N$ containing Σ, by the Equivariant Branching Lemma, generically there exist branches of equilibria of (3.12) (and so of (1.2)) with isotropy subgroups $\Sigma_1, \Sigma_2, \Sigma_3$. The solutions of equations (3.12) with Σ_1-symmetry satisfy $y = x$; those with Σ_2-symmetry satisfy $y = -2x$, and finally those with Σ_3-symmetry satisfy $y = -x/2$.

Observe that equations (3.12) correspond to the equations (1.2) restricted to $\mathrm{Fix}(\Sigma)$ in coordinates x, y corresponding to the basis $B = (B_1, B_2)$ of the fixed-point subspace $\mathrm{Fix}(\Sigma)$, where $B_1 = (-1, \ldots, -1; 0, \ldots, 0; 1, \ldots, 1)$ and $B_2 = (-1, \ldots, -1; 1, \ldots, 1; 0, \ldots, 0)$. Moreover, those equations are equivariant under the quotient group $N(\Sigma)/\Sigma$ where $N(\Sigma)$ is the normalizer of Σ in $\mathbf{S}_N$. Thus $N(\Sigma)/\Sigma \cong \mathbf{D}_3$ where $\mathbf{D}_3$ is the dihedral group of order 6.

We consider now the basis $b = \left(-\frac{2\sqrt{3}}{3}B_1 + \frac{\sqrt{3}}{3}B_2, B_2\right)$ of $\mathrm{Fix}(\Sigma)$ and denote the corresponding coordinates by X, Y. Thus $X = \left(-\sqrt{3}x\right)/2,\ Y = x/2 + y$. Identifying $z = X + iY$, we have then that the action of $N(\Sigma)/\Sigma \cong \mathbf{D}_3$ on z is given by (2.6). Moreover, equations (3.12) yield the following equation in z:

$$\frac{dz}{dt} + g(z,\lambda) = 0, \tag{3.13}$$

where

$$\begin{aligned}
g(z,\lambda) &= p(u,v,\lambda)z + q(u,v,\lambda)\overline{z}^2, \\
p(u,v,\lambda) &= -\lambda - \frac{N}{3}(3C+2D)u + \frac{\sqrt{3}}{9}N\,(E+G)\,v - \frac{N}{9}(9H+6NI+6L+4NM)u^2 \\
&\quad + \text{terms of degree } \geq 5, \qquad (3.14) \\
q(u,v,\lambda) &= \frac{\sqrt{3}}{3}NB + \frac{\sqrt{3}}{9}N(3E+2NF)u - \frac{N}{9}(H+NJ)v + \text{terms of degree } \geq 4,
\end{aligned}$$

$u = z\overline{z}$ and $v = z^3 + \overline{z}^3$.

Theorem 3.1 *Suppose that* $N = 3a$ *and* $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ *and consider (1.2) where* G *is defined by (1.3). Assume the following conditions on the coefficients of the terms of degree lower or equal to five of* G*:*

$$3C + 2D < 0,\ (3C+2D)(H+NJ) - (E+G)(3E+2NF) \neq 0 \tag{3.15}$$

and

$$B(3E + 2NF) < 0. \tag{3.16}$$

Then for sufficiently small values of $B \neq 0$*, equations (3.12) (and so (1.2)) have a secondary branch of equilibria with symmetry* Σ *bifurcating from the primary branches with symmetry* Σ_i*. This is described by:*

$$\begin{aligned}
\lambda \quad &+ \frac{N}{3}(3C+2D)(x^2+y^2+xy) - N(E+G)(xy^2+x^2y) \\
&+ \frac{N}{9}(9H+6NI+6L+4NM)(x^2+y^2+xy)^2 + \text{terms of degree } \geq 5 = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.17) \\
B \quad &+ \frac{1}{3}(3E+2NF)(x^2+y^2+xy) - (H+NJ)(x^2y+xy^2) + \text{terms of degree } \geq 4 = 0.
\end{aligned}$$

Proof: The equivariance of equations (3.12) under the group $N(\Sigma)/\Sigma \cong \mathbf{D}_3$ enables us the choice of coordinates X, Y in $\mathrm{Fix}(\Sigma)$ such that the action of $N(\Sigma)/\Sigma$ on $z \equiv X + iY$ is given by (2.6) and equations (3.12) correspond to one equation in z given by (3.13) where g is defined by (3.14). Thus we obtain $\dot{z} + g(z,\lambda) = 0$ where $g(z,\lambda) = p(u,v,\lambda)z + q(u,v,\lambda)\overline{z}^2$ and

$$\begin{aligned}
p(u,v,\lambda) &= -\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2 + \text{terms of degree } \geq 5, \\
q(u,v,\lambda) &= \beta_4 + \beta_5 u + \beta_6 v + \text{terms of degree } \geq 4,
\end{aligned} \tag{3.18}$$

where

$$\beta_1 = -\frac{N}{3}(3C+2D),\ \beta_2 = \frac{\sqrt{3}}{9}N(E+G),\ \beta_3 = -\frac{N}{9}(9H+6NI+6L+4NM),$$
$$\beta_4 = \frac{\sqrt{3}}{3}NB,\ \beta_5 = \frac{\sqrt{3}}{9}N(3E+2NF),\ \beta_6 = -\frac{N}{9}(H+NJ). \tag{3.19}$$

Note that $p(0,0,0) = 0$ and $p_\lambda(0,0,0) \neq 0$. Thus by the Equivariant Branching Lemma there are branches of steady-state solutions with symmetry $\mathbf{Z}_2$ of equation (3.13) obtained by bifurcation from the trivial equilibrium $z = 0$ at $\lambda = 0$. These correspond to the primary branches with Σ_i-symmetry, for $i = 1, 2, 3$, of equations (3.12) (and so of (1.2)). Observe that solutions of (1.2) with Σ-symmetry correspond to solutions of the $\mathbf{D}_3$-symmetric equation (3.13) with trivial symmetry. Also, note that

$$q(0,0,0) = \beta_4 = \frac{\sqrt{3}}{3}NB$$

and so $q(0,0,0) = 0$ if and only if $B = 0$.

We prove the existence of a secondary branch of solutions with trivial symmetry bifurcating from the primary branches with $\mathbf{Z}_2$-symmetry of (3.13) by showing that g as defined by (3.14) is one of the perturbations contained in the universal unfolding H in Theorem 2.2, where a secondary branch of trivial solutions exist bifurcating from the primary branches with $\mathbf{Z}_2$-symmetry. We do that by considering g with $B = 0$ and finding conditions on the corresponding coefficients such that it is $\mathbf{D}_3$-equivalent to the normal form h of Theorem 2.1.

Comparing (2.8) with (3.18) where β_4 is set to zero (thus $B = 0$), we obtain

$$\tilde{\alpha} = -1, \qquad \tilde{A} = \beta_1, \qquad \tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A} = -\beta_5, \qquad \tilde{A}\tilde{D} - \tilde{B}\tilde{C} = \beta_1\beta_6 - \beta_2\beta_5.$$

Thus g with $B = 0$ is nondegenerate if

$$\beta_1 \neq 0, \quad \beta_5 \neq 0, \quad \beta_1\beta_6 - \beta_2\beta_5 \neq 0$$

and in that case, by Theorem 2.1, it is $\mathbf{D}_3$-equivalent to

$$h(z,\lambda) = (u-\lambda)z + (\sigma u + mv)\overline{z}^2, \tag{3.20}$$

where

$$\sigma = \operatorname{sgn} \beta_5,\ m = \frac{\beta_1\beta_6 - \beta_2\beta_5}{\beta_5^2}.$$

Note that the condition $3C + 2D < 0$ implies that $\epsilon = 1 = \operatorname{sgn} \beta_1$ in the equation (2.10).

By Theorem 2.2, the function g for $\beta_4 \sim 0$ (thus $B \sim 0$), corresponds to a perturbation of (3.20) of the type

$$H(z,\lambda,\mu,\alpha) = (u-\lambda)z + (\sigma u + \mu v + \alpha)\overline{z}^2 \tag{3.21}$$

where (μ, α) varies near $(m, 0)$. Moreover, if condition (3.16) is satisfied and so $\beta_4\beta_5 < 0$, then g corresponds to a perturbation of the type as above where $\alpha\sigma < 0$ and so there is a secondary branch of solutions of trivial symmetry for $dz/dt + H(z,\lambda,\mu,\alpha) = 0$ varying λ and bifurcating from the $\mathbf{Z}_2$-branch of solutions. Observe that solutions of $H(z,\lambda,\mu,\alpha) = 0$ with trivial symmetry satisfy $\mathrm{Re}(z^3) \neq 0$ and so solving $H(z,\lambda,\mu,\alpha) = 0$ is equivalent to solving $u - \lambda = 0$, $\sigma u + \mu v + \alpha = 0$. Now for small enough values of $\alpha \neq 0$ the solutions of $\sigma u + \mu v + \alpha = 0$ (near the origin) form a circlelike curve in the XY-plane of radius approximately $\sqrt{|\alpha/\sigma|}$. It follows that in the (X, Y, λ)-space this curve intersects the $Y = 0$ plane at two points (X^-, λ^-)

and (X^+, λ^+) where $X^- < 0 < X^+$ that correspond to the intersection points of the branch with trivial isotropy (for the $\mathbf{D}_3$-problem) and solutions with isotropy $\mathbf{Z}_2$.

The branch of steady-state solutions with trivial symmetry for the $\mathbf{D}_3$-symmetric bifurcation problem $\dot{z} + g(z, \lambda) = 0$ is then given by the equations

$$
\begin{aligned}
p(u, v, \lambda) &= -\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2 + \text{terms of degree } \geq 5 = 0, && (3.22)\\
q(u, v, \lambda) &= \beta_4 + \beta_5 u + \beta_6 v + \text{terms of degree } \geq 4 = 0. && (3.23)
\end{aligned}
$$

Now recalling that $z = X + iY$ where $X = \left(-\sqrt{3}x\right)/2,\ Y = x/2 + y$, equations (3.22) and (3.23) in the x, y coordinates are given by (3.17). $\square$

Corollary 3.2 *Suppose the conditions of Theorem 3.1 and assume that*

$$(3C + 2D)(H + JN) - (E + G)(3E + 2FN) > 0. \qquad (3.24)$$

Then the secondary branch of solutions with Σ-symmetry of (1.2) where G is defined by (1.3) and guaranteed by Theorem 3.1 is stable in $\mathrm{Fix}(\Sigma)$.

Proof: We recall equations (3.13), (3.14) and the notation of (3.18), (3.19) in the proof of Theorem 3.1 corresponding to the equations (1.2) restricted to $\mathrm{Fix}(\Sigma)$. Equations (3.22) and (3.23) describe the secondary branch in the $z = X + iY$ coordinate. The stability of these solutions is determined by

$$
\begin{aligned}
\mathrm{tr}\ \left((dg)_{(z,\lambda)}\right) &= 2\left[u p_u + \tfrac{v}{2}(3p_v + q_u) + 3u^2 q_v\right]\\
&= 2\left[\beta_1 u + (3\beta_2 + \beta_5)\tfrac{v}{2} + (2\beta_3 + 3\beta_6)u^2\right] + \text{ terms of degree} \geq 5,
\end{aligned}
$$

$$
\begin{aligned}
\det\ \left((dg)_{(z,\lambda)}\right) &= 3(p_v q_u - p_u q_v)(z^3 - \overline{z}^3)^2\\
&= 12(\beta_1\beta_6 - \beta_2\beta_5 + 2\beta_3\beta_6 u)\left(\mathrm{Im}\ (z^3)\right)^2 + \text{ terms of degree} \geq 10
\end{aligned}
$$

and so the solutions (near the origin) are stable if $\beta_1 > 0$ and $\beta_1\beta_6 - \beta_2\beta_5 > 0$, that is, if conditions (3.15) and (3.24) are satisfied.

The same conclusion can be derived from the fact that $\mathbf{D}_3$-equivalence preserves the asymptotic stability of the solutions with trivial symmetry [7, Section XV4]. Note that (3.14) corresponds to a perturbation of (3.20) of the type (3.21) where $\alpha\sigma < 0$ (by (3.16)). Thus the secondary branch is stable if $\mu > 0$. As μ varies near m and $\mathrm{sgn}(m) = \mathrm{sgn}(\beta_1\beta_6 - \beta_2\beta_5)$, if condition (3.24) is satisfied then $\beta_1\beta_6 - \beta_2\beta_5 > 0$ and so $m > 0$. Thus the local bifurcation diagram of equation (3.13) corresponds to the bifurcation diagram of $dz/dt + H(z, \lambda, \mu, \alpha) = 0$, where H is defined by (3.21), that appear in Figure 1 (b). Therefore the secondary branch of steady-state solutions with trivial symmetry bifurcating from the branch of steady-state solutions with $\mathbf{Z}_2(k)$-symmetry is stable. $\square$

Observe that Theorem 3.1 guarantees the existence of the secondary branch if $q(0, 0, 0)$ is sufficiently small. We finish this section by considering (1.2) truncated to the fifth order truncation. We specify in the next corollary a sufficient condition on the coefficients of the truncated vector field that guarantees $q(0, 0, 0)$ to be sufficiently small.

Corollary 3.3 *Suppose that $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ and consider*

$$\frac{dx}{dt} = G(x, \lambda) \tag{3.25}$$

where G is defined by

$$\begin{aligned} G_i(x,\lambda) \;=\; & \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 \\ & + E(Nx_i^4 - \pi_4) + F(Nx_i^2\pi_2 - \pi_2^2) + Gx_i\pi_3 \\ & + H(Nx_i^5 - \pi_5) + I(Nx_i^3\pi_2 - \pi_3\pi_2) + J(Nx_i^2\pi_3 - \pi_3\pi_2) + Lx_i\pi_4 + Mx_i\pi_2^2 \end{aligned} \tag{3.26}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, \ldots, M \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i (and the coordinates satisfy $x_1 + \cdots + x_N = 0$). Also $\pi_j = x_1^j + \cdots + x_N^j$ for $j = 2, \ldots, 5$. Assume the following conditions on the coefficients of G:

$$3C + 2D < 0,\;\; (3C+2D)(H+NJ) - (E+G)(3E+2NF) \neq 0,\;\; B(3E+2NF) < 0, \tag{3.27}$$

and

$$H + NJ \neq 0. \tag{3.28}$$

Then for small values of $B \neq 0$ such that

$$\frac{3B}{3E+2NF} + \frac{(3E+2NF)^2}{9(H+NJ)^2} > 0 \tag{3.29}$$

equations (3.25) have a secondary branch of equilibria with symmetry Σ bifurcating from the primary branches with symmetry Σ_i. This is described by:

$$\begin{aligned} \lambda \quad & + \frac{N}{3}(3C+2D)(x^2+y^2+xy) - N(E+G)(xy^2+x^2y) \\ & + \frac{N}{9}(9H+6NI+6L+4NM)(x^2+y^2+xy)^2 = 0, \\ B \quad & + \frac{1}{3}(3E+2NF)(x^2+y^2+xy) - (H+NJ)(x^2y+xy^2) = 0. \end{aligned} \tag{3.30}$$

Proof: As before, we take coordinates X, Y in $\mathrm{Fix}(\Sigma)$ so that equations (3.25) restricted to $\mathrm{Fix}(\Sigma)$ yield one equation in $z \equiv X + iY$. This is given by $dz/dt + g(z,\lambda) = 0$, where $g(z,\lambda) = p(u,v,\lambda)z + q(u,v,\lambda)\overline{z}^2$, $p(u,v,\lambda) = -\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2$, $q(u,v,\lambda) = \beta_4 + \beta_5 u + \beta_6 v$ and $\beta_1, \ldots, \beta_6$ are defined by (3.19). By Theorem 3.1, if the conditions (3.27) are satisfied, provided $B \neq 0$ is sufficiently small, (3.25) has a secondary branch of solutions that correspond to the solutions of the $\mathbf{D}_3$-equivariant problem $dz/dt + g(z,\lambda) = 0$ with trivial symmetry. Moreover, the branch is described by the following two equations in z:

$$-\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2 = 0, \tag{3.31}$$

$$\beta_4 + \beta_5 u + \beta_6 v = 0 \tag{3.32}$$

and recall that solutions with $\mathbf{Z}_2(\kappa)$-symmetry satisfy $Y = 0$.

Set

$$r(X) = \frac{\beta_4}{\beta_5} + X^2 + 2\frac{\beta_6}{\beta_5}X^3$$

and recall that $\beta_4\beta_5 < 0$ by (3.27). We describe now generic conditions on the β_i's such that $r(X)$ has three real zeros. We have that

$$r'(X) = 2X\left(1 + \frac{3\beta_6}{\beta_5}X\right)$$

Assume (3.28) and so $\beta_6 \neq 0$. As $r(0) = \beta_4/\beta_5 < 0$ by (3.27), if $r(-\beta_5/(3\beta_6)) > 0$ we have that r has three real solutions, X^-, X^+, X^*, where $X^- < 0 < X^+$ and $X^* < -\beta_5/(3\beta_6) < X^-$ if $\beta_6/\beta_5 > 0$, or $X^* > -\beta_5/(3\beta_6) > X^+$ if $\beta_6/\beta_5 < 0$. Thus if $r(-\beta_5/(3\beta_6)) > 0$ then in the (X, Y, λ)-space the curve given by (3.32) intersects the $Y = 0$ plane at two points (X^-, λ^-) and (X^+, λ^+) where $X^- < 0 < X^+$ that correspond to the intersection points of the branch with trivial isotropy. Condition $r(-\beta_5/(3\beta_6)) > 0$ is equivalent to (3.29). □

4 Secondary Branches: Full Stability

In this section we study the stability of the solutions of the secondary branch obtained in Theorem 3.1 in the transversal directions to $\mathrm{Fix}(\Sigma)$. As before we assume that $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$.

Given an equilibrium X_0 of (1.2) in the Σ-branch obtained in Theorem 3.1, in order to analyze the stability of this solution, we need to compute the eigenvalues of the Jacobian $(dG)_{X_0}$. We use now the decomposition of V_1 into isotypic components for the action of Σ to block-diagonalize the Jacobian on V_1. We have

$$V_1 = \mathrm{Fix}(\Sigma) \oplus U_1 \oplus U_2 \oplus U_3$$

where

$$\begin{aligned}
U_1 &= \{(x_1, \ldots, x_a; 0, \ldots, 0; 0, \ldots, 0) \in V_1 : x_1 + \cdots + x_a = 0\},\\
U_2 &= \{(0, \ldots, 0; x_{a+1}, \ldots, x_{2a}; 0, \ldots, 0) \in V_1 : x_{a+1} + \cdots + x_{2a} = 0\},\\
U_3 &= \{(0, \ldots, 0; 0, \ldots, 0; x_{2a+1}, \ldots, x_{3a}) \in V_1 : x_{2a+1} + \cdots + x_{3a} = 0\}.
\end{aligned}$$

The action of Σ is absolutely irreducible on each isotypic component U_i, for $i = 1, 2, 3$ and trivial on $\mathrm{Fix}(\Sigma)$. Moreover, dim $U_i = a - 1$. Thus $(dG)_{X_0}$, when restricted to each of the U_i, has a real eigenvalue λ_i with multiplicity $a - 1$. Since $(dG)_{X_0}$ commutes with Σ,

$$(dG)_{X_0} = \begin{pmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & C_9 \end{pmatrix} \tag{4.33}$$

where the blocks correspond to the isotypic decomposition and C_1, C_5, C_9 commute with S_a.

Suppose M is a square matrix of order a with rows $l_1, \ldots, l_a$ and commuting with S_a. It follows then that $M = (l_1, (12) \cdot l_1, \cdots, (1a) \cdot l_1)^t$, where if $l_1 = (m_1, \ldots, m_a)$ then $(1i) \cdot l_1 = (m_i, m_2, \ldots, m_{i-1}, m_1, m_{i+1}, \ldots, m_a)$. Moreover, l_1 is invariant under S_{a-1} in the last $a - 1$ entries and so it has the following form: $(m_1, m_2, \ldots, m_2)$. Applying this to C_1, C_5, C_9 we get

$$C_i = \begin{pmatrix} a_i & & & \\ & \ddots & b_i & \\ & & & \\ & b_i & \ddots & \\ & & & a_i \end{pmatrix} \tag{4.34}$$

for $i = 1, 5, 9,$ where

$$a_1 = (\partial G_1/\partial x_1)_{X_0}, \quad a_5 = (\partial G_{a+1}/\partial x_{a+1})_{X_0}, \quad a_9 = (\partial G_{2a+1}/\partial x_{2a+1})_{X_0},$$

$$b_1 = (\partial G_1/\partial x_2)_{X_0}, \quad b_5 = (\partial G_{a+1}/\partial x_{a+2})_{X_0}, \quad b_9 = (\partial G_{2a+1}/\partial x_{2a+2})_{X_0}.$$

The other symmetry restrictions on the C_i, for $i \neq 1, 5, 9$, imply that the rest of the matrices each have one identical entry. From this we obtain a basis for each U_i composed by eigenvectors of $(dG)_{X_0}$: $U_1 = \{\nu_1, \ldots, \nu_{a-1}\}$, $U_2 = \{\psi_1, \ldots, \psi_{a-1}\}$, $U_3 = \{\phi_1, \ldots, \phi_{a-1}\}$ where

$$\begin{aligned}
&\nu_1 = (1, -1, 0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0),\ \nu_2 = (0, 1, -1, 0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0), \cdots,\\
&\nu_{a-1} = (0, \ldots, 0, 1, -1; 0, \ldots, 0; 0, \ldots, 0),\\
&\psi_1 = (0, \ldots, 0; 1, -1, 0, \ldots, 0; 0, \ldots, 0),\ \psi_2 = (0, \ldots, 0; 0, 1, -1, \ldots, 0; 0, \ldots, 0), \cdots,\\
&\psi_{a-1} = (0, \ldots, 0; 0, \ldots, 0, 1, -1; 0, \ldots, 0),\\
&\phi_1 = (0, \ldots, 0; 0, \ldots, 0; 1, -1, 0, \ldots, 0),\ \phi_2 = (0, \ldots, 0; 0, \ldots, 0; 0, 1, -1, 0, \ldots, 0), \cdots,\\
&\phi_{a-1} = (0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0, 1, -1).
\end{aligned}$$

Moreover the eigenvalue associated with ν_i is

$$\lambda_1 = a_1 - b_1 = (\partial G_1/\partial x_1)_{X_0} - (\partial G_1/\partial x_2)_{X_0},$$

the one associated with ψ_i is

$$\lambda_2 = a_5 - b_5 = (\partial G_{a+1}/\partial x_{a+1})_{X_0} - (\partial G_{a+1}/\partial x_{a+2})_{X_0}$$

and the one associated with ϕ_i is

$$\lambda_3 = a_9 - b_9 = (\partial G_{2a+1}/\partial x_{2a+1})_{X_0} - (\partial G_{2a+1}/\partial x_{2a+2})_{X_0}.$$

The branching conditions for Σ of Theorem 3.1 and the symmetry of G yield:

Lemma 4.1 *Let X_0 be an equilibrium of (3.12) in the Σ-branch obtained in Theorem 3.1. Then the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $(dG)_{X_0}$ are*

$$\lambda_1 = N(x+2y)(2x+y)S_2(x, -x-y),\ \lambda_2 = N(x+2y)(y-x)S_2(x, y),\ \lambda_3 = N(x-y)(2x+y)S_2(y, x), \tag{4.35}$$

where

$$S_2(x, y) = C + Ey + \left(\frac{2}{3}NI + H\right)(x^2 + y^2 + xy) + Hy^2 + \text{terms of degree} \geq 3, \tag{4.36}$$

and x and y are as in the second equation of (3.17):

$$B + \frac{1}{3}(3E + 2NF)(x^2 + y^2 + xy) - (H + NJ)(x^2y + xy^2) + \text{terms of degree } \geq 4 = 0. \tag{4.37}$$

Remark 4.2 Suppose X_0 corresponds to a solution of the primary branch with Σ_1-symmetry. Note that the isotypic decomposition of V_1 for the action of Σ_1 is

$$V_1 = W_0 \oplus W_1 \oplus W_2$$

where

$$\begin{aligned}
W_0 &= \mathrm{Fix}(\Sigma_1) = \{(-2x, \ldots; x, \ldots; x, \ldots) : x \in \mathbf{R}\},\\
W_1 &= \{(x_1, \ldots, x_a; 0, \ldots, 0) \in V_1 : x_1 + \cdots + x_a = 0\},\\
W_2 &= \{(0, \ldots, 0; x_{a+1}, \ldots, x_{3a}) \in V_1 : x_{a+1} + \cdots + x_{3a} = 0\}.
\end{aligned}$$

The action of Σ_1 is absolutely irreducible on each W_1, W_2 and trivial on W_0. It follows then that the Jacobian $(dG)_{X_0}$ has (at most) three distinct real eigenvalues, μ_j, one for each W_j, with multiplicity $\dim W_j$.

The stability in $\mathrm{Fix}(\Sigma)$ for the solution with Σ_1-symmetry is determined by the eigenvalue μ_0 associated with $W_0 = \mathrm{Fix}(\Sigma_1)$ and μ_2 since $\mathrm{Fix}(\Sigma) \bigcap W_2 \neq \{0\}$.

Suppose now that X_0 corresponds to a solution of the Σ-branch and of the Σ_1-branch. Then the eigenvalue μ_2 is zero and it is associated with the eigenspace W_2. Moreover, $U_2 \subseteq W_2$ and $U_3 \subseteq W_2$. Therefore X_0 is a zero of λ_2 and λ_3, and we have the factor $y - x$ in the expressions for λ_2 and λ_3 that appear in (4.35). Similarly, we justify the factors $x + 2y$ and $2x + y$ in those expressions. $\diamond$

Lemma 4.1 and (the proof of) Corollary 3.2 lead to the following result:

Theorem 4.3 *Assume the conditions of Theorem 3.1. Let X_0 be an equilibrium of (3.12) (and so of (1.2)) in the Σ-branch obtained in Theorem 3.1. Then the eigenvalues λ_i for $i = 1, \ldots, 5$ of $(dG)_{X_0}$ determining the stability of X_0 are λ_i for $i = 1, \ldots, 5$ where*

$$\lambda_1 = N(x+2y)(2x+y)S_2(x,-x-y),\ \lambda_2 = N(x+2y)(y-x)S_2(x,y),\ \lambda_3 = N(x-y)(2x+y)S_2(y,x),$$

$$\begin{aligned}\lambda_4\lambda_5 = {} & \frac{N^2}{9}\big[(3C+2D)(H+NJ) - (E+G)(3E+2NF)\big](x-y)^2(x+2y)^2(y+2x)^2 \\ & + \frac{2}{27}N^2(9H+6NI+6L+4NM)(H+NJ)(x^2+y^2+xy)(x-y)^2(x+2y)^2(y+2x)^2 \\ & + \text{ terms of degree } \geq 10\end{aligned}$$

$$\begin{aligned}\lambda_4 + \lambda_5 = {} & \frac{2}{3}N(3C+2D)(x^2+y^2+xy) - N(6E+2NF+3G)(x^2y+xy^2) \\ & + \frac{2}{9}N(21H+12NI+3NJ+12L+8NM)(x^2+y^2+xy)^2 + \text{ terms of degree } \geq 5\end{aligned}$$

where $S_2(x,y)$ is as in (4.36) and x, y satisfy (4.37).

We discuss now the stability of the equilibria in the secondary branch of steady-state solutions of (1.2) with symmetry Σ obtained in Theorem 3.1 for small values of $B \neq 0$.

Locally, near the origin, equation (4.37) in the x, y-plane is approximately an ellipse. Tertiary bifurcation points in the secondary branch occur if and only if the curve $S_2(x,y) = 0$ intersects the curve (4.37). Generically, the curve $S_2(x,y) = 0$ near the origin is approximately an ellipsis or an hyperbola. The distinction between these two cases depends on the sign of the product $(2NI+3H)(2NI+7H)$. It follows then that, generically, only three distinct situations can occur: the number of intersections between the curve $S_2(x,y) = 0$ and the Σ-branch in the xy-plane is zero, two or four. See Figure 2. Identifying points in the same $\mathbf{D}_3$-orbit, these correspond to zero, one and two tertiary bifurcations along the secondary branch, respectively.

We show below that the solutions of the Σ-branch are generically unstable in the first two cases. In the third case, we prove the instability of the equilibria of the secondary branch only near the secondary bifurcation points.

Theorem 4.4 *Assume the conditions of Theorem 3.1 and let X_0 be an equilibrium of the secondary branch of steady-state solutions of (1.2) with symmetry Σ obtained in Theorem 3.1 for sufficiently small values of $B \neq 0$. Then the solutions of the secondary branch near the secondary bifurcation points are generically unstable.*

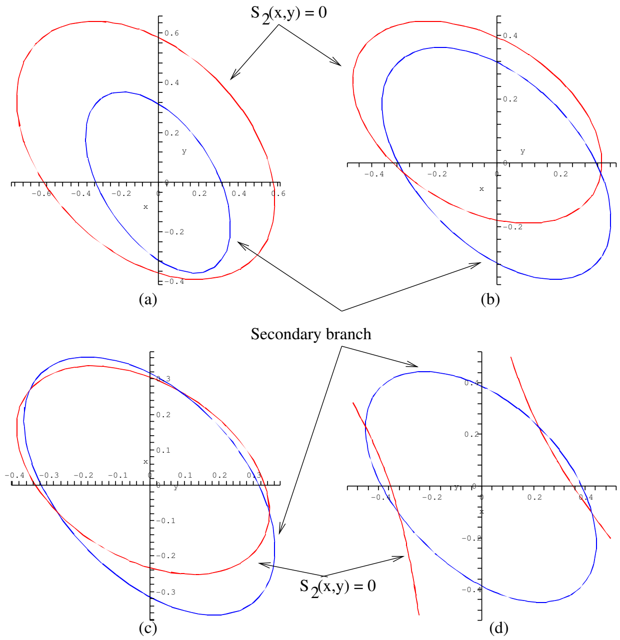

Figure 2: Intersections in the xy-plane between the Σ-branch and the curve $S_2(x, y) = 0$. (a) Zero intersections. (b) Two intersections. (c-d) Four intersections.

Proof: Under the conditions of Theorem 3.1 there is a secondary branch of equilibria of (3.12) near the origin obtained by bifurcation from the primary branches with Σ_i-symmetry for $i = 1, 2, 3$. Denote by $(x_i^-, y_i^-), (x_i^+, y_i^+)$ where $x_i^- < x_i^+$ the projections at the xy-plane of the intersections between the Σ-branch and the Σ_i-branch. Here x, y denote coordinates on Fix(Σ) (recall beginning of Section 3).

Let X_0 be an equilibrium of (3.12) in the Σ-branch not corresponding to one of the intersections between the Σ-branch and the Σ_i-branches and consider the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $(dG)_{X_0}$ as in Lemma 4.1 (defining the stability of X_0 at the isotypic components U_1, U_2, U_3 for the action of Σ).

We divide the proof in two cases. First, we suppose that $S_2(x, y) \neq 0$ along the secondary branch. Note that

$$\lambda_1\lambda_2\lambda_3 = -N^3(x + 2y)^2(2x + y)^2(y - x)^2 S_2(x, -x - y)S_2(x, y)S_2(y, x)$$

where sgn $(S_2(x, y))$ = sgn $(S_2(y, x))$ = sgn $(S_2(x, -x - y))$ since $S_2(x, y)$ does not change sign along the Σ-branch. Therefore, in order X_0 to be (linearly) stable we need sgn $(S_2(x, y)) > 0$ and $\lambda_1\lambda_2 > 0$, $\lambda_1\lambda_3 > 0$, $\lambda_2\lambda_3 > 0$. Now, the signs of these products depend on $(2x + y)(y - x)$, $(x+2y)(x-y)$, $(-1)(x+2y)(y+2x)$ and so there are no values of x, y such that these three products are positive. Thus X_0 is unstable.

Suppose now that there is an equilibrium X_0 of the secondary branch with symmetry Σ such that

$$S_2(x_0, y_0) = 0$$

where (x_0, y_0) is the projection of X_0 at the xy-plane. Generically, we can assume that X_0 is not an intersection point between the Σ-branch and one of the Σ_i-branches, for $i = 1, 2, 3$. We have then a tertiary bifurcation at $\lambda = \lambda_0$ from the secondary branch which implies the sign change of one of the eigenvalues determining the stability of the steady-state solutions of the Σ-branch near X_0. By the above discussion, generically, we have two cases: the curve $S_2(x,y) = 0$ intersects the curve (4.37) in two or four points. We have then that the three curves $S_2(y,x) = 0$, $S_2(x,y) = 0$, $S_2(x,-x-y) = 0$ intersect the curve (4.37) in six points (one $\mathbf{D}_3$-orbit) or twelve points (two $\mathbf{D}_3$-orbits), respectively. Recall situations (b-d) of Figure 2.

Denote by

$$\mathcal{R}_1 = \{(x,y) \in \mathbf{R}^2 : y - x \le 0,\ 2y + x \ge 0\},\ \mathcal{R}_2 = \{(x,y) \in \mathbf{R}^2 : y + 2x \ge 0,\ y - x \ge 0\},$$
$$\mathcal{R}_6 = \{(x,y) \in \mathbf{R}^2 : y + 2x \ge 0,\ 2y + x \le 0\}$$

and assume $(x_0, y_0) \in \mathcal{R}_1$. (The other cases are addressed in a similar way.) Then the eigenvalue λ_2 determining the stability of the equilibrium points in the Σ-branch that belong to the region $\mathcal{R}_1$ changes sign.

Observe that if $(x,y) \in \mathcal{R}_1$ is a steady-state solution in the Σ-branch then $(y,x) \in \mathcal{R}_2$ is also a solution in the Σ-branch and so with the same stability. Note that as (x,y) represents a vector

$$X = (\underbrace{-x-y,\ldots}_{a};\underbrace{y,\ldots}_{a};\underbrace{x,\ldots}_{a})$$

in $\mathrm{Fix}(\Sigma)$ then (y,x) corresponds to σX where σ is any permutation that fixes the first set of a coordinates and exchanges the second block of a coordinates with the third block of a coordinates. That is,

$$\sigma X = (\underbrace{-x-y,\ldots}_{a};\underbrace{x,\ldots}_{a};\underbrace{y,\ldots}_{a}) \in \mathrm{Fix}(\Sigma).$$

Any such σ belongs to $N(\Sigma)$.

We consider now an open set $\mathcal{O} \subset \mathcal{R}_1 \cup \mathcal{R}_2 \subset \mathbf{R}^2$ containing (x_1^+, y_1^+) such that:
(i) $\mathcal{R}_2 \cap \mathcal{O} = \sigma(\mathcal{R}_1 \cap \mathcal{O})$;
(ii) $S_2(x,y)$ does not change sign in the Σ-branch along $\mathcal{O}$.
We have then that the sign of the eigenvalue λ_2 for an equilibrium X of the secondary branch in $\mathcal{R}_1 \cap \mathcal{O}$ is opposite of the sign of λ_2 for $\sigma X \in \mathcal{R}_2 \cap \mathcal{O}$. Moreover, X and σX have the same stability. Thus, X has eigenvalues with opposite signs and so it is unstable. In Figure 3 (a) we show an example where the curve $S_2(x,y) = 0$ intersects at two points the secondary branch in the xy-plane and one of the intersections belongs to the region $\mathcal{R}_1$. Up to symmetry, there is one tertiary bifurcation along the Σ-branch. In the example of Figure 3 (b) the curve $S_2(x,y) = 0$ intersects at four points the secondary branch in the xy-plane (and one of the intersections belongs to the region $\mathcal{R}_1$). Up to symmetry, there are two tertiary bifurcations along the Σ-branch.

Similarly, taking steady-state solutions of the Σ-branch close to the point (x_3^+, y_3^+) in the region $\mathcal{R}_1$ where $S_2(x,y)$ does not vary the sign and their orbits by $\mathbf{D}_3$ in the region $\mathcal{R}_6$ we conclude the instability of the steady-state solutions of the Σ-branch close to the point (x_3^+, y_3^+) in the region $\mathcal{R}_1$.

In the example of Figure 3 (a) we have instability of equilibria in the Σ-branch. In the case of Figure 3 (b) the solutions of the secondary branch near the secondary bifurcation points are unstable. □

We show now an example illustrating the situation where solutions of the secondary branch between tertiary bifurcation points (in the region $\mathcal{R}_1$) are stable.

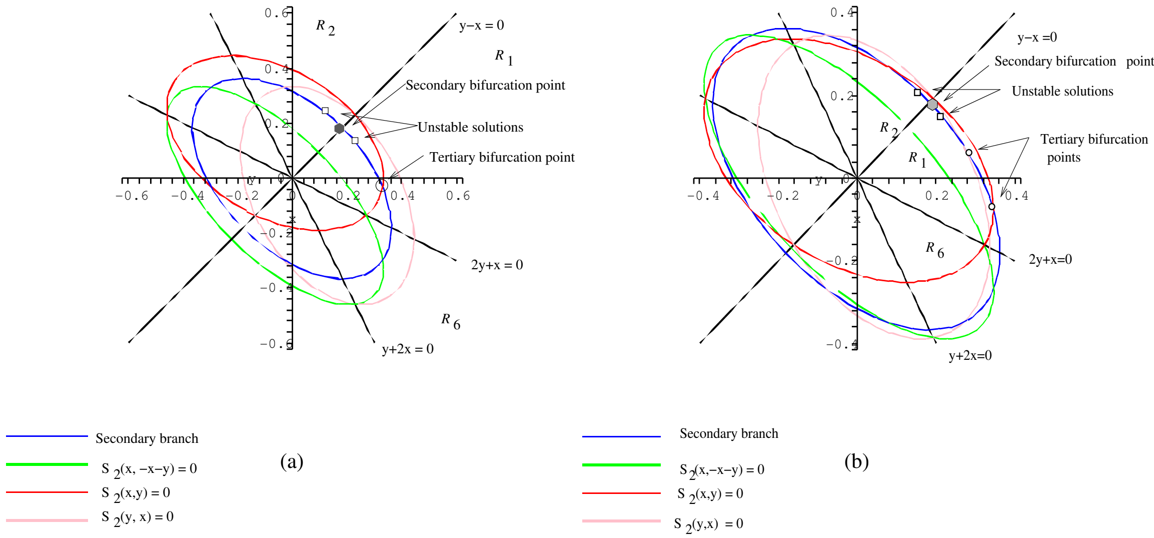

Figure 3: Examples where the curve $S_2(x,y)=0$ intersects the secondary branch and one of the intersections belongs to the region $\mathcal{R}_1$. In each example the two unstable points in the Σ-branch marked with a square are in the same $\mathbf{D}_3$-orbit. (a) There are two intersection points. (b) There are four intersection points.

Example 4.5 We consider (3.25), that is, (1.2) where G is truncated to the fifth order, $N=6$ and we assume the following parameter values:

$$B=-0.15,\ C=-1,\ D=1,\ E=0.9,\ F=0.025,\ G=-1.9,\ H=-8,\ I=4.25,\ J=1.35\,.$$

Conditions of Corollary 3.3 are satisfied. Therefore, the system (3.25) has a branch of equilibria with symmetry Σ bifurcating from the primary branches with Σ_i-symmetry, for $i=1,2,3$, which is described by (3.30). In particular, x,y satisfy

$$-0.15+x^2+y^2+xy-0.1(x^2y+xy^2)=0 \tag{4.38}$$

and the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ defined by (4.35) depend on

$$S_2(x,y)=-1+0.9y+9(x^2+y^2+xy)-8y^2.$$

In Figure 4 we show the curves $S_2(x,y)=0$ and (4.38) near the origin. Observe that $S_2(x,y)=0$ is an hyperbola intersecting (4.38) at four points. Moreover, equilibria in the Σ-branch between the tertiary bifurcation points for example in region $\mathcal{R}_1$ (following the notation of the above proof) are stable: it is clear from Figure 4 that $\lambda_1,\lambda_2,\lambda_3<0$; for the above parameter values $\lambda_4,\lambda_5<0$ using Corollary 3.2 or the expressions of Theorem 4.3. These statements are independent of the values of the parameters L,M. In Figure 5 we show the bifurcation diagram showing the amplitude and stability change of the Σ-branch with the primary bifurcation parameter λ.

$\diamond$

We state now sufficient conditions on the coefficients of G in (1.2) that imply the instability of the all Σ-branch of solutions obtained in Theorem 3.1.

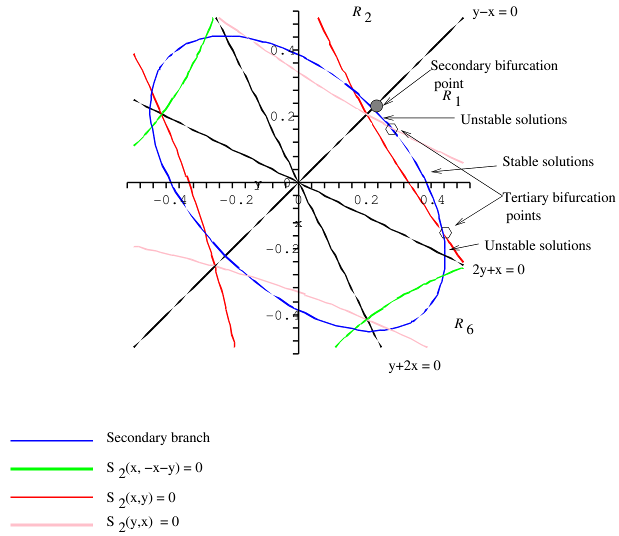

Figure 4: Example where solutions of the secondary branch between the tertiary bifurcation points (in region $\mathcal{R}_1$) are stable.

Corollary 4.6 *Suppose the conditions of Theorem 3.1 and assume $H \neq 0$. Let*

$$\Delta = E^2 - 4H\left[C - \frac{B(2NI+3H)}{3E+2NF}\right]$$

and if $\Delta > 0$ define

$$y_{\pm} = \frac{-E \pm \sqrt{\Delta}}{2H}, \quad \Delta^*_{\pm} = -3y^2_{\pm} - \frac{12B}{3E+2NF}.$$

For parameter values such that

$$(i)\ \Delta < 0, \quad or\ (ii)\ \Delta > 0,\ \Delta^*_+ < 0,\ \Delta^*_- < 0, \quad or\ (iii)\ \Delta > 0,\ \Delta^*_+\Delta^*_- < 0, \tag{4.39}$$

the solutions of the Σ-branch guaranteed by Theorem 3.1 (that do not correspond to secondary and tertiary bifurcation points) are unstable.

Proof: The instability of the solutions of the Σ-branch guaranteed by Theorem 3.1 follows directly from the proof of Theorem 4.4 if the curves $S_2(x,y) = 0$ where S_2 appears in (4.36) and (4.37), near the origin, intersect at zero or two points only. We find sufficient conditions on the coefficients of G in (1.2) that imply the above situations.

Near the origin we have

$$S_2(x,y) = 0 \Leftrightarrow C + Ey + \left(\frac{2}{3}NI + H\right)(x^2+y^2+xy) + Hy^2 + \text{terms of degree} \geq 3 = 0$$

and the equation of the Σ-branch is

$$B + \frac{1}{3}(3E+2NF)(x^2+y^2+xy) + \text{terms of degree } \geq 3 = 0.$$

We start by solving

$$\begin{cases} C + Ey + \left(\frac{2}{3}NI + H\right)(x^2+y^2+xy) + Hy^2 = 0, \\ B + \frac{1}{3}(3E+2NF)(x^2+y^2+xy). \end{cases}$$

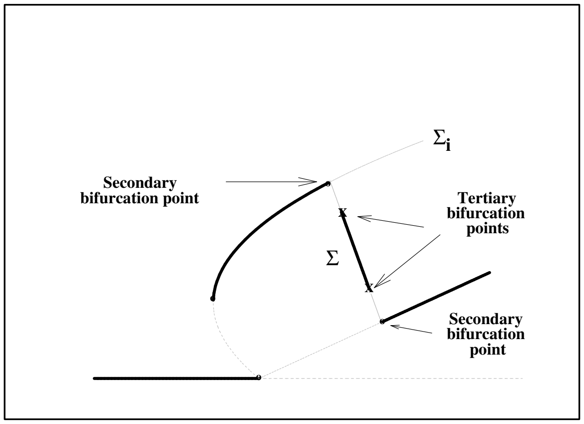

Figure 5: Bifurcation diagram showing the amplitude and stability change of the Σ-branch with the primary bifurcation parameter for $N = 6$ and the parameter values of Example 4.5. The Σ-branch solutions near the secondary bifurcation points (dashed lines) are unstable (in the transverse directions to $\mathrm{Fix}(\Sigma)$) and between the tertiary bifurcation points (solid lines) are stable (in $\mathrm{Fix}(\Sigma)$ and in the transverse directions to $\mathrm{Fix}(\Sigma)$).

Trivial calculations show that if conditions (4.39) are satisfied then this system has zero or two real solutions. Now recall that singularity theory methods were used in Theorem 3.1 to prove the existence of the Σ-branch near the origin for sufficiently small values of the parameter B. Higher order terms will not change the geometric properties of the curves $S_2(x, y) = 0$ and (4.37) from the point of view of their intersections near the origin as long as the conditions of Theorem 3.1 are satisfied. $\square$

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