Introduction to Supersymmetry([1])

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[1] **Esta é uma versão provisória, incompleta, para uso exclusivo nas sessões de trabalho do TQFT club**

Contents

1 Supersymmetry in Quantum Mechanics

1.1 The Supersymmetric Oscillator

As we will see later the *"hermitian supercharges"* Q^i_α, in the N extended SuperPoincaré Lie Algebra obey the anticommutation relations:

$$\{Q^i_\alpha, Q^j_\beta\} = 2(\gamma^m C)_{\alpha\beta}\delta^{ij}\, P_m \tag{1.1}$$

where α, β are "spinor" indices, $i, j \in \{1, \cdots, N\}$ "internal" indices and $(\gamma^m C)_{\alpha\beta}$ a bilinear form in the spinor indices α, β.

When specialized to 0-space dimensions ($(1+0)$-spacetime), then since $P_0 = H$, relations (1.1) take the form (with a little change in notations):

$$\{Q_i, Q_j\} = 2\delta_{ij}\, H \tag{1.2}$$

with N *"Hermitian charges"* Q_i, $i = 1, \cdots, N$. Let us see some imediate consequences of relations (1.2:

- *The supercharges Q_i are constants of motion.* In fact:

$$[H, Q] = [Q^2, Q] = 0 \tag{1.3}$$

where Q is any of the Q_i.

- *The Hamiltonian H is an hermitian positive operator*, and so the energy spectrum is always positive definite. In fact:

$$H = Q_1^2 = \cdots = Q_N^2 \tag{1.4}$$

So, $\forall|\psi\rangle \in \mathcal{H}$ we have:

$$\langle\psi|H|\psi\rangle = \langle\psi|Q^2|\psi\rangle = \langle\psi|Q^\dagger Q|\psi\rangle = \|Q|\psi\rangle\|^2 \geq 0$$

where Q is any of the Q_i. This also proves that:

- $$\ker H = \cap_i \ker Q_i \tag{1.5}$$

Since the Hamiltonian H is a positive operator, any eigenstate $|\psi_0>$ of H with zero eigenvalue is a *"ground state"*, and for such a ground state we have that $Q_i|\psi_0>= \mathbf{0}$, $\forall i$. We then say that the *"supersymmetry is unbroken"*. When there is no eigenstate with zero eigenvalue, then the ground state $|\psi_0\rangle$ has energy $E_{\psi_0} > 0$. This implies that $Q|\psi_0\rangle \neq \mathbf{0}$ and we then say that we have *"spontaneous susy breaking"*.

Now we focus our attention in the N = 2 model, which we call the:

"Supersymmetric Oscillator"

In this case let us define the following two *"nonhermitian supercharges"*, adjoint of each other:

$$
\begin{aligned}
S &\stackrel{\text{def}}{=} \frac{1}{2}(Q_1 + iQ_2) \\
\overline{S} = S^\dagger = &\stackrel{\text{def}}{=} \frac{1}{2}(Q_1 - iQ_2)
\end{aligned}
\tag{1.6}
$$

Then we have the following "representation" for the above ($N = 2$)-Susy algebra:

$$
\begin{aligned}
H = Q_1^2 = Q_2^2 &= \{S, \overline{S}\} \\
S^2 &= \overline{S}^2 = 0
\end{aligned}
\tag{1.7}
$$

We also have $[H, Q] = 0$, where Q is any of the Q_i, S or $\overline{S}$.

Consider the Hilbert space $\mathcal{H}$ with basis:

$$
|n_B, n_F\rangle \qquad n_B = 0, 1, 2, \cdots, \infty \qquad n_F = 0, 1 \tag{1.8}
$$

where n_B and n_F are *"boson"* and *"fermion occupation numbers"* respectivelly, and let $a, a^\dagger$ *"anihilation-creation"* bosonic operators, and $f, f^\dagger$ *"anihilation-creation"* fermionic operators, acting on $\mathcal{H}$ in the standard way. They satisfy the following commutation and anticommutation relations:

$$
\begin{aligned}
&[a, a^\dagger] = 1 \\
&\{f, f^\dagger\} = 1 \qquad f^2 = (f^\dagger)^2 = 0 \\
&[a, f] = 0
\end{aligned}
\tag{1.9}
$$

Then if we put:

$$
\begin{aligned}
S &\stackrel{\text{def}}{=} k\, a f^\dagger && \text{“destroy a boson } \otimes \text{ create a fermion”} \\
\overline{S} &\stackrel{\text{def}}{=} k\, a^\dagger f && \text{“create a boson } \otimes \text{ destroy a fermion”}
\end{aligned}
\tag{1.10}
$$

where k is a constant so that S and $\overline{S}$ are adjoints of each other ($\overline{S} = S^\dagger$), we see that:

$$
\begin{aligned}
S\,|n_B, n_F\rangle = k\, a f^\dagger |n_B, n_F> &\propto |n_B - 1, n_F + 1\rangle \\
\overline{S}\,|n_B, n_F\rangle = k\, a^\dagger f |n_B, n_F> &\propto |n_B + 1, n_F - 1\rangle
\end{aligned}
\tag{1.11}
$$

so that these operators convert a boson into a fermion and vice-versa. Moreover we can verify properties (1.7), using (1.9).

Now what about the Hamiltonian? We compute:

$$\begin{aligned} H &= \{S, \overline{S}\} \\ &= k^2(af^\dagger a^\dagger f + a^\dagger f a f^\dagger) \\ &= k^2(a^\dagger a + \frac{1}{2}) + k^2(f^\dagger f - \frac{1}{2}) \\ &\stackrel{\text{def}}{=} H_B + H_F \end{aligned} \tag{1.12}$$

So the H is the sum of two non-interacting terms: the Hamiltonian of the bosonic oscillator H_B with energy spectrum E_B, and the Hamiltonian of the fermionic oscillator H_F with energy spectrum E_F given respectivelly by:

$$\begin{aligned} H_B &= k^2(a^\dagger a + \frac{1}{2}) \qquad & E_B &= k^2(n_B + \frac{1}{2}) \qquad & n_B &= 0, 1, 2, 3, \cdots \\ H_F &= k^2(f^\dagger f - \frac{1}{2}) \qquad & E_F &= k^2(n_F - \frac{1}{2}) \qquad & n_B &= 0, 1 \end{aligned} \tag{1.13}$$

Note that:

$$n_F^2 = f^\dagger f f^\dagger f = f^\dagger \{f, f^\dagger\} f = f^\dagger f = n_F$$

and so in fact the eigenvalues of n_F are $0, 1$ which is the *"Pauli exclusion principle"*. Note also that the frequencies $\omega = k^2$ of these two oscillators are the same.

1.2 Witten Index

For the above $(N = 2)$-Susy QM model, we can define an operator:

$$(-1)^F \stackrel{\text{def}}{=} (-1)^{n_F} \mathbb{1}$$

such that:

$$\{(-1)^F, Q_i\} = 0 \qquad ((-1)^F)^2 = \mathbb{1} \qquad ((-1)^F)^\dagger = (-1)^F \tag{1.14}$$

Conversselly, given an Hilbert space $\mathcal{H}$ and hermitian operators $H, Q, (-1)^F$ such that $(-1)^F$ is bounded and:

$$H = Q^2 \qquad ((-1)^F)^2 = \mathbb{1} \qquad \{Q, (-1)^F\} = 0 \tag{1.15}$$

we can define a $(N = 2)$-Susy QM model by putting:

$$Q_1 = Q \qquad \text{and} \qquad Q_2 = i(-1)^F Q$$

We explore now the abstract data given by an Hilbert space $\mathcal{H}$ and hermitian operators $H, Q, (-1)^F$, with $(-1)^F$ bounded, and verifying conditions (1.15).

Here follows some properties of this *"abstract Susy model"*, $\{H, Q, (-1)^F\}$, which are imediate consequences of conditions (1.15):

I •

$$\begin{aligned}[(-1)^F, H] &= [(-1)^F, Q^2] \\ &= \{(-1)^F, Q\}Q - Q\{Q, (-1)^F\} = 0 \end{aligned} \tag{1.16}$$

II • We have a decomposition of $\mathcal{H}$ in eigenspaces of $(-1)^F$ corresponding to the eigenvalues $\pm$:

$$\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f$$

with:

$$\begin{aligned}\mathcal{H}_b &= \{\, \psi \in \mathcal{H} : \; (-1)^F \psi = +\psi \,\} \\ \mathcal{H}_f &= \{\, \psi \in \mathcal{H} : \; (-1)^F \psi = -\psi \,\} \end{aligned} \tag{1.17}$$

so that $(-1)^F$ acts on $\mathcal{H}$ as:

$$(-1)^F = \begin{bmatrix} \mathbb{1}_b & 0 \\ 0 & -\mathbb{1}_f \end{bmatrix}$$

III • The involution $(-1)^F$ induces also a decomposition on the algebra of operators acting on $\mathcal{H}$. If:

$$K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

acts on $\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f$, then:

- K is *"bosonic"* or *"even"* iff $[(-1)^F, K] = 0$ iff $K = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$
- K is *"fermionic"* or *"odd"* iff $\{(-1)^F, K\} = 0$ iff $K = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$

IV • In particular, since Q is hermitian and anticommutes with $(-1)^F$, we have that Q is odd and:

$$Q = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix} \tag{1.18}$$

So, applying Q to a vector $\psi = \psi_b \oplus \psi_f \in \mathcal{H}$, we have:

$$Q\psi = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix} \begin{bmatrix} \psi_b \\ \psi_f \end{bmatrix} = \begin{bmatrix} A^\dagger \psi_f \\ A\psi_b \end{bmatrix}$$

and since this belongs to $\mathcal{H}_b \oplus \mathcal{H}_f$ we must have:

$$\begin{aligned} Q\lfloor \mathcal{H}_b = A : \mathcal{H}_b \longrightarrow \mathcal{H}_f \\ Q\lfloor \mathcal{H}_f = A^\dagger : \mathcal{H}_f \longrightarrow \mathcal{H}_b \end{aligned} \tag{1.19}$$

Note also that:

$$H = \begin{bmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix} \tag{1.20}$$

V • Now we turn to the fundamental property of this Susy model. Let ψ be an eigenvalue of H with positive energy $E > 0$:

$$H\psi = E\psi \qquad E > 0$$

Then, as $[H, Q] = 0$ we have:

$$H(Q\psi) = Q(H\psi) = E\,(Q\psi)$$

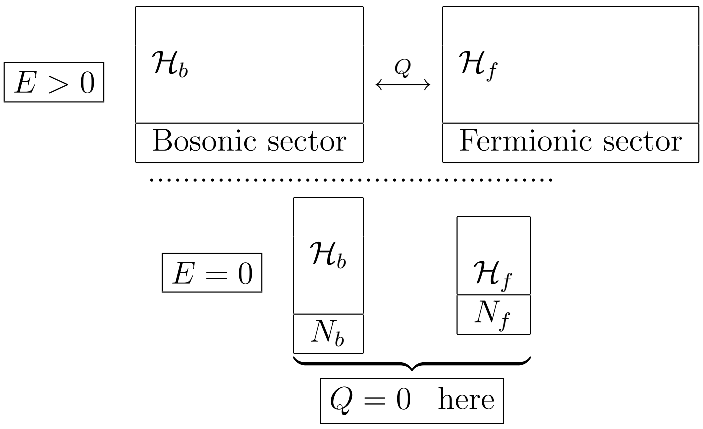
which means that $Q\psi$ is again an eigenvalue of H with the <u>same positive energy</u> $E > 0$. Note that if $E = 0$ we <u>can not</u> apply this reasoning, since $H\psi = 0$ implies that:

$$0 = \langle\psi|H|\psi\rangle = \langle\psi|Q^2|\psi\rangle = \langle\psi|Q^\dagger Q|\psi\rangle = \|Q\psi\|^2$$

and so $Q\psi = \mathbf{0}$ which is not an eigenvector.

As we have seen, if $\psi \in \mathcal{H}_b$ (resp., $\mathcal{H}_f$) then $Q\psi \in \mathcal{H}_f$ (resp., $\mathcal{H}_b$) (we call $Q\psi$ the <u>*"superpartner"*</u> of ψ), and so we conclude that "<u>*all eigenstates with energy $E > 0$ are paired"*</u>:

$$\boxed{\dim\ \ker[(H - E)\lfloor\mathcal{H}_b)] = \dim\ \ker[(H - E)\lfloor\mathcal{H}_f)] \qquad \forall E > 0} \tag{1.21}$$



Here we have put:

$$\begin{aligned} N_b &= \dim\ \ker(H\lfloor\mathcal{H}_b) \\ N_f &= \dim\ \ker(H\lfloor\mathcal{H}_f) \end{aligned} \tag{1.22}$$

If either N_b or N_f are nonzero, then there exists a state of zero energy (a ground sate) and supersymmetry is unbroken. So if we can compute N_b or N_f we can decide about Susy breaking. In general this is a difficult problem, and the only thing available is the difference $N_b - N_f$.

Thus we define the <u>*"Witten index"*</u> as:

$$\boxed{\Delta_W = N_b - N_f} \tag{1.23}$$

This has remarkable stability properties. In fact "small perturbations" of the system don't affect Δ_W, since the states of non-zero energy move always in Bose-Fermi pairs.

Since Q has the form (1.18), i.e., $Q = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix}$, with A an elliptic operator, then by (1.19), we have that:

$$\begin{aligned} \Delta_W &= N_b - N_f \\ &= \dim \ker A - \dim \ker A^\dagger \\ &= \operatorname{index}(A) \end{aligned} \tag{1.24}$$

1.3 A fundamental example: The Laplacian on forms

Assume that M is a compact oriented closed smooth n-dimensional Riemannian manifold, and let $\Omega^k(M)$ be the Hilbert space obtained by completion of the space of smooth k-forms, with respect to the usual inner product:

$$< \alpha, \beta >= \int_M \alpha \wedge *\beta$$

Now we construct an *"abstract Susy model"* $\{H, Q, (-1)^F\}$, on the Hilbert space:

$$\mathcal{H} \stackrel{\text{def}}{=} \oplus_{k=0}^n \Omega^k(M) \tag{1.25}$$

by defining:

- $H = \Delta = dd^* + d^*d$, the operator closure of the usual laplacian on smooth forms.
- $(-1)^F \lfloor \Omega^k(M) = (-1)^k \mathbb{1}$. Thus the bosonic-fermionic sectors of $\mathcal{H}$ are:

$$\begin{aligned} \mathcal{H}_b &= \bigoplus_{k \ \text{even}} \Omega^k(M) \\ \mathcal{H}_f &= \bigoplus_{k \ \text{odd}} \Omega^k(M) \end{aligned} \tag{1.26}$$

- $Q = d + d^*$, where $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the operator closure of the usual differential on forms, and $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ its adjoint (codifferential).

  So, with respect to the bosonic-fermionic grading of $\mathcal{H}$, $Q = \begin{bmatrix} 0 & d^* \\ d & 0 \end{bmatrix}$.

It's easy to see that conditions (1.15) hold, namelly:

$$Q^2 = (d + d^*)^2 = \Delta \qquad \{(-1)^F, Q\} = 0 \qquad ((-1)^F)^2 = \mathbb{1}$$

Thus in particular, property (1.21) takes, in this case, the following form:

$$\boxed{\sum_{\{k \ \text{even}\}} \dim \ker ((\Delta - E) \lfloor \Omega^k(M)) = \sum_{\{k \ \text{odd}\}} \dim \ker ((\Delta - E) \lfloor \Omega^k(M))} \tag{1.27}$$

or equivallently:

$$\boxed{\sum_{k=0}^{n} \dim \ker\left((\Delta - E)\lfloor\Omega^k(M)\right) = 0} \tag{1.28}$$

On the other hand, by Hodge theory, we know that:

$$\begin{aligned} \dim \ker(\Delta\lfloor\Omega^k(M)) &= \dim H^k(M) \\ &= b_k(M) \qquad \text{the } k\text{-Betti number of } M \end{aligned} \tag{1.29}$$

Recall also that the Euler characteristic of M is:

$$\chi(M) = \sum_{k=0}^{n} (-1)^k\, b_k(M)$$

and that Poincaré duality asserts that:

$$b_k(M) = b_{n-k}(M) \qquad \forall k = 0, \cdots, n$$

1.4 Witten's proof of Morse Inequalities

Recall that a smooth function $f : M \to \mathbf{R}$ is called a *"Morse function"* if it has finitely many critical points and each critical point is nondegenerate. Then we can prove that around each critical point $p \in M$ it's possible to choose local coordinates $\{x_i : i = 1, \cdots, n\}$, such that f has the local expression:

$$f(x_1, \cdots, x_n) = f(p) - \underbrace{x_1^2 - \cdots - x_{\mathbb{k}_p}^2}_{\mathbb{k}_p \text{ terms}} + \underbrace{x_{\mathbb{k}_p+1}^2 + \cdots + x_n^2}_{n-\mathbb{k}_p \text{ terms}} \tag{1.30}$$

where $\mathbb{k}_p$ is the index of the critical point p.

For a Morse function $f : M \to \mathbf{R}$, and for each integer $\mathbb{k} = 0, 1, \cdots, n$, let:

$$m_{\mathbb{k}}(f) \stackrel{\text{def}}{=} \text{number of critical points of } f \text{ of index } \mathbb{k}$$

Then we have the following theorem:

Morse Theorem...

Let M is a compact oriented closed smooth n-dimensional manifold, and $f : M \to \mathbf{R}$ a Morse function on M. Then we have:

(i). for each integer $\mathbb{k} = 0, 1, \cdots, n$, the "weak Morse inequalities":

$$\boxed{m_{\mathbb{k}}(f) \geq b_{\mathbb{k}}(M)}$$

(ii). for each integer $l = 0, 1, \cdots, n$, the *"strong Morse inequalities"*:

Figure 1: Examples of critical points

.

$$\boxed{\sum_{\mathbb{k}=0}^{l}(-1)^{l-\mathbb{k}}\, m_{\mathbb{k}}(f) \geq \sum_{\mathbb{k}=0}^{l}(-1)^{l-\mathbb{k}}\, b_{\mathbb{k}}(M)}$$

(iii). the *"Morse index Theorem"*:

$$\boxed{\sum_{\mathbb{k}=0}^{n}(-1)^{\mathbb{k}}\, m_{\mathbb{k}}(f) = \sum_{\mathbb{k}=0}^{n}(-1)^{\mathbb{k}}\, b_{\mathbb{k}}(M) = \chi(M)}$$

Our aim now it's to explain the main ideas of Witten's proof of this theorem.

- The first thing it's to "deform" the abstract Susy model of the previous section: $\{H = \Delta, Q = d + d^{+}, (-1)^{F}\}$, on the Hilbert space $\mathcal{H} = \oplus_{k=0}^{n}\Omega^{k}(M)$, by defining the t-dependent ($t \in \mathbf{R}$) abstract Susy model:

$$\{H_t, Q_t = d_t + d_t^{*}, (-1)^{F}\}$$

again on the same $\mathcal{H}$, where:

$$\begin{aligned} d_t &= e^{-tf}\, d\, e^{tf} \\ d_t^{*} &= e^{tf}\, d^{*}\, e^{-tf} \\ H_t &= d_t d_t^{*} + d_t^{*} d_t \end{aligned} \tag{1.31}$$

and the same involution.

As d_t is obtained from d, by conjugation with e^{tf}, the cohomology of $(\Omega(M), d)$ is the same as the cohomology of $(\Omega(M), d_t)$, and so $\ker\Delta \cong \ker H_t$, which implies for the Betti numbers that:

$$b_k(M) = \dim\, \ker(H_t\lfloor\Omega^{k}(M)$$

- Now recall that on $\mathcal{H} = \Omega(M)$ we have, for each 1-form $\alpha \in \Omega^{1}(M)$, a pair of fermionic creation-anihilation (0-order) operators, given respectivelly by exterior multiplication:

$$\varepsilon_{\alpha} : \omega \longrightarrow \varepsilon_{\alpha}(\omega) = \alpha \wedge \omega \qquad\qquad \omega \in \Omega(M)$$

and interior multiplication (or contraction with $g(\cdot,\alpha)$):

$$\iota_\alpha : \omega \longrightarrow \iota_\alpha(\omega) = (-1)^{nk+n+1} * (\alpha \wedge *\omega) \qquad\qquad \omega \in \Omega^k(M)$$

We can prove that these operators are adjoint of each other, and that:

$$\{\varepsilon_\alpha, \iota_\beta\} = g(\alpha,\beta)$$

Now we have $\forall \omega \in \mathcal{H}$:

$$\begin{aligned} d_t\omega &= e^{-tf} d\,(e^{tf}\omega) = d\omega + t\,df \wedge \omega = (d + t\,\varepsilon_{df})(\omega) \\ d_t^*\omega &= e^{tf} d^*\,(e^{-tf}\omega) = (d^* - t\,\iota_{df})(\omega) \end{aligned} \tag{1.32}$$

and so:

$$Q_t = d_t + d_t^* = d + d^* + t(\varepsilon_{df} - \iota_{df}) = Q + tB_f$$

where B_f is the endomorphism of the exterior bundle (i.e., a 0-order operator) given by $\varepsilon_{df} - \iota_{df}$.

Now it's easy to see that B_f^2 is given by multiplication by $\|df\|^2$, and that $\{Q, B_f\}$ is also a 0-order operator, say A_f. Putting all this together we have that:

$$\boxed{H_t = \Delta + t^2\|df\|^2 + tA_f} \tag{1.33}$$

In local orthonormal flat coordinates x_i we have:

$$\begin{aligned} H_t &= \Delta + t^2(\delta^{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}) + t\frac{\partial^2 f}{\partial x_i \partial x_j}[a_i^*, a_j] \\ &\stackrel{\text{def}}{=} \Delta + V_f \end{aligned} \tag{1.34}$$

where $a_i^* = \varepsilon_{dx_i}$ and $a_i = \iota_{dx_i}$.

- The above computation shows that H_t is a Schrödinger type operator with potential V_f, which for large t is dominated by the $t^2\|df\|^2$ term. When $t \to \infty$ this potential is enormeous, except at the critical points of f (where df vanishes), and so it looks like finitely many harmonic oscilators wells centered at each one of the critical points of f, and separated by large barriers.

Thus, assume that $p_1, \cdots, p_s$ are the critical points of f, each with index $p_a = \Bbbk_a$, $a = 1, \cdots, s$. Then locally, around each p_a, we can choose Morse coordinates $\{x_i\}$ where f has the local expression (1.30). By stipulating that $dx_1, \cdots, dx_n$ are orthonormal we obtain a metric in some neighborhood of p_a, and the local expression of H_t is, by (1.34):

$$\begin{aligned} H_t^{(a)} &= -\Delta + 4t^2(\sum_{i=1}^n x_i^2) - 2t\sum_{i=1}^{\Bbbk_a}[a_i^*, a_i] + 2t\sum_{i=\Bbbk_a+1}^n [a_i^*, a_i] \\ &= -\Delta + 4t^2\mathbf{x}^2 + 2t\sum_{i=1}^n \lambda_i\,[a_i^*, a_i] \end{aligned} \tag{1.35}$$

($\lambda_i = -1$, if $i = 1, \cdots, \mathbb{k}_a$, and $\lambda_i = +1$, if $i = \mathbb{k}_a + 1, \cdots, n$), and where Δ acts on k-forms as follows:

$$\Delta(h\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Since the critical points p_a of f are isolated, we can patch together such local metrics using a partition of unity, to obtain a metric on all M.

- Now we write (1.35), in the form:

$$H_t^{(a)} = \sum_{i=1}^{n} \Big(-\frac{\partial^2}{\partial x_i^2} + 4t^2\, x_i^2 + \lambda_i\, [a_i^*, a_i] \Big) \tag{1.36}$$

We see that each $-\frac{\partial^2}{\partial x_i^2} + 4t^2\, x_i^2$ is an harmonic oscillator of frequency $w = 2t$, which commutes with the 0-order operator $[a_i^*, a_i]$, and so can be simultaneously diagonalized.

Therefore, the eigenvalues of $\sum_{i=1}^n \Big(-\frac{\partial^2}{\partial x_i^2} + 4t^2\, x_i^2 \Big)$ are:

$$2t \sum_{i=1}^{n} (1 + 2\mu_i) \qquad\qquad \mu_1, \mu_2, \cdots, \mu_n = 0, 1, 2, \cdots$$

Each eigenform, is of type:

$$\psi\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \qquad\qquad 1 \leq i_1 < i_2 < \cdots < i_n \leq n$$

where ψ is an harmonic oscillator eigenfunction, and each of this eigenforms is also an eigenform of the 0-order operator $[a_i^*, a_i]$, with eigenvalue $+1$ (if $i \in \{i_1, i_2, \cdots, i_k\}$) or -1 (if $i \notin \{i_1, i_2, \cdots, i_k\}$). So the spectrum of $H_t^{(a)}$ is:

$$\text{spect}\, H_t^{(a)} = \{\, 2t \sum_{i=1}^{n} \, ((1{+}2\mu_i) + \lambda_i \epsilon_i) : \quad \mu_1, \mu_2, \cdots, \mu_n = 0, 1, 2, \cdots \quad \text{and} \quad \epsilon_i = \pm 1 \} \tag{1.37}$$

and when acting on k-forms the spectrum of $H_t^{(a)}$ is as above but with the additional restriction that exactly k of the ϵ_i are equal to $+1$.

- Now we want to make contact with Betti numbers, and so we will look for the multiplicity of the zero eigenvalue, of the restriction of $H_t^{(a)}$ to k-forms. By the above considerations we see that (since $\sum_{i=1}^n \lambda_i \epsilon_i \geq -n$) we will have zero eigenvalue iff $\mu_i = 0$, $\forall i$ (and the corresponding eigenspace is the 1-dimensional ground state of the oscillator) and $\epsilon_i = -\text{signal}\, \lambda_i$. Thus we must have exactly $\mathbb{k}_a = \text{index}\, p_a$ of the ϵ_i equal to $+1$, and so dim ker $(H_t^{(a)} \lfloor \Omega^k) = 1$ iff $k = \mathbb{k}_a = \text{index}\, p_a$, which implies that:

$$\text{dim ker}\, (\oplus_a \; H_t^{(a)} \lfloor \Omega^k M) = m_k$$

the <u>*number of critical points of index k*</u>.

But remember that we are working with an approximation! If this approximation was exact then we will have that $b_k = m_k$. Taking into account the approximation means that some of the zero eigen-k-forms may disappear in an exact computation, and so we will have the weak Morse inequalities:

$$b_k(M) \leq m_k(f)$$

Of course this deserves a more rigourous argument!... (see [1], for this and also for the proof of the strong Morse inequalities and Morse index theorem).

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2 Supergeometry and Supersymmetry

2.1 Field Theory. A quick review

- Actions. Euler equations

Fields φ in a field theory are sections of a bundle $E \to M$, with fiber F, over a smooth manifold M. We call F the *target manifold* and:

$$\Phi = \{\text{space of fields, or “histories”}\}$$

For example, In a scalar field theory F is a linear space, in a spinor field theory $E \to M$ is a spin bundle, while in Yang-Mills with gauge group G, E is an affine bundle whose sections are connections in some principal G-bundle over M. M will be flat Minkowski spacetime, Euclidean space, a Riemann surface, etc....

The *dynamics of fields* is determined by an *action functional* $S : \Phi \to \mathbf{C}$ which in general is “local”, i.e., is given by:

$$S[\varphi] = \int_M L[\varphi(x)] \qquad\qquad x \in M \quad \varphi \in F \tag{2.1}$$

where the *Lagrangian density* L is a function of $\varphi(x)$ and a finite number of its derivatives.

Example ... *Scalar field theory or nonlinear σ-model theory*

We take the trivial bundle $E = M \times F$, where M is D-dimensional flat Minkowski spacetime or Euclidean space $\mathbf{R}^D$, with cartesian coordinates x^a, $a = 1, \cdots, D$, and the target is a Riemannian manifold $\mathcal{M}$ with metric G and local coordinates φ^I. The space of fields Φ is the space of smooth maps $\varphi = (\varphi^1, \cdots, \varphi^d) : M \to F$ for which the action:

$$\begin{aligned} S[\varphi] &= -\frac{1}{2}\int_M d^D(\mathbf{x})\, \|d\varphi\|^2 \\ &= -\frac{1}{2}\int_M d^D\mathbf{x}\;\; G_{IJ}(\varphi)\, \partial^a\varphi^I \partial_a\varphi^J \qquad\qquad \partial_a = \frac{\partial}{\partial x^a}, \quad a = 1, \cdots, D \end{aligned} \tag{2.2}$$

is finite.

Of particular importance will be σ-models on complex manifolds $\mathcal{M}$ of real dimension $d = 2n$, with local real coordinates φ^I, $I = 1, \cdots, 2n$. Choose local complex coordinates w^i, $i = 1, \cdots, n$, such that:

$$\varphi^i = Re\, w^i = \frac{1}{2}(w^i + \overline{w}^i) \qquad\qquad \varphi^{i+n} = Im\, w^i = \frac{1}{2i}(w^i - \overline{w}^i)$$

If the metric G is Hermitian, then:

$$G = 2G_{i\bar{j}} dw^i\, d\overline{w}^j \qquad\qquad G_{j\bar{i}} = (G_{i\bar{j}})^*$$

and the action (2.2) is rewritten in the form:

$$S[w, \overline{w}] = -\int_M d^D\mathbf{x}\;\; G_{i\bar{j}}(w, \overline{w})\, \partial^a\overline{w}^j \partial_a w^i \tag{2.3}$$

Example ... *Bosonic string theory*

Here $M = \Sigma_h$ a Riemann surface of genus h with local smooth coordinates σ^a, $a = 1, 2$. The space of fields Φ is the space:

$$\Phi = Emb(\Sigma_h, \mathbf{R}^d) \times Met(\Sigma_h)$$

where $Emb(\Sigma_h, \mathbf{R}^d)$ is the space of smooth embeddings $\varphi : \Sigma_h \to \mathbf{R}^d$, of Σ_h into d-dimensional flat Minkowski spacetime $\mathbf{R}^d$, with cartesian coordinates X^i, $i = 1, \cdots, d$, $Met(\Sigma_h)$ is the space of Riemannin metrics g on Σ_h, and the action is the *Polyakov action*:

$$\begin{aligned} S[\varphi, g] &= \int_{\Sigma_h} d\mu_g \, \|d\varphi\|^2 \\ &= \int_{\Sigma_h} d^2\sigma \sqrt{g} \;\; g_{ab}(\sigma) \, \partial^a \varphi^i \partial^b \varphi_i \qquad \partial_a = \frac{\partial}{\partial \sigma^a}, \quad a = 1, 2 \end{aligned} \tag{2.4}$$

with $\varphi^i = X^i \circ \varphi$.

The action functional (2.1) determines the *dynamical field equations* or *Euler equations*:

$$\frac{\delta S[\varphi]}{\delta \varphi^i(x)} = 0 \tag{2.5}$$

where the *functional derivatives* are defined by:

$$\delta S[\varphi] = S[\varphi + \delta\varphi] - S[\varphi] = \int_M \delta\varphi^i(x) \frac{\delta S[\varphi]}{\delta \varphi^i(x)}$$

with $\delta\varphi^i(x) \in T_\varphi \Phi$ are arbitrary field variations.

Every solution of Euler equations is called a *dynamical field history*, and the set of all those solutions forms a subset $\Phi_o \subseteq \Phi$ called the *dynamical subspace* or *"mass shell surface"*:

$$\Phi_o = \{\varphi \in \Phi : \; \frac{\delta S[\varphi]}{\delta \varphi^i(x)} = 0\}$$

Example ... *Scalar field theory or nonlinear σ-model theory*

The Euler equations corresponding to the action (2.2) are:

$$\frac{\delta S[\varphi]}{\delta \varphi^i} = \partial^a \partial_a \varphi^i + \Gamma^i_{jk} \partial^a \varphi^j \partial_a \varphi^k = 0 \qquad i = 1, \cdots, d \tag{2.6}$$

where Γ^i_{jk} is the Levi-Civita connection of the metric G on $F = \mathbf{R}^d$. Solutions φ of (2.6) are called *harmonic maps* because they satisfy a generalized Laplace equation.

Example ... *Bosonic string theory*

The Euler equations corresponding to the action (2.4) are:

$$\begin{aligned} \frac{\delta S[\varphi]}{\delta g_{ab}} &\equiv T^{ab} = 0 \\ \frac{\delta S[\varphi]}{\delta \varphi^i} &= 0 \qquad i = 1, \cdots, d \end{aligned} \tag{2.7}$$

.

We can combine the above two models in a "string σ-model", by considering the following generalized harmonic map problem. We take a Riemannian manifold (M, G) (the target space), a sympletic form B on M (the B-field), and the action:

$$S[\varphi, g] = \int_\Sigma (d\mu_g \, \|d\varphi\|^2 + \varphi^* B) + \frac{1}{8\pi} \int_\Sigma \Psi \cdot s_g \tag{2.8}$$

where g is an arbitrary riemannian metric on a Riemann surface Σ (the worlsheet), s_g is the scalar curvature of g, and Ψ is a scalar field (the "dilaton") on Σ.

- Symmetries

In general one considers action functionals that are invariant under some symmetry group. More precisely we have an action of a (Lie) group G on the space of fields Φ:

$$(g, \varphi) \mapsto g \cdot \varphi$$

and we have that:

$$S[\varphi] = S[g \cdot \varphi] \qquad \forall g \in G$$

Traditionally we consider the infinitesimal (derived) action of the Lie algebra $\mathfrak{g}$ on Φ, given through the differential of the "orbital map" $\eta_\varphi : G \to \Phi$ (defined by $\eta_\varphi(g) = g \cdot \varphi$):

$$d\eta_\varphi : \mathfrak{g} \longrightarrow T_\varphi \Phi$$

with:

$$d\eta_\varphi(\xi) = \frac{d}{dt} \,|_{t=0} \; e^{t\xi} \cdot \varphi \equiv \delta_\xi \varphi \in T_\varphi \Phi$$

Example ... *Relativistic field theory*

In the case of a field theory in Minkowski space, the Poincaré group $\mathcal{P} = ISO(1,3)$ is assumed to act on the space of fields Φ, by means of *spacetime symmetries*, i.e., a symmetry of the base space M, represented in Φ by:

$$(g, \varphi) \mapsto g \cdot \varphi(\mathbf{x}) \equiv \varphi(g^{-1}\mathbf{x})$$

For example, translations in a flat spacetime can be written as:

$$\delta_\xi \varphi = -\xi^a \partial_a \varphi \in T_\varphi \Phi$$

When the "mass shell surface" $\Phi_o \subseteq \Phi$ is $\mathcal{P}$-invariant:

$$\frac{\delta S[\varphi]}{\delta \varphi^i(x)} = 0 \Longrightarrow \frac{\delta S[g \cdot \varphi]}{\delta \varphi^i(x)} = 0 \qquad \forall g \in \mathcal{P}$$

then we say that we have a *relativistic field theory.* This will be the case if for example the action functional is a scalar with respect to $\mathcal{P}$: $S[\varphi] = S[g \cdot \varphi]$.

In contrast to the above spacetime symmetries, we have the *internal symmetries* which act on $\varphi \in \Phi$ at each point of M, i.e., acts without spacetime derivatives. For example a $U(1)$-algebra acts on a complex field φ as:

$$\delta_\lambda \varphi = i\lambda\varphi$$

When the transformation prameters are constant over M, like the above ξ^a or λ, we say that the symmetry is *global* or *rigid.* When they are functions over M, $\lambda = \lambda(\mathbf{x})$, then the symmetry is called local, like for example, gauge transformations on Yang-Mills fields. Sometimes it is possible to promote a global symmetry to a local one. The prescription to do this is called *gauging the symmetry.*

The commutator of two infinitesimal symmetries is a symmetry and so they form a Lie algebra in general infinite dimensional. Sometimes an infinite dimensional symmetry algebra acts as a finite dimensional algebra on the dynamical fields $\varphi \in \Phi_o$. In this case we say that we have an *on shell representation* of that algebra.

Example ... *Nonlinear σ-model*

The symmetries of the action (2.2) are of two types: The spacetime ones are the isometries of flat Minkowski spacetime M, i.e., the Poincaré group $\mathcal{P}$. The internal symmetries are the isometries of the target (F, G). These are global symmetries generated by Killing vector fields of F:

$$(\delta_X \varphi)(\mathbf{x}) = (X^A K_A)(\varphi(\mathbf{x}))$$

where $K_A = k^i_A \partial_i$ is a basis for the Lie algebra of the isometry group of F.

Example ... *String theory*

The symmetries of the Polyakov action (2.4) are:

- translations in $\mathbf{R}^d$:

$$S[\varphi^i + c^i, g] = S[\varphi, g] \qquad \forall c^i \in \mathbf{R}^d$$

- the group of (orientation preserving) diffeomorphisms: $Diff^+(\Sigma_h)$:

$$S[f^*\varphi, f^*g] = S[\varphi, g] \qquad \forall f \in Diff^+(\Sigma_h)$$

- the group of conformal (pointwise) rescallings of the metric: $C^\infty(\Sigma_h, \mathbf{R})$:

$$S[\varphi, e^\lambda g] = S[\varphi, g] \qquad \forall \lambda \in C^\infty(\Sigma_h, \mathbf{R})$$

So if we quotient these symmetries, we see that the action functional is defined in the so called *moduli space* $\mathcal{M}$:

$$\mathcal{M} = \frac{Emb(\Sigma_h, \mathbf{R}^d) \times Met(\Sigma_h)}{\mathbf{R}^d \times Diff^+(\Sigma_h) \times C^\infty(\Sigma_h, \mathbf{R})}$$

2.2 SuperEuclidean Space

Consider the ($\mathbf{Z}_2$-graded) supercommutative, associative, with unit element $\mathbb{1}$, complex *"Grassmann algebra"* $\Lambda = \Lambda_L = \wedge \mathbf{C}^L$:

$$\Lambda = \Lambda_L = \Lambda_0 \oplus \Lambda_1$$

with a finite number (sufficiently large, eventually $L = \infty$,...) of generators $\{\mathbb{1}, \zeta_k : k = 1, 2, \cdots, L\}$, and with a normed topology (such that $\Lambda \cong \mathbf{C}^{2^L}$). We have for homogeneous elements:

$$\alpha\beta = (-1)^{|\alpha||\beta|}\,\beta\alpha \qquad \alpha, \beta \in \Lambda$$

where the notation $|\alpha|$ means *"grassmann parity"*, equal to 0 if $\alpha \in \Lambda_0$ and equal to 1 if $\alpha \in \Lambda_1$. In particular, elements in Λ_0 commute (Λ_0 is a commutative subalgebra of Λ) and elements in Λ_1 anticommute. Thus we call the elements in Λ_0, *"c-numbers"* and the elements in Λ_1 *"a-numbers"*, and we put:

$$\mathbf{C}_c = \Lambda_0 \qquad \mathbf{C}_a = \Lambda_1$$

Every element $\mathbf{z} \in \Lambda$ splits as:

$$\mathbf{z} = z_b + \mathbf{z}_s \in \mathbf{C} \oplus \Lambda_s$$

where $z_b \in \mathbf{C}$ is the *"body"* and $\mathbf{z}_s = \mathbf{z} - z_b \in \Lambda_s$ is the *"soul"* of $\mathbf{z}$ (its nilpotent part, because $\mathbf{z}_s^N = 0$ if $N > L$).

We define the *"SuperEuclidean space"* $\mathbf{C}^{m|n}$ of *"even dimension"* m and *"odd dimension"* n, by:

$$\mathbf{C}^{m|n} = (\mathbf{C}_c)^m \times (\mathbf{C}_a)^n \tag{2.9}$$

with a normed topology (so that $\mathbf{C}^{m|n} \cong \mathbf{C}^{(m+n)2^{L-1}}$), and denote an element of $\mathbf{C}^{m|n}$ by $(\mathbf{x}; \Theta)$, with:

$$\begin{array}{rcl} \mathbf{x} & = & (\mathbf{x}^i) = (\mathbf{x}^1, \cdots, \mathbf{x}^m) \qquad \mathbf{x}^i \in \mathbf{C}_a \quad i = 1, \cdots, m \\ \Theta & = & (\theta^\alpha) = (\theta^1, \cdots, \theta^n) \qquad \theta^\alpha \in \mathbf{C}_a \quad \alpha = 1, \cdots, n \end{array} \tag{2.10}$$

$(\mathbf{x}^1, \cdots, \mathbf{x}^m)$ are called *"bosonic coordinates"* and $(\theta^1, \cdots, \theta^n)$ *"fermionic coordinates"*. The *"body"* of $(\mathbf{x}; \Theta) \in \mathbf{R}^{m|n}$ is by definition $\mathbf{x}_b = (x_b^1, \cdots, x_b^m) \in \mathbf{C}^m$, and this defines the *"body projection"*:

$$b : \mathbf{C}^{m|n} \longrightarrow \mathbf{C}^m$$

which is a continuous open surjective map.

2.3 Reality Conditions

We define an involution * on Λ, which we call *"complex conjugation"*, as follows:

$$\begin{array}{rcll} \zeta_k^* & = & \zeta_k \quad k = 1, \cdots, L & \\ (\alpha \mathbf{z})^* & = & \overline{\alpha}\, \mathbf{z}^* & \forall \alpha \in \mathbf{C} \quad \forall \mathbf{z} \in \Lambda \\ (\mathbf{z} + \mathbf{w})^* & = & \mathbf{z}^* + \mathbf{w}^* & \forall \mathbf{z}, \mathbf{w} \in \Lambda \\ (\mathbf{z}\mathbf{w})^* & = & \mathbf{w}^* \mathbf{z}^* & \forall \mathbf{z}, \mathbf{w} \in \Lambda \end{array} \tag{2.11}$$

An element $\mathbf{z} \in \Lambda$ is called *"real"* if $\mathbf{z}^* = \mathbf{z}$, and *"imaginary"* if $\mathbf{z}^* = -\mathbf{z}$. The set of real elements in $\mathbf{C}_c$ (the real c-numbers), form a real commutative subalgebra in Λ, which is denoted by $\mathbf{R}_c$. The set of real elements in $\mathbf{C}_a$ (the real a-numbers) is denoted by $\mathbf{R}_a$. Note that the product of a real c-number and a real a-number is a real a-number, and finally the product of two real a-numbers is a bodiless imaginary c-number:

$$\mathbf{R}_c \cdot \mathbf{R}_c \subseteq \mathbf{R}_c \qquad \mathbf{R}_c \cdot \mathbf{R}_a \subseteq \mathbf{R}_a \qquad \mathbf{R}_a \cdot \mathbf{R}_a \subseteq i\mathbf{R}_a$$

The *"real SuperEuclidean space"* $\mathbf{R}^{m|n}$ of *"even dimension"* m and *"odd dimension"* n, is defined by:

$$\mathbf{R}^{m|n} = (\mathbf{R}_c)^m \times (\mathbf{R}_a)^n \tag{2.12}$$

2.4 Supersmooth functions

Given a smooth (C^∞) Λ-valued function f in an open set of $U \subseteq \mathbf{R}^m$:

$$f : U \subseteq \mathbf{R}^m \longrightarrow \Lambda$$

we define its *"Grassmann analytic continuation"*:

$$Zf : b^{-1}(U) \cap \mathbf{R}_c^m \longrightarrow \Lambda$$

by the following (finite) Taylor expansion:

$$\begin{array}{rcl} Zf(\mathbf{x}^1, \cdots, \mathbf{x}^m) & = & Zf(x_b^1 + \mathbf{x}_s^1, \cdots, x_b^m + \mathbf{x}_s^m) \\ & = & \displaystyle\sum_{j_1, \cdots, j_k} \frac{1}{j_1! \cdots j_k!} \frac{\partial^{j_1 + \cdots + j_k} f}{\partial x_b^{j_1} \cdots \partial x_b^{j_k}} (x_b^1, \cdots, x_b^m)\, (\mathbf{x}_s^1)^{j_1} \cdots (\mathbf{x}_s^m)^{j_k} \\ & = & \displaystyle\sum_J \frac{1}{J!} \frac{\partial^{|J|} f}{\partial x_b^J} (\mathbf{x}_b)\, \mathbf{x}_s^J \end{array} \tag{2.13}$$

where in the last line we have used multiindice notation: $J = (j_1, \cdots, j_k)$, $J! = j_1! \cdots j_k!$, $\mathbf{x}_b = (x_b^1, \cdots, x_b^m)$, $|J| = j_1 + \cdots + j_k$, and $\mathbf{x}_s^J = (\mathbf{x}_s^1)^{j_1} \cdots (\mathbf{x}_s^m)^{j_k}$.

Now we define a (H^∞), or *"supersmooth function"* Φ in (an open set of) $\mathbf{R}^{m|n}$ as a Λ-valued function of the form:

$$\begin{aligned}
\Phi(\mathbf{x};\Theta) &= \Phi(\mathbf{x}^1,\cdots,\mathbf{x}^m;\theta^1,\cdots,\theta^n) \\
&= \sum_\alpha \phi_\alpha(\mathbf{x})\,\Theta^\alpha \\
&\stackrel{\text{def}}{=} \sum_{\alpha=\{\alpha_1<\cdots<\alpha_k\}} Zf_\alpha(\mathbf{x})\,\Theta^\alpha \\
&= \sum_\alpha Zf_{\alpha_1,\cdots,\alpha_k}(\mathbf{x}^1,\cdots,\mathbf{x}^m)\,\theta^{\alpha_1}\cdots\theta^{\alpha_k}
\end{aligned} \tag{2.14}$$

where each *"component function"* $\phi_\alpha = Zf_\alpha$ is a Λ-valued function of the form (2.13), which depends only on the bosonic coordinates $\mathbf{x}\in(\mathbf{R}_c)^m \cong \mathbf{R}^{m|0}$. Note that the above expansion (2.14) contains only a finite number of terms.

We denote by $\mathcal{SF}(\mathbf{R}^{m|n})$ the algebra of supersmooth functions on $\mathbf{R}^{m|n}$. This is a $\mathbf{Z}_2$-graded supercommutative algebra: $\mathcal{SF} = \mathcal{SF}^+ \oplus \mathcal{SF}^-$, where $\mathcal{SF}^+$ are the $\mathbf{C}_c$-valued, or *"even supersmooth functions"*, and $\mathcal{SF}^-$ the $\mathbf{C}_a$-valued, or *"odd supersmooth functions"*.

Examples ...

(i). An even supersmooth function on $\mathbf{R}^{1|1}$ is of the form:

$$\Phi(\mathbf{t},\theta) = \phi(\mathbf{t}) + \psi(\mathbf{t})\theta \qquad \mathbf{t}\in\mathbf{R}_c \quad \theta\in\mathbf{R}_a$$

with $\phi:\mathbf{R}^{1|0}\cong\mathbf{R}_c\to\mathbf{R}_c$ and $\psi:\mathbf{R}^{1|0}\cong\mathbf{R}_c\to\mathbf{R}_a$ obtained by Z-extension: $\phi = Zf$ and $\psi = Zg$.

(ii). In $\mathbf{R}^{2|2}$, which we can think as the *superspace extension* of 2-dimensional Minkowski space-time $\mathbf{R}_{(1,1)}$, we consider the coordinates $(\mathbf{x}^1,\mathbf{x}^2;\theta^1,\theta^2)$. In supersymmetric field theories we must think of θ^1,θ^2 as coordinates with respect to a basis $\{\mathbf{Q}_1,\mathbf{Q}_2\}$ of the space $\mathcal{S}$ of (Majorana) $Spin(1)$-spinors, in such a way that the pair (θ^1,θ^2) describes a spinor of $\mathbf{R}_{(1,1)}$, with (real) a-number coefficients.

An *"even superfield"* is an even supersmooth function of the form:

$$\Phi(\mathbf{x}^1,\mathbf{x}^2;\theta^1,\theta^2) = \phi_o(\mathbf{x}^1,\mathbf{x}^2) + \phi_1(\mathbf{x}^1,\mathbf{x}^2)\,\theta^1 + \phi_2(\mathbf{x}^1,\mathbf{x}^2)\,\theta^2 + \phi_{12}(\mathbf{x}^1,\mathbf{x}^2)\,\theta^1\theta^2$$

where $\phi_o = Zf_o$ is an even function, called the *"bosonic component"* of Φ, $\phi_1 = Zf_1, \phi_2 = Zf_2$ are odd functions (of spinorial character) called the *"fermionic components"* of Φ, and $\phi_{12} = Zf_{12} = -\phi_{21}$, is even and must be viewed as a section of $\wedge^2 S$.

Consider two superfields $\Phi = \sum_\alpha \phi_\alpha(\mathbf{x})\,\Theta^\alpha \in \mathcal{SF}$, $\Psi = \sum_\alpha \psi_\beta(\mathbf{x})\,\Theta^\beta \in \mathcal{SF}$ and assume that $\phi_\alpha = Zf_\alpha, \psi_\beta = Zg_\beta$.

We define for each $i = 1,\cdots,m$:

$$\begin{aligned}
\frac{\partial\Phi}{\partial\mathbf{x}^i}(\mathbf{x};\Theta) &= \sum_\alpha \frac{\partial\phi_\alpha}{\partial\mathbf{x}^i}(\mathbf{x})\,\Theta^\alpha \\
&= \sum_\alpha \frac{\partial(Zf_\alpha)}{\partial\mathbf{x}^i}(\mathbf{x})\,\Theta^\alpha \\
&= \sum_\alpha Z(\frac{\partial f_\alpha}{\partial x_b^i})(\mathbf{x})\,\Theta^\alpha
\end{aligned} \tag{2.15}$$

where $\frac{\partial \phi_\alpha}{\partial \mathbf{x}^i}(\mathbf{x})$ is the Grassmann analytic continuation of $\frac{\partial f_\alpha}{\partial x_b^i}$ (x_b^i is the body of $\mathbf{x}^i$). Then $\frac{\partial}{\partial \mathbf{x}^i}$ is an <u>*"even derivation"*</u> on $\mathcal{SF}$:

$$\begin{aligned}
\frac{\partial(\Phi+\Psi)}{\partial \mathbf{x}^i} &= \frac{\partial \Phi}{\partial \mathbf{x}^i} + \frac{\partial \Psi}{\partial \mathbf{x}^i} \\
\frac{\partial(\lambda\Phi)}{\partial \mathbf{x}^i} &= \lambda \frac{\partial \Phi}{\partial \mathbf{x}^i} \qquad \forall \lambda \in \mathbf{R} \\
\frac{\partial(\Phi\Psi)}{\partial \mathbf{x}^i} &= \frac{\partial \Phi}{\partial \mathbf{x}^i}\Psi + \Phi \frac{\partial \Psi}{\partial \mathbf{x}^i}
\end{aligned} \tag{2.16}$$

Now, for each $\alpha = 1, \cdots, n$, we define $\frac{\partial}{\partial \theta^\alpha}$, by putting:

$$\frac{\partial}{\partial \theta^\alpha}\theta^\beta = \delta_\alpha^\beta$$

and extending this to all $\mathcal{SF}$ as an odd derivation, so that:

$$\begin{aligned}
\frac{\partial(\Phi+\Psi)}{\partial \theta^\alpha} &= \frac{\partial \Phi}{\partial \theta^\alpha} + \frac{\partial \Psi}{\partial \theta^\alpha} \\
\frac{\partial(\lambda\Phi)}{\partial \theta^\alpha} &= \lambda \frac{\partial \Phi}{\partial \theta^\alpha} \qquad \forall \lambda \in \mathbf{R} \\
\frac{\partial(\Phi\Psi)}{\partial \theta^\alpha} &= \frac{\partial \Phi}{\partial \theta^\alpha}\Psi + (-1)^{\Phi}\, \Phi \frac{\partial \Psi}{\partial \theta^\alpha}
\end{aligned} \tag{2.17}$$

for homogeneous Φ. We can prove that:

$$\begin{aligned}
[\frac{\partial}{\partial \mathbf{x}^i}, \frac{\partial}{\partial \mathbf{x}^j}] &= 0 \\
[\frac{\partial}{\partial \mathbf{x}^i}, \frac{\partial}{\partial \theta^\alpha}] &= 0 \\
\{\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \theta^\beta}\} &= 0 \qquad \alpha \neq \beta
\end{aligned} \tag{2.18}$$

Consider the graded vector space:

$$\mathbf{R}\{\frac{\partial}{\partial \mathbf{x}^i}\} \oplus \mathbf{R}\{\frac{\partial}{\partial \theta^\alpha}\} \tag{2.19}$$

and let us tensor it with the graded module $\mathcal{SF}^+ \oplus \mathcal{SF}^-$. Then we obtain the graded module of <u>*"supervector fields"*</u> on $\mathbf{R}^{m|n}$:

$$\mathfrak{X}(\mathbf{R}^{m|n}) = \mathfrak{X}^+(\mathbf{R}^{m|n}) \oplus \mathfrak{X}^-(\mathbf{R}^{m|n})$$

where:

$$\mathfrak{X}^+(\mathbf{R}^{m|n}) = \mathcal{SF}^+\{\frac{\partial}{\partial \mathbf{x}^i}\} \oplus \mathcal{SF}^-\{\frac{\partial}{\partial \theta^\alpha}\} \tag{2.20}$$

consists of the <u>*"even supervector fields"*</u>, and:

$$\mathfrak{X}^-(\mathbf{R}^{m|n}) = \mathcal{SF}^-\{\frac{\partial}{\partial \mathbf{x}^i}\} \oplus \mathcal{SF}^+\{\frac{\partial}{\partial \theta^\alpha}\} \tag{2.21}$$

consists of the *"odd supervector fields"*.

Example ...

In $\mathbf{R}^{1|1}$ the supervector field $\mathbf{D} = \frac{\partial}{\partial\theta} - \theta\,\frac{\partial}{\partial \mathbf{t}}$ is odd, and satisfy $\mathbf{D}^2 = -\frac{\partial}{\partial \mathbf{t}}$.

2.5 Supermanifolds

A (H^∞) *"Supermanifold $\mathcal{M}^{m|n}$, of dimension $(m|n)$"*, is an Hausdorff, paracompact topological space $\mathcal{M}$, locally modelled on $\mathbf{R}^{(m|n)}$, with supersmooth transition functions.

Note that every ordinary m-dimensional manifold M, can be extended to a (bosonic) supermanifold $\mathcal{M}^{m|0} = ZM^{m|0}$, by replacing each open set of M homeomorphic to an open set $U \subset \mathbf{R}^m$, by the open set $b^{-1}(U) \subset \mathbf{R}_c^m \cong \mathbf{R}^{m|0}$, and taking as transition functions between two such open sets the Z-expansion (2.13), of the transition functions of the corresponding open sets in $\mathbf{R}^m$.

2.6 Lie Superalgebras

A *"Lie Superalgebra"* is a $\mathbf{Z}_2$-graded (real or complex) vector space:

$$\mathfrak{G} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where:

(i). $\mathfrak{g}_0$ is a Lie algebra.

(ii). $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module, i.e., the carrier space of a representation of the Lie algebra $\mathfrak{g}_0$.

(iii). $\mathfrak{G}$ is endowed with a graded Lie bracket defined by the following conditions:

- This graded Lie bracket when restricted to $\mathfrak{g}_0$, is the same as the Lie bracket defined in the Lie algebra $\mathfrak{g}_0$. Thus, $\forall X, Y, Z \in \mathfrak{g}_0$:

$$[X, Y] = -[Y, X] \tag{2.22}$$

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \tag{2.23}$$

(i.e., $[X, \cdot] = ad_X$ is an even derivation on $\mathfrak{g}_0$.)

- For an element $X \in \mathfrak{g}_0$ and $\psi \in \mathfrak{g}_1$:

$$[X, \psi] \equiv -[\psi, X] = X \cdot \psi \in \mathfrak{g}_1 \tag{2.24}$$

is the element of $\mathfrak{g}_1$ given by the $\mathfrak{g}_0$-action on $\mathfrak{g}_1$. Thus, $\forall X, Y \in \mathfrak{g}_0, \forall \psi \in \mathfrak{g}_1$:

$$[[X, Y], \psi] = [X, Y] \cdot \psi = X \cdot (Y \cdot \psi) - Y \cdot (X \cdot \psi) = [X, [Y, \psi]] - [Y, [X, \psi]] \tag{2.25}$$

- The graded Lie bracket when restricted to $\mathfrak{g}_1$, is given by a bilinear symmetric mapping:

$$\{\cdot,\cdot\} : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$$

  that behaves like an anticommutator:

$$[\phi,\psi] \equiv \{\phi,\psi\} = \{\psi,\phi\} \qquad \forall\phi,\psi \in \mathfrak{g}_1 \tag{2.26}$$

  Moreover we must have the following Jacobi identities:

$$[X,\{\phi,\psi\}] = \{[X,\phi],\psi\} + \{\phi,[X,\psi]\} \tag{2.27}$$

$$\begin{aligned} [\{\phi,\psi\},\eta] &= \{\phi,\psi\}\cdot\eta \\ &= -\{\phi,\eta\}\cdot\psi - \{\psi,\eta\}\cdot\phi \\ &= [\psi,\{\phi,\eta\}] + [\phi,\{\psi,\eta\}] \end{aligned} \tag{2.28}$$

  $\forall\phi,\psi,\eta \in \mathfrak{g}_1, \forall X \in \mathfrak{g}_0$.

We can put (2.22),(2.24) and (2.26) in the short form

$$[A,B] = (-1)^{|A||B|+1}\,[B,A]$$

and the *Jacobi identities* (2.23), (2.25), (2.27) and (2.28) in the form:

$$(-1)^{|A||C|}\,[A,[B,C]] + (-1)^{|B||A|}\,[B,[C,A]] + (-1)^{|C||B|}\,[C,[A,B]] = 0$$

for homogeneous elements $A,B,C \in \mathfrak{G}$.

If $\{\mathbf{t}_a;\mathbf{T}_\alpha\}$, $a = 1,\cdots,m;\alpha = 1,\cdots,n$, is a linear basis for $\mathfrak{G} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then the structure constants of $\mathfrak{G}$ are:

- C^c_{ab} - the structure constants of the Lie algebra $\mathfrak{g}_0$.
- $C^\beta_{a\alpha}$ - where $C_a = (C^\beta_{a\alpha})$, $(a = 1,\cdots,m)$, are $n\times n$-matrices which satisfy the relations of the Lie algebra $\mathfrak{g}_0$ and generates one of its representations.
- $C^a_{\alpha\beta}$ are symmetric (in the indices α,β) structure constants, which verifies certain constraints imposed by Jacobi identities.

Example ...

Given a $\mathbf{Z}_2$ graded vector space $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ then $End(\mathcal{V})$ is a Lie superalgebra, defining the gradation $End(\mathcal{V}) = End^+(\mathcal{V}) \oplus End^-(\mathcal{V})$, by:

$$\begin{aligned} End^+(\mathcal{V}) &= Hom(\mathcal{V}^+,\mathcal{V}^+) \oplus Hom(\mathcal{V}^-,\mathcal{V}^-) \\ End^+(\mathcal{V}) &= Hom(\mathcal{V}^+,\mathcal{V}^-) \oplus Hom(\mathcal{V}^-,\mathcal{V}^+) \end{aligned} \tag{2.29}$$

and the graded bracket as the *"supercommutator"*:

$$[A,B] = AB - (-1)^{|A||B|}\,BA$$

for homogeneous elements of $End(\mathcal{V})$. In terms of a graded basis $\{\mathbf{e}_a; \mathbf{e}_\alpha\}$, $a = 1, \cdots, m; \alpha = 1, \cdots, n$, for $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$, $End^+(\mathcal{V})$ is represented by *"even supermatrices"*:

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{2.30}$$

while $End^-(\mathcal{V})$ is represented by *"odd supermatrices"*:

$$\begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \tag{2.31}$$

Example ...

The algebra $\mathcal{M}_{\mathbf{k}}(m;n)$ of ($\mathbf{k} = \mathbf{R}, \mathbf{C}, \mathbf{H}$) matrices of the form:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{2.32}$$

with even part given by even supermatrices of type (2.30), odd part given by odd supermatrices of type (2.31), and graded bracket the corresponding supercommutator.

For a supermatrix $\mathbf{M}$, of type (2.32) we define its *"supertrace"* $\mathrm{str}\,\mathbf{M}$, by:

$$\mathrm{str}\,\mathbf{M} = \mathrm{tr}\mathbf{A} - \mathrm{tr}\mathbf{D}$$

Then the subset of $\mathcal{M}_{\mathbf{k}}(m;n)$ of all matrices $\mathbf{M}$ with $\mathrm{str}\,\mathbf{M} = 0$ is a Lie subsuperalgebra, denoted by $\mathfrak{sl}_{\mathbf{k}}(m;n)$.

Example ... The *Orthosympletic algebras* $\mathfrak{osp}(2p;N)$

Consider a $\mathbf{Z}_2$ graded real vector space $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$, of dimension ($m = 2p; N$), and assume that we give a sympletic linear form Ω on $\mathcal{V}^+$, and a positive definite inner product G on $\mathcal{V}^-$. We can always choose a graded basis $\{\mathbf{e}_a; \mathbf{e}_\alpha\}$, $a = 1, \cdots, m = 2p; \alpha = 1, \cdots, N$ such that the matrices of Ω and G satisfy:

$$\Omega^2 = -\mathbb{1} \qquad \Omega^T = -\Omega \qquad \mathbf{G}^T = \mathbf{G}$$

Now we consider the supermatrix:

$$\mathbf{K} = \begin{bmatrix} \Omega & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}$$

and the subset of $\mathcal{M}_{\mathbf{R}}(2p;N)$ of all supermatrices $\mathbf{M}$ which verify:

$$\mathbf{M}^{sT}\mathbf{K} + (-1)^{\|M|}\,\mathbf{K}\mathbf{M} = \mathbf{0} \tag{2.33}$$

where the *"supertranspose"* $\mathbf{M}^{sT}$, of $\mathbf{M}$ is defined by:

$$\mathbf{M}^{sT} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{sT} = \begin{bmatrix} \mathbf{A}^T & (-1)^{\|M|}\,\mathbf{C}^T \\ (-1)^{|\mathbf{M}|+1}\,\mathbf{B}^T & \mathbf{D}^T \end{bmatrix}$$

Working the definitions, we see that if $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$ is even, then (2.33), says that:

$$\mathbf{A}^T\Omega + \Omega\mathbf{A} = \mathbf{0} \qquad \mathbf{D}^T\mathbf{G} = \mathbf{G}\mathbf{D}$$

i.e., $\mathbf{A}$ is sympletic and $\mathbf{D}$ is orthogonal. Thus the even part of $\mathfrak{osp}(2p; N)$ is the Lie algebra:

$$\mathfrak{g}_0 = \mathfrak{sp}(2p) \oplus \mathfrak{so}(N)$$

If $\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}$ is odd, then (2.33), says that:

$$\mathbf{B} = \Omega\mathbf{C}^T\mathbf{G}$$

Example ... *($N = 1, D = 2$)-Poincaré Lie Superalgebra $\mathcal{SP}(1; 2)$*

Let us begin with the construction of the *"($N = 1, D = 2$)-Poincaré Lie Superalgebra"*. Consider the Poincaré Lie algebra on ($D = 2$)-dimensional Minkowski spacetime $\mathbf{R}_{(1,1)}$, with metric η_{ab} of signature $(-, +)$, and cartesian coordinates $(x^0 = ct, x^1)$, $c = 1$:

$$\mathfrak{g}_0 = \mathfrak{so}(1, 1) \oslash \mathbf{R}^2$$

the semi-direct sum of the Lorentz Lie algebra $\mathfrak{so}(1, 1)$ with its 2-dimensional vectorial representation space $\mathbf{R}^2$. The Lie bracket in $\mathfrak{so}(1, 1) \oslash \mathbf{R}^2$ is given by:

$$[(\Lambda_1, \mathbf{x}_1), (\Lambda_2, \mathbf{x}_2)] = ([\Lambda_1, \Lambda_2], \Lambda_1\mathbf{x}_2 - \Lambda_2\mathbf{x}_1) \tag{2.34}$$

Now we choose for the odd part $\mathfrak{g}_1$ of our Lie superalgebra, the carrier space of the spinor representation of $\mathfrak{so}(1, 1)$. Recall how this is constructed.

We begin with the Clifford algebra $\mathcal{F} = Cl_{(1,1)}$ of Minkowski spacetime $\mathbf{R}_{(1,1)}$, i.e., the 2^2-dimensional real algebra generated by $\mathbb{1}$ and $\mathbf{R}^2$, subject to the relations:

$$\mathbf{xy} + \mathbf{yx} = -2\eta(\mathbf{x}, \mathbf{y})\mathbb{1} \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^2$$

$\mathcal{F} = Cl_{(1,1)}$ has a 2-dimensional real (Majorana) representation linearly generated by:

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \gamma^0 = \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \gamma^1 = i\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad \mathcal{W} = \gamma^0\gamma^1 = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$$

where γ^m, $m = 0, 1$, are the *"gamma matrices"* and $\mathcal{W} = \gamma^0\gamma^1$ is the *"Chiral (or Weyl) operator"*. Note that $\mathcal{W}^2 = \mathbb{1}$, and that $\{\mathcal{W}, \gamma^m\} = 0$.

Now we know that $\mathfrak{so}(1, 1) \cong \mathcal{F}_2 = \mathbf{R}\{\mathcal{W} = \gamma^0\gamma^1\}$, and so is 1-dimensional. Denote its generator by $\Lambda_{01} = \frac{1}{2}\gamma^0\gamma^1$. Since $[\mathcal{W}, \Lambda_{01}] = 0$, we see that this representation is Majorana-Weyl (or Chiral), and the *spinor space* $\mathfrak{g}_1 = \mathbf{R}^2$ splits into a direct sum:

$$\mathfrak{g}_1 = \mathbf{R}^2 = \mathbf{R}_l \oplus \mathbf{R}_r$$

corresponding to the eigenspaces of $\mathcal{W}$ associated to its eigenvalues ± 1, respectivelly. Elements of $\mathbf{R}_l$ are called *left spinors* and elements of $\mathbf{R}_r$ *right spinors.*

Thus, the Chiral representation of $\mathfrak{so}(1,1)$ reduces to the direct sum of two irreducible 1-dimensional representations $\Gamma = \Gamma_l \oplus \Gamma_r$, whose action on the spinor space is given by:

$$\begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} \xrightarrow{\Gamma} \frac{1}{2} \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix} = \begin{bmatrix} +\frac{1}{2}\theta^1 \\ -\frac{1}{2}\theta^2 \end{bmatrix}$$

where θ^α are coordinates with respect to a basis $\{\mathbf{Q}_\alpha\}_{\alpha=1,2}$ for $\mathfrak{g}_1$.

These $\mathbf{Q}_\alpha$ are called *"spinor charges"*, *"supercharges"*, or *"supersymmetric generators"*.

So for the moment we have defined the Lie superbracket on $\mathfrak{g}_0 = \mathfrak{so}(1,1) \oslash \mathbf{R}^2$ by (2.34), so that, if $\{\Lambda_{01}, \mathbf{P}_0, \mathbf{P}_1\}$ is a basis for $\mathfrak{g}_0$, then:

$$[\Lambda_{01}, \mathbf{P}_a] = 0 = [\mathbf{P}_a, \mathbf{P}_b]$$

Now we define, according to the previous discussion:

$$[\Lambda_{01}, \mathbf{Q}_1] = +\frac{1}{2}\mathbf{Q}_1 \qquad [\Lambda_{01}, \mathbf{Q}_2] = -\frac{1}{2}\mathbf{Q}_2 \qquad [\mathbf{P}_a, \mathbf{Q}_\alpha] = 0$$

Finally we must define the anticommutator $\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\}$ of two spinor charges. General considerations (based on the constraints imposed by Jacobi identities, together with the previous definitions) show that $\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\}$ must be a linear combination only of the linear momentum basis $\mathbf{P}_a$. So we must construct a vector with a *symmetric* combination of two spinors. Usually this is achieved by defining (if possible) a *"charge conjugation"* matrix $\mathbf{C}$, which in this particular case (where we are using the Majorana-Weyl representation) is given by:

$$\mathbf{C} = -\sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and verifies:

- $\mathbf{C}$ is antisymmetric.
- The matrices $\gamma^m \mathbf{C}$ are real and symmetric. In fact in this case $\gamma^0 \mathbf{C} = -\mathbb{1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\gamma^1 \mathbf{C} = \sigma_3 = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$.

Now we define:

$$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\} = -\sum_{m=0}^{1} (\gamma^m \mathbf{C})_{\alpha\beta} \mathbf{P}_m \qquad \alpha, \beta = 1, 2 \tag{2.35}$$

In this case we have:

$$\{\mathbf{Q}_1, \mathbf{Q}_1\} = \mathbf{P}_0 - \mathbf{P}_1 \qquad \{\mathbf{Q}_2, \mathbf{Q}_2\} = \mathbf{P}_0 + \mathbf{P}_1 \qquad \{\mathbf{Q}_1, \mathbf{Q}_2\} = 0$$

Our Lie superalgebra, *"the $(N=1, D=2)$-Poincaré superalgebra"*:

$$\mathcal{SP}(1;2) = (\mathfrak{so}(1,1) \oslash \mathbf{R}^2) \oplus \mathcal{S}$$

has real graded dimension $(3|2)$, with basis $\{\Lambda_{10}, \mathbf{P}_0, \mathbf{P}_1; \mathbf{Q}_1, \mathbf{Q}_2\}$.

Example ... *$(N=2, D=2)$-Poincaré Lie Superalgebra $\mathcal{SP}(2;2)$*

Here we simply add to the odd part of the Lie superalgebra $\mathcal{SP}(1;2)$, another spinor space $\mathcal{S}'$ with a corresponding basis $\{\mathbf{Q}'_1, \mathbf{Q}'_2\}$ of spinor charges, such that:

$$\begin{aligned} \{\mathbf{Q}'_\alpha, \mathbf{Q}'_\beta\} &= -\sum_{m=0}^{1} (\gamma^m \mathbf{C})_{\alpha\beta} \mathbf{P}_m \\ \{\mathbf{Q}_\alpha, \mathbf{Q}'_\beta\} &= 0 \qquad \alpha, \beta = 1, 2 \end{aligned} \tag{2.36}$$

Thus we put:

$$\mathcal{SP}(2;2) = (\mathfrak{so}(1,1) \oslash \mathbf{R}^2) \oplus (\mathcal{S} \oplus \mathcal{S}')$$

with real graded dimension $(3|4)$ and basis $\{\Lambda_{10}, \mathbf{P}_0, \mathbf{P}_1; \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}'_1, \mathbf{Q}'_2\}$.

2.7 Super Lie groups

Given a Lie superalgebra $\mathfrak{G} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with linear basis $\{\mathbf{t}_a; \mathbf{T}_\alpha\}$, $a = 1, \cdots, m; \alpha = 1, \cdots, n$, we consider the ordinary $2^{L-1}(m+n)$-dimensional complex Lie algebra, given by the even part of $\Lambda \otimes \mathfrak{G}$, i.e:

$$\mathfrak{G}_\Lambda \stackrel{\text{def}}{=} \mathbf{C}_c \otimes \mathfrak{g}_0 \oplus \mathbf{C}_a \otimes \mathfrak{g}_1$$

with Lie brackett:

$$\begin{aligned} &[\mathbf{x}^a \mathbf{t}_a + \theta^\alpha \mathbf{T}_\alpha, \mathbf{y}^b \mathbf{t}_b + \eta^\beta \mathbf{T}_\beta] = \\ &\quad \mathbf{x}^a \mathbf{y}^b [\mathbf{t}_a, \mathbf{t}_b] + \mathbf{x}_a \eta^\beta [\mathbf{t}_a, \mathbf{T}_\beta] + \theta^\alpha \mathbf{y}^b [\mathbf{T}_\alpha, \mathbf{t}_b] - \theta^\alpha \eta^\beta \{\mathbf{T}_\alpha, \mathbf{T}_\beta\} = \\ &\quad \mathbf{x}^a \mathbf{y}^b [\mathbf{t}_a, \mathbf{t}_b] + (\mathbf{x}_a \eta^\beta - \mathbf{y}^a \theta^\beta) [\mathbf{t}_a, \mathbf{T}_\beta] - \theta^\alpha \eta^\beta \{\mathbf{T}_\alpha, \mathbf{T}_\beta\} \end{aligned} \tag{2.37}$$

We call $\mathfrak{G}_\Lambda$ the *"Grassmann shell"* of the Lie superalgebra $\mathfrak{G}$. The associated (connected and simply connected) Lie group:

$$\mathbf{G} = \exp \mathfrak{G}_\Lambda$$

has a natural supermanifold structure and a group structure, obtained via Campbell-Haussdorff formula:

$$e^a e^b = e^{(a+b+\frac{1}{2}[a,b]+\frac{1}{12}[a,[a,b]]+\frac{1}{12}[b,[b,a]]+\cdots)} \tag{2.38}$$

which we call the *"Super Lie group"* associated with $\mathfrak{G}$. Elements of $\mathbf{G}$ take the form:

$$\exp(\mathbf{x}^a \mathbf{t}_a + \theta^\alpha \mathbf{T}_\alpha) \qquad \mathbf{x}^a \in \mathbf{C}_c, \theta^\alpha \in \mathbf{C}_a$$

Example ... *Super-Poincaré group* $\mathbf{SP}(1;2)$

The Grassmann shell of the $(N=1, D=2)$-Poincaré superalgebra $\mathcal{SP}(1;2) = (\mathfrak{so}(1,1) \oslash \mathbf{R}^2) \oplus \mathcal{S}$, with real graded dimension $(3|2)$, and basis $\{\Lambda_{10}, \mathbf{P}_0, \mathbf{P}_1; \mathbf{Q}_1, \mathbf{Q}_2\}$, has the form:

$$\mathcal{SP}(1;2)_\Lambda = \{\lambda^{01}\Lambda_{01} + \mathbf{x}^0\mathbf{P}_0 + \mathbf{x}^1\mathbf{P}_1 + \theta^1\mathbf{Q}_1 + \theta^2\mathbf{Q}_2 : \quad \lambda^{10}, \mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2 \in (\mathbf{R}_c)^3 \times (\mathbf{R}_a)^2\}$$

endowed with the Lie brackett (2.37).

Note that $\mathcal{SP}(1;2)_\Lambda$ is the semi-direct sum of two subalgebras:

$$\mathfrak{so}(1,1)_\Lambda \overset{\text{def}}{=} \{\lambda^{01}\Lambda_{01} : \lambda^{01} \in \mathbf{R}_c\}$$

and the <u>*"supersymmetric algebra"*</u>:

$$\mathfrak{m} \overset{\text{def}}{=} \{\mathbf{x}^0\mathbf{P}_0 + \mathbf{x}^1\mathbf{P}_1 + \theta^1\mathbf{Q}_1 + \theta^2\mathbf{Q}_2 : \quad \mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2 \in (\mathbf{R}_c)^2 \times (\mathbf{R}_a)^2\}$$

i.e.:

$$[\mathfrak{so}(1,1)_\Lambda, \mathfrak{so}(1,1)_\Lambda] \subseteq \mathfrak{so}(1,1)_\Lambda \qquad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m} \qquad [\mathfrak{so}(1,1)_\Lambda, \mathfrak{m}] \subseteq \mathfrak{m} \tag{2.39}$$

By definition, the elements of the Super-Poincaré group:

$$\mathbf{SP}(1;2) \overset{\text{def}}{=} \exp \mathcal{SP}(1;2)_\Lambda$$

are of the form:

$$\mathbf{g}(\lambda^{01}, \mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2) = \exp(\lambda^{01}\Lambda_{01} + \mathbf{x}^0\mathbf{P}_0 + \mathbf{x}^1\mathbf{P}_1 + \theta^1\mathbf{Q}_1 + \theta^2\mathbf{Q}_2) \tag{2.40}$$

with $(\lambda^{10}, \mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2) \in (\mathbf{R}_c)^3 \times (\mathbf{R}_a)^2 = \mathbf{R}^{3|2}$.

2.8 Rigid Superspace

We first recall some geometrical properties of homogeneous spaces. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and H a closed subgroup with Lie algebra $\mathfrak{h}$. H acts on the right on G by right translations, and as we know, G has a structure of H-principal fiber bundle over the homogeneous space of H-left cosets G/H:

$$\begin{array}{c} G \\ {\scriptstyle \pi}\downarrow \\ G/H \end{array}$$

G acts on itself by left translations $L_g : k \to gk$, and this induces a left action on G/H: $l_g : kH \to (gk)H$, since $\pi \circ L_g = l_g \circ \pi$:

$$\begin{array}{ccc} G & \xrightarrow{L_g} & G \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/H & \xrightarrow{l_g} & G/H \end{array}$$

Let $\exp : \mathfrak{g} \to G$ the exponential map of G, and define a map $\mathfrak{g} \to \mathfrak{X}(G/H)$ by:

$$\widetilde{X}_{\pi(g)} \overset{\text{def}}{=} \frac{d}{dt}\Big|_{t=0} \pi(\exp(tX)g) \qquad X \in \mathfrak{g},\ g \in G$$

so that $\widetilde{X} \in \mathfrak{X}(G/H)$ is the infinitesimal generator of the G-left action on the homogeneous space G/H. Consider also the right-invariant vector field $\widehat{X} \in \mathfrak{X}(G)$, determined by $X \in \mathfrak{g}$:

$$\widehat{X}_g \stackrel{\text{def}}{=} (R_g)_*(X)$$

Then, for all $g \in G$:

$$\pi_*\widehat{X}_g = \pi_*(R_g)_*(X) = \frac{d}{dt}\,|_{t=0}\ \pi \circ R_g \circ \exp(tX) = \widetilde{X}_{\pi(g)}$$

and so $\pi_*\widehat{X} = \widetilde{X} \circ \pi$, which means that the *right-invariant vector field* $\widehat{X}$ on G is π-related to the field $\widetilde{X}$ on G/H, determined by the *left action* of G on G/H:

$$\begin{array}{ccc} TG & \xrightarrow{\pi_*} & T(G/H) \\ {\scriptstyle \widehat{X}}\uparrow & & \uparrow{\scriptstyle \widetilde{X}} \\ G & \xrightarrow{\pi} & G/H \end{array}$$

Moreover the map $\widehat{X} \mapsto \widetilde{X}$ is a Lie algebra homomorphism from the Lie algebra of right-invariant vector fields on G, into $\mathfrak{X}(G/H)$.

On the other hand, if $Ad : G \to GL(\mathfrak{g})$ is the adjoint representation of G on its Lie algebra, then:

$$l_g\widetilde{X} = \widetilde{Ad_g X} \circ l_g$$

Assume now that $\mathfrak{m}$ is a direct sum complement to $\mathfrak{h}$ in $\mathfrak{g}$. With respect to an appropriate basis for $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the restriction to H of the adjoint representation $Ad : G \to GL(\mathfrak{g})$ takes the form:

$$Ad_h = \begin{bmatrix} A & B \\ O & C \end{bmatrix} \qquad h \in H$$

since H is a subgroup of G. The submatrix B will be O, $\forall h \in H$, iff the adjoint action of H on $\mathfrak{g}$, which is already reducible to an action on the subspace $\mathfrak{h}$ of $\mathfrak{g}$, is also reducible to an action on $\mathfrak{m}$; thus $B = O$, $\forall h \in H$ iff $Ad\lfloor H$ is reducible to the direct sum of representations of H on $\mathfrak{h}$ and $\mathfrak{m}$.

A homogeneous space G/H is called <u>*"reducible"*</u>, if there exists a vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (called a <u>*" reductive decomposition"*</u>), such that:

$$Ad_H(\mathfrak{m}) \subseteq \mathfrak{m}$$

If H is connected, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition iff:

$$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$$

Given a reducible homogeneous space G/H, with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then the $\mathfrak{h}$-component (with respect to the reductive decomposition) of the canonical Maurer-Cartan 1-form Θ on G, defines a connection on the H-principal fiber bundle $G(G/H, H)$, which is invariant by the left translations L_g on G. The corresponding horizontal subspace is $\mathfrak{m}$, under the identification $\mathfrak{g} \cong T_eG$, and the curvature form Ω of this canonical invariant connection is:

$$\Omega(X,Y) = -\frac{1}{2}[X,Y]_{\mathfrak{h}}$$

where $[X,Y]_{\mathfrak{h}}$ means $\mathfrak{h}$-component, and X, Y are arbitrary left invariant vector fields on G, belonging to $\mathfrak{m}$.

Now we apply these results to our supersymmetric situation, starting with the reductive decomposition (2.39) of the Grassmann shell $\mathfrak{g} = \mathcal{SP}(1;2)_\Lambda$.

Example ... *The rigid superspace* $\mathcal{S}^{2|2}$

By definition the *rigid superspace of graded dimension* (2|2) is the homogeneous space:

$$\mathcal{S}^{2|2} \stackrel{\text{def}}{=} \frac{\mathbf{SP}(1;2)}{\mathbf{H}} = \frac{\exp \mathcal{SP}(1;2)_\Lambda}{\exp(\mathfrak{so}(1,1)_\Lambda)}$$

where $\mathbf{H} = \exp(\mathfrak{so}(1,1)_\Lambda)$. Note that (locally) we can write every element $g \in \mathbf{SP}(1,1)$ in the form:

$$\mathbf{g} = \mathbf{g}(\mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2; \lambda^{01}) \stackrel{\text{def}}{=} \exp(\mathbf{x}^0\mathbf{P}_0 + \mathbf{x}^1\mathbf{P}_1 + \theta^1\mathbf{Q}_1 + \theta^2\mathbf{Q}_2)\exp(\lambda^{01}\Lambda_{01})) \tag{2.41}$$

with $(\mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2) \in (\mathbf{R}_c)^2 \times (\mathbf{R}_a)^2 = \mathbf{R}^{2|2}$ and $\lambda^{10} \in \mathbf{R}_c$. So, the homogeneous space $\mathcal{S}^{2|2}$ can be parametrized by local coordinates $(z^M) = (\mathbf{x}^1, \mathbf{x}^2; \theta^1, \theta^2) \in \mathbf{R}^{2|2}$, using the exponential chart (2.41). It is to be considered as a generalization of Minkowski space $\mathbf{R}_{(1,1)}$ and it is expected to have a richer structure, since now the supersymmetric algebra $\mathfrak{m}$ is not abelian.

$\mathcal{S}^{2|2}$ is a reductive homogeneous space, with reductive decomposition (see (2.39)):

$$\mathcal{SP}(1;2)_\Lambda = \mathfrak{so}(1,1)_\Lambda \oplus \mathfrak{m}$$

where, as before, $\mathfrak{m}$ is the supersymmetric algebra with generators $\{\mathbf{P}_1, \mathbf{P}_2; \mathbf{Q}_1, \mathbf{Q}_2\}$.

In fact $\mathcal{S}$ is a very particular reductive homogeneous space, since in this case $\mathfrak{m}$ is a (graded) Lie algebra (recall that $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}$). So exponentiation of $\mathfrak{m}$ give us a subgroup:

$$\mathcal{M} \stackrel{\text{def}}{=} \exp(\mathfrak{m})$$

of $\mathbf{SP}(1;2)$. Let us see their left action on $\mathbf{SP}(1;2)$, using BACH-formula (2.38), (2.37) and the supercommutation relations in $\mathcal{SP}(1;2)$:

$$\begin{aligned}
&\mathbf{g}(\mathbf{a}^0, \mathbf{a}^1; \eta^1, \eta^2; \mathbf{0})\,\mathbf{g}(\mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2; \lambda^{01}) \\
&\quad = e^{(\mathbf{a}^0\mathbf{P}_0 + \mathbf{a}^1\mathbf{P}_1 + \eta^1\mathbf{Q}_1 + \eta^2\mathbf{Q}_2)}\, e^{(\mathbf{x}^0\mathbf{P}_0 + \mathbf{x}^1\mathbf{P}_1 + \theta^1\mathbf{Q}_1 + \theta^2\mathbf{Q}_2)}\, e^{(\lambda^{01}\Lambda_{01})} \\
&\quad = e^{\left((\mathbf{a}^0 + \mathbf{x}^0 - \eta^1\theta^1 - \eta^2\theta^2)\mathbf{P}_0 + (\mathbf{a}^1 + \mathbf{x}^1 + \eta^1\theta^1 - \eta^2\theta^2)\mathbf{P}_1 + (\eta^1 + \theta^1)\mathbf{Q}_1 + (\eta^2 + \theta^2)\mathbf{Q}_2\right)}\, e^{(\lambda^{01}\Lambda_{01})} \\
= \;&\mathbf{g}(\mathbf{a}^0 + \mathbf{x}^0 - \eta^1\theta^1 - \eta^2\theta^2, \mathbf{a}^1 + \mathbf{x}^1 + \eta^1\theta^1 - \eta^2\theta^2; \eta^1 + \theta^1, \eta^2 + \theta^2)
\end{aligned}$$

So the induced $\mathcal{M}$-left action $l_{\mathbf{g}} = l_{\mathbf{g}(\mathbf{a}^0, \mathbf{a}^1; \eta^1, \eta^2; \mathbf{0})}$ on the superspace $\mathcal{S}^{2|2}$, is given in local coordinates, defined by the exponential chart (2.41), by the so called *"rigid supersymmetric translations"*:

$$z^A = (\mathbf{x}^0, \mathbf{x}^1; \theta^1, \theta^2) \mapsto z'^A = (\mathbf{x}^0 + \mathbf{a}^0 - \eta^1\theta^1 - \eta^2\theta^2, \mathbf{x}^1 + \mathbf{a}^1 + \eta^1\theta^1 - \eta^2\theta^2; \theta^1 + \eta^1, \theta^2 + \eta^2) \tag{2.42}$$

Note an important point: if we decompose each even coordinate, for example $\mathbf{x}^0$, on body and soul: $\mathbf{x}^0 = x^0_b + \mathbf{x}^0_s$, we see that the above supersymmetric translations, with $\mathbf{a}^0 = \mathbf{0}$), change the soul leaving the body invariant, i.e.::

$$x^0_b \to x^0_b \qquad \mathbf{x}^0_s \to \mathbf{x}^0_s - \eta^1\theta^1 - \eta^2\theta^2$$

so, even if $\mathbf{x}^0$ were soulles before a susy transformation, it acquires some soul afterwords!

The susy transformation (2.42) can be interpreted as infinitesimal coordinate transformations $z'^A = z^A + X^A$, generated by the super vector field:

$$\mathbf{X} = (\mathbf{a}^0 - \eta^1\theta^1 - \eta^2\theta^2)\frac{\partial}{\partial x^0} + (\mathbf{a}^1 + \eta^1\theta^1 - \eta^2\theta^2)\frac{\partial}{\partial x^1} + \eta^1\frac{\partial}{\partial \theta^1} + \eta^2\frac{\partial}{\partial \theta^2}$$

where:

$$\frac{\partial}{\partial z^A} = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta_2})$$

represents the coordinate basis on super tangent space $T_z\mathcal{S}^{2|2}$.

To determine the basis of $T_z\mathcal{S}^{2|2}$ which is induced by the (left) action of the group element $\mathbf{g}(\mathbf{a}^0, \mathbf{a}^1; \eta^1, \eta^2; \mathbf{0}) \in \mathcal{M}$, we rewrite the above super vector field $\mathbf{X}$ in the form:

$$\widetilde{\mathbf{X}} = \mathbf{a}^0\widetilde{\mathbf{P}}_0 + \mathbf{a}^1\widetilde{\mathbf{P}}_1 + \eta^1\widetilde{\mathbf{Q}}_1 + \eta^2\widetilde{\mathbf{Q}}_2$$

with:

$$\begin{aligned} \widetilde{\mathbf{P}}_0 &= \frac{\partial}{\partial x^0} \qquad \widetilde{\mathbf{P}}_1 = \frac{\partial}{\partial x^1} \\ \widetilde{\mathbf{Q}}_1 &= \frac{\partial}{\partial \theta^1} - \theta^1(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}) \\ \widetilde{\mathbf{Q}}_2 &= \frac{\partial}{\partial \theta^2} - \theta^2(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}) \end{aligned} \tag{2.43}$$

We can compute that the Lie bracket between the tangent vector fields $\{\widetilde{\mathbf{P}}_0, \widetilde{\mathbf{P}}_1, \widetilde{\mathbf{Q}}_1, \widetilde{\mathbf{Q}}_2\}$ vanish except the (graded) brackets:

$$\{\widetilde{\mathbf{Q}}_1, \widetilde{\mathbf{Q}}_1\} = \widetilde{\mathbf{P}}_0 - \widetilde{\mathbf{P}}_1 \qquad \{\widetilde{\mathbf{Q}}_2, \widetilde{\mathbf{Q}}_2\} = \widetilde{\mathbf{P}}_0 + \widetilde{\mathbf{P}}_1$$

So, while the $\{\frac{\partial}{\partial z^A}\}$ forms an holonomic frame for $T_z\mathcal{S}^{2|2}$, $\{\widetilde{\mathbf{P}}_0, \widetilde{\mathbf{P}}_1, \widetilde{\mathbf{Q}}_1, \widetilde{\mathbf{Q}}_2\}$ defines an anholonomic frame characterized by the structure constants given by the above brackets.

We call the tangent vector fields $\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{Q}_{\mathbb{1}}, \mathbf{Q}_2\}$ (we omit the hats), given by (2.43), the *supersymmetric generators* on superspace.

2.9 Covariant Derivatives

For simplicity, we continue to analyse the case of superspace $\mathcal{S}^{2|2}$, parametrized by local coordinates $(\mathbf{x}^i; \theta^\alpha) \in \mathbf{R}^{2|2}$, given by the exponential chart (2.41):

Since the rigid supersymmetric transformations are induced by left action on the group $\mathbf{SP}(1;2)$, the natural way to obtain a theory on $\mathcal{S}^{2|2}$ which is invariant under these transformations is to rely on the fact that left and right translations on a group commute. So one must express all geometric quantities on $\mathcal{S}^{2|2}$, with respect to the basis $\{\mathbf{D}_A\} = \{\partial_0, \partial_1, \mathbf{D}_{\mathbb{1}}, \mathbf{D}_2\}$ of $T_z\mathcal{S}^{2|2}$ (or its dual) which is induced by right action of $\mathcal{M}$ on $\mathbf{SP}(1;2)$. Using the same reasoning as before (BACH-formula (2.38), (2.37) and the supercommutation relations in $\mathcal{SP}(1;2)$), we deduce that this basis is given by:

$$\begin{aligned} \partial_0 &= \frac{\partial}{\partial x^0} \qquad\qquad \partial_1 = \frac{\partial}{\partial x^1} \\ \mathbf{D}_1 &= \frac{\partial}{\partial \theta^1} + \theta^1 (\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}) \\ \mathbf{D}_2 &= \frac{\partial}{\partial \theta^2} + \theta^2 (\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}) \end{aligned} \tag{2.44}$$

We can compute that the Lie bracket between the tangent vector fields $\mathbf{D}_A$ vanish except the (graded) bracket:

$$\{\mathbf{D}_1, \mathbf{D}_1\} = -\mathbf{P}_0 + \mathbf{P}_1 \qquad\qquad \{\mathbf{D}_2, \mathbf{D}_2\} = -\mathbf{P}_0 - \mathbf{P}_1$$

Moreover:

$$[\mathbf{Q}_\alpha, \mathbf{D}_A] = \mathbf{0}$$

which means that the frame $\{\mathbf{D}_A\}$ is left-invariant, i.e., invariant under rigid supersymmetric transformations on superspace. We call the left-invariant tangent vector fields $\mathbf{D}_A$ the:

"supersymmetric covariant derivatives"

on superspace. In fact, they can be considered as covariant derivatives with respect to the canonical connection on the reductive homogeneous space $\mathcal{S}^{2|2}$.

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3 APPENDIX. Clifford Algebras and Spin Groups

3.1 Clifford Algebras

Motivation. Clifford maps

"Dirac problem": Consider the Minkowski quadratic form in $\mathbf{R}^4$:

$$q(\mathbf{x}) = -t^2 + x^2 + y^2 + z^2 \qquad\qquad \mathbf{x} = (t, x, y, z) \in \mathbf{R}^4$$

and try to find a "linear function":

$$\varphi(\mathbf{x}) = \alpha t + \beta x + \gamma y + \delta z$$

such that $(\varphi(\mathbf{x}))^2 = -q(\mathbf{x})$, $\forall \mathbf{x} \in \mathbf{R}^4$, ie.:

$$(\alpha t + \beta x + \gamma y + \delta z)^2 = t^2 - x^2 - y^2 - z^2$$

A computation shows that:

$$\begin{aligned} &\alpha^2 = -\beta^2 = -\gamma^2 = -\delta^2 = \mathbb{1} \\ &\alpha\beta + \beta\alpha = \alpha\gamma + \gamma\alpha = \cdots = \mathbf{0} \end{aligned} \tag{3.1}$$

and so if there exists a solution, the coefficients of φ must belong to a noncommutative algebra. In fact, up to isomorphism, there exists only one solution which can be obtained with complex (4×4)-matrices $\alpha, \beta, \gamma, \delta$ - the Dirac matrices.

Let us generalize the above setup. Let $\mathbb{k}$ denote $\mathbf{R}$ or $\mathbf{C}$, and consider again the following:

"Dirac problem": Let $(\mathcal{V}, q)$ a $\mathbb{k}$-vector space with a non-degenerate quadratic form q, and let β the associated symmetric bilinear form. Try to find a linear map:

$$\varphi : \mathcal{V} \to \mathcal{A}$$

where $\mathcal{A}$ is an associative $\mathbf{k}$-algebra (with unit $\mathbb{1} = \mathbb{1}_{\mathcal{A}}$), such that:

$$(\varphi(\mathbf{x}))^2 = -q(\mathbf{x})\, \mathbb{1} \qquad \forall \mathbf{x} \in \mathcal{V} \tag{3.2}$$

or equivalently, such that:

$$\varphi(\mathbf{x})\varphi(\mathbf{y}) + \varphi(\mathbf{y})\varphi(\mathbf{x}) = -2\beta(\mathbf{x}, \mathbf{y})\, \mathbb{1} \qquad \forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \tag{3.3}$$

We call such a linear map $\varphi : \mathcal{V} \to \mathcal{A}$, a *"Clifford map"* from $(\mathcal{V}, q)$ to the algebra $\mathcal{A}$.

Example ... $(\mathcal{V}, q) = (\mathbf{R}, q(x) = x^2)$.

Then if $\mathcal{A} = \mathbf{C}$, considered as a real algebra, the real linear map $\varphi : \mathbf{R} \to \mathbf{C}$ defined by $\varphi(x) = ix$ is a Clifford map.

Example ... $(\mathcal{V}, q) = (\mathbf{R}^3, q(\mathbf{x}) = -(x^2 + y^2 + z^2))$.

Then if $\mathcal{A} = \mathbf{C}(2)$ is the algebra of complex (2×2)-matrices, considered as a real algebra, the real linear map $\varphi : \mathbf{R}^3 \to \mathbf{C}(2)$ defined by:

$$\varphi(\mathbf{x}) = \varphi(x, y, z) = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix}$$

is a Clifford map.

Clifford Algebras

Definition 1 ... *Let* $(\mathcal{V}, q)$ *a* $\mathbb{k}$*-vector space with a quadratic form* q*. An associative* $\mathbb{k}$*-algebra (with unit* $\mathbb{1}$*)* $Cl(\mathcal{V}, q)$ *is called a* **Clifford algebra** *of* $(\mathcal{V}, q)$*, if there exists a Clifford map:*

$$c : \mathcal{V} \to Cl(\mathcal{V}, q) \tag{3.4}$$

such that:

(i). $Cl(\mathcal{V}, q)$ *is generated by* $\mathbb{1}$ *and* $c(\mathcal{V})$*.*

(ii). The following **"universal property"** *holds: for every associative* $\mathbb{k}$*-algebra* $\mathcal{A}$ *(with unit), and every Clifford map* $\varphi : \mathcal{V} \to \mathcal{A}$*, there exists a* $\mathbb{k}$*-algebra morphism* $\Phi : Cl(\mathcal{V}, q) \to \mathcal{A}$*, such that the diagram:*

$$\begin{array}{ccc} \mathcal{V} & \stackrel{c}{\longrightarrow} & Cl(\mathcal{V}, q) \\ & \varphi & \downarrow \Phi \\ & & \mathcal{A} \end{array} \tag{3.5}$$

commutes.

Since we assume q to be a non-degenerate quadratic form, the Clifford map $c : \mathcal{V} \to Cl(\mathcal{V}, q)$ is injective, and so we identify hereater $\mathbf{x} \in \mathcal{V}$ with its image $c(\mathbf{x}) \in Cl(\mathcal{V}, q)$. So in $Cl(\mathcal{V}, q)$, we have that:

$$\mathbf{x}^2 = -q(\mathbf{x})\mathbb{1} \qquad\qquad \mathbf{xy} + \mathbf{yx} = -2\beta(\mathbf{x}, \mathbf{y})\mathbb{1} \tag{3.6}$$

$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \stackrel{c}{\hookrightarrow} Cl(\mathcal{V}, q)$. In particular we see that $\mathbf{x}$ and $\mathbf{y}$ are ortoghonal iff they anticommute in $Cl(\mathcal{V}, q)$, and that $\mathbf{x}$ is invertible in $Cl(\mathcal{V}, q)$ iff $\mathbf{x}$ is nonisotropic $q(\mathbf{x}) \neq 0$. In this case the inverse of $\mathbf{x} \in \mathcal{V}$ is:

$$\mathbf{x}^{-1} = -\frac{\mathbf{x}}{q(\mathbf{x})} \tag{3.7}$$

To construct $Cl(\mathcal{V}, q)$ we consider the tensor algebra (over $\mathbb{k}$) of $\mathcal{V}$, $\otimes\mathcal{V} = \oplus_{r\geq 0} \otimes^r \mathcal{V}$ and the two-sided ideal $\mathcal{J}_q(\mathcal{V})$ generated by all the elements of the form $\mathbf{x} \otimes \mathbf{x} + q(\mathbf{x})\mathbb{1}$, $\mathbf{x} \in \mathcal{V}$, and we put:

$$Cl(\mathcal{V}, q) = \frac{\otimes\mathcal{V}}{\mathcal{J}_q(\mathcal{V})} \tag{3.8}$$

So we may consider $Cl(\mathcal{V}, q)$ as the algebra generated by $\mathbb{1}$ and $\mathcal{V} \hookrightarrow Cl(\mathcal{V}, q)$, subject to the relations:

$$\mathbf{xy} + \mathbf{yx} = -2\beta(\mathbf{x}, \mathbf{y})\mathbb{1} \tag{3.9}$$

If $\dim \mathcal{V} = n$ and if $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a $\mathbb{k}$-basis of $\mathcal{V}$, then the 2^n elements:

$$\mathbb{1}, \quad \mathbf{e}_1, \cdots, \mathbf{e}_n, \quad \mathbf{e}_1\mathbf{e}_2, \cdots, \mathbf{e}_i\mathbf{e}_j\,(i < j), \quad \cdots, \mathbf{e}_1\mathbf{e}_2 ... \mathbf{e}_n \tag{3.10}$$

form a $\mathbb{k}$-basis of $Cl(\mathcal{V}, q)$ and so $\dim Cl(\mathcal{V}, q) = 2^n$.

Example ... If $q = \mathbf{0}$ then $Cl(\mathcal{V}, q) \cong \wedge\mathcal{V}$ the exterior algebra of $\mathcal{V}$ over $\mathbb{k}$.

Example ... $Cl(\mathbf{R}, q(x) = x^2) = \mathbf{C}$ considered as a real algebra.

Example ... $Cl(\mathcal{V}, q) = Cl(\mathbf{R}^2, q(\mathbf{x}) = x^2 + y^2) \cong \mathbf{H}$, the *"real quaternion algebra"*. In fact, let us consider a q-orthonormal real basis $\{\mathbf{i}, \mathbf{j}\}$ for $\mathbf{R}^2$. Then:

$$\mathbf{1} \quad \mathbf{i} \quad \mathbf{j} \quad \mathbf{k} \equiv \mathbf{ij}$$

is a basis for $Cl(\mathbf{R}^2, x^2 + y^2)$, which has dimension 4. The relations in $Cl(\mathbf{R}^2, x^2 + y^2)$ are:

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = -\mathbf{1} && \mathbf{k}^2 = (\mathbf{ij})^2 = \mathbf{ijij} = -\mathbf{i}^2\mathbf{j}^2 = -\mathbf{1} \\ &\mathbf{jk} = \mathbf{jij} = -\mathbf{j}^2\mathbf{i} = \mathbf{i}, && \mathbf{ki} = \mathbf{iji} = -\mathbf{i}^2\mathbf{j} = \mathbf{j} \end{aligned} \tag{3.11}$$

and we see that:

$$Cl(\mathbf{R}^2, x^2 + y^2) = \mathbf{H} \tag{3.12}$$

the *"real quaternion algebra"* $\mathbf{H}$. Let us recall some concepts about quaternions. Given a quaternion:

$$h = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{H} \tag{3.13}$$

we define:

(i). the *"conjugate"* of h:

$$h^* = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

(ii). the *"norm"* of h:

$$Q(h) = hh^* = a^2 + b^2 + c^2 + d^2$$

It's easy to see that:

$$Q(hh') = Q(h)Q(h') \qquad \forall h, h' \in \mathbf{H} \tag{3.14}$$

and that $(\mathbf{H}, Q)$ is linear isomorphic to $(\mathbf{R}^4, q_e)$, where q_e is the usual euclidean norm in $\mathbf{R}^4$. Besides, $\mathbf{H}$ is a noncommutative field.

We also use the representation of $\mathbf{H}$ as the real algebra of matrices of the form:

$$h = \begin{bmatrix} u & v \\ -\overline{v} & \overline{u} \end{bmatrix} \qquad u, v \in \mathbf{C} \tag{3.15}$$

In this representation we have that the conjugate of h is $h^* = \overline{h}^t$, the norm of h is $Q(h) = hh^* = h\overline{h}^t = (\det h)\,\mathbb{1}$, and:

$$\mathbf{1} = \mathbb{1} = \sigma_0 \qquad \mathbf{i} = i\sigma_1 \qquad \mathbf{j} = i\sigma_2 \qquad \mathbf{k} = i\sigma_3$$

where:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the Pauli matrices. Thus the real quaternion (3.13) is written in the form:

$$h = x^0\sigma_0 + i(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3) \overset{\text{def}}{=} x^0 + i\,\vec{\mathbf{x}} \cdot \vec{\sigma} \tag{3.16}$$

Note also that $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ generate the 3-dimensional space of skew-hermitian matrices of zero trace. We know that the Pauli matrices anticommute and:

$$\begin{array}{lll} \sigma_i^2 = \mathbb{1} & \sigma_1\sigma_2\sigma_3 = i\,\mathbb{1} & \\ \sigma_1\sigma_2 = i\sigma_3 & \sigma_2\sigma_3 = i\sigma_1 & \sigma_3\sigma_1 = i\sigma_2 \end{array} \tag{3.17}$$

Moreover if $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbf{R}^3$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, then we have in $\mathbf{H}$:

$$(\vec{\mathbf{x}} \cdot \vec{\sigma})(\vec{\mathbf{y}} \cdot \vec{\sigma}) = (\vec{\mathbf{x}} \cdot \vec{\mathbf{y}})\mathbb{1} + i(\vec{\mathbf{x}} \times \vec{\mathbf{y}}) \cdot \vec{\sigma}$$

The real quaternions of unit norm, form the group $SU(2)$:

$$SU(2) = \{ \begin{bmatrix} u & v \\ -\overline{v} & \overline{u} \end{bmatrix} : |u|^2 + |v|^2 = 1 \}$$

Involutions in $\mathcal{V}$

Consider the ortoghonal group of $(\mathcal{V}, q)$:

$$O(\mathcal{V}, q) = \{ f : \mathcal{V} \to \mathcal{V} : \; f^* q = q \}$$

If we take $\mathcal{A} = Cl(\mathcal{V}, q)$ and $\varphi = c \circ f$ in definition 1:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{c} & Cl(\mathcal{V}, q) \\ & \varphi = c \circ f & \downarrow \Phi = \tilde{f} \\ & & \mathcal{A} = Cl(\mathcal{V}, q) \end{array}$$

then, since $\varphi = c \circ f$ is a Clifford map ($\varphi(\mathbf{x})^2 = c(f(\mathbf{x}))^2 = -q(f(\mathbf{x}))\mathbb{1} = -q(x)\mathbb{1}$), we conclude that there exists a unique algebra morphism $\tilde{f} \in Aut(Cl(\mathcal{V}, q))$ that extends f, and so it's uniquelly determined by its action on the elements of $\mathcal{V}$. We shall see later that this embedding:

$$O(\mathcal{V}, q) \hookrightarrow Aut(Cl(\mathcal{V}, q))$$

actually lies in the subgroup of inner automorphisms.

In particular if $f(\mathbf{x}) = -\mathbf{x}, \quad \mathbf{x} \in \mathcal{V}$, then we obtain the so called *“main involution”* or *“degree involution”* $\tilde{f} = \alpha$:

$$\alpha : Cl(\mathcal{V}, q) \to Cl(\mathcal{V}, q) \tag{3.18}$$

which verifies $\alpha^2 = \mathrm{Id}$. So there exists a decomposition:

$$Cl(\mathcal{V}, q) = Cl^0(\mathcal{V}, q) \oplus Cl^1(\mathcal{V}, q) \tag{3.19}$$

where $Cl^0(\mathcal{V}, q) = \{ h \in Cl(\mathcal{V}, q) : \alpha(h) = h \}$ is the *“even part”*, which is a subalgebra, and $Cl^1(\mathcal{V}, q) = \{ h \in Cl(\mathcal{V}, q) : \alpha(h) = -h \}$ is the *“odd part”*, which is a subspace.

Note that $Cl(\mathcal{V},q)$ endows the structure of *"superalgebra"* (or $\mathbf{Z}_2$-graded algebra), i.e.:

$$Cl^i(\mathcal{V},q)Cl^j(\mathcal{V},q) \subseteq Cl^{i+j}(\mathcal{V},q) \tag{3.20}$$

where $(i+j)$ is taken $mod\, 2$. Moreover if $\dim \mathcal{V} = n$, then $\dim Cl^0(\mathcal{V},q) = \dim Cl^1(\mathcal{V},q) = 2^{n-1}$.

Now if we take $\mathcal{A} = Cl(\mathcal{V},q)^{\,op}$ and $\varphi = c$ in definition 1, we conclude that there exists a unique algebra morphism in $Aut(Cl(\mathcal{V},q))$ that extends $\varphi = c$, and that we call the *"transpose"* or *"main anti-involution"*. The image of a product $\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_k \in Cl(\mathcal{V},q)$ under this transpose is:

$$(\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_k)^t = \mathbf{x}_k\mathbf{x}_{k-1}...\mathbf{x}_1$$

and we see that:

$$(hh')^t = h'^t h^t \qquad \forall h, h' \in Cl(\mathcal{V},q)$$

Finall y we define the *"conjugation"* in $Cl(\mathcal{V},q)$, by:

$$h^* \stackrel{\text{def}}{=} \alpha(h)^t \tag{3.21}$$

so that:

$$(\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_k)^* = (-1)^k\,\mathbf{x}_k\mathbf{x}_{k-1}...\mathbf{x}_1$$

Representations

Definition 2 ... *Let* $\mathbf{K} \supseteq \Bbbk$ *a field containing* $\Bbbk$*. Then a* **K-representation** *of the Clifford algebra* $Cl(\mathcal{V},q)$ *is a* $\Bbbk$*-homomorphism:*

$$\rho : Cl(\mathcal{V},q) \longrightarrow End_{\mathbf{K}}(W)$$

into the algebra of linear transformations of a finite dimensional **K***-vector space* W*.*

W *is called a* $Cl(\mathcal{V},q)$ *-module over* **K***, and the action:*

$$\rho(h)(\mathbf{w}) \stackrel{def}{=} h \cdot \mathbf{w} \qquad h \in Cl(\mathcal{V},q) \qquad \mathbf{w} \in W$$

is called the **Clifford multiplication***.*

As usual we treat complex representations as the basic objects, viewing real and quaternionic representations as complex representations with additional structure. Thus, if $\mathcal{W}$ is a complex module, a real structure on $\mathcal{W}$ is an anti-linear $Cl(\mathcal{V},q)$-map $\mathcal{R}$ such that $\mathcal{R}^2 = \text{Id}$, while a quaternionic structure on $\mathcal{W}$ is an anti-linear G-map $\mathcal{J}$ such that $\mathcal{J}^2 = -\text{Id}$. $\mathcal{R}$ or $\mathcal{J}$ are called *"structure maps"*. A complex representation is called of *"real type"* (resp. *"quaternionic type"*, if it admits a real (resp., quaternionic) structure.

Our main interest are the cases:

- $\mathcal{V} = \mathbf{R}^n = \mathbf{R}^{r+s}$ with quadratic form:

$$q(x_1, \cdots, x_n) = \underbrace{x_1^2 + \cdots + x_r^2}_{r} - \underbrace{x_{r+1}^2 - \cdots - x_n^2}_{s=n-r} \tag{3.22}$$

The corresponding Clifford algebra will be denoted by $Cl_{r,s}$.

- $\mathcal{V} = \mathbf{C}^n$ with quadratic form:

$$q_{\mathbf{C}}(z_1, \cdots, z_n) = z_1^2 + \cdots + z_n^2 \tag{3.23}$$

The corresponding Clifford algebra will be denoted by $\mathbf{C}l_n$.

Note that the complexification of $Cl_{r,s}$ is just the Clifford algebra (over $\mathbf{C}$) corresponding to the complexified quadratic form $q \otimes \mathbf{C}$, where q is given by (3.22), i.e:

$$Cl_{r,s} \otimes_{\mathbf{R}} \mathbf{C} \cong Cl(\mathbf{C}^{r+s}, q \otimes \mathbf{C})$$

However, since all non-degenerate quadratic forms on $\mathbf{C}^n$ are equivalent, we have that:

$$\mathbf{C}l_n \cong Cl_{r,s} \otimes_{\mathbf{R}} \mathbf{C} \tag{3.24}$$

$\forall r, s : r + s = n$.

<u>**Theorem**</u> **1** ... *Assume that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ and that there exists a nondegenerate bilinear pairing between $\mathcal{V}_1$ and $\mathcal{V}_2$, denoted by $<,>$: $\mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{k}$. Consider the nondegenerate bilinear form β on $\mathcal{V}$ given by:*

$$\beta(\mathbf{v}_1 \oplus \mathbf{v}_2, \mathbf{w}_1 \oplus \mathbf{w}_2) \stackrel{def}{=} -\frac{1}{2}[< \mathbf{v}_1, \mathbf{w}_2 > + < \mathbf{w}_1, \mathbf{v}_2 >] \tag{3.25}$$

and let q be the corresponding quadratic form. Then:

$$Cl(\mathcal{V}, q) \cong End_{\mathbb{k}}(\wedge \mathcal{V}_1) \tag{3.26}$$

<u>Proof</u>...

We define for each $\mathbf{v}_1 \in \mathcal{V}_1$ a <u>*"creation operator"*</u> $\epsilon_{\mathbf{v}_1}$, in $\wedge\mathcal{V}_1$, by:

$$\epsilon_{\mathbf{v}_1} : \wedge\mathcal{V}_1 \to \wedge\mathcal{V}_1 \qquad \epsilon_{\mathbf{v}_1} \alpha = \mathbf{v}_1 \wedge \alpha \qquad \forall \alpha \in \wedge\mathcal{V}_1 \tag{3.27}$$

and for each $\mathbf{v}_2 \in \mathcal{V}_2$, an <u>*"anihilation operator"*</u> $\iota_{\mathbf{v}_2}$, again in $\wedge\mathcal{V}_1$, first defining $\iota_{\mathbf{v}_2} : \mathcal{V}_1 \to \mathbb{k}$, by $\iota_{\mathbf{v}_2}(\mathbf{w}_1) =< \mathbf{w}_1, \mathbf{v}_2 >$, and then extend this to a skew-derivation of $\wedge\mathcal{V}_1$, i.e.:

$$\iota_{\mathbf{v}_2}(\alpha \wedge \beta) = \iota_{\mathbf{v}_2}(\alpha) \wedge \beta + (-1)^{deg\,(\alpha)} \alpha \wedge \iota_{\mathbf{v}_2}(\beta) \qquad \forall \alpha, \beta \in \wedge\mathcal{V}_1$$

Then $\epsilon_{\mathbf{v}_1}$ and $\iota_{\mathbf{v}_2}$ are fermionic creation-anihilation operators, i.e.:

$$\begin{aligned} \epsilon_{\mathbf{v}_1}^2 &= 0 \\ \iota_{\mathbf{v}_2}^2 &= 0 \\ \{\epsilon_{\mathbf{v}_1}, \iota_{\mathbf{v}_2}\} &= < \mathbf{v}_1, \mathbf{v}_2 > \mathrm{Id} \end{aligned} \tag{3.28}$$

By the universal property of definition 1, to define the Clifford action on $\wedge\mathcal{V}_1$ we need only specify it on $\mathcal{V}$. We define it by:

$$\mathbf{v}\cdot\alpha = (\mathbf{v}_1\oplus\mathbf{v}_2)\cdot\alpha \overset{\text{def}}{=} (\epsilon_{\mathbf{v}_1} - \iota_{\mathbf{v}_2})\,\alpha \qquad \mathbf{v} = \mathbf{v}_1\oplus\mathbf{v}_2 \in \mathcal{V} = \mathcal{V}_1\oplus\mathcal{V}_2,\ \alpha\in\wedge\mathcal{V}_1 \tag{3.29}$$

We only need to verify if $\mathbf{v}\cdot(\mathbf{v}\cdot) = -q(\mathbf{v})\,\mathbb{1}$, which it's true since:

$$\begin{aligned}\mathbf{v}\cdot(\mathbf{v}\cdot\alpha) &= (\mathbf{v}_1\oplus\mathbf{v}_2)\cdot(\mathbf{v}_1\oplus\mathbf{v}_2)\cdot\alpha\\ &= (\mathbf{v}_1\oplus\mathbf{v}_2)\cdot(\epsilon_{\mathbf{v}_1}-\iota_{\mathbf{v}_2})\,\alpha\\ &= (\epsilon^2_{\mathbf{v}_1}+\epsilon_{\mathbf{v}_1}\iota_{\mathbf{v}_2}+\iota_{\mathbf{v}_2}\epsilon_{\mathbf{v}_1}+\iota^2_{\mathbf{v}_2})\,\alpha\\ &= <\mathbf{v}_1,\mathbf{v}_2>\mathbb{1}\alpha\\ &= -q(\mathbf{v})\mathbb{1}\alpha\end{aligned} \tag{3.30}$$

Thus this Clifford action extends to a homomorphism:

$$Cl(\mathcal{V},q)\ \to End_{\mathbb{k}}(\wedge\mathcal{V}_1) \tag{3.31}$$

Since $\dim End_{\mathbb{k}}(\wedge\mathcal{V}_1) = (2^n)^2 = 2^{2n} = \dim Cl(\mathcal{V},q)$, to show that this is an isomorphism it suffices to show that this is surjective. In fact this follows from the fact that the algebra $End_{\mathbb{k}}(\wedge\mathcal{V}_1)$ is generated by the above fermionic creation-anihilation operators, CQD.

With the same hypothesis of the previous theorem, sometimes it's useful to use another isomorphic representation, constructed as follows:

Let $\{\mathbf{e}_1,\cdots,\mathbf{e}_n\}$ be a $\mathbb{k}$-basis for $\mathcal{V}_1$, and let $\{\mathbf{f}_1,\cdots,\mathbf{f}_n\}$ be the dual basis, with respect to the duality $<.>$, so that:

$$<\mathbf{e}_i,\mathbf{f}_j>=\delta_{ij}$$

Relativelly to the bilinear form β on $\mathcal{V}$ given by (3.25), and the corresponding quadratic form q, we have:

$$q(\mathbf{e}_i)=0=q(\mathbf{f}_j) \qquad \beta(\mathbf{e}_i,\mathbf{f}_j) = -\frac{1}{2}\delta_{ij}$$

and in $Cl(\mathcal{V},q)$:

$$\mathbf{e}_i^2 = 0 = \mathbf{f}_j^2 \qquad \{\mathbf{e}_i,\mathbf{f}_j\} = \mathbf{e}_i\mathbf{f}_j+\mathbf{f}_j\mathbf{e}_i = \frac{1}{2}\delta_{ij}\mathbb{1}$$

Now we define the *"Clifford vacuum"*:

$$\Omega = \mathbf{f}_1\mathbf{f}_2\ldots\mathbf{f}_n$$

and consider the left ideal $\mathcal{S}$ in $Cl(\mathcal{V},q)$:

$$\mathcal{S} \overset{\text{def}}{=} Cl(\mathcal{V},q)\ \Omega \tag{3.32}$$

It's easy to see that $\mathcal{S}$ is in fact the subspace of $Cl(\mathcal{V},q)$ linearlly generated by all the elements of the form $\mathbf{e}_I\Omega$, i.e:

$$\mathcal{S} = \mathbb{k}\langle\,\mathbf{e}_I\,\Omega:\ \forall I = \{1\le i_1<i_2<\cdots i_r\le n,\ 1\le r\le n\},\emptyset\rangle$$

(we put $\mathbf{e}_\emptyset = \mathbb{1}$). In fact the set of all $\mathbf{e}_I\mathbf{f}_J$ is basis for $Cl(\mathcal{V},q)$, and $\mathbf{f}_J\Omega = 0$, $\forall J \neq \emptyset$. Now we consider the left action of $Cl(\mathcal{V},q)$ on $\mathcal{S} = Cl(\mathcal{V},q)\,\Omega$:

$$h \cdot (h'\Omega) \stackrel{\text{def}}{=} (hh')\Omega \tag{3.33}$$

which endows $\mathcal{S}$ with the structure of $Cl(\mathcal{V},q)$-module. Of course $\mathcal{S}$ is linearlly isomorphic to $\wedge\mathcal{V}_1\Omega \cong \wedge\mathcal{V}_1$, and the map $\alpha \mapsto \alpha\cdot\Omega$ gives an isomorphism:

$$\wedge\mathcal{V}_1 \longrightarrow \mathcal{S} = \wedge\mathcal{V}_1\cdot\Omega = Cl(\mathcal{V},q)\cdot\Omega$$

of left $Cl(\mathcal{V},q)$-modules. So we have the following:

<u>**Theorem**</u> **2** ... *Assume the same hypothesis of the previous theorem. Then the (left) $Cl(\mathcal{V},q)$-module $\wedge\mathcal{V}_1$ is isomorphic to a left ideal in $Cl(\mathcal{V},q)$. In fact, let Ω be a generator of the top exterior power $\wedge^n\mathcal{V}_2$ (the "Clifford vacuum"). Then:*

$$\mathcal{S} \stackrel{def}{=} Cl(\mathcal{V},q)\cdot\Omega \cong \wedge\mathcal{V}_1\cdot\Omega \tag{3.34}$$

and the map $\alpha \mapsto \alpha\cdot\Omega$ gives an isomorphism:

$$\wedge\mathcal{V}_1 \longrightarrow \mathcal{S} = \wedge\mathcal{V}_1\cdot\Omega = Cl(\mathcal{V},q)\cdot\Omega$$

of left $Cl(\mathcal{V},q)$-modules.

<u>**Theorem**</u> **3** ... *Let $\mathcal{V}$ be a $2n$-dimensional vector space with a nondegenerate quadratic form. Assume that there exists an involution $\Phi:\mathcal{V}\to\mathcal{V}:\ \Phi^2 = Id$, which is skew-symmetric with respect to β, i.e., $\beta(\Phi\mathbf{v},\mathbf{w}) = -\beta(\mathbf{v},\Phi\mathbf{w})$, $\forall\mathbf{v},\mathbf{w}\in\mathcal{V}$. Then $Cl(\mathcal{V},q)$ is isomorphic to $End_{\mathbb{k}}(\mathcal{V}_1)$ where $\mathcal{V}_1 = \ker(\Phi - Id)$.*

<u>Proof</u>...

Consider the (±1)-eigenspaces of Φ:

$$\begin{aligned}\mathcal{V}_1 &= \{\mathbf{v}\in\mathcal{V}:\ \Phi\mathbf{v} = \mathbf{v}\}\\ \mathcal{V}_2 &= \{\mathbf{v}\in\mathcal{V}:\ \Phi\mathbf{v} = -\mathbf{v}\}\end{aligned}$$

Then $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, with:

$$\mathbf{v} = \mathbf{v}_1\oplus\mathbf{v}_2 = \frac{1}{2}(\mathbf{v}+\Phi\mathbf{v}) + \frac{1}{2}(\mathbf{v}-\Phi\mathbf{v}) \in \mathcal{V}_1\oplus\mathcal{V}_2$$

$\mathcal{V}_1$ and $\mathcal{V}_2$ are totally isotropic with respect to q. In fact, if $\mathbf{v}_1,\mathbf{w}_1\in\mathcal{V}_1$, then, since Φ is skew:

$$\beta(\mathbf{v}_1,\mathbf{w}_1) = \beta(\Phi\mathbf{v}_1,\Phi\mathbf{w}_1) = -\beta(\Phi^2\mathbf{v}_1,\mathbf{w}_1) = -\beta(\mathbf{v}_1,\mathbf{w}_1)$$

whence $\beta(\mathbf{v}_1,\mathbf{w}_1) = 0$. Similarly $\beta(\mathbf{v}_2,\mathbf{w}_2) = 0$, $\forall\mathbf{v}_2,\mathbf{w}_2\in\mathcal{V}_2$. Now we define a bilinear pairing between $\mathcal{V}_1$ and $\mathcal{V}_2$, by:

$$<\mathbf{v}_1,\mathbf{v}_2> \stackrel{\text{def}}{=} -2\beta(\mathbf{v}_1,\mathbf{v}_2) \qquad \mathbf{v}_1\in\mathcal{V}_1,\mathbf{v}_2\in\mathcal{V}_2$$

It is nondegenerate, since β is so, and with respect to the direct sum decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, β verifies:

$$\begin{aligned}\beta(\mathbf{v}_1 \oplus \mathbf{v}_2, \mathbf{w}_1 \oplus \mathbf{w}_2) &= \beta(\mathbf{v}_1, \mathbf{w}_2) + \beta(\mathbf{w}_1, \mathbf{v}_2) \\ &= -\frac{1}{2}[< \mathbf{v}_1, \mathbf{w}_2 > + < \mathbf{w}_1, \mathbf{v}_2 >]\end{aligned}$$

and we can apply the previous theorem to conclude that $Cl(\mathcal{V}, q) \cong End_{\Bbbk}(\wedge\mathcal{V}_1)$, CQD.

Corollary 1 ...

$$\mathbf{C}l_{2n} \cong End_{\mathbf{C}}(\wedge\mathbf{C}^n)$$

Corollary 2 ...

$$Cl_{r,r} \cong End_{\mathbf{R}}(\wedge\mathbf{R}^r)$$

Theorem 4 ... *Consider the Clifford algebra $Cl(\mathcal{V}, q)$, and let $\mathbf{e} \in \mathcal{V}$ be a nonzero vector with $q(\mathbf{e}) = a \neq 0$. Consider the orthogonal $\mathcal{W} = \mathbf{e}^\perp$ and the quadratic form $q^\perp(\mathbf{y}) = a\, q(\mathbf{y}), \quad \mathbf{y} \in \mathcal{W}$.*

Then, the even subalgebra $Cl^{(0)}(\mathcal{V}, q)$ is the Clifford algebra of $(\mathcal{W}, q^\perp)$:

$$Cl^{(0)}(\mathcal{V}, q) = Cl(\mathcal{W}, q^\perp) \tag{3.35}$$

Proof...

Consider the diagram of definition 1:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{c} & Cl(\mathcal{W}, q^\perp) \\ & \varphi & \downarrow \Phi \\ & & \mathcal{A} = Cl^{(0)}(\mathcal{V}, q) \end{array}$$

with $\varphi(\mathbf{y}) = \mathbf{ye}$. Then φ is Clifford map. In fact, since $\mathbf{y}$ and $\mathbf{e}$ are q-orthogonal, they anticommute in $Cl(\mathcal{V}, q)$, and so $\forall \mathbf{y} \in \mathbf{e}^\perp = \mathcal{W}$:

$$\varphi(\mathbf{y})^2 = \mathbf{yeye} = -\mathbf{y}^2\mathbf{e}^2 = -q(\mathbf{e})q(\mathbf{y}) = -a\, q(\mathbf{y}) = -q^\perp(\mathbf{y})\mathbb{1}$$

So φ extends to a unique algebra morphism $\Phi : Cl(\mathcal{W}, q^\perp) \to Cl^{(0)}(\mathcal{V}, q)$ which it's an isomorphism, CQD.

Example ... $Cl_{0,3} = \mathbf{C}(2)$

Let $(\mathcal{V}, q) = (\mathbf{R}^3, -q_e)$, where $q_e(\mathbf{x}) = x^2 + y^2 + z^2$ is the euclidean quadratic form. We know that $Cl(\mathbf{R}^3, -q_e)$ has real dimension $8 = 2^3$. Let us apply the previous theorem, fixing an unit vector $\mathbf{e} \in \mathbf{R}^3$, with $\mathbf{q}(\mathbf{e}) = a = -1$, and considering $(\mathcal{W} = \mathbf{e}^\perp, q^\perp) \cong (\mathbf{R}^2, q^\perp = q_e \mid_{\mathcal{W}})$.

The theorem says that the even subalgebra $Cl^{(0)}(\mathbf{R}^3, -q_e)$ is the Clifford algebra of $(\mathbf{R}^2, q(\mathbf{x}) = x^2 + y^2)$, which is $\mathbf{H}$, as we have seen in a previous example, i.e.:

$$Cl^{(0)}_{0,3} = Cl_{2,0} = \mathbf{H}$$

Let us consider now an orthonormal basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{R}^3$, and the element:

$$\omega = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \in Cl_{0,3} \tag{3.36}$$

which is called the *"chirality operator"*. Note that if we choose another orthonormal basis $\hat{\mathcal{B}} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, then $\hat{\mathbf{e}}_i = g_i^j\,\mathbf{e}_j$ with $g = (g_i^j) \in O(3)$. Besides it's easy to see that:

$$\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 = (\det g)\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

and so if we choose an orientation for $\mathbf{R}^3$ we see that we can define the chirality operator by (3.36), and this definition is independent of the choice of the orthonormal basis belonging to that orientation.

Now we compute that:

$$\omega^2 = -\mathbb{1} \qquad \omega\,\mathbf{e}_i = \mathbf{e}_i\,\omega \qquad i = 1, 2, 3$$

and that the center of $Cl_{0,3}$ is the subalgebra of the elements of the form $a\mathbb{1} + b\,\omega$, thus isomorphic to $\mathbf{C}$ since $\omega^2 = -\mathbb{1}$. So we see that $Cl_{0,3}$ is a complex algebra, and since $\omega\,Cl^{(0)}_{0,3} = Cl^{(1)}_{0,3}$, then:

$$\begin{aligned} Cl_{0,3} &= Cl^{(0)}_{0,3} \oplus Cl^{(1)}_{0,3} \\ &= Cl^{(0)}_{0,3} \oplus \omega\,Cl^{(0)}_{0,3} \\ &= \mathbf{H} \oplus \omega\,\mathbf{H} \\ &= \mathbf{H}^{\mathbf{C}} \\ &= \mathbf{C}(2) \end{aligned} \tag{3.37}$$

The usual representation of $Cl_{0,3}$ by matrices of $\mathbf{C}(2)$, is the following: if $c : \mathbf{R}^3 \hookrightarrow Cl_{0,3}$ is the canonical injection, and if $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of $\mathbf{R}^3$, we put:

$$\begin{aligned} c(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) &= \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} \\ &= x\sigma_1 + y\sigma_2 + z\sigma_3 \end{aligned} \tag{3.38}$$

where:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the Pauli matrices. So $c(\mathbf{R}^3) \hookrightarrow Cl(\mathbf{R}^3, -q_e) = \mathbf{C}(2)$ is the real subspace of hermitian matrices with zero trace. We know that the Pauli matrices anticommute, and that:

$$\begin{aligned} &\sigma_i^2 = \mathbb{1} && \sigma_1\sigma_2\sigma_3 = i\,\mathbb{1} \\ &\sigma_1\sigma_2 = i\sigma_3 && \sigma_2\sigma_3 = i\sigma_1 \qquad \sigma_3\sigma_1 = i\sigma_2 \end{aligned} \tag{3.39}$$

and we see that $\{\mathbb{1}, i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis for the even subalgebra $Cl_{0,3}^{(0)}$, while $\{\mathbb{1}, i\sigma_1, i\sigma_2, i\sigma_3, i\mathbb{1},$-$\sigma_1, -\sigma_2, -\sigma_3\}$ is a real basis for $Cl_{0,3}$.

Example ... $Cl_{3,0} = Cl(\mathbf{R}^3, q_e) = \mathbf{H} \oplus \mathbf{H}$

In fact, now the chirality operator $\omega = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ verifies: $\omega^2 = \mathbb{1}$ and $\mathbf{v}\omega = \omega\mathbf{v} \quad \forall \mathbf{v} \in \mathbf{R}^3$, i.e., ω is a central element in $Cl_{3,0}$.

We can consider now the direct sum decomposition:

$$Cl_{3,0} = Cl_{3,0}^{+} \oplus Cl_{3,0}^{-}$$

where:

$$Cl_{3,0}^{+} \stackrel{\text{def}}{=} \frac{\mathbb{1}+\omega}{2} Cl_{3,0} \qquad \text{and} \qquad Cl_{3,0}^{+} \stackrel{\text{def}}{=} \frac{\mathbb{1}-\omega}{2} Cl_{3,0}$$

are isomorphic subalgebras such that $\alpha(Cl_{3,0}^{\pm}) = Cl_{3,0}^{\mp}$.

For the next theorem, assume that $(\mathcal{V}, q_{\mathcal{V}})$ and $(\mathcal{W}, q_{\mathcal{W}})$ are two vector spaces with quadratic forms. Define in $\mathcal{V} \oplus \mathcal{W}$ a quadratic form $q = q_{\mathcal{V}} \oplus q_{\mathcal{W}}$ by:

$$q(\mathbf{v} \oplus \mathbf{w}) = q_{\mathcal{V}}(\mathbf{v}) + q_{\mathcal{W}}(\mathbf{w})$$

Recall also that if $\mathcal{V} = \mathcal{V}^0 \oplus \mathcal{V}^1$ and $\mathcal{W} = \mathcal{W}^0 \oplus \mathcal{W}^1$ are two superalgebras then we define its tensor product $\mathcal{V}\hat{\otimes}\mathcal{W}$ as the superspace:

$$\begin{aligned} \mathcal{V} \otimes \mathcal{W} &= (\mathcal{V} \otimes \mathcal{W})^0 \oplus (\mathcal{V} \otimes \mathcal{W})^1 \\ &\stackrel{\text{def}}{=} (\mathcal{V}^0 \otimes \mathcal{W}^0 \oplus \mathcal{V}^1 \otimes \mathcal{W}^1) \oplus (\mathcal{V}^0 \otimes \mathcal{W}^1 \oplus \mathcal{V}^1 \otimes \mathcal{W}^0) \end{aligned} \tag{3.40}$$

together with a multiplication defined by:

$$(\mathbf{v}_1 \oplus \mathbf{w}_1)(\mathbf{v}_2 \oplus \mathbf{w}_2) = (-1)^{(deg\,\mathbf{w}_1)(deg\,\mathbf{v}_2)}\, \mathbf{v}_1\mathbf{v}_2 \otimes \mathbf{w}_1\mathbf{w}_2$$

Theorem 5 ... *There exists an isomorphism of superalgebras:*

$$Cl(\mathcal{V} \oplus \mathcal{W}, q_{\mathcal{V}} \oplus q_{\mathcal{W}}) \cong Cl(\mathcal{V}, q_{\mathcal{V}})\hat{\otimes}Cl(\mathcal{W}, q_{\mathcal{W}}) \tag{3.41}$$

Proof...

Consider the linear map $\varphi : \mathcal{V} \oplus \mathcal{W} \to Cl(\mathcal{V}, q_{\mathcal{V}})\hat{\otimes}Cl(\mathcal{W}, q_{\mathcal{W}})$ defined by:

$$\varphi(\mathbf{v} \oplus \mathbf{w}) = \mathbf{v} \otimes \mathbb{1}_{\mathcal{W}} + \mathbb{1}_{\mathcal{V}} \otimes \mathbf{w}$$

Then:

$$\begin{aligned} \varphi(\mathbf{v} \oplus \mathbf{w})^2 &= (\mathbf{v} \otimes \mathbb{1}_{\mathcal{W}} + \mathbb{1}_{\mathcal{V}} \otimes \mathbf{w})^2 \\ &= \mathbf{v}^2 \otimes \mathbb{1}_{\mathcal{W}} + \mathbf{v} \otimes \mathbf{w} - \mathbf{v} \otimes \mathbf{w} + \mathbb{1}_{\mathcal{V}} \otimes \mathbf{w}^2 \\ &= -[q_{\mathcal{V}}(\mathbf{v}) + q_{\mathcal{W}}(\mathbf{w})]\mathbb{1}_{\mathcal{V}} \otimes \mathbb{1}_{\mathcal{W}} \\ &= -(q_{\mathcal{V}} \oplus q_{\mathcal{W}})(\mathbf{v} \oplus \mathbf{w})\mathbb{1}_{\mathcal{V}} \otimes \mathbb{1}_{\mathcal{W}} \end{aligned} \tag{3.42}$$

i.e., φ is a Clifford map and so extends to an algebra morphism:

$$\tilde{\varphi} : Cl(\mathcal{V} \oplus \mathcal{W}, q_\mathcal{V} \oplus q_\mathcal{W}) \to Cl(\mathcal{V}, q_\mathcal{V}) \hat{\otimes} Cl(\mathcal{W}, q_\mathcal{W})$$

Now consider $\eta : Cl(\mathcal{V}, q_\mathcal{V}) \hat{\otimes} Cl(\mathcal{W}, q_\mathcal{W}) \to Cl(\mathcal{V} \oplus \mathcal{W}, q_\mathcal{V} \oplus q_\mathcal{W})$, defined by:

$$\eta(\mathbf{v} \otimes \mathbf{w}) = \mathbf{vw}$$

Then it's easy to prove that η is an algebra morphism such that $\eta = \varphi^{-1}$, CQD.

Now we want to compute the Clifford algebras of $(\mathcal{V} = \mathbf{R}^k, \pm q_e)$, where $q_e(\mathbf{x}) = \sum_{i=1}^k x_i^2$ is the euclidean quadratic form. But before, two useful theorems:

3.2 Pin and Spin groups

Consider again a non-degenerate quadratic space $(\mathcal{V}, q)$, and let $\mathbf{a} \in \mathcal{V}$ be a nonisotropic vector ($q(\mathbf{a}) \neq 0$). Then the reflection $s_\mathbf{a}$ with respect to $\mathbf{a}^\perp$ is the orthogonal map given by:

$$s_\mathbf{a}(\mathbf{x}) = \mathbf{x} - 2\,\frac{\beta(\mathbf{x}, \mathbf{a})}{q(\mathbf{a})}\,\mathbf{a} \tag{3.43}$$

Let us write this equality in $Cl(\mathcal{V}, q)$:

$$\begin{aligned} s_\mathbf{a}(\mathbf{x}) &= \mathbf{x} - 2\,\frac{\beta(\mathbf{x}, \mathbf{a})}{q(\mathbf{a})}\,\mathbf{a} \\ &= \mathbf{x} - (\mathbf{ax} + \mathbf{xa})\frac{-\mathbf{a}}{q(\mathbf{a})} \\ &= \mathbf{x} - (\mathbf{ax} + \mathbf{xa})a^{-1} \\ &= -\mathbf{axa}^{-1} \end{aligned} \tag{3.44}$$

By the theorem of Cartan-Dieudonné, every $f \in O(\mathcal{V}, q)$ can be written as a product of those reflections:

(i). in even number if $\det f = 1$, say $g = s_{\mathbf{a}_1} s_{\mathbf{a}_2} \cdots s_{\mathbf{a}_{2p}}$, so that in $Cl(\mathcal{V}, q)$:

$$f(\mathbf{x}) = (\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_{2p})\,\mathbf{x}\,(\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_{2p})^{-1} \tag{3.45}$$

(i). in odd number if $\det f = -1$, say $g = s_{\mathbf{a}_1} s_{\mathbf{a}_2} \cdots s_{\mathbf{a}_{2p+1}}$, so that in $Cl(\mathcal{V}, q)$:

$$f(\mathbf{x}) = -(\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_{2p+1})\,\mathbf{x}\,(\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_{2p+1})^{-1} \tag{3.46}$$

As we know every $f \in O(\mathcal{V}, q)$ extends uniquely to an algebra morphism $\tilde{f} : Cl(\mathcal{V}, q) \to Cl(\mathcal{V}, q)$. If $\det f = 1$ then $\tilde{f}$ is an inner automorphism: $\tilde{f}(\mathbf{x}) = h\mathbf{x}h^{-1}$, where h is a product of an even number of nonisotropic vectors in $\mathcal{V}$, while if $\det f = -1$ then $\tilde{f}$ is the compose of the main involution α with an inner automorphism.

This lead us to consider the so called *"Clifford group"* $\Gamma(\mathcal{V},q)$, of $(\mathcal{V},q)$, as the group of the invertible elements $h \in Cl(\mathcal{V},q)$ such that:

$$\alpha(h)\mathcal{V}h^{-1} \subseteq \mathcal{V}$$

By the above discussion, $\Gamma(\mathcal{V},q)$ contains all nonisotropic vectors in $\mathcal{V}$ as well all the elements of $Cl(\mathcal{V},q)$ that are products of nonisotropic vectors of $\mathcal{V}$.

Note that $\Gamma(\mathcal{V},q)$ come with a ready-made homomorphism:

$$\widetilde{Ad} : \Gamma(\mathcal{V},q) \longrightarrow Aut(\mathcal{V}) \tag{3.47}$$

defined by:

$$\widetilde{Ad} : g \mapsto \widetilde{Ad}_g(\mathbf{x}) \stackrel{\text{def}}{=} \alpha(g)\mathbf{x}g^{-1} \qquad g \in \Gamma(\mathcal{V},q) \qquad \mathbf{x} \in \mathcal{V} \tag{3.48}$$

which is called the *"Twisted Adjoint Representation"* of $\Gamma(\mathcal{V},q)$ on $\mathcal{V}$. This representation is nearly faithful:

Proposition 1 *([LM, prop. 2.4])... The kernel of $\widetilde{Ad} : \Gamma(\mathcal{V},q) \longrightarrow Aut(\mathcal{V})$ is $\Bbbk^\times$, the multiplicative group of nonzero scalar multiples of $\mathbb{1} \in Cl_k$.*

Consider now the *"Norm mapping"* $N : Cl_k \to Cl_k$ defined by:

$$N(h) = h\,h^*$$

where $h^* = \alpha(h^t)$ is the conjugate of h. Note that $N(\mathbf{x}) = \mathbf{x}(-\mathbf{x}) = -\mathbf{x}^2 = q(\mathbf{x})\,\mathbb{1}, \quad \forall \mathbf{x} \in \mathcal{V}$. Moreover we can prove (see ([LM, prop. 2.5])) that if $g \in \Gamma(\mathcal{V},q)$ then $N(g) \in \Bbbk^\times$, and that:

$$N : \Gamma(\mathcal{V},q) \to \Bbbk^\times$$

is an algebra homomorphism.

Proposition 2 *... For all $g \in \Gamma(\mathcal{V},q)$, the transformations $\widetilde{Ad}_g$ preserve the quadratic form q. So there is an homomorphism:*

$$\widetilde{Ad} :\ \Gamma(\mathcal{V},q) \longrightarrow O(\mathcal{V},q)$$

Proof...

Note that $N(\alpha(g)) = N(g), \forall g \in \Gamma(\mathcal{V},q)$, since $N(\alpha(g)) = \alpha(g)(\alpha(g))^* = \alpha(g)g^t = \alpha(N(g)) = N(g)$. So if we consider the subset of all the nonisotropic vectors in $\mathcal{V}$:

$$\mathcal{V}^\times = \{\mathbf{x} \in \mathcal{V} :\ q(\mathbf{x}) \neq \mathbf{0}\}$$

then for each $\mathbf{x} \in \mathcal{V}^\times \subset \Gamma(\mathcal{V},q)$, we have that $N(\widetilde{Ad}_g(\mathbf{x})) = N(\alpha(g)\mathbf{x}g^{-1}) = N(\alpha(g))N(\mathbf{x})N(g^{-1}) = N(g)N(g)^{-1}N(\mathbf{x}) = N(\mathbf{x})$, and since $N(\mathbf{v}) = q(\mathbf{v}), \forall \mathbf{v} \in \mathcal{V}$, we see that $\widetilde{Ad}_g$ preserves all non-zero q-lengths. Applying $\widetilde{Ad}_{g^{-1}}$ now shows that $\widetilde{Ad}_g(\mathcal{V}^\times) = \mathcal{V}^\times$, and so $\widetilde{Ad}_g$ leaves also invariant the set of vectors with zero q-length. Thus $\widetilde{Ad}_g$ is q-orthogonal, CQD.

Definition 3 ... *We define the* **Pin group** $Pin(\mathcal{V},q)$ *of* $(\mathcal{V},q)$, *as the subgroup of* $\Gamma(\mathcal{V},q)$ *generated by all elements* $\mathbf{v}\in\mathcal{V}$ *such that* $q(\mathbf{v})=\pm 1$:

$$Pin(\mathcal{V},q) \stackrel{def}{=} \{\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_r\in\Gamma(\mathcal{V},q) : q(\mathbf{v}_j)=\pm 1 \quad \forall j\} \tag{3.49}$$

The associated **Spin group** *of* $(\mathcal{V},q)$ *is defined by:*

$$Spin(\mathcal{V},q) \stackrel{def}{=} Pin(\mathcal{V},q)\cap Cl^{(0)}(\mathcal{V},q) \tag{3.50}$$

Example ... $Spin(4), SO(4)$

Recall that we can identify the euclidean space $(\mathbf{R}^4,q_e)$, where $q_e(\mathbf{x})=\|\mathbf{x}\|^2$ is the euclidean quadratic form, with the linear space $\mathbf{H}$ of real quaternions through the linear map:

$$\begin{aligned}
\mathbf{x}=(x^0,x^1,x^2,x^3)=(x^0,\vec{\mathbf{x}})\in\mathbf{R}^4\mapsto\mathbf{X} &= x^0\mathbf{1}-i\,\vec{\mathbf{x}}\cdot\vec{\sigma}\\
&= x^0\mathbf{1}+x^1\mathbf{i}+x^2\mathbf{j}+x^3\mathbf{k}\\
&= \begin{bmatrix} x^0-ix^3 & -x^2-ix^1\\ x^2-ix^1 & x^0+ix^3\end{bmatrix}\\
&= \begin{bmatrix} u & v\\ -\overline{v} & \overline{u}\end{bmatrix}
\end{aligned} \tag{3.51}$$

with $u=x^0-ix^3, v=-(x^2+ix^1)\in\mathbf{C}$. In this form, the conjugate of $\mathbf{X}\in\mathbf{H}$ is $\mathbf{X}^*=\overline{\mathbf{X}}^t$, the norm of $\mathbf{X}$ is $Q(\mathbf{X})=\mathbf{X}\mathbf{X}^*=\mathbf{X}\overline{\mathbf{X}}^t=(\det\mathbf{X})\,\mathbb{1}=((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)\,\mathbb{1}=\|\mathbf{x}\|^2\,\mathbb{1}$, and the real quaternions of unit norm form the group $SU(2)$.

Consider now the Clifford algebra $Cl_{4,0}\equiv Cl(\mathbf{R}^4,q_e)$. We know that $Cl_{4,0}$ has real dimension $16=2^4$. Recall that $(\mathbf{R}^4,q_e)$ is linear isomorphic to $(\mathbf{H},Q)$. The map $c:\mathbf{H}\to\mathbf{H}(2)$ defined by:

$$c(h)=\begin{bmatrix} 0 & h\\ -h^* & 0\end{bmatrix} \qquad h\in\mathbf{H}$$

where $\mathbf{H}(2)$ is the real algebra of quaternionic (2×2)-matrices, is a Clifford map, since:

$$c(h)^2=\begin{bmatrix} 0 & h\\ -h^* & 0\end{bmatrix}^2=\begin{bmatrix} -hh^* & 0\\ 0 & -h^*h\end{bmatrix}=-Q(h)\,\mathbb{1} \tag{3.52}$$

Moreover since $\mathbf{H}(2)$ is generated as a real algebra of dimension 16 by the above matrices we see that:

$$Cl_{4,0}=\mathbf{H}(2)$$

$\mathbf{R}^4\cong\mathbf{H}$ sits inside $Cl_{4,0}=\mathbf{H}(2)$ through the canonical injection c given by (3.52), and we identify $\mathbf{R}^4$ with $c(\mathbf{R}^4)$. In particular the images in $Cl_{4,0}=\mathbf{H}(2)$, under c, of the elements $\mathbf{e}_i$, $i=0,1,2,3$ of the canonical basis of $\mathbf{R}^4$ are the so called *"Dirac γ-matrices"*:

$$\gamma_0=c(\mathbf{e}_0)=\begin{bmatrix} 0 & \mathbb{1}\\ -\mathbb{1} & 0\end{bmatrix}, \qquad \gamma_k=c(\mathbf{e}_k)=\begin{bmatrix} 0 & -i\sigma_k\\ i\sigma_k & 0\end{bmatrix} \qquad k=1,2,3 \tag{3.53}$$

Now we know that the even subalgebra $Cl_{4,0}^{(0)}$ is isomorphic to $Cl_{3,0} = Cl(\mathbf{R}^3, q_e)$ where $\mathbf{R}^3$ is the subspace of $\mathbf{R}^4$ orthogonal to $\mathbf{e}_0$. But $Cl_{3,0} = \mathbf{H} \oplus \mathbf{H}$, as we have seen previously, and so $Cl_{4,0}^{(0)} = \mathbf{H} \oplus \mathbf{H} \hookrightarrow \mathbf{H}(2)$ through the map:

$$h \oplus h' \to \begin{bmatrix} h & 0 \\ 0 & h' \end{bmatrix}$$

Such an element $h \oplus h' \in Cl_{4,0}^{(0)}$ is invertible iff both h and $h' \in \mathbf{H}$ are. Moreover an invertible $h \oplus h'$ is such that:

$$Ad_{h\oplus h'}\mathbf{X} \in \mathbf{R}^4 \qquad \forall \mathbf{X} \in \mathbf{R}^4 \cong c(\mathbf{H}) \hookrightarrow CL_{4,0} = \mathbf{H}(2)$$

iff:

$$\begin{bmatrix} h & 0 \\ 0 & h' \end{bmatrix} \begin{bmatrix} 0 & \mathbf{X} \\ -\mathbf{X}^* & 0 \end{bmatrix} \begin{bmatrix} h^{-1} & 0 \\ 0 & h'^{-1} \end{bmatrix} = \begin{bmatrix} 0 & h\mathbf{X}h'^{-1} \\ -h'\mathbf{X}^*h^{-1} & 0 \end{bmatrix} \in \mathbf{R}^4$$

i.e.:

$$-h'\mathbf{X}^*h^{-1} = -(h\mathbf{X}h'^{-1})^*$$

which is equivalent to $h'h'^*\mathbf{X}^* = \mathbf{X}^*h^*h$, i.e., $(\det h')\mathbf{X}^* = (\det h)\mathbf{X}^*$, $\forall \mathbf{X} \in \mathbf{R}^4$. Thus $\det h' = \det h$, and in particular we conclude that:

$$Spin(4) = \{h \oplus h' \in Cl_4^0 = \mathbf{H} \oplus \mathbf{H} :\ \det h' = \det h = 1\} \cong SU(2) \times SU(2)$$

The above computations show also that the adjoint representation is completelly determined by the action ϕ of $Spin(4) = SU(2) \times SU(2)$ on $\mathbf{H} \cong \mathbf{R}^4$, given by:

$$\phi(h_1, h_2)\mathbf{X} = h_1\mathbf{X}h_2^{-1} \qquad h_1, h_2 \in SU(2),\ \mathbf{X} \in \mathbf{H}$$

Then $h_1\mathbf{X}h_2^{-1} \in \mathbf{H}$ and $\det(\phi(h_1, h_2)\mathbf{X}) = \det(h_1\mathbf{X}h_2^{-1}) = \det\mathbf{X}$ which give us a homomorphism:

$$\varphi : SU(2) \times SU(2) \to SO(4)$$

with kernel consists of the pairs (h_1, h_2) such that:

$$h_1\mathbf{X}h_2^{-1} = \mathbf{X} \quad \forall \mathbf{X} \in \mathbf{H}$$

This implies that $h_1 = h_2 = \lambda\mathbb{1}$ and since $\lambda\mathbb{1} \in SU(2)$ we see that $\lambda^2 = 1$ and so $\mathrm{Ker}\,\varphi = \{(\mathbb{1}, \mathbb{1}), (-\mathbb{1}, -\mathbb{1})\} = \mathbf{Z}_2$.

Thus we have the identifications:

$$Spin(4) = SU(2) \times SU(2) \tag{3.54}$$

and:

$$SO(4) = SU(2) \times SU(2)/\mathbf{Z}_2 \tag{3.55}$$

3.3 Spin Representations

We will distinguish the two copies of $SU(2)$ in $Spin(4)$, by writing:

$$Spin(4) = SU^+(2) \times SU^-(2)$$

The representations of $Spin(4)$ can be determined using this isomorphism. But first let us recall the representations of $SU(2)$: the fundamental representation $D_{1/2}$, is $SU(2)$ acting on $\mathbf{C}^2$ in the usual way, and all the others irreducible representations are symmetric powers:

$$D_{k/2} = Sym^k\, D_{1/2}$$

with $k \in \mathbf{Z}^+$. We have that $\dim_{\mathbf{C}} D_{k/2} = \dim_{\mathbf{C}} Sym^k\, D_{1/2} = k+1$, since we can identify this space with the space of homogeneous polynomials of degree k in 2 variables.

Tensor products of this representations decompose according to Clebsh-Gordon formula:

$$D_{k/2} \otimes D_{l/2} = D_{k+l/2} \oplus D_{k+l-2/2} \oplus \cdots \oplus D_{|k-l|/2}$$

The **spin representations** $D^{\pm}_{1/2}$ of $Spin(4) = SU^+(2) \times SU^-(2)$ are the representations obtained by projecting onto $SU^{\pm}(2)$ and then applying $D_{1/2}$. So any irreducible $Spin(4)$-module has the form:

$$\begin{aligned} S^{k,l} &\equiv D^+_{k/2} \otimes D^-_{l/2} \\ &= Sym^k\, D^+_{1/2} \otimes Sym^l\, D^-_{1/2} \qquad k,l \geq 0 \end{aligned} \tag{3.56}$$

which has complex dimension $(k+1)(l+1)$ and factors through $SO(4)$ iff $k+l$ is even. In particular the basic $SO(4)$-module which is $\mathbf{R}^4$, must be equal to $S^{1,1}$, i.e.:

$$\begin{aligned} (\mathbf{R}^4)^{\mathbf{C}} \cong S^{1,1} &= \mathbf{C}^2{}_+ \otimes \mathbf{C}^2{}_- \\ &\equiv \mathcal{S}^+ \oplus \mathcal{S}^- \end{aligned} \tag{3.57}$$

The spin representations $D^{\pm}_{1/2}$ generate the representation ring of $Spin(4)$.

We know that $\mathbf{C}l(4) = Cl_4 \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{H}(2) \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}(4) = End(\mathbf{C}^4)$, the algebra of complex (4×4)-matrices. The inclusion:

$$Spin(4) \subset \mathbf{C}l(4) = End(\mathbf{C}^4)$$

makes $\mathcal{S} = \mathbf{C}^4$ into a $Spin(4)$-representation. Since the chirallity operator:

$$\omega = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 \in Cl(4)$$

satisfies in this case:

$$\omega^2 = \mathbb{1}$$

we see that $\mathcal{S}$ decomposes into the ± 1 eigenspaces of ω:

$$\begin{aligned} \mathcal{S} = \mathbf{C}^4 &= \mathbf{C}^2 \oplus \mathbf{C}^2 \\ &= \mathcal{S}^+ \oplus \mathcal{S}^- \end{aligned} \tag{3.58}$$

with $\mathcal{S}^{\pm} = (\mathbb{1} + \omega)\mathcal{S}$, called the spaces of $\pm$ Majorana spinors (see [LM], prop.5.10). Moreover, since ω commutes with all the elements in the even subalgebra Cl_4^0, each of the subspaces $\mathcal{S}^+$ and $\mathcal{S}^-$ are invariant under Cl_4^0, i.e.:

$$Cl_4^0 = End(\mathcal{S}^+) \oplus End(\mathcal{S}^-)$$

as $Spin(4)$-modules. Moreover each $\mathbf{x} \in \mathbf{R}^4 \subset \mathbf{C}l_4$ gives isomorphisms through Clifford multiplication:

$$\mathbf{x} : \mathcal{S}^- \to \mathcal{S}^+ \qquad\qquad \mathbf{x} : \mathcal{S}_+ \to \mathcal{S}^- \tag{3.59}$$

which we denote by $\mathbf{x} : \psi \mapsto \mathbf{x} \cdot \psi, \quad \mathbf{x} \in \mathbf{R}^4, \psi \in \mathcal{S}^{\pm}$.

In fact, the representations $\mathcal{S}^{\pm}$ are exactly the 2-dimensional complex spin representations $D^{\pm}_{1/2}$ mentioned above.

Now let us see what happens at the Lie algebra level. We know that $spin(4) = Lie(Spin(4))$ is the Lie subalgebra of $(Cl_4, [,])$ generated by $\gamma_i\gamma_j$, $i < j$, which is of course isomorphic to $\wedge^2\mathbf{R}^4$ (see [LM], prop.6.1):

$$spin(4) = \wedge^2\mathbf{R}^4 = span_{\mathbf{R}}\{\gamma_i\gamma_j\}_{i<j}$$

through the (non canonical) linear map defined by:

$$\mathbf{e}_i \wedge \mathbf{e}_j \mapsto \iota(\mathbf{e}_i)\iota(\mathbf{e}_j) = \gamma_i\gamma_j \qquad\qquad i < j \tag{3.60}$$

Meanwhile, the Lie algebra $so(4)$ is:

$$so(4) = \{A : \mathbf{R}^4 \to \mathbf{R}^4 : \text{ A is linear and skew symmetric}\}$$

and there exists a natural isomorphism $\wedge^2\mathbf{R}^4 \cong so(4)$, induced by associating to a pair of vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^4$ the skew symmetric endomorphism "$\mathbf{v} \wedge \mathbf{w}$" defined by:

$$(\mathbf{v} \wedge \mathbf{w})(\mathbf{x}) =< \mathbf{v}, \mathbf{x} > \mathbf{w} - < \mathbf{w}, \mathbf{x} > \mathbf{v} \tag{3.61}$$

In particular $\mathbf{e}_i \wedge \mathbf{e}_j$, for $i < j$, corresponds to the elementary skew-symmetric matrix E_{ij}, with -1 in (i, j)-entry, 1 in (j, i)-entry and all others 0. These matrices form the standard basis of $so(4)$. These together with (3.60) shows that:

$$spin(4) = \wedge^2\mathbf{R}^4 = so(4) \tag{3.62}$$

Note however that the Lie algebra isomorphism:

$$\Psi : spin(4) \longrightarrow so(4)$$

induced by the adjoint representation $Ad : Spin(4) \to SO(4)$ is given explicitly on the basis elements $\{\gamma_i\gamma_j\}_{i<j}$ by (see [LM], prop. 6.2):

$$\Psi(\gamma_i\gamma_j) = 2\,\mathbf{e}_i \wedge \mathbf{e}_j \tag{3.63}$$

and consequently for $\mathbf{v}, \mathbf{w} \in \mathbf{R}^4$:

$$\Psi^{-1}(\mathbf{v} \wedge \mathbf{w}) = \frac{1}{4}[\mathbf{v}, \mathbf{w}] \tag{3.64}$$

Now recall that the Hodge star operator $* : \wedge^2 = \wedge^2\mathbf{R}^4 \to \wedge^2$ defined by:

$$\alpha \wedge *\beta = (\alpha, \beta)\,\omega \qquad \alpha, \beta \in \mathbf{R}^4$$

verifies $*^2 = 1$ and so we can decompose $\wedge^2 = \wedge^2\mathbf{R}^4$ in **self dual** and **anti-self-dual** bivectors:

$$\wedge^2 = \wedge^2_+ \oplus \wedge^2_-$$

with each of the subspaces $\wedge^2_\pm = \frac{1}{2}(1 \pm *)\wedge^2$ being (through (3.61)) a 3-dimensional space of skew symmetric matrices which we identify to $so(3) = su(2)$. The basis for $\wedge^2_\pm$ are respectivelly:

$$\{\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4, \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_4 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3\}$$

and:

$$\{\mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4, \mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_4 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3\}$$

So we have the following identifications:

$$\begin{aligned} spin(4) &= so(4) \\ &= \wedge^2 \\ &= \wedge^2_+ \oplus \wedge^2_- \\ &= su(2) \oplus su(2) \end{aligned} \tag{3.65}$$

Through (3.64) the action of an elementary transformation $\mathbf{v} \wedge \mathbf{w} \in so(4) = \wedge^2$ on the spinor space $\mathcal{S}$ is given by $\frac{1}{4}[\mathbf{v}, \mathbf{w}]\cdot$ where $\cdot$ is Clifford module multiplication on $\mathcal{S}$. In particular we can prove that:

$$(\wedge^2_\pm)^{\mathbf{C}} = [Hom(\mathcal{S}^\pm, \mathcal{S}^\pm)]^o$$

where $.^o$ denotes the component of traceless matrices. The real parts $\wedge^2_\pm$ consists of traceless skew-hermitian of $\mathcal{S}^\pm \cong \mathbf{C}^2$.

Moreover, since $\mathcal{S}^+ \cong (\mathcal{S}^+)^*$ symplectically, we also have that:

$$(\wedge^2_+)^{\mathbf{C}} = Sym^2\,\mathcal{S}^+ \tag{3.66}$$

3.4 $U(2)$, spinors and almost complex structures

If we fix a nonzero spinor $\phi \in \mathcal{S}^+$, then this gives rise to a real isomorphism $\mathbf{R}^4 \cong \mathcal{S}^- = \mathbf{C}^2$, given by Clifford multiplication: $\mathbf{x} \mapsto \mathbf{x}\cdot\phi$, and so identifies $\mathbf{R}^4$ with a complex vector space, i.e., furnishes $\mathbf{R}^4$ with a (almost) complex structure $J_\phi \in End(\mathbf{R}^4)$ wich corresponds with the multiplication by i in the cited identification $\mathbf{R}^4 \cong \mathbf{C}^2$:

$$J_\phi\mathbf{x} \cdot \phi = i(\mathbf{x} \cdot \phi) \qquad \mathbf{x} \in \mathbf{R}^4$$

This J_ϕ is compatible with the metric (is ortoghonal) and orientation. Moreover, multiplying $\phi \in \mathcal{S}^+ = \mathbf{C}^2$ by a nonzero scalar $\lambda \in \mathbf{C}^*$ defines the same complex structure: $J_{\lambda\phi} = J_\phi$. Thus the projective space:

$$P(\mathcal{S}^+) \cong \mathbf{C}P(1)$$

parametrizes a set of compatible complex structures in $\mathbf{R}^4$.

The subgroup of $Spin(4) = SU(2) \times SU(2)$ which leaves fixed ϕ up to a scalar multiple, is $S^1 \times SU(2)$, the double covering of $U(2) \subset SO(4)$. Hence the projective space $P(\mathcal{S}^+) \cong \mathbf{C}P(1)$ is naturally isomorphic to $SO(4)/U(2)$, the space of all complex structures compatible with the metric and orientation.

There exists a dual way of looking at this, where we take not the Clifford multiplication map $\mathbf{R}^4 \times \mathcal{S}^+ \to \mathcal{S}^-$ but its adjoint:

$$\Pi : \mathcal{S}^- \longrightarrow \mathbf{R}^4 \times \mathcal{S}^+ \tag{3.67}$$

defined by:

$$\Pi : \psi \mapsto \sum_i \mathbf{e}_i \cdot \psi \otimes \mathbf{e}_i \tag{3.68}$$

Now, if we are given any $\phi \in \mathcal{S}^+$, we get a map $\Pi_\phi : \mathcal{S}^- \to (\mathbf{R}^4)^{\mathbf{C}} = \mathbf{C}^4$ given by:

$$\Pi_\phi : \psi \mapsto \sum_i \epsilon(\mathbf{e}_i \cdot \psi, \phi)\, \mathbf{e}_i \tag{3.69}$$

where ϵ is the sympletic form on $\mathcal{S}^+ = \mathbf{C}^2$. The image $\Pi_\phi(\mathcal{S}^-)$ in $\mathbf{C}^4$ is the subspace of holomorphic vectors $T^{(1,0)}$ which equivalently defines the complex structure.

3.5 $Spin^c(4)$

All the preceding discussion can be extended to the complex case. We define the main involution α and the transposition $()^t$ on $Cl_4 \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{H}(2) \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}(4)$, the algebra of complex (4×4)-matrices, by:

$$\begin{aligned} \alpha(\varphi \otimes z) &= \alpha(\varphi) \otimes z \\ (\varphi \otimes z)^t &= \varphi^t \otimes \overline{z} \qquad\qquad \varphi \otimes z \in Cl_4 \otimes \mathbf{C} \end{aligned} \tag{3.70}$$

and we define $N^c(\varphi \otimes z) = N(\varphi)|z|^2, \quad \Phi \in Cl_k \otimes_{\mathbf{R}} \mathbf{C}$.

Definition **4** ...

We define Γ_4^c as the subgroup of all invertible elements $\Phi = \varphi \otimes z \in Cl_4 \otimes_{\mathbf{R}} \mathbf{C}$, for which:

$$\mathbf{x} \in \mathbf{R}^4 \quad \Longrightarrow \quad \widetilde{Ad}_\Phi(\mathbf{x}) \equiv \alpha(\Phi)\mathbf{x}\,\Phi^{-1} \in \mathbf{R}^4$$

Theorem **6** ... *([ABS], prop. 3.17)*

Let $Pin^c(4)$ be the kernel of $N^c : \Gamma_4^c \to \mathbf{C}^*$. Then we have an exact sequence:

$$\mathbb{1} \to U(1) \to Pin^c(4) \xrightarrow{\widetilde{Ad}} O(4) \to \mathbb{1}$$

where $U(1) = \{\mathbb{1} \otimes z \in Cl_4 \otimes \mathbf{C} : \ |z| = 1\}$. In particular we have a natural isomorphism:

$$Pin^c(4) \cong Pin(4) \times_{\mathbf{Z}_2} U(1) \cong Pin^c(4) \tag{3.71}$$

where $\mathbf{Z}_2$ acts on $Pin(4)$ and $U(1)$ as ± 1.

Definition 5 ...

We define the group $Spin^c(4)$ *as the inverse image of* $SO(4)$ *under the homomorphism* $Pin^c(4) \to O(4)$ *of the previous theorem. It follows that:*

$$\begin{aligned} Spin^c(4) &\cong Spin(4) \times_{\mathbf{Z}_2} U(1) \\ &= (SU(2) \times SU(2)) \times_{\mathbf{Z}_2} U(1) \end{aligned} \tag{3.72}$$

The group $Spin^c(4)$ is usefull to understand the relation between spinors and complex structures. In fact a given $U(2)$-PFbundle over a 4-manifold is an $SO(4)$-PFbundle under the natural embedding:

$$\iota : U(2) \hookrightarrow SO(4)$$

However, this mapping may not lift to $Spin(4)$. Thus the existence of a complex structure on a bundle of rank 4 does not necessarily yield a *Spin*-bundle. However it does yield a $Spin^c$-structure, a less restrictive requirement!

In fact the homomorphism:

$$l : U(2) \to SO(4) \times U(1)$$

defined by:

$$l(T) = \iota(T) \times \det T$$

does lift to $Spin^c(4)$: explicitly, the lifted map $\tilde{l} : U(2) \to Spin^c(4)$ is given as follows. Let $T \in U(2)$ be expressed relative to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbf{C}^2$ by the diagonal matrix:

$$T = \begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{bmatrix}$$

Let $\{\mathbf{e}_1, i\mathbf{e}_1, \mathbf{e}_2, i\mathbf{e}_2\}$ the corresponding basis of $\mathbf{R}^4$. Then:

$$\tilde{l}(T) = (\cos\frac{\theta_1}{2} + \sin\frac{\theta_1}{2} \cdot \mathbf{e}_1 i\mathbf{e}_1)(\cos\frac{\theta_2}{2} + \sin\frac{\theta_2}{2} \cdot \mathbf{e}_2 i\mathbf{e}_2) \times e^{\frac{i(\theta_1+\theta_2)}{2}}$$

Thus any $U(2)$-frame bundle on a 4-manifold M induces a $Spin^c(4)$-structure on M. In certain cases we shall be able to see that this $Spin^c(4)$-structure reduces to a $Spin(4)$-structure on certain real submanifolds of M. We can prove that:

Theorem 7 ...

If $H^2(M, \mathbf{Z}) = 0$ *then any* $Spin^c(4)$*-bundle can be reduced to* $Spin(4)$*-bundle over* M.

Chiral Operator. Self Duality

Definition 6 ... *Choose an orientation for* $\mathbf{R}_{r,s}$ *and let* $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ *a positively oriented q-orthonormal basis* $(n = r + s)$*. We define the associated* **"Volume element"** *by:*

$$\omega \overset{def}{=} \mathbf{e}_1 ... \mathbf{e}_n \in Cl_{r,s} \tag{3.73}$$

It's easy to see that ω doesn't depend of the choice of the positively oriented q-orthonormal basis. Moreover, we have that:

$$\omega^2 = \begin{cases} (-1)^s \mathbb{1} & \text{if} \quad n \equiv 0,3 \pmod 4 \\ (-1)^{s+1} \mathbb{1} & \text{if} \quad n \equiv 1,2 \pmod 4 \end{cases} \tag{3.74}$$

and:

$$\mathbf{x}\omega = (-1)^{n-1}\omega\mathbf{x} \qquad \forall \mathbf{x} \in \mathbf{R}^n \tag{3.75}$$

In particular, if n is odd, then ω is central, while if n is even, then:

$$h\omega = \omega\,\alpha(h) \qquad \forall h \in Cl_{r,s} \tag{3.76}$$

i.e., ω super-commutes with h. If $\rho : Cl_{r,s} \to End_{\mathbf{K}}(\mathcal{W})$ is a $\mathbf{K}$-representation, then $\Omega \stackrel{\text{def}}{=} \rho(\omega)$ is called the associated *"Chiral operator"*.

Definition 7 ... *Assume that $\omega^2 = \mathbb{1}$, in $Cl_{r,s}$. Then an element $h \in Cl_{r,s}$ is called "self-dual" if $\omega h = h$, and it's called "anti-self-dual" if $\omega h = -h$*

If we assume that $\omega^2 = \mathbb{1}$ and n odd, then ω is central, and we have a decomposition of $Cl_{r,s}$ in a direct sum:

$$Cl_{r,s} = Cl^+_{r,s} \oplus Cl^-_{r,s} \tag{3.77}$$

of isomorphic (self-dual and anti-self-dual) subalgebras:

$$Cl^{\pm}_{r,s} \stackrel{\text{def}}{=} \{h \in Cl_{r,s} : \omega\, h = \pm h\} = \frac{\mathbb{1} \pm \omega}{2}\, Cl_{r,s}$$

Moreover $\alpha(Cl^{\pm}_{r,s}) = Cl^{\mp}_{r,s}$.

Table 1... Clifford Algebras $CL_{r,s}$. In each case N is computed knowing that $r+s=n$ and the real dimension of $CL_{r,s}$ is 2^n:

$r-s \pmod 8$	$Cl_{r,s}$
0,6	$\mathbf{R}(N)$
2,4	$\mathbf{H}(N)$
1,5	$\mathbf{C}(N)$
3	$\mathbf{H}(N) \oplus \mathbf{H}(N)$
7	$\mathbf{R}(N) \oplus \mathbf{R}(N)$

Table 2... The even part $CL^{(0)}_{r,s}$ of the Clifford Algebras $CL_{r,s}$. In each case N is computed knowing that $r+s=n$ and the real dimension of $CL^{(0)}_{r,s}$ is 2^{n-1}:

$r-s \pmod 8$	$Cl^{(0)}_{r,s}$
0	$\mathbf{R}(N) \oplus \mathbf{R}(N)$
1,7	$\mathbf{R}(N)$
3,5	$\mathbf{H}(N)$
2,6	$\mathbf{C}(N)$
4	$\mathbf{H}(N) \oplus \mathbf{H}(N)$

Example ... ...

$$Cl^{(0)}_{2,0} = Cl^{(0)}_{0,2} = \mathbf{C} \qquad Cl^{(0)}_{1,1} = \mathbf{R} \oplus \mathbf{R}$$
$$Cl^{(0)}_{3,1} = Cl^{(0)}_{1,3} = \mathbf{C}(2) \quad Cl^{(0)}_{4,0} = Cl^{(0)}_{0,4} = \mathbf{H} \oplus \mathbf{H} \quad Cl^{(0)}_{2,2} = \mathbf{R}(2) \oplus \mathbf{R}(2)$$

Table 3... Decomposition in self-dual, anti-self-dual parts

$r - s \ \ (mod\, 8)$	$Cl^{(0)}_{r,s}$
0	$\mathbf{R}(N) \oplus \mathbf{R}(N)$
1,7	$\mathbf{R}(N)$
3,5	$\mathbf{H}(N)$
2,6	$\mathbf{C}(N)$
4	$\mathbf{H}(N) \oplus \mathbf{H}(N)$

Definition 8 ... *A* **"Pinnor inner product"** ϵ *is an inner product on the pinor space* $\mathcal{P}_{r,s}$ *with the property that the adjoint with respect to* ϵ *is the conjugation involution on* $Cl_{r,s}$*, i.e.:*

$$\epsilon(h \cdot \phi, \psi) = \epsilon(\phi, h^* \cdot \psi) \qquad h \in Cl_{r,s} \qquad \phi, \psi \in \mathcal{P}_{r,s} \tag{3.78}$$

In particular $\epsilon(\mathbf{x} \cdot \phi, \psi) = \epsilon(\phi, -\mathbf{x} \cdot \psi)$, $\forall \mathbf{x} \in \mathcal{V}$. We can prove that there always exists such a inner product which is unique up to a change of scale.

Now we define a symmetric bilinear $\mathcal{V}$-valued mapping on $\mathcal{S}_{r,s}$:

$$\{\, ,\, \} : \mathcal{S}_{r,s} \otimes \mathcal{S}_{r,s} \longrightarrow \mathcal{V}$$

by defining $\{\phi, \psi\} \in \mathcal{V}$ as the unique vector in $\mathcal{V}$ such that its inner product with any $\mathbf{x} \in \mathcal{V}$ is equal to $\epsilon(\mathbf{x} \cdot \phi, \psi)$:

$$< \{\phi, \psi\}, \mathbf{x} > \stackrel{\text{def}}{=} \epsilon(\mathbf{x} \cdot \phi, \psi) \qquad \forall \mathbf{x} \in \mathcal{V} \tag{3.79}$$

We can prove that $\{\, ,\, \}$ is in fact symmetric:

$$\begin{aligned} < \{\phi, \psi\}, \mathbf{x} > &= \epsilon(\mathbf{x} \cdot \phi, \psi) \\ &= \epsilon(\phi, -\mathbf{x} \cdot \psi) \\ &= \epsilon(\mathbf{x} \cdot \psi, \phi) \qquad \text{since } \epsilon \text{ is skew} \\ &= < \{\psi, \phi\}, \mathbf{x} > \qquad \forall \mathbf{x} \in \mathcal{V} \end{aligned}$$

To construct Lie superalgebras $\mathfrak{G} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, wih the even part $\mathfrak{g}_0 = \mathfrak{spin}_{r,s} \oslash \mathcal{V}$, the semi-direct sum of the Lie algebra of $Spin_{r,s}$ with its fundamental representation $\mathcal{V} \cong \mathbf{R}^{r+s}$, we choose a spinor space which is the carrier of a representation of $\mathfrak{spin}_{r,s}$ and define the anticommutator of two pinors by the bove formula. It remains to prove that:

$$\Lambda \cdot \{\phi, \psi\} = \{\Lambda \cdot \phi, \psi\} + \{\phi, \Lambda \cdot \psi\} \qquad \forall \Lambda \in \mathfrak{spin}_{r,s}, \quad \forall \phi, \psi \in \mathcal{S} \tag{3.80}$$