The Lattice of Synchrony Subspaces of a Coupled Cell Network: Characterization and Computation Algorithm

Manuela A. D. Aguiar[†]
Faculdade de Economia
Centro de Matemática [‡]
Universidade do Porto
Rua Dr Roberto Frias
4200-464 Porto, Portugal

Ana Paula S. Dias[§]
Dep. Matemática
Centro de Matemática
Universidade do Porto
Rua do Campo Alegre, 687
4169-007 Porto, Portugal

September 19, 2013

Abstract

Coupled cell systems are networks of dynamical systems (the cells), where the links between the cells are described through the network structure, the coupled cell network. Synchrony subspaces are spaces defined in terms of equalities of certain cell coordinates that are flow-invariant for all coupled cell systems associated with

[†]Correspondence to M.A.D.Aguiar. E-mail: maguiar@fep.up.pt

[‡]Research partially funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT under the projects PEst-C/MAT/UI0144/2011 and PTDC/MAT/100055/2008.

[§]E-mail: apdias@fc.up.pt

a given network structure. The intersection of synchrony subspaces of a network is also a synchrony subspace of the network. It follows then that, given a coupled cell network, its set of synchrony subspaces, taking the inclusion partial order relation, forms a lattice. In this paper we show how to obtain the lattice of synchrony subspaces for a general network and present an algorithm that generates that lattice. We prove that this problem is reduced to get the lattice of synchrony subspaces for regular networks. For a regular network we obtain the lattice of synchrony subspaces based on the eigenvalue structure of the network adjacency matrix.

AMS classification scheme numbers: 34C15 37C10 06B23 15A18

1 Introduction

Consider a *network* architecture, that is, a finite set of nodes (the cells) linked together by a finite number of arrows. Assume the cells represent individual dynamics and the arrows the interactions between the individuals. We have then the coupled cell systems, that is, the dynamical systems consistent with that network structure. See for example the approach of Stewart, Golubitsky *et al.* [15, 10, 9] that we follow in this paper or Field [6].

The network structure imposes restrictions at the dynamics that can occur for the associated coupled cell systems. In particular, it can force the existence of flow-invariant subspaces defined in terms of equalities of certain cell coordinates. These flow-invariant subspaces, called *synchrony subspaces*, have a major impact at the dynamics of the coupled cell systems associated with the given network. It is known, for example, that flow-invariant spaces favour the existence of non-generic dynamical behavior like heteroclinic cycles and networks, which leads to complicated dynamics. It follows that, knowing the set of all synchrony subspaces of a coupled cell network, can help to detect the possibility of the associated coupled cell systems to support heteroclinic behavior. See Aguiar *et al.* [2]. Another important aspect is that the restriction of the dynamics of a coupled cell system to a synchrony subspace is again a coupled cell system whose structure is consistent with a network that has fewer cells (the quotient network). We have then a coupled cell system in a lower-dimensional phase space. Moreover, it is possible that the associated quotient network has a specific structure that has been already explored from the dynamical point of view in several contexts. If that is the case, the known dynamics of the quotient network can be lifted to the original network dynamics. Although, the restriction does not give all the dynamics for the original network, it can give full information concerning the dynamics at those synchrony subspaces. See for example Aguiar *et al.* [3]. Examples of specific structures include: existence of global (quotient) network symmetries implying that the associated coupled cell systems are symmetric under a permutation symmetry group – these impose strong constrains at the dynamics that can occur, see for example [8], [9] and references therein; known bifurcation classifications of classes of networks with certain structures, see for example Leite and Golubitsky [13].

Stewart, Golubitsky and co-workers [15, 10] describe the synchrony subspaces of coupled cell systems using the associated network structure. Given a network and a space given by equalities of certain cell coordinates, we can define an equivalence relation on the set of nodes with two cells in the same class when the coordinates are equal. In [15, 10] it is proved that the space is of synchrony (for any coupled cell system associated with the network) if and only if this relation satisfies a set of conditions on the network – the relation is said to be *balanced* (see Definition 2.4). Thus the synchrony subspaces associated with a network structure are in one-to-one correspondence with the balanced equivalence relations on the set of cells of the network. Moreover, by Stewart [14] (see also Aldis [4]) the set of all such balanced equivalence relations forms a complete lattice taking the relation of refinement; recall that a *lattice* is a partially ordered set such that every pair of elements has a unique least upper bound or *join*, and a unique greatest lower bound or *meet*. Moreover, a *complete lattice* X is a lattice where every subset $Y \subseteq X$ has a unique least upper bound or join, and a unique greatest lower bound or meet. Using the one-to-one correspondence between balanced equivalence relations and synchrony subspaces, it follows that the set of synchrony subspaces associated with a network, taking the relation of inclusion $\subseteq$, is also a complete lattice [14]. See Section 3.

The aim of this paper is to describe how to obtain the lattice of synchrony subspaces of a given network. An important aspect of the fact that the set of synchrony subspaces of a network forms a finite lattice is that it is possible to generate the lattice using a subset of synchrony subspaces. See Remark 3.3. As we shall show, obtaining the lattice of synchrony subspaces of a general network reduces basically to the problem of obtaining the lattice of synchrony subspaces of regular networks. Here, we say that a network is *regular* if it has only one cell type and one arrow type and the number of arrows directed to each cell is constant. For a regular network we obtain the lattice of synchrony subspaces based on the eigenvalue structure of the network adjacency matrix (see Definition 2.1 (iv)) and we present an algorithm that generates the lattice. Our approach exploring the eigenvalue structure of the network adjacency matrices has been motivated by the work of Kamei [12] on the class of regular networks where the adjacency matrix has only simple eigenvalues. As we observe in Remark 6.4, our algorithm applies as well, directly, to all *identical-edge networks*, that is, networks with only one cell type and one edge type.

We restrict our discussion in this section to the regular case. Basically, in our framework, the key point is that a synchrony subspace of a regular network is a space that is left invariant by the corresponding network adjacency matrix and is polydiagonal, that is, it is described by a set of equality conditions on the cell coordinates.

It is known that the subspaces invariant under a linear map can be described in terms of its eigenvectors and generalized eigenvectors. Moreover, the set of such invariant subspaces forms a complete lattice under the relation of inclusion and this lattice can be described using the irreducible invariant subspaces - the Jordan subspaces - the invariant subspaces having a unique eigenvector (up to multiplication by a scalar). In this lattice the join operation is the intersection and the meet operation is the sum.

In our work, there were two fundamental difficulties that had to be overcomed: (a) how to list the possible polydiagonal subspaces that contain (generalized) eigenvectors of the network adjacency matrix; (b) how to generalize the concept of irreducible (synchrony) subspace aiming to describe the lattice of synchrony subspaces through a (small) set of irreducible synchrony subspaces.

Note that, although the lattice of synchrony subspaces, as a set, is a subset of the lattice of the invariant subspaces under the network adjacency matrix, it is not in general a sublattice - the meet operation is the same, the intersection of subspaces, but the join of two synchrony subspaces is not given in general by their sum. In fact, the join of two synchrony subspaces is given by their sum only when the sum is a polydiagonal subspace.

Concerning our first difficulty, we define the concept of *minimal synchrony subspace* associated to an eigenvector or Jordan chain - the intersection of all synchrony subspaces containing the eigenvector or Jordan chain (see Definition 5.10). We prove that the set of all minimal synchrony subspaces forms a sum-dense set for the lattice of synchrony subspaces (Theorem 5.13). That is, any synchrony subspace can be given by a sum of minimal synchrony subspaces. It follows then that, our first task - to list the possible polydiagonal subspaces containing (generalized) eigenvectors - can be reduced to the listing of the minimal synchrony subspaces.

Considering the second difficulty, we introduce the concept of *sum-irreducible synchrony subspace* (Definition 5.7) - it cannot be represented as a sum of proper synchrony subspaces. We prove then that every synchrony subspace associated with a network is a sum of sum-irreducible synchrony subspaces (Proposition 5.8). Joining the two results, as just mentioned, concerning the concepts of minimal and sum-irreducible synchrony subspaces, it follows that we can generate the lattice of synchrony subspaces associated with a regular network through the set of the minimal synchrony subspaces that are sum-irreducible. See Corollary 5.14 and Remark 5.15.

We present an algorithm (Algorithm 6.5) that outputs the lattice of synchrony subspaces, together with its irreducible sum-dense set, for a regular network. The algorithm contains four fundamental steps. Given a regular network, in the first two steps, it obtains a set of synchrony subspaces containing the minimal synchrony set for the lattice of the network synchrony subspaces. In the third step finds the irreducible sum-dense set for that lattice and in the fourth step generates the lattice.

The paper is organized in the following way. Section 2 introduces briefly a few concepts concerning networks and corresponding admissible vector fields - the coupled cell systems. It also recalls the concepts of balanced equivalence relation and synchrony subspace and the result establishing the one-to-one correspondence between the two concepts [10]. In section 3 we start by recalling a few basic concepts concerning complete lattices and the result in [14] proving that the set of all balanced equivalence relations of a network forms a complete lattice. The results in Section 4 relate the lattice of synchrony subspaces for nonhomogeneous networks and for nonregular homogeneous networks with the lattice of synchrony subspaces for their identical-edge subnetworks, a kind of regular networks. Thus, the most important question to be addressed is how to obtain the lattice of synchrony subspaces for regular networks. This is the issue addressed in Sections 5 and 6. In Section 5, we start by recalling the theory of invariant subspaces for linear maps; then we introduce the concepts of sum-irreducible and minimal synchrony subspace associated to an eigenvector or Jordan chain; finally, we prove in Proposition 5.8 that the lattice of synchrony subspaces associated with a network structure can be obtained using the sum operation of synchrony subspaces that are sum-irreducible.

In Section 6 we present an algorithm (Algorithm 6.5) that calculates the set of sum-irreducible subspaces for a given regular network and generates the corresponding lattice of synchrony subspaces. We illustrate the execution of the algorithm with three network examples in Section 6.3. Finally, in Sections 7 and 8 we show how to obtain the lattice of synchrony subspaces for general nonregular networks using Algorithm 6.5.

As a final remark, we note that it is our intention to have all these algorithms implemented and available through a free-access web page, in a near future, so that they can be executed, for any coupled cell network, by any user without having to know how the algorithms work.

2 Background

We recall briefly a few concepts concerning networks and coupled cell systems. Following Stewart, Golubitsky *et al.* [15, 10, 9], a network is a directed graph whose nodes represent the cells and the arrows (or edges) the couplings. Equivalence relations on the set of nodes and on the set of arrows can be defined symbolizing the following:

(a) Two nodes are in the same cell equivalence class if they represent individual dynamics with the same state space.

(b) Two arrows are in the same arrow equivalence class if they represent couplings of the same type.

The following consistency condition is assumed: if two arrows are of the same type then the corresponding head cells are in the same cell equivalence class and the same holds for the corresponding tail cells.

Definition 2.1 (i) Given a network, the *input set of a cell* of the network is the set of arrows directed to that cell.

(ii) Two cells of a network are said to be *(input) isomorphic* if there is an arrow-type preserving bijection between the corresponding input sets.

(iii) A *homogeneous* coupled cell network is a network in which all cells are (input) isomorphic.

(iv) A *regular* coupled cell network is a homogenous network with only one arrow type. For a regular network, the *valency* is the number of arrows of the input set of any cell and the *adjacency matrix* is the matrix where the (i, j) entry is the number of arrows from cell j to cell i, assuming the set of cells is $\{1, \ldots, n\}$. If v is the valency of a regular network then the corresponding adjacency matrix has v constant row sum.

(v) For a general coupled cell network with set of cells $\{1,\ldots,n\}$ and k arrow equivalence classes, we define k adjacency matrices, one for each arrow type, say $A_1,\ldots,A_k$, in the following way: the (i,j) entry of the matrix A_p is the number of arrows of type p from cell j to cell i. ◇

Example 2.2 Figure 1 shows two examples of 5-cell regular networks. ◇

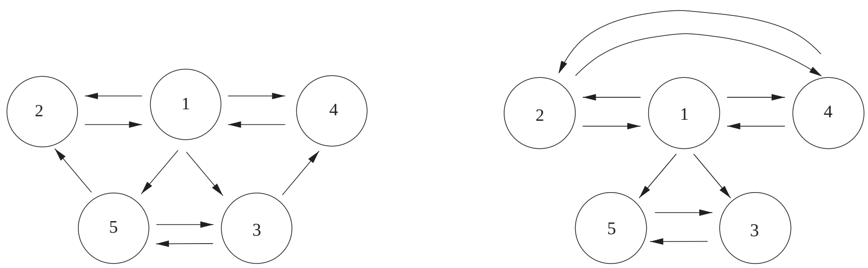

Figure 1: Two examples of 5-cell regular networks.

Following [15, 10, 9], the connection between coupled cell systems and coupled cell networks is made in the following way: to each coupled cell c is associated a choice of *cell phase space* P_c which is assumed to be a finite-dimensional real vector space, say $\mathbf{R}^k$ for some $k > 0$. If cells c and d are cell equivalent then it is required that $P_c = P_d$ and the two spaces are identified canonically. If $\mathcal{C} = \{1,\ldots,n\}$ denotes the set of cells of the network, then the *total phase space* P of the coupled cell system is the direct product of the cell phase spaces, $\prod_{c\in\mathcal{C}} P_c$, and we employ the coordinate system $x = (x_c)_{c\in\mathcal{C}}$ on P. Given a network $\mathcal{G}$ and a fixed choice of the total phase space P, we describe now the coupled cell systems that correspond to the class of the systems of ordinary differential equations, $\dot{X} = F(X),\ X \in P$, compatible with the structure of the network. The system associated with cell j has the form

$$\dot{x}_j = f_j\left(x_j; x_{i_1},\ldots,x_{i_m}\right)$$

where the first argument x_j in f_j represents the internal dynamics of the cell and each of the remaining variables x_{i_p} represents a coupling between cell i_p and cell j. Thus

$x_j \in P_j, x_{i_p} \in P_{i_p},\ p = 1, \ldots, m$ and we assume $f_j :\ P_j \times P_{i_1} \times \cdots \times P_{i_m} \to P_j$ is smooth. Moreover, identical couplings directed to cell j correspond to the invariance of f_j under permutation of the corresponding variables. Systems associated with (input) isomorphic cells are identical up to permutation of the variables accordingly to the input sets of the cells. The vector fields F are called *$\mathcal{G}$-admissible.*

For homogeneous networks (where all cells are input isomorphic) we have only one type of systems, that is, $f_j \equiv f$ for all j. For valency v regular networks, the cell systems have the form

$$\dot{x}_j = f\left(x_j; \overline{x_{i_1}, \ldots, x_{i_v}}\right)$$

where the overbar in f indicates that f is invariant under any permutation of the cell coordinates $x_{i_1}, \ldots, x_{i_v}$ (representing the cells with arrows of the same type directed to cell j).

Example 2.3 The coupled cell systems associated to the network on the left of Figure 1 satisfy

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_4}) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_5}) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_5}) \ , \\ \dot{x}_4 &= f(x_4, \overline{x_1, x_3}) \\ \dot{x}_5 &= f(x_5, \overline{x_1, x_3}) \end{aligned}$$

where $f :\ \left(\mathbf{R}^k\right)^3 \to \mathbf{R}^k$ is smooth and invariant under permutation of the last two cell coordinates. $\diamond$

2.1 Balanced equivalence relations

We recall the definition of a balanced equivalence relation on the set of cells of a network. Balanced equivalence relations of a network play a crucial role when describing the synchrony spaces of a network.

Definition 2.4 ([10] Definition 4.1) Given a network $\mathcal{G}$, an equivalence relation $\bowtie$ on the network set of cells $\mathcal{C}$ is *balanced* if for every $c, d \in \mathcal{C}$ with $c \bowtie d$, there exists an isomorphism between the input sets, $I(c)$ and $I(d)$, of c and d, respectively, say $\beta : I(c) \to I(d)$, preserving the arrow equivalence relation and such that for all $i \in I(c)$, the tail cells of i and $\beta(i)$ are in the same $\bowtie$ class. $\diamond$

Example 2.5 Consider the 5-cell regular network on the left of Figure 1. The equivalence relation on the set of cells $\mathcal{C} = \{1, \ldots, 5\}$ with classes $\{1, 2, 3\}$, $\{4, 5\}$ is balanced. $\diamond$

Definition 2.6 ([10] Definition 2.3) Given a network $\mathcal{G}$ with set of cells $\mathcal{C}$, we can define an *equivalence relation* $\sim_I$ *on* $\mathcal{C}$ in the following way: given $c, d \in \mathcal{C}$, then $c \sim_I d$ if and only if cells c and d are (input) isomorphic (recall Definition 2.1). $\diamond$

Let $\bowtie_i$ and $\bowtie_j$ be two equivalence relations on the set of cells $\mathcal{C}$ of a network $\mathcal{G}$. If for all $c \in \mathcal{C}$ we have

$$[c]_i \subseteq [c]_j,$$

where $[c]_l$ denotes the $\bowtie_l$-equivalence class of cell c, for $l = i, j$, then we say that $\bowtie_i$ *refines* $\bowtie_j$, and we write $\bowtie_i \prec \bowtie_j$.

Remark 2.7 A balanced equivalence relation on a network set of cells refines the input relation $\sim_I$. $\diamond$

2.2 Synchrony subspaces

Definition 2.8 Given a network $\mathcal{G}$, an equivalence relation $\bowtie$ on the network set of cells $\mathcal{C}$ refining the cell equivalence relation, and a choice of the total phase space P, define the *polydiagonal* subspace

$$\Delta_{\bowtie} = \{\mathbf{x} \in P : \; x_c = x_d \text{ whenever } c \bowtie d, \quad \forall c, d \in \mathcal{C}\} \; .$$

The polydiagonal subspace $\Delta_{\bowtie}$ of P is called a *synchrony subspace* if it is flow-invariant for all $\mathcal{G}$-admissible vector fields on P. $\diamond$

Example 2.9 Consider the 5-cell regular network on the left of Figure 1 with set of cells $\mathcal{C} = \{1, \ldots, 5\}$. Taking $\bowtie$ the equivalence relation on $\mathcal{C}$ with classes $\{1, 2, 3\}$, $\{4, 5\}$, then $\Delta_{\bowtie} = \{\mathbf{x} \in P : \; x_1 = x_2 = x_3, \, x_4 = x_5\}$. It follows easily from the network admissible equations (see Example 2.3) that $\Delta_{\bowtie}$ is a synchrony subspace. Moreover, their restriction to $\Delta_{\bowtie}$ is:

$$\begin{array}{l} \dot{x}_1 = f(x_1, \overline{x_1, x_4}) \\ \dot{x}_4 = f(x_4, \overline{x_1, x_1}) \end{array} .$$

$\diamond$

We present now the result of [10] relating balanced equivalence relations on the set of cells and the synchrony spaces of a network.

Theorem 2.10 ([10] Theorem 4.3) *Given a network $\mathcal{G}$, an equivalence relation $\bowtie$ on the network set of cells $\mathcal{C}$ and a choice P of the total phase space, then $\Delta_{\bowtie}$ is a synchrony subspace if and only if $\bowtie$ is balanced.*

Proof The proof of this result is divided into two steps. Check directly that $\bowtie$ being balanced is sufficient for $\Delta_{\bowtie}$ to be a synchrony subspace. The necessity is established by considering linear vector fields. $\square$

Now observe that, an equivalence relation $\bowtie$ on the network set of cells is balanced if and only if each of the adjacency matrices of the network leaves invariant the polydiagonal subspace $\Delta_{\bowtie}$ choosing all the cell phase spaces to be $\mathbf{R}$. Moreover, the polydiagonal subspace $\Delta_{\bowtie}$ is invariant under the network adjacency matrices if and only it is flow-invariant for all the linear admissible vector fields, assuming the cell phase spaces to be $\mathbf{R}$. That is, we have the following result:

Corollary 2.11 *Let $\mathcal{G}$ be a coupled cell network with set of cells $\mathcal{C}$ and let $\bowtie$ be an equivalence relation on $\mathcal{C}$. Then for any choice of the total phase space P, the subspace $\Delta_{\bowtie}$ is a synchrony subspace if and only if it is flow-invariant for* all linear *admissible vector fields choosing the cell phase spaces to be* $\mathbf{R}$.

Remark 2.12 It follows then that a polydiagonal $\Delta_{\bowtie}$ is a synchrony subspace if and only if the corresponding polydiagonal, assuming the cell phase spaces to be $\mathbf{R}$, is left invariant by the network adjacency matrices.
$\diamond$

3 Complete lattices

In this section we start by reviewing some basic facts about lattices and complete lattices. Details can be found, for example, at Davey and Priestley [5]. We then recall the results establishing that both the lattices of balanced equivalence relations and synchrony subspaces for a given network are complete lattices, taking the relation of refinement and inclusion $\subseteq$ of spaces, respectively.

3.1 Basic definitions

Given a partially ordered set X with a binary relation $\geq$ and a subset $Y \subseteq X$, an element x of X is an *upper bound* of Y if $x \geq y$ for all $y \in Y$. Further, an upper bound x of Y is said to be the *least upper bound* of Y if every upper bound x' of Y satisfies $x' \geq x$. Dually, we define *lower bound* and *greatest lower bound*.

Now recall that a *lattice* is a partially ordered set X such that every pair of elements $x, y \in X$ has a *unique least upper bound or join*, denoted by $x \vee y$, and a *unique greatest lower bound or meet*, denoted by $x \wedge y$.

A *complete lattice* is a lattice where every subset $Y \subseteq X$ has a unique least upper bound or join, and a unique greatest lower bound or meet. A complete lattice has a *top* (maximal) element, denoted $\top$, and a *bottom* (minimal) element, denoted $\bot$. Observe that every finite lattice is complete, see [5, Corollary 2.12].

Example 3.1 Given a linear map $A: \mathbf{R}^n \to \mathbf{R}^n$, the set of A-invariant subspaces is a lattice (considering the partial order $\subseteq$) with the meet operation corresponding to the intersection and the join operation given by the sum. (The sum corresponds to the subspace generated by the union.) The top element is $\mathbf{R}^n$ and the bottom element is $\{0\}$. Moreover, that lattice is either finite or uncountably infinite. See for example Gohberg *et al.* [7, Proposition 2.5.4]. $\diamond$

A *sublattice* M of a lattice X is a subset such that

$$x \in M \text{ and } y \in M \Longrightarrow x \vee y \in M \text{ and } x \wedge y \in M\,.$$

Remark 3.2 Observe that a subset of a lattice X may form a lattice according to the definition of lattice without being a sublattice of X. For example, it may happen that the greatest lower bound of x and y in the subset differs from the greatest lower bound $x \wedge y$ of x and y in X. $\diamond$

An element x in a lattice L is *join-irreducible* if

(i) $x \neq 0$ (in case L has a zero);

(ii) $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in L$.

A *meet-irreducible* element is defined dually. See for example Davey [5, Definition 8.7].

A subset S of a lattice L is called *join-dense* in L if for every element $a \in L$, there exists a subset A of S such that $a = \vee A$. The dual of join-dense is *meet-dense*. See for example [5, Definition 2.34].

Remark 3.3 It is known that, for a finite lattice, the set of join-irreducible elements is a join-dense set for the lattice. That is, every element of the lattice is the join of join-irreducible lattice elements. It follows then that the set of join-irreducible elements of the lattice generates the lattice. See Järvinen [11, Lemma 24]. $\diamond$

3.2 Complete lattice of balanced equivalence relations

Let $\mathcal{G}$ be a network with set of cells $\mathcal{C}$. Denote by $M_{\mathcal{G}}$ the set of equivalence relations on $\mathcal{C}$ and take the relation of refinement on $M_{\mathcal{G}}$. It follows that $M_{\mathcal{G}}$ is a complete lattice with the meet and join operations on the set $M_{\mathcal{G}}$ defined in the following way:

Meet operation: we have that $\bowtie_k = \bowtie_i \wedge \bowtie_j$ where if $c, d \in \mathcal{C}$ then $c \bowtie_k d$ if and only if $c \bowtie_i d$ and $c \bowtie_j d$.

Join operation: we have that $\bowtie_k = \bowtie_i \vee \bowtie_j$ where if $c, d \in \mathcal{C}$ then $c \bowtie_k d$ if and only if there exists a finite chain $c = c_q, \ldots, c_s = d$ such that for all t with $q \leq t \leq s-1$ either $c_t \bowtie_i c_{t+1}$ or $c_t \bowtie_j c_{t+1}$.

Now denote by $\Lambda_{\mathcal{G}}$ the set of balanced equivalence relations of a network $\mathcal{G}$ on the set of cells $\mathcal{C}$ and again note that $\Lambda_{\mathcal{G}}$ has a partially ordered structure, using the relation of refinement $\prec$ as defined in Section 2.1. Stewart [14] proves that the set $\Lambda_{\mathcal{G}}$ of balanced equivalence relations of a (locally finite) network forms a complete lattice. See also Aldis [4, Chapter 4].

The lattice of balanced equivalence relations $\Lambda_{\mathcal{G}}$ is not in general a sublattice of the lattice of equivalence relations $M_{\mathcal{G}}$. The join operation on equivalence relations restricts to give the join operation on balanced equivalence relations. However, in general, the meet operation on $M_{\mathcal{G}}$ does not restrict to the meet operation on $\Lambda_{\mathcal{G}}$. This follows from the fact that, using the meet operation on $M_{\mathcal{G}}$, the meet of two balanced equivalence relations is not in general a balanced equivalence relation. See Stewart [14, Example 5.5].

Remark 3.4 Consider the lattice $\Lambda_{\mathcal{G}}$ of balanced equivalence relations for a network $\mathcal{G}$.
(i) If $\mathcal{G}$ is homogeneous then the top element is the balanced equivalence relation with only one class, and the bottom element corresponds to the equivalence relation where each class is formed by a unique cell.

(ii) If $\mathcal{G}$ is nonhomogeneous, again the bottom element corresponds to the equivalence relation where each class is formed by a unique cell. However, the top element does not have to correspond to the input-equivalence relation. Following the notation of [14], the top element corresponds to a unique coarsest balanced equivalence relation, where $\bowtie_1$ is *coarser* than $\bowtie_2$ if and only if $\bowtie_2$ is finer than $\bowtie_1$. This relation refines $\sim_I$ but it is not $\sim_I$ if the input relation $\sim_I$ is not balanced. Take as an example, the chain of three identical cells $1 \leftarrow 2 \leftarrow 3$ where $\sim_I$ has two classes, $\{1, 2\}$ and $\{3\}$, and it is not balanced. Aldis [1] presents an algorithm to compute the coarsest balanced equivalence relation of a given network $\mathcal{G}$ in polynomial time (in the number of cells plus the number of edges of $\mathcal{G}$). $\diamond$

3.3 Complete lattice of synchrony subspaces

Let $V_{\mathcal{G}}^P$ the set of synchrony subspaces for $\mathcal{G}$ assuming the total phase space is P. By Theorem 2.10 there is a one-to-one correspondence between the elements of $\Lambda_{\mathcal{G}}$ and $V_{\mathcal{G}}^P$. Moreover, an equivalence relation $\bowtie$ on the network set of cells is balanced if and only if the associated polydiagonal $\Delta_{\bowtie} \subseteq P$ is left invariant by all linear network admissible vector fields. Also, for that purpose, we can take the cell phase spaces to be $\mathbf{R}$ and so the total phase space is $P = \mathbf{R}^n$ (if n is the number of cells) and check invariance of $\Delta_{\bowtie} \subseteq \mathbf{R}^n$ by all the linear admissible vector fields on $\mathbf{R}^n$ (see Corollary 2.11). From now on we denote by $V_{\mathcal{G}}$ the set of synchrony subspaces for $\mathcal{G}$ with $P = \mathbf{R}^n$. Note that $V_{\mathcal{G}}$ is a lattice taking the partial order on $V_{\mathcal{G}}$ given by inclusion $\subseteq$ of spaces.

The map $\delta : \Lambda_{\mathcal{G}} \to V_{\mathcal{G}}$ defined by $\delta(\bowtie) = \Delta_{\bowtie}$ for $\bowtie \in \Lambda_{\mathcal{G}}$ is a lattice anti-isomorphism, that is, an isomorphism that reverses order, hence interchanges meet and join, see Stewart [14]. In particular, we have

$$\Delta_{\bowtie_1 \vee \bowtie_2} = \Delta_{\bowtie_1} \cap \Delta_{\bowtie_2}.$$

Moreover, since $\Lambda_{\mathcal{G}}$ is a complete lattice, it follows that $V_{\mathcal{G}}$ is also a complete lattice.

Remark 3.5 (i) The lattice of synchrony subspaces is not in general a sublattice of the lattice of the A-invariant subspaces. (Recall Example 3.1 and Remark 3.2.) The meet operation is the same, the intersection of subspaces, but the join of two synchrony subspaces may not be given by their sum. The join of two synchrony subspaces is given by their sum only when this is a polydiagonal subspace. Note that the sum of two synchrony subspaces is always A-invariant but it may not be a polydiagonal subspace.
(ii) Apparently, there is no general form for the join operation in the lattice of synchrony subspaces. Nevertheless, the join can be defined in terms of the meet. The join of two synchrony subspaces V_1 and V_2 is the meet of all the elements in the lattice $V_{\mathcal{G}}$ that are greater than or equal to both V_1 and V_2. ◇

Remark 3.6 (i) Kamei [12] describes the lattice of balanced equivalence relations (and so the lattice of synchrony spaces) for regular networks using the eigenvalue structure of the network adjacency matrix for the cases where the adjacency matrix has only simple eigenvalues. We observe that by Corollary 2.11, the results of [12] for regular networks are valid for k-dimensional internal dynamics.
(ii) Recalling Remark 3.4, if $\mathcal{G}$ is an homogeneous network, then the top element of the lattice $\Lambda_{\mathcal{G}}$ of balanced equivalence relations (the balanced equivalence relation with only one class) corresponds to the full synchronous polydiagonal space. The bottom element (the equivalence relation where each class is formed by a unique cell) corresponds to the total asynchronous polydiagonal space. ◇

4 Description of the lattice of synchrony subspaces of a network

The question we address in this section is the description of the lattice of synchrony spaces for a network $\mathcal{G}$. As we will see, this can be done in terms of the lattice of synchrony subspaces for the identical-edge subnetworks of $\mathcal{G}$ that we define below.

Let $\mathcal{G}$ be a coupled cell network with set of cells $\mathcal{C}$ and set of arrows $\mathcal{E}$. Assume $\sim_{\mathcal{C}}$ is an equivalence relation on $\mathcal{C}$ where each $\sim_{\mathcal{C}}$-class represents a cell type. Also let $\sim_{\mathcal{E}}$ be the equivalence relation on $\mathcal{E}$ where each $\sim_{\mathcal{E}}$-class determines an edge type and denote by $\mathcal{E}_1, \ldots, \mathcal{E}_l$ the $\sim_{\mathcal{E}}$-equivalence classes. Thus $\mathcal{E} = \dot{\cup}_j \mathcal{E}_j$ where j runs through the set $\{1, \ldots, l\}$ and $\dot{\cup}$ denotes disjoint union. We can write $\mathcal{G} = \left(\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}}\right)$.

Let $I_1, \ldots, I_k \subset \mathcal{C}$ be the $\sim_I$-equivalence classes of the cells $\mathcal{C}$ in $\mathcal{G}$. For $i = 1, \ldots, k$, denote by $\mathcal{E}^{I_i}$ the subset of $\mathcal{E}$ of the edges that are directed to cells in I_i and $\mathcal{G}^{I_i} = \left(\mathcal{C}, \mathcal{E}^{I_i}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}}\right)$ where two edges $e_1, e_2 \in \mathcal{E}^{I_i}$ are equivalent if they are equivalent as edges in $\mathcal{E}$ of the network $\mathcal{G}$. Thus, each network $\mathcal{G}^{I_i}$ is a subnetwork of $\mathcal{G}$.

Let r_i be the number of edge types in $\mathcal{G}^{I_i}$. For each edge type $\mathcal{E}_{i_1}, \ldots, \mathcal{E}_{i_{r_i}}$ with $i_1, \ldots, i_{r_i} \in \{1, \ldots, l\}$, consider the subnetwork of $\mathcal{G}^{I_i}$ given by $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}} = \left(\mathcal{C}, \mathcal{E}^{I_i} \cap \mathcal{E}_{i_j}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}}\right)$. Thus, each $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}}$ is the subnetwork of $\mathcal{G}^{I_i}$ with the edges of type $\mathcal{E}_{i_j}$ directed to the cells in I_i.

Recall that $\Lambda_{\mathcal{G}}$ denotes the set of balanced equivalence relations for $\mathcal{G}$. By Remark 2.7 any $\bowtie \in \Lambda_{\mathcal{G}}$ refines $\sim_I$. Denote by $\Lambda^{I_i}_{\mathcal{G}}$ the set of balanced equivalence relations for $\mathcal{G}^{I_i}$ that also refine $\sim_I$. Finally, let $\Lambda^{\mathcal{E}_{i_j}}_{\mathcal{G}}$ be the set of balanced equivalence relations for $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}}$ (refining $\sim_I$). We have that each $\Lambda^{I_i}_{\mathcal{G}}$ and each $\Lambda^{\mathcal{E}_{i_j}}_{\mathcal{G}}$ is a complete lattice.

Example 4.1 Consider the nonhomogeneous 8-cell network of Figure 2. The network has three $\sim_I$-equivalence classes of cells, $I_1 = \{1, 2\}$, $I_2 = \{3, 4, 5\}$ and $I_3 = \{6, 7, 8\}$, and the networks $\mathcal{G}^{I_i}$ are represented in Figure 3.

$\diamond$

Theorem 4.2 *Let $\mathcal{G} = \left(\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}}\right)$ be a coupled cell network and consider the lattices $\Lambda_{\mathcal{G}}, \Lambda^{I_i}_{\mathcal{G}}, \Lambda^{\mathcal{E}_{i_j}}_{\mathcal{G}}$ and the notation defined above. Then the following holds:*
(i) The set inclusions:

$$\Lambda_{\mathcal{G}} \subseteq \Lambda^{I_i}_{\mathcal{G}} \subseteq \Lambda^{\mathcal{E}_{i_j}}_{\mathcal{G}} \, .$$

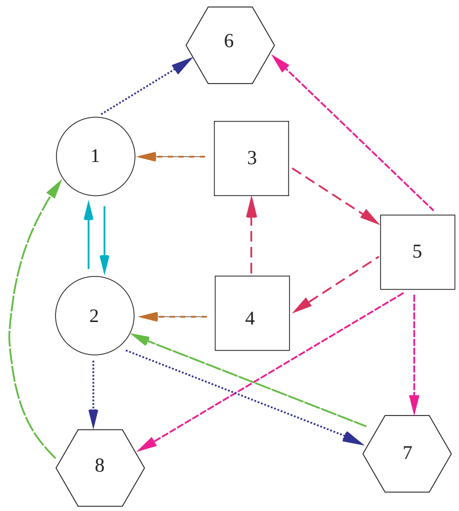

Figure 2: One example of a nonhomogeneous network of 8 cells with three $\sim_I$-equivalence classes of cells: $I_1 = \{1, 2\}$, $I_2 = \{3, 4, 5\}$ and $I_3 = \{6, 7, 8\}$.

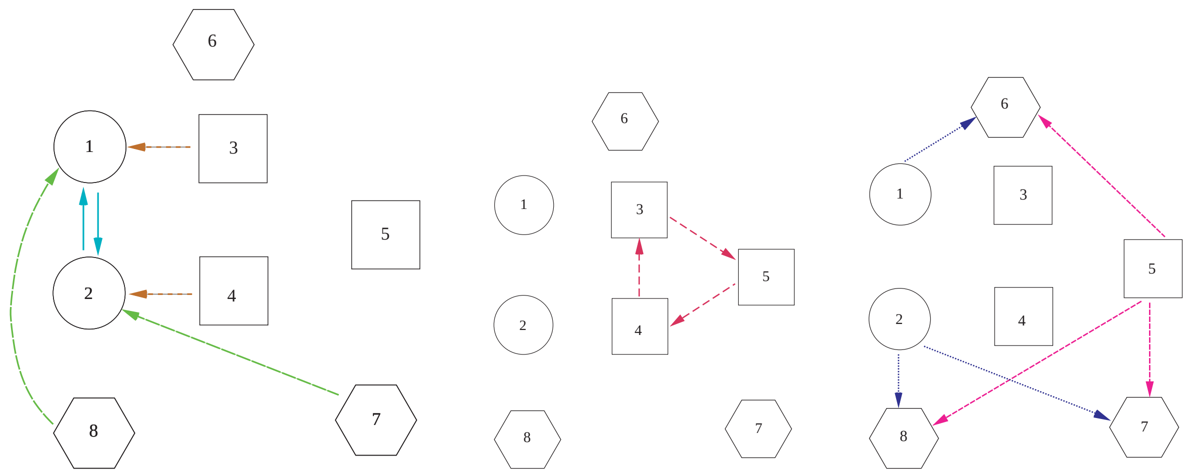

Figure 3: The three networks $\mathcal{G}^{I_i}$ for each $\sim_I$-equivalence class I_i of cells of the 8-cell nonhomogeneous network of Figure 2.

(ii) The lattice $\Lambda_{\mathcal{G}}$ *is given by:*

$$\Lambda_{\mathcal{G}} = \bigcap_{i=1}^{k} \bigcap_{j=1}^{r_i} \Lambda_{\mathcal{G}}^{\mathcal{E}_{i_j}} .$$

Proof The proof of the set inclusions in (i) comes directly from the definition of balanced equivalence relation. We prove now (ii). Let $\bowtie$ be an equivalence relation on the set of cells $\mathcal{C}$ of the network $\mathcal{G}$ refining $\sim_I$ and recall Definition 2.4 of balanced equivalence relation. Consider the subnetworks obtained from $\mathcal{G}$ by considering only edges of one type, say $\mathcal{G}_i = \left(\mathcal{C}, \mathcal{E}_i, \sim_{\mathcal{C}}, \sim_{\mathcal{E}_i}\right)$, for $i = 1, \ldots, l$, where $\sim_{\mathcal{E}_i}$ denotes the relation on $\mathcal{E}_i$ with only the class $\mathcal{E}_i$. Trivially, $\bowtie$ is balanced for $\mathcal{G}$ if and only if $\bowtie$ is balanced for all $\mathcal{G}_i$, $i = 1, \ldots, l$. Moreover, given $i \in \{1, \ldots, l\}$ the relation $\bowtie$ is balanced for the subnetwork $\mathcal{G}_i$ if and only if it is balanced for all the subnetworks $\mathcal{G}^{I_j}_{\mathcal{E}_i}$ where I_j runs through the $\sim_I$-equivalence classes of cells having edges of type $\mathcal{E}_i$ directed to them. $\square$

Corollary 4.3 *Let $\mathcal{G} = \left(\mathcal{C}, \mathcal{E}, \sim_{\mathcal{C}}, \sim_{\mathcal{E}}\right)$ be a homogeneous coupled cell network. Denote by $\mathcal{E}_1, \ldots, \mathcal{E}_l$ the $\sim_{\mathcal{E}}$-equivalence classes. For $j = 1, \ldots, l$, let $\mathcal{G}_{\mathcal{E}_j}$ be the subnetwork of $\mathcal{G}$ given by $\left(\mathcal{C}, \mathcal{E} \cap \mathcal{E}_j, \sim_{\mathcal{C}}, \sim_{\mathcal{E}}\right)$ and $\Lambda^{\mathcal{E}_j}_{\mathcal{G}}$ be the set of balanced equivalence relations for $\mathcal{G}_{\mathcal{E}_j}$. Then the following holds:*

$$\Lambda_{\mathcal{G}} = \bigcap_{j=1}^{l} \Lambda^{\mathcal{E}_j}_{\mathcal{G}} .$$

Example 4.4 Consider the homogeneous 5-cell network of Figure 4 with two types of coupling. The two networks $\mathcal{G}_{\mathcal{E}_j}$ are represented in Figure 1. $\diamond$

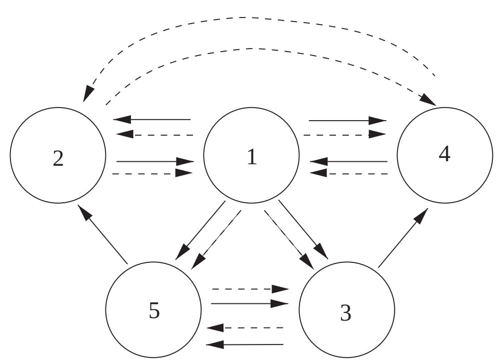

Figure 4: A 5-cell homogeneous network with two types of coupling.

From Theorem 4.2, the problem of determining the lattice of synchrony subspaces for a network $\mathcal{G}$ basically reduces to determining the synchrony subspaces of the identical-edge subnetworks $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}}$ of $\mathcal{G}$. Note that, for each identical-edge subnetwork associated with an

$\sim_I$-equivalence class I_i, the cells in I_i, receive the same number of inputs (of the same type). Roughly speaking, each such network is regular at the cells in the input class. In Sections 7 and 8 we present some results that show how to define an algorithm to obtain the lattice of synchrony subspaces for these subnetworks using Algorithm 6.5 (for regular networks), and we illustrate that with some examples.

5 Description of the lattice of synchrony subspaces of a regular network

In the next sections we concentrate our attention on how to obtain all the synchrony subspaces for a regular network. We assume the cell phase spaces to be $\mathbf{R}$ and basically describe the polydiagonals that are left invariant by the network adjacency matrix (See Remark 2.12). In particular, the synchrony subspaces form a subset of the set of all invariant subspaces under the network adjacency matrix.

5.1 Invariant subspaces and eigenvectors of the adjacency matrix

The subspaces invariant by the adjacency matrix A of a regular network $\mathcal{G}$ can be described in terms of its eigenvectors and generalized eigenvectors.

In what follows we use the following notation for eigenspaces. If $\lambda \in \mathbf{C}$ is an eigenvalue of a $n \times n$ matrix A with real entries, we define the (real) λ-*eigenspace* E_λ of A as follows:

$$E_\lambda = \begin{cases} \ker\left(A - \lambda \mathrm{Id}_n\right), & (\text{ if } \lambda \in \mathbf{R}), \\ \\ \ker\left[\left(A - \lambda \mathrm{Id}_n\right)\left(A - \overline{\lambda}\mathrm{Id}_n\right)\right], & (\text{ if } \lambda \notin \mathbf{R}). \end{cases}$$

By Gohberg *et al.* [7], every one-dimensional A-invariant subspace (one-dimensional

A_c-invariant subspace, where A_c is the complexification of A) is spanned by some eigenvector $\mathbf{v}$ of A associated with a real eigenvalue of A (eigenvector $\mathbf{u}+i\mathbf{v}$ of A_c associated with a non-real complex eigenvalue of A).

We say that A is *semisimple* when A_c is diagonalizable and, in that case, all the A-invariant subspaces are given by direct sums of some of the above one-dimensional A-invariant subspaces. If A is not semisimple, then we have to take into account generalized eigenvectors.

We denote by G_λ the (real) *generalized eigenspace* of A associated to the eigenvalue λ, which is defined as follows:

$$G_\lambda = \begin{cases} \ker\left(A - \lambda \mathrm{Id}_n\right)^p, & (\text{ if } \lambda \in \mathbf{R}), \\ \\ \ker\left[\left(A - \lambda \mathrm{Id}_n\right)^p \left(A - \overline{\lambda} \mathrm{Id}_n\right)^p\right], & (\text{ if } \lambda \notin \mathbf{R}), \end{cases}$$

with $p \geq 1$ the minimal integer such that $\ker\left(A - \lambda \mathrm{Id}_n\right)^i = \ker\left(A - \lambda \mathrm{Id}_n\right)^p$, for all $i > p$.

If λ is an eigenvalue of A, the chain of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a *Jordan chain* of A corresponding to λ if

$$\mathbf{v}_1 \neq 0, \quad A\mathbf{v}_1 = \lambda \mathbf{v}_1 \quad \text{and} \quad (A - \lambda \mathrm{Id}_n)\mathbf{v}_{j+1} = \mathbf{v}_j, \text{ for } j = 1, \ldots, k-1.$$

Thus, $\mathbf{v}_1$ is an eigenvector of A associated to λ and $\mathbf{v}_2, \ldots, \mathbf{v}_k$ are called *generalized eigenvectors* of A corresponding to the eigenvalue λ and the eigenvector $\mathbf{v}_1$. The subspace generated by a Jordan chain of A is A-invariant and the vectors in a Jordan chain are linearly independent [7, Propositions 1.3.1 and 1.3.4].

The space G_λ is also called the *root subspace* of A corresponding to λ and it contains the vectors from any Jordan chain of A corresponding to λ [7, Proposition 2.1.1].

We have that if $\lambda_1, \ldots, \lambda_r$ are all the different eigenvalues of the linear transformation

A then $\mathbf{R}^n$ is the direct sum of the generalized eigenspaces $G_{\lambda_1}, \ldots, G_{\lambda_r}$:

$$\mathbf{R}^n = G_{\lambda_1} \oplus \cdots \oplus G_{\lambda_r} .$$

Moreover, if S is an A-invariant subspace then S decomposes into a direct sum

$$S = (G_{\lambda_1} \cap S) \oplus \cdots \oplus (G_{\lambda_r} \cap S) .$$

See for example [7, Theorems 2.1.2 and 2.1.5].

5.1.1 Irreducible invariant subspaces

We start by recalling the definition of Jordan subspaces and irreducible invariant subspaces V for linear maps $A: \mathbf{R}^n \to \mathbf{R}^n$. If A has complex eigenvalues, we should read $A_c : \mathbf{C}^n \to \mathbf{C}^n$ and V_c in what follows.

Definition 5.1 (a) An A-invariant subspace V is called a *Jordan subspace* corresponding to the eigenvalue λ of A if V has a basis consisting of vectors that form a Jordan chain.
(b) An A-invariant subspace is called *irreducible* if it cannot be represented as a direct sum of nonzero A-invariant subspaces. $\diamond$

By [7, Theorem 2.5.1], an A-invariant subspace V is irreducible if and only if there is a unique eigenvector (up to multiplication by a scalar) of A in V or, equivalently, if and only if V is a Jordan subspace.

Remark 5.2 Observe that an invariant subspace V_c of $A_c : \mathbf{C}^n \to \mathbf{C}^n$ is irreducible and contains an eigenvector $\mathbf{u} + i\mathbf{v}$ (with $\mathbf{v} \neq \mathbf{0}$) if and only if the corresponding irreducible subspace over the reals contains $< \mathbf{v}, \mathbf{u} >$. See [7, Theorem 12.2.4, p. 365] for details on the interpretation of this result in case of complex eigenvalues for real operators. $\diamond$

Since every A-invariant subspace is a direct sum of irreducible A-invariant subspaces, in order to describe all the A-invariant subspaces it is sufficient to describe the irreducible

A-invariant subspaces, that is, the irreducible A-invariant subspaces (Jordan subspaces) contained in the generalized eigenspaces associated to each eigenvalue λ of A.

5.2 Polydiagonals and eigenvectors of the adjacency matrix

Definition 5.3 Let $\mathbf{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, with $\mathbf{v}_i = (v_{i1}, \ldots, v_{in})$, be a set of (generalized) eigenvectors of the adjacency matrix A. Define the following equivalence relation associated with the set $\mathbf{V}$:

$$c \bowtie_{\mathbf{V}} d \Longleftrightarrow v_{ic} = v_{id} \text{ for all } i \in \{1, \ldots, k\}.$$

Denote the corresponding polydiagonal subspace by $\Delta_{\bowtie_{\mathbf{V}}}$. ◇

Remark 5.4 Let $\mathbf{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, with $\mathbf{v}_i = (v_{i1}, \ldots, v_{in})$, be a set of (generalized) eigenvectors of the adjacency matrix A_c, considering that at least one of the vectors $\mathbf{v}_i$ is associated with a complex eigenvalue of A_c (or A) and where $\mathrm{Im}\mathbf{v}_i \neq \mathbf{0}$. Denote by $\mathrm{Re}\ \mathbf{V} = \{\mathrm{Re}\ \mathbf{v}_1, \ldots, \mathrm{Re}\ \mathbf{v}_k\}$, $\mathrm{Im}\ \mathbf{V} = \{\mathrm{Im}\ \mathbf{v}_1, \ldots, \mathrm{Im}\ \mathbf{v}_k\}$ and $\mathbf{V}_R = \mathrm{Re}\ \mathbf{V} \cup \mathrm{Im}\ \mathbf{V}$. Then $\bowtie_{\mathbf{V}} = \bowtie_{\mathbf{V}_R}$ and

$$\Delta_{\bowtie_{\mathbf{V}}} \cong \Delta_{\bowtie_{\mathbf{V}_R}}$$

interpreting $\Delta_{\bowtie_{\mathbf{V}}}$ as a complex vector space and $\Delta_{\bowtie_{\mathbf{V}_R}}$ as real vector space. This follows from the fact that

$$v_{ic} = v_{id} \text{ for all } i \in \{1, \ldots, k\} \Leftrightarrow \begin{cases} \mathrm{Re}\ v_{ic} = \mathrm{Re}\ v_{id} \\ \mathrm{Im}\ v_{ic} = \mathrm{Im}\ v_{id} \end{cases} \text{ for all } i \in \{1, \ldots, k\}.$$

◇

Definition 5.5 The polydiagonals $\mathbf{R}^n$ and $\{\mathbf{v} \in \mathbf{R}^n : \ v_i = v_j, \text{ for all } i, j\}$ are called the *trivial synchrony subspaces* of a regular network. ◇

Remark 5.6 (i) $\Delta_{\bowtie \mathbf{V}}$ can be equal to $\mathbf{R}^n$ or $\{\mathbf{v} \in \mathbf{R}^n : \ v_i = v_j, \text{ for all } i,j\}$ (the trivial synchrony subspaces).

(ii) We can have $\Delta_{\bowtie \mathbf{V}} = \Delta_{\bowtie \mathbf{W}}$ with $\mathbf{V} \neq \mathbf{W}$. Moreover, we can have $\Delta_{\bowtie \mathbf{V}} = \Delta_{\bowtie \mathbf{W}}$ with $< \mathbf{V} > \neq < \mathbf{W} >$.

(iii) Not all polydiagonal subspaces $\Delta_{\bowtie \mathbf{V}}$ are synchrony subspaces.

$\diamond$

From the discussion in Sections 5.1 and 5.2, it follows that the nontrivial synchrony subspaces associated to a regular network can be described in terms of the eigenvectors and Jordan chains of its adjacency matrix with nontrivial polydiagonal subspace.

5.3 Sum-dense set for the lattice of synchrony subspaces

Considering the lattice $V_{\mathcal{G}}$ of the synchrony subspaces of a regular network $\mathcal{G}$ with adjacency matrix A, we aim to describe a set of synchrony subspaces such that every synchrony subspace in $V_{\mathcal{G}}$ can be obtained from that set, a kind of join-dense set of join-irreducible elements in $V_{\mathcal{G}}$. (Recall end of Section 3.1.)

Note that in the case of the lattice of the A-invariant subspaces, in which the join of two subspaces is given by their sum, the set of irreducible invariant subspaces of A forms the set of join-irreducible elements which is join-dense in that lattice – see Section 5.1.1.

As observed in Remark 3.5, we cannot describe explicitly the join operation for the lattice $V_{\mathcal{G}}$ of synchrony subspaces: the sum of two synchrony subspaces may not be a synchrony subspace. However, every synchrony subspace is given by the sum of (irreducible) A-invariant subspaces. In fact, defining the concept of *sum-irreducible synchrony subspace*, more can be said.

Definition 5.7 A synchrony subspace of a regular network $\mathcal{G}$ with adjacency matrix A is called *sum-irreducible* if it cannot be represented as a sum of proper synchrony subspaces of $\mathcal{G}$. A synchrony subspace of $\mathcal{G}$ which is not irreducible is called *sum-reducible.* $\diamond$

Proposition 5.8 Every synchrony subspace associated with a regular network $\mathcal{G}$ is a sum of sum-irreducible synchrony subspaces.

Proof Let $\Delta_{\bowtie}$ be a synchrony subspace of a regular network $\mathcal{G}$ with adjacency matrix A and recall that the lattice of synchrony subspaces is a subset of the lattice of the A-invariant subspaces. If $\Delta_{\bowtie}$ is a join-irreducible A-invariant subspace, then it is a sum-irreducible synchrony subspace. If $\Delta_{\bowtie}$ is not a join-irreducible A-invariant subspace, then it is the join of proper A-invariant subspaces. The lattice of the A-invariant subspaces gives all the possible decompositions of $\Delta_{\bowtie}$ as the join of proper A-invariant subspaces, that is, as the sum of proper A-invariant subspaces. If none of these decompositions is formed by synchrony subspaces (A-invariant subspaces that are also polydiagonal) then $\Delta_{\bowtie}$ is sum-irreducible. Otherwise, $\Delta_{\bowtie}$ is sum-reducible, that is, a sum of proper synchrony subspaces. Recursively, we obtain the result. □

In analogy with the definitions of join-dense set and meet-dense set (recall Section 3.1), we define the concept of *sum-dense set*.

Definition 5.9 A *sum-dense set* for the lattice $V_{\mathcal{G}}$ of synchrony subspaces for a regular network $\mathcal{G}$ is a subset of $V_{\mathcal{G}}$ such that every nontrivial synchrony subspace in $V_{\mathcal{G}}$ can be given by the sum of elements in that subset. By Proposition 5.8, the set of sum-irreducible synchrony subspaces in $V_{\mathcal{G}}$ is a sum-dense set, that we call the *irreducible sum-dense set* of $V_{\mathcal{G}}$ and denote by $\mathcal{I}_{\mathcal{G}}$. ◇

Note that, as mentioned in Remark 5.6 (iii), not every polydiagonal subspace $\Delta_{\bowtie_{\mathbf{V}}}$ is a synchrony subspace. Nevertheless, that polydiagonal is contained in one or more synchrony subspaces. This motivates the following definition.

Definition 5.10 Given an eigenvector or Jordan chain $\mathbf{V}$, let $S_{\mathbf{V}}$ be the set of synchrony subspaces Δ_j, $j = 1, \ldots, r$ such that $\Delta_{\bowtie_{\mathbf{V}}} \subseteq \Delta_j$. The *minimal synchrony subspace*, $\mathbf{m}_{\mathbf{V}}$, associated to the eigenvector or Jordan chain $\mathbf{V}$ is the intersection of the synchrony subspaces in $S_{\mathbf{V}}$. ◇

Remark 5.11 (i) Note that $\mathbf{m_V}$ can be equal to $\mathbf{R}^n$ (or $\{\mathbf{v} \in \mathbf{R}^n : \ v_i = v_j, \ \text{for all } i,j\}$).
(ii) A polydiagonal $\Delta_{\bowtie \mathbf{V}}$ associated to an eigenvector or Jordan chain $\mathbf{V}$ is a synchrony subspace if and only if the minimal synchrony subspace $\mathbf{m_V}$ associated to $\mathbf{V}$ is $\Delta_{\bowtie \mathbf{V}}$. $\diamond$

Definition 5.12 (a) Let λ be an eigenvalue of A with algebraic and geometric multiplicities m^a and m^g, respectively.

(a.i) If $m^a = m^g$, take V_λ^Δ to be the set of the eigenvectors $\mathbf{v}$ associated with the eigenvalue λ such that $\Delta_{\bowtie \{\mathbf{v}\}}$ is not $\mathbf{R}^n$. That is,

$$V_\lambda^\Delta = \{\mathbf{v} \in E_\lambda : \ v_p = v_l \text{ for some } p \neq l\} \ .$$

(a.ii) If $m^a \neq m^g$, take V_λ^Δ to be the set of Jordan chains $\mathbf{v}_1, \ldots, \mathbf{v}_k$ corresponding to the eigenvalue λ, where $\mathbf{v}_1$ is an eigenvector associated with λ and $\Delta_{\bowtie \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}}$ is not $\mathbf{R}^n$. That is,

$$\begin{aligned} V_\lambda^\Delta = \{\mathbf{v}_1, \ldots, \mathbf{v}_k : \ \mathbf{v}_1 \in E_\lambda, \quad & (A - \lambda \mathrm{Id})\mathbf{v}_{j+1} = \mathbf{v}_j \text{ for } j = 1, \ldots, k-1 \text{ and} \\ & v_{jp} = v_{jl} \text{ for all } j = 1, \ldots, k \text{ and some } p \neq l\} \ . \end{aligned}$$

(b) Let

$$V_A^\Delta = \bigcup_{i=1}^{s} V_{\lambda_i}^\Delta$$

with λ_i, for $i = 1, \ldots, s$, where $s \leq n$, the eigenvalues of the matrix A. $\diamond$

Theorem 5.13 *The minimal synchrony set,*

$$\mathbf{m}_{\mathcal{G}} = \bigcup_{\mathbf{V}_i \in V_A^\Delta} \{\mathbf{m}_{\mathbf{V}_i}\} ,$$

associated to the eigenvectors and Jordan chains in V_A^Δ is a sum-dense set for the lattice $V_{\mathcal{G}}$ of synchrony subspaces.

Proof Let $\Delta_{\bowtie}$ be a nontrivial synchrony subspace in $V_{\mathcal{G}}$. We prove that $\Delta_{\bowtie}$ can be given by a sum of synchrony subspaces in $\mathbf{m}_{\mathcal{G}}$.

Since $\Delta_{\bowtie}$ is A-invariant, there is a decomposition of $\Delta_{\bowtie}$ into a direct sum of A-invariant irreducible subspaces, say S_i, for $i = 1, \ldots, p$,

$$\Delta_{\bowtie} = S_1 \oplus \cdots \oplus S_p,$$

where each S_i is a Jordan subspace (recall Section 5.1.1). Consider a Jordan chain $\overline{\mathbf{V}}_{\mathbf{i}} \in V_A^{\Delta}$ which forms a basis of S_i. Thus $S_i =< \overline{\mathbf{V}}_{\mathbf{i}} >\subseteq \Delta_{\bowtie_{\overline{\mathbf{V}}_{\mathbf{i}}}} \subseteq \Delta_{\bowtie}$. Moreover, by the definition of $\mathbf{m}_{\overline{\mathbf{V}}_{\mathbf{i}}}$, we have that $\Delta_{\bowtie_{\overline{\mathbf{V}}_{\mathbf{i}}}} \subseteq \mathbf{m}_{\overline{\mathbf{V}}_{\mathbf{i}}} \subseteq \Delta_{\bowtie}$. Thus

$$\Delta_{\bowtie} = \mathbf{m}_{\overline{\mathbf{V}}_{\mathbf{1}}} + \cdots + \mathbf{m}_{\overline{\mathbf{V}}_{\mathbf{p}}}\,.$$

□

Corollary 5.14 *We have the following inclusion of sum-dense sets:*

$$\mathcal{I}_{\mathcal{G}} \subseteq \mathbf{m}_{\mathcal{G}}\,.$$

Proof Let $\Delta_{\bowtie} \in \mathcal{I}_{\mathcal{G}}$. By Theorem 5.13, there are minimal synchrony subspaces associated with Jordan chains $\overline{\mathbf{V}}_{\mathbf{i}} \in V_A^{\Delta}$ such that

$$\Delta_{\bowtie} = \mathbf{m}_{\overline{\mathbf{V}}_{\mathbf{1}}} + \cdots + \mathbf{m}_{\overline{\mathbf{V}}_{\mathbf{p}}}\,.$$

Since $\Delta_{\bowtie}$ is sum-irreducible, then $\Delta_{\bowtie} = \mathbf{m}_{\overline{\mathbf{V}}_{\mathbf{i}}}$ for some i and so $\Delta_{\bowtie} \in \mathbf{m}_{\mathcal{G}}$. □

Remark 5.15 The algorithm presented in the next section to obtain the lattice $V_{\mathcal{G}}$ for a given regular network $\mathcal{G}$, is based on the Corollary 5.14. It starts by finding a subset of $V_{\mathcal{G}}$ containing $\mathbf{m}_{\mathcal{G}}$. It then extracts the subset $\mathcal{I}_{\mathcal{G}}$ from that subset. Moreover, recall that $\mathcal{I}_{\mathcal{G}}$ is a sum-dense set of $V_{\mathcal{G}}$ by Proposition 5.8. Thus, the lattice $V_{\mathcal{G}}$ is obtained from the set $\mathcal{I}_\mathcal{G}$ using the sum operation of spaces. $\diamond$

6 Algorithm for regular networks

Based on the results in the previous section, we present an algorithm to find all the nontrivial synchrony subspaces for a regular n-cell network $\mathcal{G}$, recalling that, by Corollary 2.11, we may assume that the phase space is $\mathbf{R}^n$. We also need the following:

Lemma 6.1 *Let E be a subspace of $\mathbf{R}^n$ of dimension m, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and consider a system C with s equations of the form $x_{l_1} = x_{l_2}$, where $l_1, l_2 \in \{1, \ldots, n\}$. Then:*

1. *Solving the system C in E is equivalent to solve the system $\overline{C}$ with s equations of the form $\sum_{j=1}^m \alpha_j (v_{jl_1} - v_{jl_2}) = 0$, in the unknowns $\alpha_j \in \mathbf{R}$ for $j = 1, \ldots, m$ and $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ a basis of E, where $\mathbf{v}_j = (v_{j1}, \ldots, v_{jn}) \in \mathbf{R}^n$.*

2. *If r is the rank of the matrix of the system $\overline{C}$, then there are $m - r$ linearly independent vectors in E satisfying the system C.*

Proof Let $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ be a basis of E. Then, for every vector $\mathbf{x} = (x_1, \ldots, x_n)$ in E there are unique $\alpha_j \in \mathbf{R}$ with $j = 1, \ldots, m$, such that $\mathbf{x} = \sum_{j=1}^m \alpha_j \mathbf{v}_j$. Thus, each coordinate equality condition $x_{l_1} = x_{l_2}$ is equivalent to $\sum_{j=1}^m \alpha_j v_{jl_1} = \sum_{j=1}^m \alpha_j v_{jl_2}$ and thus to $\sum_{j=1}^m \alpha_j \left(v_{jl_1} - v_{jl_2}\right) = 0$.

Note that the linear systems C and $\overline{C}$ are homogeneous. The degree of indetermination of both systems is $m - r$, with r the rank of the matrix of the system $\overline{C}$, and indicates the number of independent variables α_j in the solution of system $\overline{C}$ and the number of linearly independent vectors in E satisfying the coordinate equality conditions in system C. $\square$

Remark 6.2 Let A be a real square matrix of order n. For every eigenvalue λ_i of A, let m_i^a and m_i^g be, respectively, the algebraic and geometric multiplicity of λ_i. Let C be a set of s conditions of the form $x_{l_1} = x_{l_2}$, where $l_1, l_2 \in \{1, \ldots, n\}$. Applying Lemma 6.1, where $E = E_{\lambda_i}$ and so $m = m_i^g$ indicates the dimension of the eigenspace E_{λ_i}, we have:

(i) If $m_i^g = 1$ then there is at most one linearly independent eigenvector in E_{λ_i} satisfying the set C of s conditions.

(ii) Suppose $m_i^g > 1$. If $s < m_i^g$ then there is at least one linearly independent eigenvector satisfying the conditions. If $s \geq m_i^g$ there can be none. $\diamond$

6.1 Overview of the algorithm

Let $\mathcal{G}$ be a regular n-cell network with adjacency matrix A. In order to obtain the lattice $V_{\mathcal{G}}$ of synchrony subspaces of $\mathcal{G}$, by Proposition 5.8 and Definition 5.9, it suffices to determine the irreducible sum-dense set $\mathcal{I}_{\mathcal{G}}$ of $V_{\mathcal{G}}$. For that, by Theorem 5.13 and Corollary 5.14, it is enough to determine the minimal synchrony set $\mathbf{m}_{\mathcal{G}}$. That is, the set of minimal synchrony subspaces $\mathbf{m}_{\mathbf{V}}$ associated to the eigenvectors and Jordan chains $\mathbf{V}$ in V_A^{Δ}. Recall Definitions 5.10 and 5.12. In fact, the algorithm finds a sum-dense set of synchrony subspaces containing the minimal synchrony set $\mathbf{m}_{\mathcal{G}}$. In order to find such set, the algorithm starts by determining for each eigenvector and Jordan chain $\mathbf{V}$ in V_A^{Δ}, the polydiagonal $\Delta_{\bowtie_{\mathbf{V}}}$. See Definition 5.3. Note that for each $\mathbf{V}$ in V_A^{Δ} we have that $\Delta_{\bowtie_{\mathbf{V}}} \subseteq \mathbf{m}_{\mathbf{V}}$. Moreover, if $\Delta_{\bowtie_{\mathbf{V}}}$ is a synchrony subspace then $\mathbf{m}_{\mathbf{V}} = \Delta_{\bowtie_{\mathbf{V}}}$. Otherwise, $\mathbf{m}_{\mathbf{V}}$ is the synchrony subspace of lower dimension that contains the polydiagonal $\Delta_{\bowtie_{\mathbf{V}}}$.

The algorithm consists of four fundamental steps:

1. Find the polydiagonal subspaces $\Delta_{\bowtie_{\mathbf{V}}}$ associated with the eigenvectors and Jordan chains $\mathbf{V}$ in V_A^{Δ}.

2. Find a sum-dense set containing the minimal synchrony set $\mathbf{m}_{\mathcal{G}}$.

3. Find the irreducible sum-dense set $\mathcal{I}_{\mathcal{G}}$.

4. Generate the complete lattice $V_{\mathcal{G}}$.

We describe next with more detail the functioning of the algorithm with special emphasis at the individual steps 1 and 2 of the algorithm due to their complexity.

Step 1 *Find the polydiagonal subspaces associated with the generalized eigenvectors in* V_A^Δ

For each eigenvalue λ_i of A, Step 1 of the algorithm constructs a table with the polydiagonal subspaces $\Delta_{\bowtie\mathbf{V}}$ for the eigenvectors and Jordan chains $\mathbf{V}$ in G_{λ_i}. For each such polydiagonal, the corresponding row of the table contains information about the set of coordinate equality conditions defining it, its dimension, a basis of $G_{\lambda_i} \cap \Delta_{\bowtie\mathbf{V}}$ and the corresponding dimension.

Given a polydiagonal Δ, consider the set C of coordinates equality conditions that define it. By Lemma 6.1, the polydiagonal Δ has a nonzero intersection with the generalized eigenspace G_{λ_i} if and only if the associated homogeneous linear system $\overline{C}$ using any basis of G_{λ_i} is undetermined.

In Step 2 the information in the tables corresponding to the different eigenvalues λ_i will be crossed to check if the polydiagonal subspaces have an eigenvector basis or not. Note that a polydiagonal subspace is a synchrony subspace if and only if it has an eigenvector basis.

Given a basis of the eigenspace E_{λ_i}, in Step 1.2 the algorithm finds the set of all the coordinate equality conditions that are satisfied by all the eigenvectors of that basis and thus by all the eigenvectors in E_{λ_i}. Thus, in Step 1.2.1, if rows j and k of the matrix M, whose columns are the eigenvectors of the basis, are equal that means that $x_j = x_k$ for all vectors $\mathbf{x} = (x_1, \ldots, x_n) \in E_{\lambda_i}$ and we can eliminate one of those rows. This procedure optimizes the execution of the algorithm in the case where that set of equality conditions is nonempty. Note that, for each eigenvector $\mathbf{v} \in E_{\lambda_i}$, the polydiagonal $\Delta_{\bowtie\mathbf{v}}$ is a subspace of the polydiagonal corresponding to that set.

In Step 1.4 it is created a matrix $\overline{M}$ from M, with rows given by $r_j - r_k$, where r_j, r_k are distinct rows in M. Each row $r_j - r_k$ of $\overline{M}$ corresponds to a coordinates equality condition of the form $x_j = x_k$ for $j \neq k$. So, each submatrix of $\overline{M}$ in Step 1.5 corresponds to a system of coordinate equality conditions and thus to a polydiagonal subspace.

For each eigenvector or Jordan chain $\mathbf{V}$, the polydiagonal $\Delta_{\bowtie \mathbf{V}}$ is the polydiagonal subspace with lower dimension containing $\mathbf{V}$. Thus, we start by considering the nontrivial polydiagonal subspaces with possible lower dimension, that is, with dimension 2. In terms of matrices, this corresponds to consider the submatrices of $\overline{M}$ with $s-2$ rows, with s the number of independent coordinates (s is the number of remaining rows in M).

The two main aspects that govern Step 1.6 are: if a vector satisfies a set C of coordinate equality conditions then it satisfies every subset of the conditions in C; even if a vector does not satisfy a set C of coordinate equality conditions, it may satisfy a subset of the conditions in C. The condition that the rank of the submatrix N of $\overline{M}$ is less than m_i^g in Step 1.6.3 guarantees that there is at least one nonzero vector in E_{λ_i} satisfying the equality conditions corresponding to N, see 2. of Lemma 6.1. If the rank of N equals m_i^g then no eigenvector satisfies the equality conditions corresponding to N. However there can be eigenvectors satisfying a subset of those conditions. Thus, in Step 1.6.4, the algorithm includes, in the set S of the matrices to be analyzed, the submatrices of N, in the set S_N, obtained by eliminating one row of N and such that the corresponding set of equality conditions has not been previously analyzed. Note that the set of equality conditions corresponding to each of those submatrices in S_N includes all the equality conditions corresponding to the matrix N minus one condition.

The fundamental aspect that governs Step 1.7 and the JordanChain routine is that, given a Jordan chain $v_1, \ldots, v_{k-1}, v_k$, the generalized eigenvector v_k belongs to a polydiagonal subspace Δ only if the generalized eigenvectors $v_1, \ldots, v_{k-1}$ are in Δ. Thus, if the vectors $v_1, \ldots, v_{k-1}$ belong to a polydiagonal subspace Δ_1 defined by a set C of coordinate equality conditions, the algorithm must check if the vectors $v_1, \ldots, v_{k-1}, v_k$ belong to a polydiagonal subspace $\Delta_2 \supseteq \Delta_1$, that is, defined by the conditions in C or a subset of it.

Step 2 *Find a sum-dense set containing the minimal synchrony set* $\mathbf{m}_{\mathcal{G}}$

In Step 2 the information in the different tables are crossed to check if the polydiagonal subspaces have an eigenvector basis or not: a polydiagonal is a synchrony space if and only if the number of generalized eigenvectors found in Step 1 associated with that polydiagonal equals the dimension of the polydiagonal.

Recall that, for each $\mathbf{V}$ the polydiagonal $\Delta_{\bowtie_\mathbf{V}}$ is a subspace of the minimal synchrony subspace $\mathbf{m_V}$. If $\Delta_{\bowtie_\mathbf{V}}$ is a proper subspace of $\mathbf{m_V}$, then in Step 2.3.2, recursively, the algorithm analyzes the polydiagonals containing $\Delta_{\bowtie_\mathbf{V}}$, lowering the dimension by one, till the corresponding minimal synchrony subspace $\mathbf{m_V}$ is found. This justifies the selection of submatrices of the matrix $\overline{M}$ considered in Step 1.

At the end of Step 2, we have found a set S of synchrony subspaces containing the minimal synchrony set $\mathbf{m}_\mathcal{G}$ for the lattice $V_\mathcal{G}$ of synchrony subspaces. Recall Theorem 5.13 where it is proved that the minimal synchrony set forms a sum-dense set for the lattice $V_\mathcal{G}$. Thus, the set S is a sum-dense set for $V_\mathcal{G}$.

Step 3 *Find the irreducible sum-dense set* $\mathcal{I}_\mathcal{G}$

In this step, we extract the set $\mathcal{I}_\mathcal{G}$ of the irreducible synchrony subspaces from the sum-dense set S of synchrony subspaces obtained in step 2: these are sufficient (and necessary) to generate $V_\mathcal{G}$. Recall Remark 5.15.

Step 4 *Generate the complete lattice* $V_\mathcal{G}$

After finding the sum-dense set $\mathcal{I}_\mathcal{G}$ of irreducible synchrony subspaces, the remaining synchrony subspaces in the lattice are given by the possible sums of elements in $\mathcal{I}_\mathcal{G}$ that are polydiagonals.

Remark 6.3 The valency v of a regular network is an eigenvalue of the network adjacency matrix. In the case that its algebraic multiplicity is one, then $E_v = G_v =< (1, \ldots, 1) >$, and so the implementation of the algorithm can be optimized since the vectors in G_v satisfy all conditions of equality of coordinates. ◇

Remark 6.4 We observe that the algorithm we present below applies as well to identical-edge networks, not necessarily regular, since from Corollary 2.11, a polydiagonal subspace $\Delta_{\bowtie}$ is a synchrony subspace if and only if it is left invariant under the network adjacency matrix. $\diamond$

6.2 Algorithm

We present now an algorithm to obtain the lattice of synchrony subspaces of a regular n-cell network $\mathcal{G}$. If the adjacency matrix A of $\mathcal{G}$ has complex eigenvalues, the calculations are done on $\mathbf{C}^n$, that is, interpreting A as $A_c: \ \mathbf{C}^n \to \mathbf{C}^n$. Recall Remark 5.4.

Algorithm 6.5 Let A be the adjacency matrix of a regular n-cell network $\mathcal{G}$, with valency v. Let λ_i with $i = 1, \ldots, t$ and $t \leq n$ be the eigenvalues of A, with m_i^a and m_i^g, respectively, the algebraic and geometric multiplicities.

1 [Find polydiagonals] For each eigenvalue λ_i, $i = 1, \ldots, t$ of A:

1.1 Let $(\mathbf{v}_1, \ldots, \mathbf{v}_{m_i^g})$ be a basis of E_{λ_i}. Consider the matrix M whose columns correspond to the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_{m_i^g}$.

1.2 Let $C = \emptyset$. For every pair of rows l_j, l_k of M:

1.2.1 If $l_j = l_k$ then $C = C \cup \{x_j = x_k\}$ and eliminate row l_k of M.

1.3 Create a four-column table for E_{λ_i} with a row containing: in the first entry the set of the equality conditions C found in Step 1.2; in the second entry the corresponding polydiagonal dimension; in the third entry the basis of eigenspace E_{λ_i}; in the fourth entry the number of vectors of the basis, m_i^g.

Let s be the number of remaining rows in M. If $s = 1$ go to Step 1.

1.4 Construct a new matrix $\overline{M}$ with rows given by $r_j - r_k$ for $j = 1, \ldots, s$ and $k = j + 1, \ldots, s$ where r_j, r_k are rows in M. Thus $\overline{M}$ has $\overline{s} = s(s-1)/2$ rows, each corresponding to an equality $x_j = x_k$ with $j, k \in \{1, \ldots, n\}$.

1.5 Let S be the set of all the submatrices of $\overline{M}$ with $s-2$ rows obtained from $\overline{M}$ by elimination of rows and $\overline{S} = \emptyset$.

1.6 While $S \neq \emptyset$,

1.6.1 Let N be a submatrix in S, $S = S \setminus \{N\}$ and $\overline{S} = \overline{S} \cup \{N\}$.

1.6.2 Let r be the rank of N;

1.6.3 If $r < m_i^g$ then:

1.6.3.1 Let C_N be the set of equalities given by the rows of N and C be the set of equalities obtained in Step 1.2. Add a new row to the table containing: in the first entry $C \cup C_N$; in the second entry the corresponding polydiagonal dimension; in the third entry a basis of the subspace of the eigenvectors in E_{λ_i} that satisfy the set of equality conditions (obtained from the solution set of the homogeneous system with the coefficient matrix N), and in the fourth entry the number of vectors of the basis, $m_i^g - r$.

1.6.4 Otherwise, $r = m_i^g$:

1.6.4.1 Consider the set S_N of all the submatrices of N obtained by eliminating one row of N.

1.6.4.2 Include in the set S of submatrices of $\overline{M}$ the submatrices in the set S_N such that there is no submatrix in the set $\overline{S}$ with the same corresponding set of coordinates equality conditions.

1.7 If $m_i^g < m_i^a$ then:

1.7.1 Compute a basis of $\mathrm{Im}\,(A - \lambda_i \mathrm{Id}_n)$.

1.7.2 For each row in the table for E_{λ_i},

1.7.2.1 If the intersection of the subspace corresponding to the basis in that row with $\mathrm{Im}\,(A - \lambda_i \mathrm{Id}_n)$ is a nonzero subspace then:

Let B_1 be a basis of that intersection;

Let C be the first entry of the row (the set of equality conditions);

JordanChain$(B_1, C, 2)$.

2 [Find sum-dense set] Consider the empty set S. For each table, for each row of the table:

2.1 Let C be the set of equality conditions in that row and d the dimension of the polydiagonal subspace $\Delta_{\bowtie}$ given by those conditions.

2.2 If the number of vectors in that row of the table equals $d-1$ then there is an eigenvector basis of $\Delta_{\bowtie}$ (considering also the eigenvector $(1, \ldots 1)$) and thus $\Delta_{\bowtie}$ is a synchrony subspace. Let $S = S \cup \{\Delta_{\bowtie}\}$.

2.3 If the number of vectors in that row of the table is less than $d-1$ and there are more tables, then look at the other tables to find all the rows whose equality conditions include the set C of equality conditions.

2.3.1 If the total sum of the number of collected vectors equals d then the collected vectors form an eigenvector basis of $\Delta_{\bowtie}$ and thus $\Delta_{\bowtie}$ is a synchrony subspace. Let $S = S \cup \{\Delta_{\bowtie}\}$.

2.3.2 If the total sum of the number of collected vectors is still less than d then execute the following changes in the table for the eigenvalue λ_i:

2.3.2.1 Eliminate that row of the table.

2.3.2.2 Let $c = \#C$. If $c > 1$ then, for each subset of $c-1$ conditions of the initial set C of c conditions:

If there is no row at the table with that set of $c-1$ conditions then add a new row to the end of the table differing from the deleted row only at the first and second entries: the first entry contains the set of the $c-1$ conditions and the second entry is $n-c+1$, the dimension of the corresponding polydiagonal.

Otherwise, change the corresponding row: replacing the third entry by a basis of the subspace generated by the union of the bases in this row and the one in the deleted row; changing the fourth entry by the number of vectors of that basis. Move that row to the end of the table.

3 [Find the irreducible sum-dense set] Decompose S into the disjoint union $\cup_{i=1}^{r} S_{j_i}$, where each set S_{j_i} contains the synchrony subspaces in S of dimension j_i, with $j_{i-1} < j_i$, for $i = 2, \ldots, r$. Let $\mathcal{I}_{\mathcal{G}} = S_{j_2}$.

3.1 For $i = 3$ to r:

3.1.1 For each subspace E in S_{j_i}, if it is not a sum of subspaces in $\mathcal{I}_{\mathcal{G}}$, then let $\mathcal{I}_{\mathcal{G}} = \mathcal{I}_{\mathcal{G}} \cup \{E\}$.

4 [Find the lattice] Let $V_{\mathcal{G}} = \mathrm{Sum}(\mathcal{I}_{\mathcal{G}})$. Return$(\mathcal{I}_{\mathcal{G}}, V_{\mathcal{G}})$

$\diamond$

Algorithm 6.6 [JordanChain(B_{k-1}, C, k)]

1 Let V_{k-1} be the subspace generated by the basis B_{k-1}.

2 Let V_k be the subspace of vectors v_k that satisfy $(A - \lambda_i \mathrm{Id}_n)\, v_k = v_{k-1}$ for some $v_{k-1} \in V_{k-1}$. [1]

3 Let B_C be the basis at the third entry in the table for E_{λ_i} corresponding to the set of equality conditions C.

4 Complete the basis B_C with a set $\overline{B}_k$ of vectors forming a basis of V_k.[2]

5 Consider the matrix M whose columns are the vectors of the basis $\overline{B}_k$.

[1] V_k is a subspace of $\ker (A - \lambda_i \mathrm{Id}_n)^k$.

[2] $< B_C > \subseteq \ker (A - \lambda_i \mathrm{Id}_n)^{k-1} \subseteq V_k$.

6 Construct a new matrix $\overline{M}$ with rows given by $r_j - r_k$, with r_j and r_k rows in M, whenever $x_j = x_k$ is in C. [3]

7 Let S be the set of all submatrices of $\overline{M}$ of rank less than $\#\overline{B}_k$ obtained from $\overline{M}$ by elimination of rows.

8 Decompose S into disjoint union $\bigcup_{i=0}^{\#\overline{B}_k-1} S_i$, where each S_i is the set of all matrices in S with rank i. For each S_i remove any matrix N that is a submatrix of a matrix in S_i different from N.

9 If $S \neq \emptyset$ then, for $i = 0$ to $\#\overline{B}_k - 1$:

9.1 While $S_i \neq \emptyset$ do:

9.1.1 Let $N \in S_i$ and $S_i = S_i \setminus \{N\}$.

9.1.2 Let $\overline{\overline{B}}_k$ be a basis of the subspace of $< \overline{B}_k >$ obtained from the solution set of the homogeneous system with the coefficient matrix N.

9.1.3 [4]Let C_N be the set of equality conditions corresponding to the rows of N.

9.1.4 [5] If $C_N = C$, then change the row corresponding to the set C: replacing the third entry by the basis $B = B_C \cup \overline{\overline{B}}_k$ and the fourth entry by $\#B$.

Otherwise,

If there is no row at the table with the set of conditions C_N, then add a new row at the top of the table containing: in the first entry C_N; in the second entry the corresponding polydiagonal dimension; in the third entry the basis $B = B_C \cup \overline{\overline{B}}_k$; in the fourth entry $\#B$.

Else, go to Step 9.1.

[3]If row $r_j - r_k$ of $\overline{M}$ is zero, that means that $x_j = x_k$ for all vectors in V_k.
[4]Equivalently, C_N is the set of equality conditions satisfied by the vectors in $< \overline{\overline{B}}_k >$.
[5]See Remark 6.7.

9.1.5 If the intersection of the subspace corresponding to the basis $\overline{\overline{B}}_k$ with $\mathrm{Im}\,(A - \lambda_i \mathrm{Id}_n)$ is a nonzero subspace then:

9.1.5.1 Let B_k be a basis of the intersection $< B > \cap \mathrm{Im}\,(A - \lambda_i \mathrm{Id}_n)$.

9.1.5.2 JordanChain$(B_k, C_N, k+1)$.

$\diamond$

Remark 6.7 In the JordanChain routine, in Step 7, the matrix $\overline{M}$ is constructed using the equality conditions in C that are satisfied by all the vectors in $< B_C >$. From the way C_N is defined, we have that $C_N \subseteq C$ and so, trivially, all vectors in $< B_C >$ satisfy them. In Step 9.1.4, if $C_N = C$, then the vectors in $< \overline{\overline{B}}_k >$ also satisfy the conditions in C, and so, we have to join that information to the corresponding row and proceed with Step 9.1.5. If $C_N \subsetneqq C$ then two distinct cases can occur. If there is no row corresponding to the set of conditions C_N then a new row is created at the top of the table and we proceed with Step 9.1.5. In case there is already a row in the table corresponding to the set of conditions C_N then nothing should be done, because that row has already been treated or it will be in Step 1.7.2, thus proceeding with Step 9.1.

$\diamond$

Algorithm 6.8 [Sum($\mathcal{I}_\mathcal{G}$)]

The set $\mathcal{I}_\mathcal{G}$ contains the irreducible sum-dense set of the lattice $V_\mathcal{G}$.

1 Let $V_\mathcal{G} = \mathcal{I}_\mathcal{G}$.

2 Let $s = \#\mathcal{I}_\mathcal{G}$.

3 For $i = 2$ to s,

3.1 For every (possible) subset $\{\Delta_{\bowtie_{j_1}}, \ldots, \Delta_{\bowtie_{j_i}}\}$, with $j_k \neq j_l$, of i synchrony subspaces in $\mathcal{I}_\mathcal{G}$,

3.1.1 Let $\Delta_{\bowtie} = \Delta_{\bowtie_{j_1}} + \cdots + \Delta_{\bowtie_{j_i}}$, 3.1.2 If $\Delta_{\bowtie}$ is a polydiagonal subspace then let $V_{\mathcal{G}} = V_{\mathcal{G}} \cup \{\Delta_{\bowtie}\}$.

4 Return $V_{\mathcal{G}} \cup \{\Delta_0\} \cup \{P\}$.

$\diamond$

Remark 6.9 As observed in Remark 6.4, Algorithm 6.5 applies, as well, directly to nonregular identical-edge networks, since a polydiagonal subspace $\Delta_{\bowtie}$ is a synchrony subspace if and only if it is left invariant under the network adjacency matrix. We just point out that, in case we have an identical-edge network which is not regular, then the vector $(1, \ldots, 1)$ is not an eigenvector of the network adjacency matrix. That is, the polydiagonal $\{\mathbf{v} \in \mathbf{R}^n : \ v_i = v_j, \text{ for all } i, j\}$ is not a synchrony subspace. It follows then that:

At Step 2 of the Algorithm 6.5, Steps 2.2 and 2.3 have to be:

2.2 If the number of vectors in that row of the table equals d then there is an eigenvector basis of $\Delta_{\bowtie}$ and thus $\Delta_{\bowtie}$ is a synchrony subspace. Let $S = S \cup \{\Delta_{\bowtie}\}$.

2.3 If the number of vectors in that row of the table is less than d and there are more tables, then look at the other tables to find all the rows whose equality conditions include the set C of equality conditions.

At the beginning ot Step 3 it has to be:

Let $\mathcal{I}_{\mathcal{G}} = S_{j_1}$.

Finally, Step 4 of the Algorithm 6.8 is:

4 Return $V_{\mathcal{G}} \cup \{P\}$.

$\diamond$

6.3 Examples

In this section we illustrate the implementation of the Algorithm 6.5 with three regular network examples: two where the adjacency matrix is semi-simple and one where it is not the case.

6.3.1 Semi-simple adjacency matrix

Example 6.10 Consider the 5-cell regular network $\mathcal{G}$ on the left of Figure 1 and recall the associated coupled cell systems at Example 2.3. Using Algorithm 6.5, we obtain all the nontrivial synchrony spaces associated to the network $\mathcal{G}$, see Table 1.

$\Delta_1 = \{\mathbf{x}:\ x_2 = x_3\}$ $\Delta_2 = \{\mathbf{x}:\ x_3 = x_5\}$ $\Delta_3 = \{\mathbf{x}:\ x_4 = x_5\}$	$\Delta_4 = \{\mathbf{x}:\ x_2 = x_3 = x_5\}$ $\Delta_5 = \{\mathbf{x}:\ x_3 = x_4 = x_5\}$ $\Delta_6 = \{\mathbf{x}:\ x_1 = x_2,\ x_4 = x_5\}$ $\Delta_7 = \{\mathbf{x}:\ x_1 = x_4,\ x_2 = x_3\}$ $\Delta_8 = \{\mathbf{x}:\ x_2 = x_3,\ x_4 = x_5\}$ $\Delta_9 = \{\mathbf{x}:\ x_2 = x_4,\ x_3 = x_5\}$	$\Delta_{10} = \{\mathbf{x}:\ x_1 = x_2,\ x_3 = x_4 = x_5\}$ $\Delta_{11} = \{\mathbf{x}:\ x_1 = x_4,\ x_2 = x_3 = x_5\}$ $\Delta_{12} = \{\mathbf{x}:\ x_1 = x_2 = x_3,\ x_4 = x_5\}$ $\Delta_{13} = \{\mathbf{x}:\ x_1 = x_4 = x_5,\ x_2 = x_3\}$ $\Delta_{14} = \{\mathbf{x}:\ x_2 = x_3 = x_4 = x_5\}$

Table 1: Nontrivial synchrony subspaces for the network on the left of Figure 1.

We illustrate now the implementation of Algorithm 6.5.

Step 1. The adjacency matrix of $\mathcal{G}$ has eigenvalues $2, -1, 0$, with algebraic (and geometric) multiplicities $1, 2, 2$, respectively. The associated eigenspaces (in $\mathbf{R}^5$) are $E_2 =< (1, 1, 1, 1, 1) >$,

$$E_{-1} =< (1, -1, -1, 0, 0), (1, 0, 0, -1, -1) > \quad \text{and} \quad E_0 =< (1, 0, -1, 0, -1), (0, 1, 0, -1, 0) > .$$

Steps 1.1-1.3 for E_2: Table 2 is constructed.

Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
$x_1 = x_2 = x_3 = x_4 = x_5$	1	$((1, 1, 1, 1, 1))$	1

Table 2: Table for E_2.

Steps 1.1-1.6 for E_{-1}: let

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}.$$

As rows $2, 3$ are equal and rows $4, 5$ are equal we have that vectors $\mathbf{x} = (x_1, \ldots, x_5) \in E_{-1}$ satisfy $x_2 = x_3$ and $x_4 = x_5$. We eliminate rows 3 and 5 of M and create a table with one row, the first row in Table 3. Thus,

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \overline{M} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

As M has three rows, we have $s = 3$, and so, it is sufficient to consider the submatrices of $\overline{M}$ with $3 - 2 = 1$ row. The first row of $\overline{M}$ corresponds to the equality $x_1 = x_2$, the second row to $x_1 = x_4$ and the third to $x_2 = x_4$. Considering all the submatrices formed by one row of $\overline{M}$, that is,

$$S = \{[2\,1],\ [1\,2],\ [-1\,1]\},$$

we obtain Table 3.

Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
$x_2 = x_3$, $x_4 = x_5$	3	$((1, -1, -1, 0, 0), (1, 0, 0, -1, -1))$	2
$x_1 = x_2 = x_3$, $x_4 = x_5$	2	$((1, 1, 1, -2, -2))$	1
$x_1 = x_4 = x_5$, $x_2 = x_3$	2	$((1, -2, -2, 1, 1))$	1
$x_2 = x_3 = x_4 = x_5$	2	$((2, -1, -1, -1, -1))$	1

Table 3: Table for E_{-1}.

Steps 1.1-1.6 for E_0: let

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and observe that row 3 is equal to row 5. Therefore $x_3 = x_5$ for the eigenvectors $\mathbf{x}$ in E_0. We then eliminate row 5 of M creating a table with one row, the first row of Table 4, and obtaining

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \overline{M} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}.$$

We have $s = 4$, and so, it is sufficient to consider the submatrices of $\overline{M}$ with $4 - 2 = 2$ rows. The rows of $\overline{M}$ correspond, respectively, to the equalities $x_1 = x_2$, $x_1 = x_3$, $x_1 = x_4$, $x_2 = x_3$, $x_2 = x_4$ and $x_3 = x_4$. The submatrices of $\overline{M}$ with 2 rows and rank less than 2 are $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. These submatrices correspond, respectively, to the conditions $x_1 = x_2, x_3 = x_4 = x_5$ and $x_1 = x_4, x_2 = x_3 = x_5$. We have then that the set S in step 1.6 is

$$S = \left\{ [1\ -1],\ [2\,0],\ [1\,1],\ [1\,1],\ [0\,2],\ [-1\,1],\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

Considering this information, we obtain Table 4.

Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
$x_3 = x_5$	4	$((1,0,-1,0,-1),(0,1,0,-1,0))$	2
$x_1 = x_2,\ x_3 = x_5$	3	$((1,1,-1,-1,-1))$	1
$x_1 = x_3 = x_5$	3	$((0,1,0,-1,0))$	1
$x_1 = x_4,\ x_3 = x_5$	3	$((1,-1,-1,1,-1))$	1
$x_2 = x_3 = x_5$	3	$((1,-1,-1,1,-1))$	1
$x_2 = x_4,\ x_3 = x_5$	3	$((1,0,-1,0,-1))$	1
$x_3 = x_4 = x_5$	3	$((1,1,-1,-1,-1))$	1
$x_1 = x_2,\ x_3 = x_4 = x_5$	2	$((1,1,-1,-1,-1))$	1
$x_1 = x_4,\ x_2 = x_3 = x_5$	2	$((1,-1,-1,1,-1))$	1

Table 4: Table for E_0.

Steps 2.1-2.3 From Table 2 we obtain the trivial synchrony subspace $\Delta_0 = \{\mathbf{x} :\ x_1 = x_2 = x_3 = x_4 = x_5\}$.

Consider the first row in Table 3 of E_{-1}. The polydiagonal subspace of $\mathbf{R}^5$ of the vectors $(x_1, x_2, x_3, x_4, x_5)$ satisfying the conditions $x_2 = x_3, x_4 = x_5$ is 3-dimensional. The two linearly independent eigenvectors of E_{-1} that verify those conditions and the eigenvector $(1, 1, \ldots, 1)$ of Table 2 form a basis of that polydiagonal subspace. Thus, it is a synchrony subspace. The same holds for the other three rows of Table 3 obtaining so the synchrony subspaces Δ_8, Δ_{12}, Δ_{13} and Δ_{14} of Table 1.

The polydiagonal subspace F of the vectors $(x_1, x_2, x_3, x_4, x_5)$ (in $\mathbf{R}^5$) satisfying the condition $x_3 = x_5$ which appears in the first row of Table 4 for E_0 is 4-dimensional. Besides the eigenvector $(1, 1, \ldots, 1)$ there are two linearly independent eigenvectors of E_0 in F and one linearly independent eigenvector of E_{-1} (satisfying $x_2 = x_3 = x_4 = x_5$) in F (fourth row of Table 3). Thus, there is an eigenvector basis for the subspace and so, it is a synchrony subspace, Δ_2.

The subspace F of the vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbf{R}^5$ satisfying the conditions $x_1 = x_2,\ x_3 = x_5$ in the second row of Table 4 for E_0 is 3-dimensional. Besides the eigenvector $(1, 1, \ldots, 1)$, there is only one linearly independent eigenvector of E_0 in F and no eigenvector of E_{-1} in F. Thus, there is no eigenvector basis for the subspace and so, it is not a synchrony subspace. The same happens for the next two rows of Table 4 for E_0. Applying the step 2.3.2 to these three rows, we have to add the following four rows to the Table 4:

Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
$x_1 = x_2$	4	$((1,1,-1,-1,-1))$	1
$x_1 = x_3$	4	$((0,1,0,-1,0))$	1
$x_1 = x_5$	4	$((0,1,0,-1,0))$	1
$x_1 = x_4$	4	$((1,-1,-1,1,-1))$	1

However, applying the steps 2.1-2.3 to these rows we obtain no synchrony subspaces.

The subspace F of the vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbf{R}^5$ satisfying the conditions $x_2 = x_3 = x_5$ in row five of Table 4 for E_0 is 3-dimensional. Besides the eigenvector $(1, 1, \ldots, 1)$ there is one linearly independent eigenvector of E_0 in F and one linearly independent eigenvector of E_{-1} (satisfying $x_2 = x_3 = x_4 = x_5$) in F. Thus,there is an eigenvector basis for the subspace and so, it is a synchrony subspace, Δ_4. The same happens for the subspaces of $\mathbf{R}^5$ defined by the conditions in the next two rows of the table for E_0: we obtain Δ_9 and Δ_5.

For the last two rows of Table 4 for E_0, the subspace of $\mathbf{R}^5$ of the vectors $(x_1, x_2, x_3, x_4, x_5)$ satisfying the conditions in each row is 2-dimensional. Besides the eigenvector $(1, 1, \ldots, 1)$ there is one more linearly independent eigenvector in E_0 that verifies those conditions. Thus, it is a synchrony subspace. We have then more six synchrony subspaces: Δ_2, Δ_4, Δ_5, Δ_9, Δ_{10} and Δ_{11} of Table 1.

At the end of the step 2.3, the sum-dense set S of synchrony subspaces (which contains the minimal synchrony set $\mathbf{m}_{\mathcal{G}}$ as defined in Theorem 5.13), obtained directly from Tables 3 and 4, is

$$S = \{\Delta_0, \Delta_2,\ \Delta_4,\ \Delta_5,\ \Delta_8,\ \Delta_9,\ \Delta_{10},\ \Delta_{11},\ \Delta_{12},\ \Delta_{13},\ \Delta_{14}\}\ .$$

Step 3 Observe that each of the synchrony subspaces $\Delta_{10}, \ldots, \Delta_{14}$ is two-dimensional and so it is sum-irreducible. Moreover,

$$\Delta_2 = \Delta_{10} + \Delta_{11} + \Delta_{14}, \quad \Delta_4 = \Delta_{11} + \Delta_{14}, \quad \Delta_5 = \Delta_{10} + \Delta_{14} \quad \text{and} \quad \Delta_8 = \Delta_{12} + \Delta_{14}\,.$$

It follows then that at the end of step 3, we get the irreducible sum-dense set

$$\mathcal{I}_{\mathcal{G}} = \{\Delta_9,\, \Delta_{10},\, \Delta_{11},\, \Delta_{12},\, \Delta_{13},\, \Delta_{14}\}\ .$$

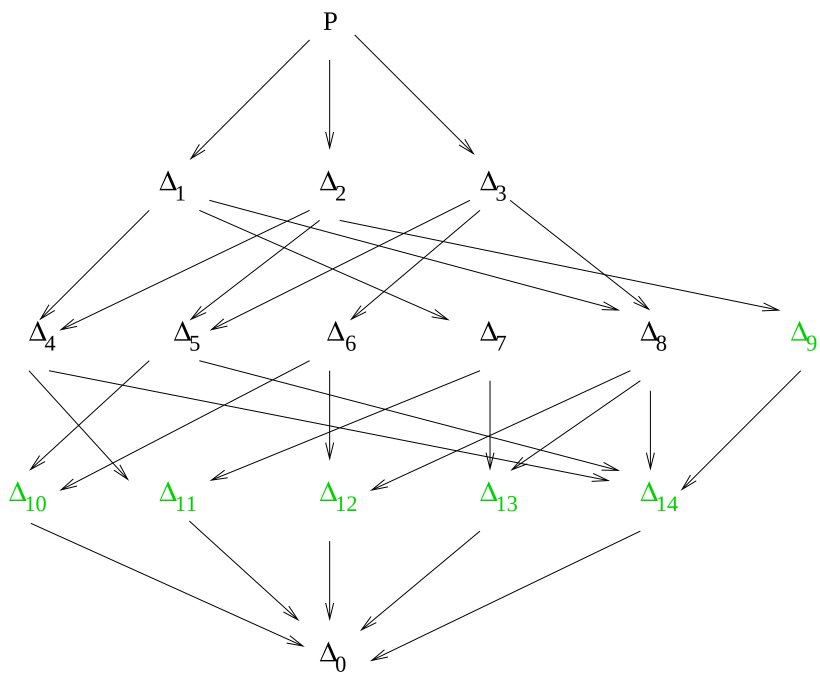

Figure 5: The lattice of synchrony subspaces for the 5-cell regular network $\mathcal{G}$ on the left of Figure 1: the nontrivial synchrony subspaces Δ_i, for $i = 1, \ldots, 14$, are listed in Table 1. The top element is the total phase space P (the total asynchronous polydiagonal space) and the bottom element Δ_0 is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense set $\mathcal{I}_{\mathcal{G}}$ are in green.

Step 4 Applying $\mathrm{Sum}\left(\mathcal{I}_{\mathcal{G}}\right)$, we get the lattice of synchrony subspaces listed in Table 1. See the lattice in Figure 5. $\diamond$

Example 6.11 Consider the 5-cell regular network $\mathcal{G}$ in Figure 6. Using Algorithm 6.5, we obtain all the nontrivial synchrony spaces associated to the network $\mathcal{G}$, see Table 5.

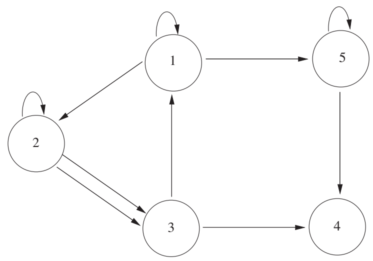

Figure 6: A 5-cell regular network.

$\Delta_1 = \{\mathbf{x} : \ x_2 = x_5\}$	$\Delta_2 = \{\mathbf{x} : \ x_1 = x_2 = x_3\}$	$\Delta_3 = \{\mathbf{x} : \ x_1 = x_2 = x_3 = x_5\}$ $\Delta_4 = \{\mathbf{x} : \ x_1 = x_2 = x_3, \, x_4 = x_5\}$

Table 5: Nontrivial synchrony subspaces for the network in Figure 6.

We illustrate very briefly the implementation of Algorithm 6.5.

Step 1. The adjacency matrix of $\mathcal{G}$ has eigenvalues $2, 0, 1$ and $\pm i$. We consider thus the complexification A_c of A (see Remark 5.4). The associated eigenspaces (in $\mathbf{C}^5$) are $E_2 =< (1,1,1,1,1) >$ and

$$E_0 =< (0,0,0,1,0) >, \; E_1 =< (0,0,0,1,1) >, \; E_i =< (-1+i, 1, -2i, -2-i, 1) >, \; E_{-i} =< (-1-i, 1, 2i, -2+i, 1) > \; .$$

Steps 1.1-1.6: at the end of Step 1 we get five tables, each with one row, whose information is collected in Table 6.

Table	Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
E_2	$x_1 = x_2 = x_3 = x_4 = x_5$	1	$((1,1,1,1,1))$	1
E_0	$x_1 = x_2 = x_3 = x_5$	2	$((0,0,0,1,0))$	1
E_1	$x_1 = x_2 = x_3, \, x_4 = x_5$	2	$((0,0,0,1,1))$	1
E_i	$x_2 = x_5$	4	$((-1+i, 1, -2i, -2-i, 1))$	1
E_{-i}	$x_2 = x_5$	4	$((-1-i, 1, 2i, -2+i, 1))$	1

Table 6: Table collecting the information of the tables for $E_2, E_0, E_1, E_i, E_{-i}$.

Steps 2.1-2.3 Since all the polydiagonals in Table 6 have an eigenvector basis, at the end of Step 2 we get the synchrony subspaces Δ_0, Δ_1, Δ_3 and Δ_4.

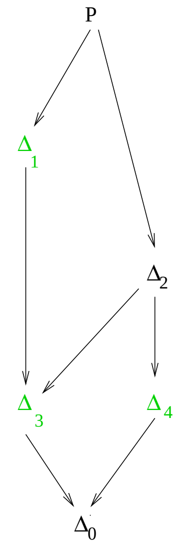

Figure 7: The lattice of synchrony subspaces for the 5-cell regular network $\mathcal{G}$ of Figure 6: the nontrivial synchrony subspaces Δ_i, for $i = 1, \ldots, 4$, are listed in Table 5. The top element is the total phase space P (the total asynchronous polydiagonal space) and the bottom element Δ_0 is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense set $\mathcal{I}_{\mathcal{G}}$ are in green.

Steps 3 and 4 We get the irreducible sum-dense set

$$\mathcal{I}_{\mathcal{G}} = \{\Delta_1,\, \Delta_3,\, \Delta_4\}\ .$$

Applying $\mathrm{Sum}\left(\mathcal{I}_{\mathcal{G}}\right)$, we get the lattice of synchrony subspaces listed in Table 5. See the lattice in Figure 7. ◇

6.3.2 Non semi-simple adjacency matrix

Example 6.12 Consider the 6-cell regular network $\mathcal{G}$ of Figure 8. Using Algorithm 6.5, we obtain all the nontrivial synchrony spaces associated to the network $\mathcal{G}$, see Table 7.

We illustrate briefly the implementation of Algorithm 6.5.

Step 1. The adjacency matrix A of $\mathcal{G}$ has eigenvalues 2 and 0, with algebraic multiplicities

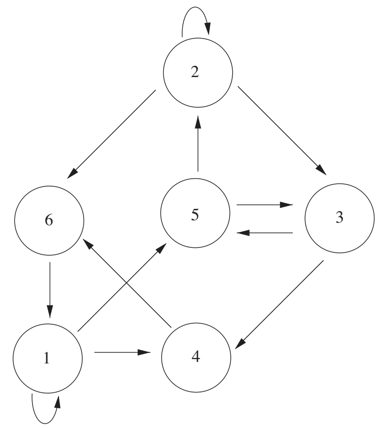

Figure 8: A 6-cell regular network.

$\Delta_1 = \{\mathbf{x}: \ x_2 = x_3\}$ $\Delta_2 = \{\mathbf{x}: \ x_4 = x_5\}$	$\Delta_3 = \{\mathbf{x}: \ x_2 = x_3, \ x_4 = x_5\}$ $\Delta_4 = \{\mathbf{x}: \ x_2 = x_6, \ x_4 = x_5\}$ $\Delta_5 = \{\mathbf{x}: \ x_3 = x_6, \ x_4 = x_5\}$
$\Delta_6 = \{\mathbf{x}: \ x_1 = x_4 = x_5, \ x_3 = x_6\}$ $\Delta_7 = \{\mathbf{x}: \ x_2 = x_3 = x_6, \ x_4 = x_5\}$	$\Delta_8 = \{\mathbf{x}: \ x_1 = x_2, \ x_3 = x_4 = x_5 = x_6 \}$ $\Delta_9 = \{\mathbf{x}: \ x_1 = x_4 = x_5, \ x_2 = x_3 = x_6 \}$

Table 7: Nontrivial synchrony subspaces for the network of Figure 8.

1 and 5, respectively. The associated eigenspaces (in $\mathbf{R}^6$) are $E_2 =< (1,1,1,1,1,1) >$ and

$$E_0 = \ker A =< (1,0,-1,0,0,-1), (0,1,0,-1,-1,0) > \ .$$

Steps 1.1-1.3 for E_2: Table 8 is constructed.

Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
$x_1 = x_2 = x_3 = x_4 = x_5 = x_6$	1	$((1,1,1,1,1,1))$	1

Table 8: Table for E_2.

Steps 1.1-1.6 for E_0: we get Table 9 and we identify the synchrony subspaces Δ_8 and Δ_9 (first two lines of the table).

Step 1.7

Step 1.7.1 Since Im$A = \{\mathbf{x}: \ x_2 = x_3, \, x_4 = x_5\}$, we can take the following basis for

Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
$x_1 = x_2,\ x_3 = x_4 = x_5 = x_6$	2	$((1, 1, -1, -1, -1, -1))$	1
$x_1 = x_4 = x_5,\ x_2 = x_3 = x_6$	2	$((1, -1, -1, 1, 1, -1))$	1
$x_1 = x_2,\ x_3 = x_6,\ x_4 = x_5$	3	$((1, 1, -1, -1, -1, -1))$	1
$x_1 = x_3 = x_6,\ x_4 = x_5$	3	$((0, 1, 0, -1, -1, 0))$	1
$x_1 = x_4 = x_5,\ x_3 = x_6$	3	$((1, -1, -1, 1, 1, -1))$	1
$x_2 = x_3 = x_6,\ x_4 = x_5$	3	$((1, -1, -1, 1, 1, -1))$	1
$x_2 = x_4 = x_5,\ x_3 = x_6$	3	$((1, 0, -1, 0, 0, -1))$	1
$x_3 = x_4 = x_5 = x_6$	3	$((1, 1, -1, -1, -1, -1))$	1
$x_3 = x_6,\ x_4 = x_5$	4	$((1, 0, -1, 0, 0, -1), (0, 1, 0, -1, -1, 0))$	2

Table 9: Table for E_0.

ImA:

$$((1, 0, 0, 0, 0, 0),\ (0, 1, 1, 0, 0, 0),\ (0, 0, 0, 1, 1, 0),\ (0, 0, 0, 0, 0, 1))\ .$$

Step 1.7.2 Observe that only the rows $2, 5, 6, 9$ of Table 9 for E_0 are such that the intersection of the subspace corresponding to the basis in the row with ImA is a nonzero subspace.

Step 1.7.2.1 for row 2 of Table 9

Take the following basis for the intersection of the subspace corresponding to the basis in the row with ImA,

$$B_1 = ((1, -1, -1, 1, 1, -1))\,,$$

and call:

JordanChain$(((1, -1, -1, 1, 1, -1))\,, \{x_1 = x_4 = x_5,\ x_2 = x_3 = x_6\}, 2)$

Steps 1-5 of the JordanChain We obtain the following data:

$$V_1 =< (1, -1, -1, 1, 1, -1) >,\ B_C = ((1, -1, -1, 1, 1, -1))\ ;$$

The subspace V_2 of the vectors $v_2 \in \ker A^2$ satisfying $Av_2 \in V_1$ is $\ker A^2$:

$$V_2 = \{(\beta, \gamma, \alpha - \beta, -\alpha - \gamma, -\alpha - \gamma, \alpha - \beta) : \ \alpha, \beta, \gamma \in \mathbf{R}\}.$$

Choosing

$$\overline{B}_2 = ((1, 0, -1, 0, 0, -1), (0, 0, 1, -1, -1, 1)),$$

we have that

$$B_C \cup \overline{B}_2 = ((1, -1, -1, 1, 1, -1),\ (1, 0, -1, 0, 0, -1),\ (0, 0, 1, -1, -1, 1)\ )$$

is a basis of V_2 and

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}.$$

Steps 6-8 of JordanChain The matrix $\overline{M}$ for the set of equality conditions $C = \{x_1 = x_4 = x_5,\ x_2 = x_3 = x_6\}$ is

$$\overline{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \quad \text{rank}\overline{M} = 2;$$

Taking the set $S = S_1 \cup S_0$ of the submatrices of $\overline{M}$ with rank 1 or 0, as described in Step 8 of the JordanChain routine, we have

$$S_1 = \left\{ N_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; N_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

and

$$S_0 = \left\{ N_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

N_1 **Step 9.1** The rows of N_1 correspond to the set of equality conditions $C_{N_1} = \{x_1 = x_4 = x_5, \, x_3 = x_6\}$ of row 5 of Table 9 for E_0.

N_2 **Step 9.1** The rows of N_2 correspond to the set of the equality conditions $C_{N_2} = \{x_2 = x_3 = x_6, \, x_4 = x_5\}$ of row 6 of Table 9 for E_0.

N_3 **Step 9.1** The rows of N_3 correspond to the set of equality conditions $C_{N_3} = \{x_3 = x_6, \, x_4 = x_5\}$ of row 9 of Table 9 for E_0.

Step 1.7.2.1 for row 5 of Table 9

Take the following basis for the intersection of the subspace corresponding to the basis in the row with ImA,

$$B_1 = ((1, -1, -1, 1, 1, -1)),$$

and call:

JordanChain$(((1, -1, -1, 1, 1, -1)), \{x_1 = x_4 = x_5, \, x_3 = x_6\}, 2)$

Steps 1-5 of the JordanChain We obtain the same data as for row 2.

Steps 6-8 of JordanChain The matrix $\overline{M}$ for the set of equality conditions $C = \{x_1 = x_4 = x_5,\ x_3 = x_6\}$ is

$$\overline{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{rank}\overline{M} = 1 < 2\,.$$

Taking the set $S = S_1 \cup S_0$ of the relevant submatrices of $\overline{M}$ with rank 1 or 0,

$$S_1 = \left\{ N_1 = \overline{M} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

and

$$S_0 = \left\{ N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

N_1 **Step 9.1** The rows of N_1 correspond to the set of equality conditions $C_{N_1} = C = \{x_1 = x_4 = x_5, x_3 = x_6\}$ of row 5 of Table 9 for E_0. We take

$$\overline{\overline{B}}_2 = ((1, 0, -2, 1, 1, -2))$$

and we change that row: taking the basis

$$B = ((1, -1, -1, 1, 1, -1), (1, 0, -2, 1, 1, -2))$$

in the third entry and $\#B = 2$ in the fourth entry - see row 5 of the Old group of rows of the Table 10. We also identify the synchrony subspace Δ_6 and we don't call the JordanChain routine at Step 9.1.5 since

$$< \overline{\overline{B}}_2 > \cap \text{Im}A = \{0\}\,.$$

N_2 **Step 9.1** The rows of N_2 correspond to the set of equality conditions $C_{N_2} = \{x_3 = x_6,\, x_4 = x_5\}$ of row 9 of Table 9 for E_0.

Step 1.7.2.1 for row 6 of Table 9

Take the following basis for the intersection of the subspace corresponding to the basis in the row with ImA,

$$B_1 = ((1, -1, -1, 1, 1, -1))\,,$$

and call:

$$\textbf{JordanChain}(((1, -1, -1, 1, 1, -1))\,, \{x_2 = x_3 = x_6,\, x_4 = x_5\}, 2)$$

Steps 1-5 of the JordanChain We obtain the same data as for row 2.

Steps 6-8 of JordanChain The matrix $\overline{M}$ for the set of equality conditions $\{x_2 = x_3 = x_6,\, x_4 = x_5\}$ is

$$\overline{M} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{rank}\overline{M} = 1 < 2\,.$$

Taking the set $S = S_1 \cup S_0$ of the relevant submatrices of $\overline{M}$ with rank 1 or 0, we have

$$S_1 = \left\{ N_1 = \overline{M} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

and

$$S_0 = \left\{ N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

N_1 **Step 9.1** The rows of N_1 correspond to the set of equality conditions $C_{N_1} = C = \{x_2 = x_3 = x_6,\ x_4 = x_5\}$ of row 6 of Table 9 for E_0. We take

$$\overline{\overline{B}}_2 = ((1, 0, 0, -1, -1, 0))$$

and we change that row: taking the basis

$$B = ((1, -1, -1, 1, 1, -1), (1, 0, 0, -1, -1, 0))$$

in the third entry and $\#B = 2$ in the fourth entry - see row 6 of the Old group of rows of the Table 10. We identify the synchrony subspace Δ_7. As

$$< \overline{\overline{B}}_2 > \cap \mathrm{Im} A = < \overline{\overline{B}}_2 >,$$

and $< B > \subseteq \mathrm{Im} A$, we have that $< B > \cap \mathrm{Im} A = < B >$. Take

$$B_2 = B = ((1, -1, -1, 1, 1, -1)\ , (1, 0, 0, -1, -1, 0))\,,$$

and execute:

JordanChain$(B_2, C, 3)$

We obtain the following data:

$$B_2 = ((1,-1,-1,1,1,-1)\ , (1,0,0,-1,-1,0))\,;$$

$$V_2 =< B_2 >;$$

$$\begin{aligned} V_3 &= \{v_3 \in \ker A^3 :\ Av_3 \in V_2\} = \ker A^3 \\ &= \{(\alpha+\beta+\gamma+\tau, -\alpha, -2\beta, -\gamma, -\gamma, -2\tau) :\ \alpha,\beta,\gamma,\tau \in \mathbf{R}\}\,; \end{aligned}$$

$$B_C = ((1,-1,-1,1,1,-1)\ , (1,0,0,-1,-1,0))\,;$$

$$\overline{B}_3 = ((1,0,-1,0,0,-1), (2,-1,-2,0,0,0))\,;$$

$B_C \cup \overline{B}_3$ is a basis of V_3;

$$M = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & -2 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

The matrix $\overline{M}$ for the set of equality conditions $C_{N_1} = C = \{x_2 = x_3 =$

x_6, $x_4 = x_5\}$ is

$$\overline{M} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} \text{ and } \quad \text{rank}\overline{M} = 2;$$

Taking the set $S = S_1 \cup S_0$ of the relevant submatrices of $\overline{M}$ with rank 1 or 0,

$$S_1 = \left\{ N_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, N_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, N_{13} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \right\}$$

and

$$S_0 = \left\{ N_{14} = \begin{bmatrix} 0 & 0 \end{bmatrix} \right\}.$$

N_{11} **Step 9.1** The rows of N_{11} correspond to the set of equality conditions $\{x_2 = x_3,\, x_4 = x_5\}$. We add one row to the Table 9, obtaining the fourth row of the New group of rows of the Table 10, identifying the synchrony subspace Δ_3 and we have to execute

JordanChain$(((1,-1,-1,1,1,-1),(1,0,0,-1,-1,0),(-1,1,1,0,0,-1)), \{x_2 = x_3, x_4 = x_5\}, 4)$

We have

$$B_3 = ((1,-1,-1,1,1,-1),(1,0,0,-1,-1,0),(-1,1,1,0,0,-1))\,;$$

$$V_3 =< B_3 >;$$

$$\begin{aligned} V_4 &= \{v_4 \in \ker A^4 : \;\; Av_4 \in V_3\} = \ker A^4 \\ &= \{\mathbf{x} : \;\; x_4 = -4x_1 - 4x_2 - 2x_3 - 3x_5 - 2x_6\}; \end{aligned}$$

$$B_C = B_3 = ((1,-1,-1,1,1,-1),(1,0,0,-1,-1,0),(-1,1,1,0,0,-1))\,;$$

$$\overline{B}_4 = ((0,0,0,-2,0,1),(0,0,1,-7,1,1))\,;$$

$B_C \cup \overline{B}_4$ is a basis of V_4;

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -2 & -7 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The matrix $\overline{M}$ for the set of equality conditions $\{x_2 = x_3,\; x_4 = x_5\}$ is

$$\overline{M} = \begin{bmatrix} 0 & -1 \\ -2 & -8 \end{bmatrix} \text{and} \quad \text{rank}\overline{M} = 2.$$

We need to consider the submatrices $[0,-1]$ and $[-2,-8]$ of $\overline{M}$, of rank 1. From the first one, we add one row to the Table 9, obtaining the third row of the New group of rows of the Table 10, and identifying the synchrony subspace Δ_1. From the second one, we add one row to the Table 9, obtaining the second row of the New group of rows of the Table 10, identifying the synchrony subspace Δ_2 and we don't call the JordanChain routine in Step 9.1.5.

N_{12} **Step 9.1** The rows of N_{12} correspond to the set of equality conditions $x_2 = x_6, x_4 = x_5$. We add one row to the Table 9, obtaining the first row of the New group of rows of the Table 10, identifying the synchrony subspace Δ_4 and we don't call the JordanChain routine in Step 9.1.5.

N_{13} **Step 9.1** The rows of N_{13} correspond to the set of equality conditions $x_3 = x_6, x_4 = x_5$ of row 9 of Table 9.

N_{14} **Step 9.1** The row of N_{14} corresponds to the equality condition $x_4 = x_5$, which is the condition in the second row of the New group of rows of the Table 10.

N_2 **Step 9.1** The rows of N_2 correspond to the set of equality conditions $C_{N_2} = \{x_3 = x_6,\, x_4 = x_5\}$ of row 9 of Table 9 for E_0.

Step 1.7.2.1 for row 9 of Table 9

Take the following basis for the intersection of the subspace corresponding to the basis in the row with ImA,

$$B_1 = ((1, -1, -1, 1, 1, -1))\,,$$

and call:

JordanChain$(((1, -1, -1, 1, 1, -1))\,, \{x_3 = x_6,\, x_4 = x_5\}, 2)$

Steps 1-8 of the JordanChain We obtain the following data:

$$V_1 =< (1, -1, -1, 1, 1, -1) >;$$

$$B_C = ((1, 0, -1, 0, 0, -1), (0, 1, 0, -1, -1, 0))\,;$$

$$\begin{aligned} V_2 &= \{v_2 \in \ker A^2 : \; Av_2 \in V_1\} = \ker A^2 \\ &= \{(\beta, \gamma, \alpha - \beta, -\alpha - \gamma, -\alpha - \gamma, \alpha - \beta) : \; \alpha, \beta, \gamma \in \mathbf{R}\}; \end{aligned}$$

$$\overline{B}_2 = ((0, 0, 1, -1, -1, 1))\,;$$

$B_C \cup \overline{B}_2$ is a basis of V_2;

$$M = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \overline{M} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

As rank$\overline{M} = 0$, we have $S = S_0$ and in S_0 we just have to take into consideration the matrix $N = \overline{M}$ that corresponds to the set of equality conditions $C_N = C = \{x_3 = x_6,\, x_4 = x_5\}$ of row 9 of Table 9 for E_0. We change that row, taking the basis

$$B = B_C \cup \overline{B}_2$$

in the third entry and $\#B = 3$ in the fourth entry - see row 9 of the Old group of rows of the Table 10. We identify the synchrony subspace Δ_5 and we don't call the JordanChain routine since

$$< \overline{B}_2 > \cap \mathrm{Im} A = \{0\}\,.$$

There are no more rows in Table 9 such that the corresponding basis intersects nontrivially ImA. Thus step 1.7.2 is concluded and we proceed to step 2 of the Algorithm 6.5 considering now Table 10 that was obtained from Table 9, with changes at rows 5,6 and 9, and four new rows.

Conditions	Polydiag. dim.	Basis	Nr. l.i. vectors
New			
$x_2 = x_6,\ x_4 = x_5$	4	$((1,-1,-1,1,1,-1),\ (1,0,0,-1,-1,0),$ $(3,-1,-3,0,0,-1)\ )$	3
$x_4 = x_5$	5	$((1,-1,-1,1,1,-1),\ (1,0,0,-1,-1,0),$ $(-1,1,1,0,0,-1),\ (0,0,1,1,1-3)\ )$	4
$x_2 = x_3$	5	$((1,-1,-1,1,1,-1),\ (1,0,0,-1,-1,0),$ $(-1,1,1,0,0,-1),\ (0,0,0,-2,0,1)\ )$	4
$x_2 = x_3,\ x_4 = x_5$	4	$((1,-1,-1,1,1,-1),\ (1,0,0,-1,-1,0),$ $(-1,1,1,0,0,-1)\ )$	3
Old			
$x_1 = x_2,\ x_3 = x_4 = x_5 = x_6$	2	$((1,1,-1,-1,-1,-1))$	1
$x_1 = x_4 = x_5,\ x_2 = x_3 = x_6$	2	$((1,-1,-1,1,1,-1))$	1
$x_1 = x_2,\ x_3 = x_6,\ x_4 = x_5$	3	$((1,1,-1,-1,-1,-1))$	1
$x_1 = x_3 = x_6,\ x_4 = x_5$	3	$((0,1,0,-1,-1,0))$	1
$x_1 = x_4 = x_5,\ x_3 = x_6$	3	$((\mathbf{1},\mathbf{-1},\mathbf{-1},\mathbf{1},\mathbf{1},\mathbf{-1}),\ (\mathbf{1},\mathbf{0},\mathbf{-2},\mathbf{1},\mathbf{1},\mathbf{-2}))$	**2**
$x_2 = x_3 = x_6,\ x_4 = x_5$	3	$((\mathbf{1},\mathbf{-1},\mathbf{-1},\mathbf{1},\mathbf{1},\mathbf{-1}),\ (\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{-1},\mathbf{-1},\mathbf{0}))$	**2**
$x_2 = x_4 = x_5,\ x_3 = x_6$	3	$((1,0,-1,0,0,-1))$	1
$x_3 = x_4 = x_5 = x_6$	3	$((1,1,-1,-1,-1,-1))$	1
$x_3 = x_6,\ x_4 = x_5$	4	$((\mathbf{1},\mathbf{0},\mathbf{-1},\mathbf{0},\mathbf{0},\mathbf{-1}),\ (\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{-1},\mathbf{-1},\mathbf{0}),$ $(\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{-1},\mathbf{-1},\mathbf{1})\ )$	**3**

Table 10: At the end of the execution of the Step 1 of the Algorithm 6.5, rows 5,6,9 of the Table 9 for E_0 were changed and four rows were added (the first four rows of this table).

Step 2. We identify the sum-dense set of synchrony subspaces

$$S = \{\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9\}\,.$$

Step 3. From the set S, we have that $\Delta_5 = \Delta_6 + \Delta_8$ and $\Delta_2 = \Delta_3 + \Delta_4$. Moreover, all the others synchrony subspaces are sum-irreducible. It follows then that

$$\mathcal{I}_{\mathcal{G}} = \{\Delta_1, \Delta_3, \Delta_4, \Delta_6, \Delta_7, \Delta_8, \Delta_9\}\,.$$

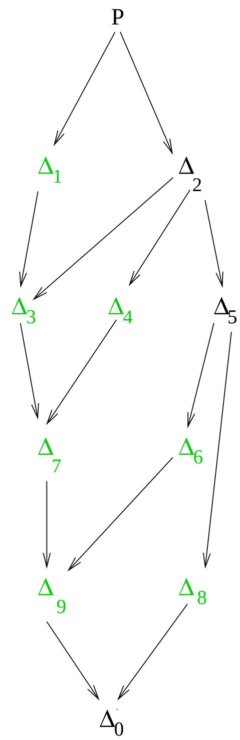

Figure 9: The lattice of synchrony subspaces for the 6-cell regular network $\mathcal{G}$ of Figure 8. The synchrony subspaces Δ_i, for $i = 1, \ldots, 9$, are listed in Table 7. The top element is the total phase space P (the total asynchronous polydiagonal space) and the bottom element Δ_0 is the full synchronous polydiagonal space. The sum-irreducible synchrony subspaces of the sum-dense $\mathcal{I}_{\mathcal{G}}$ are in green.

Step 4. We obtain the lattice $V_{\mathcal{G}}$ formed by the synchrony subspaces in Figure 9 and listed in Table 7.

$\diamond$

7 More on the lattice of synchrony subspaces for non-regular homogeneous networks

Recall Corollary 4.3 in Section 4 which shows that, a polydiagonal subspace for a homogeneous network $\mathcal{G}$ with edge-types $\mathcal{E}_1, \ldots, \mathcal{E}_l$, is of synchrony, if and only if it is a synchrony subspace for all its regular subnetworks $\mathcal{G}_{\mathcal{E}_j}$. Thus we can use Algorithm 6.5 to obtain the lattice $V_{\mathcal{G}_{\mathcal{E}_j}}$ for the subnetworks $\mathcal{G}_{\mathcal{E}_j}$ and then the lattice $V_{\mathcal{G}}$ is given by the intersection of those lattices.

Example 7.1 Consider the 5-cell homogeneous network $\mathcal{G}$ in Figure 4. The coupled cell systems associated to the network $\mathcal{G}$ satisfy

$$\begin{aligned}
\dot{x}_1 &= f(x_1, \overline{x_2, x_4}, \overline{x_2, x_4}) \\
\dot{x}_2 &= f(x_2, \overline{x_1, x_5}, \overline{x_1, x_4}) \\
\dot{x}_3 &= f(x_3, \overline{x_1, x_5}, \overline{x_1, x_5}) \ , \\
\dot{x}_4 &= f(x_4, \overline{x_1, x_3}, \overline{x_1, x_2}) \\
\dot{x}_5 &= f(x_5, \overline{x_1, x_3}, \overline{x_1, x_3})
\end{aligned}$$

where $f(u, v, w, z, t)$ is a smooth function invariant under permutation of the variables v and w and under the permutation of the variables z and t. There are two edge types, $\mathcal{E}_1$ and $\mathcal{E}_2$ with adjacency matrices, respectively,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that, the subnetwork $\mathcal{G}_{\mathcal{E}_1}$, with adjacency matrix A_1, is the one in the Example 6.10

and the set of the nontrivial synchrony subspaces for the subnetwork $\mathcal{G}_{\mathcal{E}_1}$ is given in Table 1. Using Algorithm 6.5 to obtain the lattice $V_{\mathcal{G}_{\mathcal{E}_2}}$ for the subnetwork $\mathcal{G}_{\mathcal{E}_2}$, we obtain the set of the nontrivial synchrony subspaces in Table 11. By Corollary 4.3, the nontrivial synchrony subspaces for $\mathcal{G}$ are the ones listed in Table 12.

$\Delta_1 = \{\mathbf{x}:\ x_1 = x_2\}$ $\Delta_2 = \{\mathbf{x}:\ x_1 = x_4\}$ $\Delta_3 = \{\mathbf{x}:\ x_2 = x_4\}$ $\Delta_4 = \{\mathbf{x}:\ x_3 = x_5\}$	$\Delta_5 = \{\mathbf{x}:\ x_1 = x_2 = x_4\}$ $\Delta_6 = \{\mathbf{x}:\ x_2 = x_3,\ x_4 = x_5\}$ $\Delta_7 = \{\mathbf{x}:\ x_1 = x_2,\ x_3 = x_5\}$ $\Delta_8 = \{\mathbf{x}:\ x_2 = x_4,\ x_3 = x_5\}$ $\Delta_9 = \{\mathbf{x}:\ x_2 = x_5,\ x_3 = x_4\}$ $\Delta_{10} = \{\mathbf{x}:\ x_1 = x_4, x_3 = x_5\}$	$\Delta_{11} = \{\mathbf{x}:\ x_1 = x_2 = x_3,\ x_4 = x_5\}$ $\Delta_{12} = \{\mathbf{x}:\ x_1 = x_2 = x_4,\ x_3 = x_5\}$ $\Delta_{13} = \{\mathbf{x}:\ x_1 = x_2 = x_5,\ x_3 = x_4\}$ $\Delta_{14} = \{\mathbf{x}:\ x_1 = x_3 = x_4,\ x_2 = x_5\}$ $\Delta_{15} = \{\mathbf{x}:\ x_1 = x_4 = x_5,\ x_2 = x_3\}$ $\Delta_{16} = \{\mathbf{x}:\ x_2 = x_3 = x_4 = x_5\}$

Table 11: Nontrivial synchrony subspaces for the network with adjacency matrix A_2.

$\{\mathbf{x}:\ x_3 = x_5\}$	$\{\mathbf{x}:\ x_2 = x_3,\ x_4 = x_5\}$ $\{\mathbf{x}:\ x_2 = x_4,\ x_3 = x_5\}$	$\{\mathbf{x}:\ x_1 = x_2 = x_3,\ x_4 = x_5\}$ $\{\mathbf{x}:\ x_1 = x_4 = x_5,\ x_2 = x_3\}$ $\{\mathbf{x}:\ x_2 = x_3 = x_4 = x_5\}$

Table 12: Nontrivial synchrony subspaces for the network of Figure 4.

$\diamond$

One way to implement an efficient algorithm to obtain the lattice $V_{\mathcal{G}}$ is to find the lattice $V_{\mathcal{G}_{\mathcal{E}_j}}$ for some subnetwork $\mathcal{G}_{\mathcal{E}_j}$ executing Algorithm 6.5, and then finding the subset of subspaces in $V_{\mathcal{G}_{\mathcal{E}_j}}$ that are left invariant by the adjacency matrices of the other subnetworks $\mathcal{G}_{\mathcal{E}_k}$.

Another way to implement an optimized algorithm to obtain the lattice $V_{\mathcal{G}}$ is not to make distinction of the edge-types and consider a 'global adjacency matrix' (that includes the arrows of all types). Obviously, the synchrony subspaces in $V_{\mathcal{G}}$ are synchrony subspaces for the regular network $\overline{\mathcal{G}}$ with adjacency matrix given by that global matrix. Since the reverse is not true, after executing Algorithm 6.5 for $\overline{\mathcal{G}}$, it is then necessary to find the subset of subspaces in $V_{\overline{\mathcal{G}}}$ that are left invariant by all the adjacency matrices of $\mathcal{G}$.

8 More on the lattice of synchrony subspaces for nonhomogeneous networks

Theorem 4.2 in Section 4 relates the lattice of balanced equivalence relations of a nonhomogeneous network $\mathcal{G}$ with the lattices of balanced equivalence relations of its subnetworks by input class and edge-type class $\mathcal{G}^{I_i}$ and $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}}$ (recall the definition in Section 4). Here we go further and present results that define a process to obtain the nontrivial balanced equivalence relations in $\Lambda_{\mathcal{G}}$ given the balanced equivalence relations in $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}}$. For that, we need to introduce the following definition.

Definition 8.1 Let H be a subnetwork of a network $\mathcal{G}$ and $\bowtie$ be an equivalence relation on the set $\mathcal{C}_H$ of the cells of H. Using $\bowtie$ we define an equivalence relation $\breve{\bowtie}$ on the set $\mathcal{C} \supseteq \mathcal{C}_H$ of the cells of $\mathcal{G}$ by $[c]_{\breve{\bowtie}} = [c]_{\bowtie}$, for $c \in \mathcal{C}_H$ and $[c]_{\breve{\bowtie}} = \{c\}$ for $c \in \mathcal{C} \setminus \mathcal{C}_H$. We say that $\breve{\bowtie}$ is the *unfolding* of $\bowtie$ to $\mathcal{C}$. Analogously, if $\bowtie$ is an equivalence relation on the set $\mathcal{C}$, then the equivalence relation $\widehat{\bowtie}$ on the set of cells $\mathcal{C}_H \subset \mathcal{C}$ of the cells in H defined by $[c]_{\widehat{\bowtie}} = [c]_{\bowtie} \cap \mathcal{C}_H$ is called the *restriction* of $\bowtie$ to $\mathcal{C}_H$. $\diamond$

Let $\tilde{\mathcal{G}}^{I_i}$ and $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$ be the subnetworks of $\mathcal{G}^{I_i}$ and $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}}$, considering only the cells in I_i and in their input sets in $\mathcal{G}^{I_i}$ and $\mathcal{G}^{I_i}_{\mathcal{E}_{i_j}}$, respectively.

Theorem 8.2 *Let $\mathcal{G}$ be a nonhomogeneous network. For an input equivalence class I_i of $\mathcal{G}$, consider the corresponding subnetwork $\tilde{\mathcal{G}}^{I_i}$ and the associated identical-edge subnetworks $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$, for $j = 1, \ldots, r_i$. Let $\tilde{\mathcal{B}}_i$ be the set of balanced equivalence relations $\bowtie_i$ on the cells of $\tilde{\mathcal{G}}^{I_i}$ such that there is at least one cell c in I_i with $\#[c]_{\bowtie_i} > 1$ and not admitting a balanced refinement $\bowtie_i^r$ such that for $c \in I_i$ we have $[c]_{\bowtie_i} = [c]_{\bowtie_i^r}$. Let $\tilde{\mathcal{B}}_{i_j}$ be the set of balanced equivalence relations $\bowtie_{i_j}$ on the cells of $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$ such that there is at least one cell c in I_i with $\#[c]_{\bowtie_{i_j}} > 1$ and not admitting a balanced refinement $\bowtie_{i_j}^r$ such that for $c \in I_i$ we have $[c]_{\bowtie_{i_j}} = [c]_{\bowtie_{i_j}^r}$. For each $\bowtie_{i_j} \in \tilde{\mathcal{B}}_{i_j}$, consider its unfolding $\breve{\bowtie}_{i_j}$ to the set of cells of $\tilde{\mathcal{G}}^{I_i}$.*

An equivalence relation $\bowtie_i$ is in $\tilde{\mathcal{B}}_i$ if and only if

$$\bowtie_i = \breve{\bowtie}_{i_1} \vee \cdots \vee \breve{\bowtie}_{i_{r_i}}$$

for $\bowtie_{i_1} \in \tilde{\mathcal{B}}_{i_1}, \ldots, \bowtie_{i_{r_i}} \in \tilde{\mathcal{B}}_{i_{r_i}}$ satisfying $[c]_{\bowtie_{i_j}} = [c]_{\bowtie_{i_k}}$, for all $j, k \in \{1, \ldots, r_i\}$ and $c \in I_i$.

Proof Let $\bowtie_i \in \tilde{\mathcal{B}}_i$ and $\bowtie_{i_j}$, for $j = 1, \ldots, r_i$, be the restriction of $\bowtie_i$ to the set of cells of $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$. Trivially each $\bowtie_{i_j}$ is a balanced equivalence relation in $\tilde{\mathcal{B}}_{i_j}$ and $\bowtie_i = \breve{\bowtie}_{i_1} \vee \cdots \vee \breve{\bowtie}_{i_{r_i}}$, with $\breve{\bowtie}_{i_j}$ the unfolding of $\bowtie_{i_j}$ to the cells of $\tilde{\mathcal{G}}^{I_i}$. Moreover, since for all $j = 1, \ldots, r_i$ the input equivalence class I_i is a subset of the set of cells of $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$, we have $[c]_{\bowtie_{i_j}} = [c]_{\bowtie_{i_k}}$, for all $j, k \in \{1, \ldots, r_i\}$ and $c \in I_i$.

Now, let $\bowtie_{i_1} \in \tilde{\mathcal{B}}_{i_1}, \ldots, \bowtie_{i_{r_i}} \in \tilde{\mathcal{B}}_{i_{r_i}}$ satisfying $[c]_{\bowtie_{i_j}} = [c]_{\bowtie_{i_k}}$, for all $j, k \in \{1, \ldots, r_i\}$ and $c \in I_i$. First, we prove that $\bowtie_i = \breve{\bowtie}_{i_1} \vee \cdots \vee \breve{\bowtie}_{i_{r_i}}$ is a balanced equivalence relation on the cells of $\tilde{\mathcal{G}}^{I_i}$. For that we have to show that given any two cells c, d of $\tilde{\mathcal{G}}^{I_i}$ such that $c \bowtie_i d$, there is an edge-type preserving isomorphism $\beta_i(c,d) : I(c) \to I(d)$, between the input sets of c and d in $\tilde{\mathcal{G}}^{I_i}$, respectively, such that for all $\alpha \in I(c)$, the tail cells of α and $\beta_i(c,d)(\alpha)$ are in the same $\bowtie_i$ class. Note that $c \bowtie_i d$ only if c and d belong to the same input equivalence class. Consider first the case $c, d \in I_i$. If $c \bowtie_i d$, then $c \bowtie_{i_j} d$ (and thus $c \,\breve{\bowtie}_{i_j} d$), for all $j = 1, \ldots, r_i$. Moreover, since $\bowtie_{i_j}$ is balanced, as $[c]_{\bowtie_{i_j}} = [c]_{\bowtie_{i_k}}$, we define the isomorphism $\beta_i(c,d)$ as $\beta_i(c,d)(e) = \beta_{i_j}(c,d)(e)$, with i_j the edge type of e. Since for every $\beta_{i_j}(c,d)$, for $j = 1, \ldots, r_i$, the tail cells of α and $\beta_{i_j}(c,d)(\alpha)$ are in the same $\bowtie_{i_j}$ class, the same follows for $\beta_i(c,d)$ and $\bowtie_i$. In the case $c \bowtie_i d$ with $c, d \in I_l$, for $l \neq i$, we have $I(c) = I(d) = \emptyset$ and thus there is nothing to prove. That $\bowtie_i$ belongs to $\tilde{\mathcal{B}}_i$ follows trivially from the fact that $\bowtie_{i_j} \in \tilde{\mathcal{B}}_{i_j}$, for $j = 1, \ldots, r_i$. $\square$

Theorem 8.3 *Let $\mathcal{G}$ be a nonhomogeneous network and for $i = 1, \ldots, k$, consider $\tilde{\mathcal{G}}^{I_i}$ the subnetwork of $\mathcal{G}$ corresponding to the input equivalence class I_i. Let $\mathcal{B}_i$ be the set of the unfoldings $\breve{\bowtie}_i$ to $\mathcal{C}$ of all the balanced equivalence relations $\bowtie_i \in \tilde{\mathcal{B}}_i$ defined in Theorem 8.2,* *together with the trivial relation (where each class is formed by a unique cell), on the set of cells of* $\tilde{\mathcal{G}}^{I_i}$.

An equivalence relation $\bowtie$ *on the set of cells of* $\mathcal{G}$ *is balanced if and only if*

$$\bowtie = \breve{\bowtie}_1 \vee \cdots \vee \breve{\bowtie}_k,$$

with $\breve{\bowtie}_i \in \mathcal{B}_i$ *such that* $[c]_{\breve{\bowtie}_j} \subseteq [c]_{\breve{\bowtie}_l}$, *for* $c \in I_l$, *for all* $1 \leq j, l \leq k$.

Proof Let $\bowtie$ be a balanced equivalence relation on the set of cells of $\mathcal{G}$. Let $\bowtie_i$, for $i = 1, \ldots, k$, be the restriction of $\bowtie$ to the set of cells of $\tilde{\mathcal{G}}^{I_i}$. Trivially each $\bowtie_i$ is a balanced equivalence relation on the cells of $\tilde{\mathcal{G}}^{I_i}$, for $i = 1, \ldots, k$. If for all $c \in I_i$ we have $\#[c]_{\bowtie_i} = 1$ then we can take $\bowtie_i$ to be the trivial relation on $\mathcal{C}$. Otherwise, if $\bowtie_i \notin \tilde{\mathcal{B}}_i$ then let $\bowtie_i^r \in \tilde{\mathcal{B}}_i$ such that $\bowtie_i^r \prec \bowtie_i$ and take $\bowtie_i = \bowtie_i^r$. It follows trivially that $\bowtie = \breve{\bowtie}_1 \vee \cdots \vee \breve{\bowtie}_k$. Moreover, observe that $\tilde{\mathcal{G}}^{I_l}$ contains the cells in I_l and that $\bowtie$ refines $\sim_I$ since it is balanced. Thus, for $c \in I_l$, we have that $[c]_{\bowtie} \subseteq I_l$ and so $[c]_{\bowtie} = [c]_{\bowtie_l} = [c]_{\breve{\bowtie}_l}$. If $j \neq l$ then $[c]_{\breve{\bowtie}_j} \subseteq [c]_{\breve{\bowtie}_l}$.

Let $\breve{\bowtie}_i \in \mathcal{B}_i$ such that $[c]_{\breve{\bowtie}_j} \subseteq [c]_{\breve{\bowtie}_l}$, for $c \in I_l$, for all $1 \leq j, l \leq k$. To prove that $\bowtie = \breve{\bowtie}_1 \vee \ldots \vee \breve{\bowtie}_k$ is a balanced equivalence relation on the cells of $\mathcal{G}$ we have to show that given any two cells c, d of $\mathcal{G}$ such that $c \bowtie d$, there is an edge-type preserving isomorphism $\beta(c, d) : \ I(c) \to I(d)$, between the input sets of c and d, respectively, such that for all $\alpha \in I(c)$, the tail cells of α and $\beta(c, d)(\alpha)$ are in the same $\bowtie$ class.

Note that $c \bowtie d$ only if c and d belong to the same input equivalence class since each $\breve{\bowtie}_l \in \mathcal{B}_l$ refines $\sim_I$. If $c, d \in I_l$, for $l \in \{1, \ldots, k\}$, and $c \bowtie d$, then $c \, \breve{\bowtie}_l d$, and thus $c \bowtie_l d$, since, for all $j \neq l$, $[c]_{\breve{\bowtie}_j} \subseteq [c]_{\breve{\bowtie}_l}$. As $\bowtie_l$ is balanced, there is an edge-type preserving isomorphism $\beta_l(c, d) : \ I(c) \to I(d)$ such that for all $\alpha \in I(c)$, the tail cells of α and $\beta_l(c, d)(\alpha)$ are in the same $\bowtie_l$ class. We define $\beta(c, d)(e) = \beta_l(c, d)(e)$. Since the input set of the cells in I_l is empty in the subnetworks $\tilde{\mathcal{G}}^{I_j}$, for $j \neq l$, we conclude that the isomorphism $\beta(c, d)$ satisfies for all $\alpha \in I(c)$, that the tail cells of α and $\beta(c, d)(\alpha)$ are in the same $\bowtie$ class.

□

Using the above results and Algorithm 6.5, together with the one-to-one correspondence between balanced equivalence relations and synchrony subspaces, it is easy to generate an algorithm to obtain the lattice of synchrony subspaces of a nonhomogeneous network. More specifically, given a nonhomogeneous network $\mathcal{G}$, consider the corresponding subnetworks $\tilde{\mathcal{G}}^{I_i}$ and the associated identical-edge subnetworks $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$. Note that the subnetworks $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$ may be not regular, even though they have only one edge type. For the purpose of getting the lattice of synchrony subspaces, we consider the network adjacency matrix. Thus, for each identical-edge subnetwork $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$, its lattice of synchrony subspaces can be obtained using Algorithm 6.5 taking into account Remark 6.9.

The lattices of balanced equivalence relations for the subnetworks $\tilde{\mathcal{G}}^{I_i}_{\mathcal{E}_{i_j}}$ are obtained from the corresponding lattices of synchrony subspaces. The lattices of balanced equivalence relations for the subnetworks $\tilde{\mathcal{G}}^{I_i}$ are then obtained using the result in Theorem 8.2. Finally, the lattice of balanced equivalence relations for the network $\mathcal{G}$ is obtained using the result in Theorem 8.3 and from that lattice we get the corresponding lattice of synchrony subspaces of $\mathcal{G}$.

We illustrate the above explanation with a network example.

Example 8.4 Consider the nonhomogeneous network $\mathcal{G}$ in Figure 2. The coupled cell systems associated to the network $\mathcal{G}$ satisfy

$$\begin{aligned}
\dot{x}_1 &= f(x_1, x_2, x_3, x_8)\\
\dot{x}_2 &= f(x_2, x_1, x_4, x_7)\\
\dot{x}_3 &= g(x_3, x_4)\\
\dot{x}_4 &= g(x_4, x_5)\\
\dot{x}_5 &= g(x_5, x_3)\\
\dot{x}_6 &= h(x_6, x_1, x_5)\\
\dot{x}_7 &= h(x_7, x_2, x_5)\\
\dot{x}_8 &= h(x_8, x_2, x_5)
\end{aligned},$$

where $f(u, v, w, z, t)$ is a smooth function.

The nontrivial synchrony subspaces for the network $\mathcal{G}$ are given in Table 13 and we show now how to use the above results in order to obtain this table.

$\Delta_1 = \{\mathbf{x}: \ x_7 = x_8\}$	$\Delta_4 = \{\mathbf{x}: \ x_1 = x_2, \ x_3 = x_4 = x_5, \ x_7 = x_8\}$
$\Delta_2 = \{\mathbf{x}: \ x_3 = x_4 = x_5\}$	$\Delta_5 = \{\mathbf{x}: \ x_1 = x_2, \ x_3 = x_4 = x_5, \ x_6 = x_7 = x_8\}$
$\Delta_3 = \{\mathbf{x}: \ x_3 = x_4 = x_5, \ x_7 = x_8\}$	

Table 13: Nontrivial synchrony subspaces for the network of Figure 2.

The $\sim_I$- equivalence classes of $\mathcal{G}$ are

$$I_1 = \{1, 2\}, \quad I_2 = \{3, 4, 5\} \quad \text{and } I_3 = \{6, 7, 8\}.$$

The subnetworks $\mathcal{G}^{I_i}$, for $i = 1, 2, 3$, of $\mathcal{G}$ are shown in Figure 3. Recall that each subnetwork $\mathcal{G}^{I_i}$ is obtained from $\mathcal{G}$ considering only the edges that are directed to cells in I_i. Looking at $\mathcal{G}^{I_1}$ we see that: there are three edge-types with head cells in I_1 - let $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ be the classes of the edges with head cells in I_1 and tail cells in I_1, I_2 and I_3, respectively; there is only one type of edge with head cells in I_2 - let $\mathcal{E}_4$ be the class of the edges with head and tail cells in I_2; there are two edge-types with head cells in I_3 -

let $\mathcal{E}_5$ and $\mathcal{E}_6$ be the classes of the edges with head cells in I_3 and tail cells in I_1 and I_2, respectively.

Let's consider now the subnetworks $\tilde{\mathcal{G}}^{I_i}$ of $\mathcal{G}^{I_i}$, for $i = 1, 2, 3$. Thus each $\tilde{\mathcal{G}}^{I_i}$ is obtained from $\mathcal{G}^{I_i}$ considering only the cells in I_i and in their input sets. It follows then that the subnetwork $\tilde{\mathcal{G}}^{I_1}$ contains the cells $1, 2, 3, 4, 7$ and 8; the subnetwork $\tilde{\mathcal{G}}^{I_2}$ contains only the cells in I_2; and $\tilde{\mathcal{G}}^{I_3}$ contains the cells $6, 7, 8, 1, 2$ and 5.

We get

$$
\begin{aligned}
V_{\tilde{\mathcal{G}}^{I_1}_{\mathcal{E}_1}} &= \{\{\mathbf{x}: \; x_1 = x_2\}\}, \\
V_{\tilde{\mathcal{G}}^{I_1}_{\mathcal{E}_2}} &= \{\{\mathbf{x}: \; x_1 = x_2, \; x_3 = x_4\}\}, \\
V_{\tilde{\mathcal{G}}^{I_1}_{\mathcal{E}_3}} &= \{\{\mathbf{x}: \; x_1 = x_2, \; x_7 = x_8\}\},
\end{aligned}
$$

using Algorithm 6.5 together with Remark 6.9. By Theorem 8.2, taking

$$
\begin{aligned}
\Delta_1^1 &= \{\mathbf{x}: \; x_1 = x_2\} \cap \{\mathbf{x}: \; x_1 = x_2, \; x_3 = x_4\} \cap \{\mathbf{x}: \; x_1 = x_2, \; x_7 = x_8\} \\
&= \{\mathbf{x}: \; x_1 = x_2, \; x_3 = x_4, \; x_7 = x_8\},
\end{aligned}
$$

we have that

$$V_{\tilde{\mathcal{G}}^{I_1}} = \{\Delta_1^1\}.$$

Analogously, we obtain

$$V_{\tilde{\mathcal{G}}^{I_2}} = \{\Delta_1^2\}$$

with

$$\Delta_1^2 = \{\mathbf{x} : x_3 = x_4 = x_5\}.$$

Moreover, we have that

$$V_{\tilde{\mathcal{G}}^{I_3}} = \{\Delta_1^3, \; \Delta_2^3, \; \Delta_3^3, \; \Delta_4^3, \; \Delta_5^3\},$$

taking

$$\Delta_1^3 = \{\mathbf{x} : x_1 = x_2,\, x_6 = x_7\},$$

$$\Delta_2^3 = \{\mathbf{x} : x_1 = x_2,\, x_6 = x_8\},$$

$$\Delta_3^3 = \{\mathbf{x} : x_1 = x_2,\, x_7 = x_8\},$$

$$\Delta_4^3 = \{\mathbf{x} : x_1 = x_2,\, x_6 = x_7 = x_8\},$$

$$\Delta_5^3 = \{\mathbf{x} : x_7 = x_8\}\,.$$

By Theorem 8.3, we get the nontrivial synchrony subspaces in $V_{\mathcal{G}}$:

$$\begin{aligned}
\Delta_4 &= \Delta_1^1 \cap \Delta_1^2 \cap \Delta_3^3,\\
\Delta_5 &= \Delta_1^1 \cap \Delta_1^2 \cap \Delta_4^3,\\
\Delta_3 &= \Delta_0^1 \cap \Delta_1^2 \cap \Delta_5^3,\\
\Delta_2 &= \Delta_0^1 \cap \Delta_1^2 \cap \Delta_0^3,\\
\Delta_1 &= \Delta_0^1 \cap \Delta_0^2 \cap \Delta_5^3\,.
\end{aligned}$$

Here, each Δ_0^i corresponds to the total asynchronous polydiagonal subspace (corresponding to the trivial relation where each class is formed by a unique cell). $\diamond$

Acknowledgments

The authors thank the referees for their comments that improved the paper presentation.

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