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# COUPLED CELL NETWORKS: HOPF BIFURCATION AND INTERIOR SYMMETRY

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Abstract. We consider interior symmetric coupled cell networks where a group of permutations of a subset of cells partially preserves the network structure. In this setup, the full analogue of the Equivariant Hopf Theorem for networks with symmetries was obtained by Antoneli, Dias and Paiva (Hopf bifurcation in coupled cell networks with interior symmetries, *SIAM J. Appl. Dynam. Sys.* **7** (2008) 220–248). In this work we present an alternative proof of this result using center manifold reduction.

1. **Introduction.** Coupled cell systems are networks of dynamical systems (the cells) that are coupled together. Relevant aspects in the study of the dynamics of these systems can be encoded by a directed graph (*coupled cell network*): the nodes represent the cells and the edges indicate which cells are coupled and if the couplings are of the same type. We consider a special class of non-symmetric networks – the *interior symmetric* coupled cell networks. These networks admit a subset $\mathcal{S}$ of the cells such that the cells in $\mathcal{S}$ together with all the edges directed to them form a subnetwork which possesses a non-trivial symmetry group $\Sigma_{\mathcal{S}}$. Here, we follow the theory of Stewart *et al.*[4, 6, 9].

The local synchrony-breaking bifurcations in a coupled cell system occur when a synchronous state loses stability and bifurcates to a state with less synchrony. Such bifurcations can be considered to be a generalisation of symmetry-breaking bifurcations in symmetric coupled cell systems. See Golubitsky, Stewart and Schaeffer [5]. An analogue of the Equivariant Hopf Theorem for coupled cell systems with interior symmetries was obtained by Golubitsky, Pivato and Stewart [4] proving

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the existence of states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatial symmetries*. Antoneli, Dias and Paiva [1] extended this result obtaining states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatio-temporal symmetries*, that is, corresponding to *interiorly* $\mathbf{C}$-*axial* subgroups of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$. The proof of this result uses a modification of the Lyapunov-Schmidt reduction to arrive at a situation where the proof of the Standard Hopf Bifurcation Theorem can be applied. In this work, we present an alternative proof using center manifold reduction. This approach can be useful in the development of normal form theory aiming at the study of the stability of such periodic solutions (see Antoneli, Dias and Paiva [2]).

In Sections 2-5 we recall the definition of interior symmetry and the structure of coupled cell systems associated with interior symmetric networks. In Section 6 we state the Interior Symmetry-Breaking Hopf Bifurcation Theorem and prove it using Center Manifold Reduction.

2. **Coupled Cell Networks with Interior Symmetry.** Given a coupled cell network $\mathcal{G}$, the associated coupled cell systems are dynamical systems compatible with the architecture of $\mathcal{G}$. More specifically, each cell $c$ is equipped with a phase space $P_c$, and the total phase space of the network is the cartesian product $P = \prod_c P_c$. Call the set of edges directed to a cell $c$ by the *input set* of $c$. A vector field $f$ is called *admissible* if its component $f_c$ for cell $c$ depends only on variables associated with the input set of $c$ (*domain condition*), and if its components for cells $c$, $d$ that have isomorphic input sets are identical up to a suitable permutation of the relevant variables (*pull-back condition*). See Golubitsky *et al.*[6] for the formal definitions of coupled cell network and admissible vector fields.

Consider a subset $\mathcal{S}$ of the set of cells of $\mathcal{G}$ and let $\mathcal{G}_{\mathcal{S}}$ be the sub-network of $\mathcal{G}$ formed by the cells in $\mathcal{G}$ and the edges that are directed to cells in $\mathcal{S}$. By Antoneli *et al.* [1] (Proposition 3.3), the group of interior symmetries of $\mathcal{G}$ (on the subset $\mathcal{S}$) can be canonically identified with the group of symmetries of $\mathcal{G}_{\mathcal{S}}$. See Golubitsky *et al.*[4] for the original definition of interior symmetry and Antoneli *et al.* [1, 8] for the details about its identification with the group of symmetries of $\mathcal{G}_{\mathcal{S}}$.

**Example 1.** Figure 1 shows three networks: $\mathcal{G}_1$ (left), $\mathcal{G}_2$ (right) and $\mathcal{G}_{\mathcal{S}}$ (center) where $\mathcal{S} = \{1, 2, 3, 4\}$. The network $\mathcal{G}_{\mathcal{S}}$ obtained from $\mathcal{G}_1$ is the same as the one obtained from $\mathcal{G}_2$. Observe that for the three networks the arrows coming from the set $\mathcal{S} = \{1, 2, 3, 4\}$ and directed to its complement $\mathcal{C} \setminus \mathcal{S} = \{5\}$ are different.

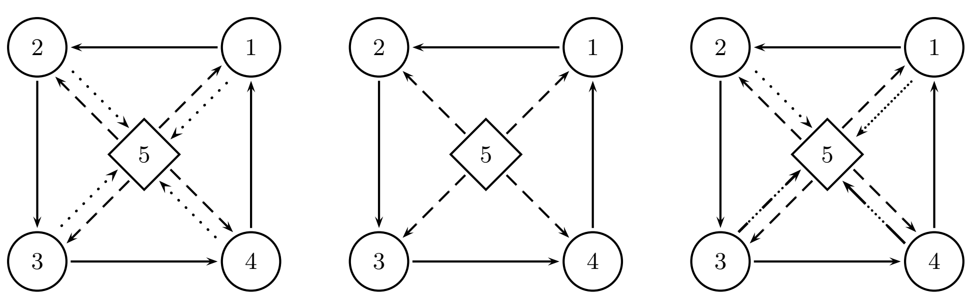


FIGURE 1. (Left) Network $\mathcal{G}_1$ with exact $\mathbf{Z}_4$-symmetry. (Center) Network $\mathcal{G}_{\mathcal{S}}$, where $\mathcal{S} = \{1, 2, 3, 4\}$. (Right) Network $\mathcal{G}_2$ with $\mathbf{Z}_4$-interior symmetry.

3. **Vector Fields with Interior Symmetry.** Suppose that $\mathcal{G}$ admits a nontrivial group of interior symmetries $\Sigma_{\mathcal{S}}$ on a subset of cells $\mathcal{S}$. We can decompose the phase space $P$ as a cartesian product $P = P_{\mathcal{S}} \times P_{\mathcal{C}\setminus\mathcal{S}}$ where $P_{\mathcal{S}} = \prod_{s\in\mathcal{S}} P_s$ and $P_{\mathcal{C}\setminus\mathcal{S}} = \prod_{c\in\mathcal{C}\setminus\mathcal{S}} P_c$. For any $x \in P$ we write $x = (x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}})$ where $x_{\mathcal{S}} \in P_{\mathcal{S}}$ and $x_{\mathcal{C}\setminus\mathcal{S}} \in P_{\mathcal{C}\setminus\mathcal{S}}$ and we can take the action of $\Sigma_{\mathcal{S}}$ on $P$ given by:

$$\sigma\left(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}\right) = \left(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}\right) \quad (\sigma \in \Sigma_{\mathcal{S}})\ . \tag{1}$$

Here $\Sigma_{\mathcal{S}}$ acts on $x_{\mathcal{S}}$ by permuting the coordinates corresponding to the cells in $\mathcal{S}$.

For a subgroup $\Xi \subseteq \Sigma_{\mathcal{S}}$ define

$$\mathrm{Fix}_P(\Xi) = \{(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) : \sigma x_{\mathcal{S}} = x_{\mathcal{S}}\ \ \forall\, \sigma \in \Xi\}.$$

**Proposition 1.** *Let $\Xi$ be a subgroup of $\Sigma_{\mathcal{S}}$. Then $\mathrm{Fix}_P(\Xi)$ is flow invariant under all $\mathcal{G}$-admissible vector fields.*

*Proof.* See Golubitsky *et al.* [4, p. 397]. □

By Proposition 1 the subspace $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is flow-invariant under any admissible vector field on $P$. Since $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is $\Sigma_{\mathcal{S}}$-invariant and $\Sigma_{\mathcal{S}}$ acts trivially on the cells in $\mathcal{C} \setminus \mathcal{S}$ we have that $P_{\mathcal{C}\setminus\mathcal{S}} \subset \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. The action of the group $\Sigma_{\mathcal{S}}$ decomposes the set $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_k,$$

where the sets $\mathcal{S}_i$ $(i = 1, \ldots, k)$ are the orbits of the $\Sigma_{\mathcal{S}}$-action. Let

$$W = \left\{ x \in P \ :\ x_c = 0\ \ \forall\, c \in \mathcal{C} \setminus \mathcal{S} \ \text{ and } \sum_{s\in\mathcal{S}_i} x_s = 0 \ \text{ for } 1 \le i \le k \right\}.$$

Since $W$ is a $\Sigma_{\mathcal{S}}$-invariant subspace of $P_{\mathcal{S}}$ and $W \cap \mathrm{Fix}_P(\Sigma_{\mathcal{S}}) = \{0\}$ we can decompose the phase space $P$ as a direct sum of $\Sigma_{\mathcal{S}}$-invariant subspaces:

$$P = W \oplus \mathrm{Fix}_P(\Sigma_{\mathcal{S}}). \tag{2}$$

Consider a coupled cell system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

with interior symmetry group $\Sigma_{\mathcal{S}}$ on $\mathcal{S}$. Let $U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. We can choose coordinates $(w, u)$ with $w \in W$ and $u \in U$ adapted to the decomposition (2) and write the admissible vector field $f$ as

$$f(w,u) = \begin{bmatrix} f_W(w,u) \\ f_U(w,u) \end{bmatrix} + \begin{bmatrix} 0 \\ h(w,u) \end{bmatrix}, \tag{3}$$

where $f_U$, $h : P \to U$, $f_W : P \to W$ and

$$\tilde{f}(w,u) = (f_W(w,u), f_U(w,u))$$

is the $\Sigma_{\mathcal{S}}$-*equivariant part* of $f$. That is,

$$\begin{bmatrix} \sigma f_W(w,u) \\ f_U(w,u) \end{bmatrix} = \begin{bmatrix} f_W(\sigma w,u) \\ f_U(\sigma w,u) \end{bmatrix} \qquad (\forall\ \sigma \in \Sigma_{\mathcal{S}})\,, \tag{4}$$

since $\Sigma_{\mathcal{S}}$ acts trivially on $U$. Thus we may write an admissible vector field $f$ as

$$f(w,u) = \tilde{f}(w,u) + h(w,u)$$

where $\tilde{f}$ is $\Sigma_{\mathcal{S}}$-equivariant.

4. **Linear Maps with Interior Symmetry.** In the linear case, we may choose a basis of $P$ adapted to the decomposition (2) and then a $\mathcal{G}$-admissible linear vector field $L$ can be written as

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{5}$$

where $B = L|_U : U \to U$, $C : W \to U$ and $A : W \to W$ satisfies, by (4),

$$A\,\sigma = \sigma A \qquad (\forall\, \sigma \in \Sigma_{\mathcal{S}})\,.$$

The spectral properties of $L$ in (5) are summarized in the following proposition.

**Proposition 2.** *Let $\mathcal{G}$ be a network admitting a non-trivial group of interior symmetries $\Sigma_{\mathcal{S}}$ and fix a total phase space $P$. Let $L : P \to P$ be a $\mathcal{G}$-admissible linear vector field and consider the decomposition of $L$ given by (5). Then*

(i) *The eigenvalues of $L$ are the eigenvalues of $A$ together with the eigenvalues of $B$.*
(ii) *A vector $u \in U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is an eigenvector of $B$ with eigenvalue $\nu$ if and only if $u$ is an eigenvector of $L$ with eigenvalue $\nu$.*
(iii) *If $w \in W$ is an eigenvector of $A$ with eigenvalue $\mu$, then there exists an eigenvector $v$ of $L$ with eigenvalue $\mu$ of the form*

$$v = w + u$$

*where $u \in U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$.*
(iv) *All eigenspaces of $A$ are $\Sigma_{\mathcal{S}}$-invariant.*

*Proof.* See Golubitsky *et al.* [4, p. 399]. □

5. **Bifurcations in Systems with Interior Symmetry.** Consider a 1-parameter family of coupled cell systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \tag{6}$$

with interior symmetry group $\Sigma_{\mathcal{S}}$ on $\mathcal{S}$ and suppose that it undergoes a codimension-one synchrony-breaking bifurcation at a synchronous equilibrium $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ when $\lambda = \lambda_0$. Let $L = (df)_{(x_0,\lambda_0)}$ be written as in (5). As defined in Antoneli *et al.* [1], we say that $f$ undergoes a *codimension-one interior symmetry-breaking Hopf bifurcation* if the following conditions hold:

(a) All the critical eigenvalues $\mu$ of $L$ come from the $\Sigma_{\mathcal{S}}$-equivariant sub-block $A$ of $L$.
(b) The critical eigenvalues $\mu$ extend uniquely and smoothly to eigenvalues $\mu(\lambda)$ of $(\mathrm{d}f)_{(x_0,\lambda)}$ for $\lambda$ near $\lambda_0$.
(c) The *eigenvalue crossing condition*: if $\sigma(\lambda) = \mathrm{Re}(\mu(\lambda))$ then $\sigma'(\lambda_0) \neq 0$.
(d) The matrix $A$ is non-singular and (after rescaling time if necessary) all the critical eigenvalues have the form $\pm i$ and the associated center subspace is given by $E_i(A) = \{x \in W : (A^2 + 1)x = 0\}$.

Assume that $L$ as in (5) has $\pm i$ as eigenvalues that come only from the sub-block $A$ of $L$ and that they are the only critical eigenvalues of $L$. Consider $A^c = A|_{E_i(A)}$. As $A$ has $\pm i$ as eigenvalues there is a natural action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $P$, where $\mathbf{S}^1$ acts on $E_i(A)$ by $\exp(s(A^c)^t)$ and trivially on $P \setminus E_i(A)$. The action of $\Sigma_{\mathcal{S}}$ on $P$ is given by (1).

**Remark 1.** *Observe that, in general, there is no action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on the center subspace $E^c(L)$, which is not a $\Sigma_{\mathcal{S}}$-invariant subspace. Even in the (non-generic) case where $E^c(L) = E^c(A)$ there is a natural action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $E^c(L)$, since the center subspace $E^c(A)$ is $\Sigma_{\mathcal{S}}$-invariant; however, the reduced equation – obtained by either Lyapunov-Schmidt reduction or center manifold reduction – is not necessarily $\Sigma_{\mathcal{S}}$-equivariant with respect to this induced action, since the original vector field $f$ is not $\Sigma_{\mathcal{S}}$-equivariant. In any case, the reduced equation does not satisfy the requirements of Equivariant Hopf Theorem and thus it is not straightforward to apply the standard results of equivariant bifurcation theory.*

Now suppose the family (6) undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at the equilibrium $x_0$ when $\lambda = \lambda_0$. Then the center subspace $E^c(A) \equiv E_i(A)$ of the $\Sigma_{\mathcal{S}}$-equivariant sub-block $A$ of the linearization $L = (\mathrm{d}f)_{(x_0,\lambda_0)}$ of $f$ at $(x_0, \lambda_0)$ is a $\Sigma_{\mathcal{S}}$-invariant subspace of $W$. Therefore, the action of the circle group $\mathbf{S}^1$ defined by $\exp(s(A^c)^t)$ commutes with the action of $\Sigma_{\mathcal{S}}$. Thus $E^c(A)$ is a $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-invariant subspace and so there is a well-defined action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $E^c(A)$ (and $W$). Following Antoneli *et al.* [1] (Definition 4.6), we say that an isotropy subgroup $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ is *interiorly* $\mathbf{C}$*-axial (on $E^c(A)$)* if

$$\dim_{\mathbf{R}} \mathrm{Fix}_{E^c(A)}(\Delta) = 2.$$

6. **Interior Symmetry-breaking Hopf Bifurcation Theorem.** In this section we shall give an alternative proof the main result of Antoneli *et al.* [1] (Theorem 4.8) using a center manifold reduction approach.

The original proof of this theorem in [1] is through a modified version of the Lyapunov-Schmidt reduction. The new proof presented below combines an extra condition that the vector field is in "interior normal form up to all orders" with a "hidden symmetry" center manifold reduction. Let us start by giving a precise definition of the notion of "interior normal form".

**Definition 1.** *We say that $f$ is in* interior normal form (to all orders)*, with respect to $L = (\mathrm{d}f)_{(x_0,0)}$ near $\lambda_0$, if $\tilde{f}(\cdot, \lambda)$ is in normal form (to all orders) with respect to $\tilde{L} = (\mathrm{d}\tilde{f})_{(x_0,0)}$ near $\lambda_0$, that is, $\tilde{f}$ commutes with the action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $P$ defined above, for $\lambda$ near $\lambda_0$.*

It is well known that the definition of normal form as used above is equivalent to the following definition.

**Definition 2.** *A smooth family of vector fields $f : \mathbf{R}^n \times \mathbf{R}^l \to \mathbf{R}^n$ is in normal form up to some order $k > 1$, with respect to $L = (\mathrm{d}f)_{(x_0,\lambda_0)}$ for $\lambda$ in some neighborhood $\Lambda$ of $\lambda_0$ in $\mathbf{R}^l$, if we can write $f$ in the form*

$$f(x, \lambda) = f_{NF}(x, \lambda) + r(x, \lambda)\,,$$

*where the mappings $f_{NF}$ and $r$ satisfy:*

*(i)* $[f_{NF}(\cdot, \lambda), L] = 0$ *for all* $\lambda \in \Lambda$,
*(ii)* $r(x, \lambda) = O(\|x\|^{k+1})$ *as* $x \to 0$*, uniformly for* $\lambda \in \Lambda$.

*In (i) we use the* Lie bracket *of two vector fields:*

$$[g, h](x) = Dh(x)\, g(x) - Dg(x)\, h(x)\ .$$

Therefore, if $\tilde{f}$ is in normal form (up to all orders) then $\tilde{f} = \tilde{f}_{NF}$.

**Theorem 1.** *Let $\mathcal{G}$ be a coupled cell network admitting a non-trivial group of interior symmetries $\Sigma_{\mathcal{S}}$ relative to a subset $\mathcal{S}$ of cells and fix a phase space $P$. Consider a smooth 1-parameter family of $\mathcal{G}$-admissible vector fields $f : P \times \mathbf{R} \to P$ in interior normal form up to all orders. Suppose that the bifurcation problem*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \tag{7}$$

*undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at an equilibrium point $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ when $\lambda = 0$. Let $L = (\mathrm{d}f)_{(x_0,0)}$ be written as in (5) and $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be an interiorly $\mathbf{C}$-axial subgroup (on $E^c(A)$). Then generically there exists a family of small amplitude periodic solutions of (7) bifurcating from $(x_0, 0)$ and having period near $2\pi$. Moreover, to lowest order in the bifurcation parameter $\lambda$, the solution $x(t)$ is of the form*

$$x(t) \approx w(t) + u(t),$$

*where $w(t) = \exp(tL)w_0$ ($w_0 \in \mathrm{Fix}_W(\Delta)$) has exact spatio-temporal symmetry $\Delta$ on the cells in $\mathcal{S}$ and $u(t) = \exp(tL)u_0$ ($u_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$) is synchronous on the $\Sigma_{\mathcal{S}}$-orbits of cells in $\mathcal{S}$.*

In the proof of Theorem 1 we will need two lemmas and one proposition that we introduce next.

**Lemma 1.** *Let $f : P \times \mathbf{R}^l \to P$ be a smooth $l$-parameter family of $\mathcal{G}$-admissible vector fields in interior normal form (up to all orders). Then for every subgroup $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ we have that*

$$f(\mathrm{Fix}_P(\Delta) \times \mathbf{R}^l) \subseteq \mathrm{Fix}_P(\Delta)\ . \tag{8}$$

*Proof.* Let $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be a subgroup. Since $f$ is in interior normal form up to all orders, $\tilde{f}(\cdot, \lambda)$ is $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-equivariant for all $\lambda \in \mathbf{R}^l$ and so $\tilde{f}(\mathrm{Fix}_P(\Delta) \times \mathbf{R}^l) \subseteq \mathrm{Fix}_P(\Delta)$. Since $h : P \times \mathbf{R}^l \to P$ maps into $P_{\mathcal{C}\setminus\mathcal{S}}$ and $P_{\mathcal{C}\setminus\mathcal{S}} \subseteq \mathrm{Fix}_P(\Delta)$, we arrive at (8). □

Finally, let us recall a result from Leite and Golubitsky [7], which may be seen as a center manifold reduction for vector fields with "hidden" symmetries.

**Proposition 3.** *Let the vector field $f : \mathbf{R}^n \to \mathbf{R}^n$ have a flow invariant subspace $V$ with an equilibrium at $x_0 \in V$. Let $E^c$ be the center subspace at $x_0$. Then a center manifold reduction $f^c : E^c \to E^c$ can be chosen so that the subspace $E^c \cap V$ is flow invariant for $f$. Moreover, if $\sigma : V \to V$ is a symmetry of $f|_V$ that leaves $E^c \cap V$ invariant, then the center manifold reduction $f^c$ may be chosen so that $\sigma|_{E^c \cap V}$ is a symmetry for $f^c|_{E^c \cap V}$.*

*Proof.* See Leite and Golubitsky [7, p. 2346]. □

We start the proof of Theorem 1 with the following lemma.

**Lemma 2.** *Consider $L = (\mathrm{d}f)_{(x_0,0)}$ in the conditions of Theorem 1 and written as in (5). Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be an isotropy subgroup for the action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ as defined in the previous section. Then $\dim(E_i(A)) = \dim(E_i(L))$ and $\dim(\mathrm{Fix}_{E_i(A)}(\Delta)) = \dim(\mathrm{Fix}_P(\Delta) \cap E_i(L))$.*

*Proof.* Consider $x = (w, u) \in P$ where $w \in W, u \in \mathrm{Fix}_P(\Sigma_S)$. Assume $\dim W = k$ and $\dim \mathrm{Fix}_P(\Sigma_S) = l$. As

$$(L^2 + I_{k+l})x = 0 \quad \Longleftrightarrow \quad \begin{bmatrix} A^2 + I_k & 0 \\ CA + BC & B^2 + I_l \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $B$ does not have $\pm i$ as eigenvalues, we get

$$E_i(L) = \{(w, -(B^2 + I_l)^{-1}(CA + BC)w), w \in E_i(A)\}.$$

In particular, it follows that $\dim(E_i(A)) = \dim(E_i(L))$. As $\mathrm{Fix}_P(\Delta) = \mathrm{Fix}_W(\Delta) \oplus \mathrm{Fix}_P(\Sigma_S)$, we have

$$\mathrm{Fix}_P(\Delta) \cap E_i(L) = \{(v, -(B^2 + I_l)^{-1}(CA + BC)v), v \in \mathrm{Fix}_{E_i(A)}(\Delta)\}$$

and so $\dim(\mathrm{Fix}_P(\Delta) \cap E_i(L)) = \dim(\mathrm{Fix}_{E_i(A)}(\Delta))$. □

*Proof of Theorem 1.* Consider $f$ written in the coordinates $(w, u)$ adapted to the decomposition (2) as in (3), that is $\tilde{f}(w, u, \lambda)$ is $\Sigma_{\mathcal{S}}$-equivariant. Now let us assume that $f$ is in interior normal form (up to all orders) near $\lambda = 0$, that is, $\tilde{f} = \tilde{f}_{NF}$ and $\tilde{f}$ comutes with the action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$. Then by Lemma 1 we have

$$f(\mathrm{Fix}_P(\Delta) \times \mathbf{R}) \subseteq \mathrm{Fix}_P(\Delta). \tag{9}$$

In our case, $E^c(L) = E_i(L)$ since the only critical eigenvalues of $L$ are $\pm i$ and these come only from the sub-block $A$ of $L$. Then, under the condition (9), Proposition 3 grants that a center manifold reduction $f^c : E_i(L) \to E_i(L)$ can be chosen so that $f^c(E_i(L) \cap \mathrm{Fix}_P(\Delta)) \subseteq E_i(L) \cap \mathrm{Fix}_P(\Delta)$. By hypothesis $\dim(\mathrm{Fix}_{E_i(A)}(\Delta)) = 2$. Then, by Lemma 2, it follows that $\dim(E_i(L) \cap \mathrm{Fix}_P(\Delta)) = 2$. Now we may apply the standard Hopf Theorem to get the result. □

The argument that the Theorem holds in the case where $f$ is in interior normal form may be regarded as the statement that the branch of periodic solutions is persistent under a certain kind of perturbation. The Equivariant Hopf Theorem applied to $\tilde{f}_{NF}$ provides the existence of a branch of periodic solutions emanating from $(x_0, 0)$, with period near $2\pi$, such that, to lowest order in the bifurcation parameter $\lambda$, the solution $x(t)$ is of the form $x(t) \approx \exp(tL)w_0$ where $w_0 \in \mathrm{Fix}_W(\Delta)$. Now our argument shows that the addition of the term $h$ to $\tilde{f}_{NF}$ does not destroy the branch of periodic solutions, it just breaks the exact spatio-temporal symmetry of the periodic solution. On the other hand, Field [3, p. 163] shows that a branch of periodic solutions with maximal isotropy symmetry group of a *generic* $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-equivariant vector field persists under perturbations that breaks the $\mathbf{S}^1$ symmetry. Now, in the general case, the vector field can be written as

$$f(x, \lambda) = \tilde{f}_{NF}(x, \lambda) + \tilde{r}(x, \lambda) + h(x, \lambda)$$

where, $\tilde{f}_{NF}$ is $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-equivariant and $\tilde{r}(x, \lambda)$ is $\Sigma_{\mathcal{S}}$-equivariant. Thus, one is tempted to argue that since the periodic solution independently persists under both types of perturbation, namely $\tilde{r}$ and $h$, it will persist under their sum as well. Unfortunately, this argument does not hold in general, since a $\mathcal{G}_{\mathcal{S}}$-admissible vector field may *never* be a generic $\Sigma_{\mathcal{S}}$-equivariant vector field, that is, the structure of network forces the vector field to be more degenerate than it would be expected if it were a $\Sigma_{\mathcal{S}}$-equivariant vector field. Nevertheless, it is true if one can show that, for a particular network under consideration, the set of admissible vector fields and the equivariant vector fields are equal.

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