Synchrony in Lattice Differential Equations*

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Abstract

We survey recent results on patterns of synchrony in lattice differential equations on a square lattice. Lattice differential equations consist of choosing a phase space $\mathbf{R}^m$ for each point in a lattice and a system of differential equations on each of these phase spaces such that the whole system is translation invariant. The architecture of a lattice differential equation is the specification of which sites are coupled to which (nearest neighbor coupling is a standard example). A polydiagonal is a finite-dimensional subspace obtained by setting coordinates in different phase spaces equal. A polydiagonal Δ has k colors if points in Δ have at most k unequal cell coordinates. A *pattern of synchrony* is a polydiagonal that is flow-invariant for every lattice differential equation with a given architecture. We survey two main results: the classification of two-color patterns of synchrony and the fact that every pattern of synchrony for a fixed architecture is spatially doubly

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periodic assuming that the architecture includes both nearest and next nearest neighbor couplings.

1 Introduction

In this paper we survey recent results on lattice differential equations (LDE) [6, 11, 12, 2, 1] that show that certain characteristics of the dynamics of LDE depend significantly on whether the coupling is nearest neighbor coupling or a more general coupling. In our exposition we focus on planar square lattices. We show that patterns of synchrony that occur naturally in these systems can be aperiodic in the case of nearest neighbor (NN) coupling and must be spatially doubly periodic when both nearest and next nearest coupling (NNN) are present. The framework of our work is a general theory of patterns of synchrony in coupled systems developed by Stewart *et al.* [10, 9]. The dynamics of LDE are also discussed in Chow *et al.* [3, 4, 5].

We index the planar square lattice $\mathcal{L} \cong \mathbf{Z}^2$ by pairs of integers (i, j) and call each lattice point a *cell*. The architecture of a lattice dynamical system specifies which cells are coupled to which, which cells have the same dynamics, and which couplings are the same. The set $I(c)$, the *input set* of a cell $c \in \mathcal{L}$, is the set of all cells coupled to c. For example, with NN coupling the input set of cell (i, j) is

$$I(i,j) = \{(i+1,j),(i-1,j),(i,j+1),(i,j-1)\}$$

whereas, with NNN coupling the input set is

$$\begin{aligned} I(i,j) = \{&(i+1,j),(i-1,j),(i,j+1),(i,j-1)\} \cup \\ \{&(i+1,j+1),(i-1,j+1),(i+1,j-1),(i-1,j-1)\} \end{aligned}$$

See Figure 1.1.

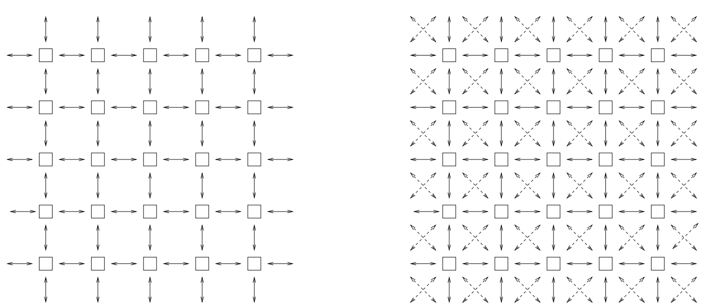

Figure 1.1: (Left) square lattice network with NN coupling (—). (Right) square lattice network with NN and NNN couplings (- - -).

We assume that the phase space of cell c is $\mathbf{R}^m$ and we denote the coordinates in this phase space by x_c. We consider only those LDE that are equivariant with respect to the symmetries of the lattice. Square lattice symmetries are generated by translations within the lattice ($\mathcal{L}$ itself) and rotations and reflections of the origin that preserve the lattice. The latter group is isomorphic to $\mathbf{D}_4$, the eight element symmetry group of the square. Thus the group of symmetries of the square lattice is

$$\mathcal{G}_{\mathcal{L}} = \mathbf{D}_4 \dot{+} \mathcal{L}$$

Equivariance with respect to $\mathcal{L}$ requires that m is constant for all cells.

Suppose that the input set $I(c) = \{c_1, \ldots, c_p\}$. Then we denote the coordinates in the input set, the *coupling variables*, by

$$x_{I(c)} = (x_{c_1}, \ldots, x_{c_p}) \in (\mathbf{R}^m)^p$$

With this notation a *lattice differential equation* is a coupled system of differential equations of the form

$$\dot{x}_{i,j} = g_{i,j}(x_{i,j}, x_{I(i,j)})$$

where $(i,j) \in \mathcal{L}$, $x_{i,j} \in \mathbf{R}^m$, and $g_{i,j}$ is a function that depends on the internal cell variable $x_{i,j}$ and the coupling variables $x_{I(i,j)}$. Moreover, equivariance with respect to translations implies that $g_{i,j} = g$ is independent of the cell.

With NN coupling the square symmetry $\mathbf{D}_4$ forces g to be invariant under permutations of the coupling variables. Thus square lattice differential equations with nearest neighbor coupling have the form

$$\dot{x}_{i,j} = g(x_{i,j}, \overline{x_{i+1,j}, x_{i-1,j}, x_{i,j+1}, x_{i,j-1}}) \tag{1.1}$$

where g is invariant under all permutations of the coupling variables under the bar.

Synchrony is one of the most interesting features of coupled cell systems and in order to study it, we need to formalize the concept. We use a strong form of network synchrony, which we now define. A *polydiagonal* Δ is a subspace of the phase space of a coupled cell system that is defined by equality of cells coordinates.

Definition 1.1. A polydiagonal Δ is a *pattern of synchrony* if Δ is flow-invariant for every coupled cell system with the given network architecture.

Solutions in a flow-invariant polydiagonal Δ have sets of coordinates that remain equal for all time, and hence have coordinates that are synchronous in this very strong way. Coloring two cells the same when

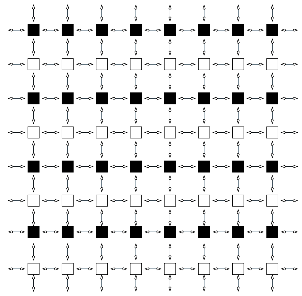

Figure 1.2: Horizontal two-color pattern of synchrony in square lattice with NN coupling.

the coordinates in Δ are required to be equal permits us to visualize the pattern of synchrony associated with a flow-invariant polydiagonal. A k-color pattern of synchrony is one which is defined by exactly k colors.

Symmetry generates many patterns of synchrony. Let $\Sigma \subset \mathcal{G}_{\mathcal{L}}$ be a subgroup and define the *fixed-point subspace*

$$\mathrm{Fix}(\Sigma) = \{x : \sigma x = x \quad \forall\, \sigma \in \Sigma\}$$

Fixed-point subspaces are well-known to be flow-invariant for any equivariant system. See [8, Lemma XIII 2.1] or [7, Theorem 1.17]. Moreover, fixed-point subspaces of subgroups of $\mathcal{G}_{\mathcal{L}}$ are polydiagonals; so, for lattice differential equations, fixed-point subspaces are patterns of synchrony. As an example, let Σ be generated by two translations $\sigma_1(i,j) = (i+1,j)$ and $\sigma_2(i,j) = (i,j+2)$. Then $\mathrm{Fix}(\Sigma)$ is $2m$-dimensional and consists of points where $x_{i,j} = x_{k,l}$ whenever $j \equiv l \mod 2$. In this case $\mathrm{Fix}(\Sigma)$ is a 2-color pattern of synchrony on the square lattice with NN coupling. See Figure 1.2.

It is not the case that every pattern of synchrony in a coupled cell system is a fixed-point subspace. Stewart *et al.* [10, Theorem 6.1] prove that a polydiagonal is a pattern of synchrony if and only if the coloring associated to the polydiagonal is balanced.

Definition 1.2. A coloring is *balanced* if for for every pair of cells c and d that have the same color there is a bijection from $I(c)$ to $I(d)$ that preserves both color and coupling type.

In the horizontal two-color pattern with nearest neighbor coupling in Figure 1.2, all couplings are identical and every black cell receives two black inputs and two white inputs and every white cell receives two black inputs and two white inputs. So this pattern is balanced.

In Section 2 we describe a result from [6] that shows that with NN coupling there are continua of balanced two-colorings most of which are spatially aperiodic. We also describe a result of Wang [11, 12] that classifies all balanced two-colorings up to symmetries in $\mathcal{G}_{\mathcal{L}}$.

There are some subtleties concerning the definition of square lattice LDE with NNN coupling, which we now describe. First we define *square lattice differential equations with nearest and next nearest neighbor coupling*, the NNN case, to be those equations that have the form

$$\begin{aligned}\dot{x}_{i,j} = g(&x_{i,j}, \overline{x_{i+1,j}, x_{i-1,j}, x_{i,j+1}, x_{i,j-1}}, \\ &\overline{x_{i+1,j+1}, x_{i-1,j+1}, x_{i+1,j-1}, x_{i-1,j-1}})\end{aligned} \tag{1.2}$$

where $(i, j) \in \mathbf{Z}^2$, $x_{i,j} \in \mathbf{R}^m$, and g is invariant under independent permutation of the nearest neighbor cells and of the next nearest neighbor cells.

Lattice differential equations can arise from the discretization of systems of partial differential equations. For example, the discretization of a planar reaction-diffusion system leads to

$$\dot{u}_{i,j} \;=\; -\beta^{+}\,\triangle^{+}u_{i,j} - \beta^{\times}\,\triangle^{\times}u_{i,j} - f(u_{i,j})\,, \quad (i,j) \in \mathbf{Z}^2 \tag{1.3}$$

where $\beta^{+} > \beta^{\times} > 0$ and $\triangle^{+}$ and $\triangle^{\times}$ are the *discrete* 2*-dimensional Laplace operators* on $\mathbf{Z}^2$ based on nearest neighbors and next nearest neighbors, respectively, and are given by

$$\begin{aligned}\triangle^{+}u_{i,j} &= u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \\ \triangle^{\times}u_{i,j} &= u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} - 4u_{i,j}\,.\end{aligned}$$

In these equations the parameters β^{+} and $\beta^{\times}$ are coupling parameters, and f is a nonlinear function that represents the “internal dynamics.” When $\beta^{\times} = 0$ this system has only nearest neighbor coupling, which is a discretization that is frequently used. When the nine-point footprint of the discretized Laplacian is used (β^{+} and $\beta^{\times}$ are both nonzero), then we have an example of square lattice differential equations with nearest and next nearest neighbor coupling, since this system is independently invariant under permutation of the nearest neighbor cells and the next nearest neighbor cells.

Note that (1.3) is equivariant with respect to rotations, reflections, and translations that preserve the lattice. There are, however, examples of $\mathcal{G}_{\mathcal{L}}$-equivariant systems that are not in the form of square lattice differential equations with nearest and next nearest neighbor coupling; that is, g is not invariant under independent permutation of the nearest and the next nearest neighbor cells. For example, consider the $\mathbf{D}_4$-invariant function

$$\begin{aligned}g = &x_{1,0}(x_{1,1} + x_{1,-1}) + x_{0,1}(x_{-1,1} + x_{1,1}) + \\ &x_{-1,0}(x_{-1,-1} + x_{-1,1}) + x_{0,-1}(x_{1,-1} + x_{-1,-1})\end{aligned} \tag{1.4}$$

where g is centered at cell $(0,0)$, i.e. $\dot{x}_{0,0} = g$. So, in fact, there are two reasonable and distinct classes of LDE with nearest and next nearest neighbor coupling: NNN equations and $\mathbf{D}_4$-invariant g. Each NNN system is $\mathbf{D}_4$-equivariant, but the converse is false.

In Section 3 we review results in [2] that show that every balanced k-coloring for square lattice NNN equations is spatially doubly periodic and that there are only a finite number of k-colorings for each k. These results show that there are huge differences in patterns of synchrony between the NN and NNN cases. The techniques we develop are general enough to prove similar theorems for other lattices (in particular the other planar lattices); the general principle seems to be that if there is enough coupling, then balanced k-colorings are spatially periodic and finite in number.

Note that every balanced coloring for the $\mathbf{D}_4$-equivariant case is automatically a balanced coloring for the NNN case. So once the finiteness and spatial periodicity theorems about balanced k-colorings are proved for the NNN case, they are automatically valid for the $\mathbf{D}_4$-equivariant case as well. But there may be fewer balanced colorings for the $\mathbf{D}_4$ case than exist for the NNN case, and in fact this happens. See Example 3.11.

2 Square Lattice with NN Coupling

Examples show that with NN coupling there are a continuum of balanced two-colorings, most of which are spatially aperiodic. These examples are easily understood using the trick of interchanging colors along diagonals discovered in [6].

Consider the balanced horizontal two-coloring in Figure 1.2 and the diagonal line drawn in Figure 2.1 (left). If we interchange colors along that diagonal we arrive at the two-coloring in Figure 2.1 (right). Observe that this new coloring is still balanced. For example each black cell still receives two black inputs and two white inputs. The only difference is that the black inputs are now not necessarily along the horizontal. Also note that the new pattern of synchrony is not spatially doubly periodic, even though the original pattern is.

Next we observe that this diagonal trick can be repeated independently on as many diagonals as one wishes. This process leads to patterns of synchrony like those in Figure 2.2. In fact there are a continuum of such patterns of synchrony. To see this, fix any horizontal line H and any binary sequence of colors black and white along H. Now by performing the diagonal trick along each diagonal, if needed, we can arrange for the balanced coloring restricted to H to be the designated binary sequence. Since the number of bi-infinite binary sequences is the same as the real numbers there are a continuum of balanced two-colorings. Of

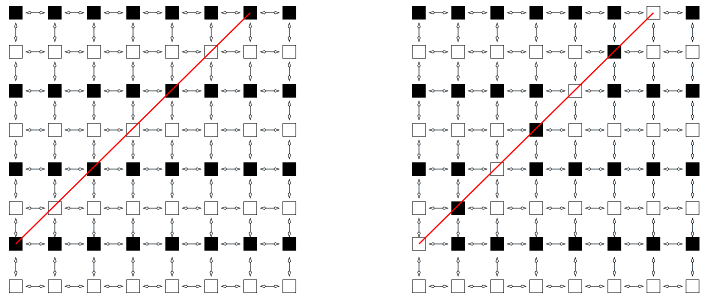

Figure 2.1: Alternating diagonal trick for horizontal two-coloring.

course many of these are identical up to symmetry. But there are only a countable number of symmetries in $\mathcal{G}_{\mathcal{L}}$; so, up to symmetry, there are still a continuum of different balanced two-colorings.

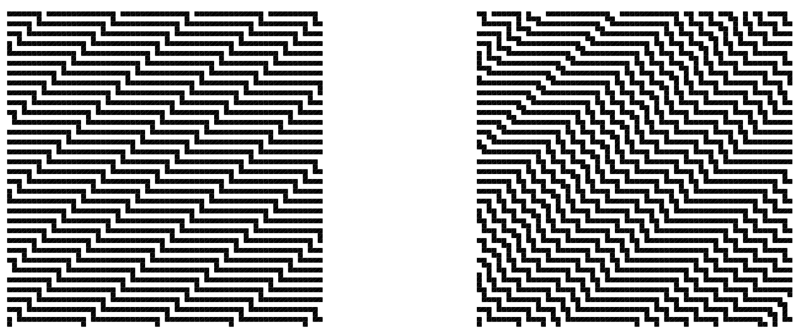

Figure 2.2: Repeating the alternating diagonal trick for horizontal two-coloring.

There is an interesting observation that follows from these results. Suppose that an LDE has an equilibrium inside the horizontal pattern of synchrony in Figure 1.2. Then there is a corresponding equilibrium in every one of the continuum of balanced two-colorings that can obtained from the horizontal one using this diagonal trick. The reason is that LDE's restricted to each of these flow-invariant spaces are identical, and hence each has an equilibrium. (This point can be checked directly from the LDE, but it also follows from the notion of quotient networks discussed in [10, 9].) So symmetry forces a countable number of equilibria with patterns of synchrony that are identical up to symmetry, but network architecture (NN coupling) forces a continuum of equilibria corresponding to patterns of synchrony that are quite different.

Wang [11, 12] classifies, up to symmetry, all balanced two-colorings with NN coupling. There are eight isolated examples, shown in Figure 2.3, and two infinite families, shown in Figure 2.4.

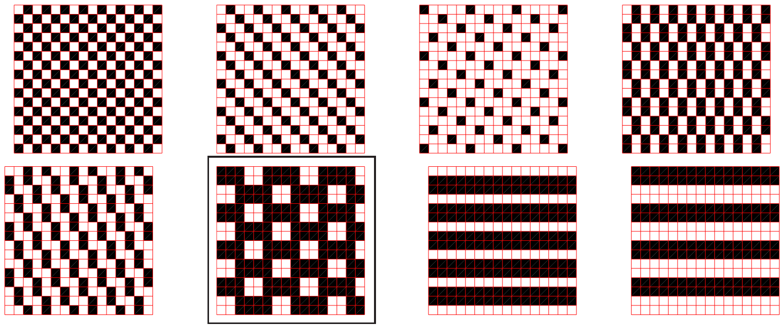

Figure 2.3: Isolated balanced two-colorings on square lattice with NN coupling. The pattern in the box is not a fixed-point subspace of a subgroup of $\mathcal{G}_{\mathcal{L}}$.

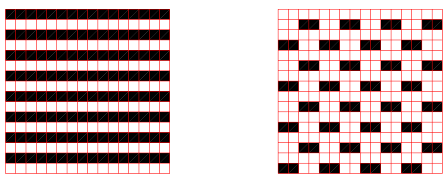

Figure 2.4: Balanced two-colorings on square lattice with NN coupling that generate a continuum of patterns of synchrony by the diagonal trick of interchanging colors on diagonals along which colors alternate.

There is an interesting question: Does the existence of a continuum of balanced two-colorings in the NN case tell us about spurious solutions to discretizations of planar systems of PDE. By themselves, the answer is probably no since two-color solutions are highly oscillatory solutions. If there is to be some such correspondence, we would have to find continua of balanced k-colorings that would, in some sense, converge to a variety of patterns in a sensible limit. For now we just present a continua of balanced k-colorings in Figure 2.5.

3 Square Lattice with NNN Coupling

In this section we discuss the following result proved in [2].

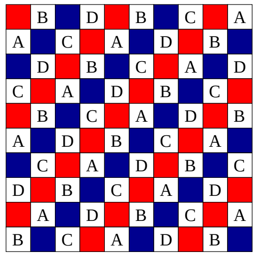

Figure 2.5: A double continua of balanced 6-colorings where diagonals with colors AB and diagonals with colors CD can be interchanged. Similarly, a balanced 5-coloring can be obtained by setting $C = D$. Both examples generalize to examples for any k.

Theorem 3.1. *On the square lattice with both nearest and next nearest coupling, every pattern of synchrony is spatially doubly periodic. Moreover, for fixed k, there are, up to symmetry, at most a finite number of balanced k-colorings.*

Definition 3.2. Let $U \subset \mathcal{L}$ be a subset. The *closure* of U consists of all cells that are connected by some arrow to a cell in U. The *boundary* of U is the set

$$\mathrm{bd}(U) = \mathrm{cl}(U) \smallsetminus U$$

There is a natural expanding sequence of finite subsets that covers the lattice $\mathcal{L}$ and which depends on the kind of coupling. Let

$$W_0 = \{0\} \quad \text{and} \quad W_{i+1} = \mathrm{cl}(W_i) \tag{3.1}$$

for $i \geqslant 0$. Note that for any coloring of a lattice $\mathcal{L}$ by k colors, there is some j such that all k colors are represented by cells in W_j. In fact, more is true for balanced colorings.

Lemma 3.3. *In any balanced k-coloring W_{k-1} contains all k colors.*

Proof. We claim that if $\ell < k$, then W_ℓ contains at least $\ell + 1$ colors. The proof proceeds by induction on W_ℓ. $W_0 = \{0\}$ contains one cell and one color.

Assume that the statement is true for $\ell < k - 1$; we prove that it is also true for $\ell + 1$. Suppose that the number m of colors contained in $W_{\ell+1} = \mathrm{cl}(W_\ell)$ is the same as the number of colors in W_ℓ. Then every cell $c \in W_{\ell+1}$ has a color that is the same as the color of a cell d in W_ℓ. So, all cells connected to d lie in $W_{\ell+1}$ and are colored by the m colors. Therefore, balanced implies that the cells connected to c must also be colored by one of the m colors. It follows that the cells in

$W_{\ell+2} = \mathrm{cl}(W_{\ell+1})$ are also colored by these m colors. By induction the entire lattice is colored by m colors; hence $m = k$. So if $m < k$, the number of colors in $W_{\ell+1}$ must be greater than the number of colors in W_ℓ. That is, $W_{\ell+1}$ contains at least $\ell + 2$ colors. It follows that W_{k-1} contains all k colors. □

It follows from Lemma 3.3 that if we fix a balanced k-coloring to W_k, then there exists a cell e with that color in W_{k-1}. Moreover, all the neighbors of e are in W_k and so all their colors are known. We can ask now about the extension of the pattern from W_k to the whole lattice, more specifically, about the extension of the pattern from W_k to W_{k+1}. Recall that in a balanced k-coloring, for any two cells c and d of the same color, there is a bijection between $I(c)$ and $I(d)$ that preserves arrow type and color. In particular, if we know the color of all cells in $I(c)$ of a certain coupling type and we know the color of all cells of the same coupling type except one in $I(d)$, then since the coloring is balanced we can determine the color of the last cell with that coupling type in $I(d)$.

Definition 3.4. Let $U \subset \mathcal{L}$ be a finite set.

(a) Every cell $c \in U$ is called 0-determined.

(b) A cell $c \in \mathrm{bd}(U)$ is *p-determined*, where $p \geq 1$ if there is a cell $d \in U$ such that c is in the input set of d and each cell in the input set of d that has the same coupling type as c, except c itself, is q-determined for some $q < p$.

(c) A cell $c \in \mathrm{bd}(U)$ is *determined* if it is *p-determined* for some p.

(d) The set U *determines* its boundary if all cells in $\mathrm{bd}(U)$ are determined.

Definition 3.5. The set W_{i_0} is a *window* if W_i determines its boundary for all $i \geqslant i_0$.

Remark 3.6. Note that if there are no 1-determined cells then, by induction, there are no p-determined cells for any p. In particular, if there are no 1-determined cells, then windows do not exist.

Example 3.7. Let $\mathcal{L}$ be the square lattice with nearest neighbor coupling. Then this network has no window.

We claim that no set W_i is a window. By Remark 3.6 it is sufficient to show that there are no 1-determined cells. For example, consider W_2 and its boundary as shown in Figure 3.1. Since the cells on the boundary are in a diagonal line it is not possible for them to be the only cell in the input set of a cell in W_2 that is not in W_2. Note that when $i > 2$ the set W_i has the same "diamond shape" as W_2. So there are no 1-determined cells in $\mathrm{bd}(W_i)$. By Remark 3.6, this network has no window. ◇

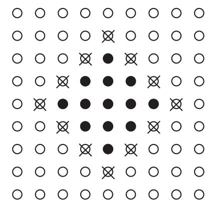

Figure 3.1: The set W_2 (black cells) and its boundary (white cells with a cross).

Example 3.8. Consider $\mathcal{L}$ with four nearest neighbors and four next nearest neighbors. In this case the set W_i is a square of size $2i+1$. Note that all the cells on each side, except the last two on both extremes, are 1-determined, since they are the only nearest neighbor cells outside the square (Figure 3.2). We show that the sets W_i for $i \geqslant 2$ determine their boundaries. To do this, we need (by symmetry) to analyze just one of the four corners.

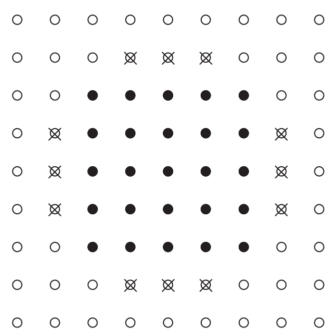

Figure 3.2: The set W_2 (black cells) and the 1-determined cells in its boundary (white cells with a cross).

The three cells in the corners of the square are 2-determined using the next nearest neighbors coupling as long as the square has size greater than 3. See Figure 3.3. $\diamond$

Lemma 3.9. *Assume that W_{i_0} is a window. Suppose that a balanced k-coloring restricted to W_{i-1} for some $i \geqslant i_0$ contains all k colors. Then the k-coloring is uniquely determined on the whole lattice by its restriction to W_i.*

The proof of this lemma [2, Lemma 3.11] uses determinacy.

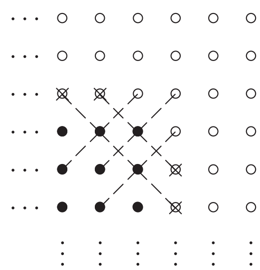

Figure 3.3: The corner of a set W_i (black cells), the 1-determined cells (white cells with a cross) and 2-determined cells (white cells connected to black cells by dashed lines).

Theorem 3.10. *Suppose that a lattice network has a window. Fix $k \geqslant 1$. Then there are a finite number of balanced k-colorings on $\mathcal{L}$ and each balanced k-coloring is spatially multiply-periodic.*

Proof. Let W_j be a window for $G_{\mathcal{L}}$ where $j \geqslant k$. By Lemma 3.3, the interior of W_j contains all k colors. Then by Lemma 3.9, a balanced k-coloring is uniquely determined by its restriction to W_j. Since there is only a finite number of possible ways to distribute k colors on the cells in W_j it follows that there are only a finite number of balanced k-colorings.

Let K be a balanced k-coloring on $G_{\mathcal{L}}$ and let $v \in \mathcal{L}$. Let $T_v(K)$ be the coloring obtained by shifting the coloring K by v, that is, the color of cell c in $T_v(K)$ is the same as the color of cell $c - v$ in K. Since translations are symmetries of the lattice network $T_v(K)$ is also a balanced coloring.

Let v be a generator of the lattice and consider all translates of K in the direction of v. Since there are only a finite number of balanced k-colorings and an infinite number of translates of K, there must exist $N \in \mathbf{Z}^+$, such that K and $T_{Nv}(K)$ exhibit exactly the same coloring. It follows that K is invariant under the translation T_{Nv}. Hence K is periodic in the direction of v. The same argument can be applied to all the generators of the lattice, thus all balanced k-colorings are spatially multiply-periodic. $\square$

Example 3.11. It is straightforward to check that the two-coloring in Figure 3.4 (left) is balanced in the NNN case. Note that up to symmetry there are three different kinds of black cells; see Figure 3.4 (right). Each of these black cells has two white and two black nearest neighbors and three white and one black next nearest neighbors; hence the black cells are NNN balanced. An analogous calculation works for the three types of white cell.

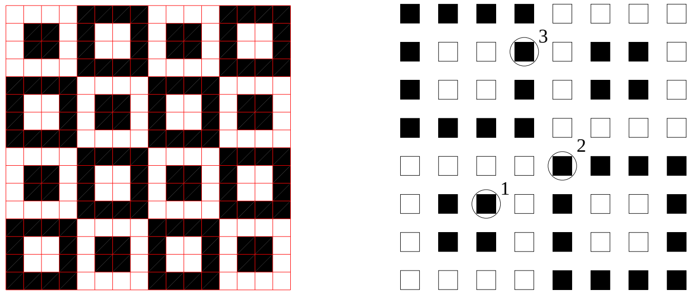

Figure 3.4: A pattern of synchrony of square lattices with NNN coupling that is not flow-invariant for all $\mathbf{D}_4$-equivariant LDE.

We claim that this pattern is not flow-invariant for all $\mathbf{D}_4$-equivariant LDE. Indeed, we can just check this statement using the $\mathbf{D}_4$-invariant function g given in (1.4), which we recall here as

$$\begin{aligned} \dot{x}_{ij} = x_{i+1,j}(x_{i+1,j+1} + x_{i+1,j-1}) + x_{i,j+1}(x_{i+1,j+1} + x_{i-1,j+1}) + \\ x_{i-1,j}(x_{i-1,j+1} + x_{i-1,j-1}) + x_{i,j-1}(x_{i-1,j-1} + x_{i+1,j-1}) \end{aligned} \tag{3.2}$$

Let the black cells have the coordinate x_B and the white cells have the coordinate x_W. Then the differential equation associated to black cell 1 in Figure 3.4 (right) has the form

$$\begin{aligned} \dot{x}_B = x_W(x_W + x_W) + x_W(x_W + x_W) + \\ x_B(x_W + x_B) + x_B(x_B + x_W) \end{aligned} \tag{3.3}$$

and the differential equation associated to the black cell 2 has the form

$$\begin{aligned} \dot{x}_B = x_B(x_W + x_W) + x_W(x_W + x_B) + \\ x_W(x_B + x_W) + x_B(x_W + x_W). \end{aligned} \tag{3.4}$$

However, the right hand side of (3.4) is not equal to the right hand side of (3.3). Hence, the subspace corresponding to Figure 3.4 (left) is not a flow invariant subspace of (3.2). ◇

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