Chen Integrals, Generalized Loops and Loop Calculus

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Abstract

We use Chen iterated line integrals to construct a topological algebra $\mathcal{A}_p$ of separating functions on the *Group of Loops* $\mathbf{L}\mathcal{M}_p$. $\mathcal{A}_p$ has an Hopf algebra structure which allows the construction of a group structure on its spectrum. We call this topological group, the group of generalized loops $\widetilde{\mathbf{L}\mathcal{M}_p}$.

Then we develope a *Loop Calculus*, based on the *Endpoint* and *Area Derivative Operators*, providing a rigorous mathematical treatment of early heuristic ideas of Gambini, Trias and also Mandelstam, Makeenko and Migdal. Finally we define a natural action of the "pointed" diffeomorphism group $Diff_p(\mathcal{M})$ on $\widetilde{\mathbf{L}\mathcal{M}_p}$, and consider a *Variational Derivative* which allows the construction of homotopy invariants.

This formalism is useful to construct a mathematical theory of *Loop Representation* of Gauge Theories and Quantum Gravity.

Keywords

Iterated (line) Integral. Generalized loop. Group of Loops.
Endpoint Derivative. Area Derivative.
Variational Derivative.

1 Introduction

Let $\mathcal{PM}$ (resp. $\mathcal{LM}$) be the Path space (resp., the Loop space) of a smooth manifold $\mathcal{M}$. On $\mathcal{PM}$ (resp. $\mathcal{LM}$), we consider a "sufficiently large" class of functionals, constructed through the so called Chen iterated (line) integrals. This kind of integrals have been introduced, a few years ago, by K.T.Chen, who used them extensively as an useful tool in studying several algebraic topological aspects of Path spaces (see [Chen 1,2,3,4]).

Here we consider them as a class of separating functions on the so called *Group of Loops* $\mathbf{L}\mathcal{M}_p$, consisting of equivalence classes of picewise regular loops based at p, under a *"retracing"* equivalence. This group has been considered many years ago by Lefschetz and mainly by Teleman, who has studied its representations to obtain a reconstruction theory for principal fiber bundles with connection (see [T]). In this approach, $\mathbf{L}\mathcal{M}_p$ plays the role of the fundamental group in classification of flat bundles. More recently, Barrett and Lewandowsky have improved Teleman's work establishing interesting connections with gauge theories (see [B],[Lew]).

We endow the algebra $\mathcal{A}_p$, generated by the above mentioned iterated integrals, with the structure of topological algebra. This algebra has also a natural Hopf algebra structure, which allows to introduce on its spectrum, a structure of topological group. We call this group the *Group of Generalized Loops* and we denote it by $\widetilde{\mathbf{L}\mathcal{M}_p}$. It follows that $\mathbf{L}\mathcal{M}_p$ is now embedded as a subgroup of $\widetilde{\mathbf{L}\mathcal{M}_p}$, and so it is a topological group, with the induced topology. This topology is well fited to traduce the intuitive idea that two loops are close if they differ by a "small area", and gives a correct setting to develope several "directional" derivatives that are defined in section 4. The relation of generalized loops to smooth loops resembles that of distributions to smooth functions or that of DeRham currents to smooth chains.

The concept of generalized loop appears already in [Chen 5], where some related algebraic aspects are considered. Here, however, we use affine group theory [Ab], which substancially simplifies the treatment and allows us to carry on much farther the analogy with Lie Group Theory.

During the preparation of this work, we have received a preprint of Di Bartolo, Gambini, Griego (see [BGG]), in which it is constructed an *Extended Loop Group* that, apparently at least, has similarities with our $\widetilde{\mathbf{L}\mathcal{M}_p}$. However (we believe that) these authors use a substantially different formalism from ours, and thus, for the moment it is not very clear for us, wether the two objects are really the same.

This work has grown as an effort to understand, from a mathematical viewpoint, early work developed by Mandelstam, Makeenko, Migdal and mainly Gambini, Trias (see [Man],[MM],[GT1],[GT Thus, conceptually, most of the ideas here formalized, are present in one or another form in the work of the above mentioned authors. Here, however, the emphasis is in a mathematical framework which can be used to construct a mathematical theory of loop representation of gauge theories (see [L], for a recent review) and quantum gravity (in Ashtekar-Rovelli-Smolin formulation, see [Ash],[RS],[G2]). In particular, our constructions are explicitly free of any background metric structure and are really intrinsic.

The present paper is organized as follows. In section 2, we briefly review the definition and the main properties of Chen iterated line integrals, establishing as well, their relationship with geometry, via the holonomy of a connection in a principal fiber bundle, following closely the exposition of Chen's ideas in [Hain]. In section 3, we define a topological algebra $\mathcal{A}_p$, generated by the iterated integrals, which furthermore has a natural Hopf algebra structure. Then, inspired by the theory of affine groups, we define the group of generalized loops $\widetilde{\mathbf{L}\mathcal{M}_p}$, as well as its Lie algebra $\widetilde{l\mathcal{M}_p}$. In section 4, we develope a *Loop Calculus* based on the introduction of several operators. So, we define *Endpoint Derivatives* and *Area Derivatives* and establish sufficient machinery that turn them useful for efective computations. At a conceptual level, most of these operators appear heuristically formulated in early work of Mandelstam, Makeenko, Migdal and mainly Gambini, Trias (we feel that our approach, in section 4, is more close to that of these last two authors). However, here we put emphasis in the mathematical foundations of what we call *Loop Calculus*. In fact, our aim is to apply these loop calculus in a well formulated mathematical theory of loop representation of gauge theories and quantum gravity, subjects that are now under investigation, and will be published elsewhere. Finally, in section 5, we analyze the natural action of the "pointed" diffeomorphism group $Diff_p(\mathcal{M})$ on $\widetilde{\mathbf{L}\mathcal{M}_p}$, and consider a *Variational Derivative* which allows the construction of some simple homotopy invariants. We hope that this framework will be also useful in reconstruction theory of principal fiber bundles with connection and (why not?!) knot theory.

2 Chen Iterated Line Integrals. Definition and main Properties

Let $\mathcal{M}$ be a smooth n-dimensional manifold. Denote by $\mathcal{PM}$ the set of picewise smooth paths $\gamma : I \to \mathcal{M}$. Given real 1-forms $\omega_1, ..., \omega_r$ in $\bigwedge^1 \mathcal{M}$ and a path $\gamma \in \mathcal{PM}$, we define the iterated (line) integrals inductivelly, as follows:

$$\begin{aligned}
\int_\gamma \omega_1 &= \int_0^1 \omega_1(t)dt \\
\int_\gamma \omega_1\omega_2 &= \int_0^1 (\int_0^t \omega_1(s)ds)\omega_2(t)dt \\
&= \int_0^1 (\int_{\gamma^t} \omega_1)\omega_2(t)dt \\
&\cdots\cdots\cdots \\
\int_\gamma \omega_1...\omega_r &= \int_0^1 (\int_{\gamma^t} \omega_1...\omega_{r-1})\omega_r(t)dt \qquad (1)
\end{aligned}$$

where we have used the notations $\omega_k(t) \equiv \omega_k(\gamma(t)).\dot\gamma(t)$ and $\gamma^t : I \to \mathcal{M}$ defined by $\gamma^t(s) \equiv \gamma(ts)$, for a fixed $t \in I$.

Each iterated integral will be considered as a function $X^{\omega_1...\omega_r} : \mathcal{PM} \to R$ defined by:

$$X^{\omega_1...\omega_r}(\gamma) = \int_\gamma \omega_1...\omega_r. \tag{2}$$

Note that the above definitions work equally well for 1-forms on $\mathcal{M}$ with values in an associative algebra $\mathbf{A}$ (p.ex., C or $gl(p) = gl(p, C)$, the algebra of $p \times p$ complex matrices). Of course in this case the functions $X^{\omega_1...\omega_r}$ take values on $\mathbf{A}$. So, for example, $X^{\omega_1\omega_2}(\alpha) = \int_\alpha \omega_1\omega_2$, with $\omega_1, \omega_2 \in \bigwedge^1 \mathcal{M} \otimes gl(p)$ (i.e., ω_1, ω_2 are two matrices of 1-forms in $\mathcal{M}$), denotes the matrix in $gl(p)$:

$$\begin{aligned} \left(\int_\alpha \omega_1\omega_2\right)^i_j &= \int_\alpha (\omega_1)^i_k \otimes (\omega_2)^k_j \\ &= \int_\alpha (\omega_1)^i_k (\omega_2)^k_j \end{aligned} \tag{3}$$

and analousgly for $\int \omega_1\omega_2...\omega_r$. (We have used notation (5) in (3)).

Let us now state the main properties of those iterated integrals.

2.1 Proposition

$\int_\gamma \omega_1...\omega_r$ is independent on orientation preserving reparametrizations of γ.

Before stating other properties, let us define the so called *Shuffle Algebra of $\mathcal{M}$*. Let $\mathbf{k} = R$ or C. Consider the $\mathbf{k}$-vector space $\bigwedge^1 \mathcal{M}$ of $\mathbf{k}$-1-forms on $\mathcal{M}$, and the tensor algebra (over $\mathbf{k}$) of $\bigwedge^1 \mathcal{M}$:

$$\mathcal{T}(\bigwedge{}^1 \mathcal{M}) = \bigoplus_{r\geq 0} (\bigotimes^r \bigwedge{}^1 \mathcal{M}) \tag{4}$$

For simplicity we use the notation:

$$\omega_1...\omega_r = \omega_1 \otimes ... \otimes \omega_r \in \bigotimes^r \bigwedge{}^1 \mathcal{M} \tag{5}$$

for $r \geq 1$, and set $\omega_1...\omega_r = 1$, when $r = 0$. Now we replace the tensor multiplication in $\mathcal{T}(\bigwedge^1 \mathcal{M})$ by the *shuffle multiplication* $\bullet$, defined by:

$$\omega_1...\omega_r \bullet \omega_{r+1}...\omega_{r+s} = \sum\nolimits'_\sigma \omega_{\sigma(1)}...\omega_{\sigma(r)} \tag{6}$$

where $\sum'_\sigma$ denotes sum over all (r, s)-shuffles, i.e., permutations σ of $r + s$ letters with $\sigma^{-1}(1) < ... < \sigma^{-1}(r)$ and $\sigma^{-1}(r + 1) < ... < \sigma^{-1}(r + s)$.

For example:

$$\begin{aligned} \omega_1 \bullet \omega_2 &= \omega_1\omega_2 + \omega_2\omega_1 \\ \omega_1 \bullet \omega_2\omega_3 &= \omega_1\omega_2\omega_3 + \omega_2\omega_1\omega_3 + \omega_2\omega_3\omega_1. \end{aligned}$$

$(\mathcal{T}(\bigwedge^1\mathcal{M}),\bullet)$ is then an associative, graded commutative $\mathbf{k}$-algebra, with unity $1\in\mathbf{k}\subset\mathcal{T}(\bigwedge^1\mathcal{M})$, which is called the *Shuffle Algebra* of $\mathcal{M}$ and is denoted by $Sh(\mathcal{M})$, or simply by Sh.

2.2 Proposition

$$\int_\gamma \omega_1...\omega_r.\int_\gamma \omega_{r+1}...\omega_{r+s} = \int_\gamma \omega_1...\omega_r\bullet\omega_{r+1}...\omega_{r+s} \tag{7}$$

for $\omega_1,...,\omega_r\in\bigwedge^1\mathcal{M}$.

For example, we have:

$$\begin{aligned}
\int_\gamma\omega_1.\int_\gamma\omega_2 &= \int_\gamma\omega_1\bullet\omega_2\\
&= \int_\gamma\omega_1\omega_2+\omega_2\omega_1\\
\int_\gamma\omega_1.\int_\gamma\omega_2\omega_3 &= \int_\gamma\omega_1\bullet\omega_2\omega_3\\
&= \int_\gamma\omega_1\omega_2\omega_3+\omega_2\omega_1\omega_3+\omega_2\omega_3\omega_1.
\end{aligned}$$

When $\omega_1,...,\omega_r\in\bigwedge^1\mathcal{M}\otimes gl(p)$, the same formula (7) holds, but in the RHS of (7), $\omega_1...\omega_r\bullet\omega_{r+1}...\omega_{r+s}$ means the product of the matrix $\omega_1...\omega_r$ by the matrix $\omega_1...\omega_{r+s}$, the entries being multiplied through the shuffle product $\bullet$ ($\omega_1...\omega_r$ means in turn, the product of the matrices $\omega_1,\omega_2,...,\omega_r$, the entries being now multiplied through $\otimes$).

From the equality $f(\gamma(t))=f(\gamma(0))+\int_{\gamma^t}df$, we deduce the following:

2.3 Proposition

For any $f\in C^\infty\mathcal{M}$ (resp. $f\in C^\infty\mathcal{M}\otimes gl(p)$), we have:

$$\int_\gamma df.\omega_1...\omega_r = \int_\gamma (f.\omega_1)\omega_2...\omega_r - f(\gamma(0)).\int_\gamma \omega_1...\omega_r \tag{8}$$

$$\int_\gamma \omega_1...\omega_r.df = \Big(\int_\gamma \omega_1...\omega_r\Big).f(\gamma(1)) - \int_\gamma \omega_1...\omega_{r-1}.(\omega_r.f) \tag{9}$$

$$\begin{aligned}\int_\gamma \omega_1...\omega_{i-1}.(df).\omega_{i+1}...\omega_r &= \int_\gamma \omega_1...\omega_{i-1}.(f.\omega_{i+1}).\omega_{i+2}...\omega_r \\ &\quad - \int_\gamma \omega_1....(\omega_{i-1}.f).\omega_i...\omega_r \end{aligned}\tag{10}$$

$$\begin{aligned}\int_\gamma \omega_1...\omega_{i-1}.(f.\omega_i).\omega_{i+1}...\omega_r &= f(\gamma(0)).\int_\gamma \omega_1...\omega_r \\ &\quad + \int_\gamma ((\omega_1...\omega_{i-1}) \bullet df).\omega_i...\omega_r \end{aligned}\tag{11}$$

for $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M}$ (resp., $\in \bigwedge^1 \mathcal{M} \otimes gl(p)$).

2.4 Proposition

If $\alpha, \beta \in \mathcal{PM}$, with $\alpha(1) = \beta(0)$, then:

$$\int_{\alpha.\beta} \omega_1...\omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1...\omega_i.\int_\beta \omega_{i+1}...\omega_r \tag{12}$$

and,

$$\int_{\alpha^{-1}} \omega_1...\omega_r = (-1)^r \int_\alpha \omega_r...\omega_1 \tag{13}$$

for $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M}$, and with the convention that $\int_\gamma \omega_1...\omega_r = 1$, if $r = 0$.

Moreover formula (12) holds with $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M} \otimes gl(p)$, while formula (13) must be substituted by:

$$\int_{\alpha^{-1}} \omega_1...\omega_r = (-1)^r \int_\alpha [\omega_r^T...\omega_1^T]^T \tag{14}$$

where ω^T means transpose of the matrix ω (note that here $A^T B^T$ is not equal to $[BA]^T$).

Now we analyse the separating properties of the functions $X^{\omega_1...\omega_r} : \mathcal{PM} \to R$, defined by (2). We shall not distinguish two paths in $\mathcal{PM}$, which differ only by orientation preserving reparametrization.

Two paths are called *elementary equivalent* if one of them can be written in the form $\alpha\beta\beta^{-1}\gamma$, and the other in the form $\alpha\gamma$. If $\alpha_1, ..., \alpha_s$ is a finite sequence of paths, such that α_i and α_{i+1} are elementary equivalent ($i = 1, ..., s-1$), then we say that α_1 is *equivalent* to α_s. We denote by $[\alpha]$ the equivalence class of α. A *picewise regular path* is a path in $\mathcal{PM}$ with nonvanishing tangent vectors. Finally, a *reduced path* is a picewise regular path that is not of the type $\alpha\beta\beta^{-1}\gamma$, for any β.

It's easy to see that the functions $X^{\omega_1...\omega_r}$ depend only on the equivalence class $[\alpha]$ of the path α. The following lemma is proved in [Chen 3]:

2.5 Lemma

Let α be a nonempty reduced picewise regular path in $\mathcal{PM}$. Then there exists 1-forms $\omega_1, ..., \omega_r$ in $\bigwedge^1 \mathcal{M}$, $r \geq 1$, such that:

$$X^{\omega_1 ... \omega_r}(\alpha) \neq 0$$

As a consequence, it was also proved that every picewise regular path is equivalent to one and only one reduced path. The next theorem answers the question how well iterated integrals separate paths (see [Chen 4]):

2.6 Theorem

Two picewise regular paths α, β are equivalent if and only if

$$X^{\omega_1 ... \omega_r}(\alpha) = X^{\omega_1 ... \omega_r}(\beta)$$

for any 1-forms $\omega_1, ..., \omega_r$ in $\bigwedge^1 \mathcal{M}$, $\quad r \geq 1$.

Proof...

($\Rightarrow$)...definitions.

($\Leftarrow$)... By the above lemma, we may assume that $X^{\omega_1 ... \omega_r}(\alpha) \neq 0$, for some $\omega_1, ..., \omega_r$ in $\bigwedge^1 \mathcal{M}$, $r \geq 1$. Otherwise, both α and β are equivalent to the empty reduced path, and the theorem follows.

Given any $f \in C^\infty \mathcal{M}$, we obtain from (8):

$$\int_\alpha (f\omega_1)\omega_2 ... \omega_r = \int_\alpha df \omega_1 ... \omega_r + f(\alpha(0)) \int_\alpha \omega_1 ... \omega_r \tag{15}$$

and an analogous formula for β holds, by the hypothesis. It follows that $f(\alpha(0)) = f(\beta(0))$, $\forall f \in C^\infty \mathcal{M}$, and so $\alpha(0) = \beta(0)$.

Now, $\beta^{-1}\alpha$ is a picewise regular path and using (12-13), we verify that every iterated integral vanishes along $\beta^{-1}\alpha$, which implies, again by the lemma, that $\beta^{-1}\alpha$ is equivalent to the reduced empty path, and so $\alpha \sim \beta$, QED.

There is an interesting relation between iterated integrals and geometry, via the parallel transport of a connection on a trivial bundle, that we can use to prove the above propositions (see [Hain]). In fact, assume that ∇ is a connection on the trivial bundle $R^p \times \mathcal{M} \to \mathcal{M}$, over $\mathcal{M}$. Sections of this bundle are identified with functions $s : \mathcal{M} \to R^p$, and its canonical framing e is given by the p constant functions:

$$e_i : M \to R^p \tag{16}$$

$i = 1, ..., p$, which take each point of $\mathcal{M}$ to the i^{th} standard basis vector of R^p. So, for a section $s : \mathcal{M} \to R^p$, we can write $s = \sum s^i e_i$, with $s^i \in C^\infty \mathcal{M}$.

As ∇e_i must be an R^p -valued 1-form on $\mathcal{M}$, we can write:

$$\nabla e_i = -\sum \omega_i^j e_j \tag{17}$$

where $\omega = (\omega_i^j)$, a $p \times p$ matrix of 1-forms on $\mathcal{M}$, is the connection form (associated to the framing e and the connection ∇). Now, for a section s as above, we have:

$$\begin{aligned}
\nabla s &= \nabla(s^i e_i) = s^i \nabla e_i + ds^i.e_i \\
&= s^i(-\omega_i^j e_j) + ds^i.e_i \\
&= (ds^j - s^i\omega_i^j)e_j \\
&= (ds - s.\omega)^j e_j
\end{aligned} \tag{18}$$

i.e., in matrix notation (s as a line-vector):

$$\nabla s = ds - s.\omega. \tag{19}$$

Conversely, if $\omega \in \bigwedge^1 \mathcal{M} \otimes gl(p)$ is a $p \times p$ matrix of 1-forms, we can define a connection on $R^p \times \mathcal{M} \to \mathcal{M}$, by (21).

This connection lifts to the principal fiber bundle $Gl(p) \times \mathcal{M} \to \mathcal{M}$, by defining:

$$\nabla S = dS - S.\omega \tag{20}$$

where $S : \mathcal{M} \to Gl(p)$. A section $S : I \to Gl(p)$, along a path $\gamma : I \to \mathcal{M}$, is called *horizontal* if:

$$dS(t) = S(t).\gamma^*\omega \tag{21}$$

$\forall t \in I$. Denoting by $A(t)dt = \gamma^*\omega$, (23) becomes:

$$S'(t) = S(t).A(t). \tag{22}$$

We define the parallel transport as the map $U : \mathcal{PM} \to Gl(p)$, $\gamma \mapsto U_\gamma$, defined as follows: $U_\gamma = S(1)$, where $S : I = [0,1] \to Gl(p)$ is the unique horizontal section along γ, i.e., the unique solution of the linear (nonautonomous) differential equation (24), satisfying $S(0) = Id$. By the general theory of linear ordinary differential equations, we know that U_γ is independent of the parametrization of γ and, if $\alpha, \beta \in \mathcal{PM}$, with $\alpha(1) = \beta(0)$, then $U_{\alpha.\beta} = U_\alpha U_\beta$.

We can give a formula for U_γ in terms of iterated integrals of the connection form $\omega = (\omega_j^i)$. First it's easy to prove that, for $\gamma \in \mathcal{PM}$, there exists a constant $M > 0$ such that:

$$\| \int_\gamma \underbrace{\omega\omega\cdots\omega}_{r} \| = O(\frac{M^r}{r!}) \tag{23}$$

and so, the series:

$$Id + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + ... \tag{24}$$

converges in $Gl(p)$. Moreover, U_γ is given exactly by this "chronological series" of iterated integrals (see [DF] and [Hain] for all this):

$$U_\gamma = Id + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + ... \tag{25}$$

Of special importance is the case where $\omega \in \bigwedge^1 \mathcal{M} \otimes \mathcal{N}_r$, where $\mathcal{N}_r$ denotes the Lie algebra of nilpotent upper triangular $(r+1)\times(r+1)$ matrices. In this case, the sum in (27) is finite. For example, if $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M}$ and:

$$\omega = \begin{bmatrix} 0 & \omega_1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \omega_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \omega_r \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

then,

$$U = \begin{bmatrix} 1 & \int \omega_1 & \int \omega_1\omega_2 & \int \omega_1\omega_2\omega_3 & \dots & \int \omega_1\omega_2...\omega_r \\ 0 & 1 & \int \omega_2 & \int \omega_2\omega_3 & \dots & \int \omega_2...\omega_r \\ 0 & 0 & 1 & \int \omega_3 & \dots & \int \omega_3...\omega_r \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \int \omega_r \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

We can use this, to prove most of the above mentioned properties of iterated integrals.

If $P(\mathcal{M}, G)$ is a principal fiber bundle over $\mathcal{M}$, with structure group G (a subgroup of $Gl(p)$), we can apply the same reasoning in each trivializing chart of that bundle. If γ is a loop in $\mathcal{M}$, we obtain, by the above construction, a (gauge dependent) transformation $U_\gamma \in G$, which is called the *Holonomy* of the connection ω around the loop γ.

Under a gauge transformation $g : \mathcal{U} \subseteq \mathcal{M} \to G$, we have that:

$$U^g_\gamma = g_x^{-1} U_\gamma g_x \tag{26}$$

where $x = \gamma(o)$, and so, we obtain a gauge independent loop functional $\mathcal{W} : \mathcal{LM} \to C$, defined by:

$$\mathcal{W}(\gamma) = Trace\, U_\gamma \tag{27}$$

which is usually called *Wilson loop variable.*

By continuity of the trace, we obtain from (27):

$$\mathcal{W}(\gamma) = \sum_{r\geq 0} Trace \int_\gamma \underbrace{\omega\omega\cdots\omega}_{r} \tag{28}$$

with the convention $\int_\gamma \underbrace{\omega\omega\cdots\omega}_{r} = Id$, if $r = 0$. So $\mathcal{W}(\gamma)$ is given by a convergent series of cyclic combinations of iterated integrals of the 1-forms ω^i_j in the connection matrix $\omega = (\omega^i_j)$.

3 The Group of Generalized Loops and its Lie Algebra

3.1 The Shuffle Algebra over $\bigwedge^1 \mathcal{M}$, as an Hopf Algebra

Recall that we have defined the *Shuffle Algebra* of $\mathcal{M}$, denoted by $Sh(\mathcal{M})$, or simply by Sh, as $(\mathcal{T}(\bigwedge^1\mathcal{M}), \bullet)$. Sh is then an associative, graded, commutative $\mathbf{k}$-algebra, with unity $1 \in \mathbf{k} \subset \mathcal{T}(\bigwedge^1\mathcal{M})$.

Sh has also a $\mathbf{k}$-Hopf algebra structure. This means (see [Ab], [Sw, Chp.XII]) that, in adition to the above $\mathbf{k}$-algebra stucture, we have two $\mathbf{k}$-linear maps $\Delta : Sh \to Sh \otimes Sh$, called *comultiplication*, and $\epsilon : Sh \to \mathbf{k}$, called *counity*, defined respectivelly, by the formulas:

$$\begin{aligned} \Delta(\omega_1 ... \omega_r) &= \sum_{i=0}^{r} \omega_1 ... \omega_i \otimes \omega_{i+1} ... \omega_r && (29) \\ \epsilon(\omega_1 ... \omega_r) &= 0, \quad if \;\; r \geq 1 && (30) \\ &= 1, \quad if \;\; r = 0 && (31) \end{aligned}$$

which verifies the following identities:

$$\begin{aligned} (\Delta \otimes 1) \circ \Delta &= (1 \otimes \Delta) \circ \Delta \qquad \text{(Coassociative law)} && (32) \\ (1 \otimes \epsilon) \circ \Delta &= (\epsilon \otimes 1) \circ \Delta = 1 \qquad \text{(counitary property)} && (33) \\ \Delta(\mathbf{u} \bullet \mathbf{v}) &= \Delta(\mathbf{u}) \bullet \Delta(\mathbf{v}) \qquad (\Delta \text{ is an algebra morphism}) && (34) \\ \epsilon(\mathbf{u} \bullet \mathbf{v}) &= \epsilon(\mathbf{u}) \bullet \epsilon(\mathbf{v}) \qquad (\epsilon \text{ is an algebra morphism}) && (35) \end{aligned}$$

$\forall \mathbf{u}, \mathbf{v} \in Sh$.

Moreover there is also a $\mathbf{k}$-linear map $J : Sh \to Sh$, called *antipode*, defined by:

$$J(\omega_1 ... \omega_r) = (-1)^r \omega_r ... \omega_1 \qquad (36)$$

which verifies:

$$\begin{aligned} s \circ (J \otimes 1) \circ \Delta &= s \circ (1 \otimes J) \circ \Delta = \eta \circ \epsilon && (37) \\ J(\mathbf{u} \bullet \mathbf{v}) &= J(\mathbf{v}) \bullet J(\mathbf{u}) && (38) \\ J(1) &= 1 && (39) \\ \epsilon \circ J &= \epsilon && (40) \\ \tau \circ (J \otimes J) \circ \Delta &= \Delta \circ J && (41) \\ J^2 &= 1 && (42) \end{aligned}$$

$\forall \mathbf{u}, \mathbf{v} \in Sh$. In (39), $s : Sh \otimes Sh \to Sh$ denotes shuffle multiplication, and $\eta : \mathbf{k} \to Sh$ the unit map. Finally, in (43), $\tau : Sh \otimes Sh \to Sh \otimes Sh$ is the transposition map $\tau(\mathbf{u} \otimes \mathbf{v}) = \mathbf{v} \otimes \mathbf{u}$.

Note that (39) means explicitly the following identity:

$$\begin{aligned} \sum_{i=0}^{r} (-1)^i \omega_i ... \omega_1 \bullet \omega_{i+1} ... \omega_r &= \sum_{i=0}^{r} (-1)^{r-i} \omega_1 ... \omega_i \bullet \omega_r ... \omega_{i+1} \\ &= \epsilon(\omega_1 ... \omega_r) && (43) \end{aligned}$$

We endow $Sh(\mathcal{M})$ with the structure of nuclear LMC algebra (see [Mal]) in the following way.

First we topologize the vector space $\bigwedge^1 \mathcal{M}$. Given a local chart $(\mathcal{U}, x)$ of $\mathcal{M}$, choose a nested sequence of compacts $\{K_m^{\mathcal{U}}\}_{m \geq 1}$, in $\mathcal{U}$, such that $\cup_{m \geq 1} K_m^{\mathcal{U}} = \mathcal{U}$ (this is always possible). Then we topologize the vector space $\bigwedge^1 \mathcal{U}$ through the familly of seminorms:

$$N_m^{\mathcal{U}}(\omega) = \max_{1 \leq i \leq n} \rho_m^{\mathcal{U}}(\omega_i) \qquad \omega \in \bigwedge^1 \mathcal{U} \qquad (44)$$

where m is a positive integer, $\omega = \sum_{i=1}^n \omega_i dx^i$ and:

$$\rho_m^{\mathcal{U}}(\omega_i) = \sup_{|p|\leq m} \big(\sup_{x\in K_m^{\mathcal{U}}} |D^p\omega_i(x)| \big) \tag{45}$$

for $\omega_i \in C^\infty\mathcal{U}$.

Each $\bigwedge^1\mathcal{U}$ becomes in this way a nuclear locally convex topological vector space.

Now, let $\{\mathcal{U}_k\}_{k\in N}$ be a basis for the topology of $\mathcal{M}$, consisting of local charts (this is always possible). The inclusions $i_k : \mathcal{U}_k \to \mathcal{M}$, induce linear maps $i_k^* : \bigwedge^1\mathcal{M} \to \bigwedge^1\mathcal{U}_k$. We endow $\bigwedge^1\mathcal{M}$ with the initial topology defined by this maps. By definition this is the weakest topology for which all the maps i_k^* are continuous, and a local basis consists of sets of the form $\cap_{j=1}^r (i_{k_j}^*)^{-1}(\mathcal{O}_{k_j})$, where the sets $\mathcal{O}_{k_j}$ run over local basis of $\bigwedge^1\mathcal{U}_{k_j}$. So, $\bigwedge^1\mathcal{M}$ becomes in this way a nuclear locally convex topological vector space, whose topology can be described by the family of seminorms:

$$p_{k,m,l}(\omega) = \max_{1\leq j\leq l} N_m^{\mathcal{U}_{k_j}} \left(i_{\mathcal{U}_{k_j}}^* \omega \right) \tag{46}$$

In this topology, a sequence of 1-forms $(\omega_k)_{k\geq 1}$, in $\bigwedge^1\mathcal{M}$, converges to zero iff, in a neighbourhood of each point of $\mathcal{M}$, each derivative of each coeficient of ω_k converges uniformly to zero.

Then, each tensor power $\bigotimes^r \bigwedge^1\mathcal{M}$ is topologized through the so called projective tensor product topology, i.e., through the seminorms $N_{k,m,l}^{(r)}$ which are the tensor product of the above ones (see [Mal], chpt X). For example, for an $\mathbf{u} \in \bigwedge^1\mathcal{M} \otimes \bigwedge^1\mathcal{M}$ we have:

$$N_{k,m,l}^{(2)}(\mathbf{u}) = inf \sum_{i=1}^n p_{k,m,l}(\omega^i).p_{k,m,l}(\eta^i) \tag{47}$$

where inf is taken over all expressions of the element $\mathbf{u}$ in the form $\mathbf{u} = \sum_{i=1}^n \omega^i \otimes \eta^i$. Finally, since an element in $\bigoplus_{r\geq 0}(\bigotimes^r \bigwedge^1\mathcal{M})$ is a finite sum $\mathbf{u} = \sum_r \mathbf{u}_r$, with $\mathbf{u}_r \in \bigotimes^r \bigwedge^1\mathcal{M}$, we use the seminorms:

$$\mathbf{N}_{k,m,l}(\mathbf{u}) = \sum_r N_{k,m,l}^{(r)}(\mathbf{u}_r) \tag{48}$$

to put a locally convex topology in $\mathcal{T}(\bigwedge^1\mathcal{M})$. Since the shuffle product is continuous, we obtain in this way a commutative LMC algebra which we continue to denote by $Sh(\mathcal{M})$, or simply by Sh.

3.2 The group $\mathrm{L}\mathcal{M}_p$ of Loops and the Algebra of Iterated Integrals

Fix a point $p \in \mathcal{M}$, and consider the Loop Space $\mathcal{LM}_p$ of picewise smooth loops based at p. $\mathcal{LM}_p$ is a semigroup with respect to the operation of justaposition of loops: $\alpha.\beta$, with $\alpha, \beta \in \mathcal{LM}_p$. Recall that $[\alpha]$ denotes the equivalence class of the loop α, under the equivalence relation $\sim$, defined in section 2.

Define an operation on the set $\mathcal{LM}_p/\sim$, by:

$$[\alpha] \diamond [\beta] = [\alpha.\beta] \tag{49}$$

which endows $\mathcal{LM}_p/\sim$ with the structure of group. The inverse of an element $[\alpha] \in \mathcal{LM}_p/\sim$, is given by $[\alpha]^{-1} = [\alpha^{-1}]$, and the unit element is given by $[p]$, the class of the constant loop equal to the point p. This group $(\mathcal{LM}_p/\sim, \diamond)$ is called the *Group of loops* of the manifold $\mathcal{M}$, based at p, and is denoted by $\mathbf{L}\mathcal{M}_p$.

Let $\mathcal{A}_p$ denote the algebra generated by all the functions $X^{\omega_1 \cdots \omega_r}$, defined by (2) and considered as functions on $\mathbf{L}\mathcal{M}_p$. Then proposition 2.2 says that the surjective map $Sh(\mathcal{M}) \to \mathcal{A}_p$, defined by $1 \mapsto 1$ and $\omega_1 ... \omega_r \mapsto X^{\omega_1 \cdots \omega_r}$ is a morphism of algebras. Moreover, Proposition 2.3 implies that the Kernel of this morphism, contains the ideal $\mathbf{I}_p$ generated by all the elements of type:

$$\omega_1 ... \omega_{i-1}(f\omega_i)\omega_{i+1} ... \omega_r - f(p)\omega_1 ... \omega_r - ((\omega_1 ... \omega_{i-1}) \bullet df)\omega_i ... \omega_r. \tag{50}$$

or, in short, by the elements of type:

$$\mathbf{u}(f\omega)\mathbf{v} - (\mathbf{u} \bullet df)\omega\mathbf{v} - f(p)\mathbf{u}\omega\mathbf{v} \tag{51}$$

for $\mathbf{u}, \mathbf{v} \in Sh, \quad \omega \in \bigwedge^1 \mathcal{M}, \quad f \in C^\infty \mathcal{M}$.

On the other hand, since $\int_\gamma df = 0$, for a $\gamma \in \mathcal{LM}$, this kernel also contains $dC^\infty(\mathcal{M})$. Denoting by $< dC >$ the ideal generated by $dC^\infty(\mathcal{M})$, in $Sh(\mathcal{M})$, and putting:

$$\mathbf{J}_p = \mathbf{I}_p + < dC > \tag{52}$$

then we have an algebra isomorphism (see [Chen 4]):

$$Sh(\mathcal{M})/\mathbf{J}_p \simeq \mathcal{A}_p. \tag{53}$$

The algebra $\mathcal{A}_p$ admits also a $\mathbf{k}$-Hopf Algebra structure, by defining the comultiplication $\Delta : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$, the counity $\epsilon : \mathcal{A}_p \to \mathbf{k}$ and the antipode $J : \mathcal{A}_p \to \mathcal{A}_p$, respectivelly by:

$$\begin{aligned}
\Delta\big(X^{\omega_1 \cdots \omega_r}\big) &= \sum_{i=0}^{r} X^{\omega_1 \cdots \omega_i} \otimes X^{\omega_{i+1} \cdots \omega_r} && (54)\\
\epsilon\big(X^{\omega_1 \cdots \omega_r}\big) &= 0 \qquad if \quad r \geq 1 && (55)\\
&= 1 \qquad if \quad r = 0 && (56)\\
J\big(X^{\omega_1 \cdots \omega_r}\big) &= (-1)^r X^{\omega_r \cdots \omega_1} && (57)
\end{aligned}$$

which verify the above formal properties (34) to (37), and (39) to (44).

Finally, we topologize the algebra $\mathcal{A}_p$ through the isomorphism (55), obtaining, in this way, a commutative LMC algebra, generated by the functions $X^{\omega_1 \cdots \omega_r} : \mathbf{L}\mathcal{M}_p \to \mathbf{k}$, which, moreover, has an aditional (commutative, noncocomutative) Hopf algebra structure. This is all we need to define the group of generalized loops, in the next section.

3.3 The Group $\widetilde{\mathbf{L}\mathcal{M}_p}$ of Generalized Loops

Let us consider the spectrum $\mathbf{\Delta}_p$ (or Gelfand space) of the algebra $\mathcal{A}_p$. By definition, $\mathbf{\Delta}_p$ consists of all nonnull continuous algebra homomorphisms (characters) $\phi : \mathcal{A}_p \to C$, endowed with the induced weak $\star$-topology (or Gelfand topology). Equivalently, a character of $\mathcal{A}_p$, is a continuous complex algebra homomorphism $\tilde{\alpha} : Sh(\mathcal{M}) \to C$, that vanishes on the closed ideal $\mathbf{J}_p$, given by (54).

3.3.1 Definition

A *Generalized Loop* based at $p \in \mathcal{M}$ is a character of the algebra $\mathcal{A}_p$ or, equivalently, a continuous complex algebra homomorphism $\tilde{\alpha} : Sh(\mathcal{M}) \to C$, that vanishes on the ideal $\mathbf{J}_p$.

So, the set of all generalized loops is $\mathbf{\Delta}_p$, with the Gelfand topology: a sequence $(\tilde{\alpha}_n)$ converges to $\tilde{\alpha}$, in $\mathbf{\Delta}_p$, iff:

$$\lim_{n\to\infty} \tilde{\alpha}_n(X^{\mathbf{u}}) = \tilde{\alpha}(X^{\mathbf{u}}) \qquad \forall \mathbf{u} \in Sh(\mathcal{M}). \tag{58}$$

We have a natural embedding of $\mathbf{L}\mathcal{M}_p$ into $\mathbf{\Delta}_p$, given by the "Dirac map" $\delta : \mathbf{L}\mathcal{M}_p \to \mathbf{\Delta}_p$, $[\alpha] \mapsto \delta_{[\alpha]}$ defined by:

$$\delta_{[\alpha]}(X^{\omega_1 \ldots \omega_r}) = X^{\omega_1 \ldots \omega_r}([\alpha]) \quad [\alpha] \in \mathbf{L}\mathcal{M}_p. \tag{59}$$

Since the functions $X^{\omega_1 \ldots \omega_r}$ separate "points" in $\mathbf{L}\mathcal{M}_p$ (Theorem 2.6), we see that this is an injective embedding. So, we identify $\mathbf{L}\mathcal{M}_p$ with its image, under δ, in $\mathbf{\Delta}_p$, and endow $\mathbf{L}\mathcal{M}_p$ with the induced topology. In this topology, a sequence $([\alpha]_n)$ converges to $[\alpha]$, in $\mathbf{L}\mathcal{M}_p$ iff $\lim_{n\to\infty} X^{\mathbf{u}}([\alpha]_n) = X^{\mathbf{u}}([\alpha]), \quad \forall \mathbf{u} \in Sh\mathcal{M}$.

Now we want to define a group structure in the set $\mathbf{\Delta}_p$ of generalized loops, and, for this, we use some facts about $\mathbf{k}$-affine groups that can be seen in detail in [Ab].

Thus, we define the group product $\tilde{\alpha} \star \tilde{\beta}$ through the so called *convolution* of the two elements $\tilde{\alpha}, \tilde{\beta} \in \mathbf{\Delta}_p$. Let us explain what this means. Denoting by $\mathcal{A}_p^*$ the $\mathbf{k}$-dual linear space of $\mathcal{A}_p$, we define a multiplication on $\mathcal{A}_p^*$ in the following way. First note that $\mathcal{A}_p^* \otimes \mathcal{A}_p^* \cong \big(\mathcal{A}_p \otimes \mathcal{A}_p\big)^*$ and then, compose with the dual of the comultiplication map, to obtain the required multiplication as follows:

$$\mathcal{A}_p^* \otimes \mathcal{A}_p^* \cong \big(\mathcal{A}_p \otimes \mathcal{A}_p\big)^* \xrightarrow{\Delta^*} \mathcal{A}_p^* \tag{60}$$

This multiplication is what we call the *convolution*, and is given by:

$$\tilde{\alpha} \star \tilde{\beta} = (\tilde{\alpha} \otimes \tilde{\beta}) \circ \Delta \tag{61}$$

where we have used the identification $\mathbf{k} \otimes \mathbf{k} \simeq \mathbf{k}$. More explicitly:

$$\tilde{\alpha} \star \tilde{\beta}(X^{\omega_1 \ldots \omega_r}) = \sum_{i=0}^{r} \tilde{\alpha}(X^{\omega_1 \ldots \omega_i}).\tilde{\beta}(X^{\omega_{i+1} \ldots \omega_r}). \tag{62}$$

We define also the inverse of $\tilde{\alpha} \in \mathbf{\Delta}_p$, by $\tilde{\alpha} \circ J$, i.e.:

$$\tilde{\alpha}^{-1}(\omega_1 \ldots \omega_r) = (-1)^r \tilde{\alpha}(\omega_r \ldots \omega_1) \tag{63}$$

and take ϵ, given by (57-58), as the unit element

Now, the algebraic part of the following theorem follows directly from [Ab, Th.2.1.5], while the topological part is of easy verification.

3.3.2 Theorem

$\tilde{\alpha} \star \tilde{\beta}$, $\tilde{\alpha}^{-1}$ and ϵ are generalized loops based at p, i.e., they are continuous characteres on the algebra $\mathcal{A}_p$, or equivalently, continuous characteres on the algebra $Sh(\mathcal{M})$ that vanish on the ideal $\mathbf{J}_p$.

Moreover, $(\mathbf{\Delta}_p, .)$ has the structure of topological (Hausdorff and completely regular) group.

3.3.3 Definition

We call the above mentioned topological group $(\mathbf{\Delta}_p, .)$, the *Group of Generalized Loops* of $\mathcal{M}$, based at $p \in \mathcal{M}$, and we denote it by $\widetilde{\mathbf{L}\mathcal{M}}_p$.

Note that under the identification given by the "Dirac map" $\delta : \mathbf{L}\mathcal{M}_p \to \mathbf{\Delta}_p$, (see (61)), $\mathbf{L}\mathcal{M}_p$ is a topological subgroup of $\widetilde{\mathbf{L}\mathcal{M}}_p$.

3.3.4 Note

Each $X^{\mathbf{u}} \in \mathcal{A}_p$ defines a continuous function $F_{X^{\mathbf{u}}} : \widetilde{\mathbf{L}\mathcal{M}}_p \to \mathbf{k}$ through:

$$F_{X^{\mathbf{u}}}(\tilde{\alpha}) \equiv X^{\mathbf{u}}(\tilde{\alpha})$$

Each such function is a *representative function* of $\widetilde{\mathbf{L}\mathcal{M}}_p$, i.e., the linear space generated by all left translates $\tilde{\alpha}.F_{X^{\mathbf{u}}}$ is finite dimensional. In fact:

$$\tilde{\alpha}.F_{X^{\mathbf{u}}} = F_{(1\otimes\tilde{\alpha})\Delta X^{\mathbf{u}}}$$

and, for each $X^{\mathbf{u}}$ fixed, the linear space generated by:

$$\{(1 \otimes \tilde{\alpha})\Delta X^{\mathbf{u}} : \tilde{\alpha} \in \widetilde{\mathbf{L}\mathcal{M}}_p\}$$

is finite dimensional.

3.3.5 Example

Let $\mathcal{M}$=S^1. Then it's easy to see that $\mathbf{L}S^1_p = Z$. However $\widetilde{\mathbf{L}\mathcal{S}}^1_p = R$. In fact, since $H^1(S^1, R) = R$, any 1-form ω in S^1 is equal to a constant multiple of $\omega_0 \equiv d\theta$ (the usual volume form in S^1), modulo an exact form: $\omega = c\omega_0 + df, c \in R$. So:

$$\wedge^1 S^1 = R\omega_0 \oplus dC^\infty(S^1) \tag{64}$$

From this fact, and using the relations (5-8), we can prove that $\mathcal{A}_p$ is isomorphic, as an Hopf algebra, to $R[t]$, the polynomial ring in one variable $t \leftrightarrow X^{\omega_0}$ (see [Chen5]). The Hopf operations on $R[t]$ are:

$$\Delta(t) = 1 \otimes t + t \otimes 1 \quad J(t) = -t \quad \epsilon(t) = 0 \tag{65}$$

Now, $\widetilde{\mathbf{L}\mathcal{S}}^1_p = R$, follows from Example 4.1 in [Ab, pag.172].

3.3.6 Note

Consider the Path space $\mathcal{PM}_p$ of paths based at $p \in \mathcal{M}$, and the algebra $\mathcal{B}_p$ generated by all the functions $X^{\omega_1 \cdots \omega_r}$, considered now as functions on $\mathcal{PM}_p$. In exactly the same way as in the previous case, (see [Chen 4]) there is an algebra isomorphism $Sh(\mathcal{M})/\mathbf{I}_p \simeq \mathcal{B}_p$, which allows to consider $\mathcal{B}_p$ as an LMC algebra and define *generalized paths*, based at p, as continuous characteres on $Sh(\mathcal{M})$, that vanish on $\mathbf{I}_p$.

3.4 The Lie Algebra of the Group $\widetilde{\mathbf{L}\mathcal{M}_p}$

Consider the topological algebra $\mathcal{A}_p$ of iterated integrals with the structure of commutative Hopf algebra described above. A $\mathbf{k}$-linear map $D : \mathcal{A}_p \to \mathcal{A}_p$ is called a *Left Invariant Derivation* (resp., *Right Invariant Derivation*) on $\mathcal{A}_p$ if D verifies the following two conditions:

$$\begin{aligned} D\big(X^{\mathbf{u}}X^{\mathbf{v}}\big) &= X^{\mathbf{u}}D\big(X^{\mathbf{v}}\big) + D\big(X^{\mathbf{u}}\big)X^{\mathbf{v}} &(66)\\ \Delta \circ D &= (1 \otimes D) \circ \Delta. &(67) \end{aligned}$$

(resp., $\Delta \circ D = (D \otimes 1) \circ \Delta$), for all $\mathbf{u}, \mathbf{v} \in Sh$.

The next definition follows from the general theory of affine $\mathbf{k}$-groups (see [Ab, Chp.4.3]):

3.4.1 Definition

We define the Lie Algebra of the Group $\widetilde{\mathbf{L}\mathcal{M}_p}$ as the $\mathbf{k}$-linear space $\widetilde{l\mathcal{M}_p}$ of all continuous Left Invariant Derivations on $\mathcal{A}_p$.

Of course the brackett in $\widetilde{l\mathcal{M}_p}$ is the usual commutator of derivations:

$$[D_1, D_2] = D_1 D_2 - D_2 D_1 \tag{68}$$

Let us give another (equivalent) description of this Lie algebra. Consider the convolution product $f \star g$, of two elements $f, g \in \mathcal{A}_p^*$ (the topological (weak) dual of $\mathcal{A}_p$):

$$\begin{aligned} f \star g(X^{\omega_1 \cdots \omega_r}) &= (f \otimes g) \circ \Delta(X^{\omega_1 \cdots \omega_r})\\ &= \sum_{i=0}^{r} f(X^{\omega_1 \cdots \omega_i}).g(X^{\omega_{i+1} \cdots \omega_r}) &(69) \end{aligned}$$

We need the following Lemma (see [Ab]):

3.4.2 Lemma

$(\mathcal{A}_p^*, \star)$ is a topological $\mathbf{k}$-algebra, isomorphic (resp., antiisomorphic) to the topological algebra $End^{ll}(\mathcal{A}_p)$ (resp., $End^{rl}(\mathcal{A}_p)$) of all left (resp., right) invariant $\mathbf{k}$-linear endomorphisms of $\mathcal{A}_p$ (i.e., $\mathbf{k}$-linear morphisms $\sigma : \mathcal{A}_p \to \mathcal{A}_p$ that verify the left (resp., right) invariance condition:

$$\Delta \circ \sigma = (1 \otimes \sigma) \circ \Delta \tag{70}$$

(resp., $\Delta \circ \sigma = (\sigma \otimes 1) \circ \Delta$), and endowed with the topology of pointwise convergence. Moreover, each element of $End^{ll}(\mathcal{A}_p)$ commutes with each element of $End^{rl}(\mathcal{A}_p)$

Proof...

It's a standard fact that $(\mathcal{A}_p^*, \star)$ is a topological $\mathbf{k}$-algebra.

Define now $\mathbf{k}$-linear maps $\Phi : End^{ll}(\mathcal{A}_p) \to \mathcal{A}_p^*$, $\Psi : \mathcal{A}_p^* \to End^{ll}(\mathcal{A}_p)$, and $\Lambda : \mathcal{A}_p^* \to End^{rl}(\mathcal{A}_p)$ respectivelly by:

$$\Phi : \sigma \to \Phi(\sigma) \equiv f_\sigma \equiv \epsilon \circ \sigma \tag{71}$$

$$\Psi : f \to \Psi(f) \equiv \sigma_f \equiv (1 \otimes f) \circ \Delta \tag{72}$$

and

$$\Lambda : f \to \Lambda(f) \equiv \rho_f \equiv (f \otimes 1) \circ \Delta \tag{73}$$

Let us verify, for example, that Ψ is well defined and it's an algebra morphism. In fact:

$$\begin{aligned}
\big(1 \otimes \Psi(f)\big)\Delta &= \big(1 \otimes (1 \otimes f) \circ \Delta\big)\Delta \\
&= (1 \otimes 1 \otimes f)(1 \otimes \Delta)\Delta \\
&= (1 \otimes 1 \otimes f)(\Delta \otimes 1)\Delta \\
&= \Delta(1 \otimes f)\Delta = \Delta\Psi(f)
\end{aligned} \tag{74}$$

which proves that $\Psi(f)$ is left invariant. On the other hand, composing with $1 \otimes g$, $g \in \mathcal{A}_p^*$, we obtain:

$$\begin{aligned}
\Psi(g)\Psi(f) &= (1 \otimes g)\Delta\Psi(f) \\
&= (1 \otimes g)(1 \otimes \Psi(f))\Delta \\
&= (1 \otimes g\Psi(f))\Delta \\
&= (1 \otimes g \star f)\Delta \\
&= \Psi(g \star f)
\end{aligned} \tag{75}$$

The rest of the proof follows directly from the definitions , QED.

3.4.3 Example

Let $\alpha \in \mathbf{L}\mathcal{M}_p$ and $\delta_\alpha \in \mathcal{A}_p^*$ as in (61). Then Ψ_{δ_α} is the automorphism $X^{\mathbf{u}} \to \alpha \cdot X^{\mathbf{u}}$ corresponding to the action of α, on $\mathbf{L}\mathcal{M}_p$ from the right.

In fact, the right action of $\mathbf{L}\mathcal{M}_p$ on itself, through right translations $r_\alpha : \beta \mapsto \beta.\alpha$, induces a left action of $\mathbf{L}\mathcal{M}_p$ on $\mathcal{A}_p$ by:

$$(\alpha \cdot X^{\mathbf{u}})(\beta) \equiv X^{\mathbf{u}}(\beta.\alpha) \tag{76}$$

By the identification $\alpha \to \delta_\alpha$, we can write the RHS of the above equation in the form:

$$\begin{aligned}
X^{\mathbf{u}}(\beta.\alpha) &= \delta_{\beta.\alpha}(X^{\mathbf{u}}) = \delta_\beta \star \delta_\beta(X^{\mathbf{u}}) \\
&= \delta_\beta\big((1 \otimes \delta_\alpha)\Delta X^{\mathbf{u}}\big) = \delta_\beta\big(\Psi_{\delta_\alpha}(X^{\mathbf{u}}\big)
\end{aligned} \tag{77}$$

while the LHS is simply $\delta_\beta(\alpha \cdot X^{\mathbf{u}})$, which allows the above mentioned identification $\Psi_{\delta_\alpha} \simeq \alpha \cdot X^{\mathbf{u}}$.

In the same way we can prove that Λ_{δ_α} is the automorphism $X^{\mathbf{u}} \to X^{\mathbf{u}} \cdot \alpha$, corresponding to the action of α, on $\mathbf{L}\mathcal{M}_p$ from the left.

Now, if moreover $\sigma = D$ is a left invariant derivation, then the corresponding $\Phi(D) = f_D = \epsilon \circ D \in \mathcal{A}_p^*$ verifies:

$$\begin{aligned} f_D(X^{\mathbf{u}}X^{\mathbf{v}}) &= \epsilon D(X^{\mathbf{u}}X^{\mathbf{v}}) \\ &= \epsilon(X^{\mathbf{u}}DX^{\mathbf{v}} + DX^{\mathbf{u}}X^{\mathbf{v}}) \\ &= \epsilon(X^{\mathbf{u}})f_D(X^{\mathbf{v}}) + f_D(X^{\mathbf{u}})\epsilon(X^{\mathbf{v}}) \end{aligned} \tag{78}$$

and we see that the Lie algebra $\widetilde{l\mathcal{M}_p}$ is isomorphic, as $\mathbf{k}$-linear space, to the subspace of $\mathcal{A}_p^*$ consisting of the so called *point derivations* at ϵ, i.e.:

$$\widetilde{l\mathcal{M}_p} \cong \{\delta \in \mathcal{A}_p^* : \quad \delta(X^{\mathbf{u}}X^{\mathbf{v}}) = \epsilon(X^{\mathbf{u}})\delta(X^{\mathbf{v}}) + \delta(X^{\mathbf{u}})\epsilon(X^{\mathbf{v}})\}. \tag{79}$$

This $\mathbf{k}$-linear space of point derivations at ϵ is called the *Tangent Space*, at ϵ, to the group $\widetilde{\mathbf{L}\mathcal{M}_p}$ and is denoted by $T_\epsilon\widetilde{\mathbf{L}\mathcal{M}_p}$. The reason for this terminology can be explained in the following more familiar terms. Let $\tilde{\alpha}_t$ be a curve of generalized loops such that:

$$\tilde{\alpha}_0 = \epsilon \tag{80}$$

$$\lim_{t\to 0} \tilde{\alpha}_t = \epsilon \tag{81}$$

$$\lim_{t\to 0} \frac{\tilde{\alpha}_t - \epsilon}{t} = \delta \quad \in \mathcal{A}_p^* \tag{82}$$

where the limits in (83) and (84) are taken in the weak sense (for example, (83) means that $\lim_{t\to 0} \tilde{\alpha}_t(X^{\mathbf{u}}) = \epsilon(X^{\mathbf{u}}), \forall \mathbf{u} \in Sh$, and an analogous condition for (84)). Then:

$$\begin{aligned} \delta(X^{\mathbf{u}}X^{\mathbf{v}}) &= \lim_{t\to 0} \frac{\tilde{\alpha}_t(X^{\mathbf{u}}X^{\mathbf{v}}) - \epsilon(X^{\mathbf{u}}X^{\mathbf{v}})}{t} \\ &= \lim_{t\to 0} \Big(\tilde{\alpha}_t(X^{\mathbf{u}})\frac{\tilde{\alpha}_t(X^{\mathbf{v}}) - \epsilon(X^{\mathbf{v}})}{t} + \frac{\tilde{\alpha}_t(X^{\mathbf{u}}) - \epsilon(X^{\mathbf{u}})}{t}\epsilon(X^{\mathbf{v}})\Big) \\ &= \epsilon(X^{\mathbf{u}})\delta(X^{\mathbf{v}}) + \delta(X^{\mathbf{u}})\epsilon(X^{\mathbf{v}}) \end{aligned} \tag{83}$$

Thus, we have a $\mathbf{k}$-linear isomorphism:

$$T_\epsilon\widetilde{\mathbf{L}\mathcal{M}_p} \cong \widetilde{l\mathcal{M}_p} \tag{84}$$

given by $\delta \to D_\delta = (1 \otimes \delta) \circ \Delta$. We endow $T_\epsilon\widetilde{\mathbf{L}\mathcal{M}_p}$ with a Lie brackett, by defining:

$$[\delta, \eta] \equiv \epsilon \circ [D_\delta, D_\eta]. \tag{85}$$

Using lemma 3.4.2, we see that:

$$\begin{aligned} [\delta, \eta] &= \epsilon \circ [D_\delta, D_\eta] \\ &= \epsilon(D_\delta D_\eta - D_\eta D_\delta) \\ &= \Phi(D_\delta D_\eta - D_\eta D_\delta) \\ &= \Phi(D_\delta) \star \Phi(D_\eta) - \Phi(D_\eta) \star \Phi(D_\delta) \\ &= \delta \star \eta - \eta \star \delta \end{aligned} \tag{86}$$

Note that any point derivation δ, at ϵ, verifies:

$$\delta(X^{\omega_1\ldots\omega_r}X^{\omega_{r+1}\ldots\omega_{r+s}}) = 0 \tag{87}$$

$\forall r \geq 1, \forall s \geq 1$, and from this we can deduce that:

$$\delta^n(X^{\omega_1\ldots\omega_r}) = 0 \qquad \forall n > r \geq 0 \tag{88}$$

where $\delta^n \equiv \delta^n \star \delta, \quad \forall n \geq 1$.

Now, for each $\delta \in T_\epsilon\widetilde{\mathbf{L}\mathcal{M}_p} \cong \widetilde{l\mathcal{M}_p}$, define $\exp\delta$ by:

$$\exp\delta \equiv \epsilon + \sum_{n\geq 1}\frac{\delta^n}{n!} \tag{89}$$

where, as always, this means that, for each $X^{\omega_1\ldots\omega_r}$, $\exp\delta(X^{\omega_1\ldots\omega_r})$ is defined by:

$$\Big(\epsilon + \sum_{n\geq 1}\frac{\delta^n}{n!}\Big)(X^{\omega_1\ldots\omega_r}) \tag{90}$$

if, of course, this series converges.

Now, it follows from (90) that the series (92) is in fact a finite sum, and so $\exp\delta$ is well defined, in the above sense. Moreover, we can prove that $\exp\delta$ is a generalized loop (definition 3.3.1).

Conversely, given $\tilde{\alpha} \in \widetilde{\mathbf{L}\mathcal{M}_p}$, we define:

$$\log\tilde{\alpha} \equiv \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}(\tilde{\alpha}-\epsilon)^n \tag{91}$$

where $(\tilde{\alpha}-\epsilon)^n \equiv (\tilde{\alpha}-\epsilon)^{n-1} \star (\tilde{\alpha}-\epsilon), \quad \forall n \geq 1$.

Since, $(\tilde{\alpha}-\epsilon)^n(X^{\omega_1\ldots\omega_r}) = 0, \quad \forall n > r \geq 0$, $\log\tilde{\alpha}$ is also a well defined element in the above sense, which moreover, belongs to $T_\epsilon\widetilde{\mathbf{L}\mathcal{M}_p} \cong \widetilde{l\mathcal{M}_p}$. By the calculus of formal power series, we know that:

$$\begin{aligned} \exp(k\log\tilde{\alpha}) &= \tilde{\alpha}^k \qquad \forall k \in Z \\ \log(\exp\delta) &= \delta \end{aligned}$$

Let us define, for each $t \in \mathbf{k}$:

$$\tilde{\alpha}^t \equiv \exp(t\log\tilde{\alpha}) \tag{92}$$

Then we can easilly prove that $t \mapsto \tilde{\alpha}^t$ is a one-parameter subgroup of $\widetilde{\mathbf{L}\mathcal{M}_p}$, generated by $\log\tilde{\alpha}$, i.e.:

$$\begin{aligned} \tilde{\alpha}^0 &= \epsilon \\ \tilde{\alpha}^t \star \tilde{\alpha}^s &= \tilde{\alpha}^{t+s} \\ \lim_{t\to 0}\frac{\tilde{\alpha}^t-\epsilon}{t} &= \log\tilde{\alpha} \end{aligned}$$

this last limit in the above (weak) sense. Similar results seem to be obtained in [BGG], using however a quite different formalism.

4 Loop Calculus. Endpoint and Area Derivative Operators

In this section we try to develope a rigorous mathemathical *Loop Calculus* which can be used to formalize early heuristic ideas mainly due to Gambini and Trias (see [GT1],[GT2],[GT3]), and also Makeenko, Migdal (see [MM]).

4.1 Endpoint Derivatives

Consider a path $\gamma \in \mathcal{PM}$, and a tangent vector field $V \in \mathcal{XU}$, defined in a neighbourhood $\mathcal{U}$ of $q = \gamma(1)$ and such that $V(\gamma(1)) = v \in T_{\gamma(1)}\mathcal{M}$. Let us put $\eta_s^V = \Phi^V(s)(q)$ for the integral curve of V, starting at $q = \gamma(1)$, at $s = 0$, with velocity v. Finally put $\gamma_s = \gamma.\eta_s^V$ and $q_s = \gamma_s(1)$ (see figure 1).

Figure 1: $\gamma_s = \gamma.\eta_s^V$ and $q_s = \gamma_s(1)$

.

We use these notations in the following:

4.1.1 Definition

Let Ψ be a path functional on $\mathcal{PM}$, with values in R (resp., $C; gl(m)$). We define the *Terminal Covariant Endpoint Derivative* $\nabla_V^T(q_s)\Psi(\gamma)$, of Ψ, at γ, in the direction of V, as the limit:

$$\nabla_V^T(q_s)\Psi(\gamma) = \lim_{h\to 0} \frac{\Psi(\gamma_{s+h}) - \Psi(\gamma_s)}{h}. \tag{93}$$

4.1.2 Note

With the obvious modifications (in particular γ_s now means $\gamma_s = (\eta_s^V)^{-1}.\gamma$), we also define the *Initial Covariant Endpoint Derivative* $\nabla_V^I(q_s)\Psi(\gamma)$, as the limit given by the same formal expression. Hereafter, to simplify the discussion, we treat only the *Terminal Endpoint* case. Similar formulas can be obtained for the *Initial Endpoint* case, with the obvious modifications.

Note that the limit (95) (if it exists) defines $\nabla_V^T(q_s)\Psi(\gamma)$ "near" the endpoint $q = \gamma(1)$, more precisely at the point $q_s = \gamma_s(1)$, and obviously depend on the vector field V. When $s = 0$, we define:

4.1.3 Definition

Let Ψ be a path functional on $\mathcal{PM}$, with values in R (resp., $C; gl(m)$). We define the *Terminal Endpoint Derivative* $\partial_v^T \Psi(\gamma)$, of Ψ, at γ, in the direction of $v \in T_{\gamma(1)}\mathcal{M}$, as the limit:

$$\partial_v^T \Psi(\gamma) = \lim_{h \to 0} \frac{\Psi(\gamma_h) - \Psi(\gamma)}{h} \tag{94}$$

provided this limit exists independently of the choice of the vector field $V \in \mathcal{XM}$, such that $V(\gamma(1)) = v$.

As a simple example, let us take a smooth function $f \in C^\infty\mathcal{M}$, and define a path functional Ψ_f, by $\Psi_f(\gamma) = f(\gamma(1))$. Then it's easy to see that:

$$\nabla_V^T(q_s)\Psi_f(\gamma) = V.f(q_s) = df(V_{q_s}). \tag{95}$$

Moreover,

$$\partial_v^T \Psi_f(\gamma) = V.f(\gamma(1)) = df(v) \tag{96}$$

which depends only on the vector v, and not of the particular extension V.

4.1.4 Definition

Let Ψ be a path functional and $f \in C^\infty\mathcal{M}$. We define the *Marked Path Functional* $f \odot \Psi$, by:

$$(f \odot \Psi)(\gamma) = f(\gamma(1))\Psi(\gamma). \tag{97}$$

The following lemma is proved as usual, following the definitions.

4.1.5 Lemma. Leibniz Rule

Assume that Ψ is a path functional for which the limit (95) exists, and which verifies the "continuity condition" $\lim_{h\to 0} \Psi(\gamma_{s+h}) = \Psi(\gamma_s), \forall s \geq 0$. Then we have the following *Leibniz Rule*:

$$\begin{aligned} \nabla_V^T(q_s)\big(f \odot \Psi\big)(\gamma) &= V.f(q_s)\Psi(\gamma_s) + f(q_s)\nabla_V^T(q_s)\Psi(\gamma) \\ &= \nabla_V^T(q_s)f(\gamma)\Psi(\gamma_s) + f(\gamma_s)\nabla_V^T(q_s)\Psi(\gamma) \end{aligned} \tag{98}$$

where we have put $q_s = \gamma_s(1)$, and used the notation f for Ψ_f.

In particular, if $\partial_v^T \Psi(\gamma)$ exists in the sense of definition 4.1.3, we have at the terminal endpoint $q = \gamma(1)$:

$$\partial_v^T\big(f \odot \Psi\big)(\gamma) = \partial_v^T f(\gamma)\Psi(\gamma) + f(\gamma)\partial_V^T \Psi(\gamma) \tag{99}$$

which depends only on the vector v, and not of the particular extension V.

Our aim now, is to prove that the path functionals $X^{\omega_1 \cdots \omega_r}$, defined by (2), have well defined endpoint derivatives in the above sense. For this, we need the following lemma which is easilly proved in local coordinates.

4.1.6 Lemma

Let $\eta_s = \eta_s^V$. Then:

$$\lim_{s\to 0} \frac{\int_{\eta_s} \omega}{s} = \omega(v) \tag{100}$$

$$\lim_{s\to 0} \frac{\int_{\eta_s} \omega_1 ... \omega_r}{s} = 0 \quad \forall r \geq 2 \tag{101}$$

where $\omega, \omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M}$ (resp., $\omega, \omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M} \otimes gl(m)$).

Then, using the above lemma, Proposition 2.4 and induction on $r \geq 1$, we compute that:

$$\nabla_V^T(q_s) X^{\omega_1 ... \omega_r}(\gamma) = X^{\omega_1 ... \omega_{r-1}}(\gamma_s).\omega_r(V_{q_s}) \quad \forall r \geq 1 \tag{102}$$

and:

$$\partial_v^T X^{\omega_1 ... \omega_r}(\gamma) = X^{\omega_1 ... \omega_{r-1}}(\gamma).\omega_r(v) \quad \forall r \geq 1 \tag{103}$$

In the same way (see Note 4.1.2), we can compute that:

$$\nabla_V^I(q_s) X^{\omega_1 ... \omega_r}(\gamma) = -\omega_1(V_{q_s}).X^{\omega_2 ... \omega_r}(\gamma_s) \quad \forall r \geq 1 \tag{104}$$

and:

$$\partial_v^I X^{\omega_1 ... \omega_r}(\gamma) = -\omega_1(v).X^{\omega_2 ... \omega_r} \quad \forall r \geq 1 \tag{105}$$

where $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M}$ (resp., $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M} \otimes gl(m)$).

4.1.7 Note

It's easy to see that ∂_v^T (resp., ∂_v^I) is a derivation in the algebra of iterated integrals. In fact, this follows from the fact that $\int$ is an algebra homomorphism and $\partial_v^T : \omega_1 ... \omega_r \mapsto \omega_1 ... \omega_{r-1}.(\omega_r(v))$ (resp., $\partial_v^I : \omega_1 ... \omega_r \mapsto -\omega_1(v).\omega_2 ... \omega_r$) is a derivation in $Sh(\mathcal{M})$ (we put also ∂_v^T (resp., ∂_v^I): $1 \mapsto 0$, when $r = 0$).

In this spirit, we can compute the (algebraic) commutator of two endpoint derivatives (considered as derivations on the algebra of iterated integrals), applied to a function $X^{\omega_1 ... \omega_r}$. It is given by:

$$[\partial_u^T, \partial_v^T] X^{\omega_1 ... \omega_r}(\gamma) = X^{\omega_1 ... \omega_{r-2}}(\gamma).(\omega_{r-1} \wedge \omega_r)(u \wedge v). \tag{106}$$

Note that $\nabla_V^T(q_s) X^{\omega_1 ... \omega_r}(\gamma)$, given by (104), is a Marked Path Functional, in the sense of definition 4.1.4 (f is the function $\omega_r(V)$). So, if U, V are two vector fields locally defined around $q = \gamma(1)$, we can apply Leibniz rule (101) and the well known formula $d\omega(U, V) = U.\omega(V) - V.\omega(U) - \omega([U, V])$, to compute that, at q:

$$\left[\nabla^T_U(q),\nabla^T_V(q)\right]X^{\omega_1\ldots\omega_r}(\gamma) = X^{\omega_1\ldots\omega_{r-1}}(\gamma).d\omega_r(u\wedge v) + X^{\omega_1\ldots\omega_{r-2}}(\gamma).(\omega_{r-1}\wedge\omega_r)(u\wedge v). \tag{107}$$

As the RHS of the above formula, depends only on the vectors u, v, and not of the particular extensions U, V, we write the LHS as $\left[\nabla^T_u,\nabla^T_v\right]X^{\omega_1\ldots\omega_r}(\gamma)$.

For the parallel transport path functional $U : \mathcal{PM}\to Gl(p)$, given by the series (27), we compute that:

$$\partial^T_v U_\gamma = U_\gamma.\omega(v), \tag{108}$$

$$\partial^I_v U_\gamma = -\omega(v).U_\gamma, \tag{109}$$

while, for the commutator:

$$\begin{aligned}\left[\nabla^T_u,\nabla^T_v\right]U_\gamma &= U_\gamma.(d\omega+\omega\wedge\omega)(u\wedge v)\\ &= U_\gamma.\Omega(u\wedge v)\end{aligned} \tag{110}$$

where Ω is the curvature of the connection ω.

Let us give a last example, to finish this section. Given a path $\lambda\in\mathcal{PM}_p$ and a function $f\in C^\infty\mathcal{M}$ (resp., $\in C^\infty\mathcal{M}\otimes gl(m)$), we define a (marked) path functional through:

$$Z^{\omega_1\ldots\omega_r}_{(i)}(\lambda;f)\equiv X^{\omega_1\ldots\omega_i}(\lambda)f(\lambda(1))X^{\omega_{i+1}\ldots\omega_r}(\lambda^{-1}) \tag{111}$$

where $\omega_1,\ldots,\omega_r\in\bigwedge^1\mathcal{M}$ (resp., $\omega_1,\ldots,\omega_r\in\bigwedge^1\mathcal{M}\otimes gl(m)$).

Using Leibniz rule, we compute that:

$$\begin{aligned}\nabla^T_v Z^{\omega_1\ldots\omega_r}_{(i)}(\lambda;f) &= X^{\omega_1\ldots\omega_i}(\lambda).df_q(v).X^{\omega_{i+1}\ldots\omega_r}(\lambda^{-1})\\ &\quad+X^{\omega_1\ldots\omega_{i-1}}(\lambda).\omega_i(v).f(q).X^{\omega_{i+1}\ldots\omega_r}(\lambda^{-1})\\ &\quad+X^{\omega_1\ldots\omega_i}(\lambda).f(q).\omega_{i+1}(v).X^{\omega_{i+2}\ldots\omega_r}(\lambda^{-1})\end{aligned} \tag{112}$$

Where $q=\lambda(1)$.

Finally, let us define, for a connection 1-form ω, a (marked) path functional Ψ through:

$$\Psi(\lambda;f)\equiv U_\lambda.f(q).U_{\lambda^{-1}} \tag{113}$$

where $q=\lambda(1)$, U is the parallel transport operator of the connection ω and $f\in C^\infty\mathcal{M}\otimes gl(m)$. Then, using (114), we compute that:

$$\begin{aligned}\nabla^T_v\Psi(\lambda;f) &= U_\lambda.\big(df_q(v)+[\omega,f](v)\big).U_{\lambda^{-1}}\\ &= U_\lambda.D^\omega_q f(v).U_{\lambda^{-1}}\end{aligned} \tag{114}$$

where $D^\omega_q f(v)\equiv df_q(v)+[\omega,f](v)$ denotes the covariant derivative of f. This is the reason why we call the operator ∇^T_v, Terminal Endpoint *Covariant* Derivative.

4.2 Area Derivative

4.2.1 Definition and Main Properties

Consider a loop $\gamma \in \mathcal{LM}_p$, a point $q \in \mathcal{M}$ and a path $\lambda \in \mathcal{PM}_p$, going from p to $q = \lambda(1)$.

Given an ordered pair (u, v) of tangent vectors $u, v \in T_q\mathcal{M}$, we extend them by two commuting vector fields $U, V \in \mathcal{XU}$, defined in a small neighbourhood $\mathcal{U}$ of $q = \lambda(1)$ (this is always possible). Then, we consider the "small" loop $\Box_t^{(U,V)}$, based at q, defined by:

$$\Box_t^{(U,V)} = \Phi^V(-t)\Phi^U(-t)\Phi^V(t)\Phi^U(t)(q) \tag{115}$$

where Φ^U (resp., Φ^V) denotes the local flow of U (resp.,V).

Denote by λ_t the (t-dependent) loop $\lambda.\Box_t^{(U,V)}.\lambda^{-1}$ (see figure 2).

Figure 2: $\lambda_t \equiv \lambda.\Box_t^{(U,V)}.\lambda^{-1}$

.

Note that $\lim_{t\to 0} \lambda_t = \epsilon$, where ϵ denotes the unity in the group $\mathbf{L}\mathcal{M}_p$ of (equivalence classes) of loops based at $p \in \mathcal{M}$ (i.e., $\epsilon \equiv [p]$, the equivalence class of the trivial loop reduced to the point p), and the limit is taken in weak sense, i.e.:

$$\lim_{t\to 0} \lambda_t(X^{\mathbf{u}}) = \lim_{t\to 0} X^{\mathbf{u}}(\lambda_t) = \epsilon(X^{\mathbf{u}}) \tag{116}$$

$\forall \mathbf{u} \in Sh(\mathcal{M})$ (see section 3.3). So, $\lambda_t.\gamma$ represents a "small" deformation of the loop γ, in the topology of $\mathbf{L}\mathcal{M}_p$, defined in section 3.3.

With all these notations we now formulate the following:

4.2.2 Definition

Given a loop functional Ψ on $\mathbf{L}\mathcal{M}_p$, with values in R (resp., $C; gl(m)$), we define its *Area Derivative*, denoted by $\triangle_{\lambda;(u,v)}(q).\Psi(\gamma)$, as the limit:

$$\triangle_{\lambda;(u,v)}(q)\Psi(\gamma) = \lim_{t\to 0} \frac{\Psi(\lambda_t.\gamma) - \Psi(\gamma)}{t^2} \tag{117}$$

provided this limit exists independently of the choice of the vector fields $U, V \in \mathcal{XU}$, considered above.

We want to prove that every function $X^{\omega_1 \cdots \omega_r} : \mathbf{L}\mathcal{M}_p \to R$ (resp., $C; gl(m)$) admits a well defined Area derivative, in the above sense. For this, we first state a lemma which is, once more, easily proved in local coordinates:

4.2.3 Lemma

Denote simply by $\square_t$, the above mentioned loop $\square_t^{(U,V)}$. Then:

$$\lim_{t\to 0}\frac{\int_{\square_t}\omega}{t^2} = d\omega(u\wedge v) \tag{118}$$

$$\lim_{t\to 0}\frac{\int_{\square_t}\omega_1\omega_2}{t^2} = (\omega_1\wedge\omega_2)(u\wedge v) \tag{119}$$

$$\lim_{t\to 0}\frac{\int_{\square_t}\omega_1...\omega_r}{t^2} = 0, \qquad \forall r\geq 3 \tag{120}$$

where $\omega,\omega_1,...,\omega_r\in\bigwedge^1\mathcal{M}$ (resp., $\omega,\omega_1,...,\omega_r\in C;\bigwedge^1\mathcal{M}\otimes gl(m)$).

To give a compact form for the expression of the area derivative of the path functionals $X^{\omega_1...\omega_r}$, we introduce some more notations.

First, for $u\wedge v\in\bigwedge^2 T_q\mathcal{M}$, we define a derivation $D_{u\wedge v}(q)$, in the algebra of iterated integrals, by the formula:

$$D_{u\wedge v}(q)X^{\omega_1...\omega_r} = X^{\omega_1...\omega_{r-1}}.d\omega_r(u\wedge v) \tag{121}$$

Second, recalling formula (108) for the (algebraic) commutator $[\partial_u^T,\partial_v^T]$ of two terminal endpoint derivatives at q, we define a new derivation $\mathcal{D}_{u\wedge v}(q)$ by:

$$\mathcal{D}_{u\wedge v}(q) = D_{u\wedge v}(q) + [\partial_u^T,\partial_v^T] \tag{122}$$

Let us evaluate first, the area derivative of $X^{\omega_1...\omega_r}$ at ϵ. Using lemma 4.2.3, Proposition 2.4 and induction in r, we can prove the following lemma:

4.2.4 Lemma

$$\triangle_{\lambda;(u,v)}(q)X^{\omega_1...\omega_r}(\epsilon) = \sum_{i=1}^{r}\Big(\mathcal{D}_{u\wedge v}(q)X^{\omega_1...\omega_i}(\lambda)\Big)\Big(X^{\omega_{i+1}...\omega_r}(\lambda^{-1})\Big) \tag{123}$$

where $\omega_1,...,\omega_r\in\bigwedge^1\mathcal{M}$ (resp., $\in C,\bigwedge^1\mathcal{M}\otimes gl(m)$).

Note that $\triangle_{\lambda;(u,v)}(q)X^{\omega_1...\omega_r}(\epsilon)$ depends on the pair (u,v) through $u\wedge v$. So we write it in the form $\triangle_{(\lambda;u\wedge v)}(q)X^{\omega_1...\omega_r}(\epsilon)$.

For example:

$$\begin{aligned}\triangle_{(\lambda;u\wedge v)}(q)X^{\omega}(\epsilon) &= d\omega(u\wedge v)\\ \triangle_{(\lambda;u\wedge v)}(q)X^{\omega_1\omega_2}(\epsilon) &= d\omega_1(u\wedge v).X^{\omega_2}(\lambda^{-1}) + X^{\omega_1}(\lambda).d\omega_2(u\wedge v) +\\ &\quad +(\omega_1\wedge\omega_2)(u\wedge v)\end{aligned} \tag{124}$$

and, more generally:

$$\triangle_{(\lambda;u\wedge v)}(q)X^{\omega_1\ldots\omega_r}(\epsilon)=\sum_{i=1}^{r}X^{\omega_1\ldots\omega_{i-1}}(\lambda).d\omega_i(u\wedge v).X^{\omega_{i+1}\ldots\omega_r}(\lambda^{-1})+$$
$$+\sum_{i=2}^{r}X^{\omega_1\ldots\omega_{i-2}}(\lambda).(\omega_{i-1}\wedge\omega_i)(u\wedge v).X^{\omega_{i+1}\ldots\omega_r}(\lambda^{-1}) \qquad (125)$$

Recall that $\lim_{t\to 0}\lambda_t=\epsilon$. We can also prove that $\lim_{t\to 0}\frac{\lambda_t-\epsilon}{t}=0$, while $\lim_{t\to 0}\frac{\lambda_t-\epsilon}{t^2}$ exists and is our area derivative . To be more specific, let us define an operator $\delta_{(\lambda;u\wedge v)}$ in the algebra of iterated integrals $\mathcal{A}_p$, through the formula:

$$\begin{aligned}\delta_{(\lambda;u\wedge v)}X^{\mathbf{u}} &\equiv \triangle_{(\lambda;u\wedge v)}(q)X^{\mathbf{u}}(\epsilon)\\ &= (\lambda\otimes\lambda)\big((\mathcal{D}_{u\wedge v}(q)\otimes J)\circ\Delta\big)X^{\mathbf{u}}\end{aligned} \qquad (126)$$

$\forall\mathbf{u}\in Sh$. Note that the last equality in (127), is nothing else that (126), written using the definitions from section 3.

The above formula shows that $\delta_{(\lambda;u\wedge v)}:\mathcal{A}_p\to\mathbf{k}$ is linear and we can also prove easilly that:

$$\delta_{(\lambda;u\wedge v)}(X^{\mathbf{u}}X^{\mathbf{v}})=\delta_{(\lambda;u\wedge v)}(X^{\mathbf{u}})\epsilon(X^{\mathbf{v}})+\epsilon(X^{\mathbf{u}})\delta_{(\lambda;u\wedge v)}(X^{\mathbf{v}}) \qquad (127)$$

$\forall\mathbf{u},\mathbf{v}\in Sh$. So, acording to section 3.4, $\delta_{(\lambda;u\wedge v)}$ is a point derivation at ϵ. We call it a tangent vector to the group $\mathbf{L}\mathcal{M}_p$, at ϵ.

By definition, the Tangent Space $T_\epsilon\mathbf{L}\mathcal{M}_p$, to the group $\mathbf{L}\mathcal{M}_p$, at ϵ, is the $\mathbf{k}$-linear subspace of $\mathcal{A}_p^*$, generated by all the $\delta_{(\lambda;u\wedge v)}$. We can give a geometrical interpretation to the adition and scalar multiplication on $T_\epsilon\mathbf{L}\mathcal{M}_p$.

Consider now a loop $\gamma\in\mathbf{L}\mathcal{M}_p$ and let us compute the area derivative $\triangle_{(\lambda;u\wedge v)}(q)X^{\omega_1\ldots\omega_r}(\gamma)$, acording to definition (see figure 3).

Figure 3: $\triangle_{(\lambda;u\wedge v)}(q)X^{\omega_1\ldots\omega_r}(\gamma)$

.

We have, after an easy computation, the following lemma:

4.2.5 Lemma

Let $\gamma\in\mathbf{L}\mathcal{M}_p$, $\lambda\in\mathcal{P}\mathcal{M}_p$ and $u\wedge v\in\wedge^2T_{\lambda(1)}\mathcal{M}$. Then:

$$\begin{aligned}\triangle_{(\lambda;u\wedge v)}(q)X^{\omega_1\ldots\omega_r}(\gamma) &= \sum_{i=1}^{r}\triangle_{(\lambda;u\wedge v)}(q)X^{\omega_1\ldots\omega_i}(\epsilon)X^{\omega_{i+1}\ldots\omega_r}(\gamma)\\ &= \gamma\circ(\delta_{(\lambda;u\wedge v)}\otimes 1)\circ\Delta(X^{\omega_1\ldots\omega_r})\end{aligned} \qquad (128)$$

where we have used, in the last equation, the formalism of section 3.

Recall that $(\delta_{(\lambda;u\wedge v)}\otimes 1)\circ\Delta$ is the right invariant derivation on the algebra $\mathcal{A}_p$, associated to the tangent vector $\delta_{(\lambda;u\wedge v)}$ (see section 3.4). So, it is natural to call the map $\Lambda^R_{(\lambda;u\wedge v)}$: $\mathbf{L}\mathcal{M}_p \to \mathcal{A}_p^*$, given by:

$$\Lambda^R_{(\lambda;u\wedge v)}(\gamma) \equiv \gamma\circ(\delta_{(\lambda;u\wedge v)}\otimes 1)\circ\Delta \tag{129}$$

the *Right Invariant "Vector Field"*, on $\mathbf{L}\mathcal{M}_p$, determined by $\delta_{(\lambda;u\wedge v)}$.

In the special case in which $\lambda=\epsilon$, we call $\triangle_{(\epsilon;u\wedge v)}(p)$ the *Initial Endpoint Area Derivative* and we denote it by $\triangle^I_{(\epsilon;u\wedge v)}(p)$ (see figure 4).

Figure 4: $\triangle^I_{(\epsilon;u\wedge v)}(p)$

.

Then, the above formula reduces to:

$$\begin{aligned}\triangle^I_{(\epsilon;u\wedge v)}(p)X^{\omega_1\ldots\omega_r}(\gamma) &= d\omega_1(u\wedge v).X^{\omega_2\ldots\omega_r}(\gamma)\\ &\quad +(\omega_1\wedge\omega_2)(u\wedge v).X^{\omega_3\ldots\omega_r}(\gamma).\end{aligned} \tag{130}$$

Consider now the case in which $\lambda=\gamma.\eta$, $\gamma\in\mathbf{L}\mathcal{M}_p$, $\eta\in\mathcal{P}\mathcal{M}_p$, and $u\wedge v\in\wedge^2 T_{\eta(1)}\mathcal{M}$.
Then $\lambda_t.\gamma\equiv(\lambda.\Box_t^{(U,V)}.\lambda^{-1}).\gamma=\gamma.\eta.\Box_t^{(U,V)}.\gamma.\eta^{-1}.\gamma=\gamma.(\eta.\Box_t^{(U,V)}.\eta^{-1})\equiv\gamma.\eta_t$ (see figure 5).

Figure 5: $\triangle^E_{(\eta;u\wedge v)}(q)$

In this case, we call the corresponding area derivative *Endpoint Area Derivative* and we denote it by $\triangle^E_{(\eta;u\wedge v)}(q)$. We compute that:

$$\begin{aligned}\triangle^E_{(\eta;u\wedge v)}(q)X^{\omega_1\ldots\omega_r}(\gamma) &= \sum_{i=1}^r X^{\omega_1\ldots\omega_i}(\gamma)\triangle_{(\eta;u\wedge v)}(q)X^{\omega_{i+1}\ldots\omega_r}(\epsilon)\\ &= \gamma\circ(1\otimes\delta_{(\eta;u\wedge v)})\circ\Delta(X^{\omega_1\ldots\omega_r})\end{aligned} \tag{131}$$

As before, since $(1 \otimes \delta_{(\eta;u\wedge v)}) \circ \Delta$ is the left invariant derivation associated to $\delta_{(\eta;u\wedge v)}$, it is natural to call the map $\Lambda^L_{(\eta;u\wedge v)} : \mathbf{L}\mathcal{M}_p \to \mathcal{A}^*_p$, given by:

$$\Lambda^L_{(\eta;u\wedge v)}(\gamma) \equiv \gamma \circ (1 \otimes \delta_{(\eta;u\wedge v)}) \circ \Delta \tag{132}$$

the *Left Invariant "Vector Field"*, on $\mathbf{L}\mathcal{M}_p$, determined by $\delta_{(\eta;u\wedge v)}$.

Note the particular case $\eta = \epsilon$ (see figure 6), in which the above formula reduces to:

$$\begin{aligned} \triangle^E_{(\epsilon;u\wedge v)}(p) X^{\omega_1 \dots \omega_r}(\gamma) &= \mathcal{D}_{u\wedge v}(p) X^{\omega_1 \dots \omega_r} \\ &= X^{\omega_1 \dots \omega_{r-1}}(\gamma).d\omega_r(u \wedge v) \\ &\quad + X^{\omega_1 \dots \omega_{r-2}}(\gamma).(\omega_{r-1} \wedge \omega_r)(u \wedge v) \end{aligned} \tag{133}$$

Figure 6: $\triangle^E_{(\epsilon;u\wedge v)}(p)$

.

Note that, in this case we have that (see (109)):

$$\triangle_{(\epsilon;u\wedge v)}(q) X^{\omega_1 \dots \omega_r}(\gamma) = \left[\nabla^T_u, \nabla^T_v\right] X^{\omega_1 \dots \omega_r}(\gamma) \tag{134}$$

which is an expected relation between the endpoint area derivative and the commutator of two terminal endpoint covariant derivatives.

For example, for a loop γ, based at $p \in \mathcal{M}$, the endpoint area derivative of the holonomy U_γ is given by:

$$\begin{aligned} \triangle^E_{(\epsilon;u\wedge v)}(p) U_\gamma &= U_\gamma.(d\omega + \omega \wedge \omega)(u \wedge v) \\ &= U_\gamma.\Omega(u \wedge v) \end{aligned} \tag{135}$$

where Ω is the curvature of the connection ω. The endpoint area derivative of the Wilson loop variable $\mathcal{W}$ is given by:

$$\begin{aligned} \triangle^E_{(\epsilon;u\wedge v)}(p) \mathcal{W}(\gamma) &= Trace\big((d\omega + \omega \wedge \omega)(u \wedge v).U_\gamma\big) \\ &= Trace\big(\Omega(u \wedge v).U_\gamma\big). \end{aligned} \tag{136}$$

These formulas are known as Mandelstam formulas.

Another such relation is given by the so called *Bianchi Identity*:

4.2.6 Bianchi Identity

$$\sum_{cycl\{u,v,w\}} \nabla_u^T(\lambda(1))\delta_{(\lambda;v\wedge w)} = 0 \tag{137}$$

Proof...
It suffices to prove that:

$$\sum_{cycl\{u,v,w\}} \nabla_u^T(\lambda(1))\delta_{(\lambda;v\wedge w)}(X^{\omega_1\ldots\omega_r}) = 0 \tag{138}$$

$\forall r \geq 1$. This follows by direct computation, using Leibniz Rule (96) and the identity:

$$\sum_{cycl\{U,V,W\}} U.d\omega(V,W) = 0 \tag{139}$$

QED.

Finally, let us compute the commutator $[\delta_{(\lambda;a\wedge b)}, \delta_{(\eta;u\wedge v)}]$, of two tangent vectors considered as elements of the Lie Algebra $l\widetilde{\mathcal{M}_p}$ (see section 3.4).

By (88), we have that:

$$[\delta_{(\lambda;a\wedge b)}, \delta_{(\eta;u\wedge v)}] = \delta_{(\lambda;a\wedge b)} \star \delta_{(\eta;u\wedge v)} - \delta_{(\eta;u\wedge v)} \star \delta_{(\lambda;a\wedge b)} \tag{140}$$

Using (87) and (132), we can write:

$$\begin{aligned}
&[\delta_{(\lambda;a\wedge b)}, \delta_{(\eta;u\wedge v)}]X^{\omega_1\ldots\omega_r} = \epsilon \circ \Big(\big(1 \otimes \delta_{(\lambda;a\wedge b)}\big)\Delta\big(1 \otimes \delta_{(\eta;u\wedge v)}\big)\Delta\Big)X^{\omega_1\ldots\omega_r} + \\
&\quad -\epsilon \circ \Big(\big(1 \otimes \delta_{(\eta;u\wedge v)}\big)\Delta\big(1 \otimes \delta_{(\lambda;a\wedge b)}\big)\Delta\Big)X^{\omega_1\ldots\omega_r} \\
&\qquad = \triangle^E_{(\lambda;a\wedge b)}(\lambda(1))\Big(\big(1 \otimes \delta_{(\eta;u\wedge v)}\big)\Delta X^{\omega_1\ldots\omega_r}\Big)(\epsilon) + \\
&\quad -\triangle^E_{(\eta;u\wedge v)}(\eta(1))\Big(\big(1 \otimes \delta_{\lambda;a\wedge b)}\big)\Delta X^{\omega_1\ldots\omega_r}\Big)(\epsilon) \\
&\qquad = \triangle^E_{(\lambda;a\wedge b)}(\lambda(1))\Big(\sum_{i=0}^{r} X^{\omega_1\ldots\omega_i}\delta_{(\eta;u\wedge v)}(X^{\omega_{i+1}\ldots\omega_r})\Big)(\epsilon) + \\
&\quad -\triangle^E_{(\eta;u\wedge v)}(\eta(1))\Big(\sum_{i=0}^{r} X^{\omega_1\ldots\omega_i}\delta_{(\lambda;a\wedge b)}(X^{\omega_{i+1}\ldots\omega_r})\Big)(\epsilon) \\
&= \sum_{i=0}^{r}\sum_{k=o}^{i}\big(\mathcal{D}_{a\wedge b}(\lambda(1))X^{\omega_1\ldots\omega_k}(\lambda)\big)\big(X^{\omega_{k+1}\ldots\omega_i}(\lambda^{-1})\big)\delta_{(\eta;u\wedge v)}(X^{\omega_{i+1}\ldots\omega_r}) + \\
&-\sum_{i=0}^{r}\sum_{k=o}^{i}\big(\mathcal{D}_{u\wedge v}(\eta(1))X^{\omega_1\ldots\omega_k}(\eta)\big)\big(X^{\omega_{k+1}\ldots\omega_i}(\eta^{-1})\big)\delta_{(\lambda;a\wedge b)}(X^{\omega_{i+1}\ldots\omega_r})
\end{aligned} \tag{141}$$

which give the "structure constants" of $l\widetilde{\mathcal{M}_p}$.

5 Variational Calculus

5.1 The action of the Diffeomorphism Group

Denote by $Diff(\mathcal{M})$ the Diffeomorphism group of $\mathcal{M}$. If $\varphi \in Diff(\mathcal{M})$ and $\gamma \in \mathcal{PM}$, we denote by $\varphi.\gamma$ the image of the path γ under φ.

Then we easilly prove that:

$$X^{\omega_1 \dots \omega_r}(\varphi.\gamma) = X^{\varphi^*\omega_1 \dots \varphi^*\omega_r}(\gamma) \tag{142}$$

where $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M}$ (resp., $\in C; \bigwedge^1 \mathcal{M} \otimes gl(m)$).

Infinitesimally, if φ_s is a 1-parameter group of diffeomorphisms, with infinitesimal generator $Y \in \mathcal{XM}$, we deduce from the previous formula, that:

$$\begin{aligned} D_V X^{\omega_1 \dots \omega_r}(\gamma) &\equiv \frac{d}{ds}|_{s=0} X^{\omega_1 \dots \omega_r}(\varphi_s.\gamma) \\ &= \sum_{i=1}^{r} X^{\omega_1 \dots \omega_{i-1} (L_Y \omega_i) \omega_{i+1} \dots \omega_r}(\gamma) \end{aligned} \tag{143}$$

where $\omega_1, ..., \omega_r \in \bigwedge^1 \mathcal{M}$ (resp., $\in C; \bigwedge^1 \mathcal{M} \otimes gl(m)$), and $L_Y \omega$ means Lie derivative of the 1-form ω, in the direction of Y. In the above formula, $D_V X^{\omega_1 \dots \omega_r}(\gamma)$ denotes the "Fréchet" Derivative of $X^{\omega_1 \dots \omega_r}$, at γ, in the direction of the "tangent vector" $V = Y \circ \gamma \in \gamma^* T\mathcal{M}$, and will be treated in detail in the next section.

Using Cartan formula $L_Y = \iota_Y d + d\iota_Y$ as well as (8-11), we can deduce the following more explicit formula:

$$\begin{aligned} D_V X^{\omega_1 \dots \omega_r}(\gamma) &= \sum_{i=1}^{r} X^{\omega_1 \dots \omega_{i-1} . (\iota_Y d\omega_i) . \omega_{i+1} \dots \omega_r}(\gamma) \\ &+ \sum_{i=2}^{r} X^{\omega_1 \dots \omega_{i-2} . \iota_Y(\omega_{i-1} \wedge \omega_i) . \omega_{i+1} \dots \omega_r}(\gamma) \\ &+ \omega_r(V(1)) X^{\omega_1 \dots \omega_{r-1}}(\gamma) - \omega_1(V(0)) X^{\omega_2 \dots \omega_r}(\gamma) \end{aligned} \tag{144}$$

Consider now, the "pointed" Diffeomorphism Group $Diff_p(\mathcal{M})$, consisting of the diffeomorphisms φ that fix the point p. Its "Lie algebra" $\mathcal{X}_p(\mathcal{M})$, consists of the vector fields Y that vanish on p.

$Diff_p(\mathcal{M})$ acts naturally on $\mathcal{A}_p$ through:

$$\begin{aligned} (\varphi, X^{\omega_1 \dots \omega_r}) &\mapsto \varphi . X^{\omega_1 \dots \omega_r} \\ &\equiv X^{\varphi^*\omega_1 \dots \varphi^*\omega_r} \end{aligned} \tag{145}$$

Each such φ is an Hopf algebra automorphism, i.e.:

$$\varphi.(X^{\mathbf{u}} X^{\mathbf{v}}) = (\varphi.X^{\mathbf{u}})(\varphi.X^{\mathbf{v}}) \quad and \quad \Delta \circ \varphi = (\varphi \otimes \varphi) \circ \Delta \tag{146}$$

Thus, φ induces an automorphism of $\widetilde{\mathbf{L}\mathcal{M}_p}$, through:

$$\begin{aligned}\varphi.\tilde{\alpha}(X^{\omega_1\ldots\omega_r}) &\equiv \tilde{\alpha}(\varphi.X^{\omega_1\ldots\omega_r})\\ &= \tilde{\alpha}(X^{\varphi^*\omega_1\ldots\varphi^*\omega_r}) \end{aligned}\tag{147}$$

Considered as an element of $Aut(\widetilde{\mathbf{L}\mathcal{M}_p})$, φ has a "differential" $d\varphi : \widetilde{l\mathcal{M}_p} \to \widetilde{l\mathcal{M}_p}$, defined by:

$$d\varphi(\delta)(X^{\omega_1\ldots\omega_r}) \equiv \delta(X^{\varphi^*\omega_1\ldots\varphi^*\omega_r})\tag{148}$$

and so, $\varphi \mapsto d\varphi$ give us a linear representation of $Diff_p(\mathcal{M})$ on $\widetilde{l\mathcal{M}_p}$. The corresponding infinitesimal action of $Y \in \mathcal{X}_p(\mathcal{M})$ on δ, denoted by $Y \cdot \delta$, is given by:

$$\big(Y\cdot\delta\big)(X^{\omega_1\ldots\omega_r}) = \sum_{i=1}^{r} \delta(X^{\omega_1\ldots\omega_{i-1}.(L_Y\omega_i).\omega_{i+1}\ldots\omega_r})\tag{149}$$

Recall that $Y(0) = 0 = Y(1)$. Using this, together with Cartan formula and relations (52-54), defining the ideal $\mathbf{J}_p$, we compute that:

$$\begin{aligned}\big(Y\cdot\delta\big)(X^{\omega_1\ldots\omega_r}) &= \sum_{i=1}^{r} \delta(X^{\omega_1\ldots\omega_{i-1}.(\iota_Y d\omega_i).\omega_{i+1}\ldots\omega_r})\\ &+ \sum_{i=2}^{r} \delta(X^{\omega_1\ldots\omega_{i-2}.\iota_Y(\omega_{i-1}\wedge\omega_i).\omega_{i+1}\ldots\omega_r}) \end{aligned}\tag{150}$$

5.2 Variational Derivative. Relation with Area Derivative. Homotopy Invariants

Consider a path $\gamma \in \mathcal{PM}_p$, based at p, and the "tangent space" $T_\gamma\mathcal{PM}_p$, to $\mathcal{PM}_p$ at γ. By definition, $T_\gamma\mathcal{PM}_p$ consists of sections of the pull-back bundle $\gamma^*T\mathcal{M}$, i.e., of vector fields along γ, that vanish on p. Fix a "tangent vector" $V \in T_\gamma\mathcal{PM}_p$, and let $s \mapsto \gamma_s$ be a curve of paths in $\mathcal{PM}_p$, starting at γ, in "time" $s = 0$, with velocity V, i.e.:

$$\gamma_0 = \gamma\tag{151}$$

$$V(t) = \frac{\partial}{\partial s}|_{s=0}\gamma_s(t)\tag{152}$$

$$V(0) = 0.\tag{153}$$

We call $s \mapsto \gamma_s$ a *Variation* of $\gamma = \gamma_0$, with associated *Variational Vector Field* V. For example, in the previous section we have consider the special case where $\gamma_s = \varphi_s \circ \gamma$ and $V = Y \circ \gamma$.

We may compute the "Fréchet" derivative of the path functionals $X^{\omega_1\ldots\omega_r}$, at $\gamma \in \mathcal{PM}_p$. By definition, this derivative is the linear map $D.X^{\omega_1\ldots\omega_r}(\gamma) : T_\gamma\mathcal{LM} \to R$ (resp.,$C, gl(m)$), given by:

$$D_V X^{\omega_1 \ldots \omega_r}(\gamma) \equiv \frac{d}{ds}|_{s=0} X^{\omega_1 \ldots \omega_r}(\gamma_s) \tag{154}$$

Of course, if V is induced by restriction to γ of a vector field $Y \in \mathcal{X}_p\mathcal{M}$, i.e., $V = Y \circ \gamma$ (for example, if γ is embedded), then we have the situation considered in the previous section. In the general case, to compute the RHS of (156), we use the following Lemma (see [H], chpt.12):

5.2.1 Lemma

Let N be a manifold (in our case, N will be I or S^1), $\gamma : N \to \mathcal{M}$ an imersion, and ω a differential form in $\mathcal{M}$.

Assume that $\Gamma : N \times [0, \epsilon] \to \mathcal{M}$ is a smooth variation of γ, with *Variational Vector Field* V. That is, putting $\gamma_s(t) = \Gamma(t, s), \quad \forall (t, s) \in N \times [0, \epsilon]$, we have $\gamma_o = \gamma$ and $V(t) = \frac{\partial}{\partial s}|_{s=0}\Gamma(t, s) = \Gamma_{\star(t,0)}(\frac{\partial}{\partial s}|_{(t,0)}), \quad \forall t \in N$.

Then, as differential forms on $N = N \times \{0\}$ we have:

$$\begin{aligned} \frac{d}{ds}|_{s=0} {\gamma_s}^* \omega &= \gamma^*\big(\iota_V d\omega + d(\iota_V \omega)\big) \\ &= \gamma^*(\iota_V d\omega) + d\big(\gamma^*(\iota_V \omega)\big). \end{aligned} \tag{155}$$

Notational Convention...For each $t \in N$, $\iota_{V(t)}\omega$ is the contraction of $\omega(\gamma(t))$ with $V(t) \in T_{\gamma(t)}\mathcal{M}$. So, it's a form in $T_{\gamma(t)}\mathcal{M}$. $\gamma^*(\iota_{V(t)}\omega)$ is the pull-back of this form, to give a form on $T_t N$, as t varies on N. This defines a differential form on N (denoted by $\gamma^*(\iota_V \omega)$, in the above formula (157)) to which we apply d. This is the meaning of $d\big(\gamma^*(\iota_V\omega)\big)$ in formula (157). The other term $\gamma^*(\iota_V d\omega)$ is similarly interpreted, with $d\omega$ replacing ω.

Now, if $\gamma : I \to \mathcal{M}$ is an imersed path, based at p, γ_s a variation, with variational vector field V, we can apply the above Lemma to compute that:

$$\begin{aligned} \frac{d}{ds}|_{s=0}\Big(\int_{\gamma_s} \omega\Big) &= \frac{d}{ds}|_{s=0}\Big(\int_I {\gamma_s}^* \omega\Big) \\ &= \int_I \gamma^*\big(\iota_V d\omega + d(\iota_V\omega)\big) \\ &= \int_I \gamma^*(\iota_V d\omega) + \int_{\partial I} \gamma^* d(\iota_V \omega) \\ &= \int_I \gamma^*(\iota_V d\omega) + \omega(V(1)) - \omega(V(0)) \\ &= \int_\gamma \iota_V d\omega + \omega(V(1)) \end{aligned} \tag{156}$$

where in the last equality we have used the notation $\int_\gamma \iota_V d\omega$ for $\int_I \gamma^*(\iota_V d\omega)$.

In particular, for a loop $\gamma \in \mathcal{LM}_p$, using that notation and the fact that $V(0) = 0 = V(1)$ we have:

$$D_V X^{\omega}(\gamma) = X^{\iota_V d\omega}(\gamma) = \int_\gamma \iota_V d\omega. \tag{157}$$

Finally, using Lemma 5.2.1, (8-9) and induction, we can deduce that:

$$\begin{aligned} D_V X^{\omega_1 \ldots \omega_r}(\gamma) &= \sum_{i=1}^{r} \int_\gamma \omega_1 \ldots \omega_{i-1} . \iota_V (d\omega_i) . \omega_{i+1} \ldots \omega_r \\ &+ \sum_{i=2}^{r} \int_\gamma \omega_1 \ldots \omega_{i-2} . \iota_V (\omega_{i-1} \wedge \omega_i) . \omega_{i+1} \ldots \omega_r \\ &+ \Big(\int_\gamma \omega_1 \ldots \omega_{r-1} \Big) . \omega_r (V(1)) \end{aligned} \tag{158}$$

where we used the above mentioned notational conventions.

For an imersed loop $\gamma \in \mathcal{LM}_p$, we consider the restricted class of variations V, that keep the base point p fixed:

$$\mathcal{V}_p \equiv \{ V \in \gamma^* T\mathcal{M} : \quad V(0) = 0 = V(1) \} \tag{159}$$

Any solution Ψ of the equation:

$$D_\gamma \Psi(V) = 0 \qquad \forall V \in \mathcal{V}_p \tag{160}$$

is called a (relative) *Homotopy Invariant* of the loop γ.

It's useful, in this moment, to analyze the relationship between the above mentioned "Fréchet" Derivative and the Area Derivative introduced in section 4.2. Thus, define an element of $\widetilde{l\mathcal{M}_p}$, by:

$$\Theta(\gamma; V) \equiv \int_0^1 \delta_{(\gamma_o^t; V(t) \wedge \dot\gamma(t))}(\gamma(t)) dt \tag{161}$$

where $V \in \mathcal{V}_p$ and γ_o^t denotes the portion of γ, from $\gamma(0)$ to $\gamma(t)$. The meaning of the above expression is, as usual, the following: for each $X^{\mathbf{u}}$:

$$\Theta(\gamma; V)(X^{\mathbf{u}}) \equiv \int_0^1 \delta_{(\gamma_o^t; V(t) \wedge \dot\gamma(t))}(\gamma(t))(X^{\mathbf{u}}) dt \tag{162}$$

if, of course, the RHS is well defined.

Now, using the notations of section 4.2, we can prove, after a tedious computation, the following formula (see figure 7):

$$\begin{aligned} D_V X^{\mathbf{u}}(\gamma) &= \int_0^1 \triangle_{(\gamma_o^t; V \wedge \dot\gamma)}(\gamma(t)) X^{\mathbf{u}} dt & (163) \\ &= \gamma \circ \Big(\Theta(\gamma; V) \otimes 1 \Big) \circ \Delta X^{\mathbf{u}} & (164) \end{aligned}$$

Figure 7: $\int_\gamma \Delta_{(\gamma_o^t;V\wedge\dot\gamma)}(\gamma(t))dt$

.

For example, if $U : \mathbf{L}\mathcal{M}_p \to gl(n;R)$ is the holonomy of a connection ω on the trivial bundle $R^n \times \mathcal{M} \to \mathcal{M}$ (see section 2), then, using the above formulas, we can compute that:

$$D_V U(\gamma) = U_\gamma . \Big(\int_0^1 U_{\gamma_0^t} . \Omega_{\gamma(t)}\big(V(t) \wedge \dot\gamma(t)\big) . U_{(\gamma_0^t)^{-1}} \Big) \tag{165}$$

a formula that it's called sometimes the "Non-Abelian Stokes Theorem" (see [FGK]).

Incidently, the same computations show that:

$$\delta_{(\lambda;u\wedge v)} U_\lambda = U_\lambda . \Omega_{\lambda(1)}(u \wedge v) . U_{\lambda^{-1}} \tag{166}$$

Using definition (162) as well as the above "Non-abelian Stokes Theorem" (167), we can produce, in a constructive way, several such homotopy invariants, which can be useful for example for loop representation of Chern-Simmons Theory. Let us detail this point.

Consider a system of 1-forms on $\mathcal{M}$, of type:

$$\begin{array}{cccccc}
\omega_1 & \omega_2 & \omega_3 & \ldots & \ldots & \omega_r \\
\omega_{12} & \omega_{23} & \omega_{34} & \ldots & \ldots & \omega_{r-1\,r} \\
\omega_{123} & \omega_{234} & \omega_{345} & \ldots & \omega_{r-2\,r-1\,r} & \\
\ldots & \ldots & \ldots & & & \\
\omega_{12\ldots r} & & & & &
\end{array}$$

and, with them, construct the nilpotent connection 1-form (see section 2):

$$\omega = \begin{bmatrix}
0 & \omega_1 & \omega_{12} & \omega_{123} & \ldots & \omega_{12\ldots r} \\
0 & 0 & \omega_2 & \omega_{23} & \ldots & \omega_{2\ldots r} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 & \omega_r \\
0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}$$

The holonomy of this connection is given by:

$$U = \begin{bmatrix}
1 & \int \mathbf{u}_1 & \int \mathbf{u}_{12} & \int \mathbf{u}_{123} & \ldots & \int \mathbf{u}_{12\ldots r} \\
0 & 1 & \int \mathbf{u}_2 & \int \mathbf{u}_{23} & \ldots & \int \mathbf{u}_{23\ldots r} \\
0 & 0 & 1 & \int \mathbf{u}_3 & \ldots & \int \mathbf{u}_{3\ldots r} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 & \int \mathbf{u}_r \\
0 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}$$

where:

$$\begin{array}{rcl}
\mathbf{u}_1 & = & \omega_1 \\
\mathbf{u}_{12} & = & \omega_1\omega_2 + \omega_{12} \\
\mathbf{u}_{123} & = & \omega_1\omega_2\omega_3 + \omega_{12}\omega_3 + \omega_1\omega_{23} + \omega_{123} \\
& \ldots\ldots\ldots & \\
\mathbf{u}_{12\ldots r} & = & \omega_1\ldots\omega_r + \omega_{12}\omega_{3\ldots r} + \ldots + \omega_{12\ldots r} \\
\mathbf{u}_2 & = & \omega_2 \\
\mathbf{u}_{23} & = & \omega_2\omega_3 + \omega_{23} \\
& \ldots\ldots\ldots &
\end{array}$$

and the curvature of ω is:

$$\Omega = \left[\begin{array}{cccccc}
0 & W_1 & W_{12} & W_{123} & \ldots & W_{12\ldots r} \\
0 & 0 & W_2 & W_{23} & \ldots & W_{2\ldots r} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 & W_r \\
0 & 0 & 0 & \ldots & 0 & 0
\end{array}\right]$$

where:

$$\begin{array}{rcl}
W_1 & = & d\omega_1 \\
W_{12} & = & \omega_1 \wedge \omega_2 + d\omega_{12} \\
W_{123} & = & \omega_1 \wedge \omega_{23} + \omega_{12} \wedge \omega_3 + d\omega_{123} \\
& & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
W_{12\ldots r} & = & \omega_1 \wedge \omega_{2\ldots r} + \omega_{12} \wedge \omega_{3\ldots r} + \ldots + d\omega_{12\ldots r} \\
W_2 & = & d\omega_2 \\
W_{23} & = & \omega_2 \wedge \omega_3 + d\omega_{23} \\
& & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

As is well known (see [KN]) the holonomy U is a homotopy invariant, iff the curvature is zero. In fact, this can be proved directly, using equation (162) as well as the nonabelian Stokes theorem (167). So, forcing the curvature Ω to be 0, i.e., forcing the above 2-forms $W_{12\ldots r}$ to be 0, we have that each entry in the above matrix U gives an homotopy invariant.

To construct such a *Flat Nilpotent Connection*, assume that $H^2(\mathcal{M}, R) = 0$ and start with closed 1-forms $\omega_1, \ldots, \omega_r$. Then, $\omega_1 \wedge \omega_2$,...,$\omega_{r-1} \wedge \omega_r$ are also closed 2-forms and we can solve the equations obtained equalizing to zero each entry of the matrix Ω, for ω_{12},...,$\omega_{r-1\,r}$,...,$\omega_{1\ldots r}$, step by step.

In fact, since $\omega_1 \wedge \omega_2$,...,$\omega_{r-1} \wedge \omega_r$ are closed 2-forms, they are also exact, and thus, we can find $\omega_{12}, \ldots, \omega_{r-1\,r}$. Now:

$$d(\omega_1 \wedge \omega_{23} + \omega_{12} \wedge \omega_3) = \omega_1 \wedge \omega_2 \wedge \omega_3 - \omega_1 \wedge \omega_2 \wedge \omega_3 = 0$$

and we can find ω_{123}, and so on.

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