Hopf Bifurcation on Hemispheres

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December 8, 2005

Abstract

Field, Golubitsky and Stewart (Bifurcations on Hemispheres. *J. Nonlinear Sci.* **1** (1991) 201-223.) consider the steady-state bifurcations of reaction-diffusion equations defined on the hemisphere with Neumann boundary conditions on the equator. We consider Hopf bifurcations for these equations. We show the effect of the hidden symmetries on spherical domains for the type of Hopf bifurcations that can occur. We obtain periodic solutions for the hemisphere problem by extending the problem to the sphere and finding then periodic solutions with spherical spatial symmetries containing the reflection across the equator. The equations on the hemisphere have $\mathbf{O}(2)$-symmetry and the equations on the sphere have spherical symmetry. We find orbits of periodic solutions for the sphere problem containing multiple periodic solutions that restrict to periodic solutions of the Neumann boundary value problem on the hemisphere lying on different $\mathbf{O}(2)$-orbits.

Mathematics Subject Classification: 35B32, 35K57, 35B10

1 Introduction

It is known that the bifurcation behavior of reaction-diffusion equations on certain domains with certain boundary conditions can be nongeneric taking into account

[*]CMUP is supported by FCT through POCTI and POSI of Quadro Comunitário de Apoio III (2000-2006) with FEDER and national fundings.

only the symmetries of the equations. Let $u_t = \mathcal{P}(u)$ be a system of reaction-diffusion equations posed on a bounded domain $N \subset \mathbf{R}^n$ with Neumann boundary conditions (NBC) on the boundary of N. It follows that the group of the physical symmetries of the equations is the compact subgroup Γ of $\mathbf{E}(n)$ that preserves the domain N and the boundary conditions. For certain domains these problems possess more symmetry than it is immediately apparent, and these 'hidden symmetries' in $\mathbf{E}(n)\backslash\Gamma$ can be responsible for subtle changes in the generic bifurcations of such systems. These extra symmetries naturally appear when the equations may be extended to larger domains with larger symmetry groups. This was first noticed by Fujii *et al.* [7] and formalized by Armbruster and Dangelmayr [1] for reaction-diffusion equations on an interval with NBC. See the review paper of Gomes *et al.* [10] and references therein.

Field *et al.* [6] studied hidden symmetries on hemispherical domains, and described a general setting for hidden symmetries induced by Neumann and Dirichlet boundary conditions on a large class of spatial domains. Moreover, these authors considered steady-state bifurcations for reaction-diffusion equations defined on the hemisphere with NBC along the equator. As they pointed out, such equations have a natural $\mathbf{O}(2)$-symmetry but may be extended to the full sphere where the natural symmetry group is $\mathbf{O}(3)$. They show that the expected bifurcations are governed not by circular symmetry but by spherical symmetry. More precisely, solutions to the Neumann boundary problem on the hemisphere can be found by first finding solutions to the extended problem on the sphere invariant by the reflection across the equator (the boundary of the hemisphere). Their results were recently applied when modelling the growth of plants, see Nagata *et al.* [13]. Much of the growth of plants occurs by the elongation of cylindrical stalks or roots by action mainly at a dome-shaped tip, as referred in [13]. This fact suggested the authors that a hemisphere would be a reasonable working approximation for mathematical study. Other applications include elastic buckling of hemispherical shells, see Bauer *et al.* [2, 3].

In this paper we show the effects of the hidden symmetries on spherical domains for the type of Hopf bifurcations that can occur. Specifically, we consider the Hopf bifurcations of reaction-diffusion equations defined on the hemisphere with NBC along the equator.

Golubitsky and Stewart [8] give a list of those conjugacy classes of isotropy subgroups of $\mathbf{O}(3) \times \mathbf{S}^1$ (action on $V_l \oplus V_l$ for each l) that have two-dimensional fixed-point subspaces. Here, V_l denotes the space of spherical harmonics of order l. See Tables 2 and 3 for the natural representation of $\mathbf{O}(3)$. The Equivariant Hopf Theorem guarantees the existence, in generic Hopf bifurcation problems with $\mathbf{O}(3)$ symmetry, of a branch of periodic solutions for each isotropy subgroup in each of the conjugacy classes.

We obtain periodic solutions for the hemisphere problem by extending the problem to the sphere and finding then periodic solutions with spherical spatial symmetries containing the reflection across the equator. Namely, we determine group orbits of periodic solutions in generic $\mathbf{O}(3)$-equivariant Hopf bifurcations problems that have representative solutions with symmetry containing the above reflection. Each orbit contains multiple periodic solutions that restrict to periodic solutions of the Neumann boundary value problem on the hemisphere lying on different $\mathbf{O}(2)$-orbits. See Theorems 4.1, 4.2, 5.2, 5.5. In this way we find solutions to the hemisphere problem that would not have been expected if we would have considered only the circular symmetry of the hemispherical domain.

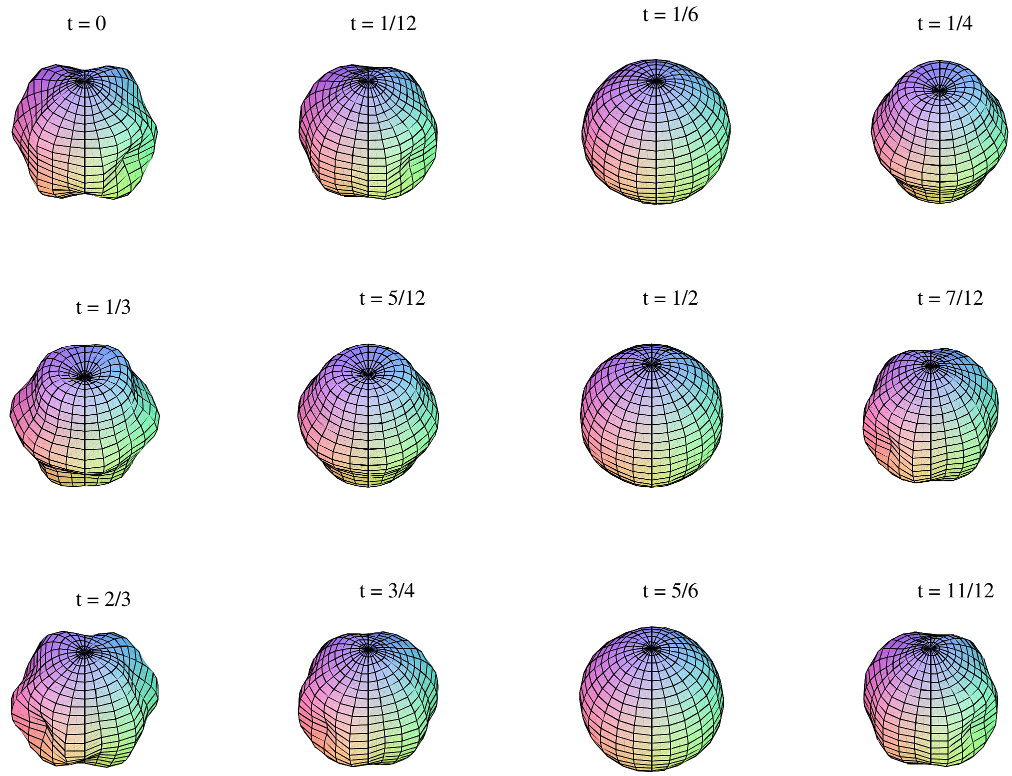

Figure 1: Oscillations of a sphere deformed by spherical harmonics of order $l = 6$: spatial symmetry $\mathbf{D}_2 \oplus \mathbf{Z}_2^c$ and spatio-temporal symmetry $(\mathbf{T} \oplus \mathbf{Z}_2^c)^\theta$ (and twist type $\mathbf{Z}_3$).

We conclude the introduction by illustrating two examples contained in our results. The Equivariant Hopf Theorem guarantees the generic existence for each l of an orbit of periodic solutions with an axis of rotation for the sphere problem. In Theorem 4.1, we prove that when l is even, this orbit of periodic solutions restricts to solutions of the hemisphere problem as follows: an isolated solution with $\mathbf{O}(2)$-symmetry, and an $\mathbf{O}(2)$-orbit of solutions with $\mathbf{D}_2^-$-symmetry. The Equivariant Hopf

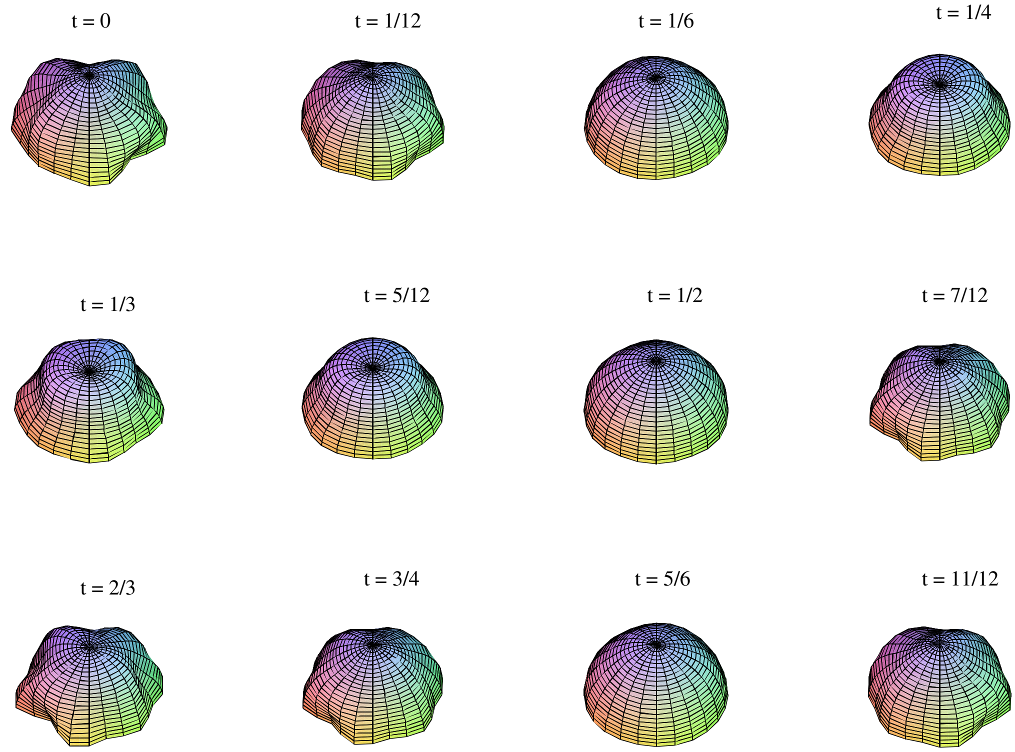

Figure 2: (Orbit 1) Restriction of solution of Figure 1 to hemisphere with spatial symmetry $\mathbf{D}_2^-$.

Theorem also guarantees the existence of an orbit of periodic solutions with twisted tetrahedral symmetry of twist type $\mathbf{Z}_3$, when $l = 2, 4, 6$ (see entry 6 of Table 2). In Theorem 5.2 (e), we show that this orbit of periodic solutions restricts to the hemisphere problem to three $\mathbf{O}(2)$-orbits of solutions with spatial dihedral symmetry $\mathbf{D}_2^-$ of trivial twist type, that is, they have no spatio-temporal symmetries. We show in Figures 1, 2 periodic one-parameter families of deformations of the sphere with twisted tetrahedral symmetry and their restriction to the hemisphere for $l = 6$. See Section 6 for details.

This paper is organized in the following way. In Section 2 we state the main results on Hopf bifurcation with spherical symmetry. The problem of Hopf bifurcations of reaction-diffusion equations defined on the hemisphere with Neumann boundary conditions along the equator is described in Section 3. The main results of this paper are obtained in Sections 4 and Section 5. Theorems 4.1 and 4.2 describe solutions of the sphere problem with an axis of rotation, for l even and odd respectively, that restrict to solutions of the hemisphere problem. In Theorems 5.2 and 5.5 we consider periodic solutions with finite spatial symmetry, for l even and odd respectively. Figures illustrating the results obtained in the previous theorems are presented in Section 6. A more abstract formulation of the extension problem is presented in Section 7.

2 Hopf Bifurcation with $\mathbf{O}(3)$-symmetry

In this section, we study the existence of branches of periodic solutions for $\mathbf{O}(3)$-equivariant bifurcation problems. We state, without proofs, the main results on Hopf bifurcation with spherical symmetry ([8], see also [9, Section XVIII 5]). Namely, we give, for each irreducible representation V_l of the orthogonal group $\mathbf{O}(3)$, a list of isotropy subgroups $\Sigma \subset \mathbf{O}(3) \times \mathbf{S}^1$ for which the Equivariant Hopf Theorem proves the existence of a branch of periodic solutions. Specifically, we give a classification of those isotropy subgroups Σ for which $\dim \mathrm{Fix}_{V_l \oplus V_l}(\Sigma) = 2$, when V_l is any irreducible representation of $\mathbf{O}(3)$.

Consider the smooth $\mathbf{O}(3)$-equivariant system of ordinary differential equations

$$\frac{dx}{dt} = F(x, \lambda) \tag{2.1}$$

and assume that the system has a generic Hopf bifurcation at the origin. Generically Hopf bifurcation in such systems occurs when the center subspace $E = V \oplus V$, where V is an absolutely irreducible representation of $\mathbf{O}(3)$. Moreover, after an equivariant change of coordinates and a rescaling of time one can assume that $(dF)_{0,0}|E$ has the form

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

where $m = \dim(V)$.

Observe that $\mathbf{O}(3) \times \mathbf{S}^1$ acts on $V \oplus V$, where the action of $\theta \in \mathbf{S}^1$ is given by $e^{i\theta J}$. Generically, we can also assume that the complex eigenvalues that extend $\pm i$ when $\lambda = 0$ cross the imaginary axis with nonzero speed.

The subgroup $\Sigma \subset \mathbf{O}(3) \times \mathbf{S}^1$ is $\mathbf{C}$-axial if Σ is an isotropy subgroup and if $\dim\, \mathrm{Fix}_E(\Sigma) = 2$. The Equivariant Hopf Theorem states that for every $\mathbf{C}$-axial Σ there exists a branch of small amplitude periodic solutions with spatiotemporal symmetries Σ. That is, if $x(t)$ is on the branch of solutions and if $(\sigma, \theta) \in \Sigma$, then

$$\sigma x(t) = x(t + \theta)$$

Finally we observe that every isotropy subgroup Σ is a twisted group, that is, there is a homomorphism $\theta : H \to \mathbf{S}^1$, where H is a subgroup of $\mathbf{O}(3)$, such that

$$\Sigma = \{(h, \theta(h)) : h \in H\}$$

Let K denote the group of spatial symmetries of a solution $x(t)$, that is, $\gamma x(t) = x(t)$ for every t. It follows that $K = \Sigma \cap \mathbf{O}(3)$.

Representations of the group $\mathbf{O}(3)$

The *orthogonal group* $\mathbf{O}(3)$ consists of all 3×3 matrices A such that $A^t = A^{-1}$. That is $\det(A) = \pm 1$. The *special orthogonal group* $\mathbf{SO}(3)$ consists of the elements in $\mathbf{O}(3)$ with positive determinant. Algebraically, the orthogonal group is just a direct sum

$$\mathbf{O}(3) = \mathbf{SO}(3) \oplus \mathbf{Z_2}^c$$

where $\mathbf{Z_2}^c = \{\pm I\}$. Each irreducible representation of $\mathbf{SO}(3)$ gives rise to two irreducible representations of $\mathbf{O}(3)$ corresponding to the possibilities of $-I$ acting trivially or as minus the identity. In the *natural action* (or *standard action*) of $\mathbf{O}(3)$ the element $-I$ acts trivially if l is even and nontrivially when l is odd.

The irreducible representations of the rotation group $\mathbf{SO}(3)$ have dimension $2l+1$, $l = 0, 1, 2, \ldots$ Up to isomorphism, for each l there is only one such representation, denoted by V_l, the spherical harmonics of order l and the action of $\mathbf{SO}(3)$ on V_l is induced from the standard action on $\mathbf{R}^3$. See for example Miller [12].

Subgroups of $\mathbf{O}(3)$

The closed subgroups of $\mathbf{O}(3)$ fall in three classes [12]:
Class I. Subgroups of $\mathbf{SO}(3)$.
Class II. Subgroups of $\mathbf{O}(3)$ that contain $-I$.
Class III. Subgroups of $\mathbf{O}(3)$ that do not fall into the classes I and II.

The subgroups of class I consist of the planar subgroups $\mathbf{O}(2)$, $\mathbf{SO}(2)$, $\mathbf{D}_m$ $(m \geq 2)$, $\mathbf{Z}_m$ $(m \geq 1)$, and the exceptional subgroups $\mathbf{I}$, $\mathbf{O}$, $\mathbf{T}$. The planar subgroups are the symmetry groups of the unoriented and oriented circle and m-gon respectively. The exceptional subgroups are the rotation groups of the icosahedron, octahedron and tetrahedron.

The subgroups of class II are of the form $\Sigma \oplus \mathbf{Z_2}^c$ where Σ is a subgroup of class I.

From [11, Lemma 2.7] each subgroup Σ of class III is determined by two subgroups K and L of $\mathbf{SO}(3)$ where K is isomorphic to Σ, and L is of index 2 in K. Moreover, $K = \pi(\Sigma)$, $L = \Sigma \cap \mathbf{SO}(3)$, where π is the projection

$$\pi : \ \mathbf{O}(3) \to \mathbf{SO}(3) \quad \pi(\pm I\gamma) = \gamma \quad \text{for all } \gamma \in \mathbf{SO}(3)$$

See Table 1. Note that

$$\mathbf{O}(2)^- = \mathbf{SO}(2) \ \dot{\cup}\ (-\kappa)\mathbf{SO}(2)$$

where

$$-\kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Σ	K	L	Σ	K	L
$\mathbf{O}(2)^-$	$\mathbf{O}(2)$	$\mathbf{SO}(2)$	$\mathbf{D}^d_{2m}$ $(m \geq 2)$	$\mathbf{D}_{2m}$	$\mathbf{D}_m$
$\mathbf{O}^-$	$\mathbf{O}$	$\mathbf{T}$	$\mathbf{D}^-_m = \mathbf{D}^z_m$ $(m \geq 2)$	$\mathbf{D}_m$	$\mathbf{Z}_m$
			$\mathbf{Z}^-_{2m}$ $(m \geq 1)$	$\mathbf{Z}_{2m}$	$\mathbf{Z}_m$

Table 1: The subgroups Σ of $\mathbf{O}(3)$ of class III.

and $\dot\cup$ denotes disjoint union.

If Σ is a subgroup of $\mathbf{O}(3)$, its *normalizer* $N_{\mathbf{O}(3)}(\Sigma)$ is the group

$$N_{\mathbf{O}(3)}(\Sigma) = \{\gamma \in \mathbf{O}(3) :\ \gamma^{-1}\Sigma\gamma = \Sigma\}$$

This is the largest subgroup of $\mathbf{O}(3)$ in which Σ is normal. Moreover,

$$N_{\mathbf{O}(3)}(\Sigma) = N_{\mathbf{SO}(3)}(\pi(\Sigma)) \oplus {\mathbf{Z}_2}^c$$

In particular, we have

$$N_{\mathbf{O}(3)}(\mathbf{O}(2)^-) = \mathbf{O}(2) \oplus {\mathbf{Z}_2}^c, \qquad N_{\mathbf{O}(3)}(\mathbf{O}(2) \oplus {\mathbf{Z}_2}^c) = \mathbf{O}(2) \oplus {\mathbf{Z}_2}^c$$

Isotropy Subgroups of $\mathbf{O}(3) \times \mathbf{S}^1$

Recall that in the natural representation of $\mathbf{O}(3)$, arising in most applications, the element $-I$ acts trivially if l is even and nontrivially when l is odd. (These are the natural representations induced on spherical harmonics by the standard actions of $\mathbf{O}(3)$ on the 2-sphere in $\mathbf{R}^3$.) It is the natural representation that we shall consider in this paper.

Golubitsky and Stewart [8] give a list of those conjugacy classes of isotropy subgroups of $\mathbf{O}(3) \times \mathbf{S}^1$ (action on $V_l \oplus V_l$ for each l) that have two-dimensional fixed-point subspaces. The Equivariant Hopf Theorem guarantees the existence of a branch of periodic solutions, in generic Hopf bifurcation problems with $\mathbf{O}(3)$ symmetry, for each isotropy subgroup in each of those conjugacy classes. We reproduce their results for the natural representation in Tables 2 (even l) and 3 (odd l). Recall that any isotropy subgroup of $\mathbf{O}(3) \times \mathbf{S}^1$ is a twisted subgroup H^θ where H is the projection of Σ on $\mathbf{O}(3)$ and $\theta :\ H \to \mathbf{S}^1$ is a group homomorphism. Denote by $K = \ker(\theta)$. Suppose now that H^θ is an isotropy subgroup of $\mathbf{O}(3) \times \mathbf{S}^1$ with two-dimensional fixed-point space. It turns out that H must be a closed subgroup of $\mathbf{O}(3)$ of type II, so that $H = J \oplus {\mathbf{Z}_2}^c$ where $J \subset \mathbf{SO}(3)$. To see this note that for the plus representation, $(-I,0)$ lies in every isotropy subgroup, and for the minus representation, $(-I,\pi)$ lies in every isotropy subgroup. Therefore H is a closed subgroup of $\mathbf{O}(3)$ that contains $\mathbf{Z}_2{}^c$ and so it is of type II. It also follows that for the plus representation, K is of type II, and for the minus representation, K is of type I or III. The strategy of Golubitsky and Stewart [8] for finding the isotropy subgroups with two-dimensional fixed-point subspaces was to classify first by twist type, and second by the type (I,II, or III) of K.

	J	K	Twist $\theta(H)$	Value of l Plus Representation
1	$\mathbf{O}(2)$	$\mathbf{O}(2)\oplus\mathbf{Z}_2{}^c$	1	even l
2	$\mathbf{SO}(2)$	$\mathbf{Z}_k\oplus\mathbf{Z}_2{}^c$	$\mathbf{S}^1$ $[k=1,\ldots,l]$	even l
3	$\mathbf{I}$	$\mathbf{I}\oplus\mathbf{Z}_2{}^c$	1	$6,10,12,16,18,$ $20,22,24,26,28$ $32,34,38,44$
4	$\mathbf{O}$	$\mathbf{O}\oplus\mathbf{Z}_2{}^c$	1	$4,6,8,10,14$
5	$\mathbf{O}$	$\mathbf{T}\oplus\mathbf{Z}_2{}^c$	$\mathbf{Z}_2$	$6,10,12,14,16,20$
6	$\mathbf{T}$	$\mathbf{D}_2\oplus\mathbf{Z}_2{}^c$	$\mathbf{Z}_3$	$2,4,6$
7	$\mathbf{D}_n$ (even n)	$\mathbf{D}_{n/2}\oplus\mathbf{Z}_2{}^c$	$\mathbf{Z}_2$	$l<n\leq 2l$ even l

Table 2: Isotropy subgroups $\Sigma=H^\theta$ of $\mathbf{O}(3)\times\mathbf{S}^1$ on $V_l\oplus V_l$ when l is even, for the natural representation of $\mathbf{O}(3)$, having two-dimensional fixed-point subspaces. Here $H=J\oplus\mathbf{Z}_2{}^c$ where $J\subset\mathbf{SO}(3)$ and $K=\ker(\theta)$. For $\mathbf{S}^1$ twists, $\theta:\ \mathbf{SO}(2)\oplus\mathbf{Z}_2{}^c\to\mathbf{S}^1$ is given by $\theta(\psi)=k\psi$ for $\psi\in\mathbf{SO}(2)$ and $k=1,\ldots,l$ occur; also $\theta(-I)=0$.

3 Hopf Bifurcation on Hemispheres

In this paper we study Hopf bifurcations of reaction-diffusion equations defined on the hemisphere with Neumann boundary conditions along the equator. In [6] the authors studied steady-state bifurcations for the same class of equations. As they pointed out, such equations have a natural $\mathbf{O}(2)^-$-symmetry but may be extended to the full sphere where the natural symmetry group is $\mathbf{O}(3)$. Field *et al.* [6] show that the expected bifurcations are governed not by circular symmetry but by spherical symmetry – subject to a final restriction back to the hemispherical domain. We show that the same is true for Hopf bifurcations.

	J	K	Twist $\theta(H)$	Value of l Minus Representation
1	$\mathbf{O}(2)$	$\mathbf{O}(2)^-$	$\mathbf{Z}_2$	odd l
2	$\mathbf{SO}(2)$	$\mathbf{Z}_{2k}^-$	$\mathbf{S}^1$ $[k=1,\ldots,l]$	odd l
3	$\mathbf{I}$	$\mathbf{I}$	$\mathbf{Z}_2$	$21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59$
4	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{Z}_2$	$9, 13, 15, 17, 19, 23$
5	$\mathbf{O}$	$\mathbf{O}^-$	$\mathbf{Z}_2$	$3, 7, 9, 11, 13, 17$
6	$\mathbf{T}$	$\mathbf{D}_2$	$\mathbf{Z}_6$	$5, 7, 9$
7	$\mathbf{D}_n$	$\mathbf{D}_n$	$\mathbf{Z}_2$	$l/2 < n \leq l$ odd l

Table 3: Isotropy subgroups $\Sigma = H^\theta$ of $\mathbf{O}(3) \times \mathbf{S}^1$ on $V_l \oplus V_l$ when l is odd, for the natural representation of $\mathbf{O}(3)$, having two-dimensional fixed-point subspaces. Here $H = J \oplus {\mathbf{Z}_2}^c$ where $J \subset \mathbf{SO}(3)$ and $K = \ker(\theta)$. For $\mathbf{S}^1$ twists, $\theta : \mathbf{SO}(2) \oplus {\mathbf{Z}_2}^c \to \mathbf{S}^1$ is given by $\theta(\psi) = k\psi$ for $\psi \in \mathbf{SO}(2)$ and $k = 1, \ldots, l$ occur; also $\theta(-I) = \pi$.

Denote the coordinates on $\mathbf{R}^3$ by (x_1, x_2, x_3), the unit sphere in $\mathbf{R}^3$ by S and the upper hemisphere of S, $\{x \in S : \ x_3 \geq 0\}$ by H. Let Δ denote the Laplacian on S and $f : \ \mathbf{R}^2 \to \mathbf{R}$ be a smooth map. Consider the reaction-diffusion equation defined on H by

$$\frac{\partial u}{\partial t} = \Delta u + f(u, \lambda) \tag{3.2}$$

where $u : \ H \times [0, +\infty[\to \mathbf{R}$. Assume (3.2) satisfies Neumann boundary conditions on $\partial H = \{(x_1, x_2, x_3) \in H : \ x_3 = 0\}$:

$$\frac{\partial u}{\partial x_3}(x_1, x_2, 0, t) = 0 \quad \text{on} \quad \partial H \times \mathbf{R}_0^+ \tag{3.3}$$

Solutions of (3.2) on H that satisfy the boundary condition (3.3) can be extended to solutions of (3.2) on S by defining u on the lower hemisphere by reflection. Namely, if $\tau : \ S \to S$ denotes the reflection across ∂H defined by

$$\tau(x_1, x_2, x_3) = (x_1, x_2, -x_3) \tag{3.4}$$

then we can define u on the lower hemisphere by

$$u(\tau(x), t) \equiv u(x, t) \quad \forall x \in H, \ \forall t \in \mathbf{R}_0^+$$

A function u on S is τ-*invariant* if and only if

$$u(\tau(x), t) = u(x, t) \quad \forall x \in S,\ \forall t \in \mathbf{R}_0^+$$

The extension u defined on S is τ-invariant. See Theorem 7.2 for the regularity of the extended solution along ∂H. Conversely, suppose that u is a τ-invariant solution for the reaction-diffusion equation (3.2) on S. Then $u|_H$ is a solution for the Neumann boundary value problem (3.2) on H. We follow the approach of Field *et al.* [6] for finding solutions to the Neumann problem on the hemisphere by first finding solutions to the extended problem on S that are τ-invariant.

Observe that the equation (3.2) defined on the hemisphere H and satisfying boundary conditions (3.3) has symmetry group $\mathbf{O}(2)^-$, whereas (3.2) defined on the sphere S has symmetry group $\mathbf{O}(3)$.

Suppose that

$$f(0, \lambda) \equiv 0$$

Thus, equation (3.2) has the trivial group-invariant steady-state solution $u = 0$. If a Hopf bifurcation occurs then let E be the imaginary eigenspace of the linearization of (3.2) about $u = 0$ at $\lambda = 0$. It follows then that the group of symmetries of the equation leaves the space E invariant. Moreover, generically the action of the symmetry group $\mathbf{O}(2)^-$ on E is $\mathbf{O}(2)^-$-simple. See [9, Proposition XVI 1.4]. Let E_S be the imaginary eigenspace of the linearization of (3.2) on the sphere (about $u = 0$ at $\lambda = 0$). Then E consists of those eigenfunctions in E_S that are τ-invariant. The irreducible representations of $\mathbf{O}(2)^-$ have dimension either one or two. A direct application of the general $\mathbf{O}(2)^-$-symmetric Hopf theory would imply that generically we should expect the dimension of E to be two or four. However, from the general $\mathbf{O}(3)$-symmetric Hopf theory for the reaction-diffusion equation on the full sphere, we expect the action of $\mathbf{O}(3)$ on E_S to be $\mathbf{O}(3)$-simple. That is, the direct sum of two isomorphic absolutely $\mathbf{O}(3)$-irreducible spaces. The irreducible representations of $\mathbf{O}(3)$ correspond to the action of $\mathbf{O}(3)$ on the spherical harmonics of order l which have dimension $2l + 1$. Moreover, the vectors in E_S that are τ-invariant form a subspace of dimension approximately $(1/2) \dim E_S$. Thus the space E may be of higher dimension than would have been expected from the $\mathbf{O}(2)^-$-symmetric Hopf bifurcation problem. We show in the next sections that periodic solutions to the hemisphere problem that would not be expected if the extension property was not valid can exist.

$\mathbf{O}(3)$-symmetric Hopf bifurcation problems have been studied [8,9]. In particular, subgroups of $\mathbf{O}(3) \times \mathbf{S}^1$ that are known to support branches of periodic solutions. Recall Tables 2, 3 and Section 2. Using that, we can determine group orbits of periodic solutions in generic $\mathbf{O}(3)$-equivariant Hopf bifurcations problems that have representative solutions with isotropy containing τ. Observe that a solution to the $\mathbf{O}(3)$-symmetric equation (3.2) on S restricts to the hemisphere if and only if it is invariant under the reflection τ. Specifically, we determine the subgroups in the conjugacy classes of the groups H^θ in Tables 2 and 3 that support periodic solutions on the hemisphere problem. That is, those subgroups Σ of $\mathbf{O}(3) \times \mathbf{S}^1$ with two-dimensional fixed-point subspace such that $\tau \in \Sigma$. We classify such solutions up to $\mathbf{O}(2)^-$-symmetry, the symmetry group of the hemisphere. We find that some group orbits of periodic solutions contain multiple periodic solutions with symmetry τ. Each of these periodic solutions then restricts to solutions of the Neumann boundary value problem on the hemisphere lying on different $\mathbf{O}(2)^-$-orbits.

Remark 3.1 Let $\Sigma = H^\theta$ be an isotropy subgroup of $\mathbf{O}(3) \times \mathbf{S}^1$ (for the action of $\mathbf{O}(3)\times\mathbf{S}^1$ on $V_l \oplus V_l$, for a given value of l). Then $\tau \in \Sigma$ if and only if $\tau \in K = \ker(\theta)$. Moreover, we have that τ belongs to an isotropy subgroup of $\mathbf{O}(3) \times \mathbf{S}^1$ in the conjugacy class of H^θ if and only if τ belongs to a subgroup of $\mathbf{O}(3)$ in the conjugacy class of $K = \ker(\theta)$. Note that if $\tau \in \gamma K \gamma^{-1}$ for some $\gamma \in \mathbf{O}(3)$, and $K = \ker(\theta)$ for some homomorphism $\theta : \ H \to \mathbf{S}^1$ such that H^θ is a two-dimensional isotropy subgroup of $\mathbf{O}(3) \times \mathbf{S}^1$, then $\tau \in \gamma H^\theta \gamma^{-1}$ and $\gamma H^\theta \gamma^{-1}$ is conjugate to H^θ. $\diamond$

Definition 3.2 Denote the identity map on $\mathbf{R}^3$ by $I_{\mathbf{R}^3}$. Given a map $f : \ \mathbf{R}^3 \to \mathbf{R}^3$, let $\mathrm{Fix}(f) = \{x \in \mathbf{R}^3 : \ f(x) = x\}$ denote the *fixed-point set* of f. We define:

(i) A linear map $\sigma : \ \mathbf{R}^3 \to \mathbf{R}^3$ is an *involution* if $\sigma \neq I_{\mathbf{R}^3}$ and $\sigma^2 = I_{\mathbf{R}^3}$.

(ii) An involution $\sigma \in \mathbf{O}(3)$ is a *reflection* if the fixed-point set of σ is two-dimensional. $\diamond$

Remark 3.3 If $\sigma \in \mathbf{O}(3)$ is an involution which is not a reflection and not equal to $-I_{\mathbf{R}^3}$, then $-\sigma$ is a reflection. $\diamond$

Note that τ is a reflection in $\mathbf{O}(3)$. Moreover, we have:

Lemma 3.4 *An involution $\sigma \in \mathbf{O}(3)$ is a reflection if and only if σ is conjugate to τ. That is, if and only if there exists $\gamma \in \mathbf{O}(3)$ such that $\sigma = \gamma\tau\gamma^{-1}$.*

Proof This is [6, Lemma 2.3]. Let $\sigma \in \mathbf{O}(3)$ be an involution. Then it has eigenvalues ± 1, and at least one is -1. Moreover, it is a reflection if and only if it has precisely one eigenvalue equal to -1. Hence, σ is a reflection if and only if it is conjugate to τ. $\square$

4 Axisymmetric Solutions

There are two types of twisted isotropy subgroups of $\mathbf{O}(3) \times \mathbf{S}^1$ that contain $\mathbf{SO}(2)$: those with spatial group $\mathbf{O}(2) \oplus \mathbf{Z}_2{}^c$ (when l is even) and $\mathbf{O}(2)^-$ (when l is odd). See entries 1 of Tables 2 and 3 respectively. We call those solutions *axisymmetric* since they have an axis of rotation. We discuss now solutions to the hemisphere problem having axisymmetric symmetry.

Theorem 4.1 *Suppose we have an orbit of axisymmetric periodic solutions to the sphere problem with spatial symmetry group* $\mathbf{O}(2) \oplus \mathbf{Z}_2{}^c$ *(entry 1 of Table 2 for l even). On restriction we obtain the following orbits of solutions to the hemisphere problem:*

(a) An isolated axisymmetric solution with the x_3-axis as axis of symmetry. The symmetry of the solution is $\mathbf{O}(2)^-$.

(b) A unique orbit of solutions with spatial symmetry group $\mathbf{D}_2^-$ *(inside* $\mathbf{O}(2)^-$*).*

Proof See [6, Theorem 3.1]. We include a more detailed proof. We begin by determining those subgroups of $\mathbf{O}(3)$ conjugate to $\mathbf{O}(2) \oplus \mathbf{Z}_2{}^c$ (when l is even) that also contain τ. An element in the conjugacy class of $\mathbf{O}(2) \oplus \mathbf{Z}_2{}^c$ in $\mathbf{O}(3)$ is $\widetilde{\mathbf{O}(2)} \oplus \mathbf{Z}_2{}^c$, where $\widetilde{\mathbf{O}(2)} = \widetilde{\mathbf{SO}(2)} \ \cup \ (-\widetilde{\sigma})\widetilde{\mathbf{SO}(2)}$ and $\widetilde{\mathbf{SO}(2)}$ contains all rotations of the plane orthogonal to the axis of symmetry that fix this axis. Here we can take $\widetilde{\sigma}$ any reflection with fixed point set containing the axis of symmetry of $\widetilde{\mathbf{SO}(2)}$. Therefore $\widetilde{\mathbf{O}(2)}$ is generated by $\widetilde{\mathbf{SO}(2)}$ and $-\widetilde{\sigma}$.

Let $\Sigma = \widetilde{\mathbf{O}(2)} \oplus \mathbf{Z}_2{}^c$ be such that $\tau \in \Sigma$. Since τ reverses orientation ($\det \tau = -1$), then $\tau \notin \widetilde{\mathbf{SO}(2)}$. Since $\tau \in \Sigma$, then τ leaves the axis of rotation of $\widetilde{\mathbf{SO}(2)}$ invariant (all the elements of Σ fix or transform in $-v$ an element v in the axis of rotation of $\widetilde{\mathbf{SO}(2)}$). That is, τ is in the normalizer of $\widetilde{\mathbf{SO}(2)}$. As $\tau(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ and τ maps the axis of rotation of $\widetilde{\mathbf{SO}(2)}$ into itself, then this axis is either: **(a)** the x_3-axis or **(b)** perpendicular to the x_3-axis.

Case (a) In this case $\Sigma = \mathbf{O}(2) \oplus \mathbf{Z}_2{}^c$, where $\mathbf{O}(2) = \mathbf{SO}(2) \ \cup \ \kappa\mathbf{SO}(2)$. The elements of Σ are of one of the following types: $R_\theta \in \mathbf{SO}(2) \subset \mathbf{SO}(3)$, $\kappa R_\theta \in \kappa\mathbf{SO}(2) \subset \mathbf{SO}(3)$, $-R_\theta \in -I\mathbf{SO}(2)$, $-\kappa R_\theta \in -\kappa\mathbf{SO}(2)$. Note that $\tau \notin \mathbf{SO}(3)$ since $\det \tau = -1$. Moreover $\tau = -R_\pi$ (where $R_\pi \in \mathbf{SO}(2)$) (and $-\tau = R_\pi$). Thus $\tau \in \Sigma$. Moreover,

$$(\mathbf{O}(2) \oplus \mathbf{Z}_2{}^c) \cap \mathbf{O}(2)^- = \mathbf{O}(2)^-$$

since $\mathbf{O}(2)^- = \mathbf{SO}(2) \ \cup \ (-\kappa)\mathbf{SO}(2)$.

Case (b) Suppose the axis of rotation is $\mathbf{R}\{a_1\}$ where $a_1 = (a, b, 0)$ (and $a^2 + b^2 \neq 0$). Now $\tau a_1 = a_1$. Thus τ is a reflection fixing the axis of rotation and so $-\tau \in \widetilde{\mathbf{O}(2)}$. We conclude that $\tau \in \Sigma = \widetilde{\mathbf{O}(2)} \oplus \mathbf{Z}_2{}^c$.

We show that

$$(\widetilde{\mathbf{O}(2)} \oplus \mathbf{Z}_2{}^c) \cap \mathbf{O}(2)^- = \mathbf{D}_2^-$$

To see this, note that the elements of Σ are of one of the following four types:

(b.1) $\widetilde{R_\theta} \in \widetilde{\mathbf{SO}(2)} \subset \mathbf{SO}(3)$ with positive determinant;

(b.2) $-\widetilde{\sigma}\widetilde{R_\theta} \in \widetilde{\mathbf{O}(2)} \setminus \widetilde{\mathbf{SO}(2)} \subset \mathbf{SO}(3)$ with positive determinant;

(b.3) $-\widetilde{R_\theta} \in \mathbf{O}(3)$ with negative determinant;

(b.4) $\widetilde{\sigma}\widetilde{R_\theta} \in \mathbf{O}(3)$ with negative determinant.

Recall that $\widetilde{\sigma}$ fixes the rotation axis a_1 and $\mathbf{O}(2)^- = \mathbf{SO}(2) \cup (-\kappa)\mathbf{SO}(2)$, where $\det(R_\theta) = 1$ and $\det(-\kappa R_\theta) = -1$, for $R_\theta \in \mathbf{SO}(2)$. We consider the four cases, each corresponding to the intersection of $\mathbf{O}(2)^-$ with the set of elements of one of the four types that we have in Σ.

Case (b.1): the intersection of $\mathbf{O}(2)^-$ with the set of elements of type $\widetilde{R_\theta}$ is formed by the elements of the type $\widetilde{R_\theta} \in \widetilde{\mathbf{SO}(2)} \subset \mathbf{SO}(3)$ that are also of type $R_\theta \in \mathbf{SO}(2) \subset \mathbf{SO}(3)$. One element in these conditions would have to fix the a_1 axis and the x_3 axis of rotation of $\mathbf{SO}(2)$, so it can only be the identity.

Case (b.2): the intersection of $\mathbf{O}(2)^-$ with the set of elements of type $-\widetilde{\sigma}\widetilde{R_\theta}$ is given by the elements of type $-\widetilde{\sigma}\widetilde{R_\theta} \in \widetilde{\mathbf{O}(2)} \subset \mathbf{SO}(3)$ that are of type $R_\theta \in \mathbf{SO}(2) \subset \mathbf{SO}(3)$ since the elements $-\widetilde{\sigma}\widetilde{R_\theta}$ have positive determinant. The second ones fix the x_3-axis, so they have 1 as an eigenvalue. The first ones are such that $-\widetilde{\sigma}\widetilde{R_\theta}(a_1) = -a_1$ because $-\widetilde{\sigma}\widetilde{R_\theta} \in \widetilde{\mathbf{O}(2)} \setminus \widetilde{\mathbf{SO}(2)}$. They have -1 as an eigenvalue. The third eigenvalue must be -1 because these elements are in $\mathbf{SO}(3)$. Therefore $\theta = \pi$ and $R_\theta = R_\pi$. Moreover, taking the orthogonal basis $(a_1, b_1, (0,0,1))$, where $b_1 = (-b, a, 0)$, we can take $\widetilde{R}_\theta = I$ and $\widetilde{\sigma}$ such that $-\widetilde{\sigma} = R_\pi$ in this basis (and in the canonical basis).

Case (b.3): the intersection of $\mathbf{O}(2)^-$ with elements of type $-\widetilde{R_{\theta_1}} \in \mathbf{O}(3)$. In this case we must find θ such that $\gamma = -\kappa R_\theta = -\widetilde{R_{\theta_1}}$. Since γ fixes the x_3-axis and maps a_1 into $-a_1$, where a_1 is the rotation axis of $\widetilde{\mathbf{O}(2)}$, then as $\det(\gamma) = -1$, it follows that $-\widetilde{R}_{\theta_1} = -\widetilde{R}_\pi$ (and so $\theta_1 = \pi$). Moreover, γ has eigenvalues $-1, 1, 1$ and

$$-\kappa R_\theta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the canonical coordinates x_1, x_2, x_3. Recall that this is a reflection on the line $x_2 = \tan(\pi/2 - \theta/2)x_1$ if $\theta \neq 0$ and $x_1 = 0$ otherwise (in the x_1x_2-plane). On the other hand $-\widetilde{R}_\pi$ is a reflection along the axis $(-b, a, 0)$. This axis must coincide with the line $x_2 = \tan(\pi/2 - \theta/2)x_1$ if $\theta \neq 0$, otherwise with $x_1 = 0$. If $b = 0$ we take $\theta = 0$ and $\gamma = -\kappa R_0 = -\kappa = -\widetilde{R}_\pi$. If not, then θ is the angle such that

$$\tan\left(\frac{\pi - \theta}{2}\right) = -\frac{a}{b}$$

In any case,

$$\gamma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the $(a_1, b_1, (0,0,1))$-basis.
Case (b.4): the intersection of $\mathbf{O}(2)^-$ with elements of type $\widetilde{\sigma}\widetilde{R_{\theta_1}} \in \mathbf{O}(3)$. We must find θ such that $\gamma = -\kappa R_\theta = \widetilde{\sigma}\widetilde{R_{\theta_1}}$. As $\gamma = -\kappa R_\theta$ fixes the x_3-axis and $\gamma = \widetilde{\sigma}\widetilde{R_{\theta_1}}$ fixes the a_1-axis, and $\det(\gamma) = -1$, it follows that the eigenvalues of γ are $1, 1, -1$. In the basis $(a_1, b_1, (0,0,1))$ we have

$$\widetilde{\sigma}\widetilde{R_\pi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall that in case **(b.2)** we chose

$$\widetilde{\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note that $\widetilde{\sigma}\widetilde{R_\pi}$ is a reflection on the a_1-axis (in the x_1x_2-plane). Also $-\kappa R_\theta$ is a reflection on the line $x_2 = \tan(\pi/2 - \theta/2)x_1$ if $\theta \neq 0$ and $x_1 = 0$ otherwise (in the same plane). If $a = 0$ we take $\theta = 0$ and $\gamma = -\kappa R_0 = -\kappa = \widetilde{\sigma}\widetilde{R_\pi}$. If $a \neq 0$ we choose θ such that

$$\tan\left(\frac{\pi - \theta}{2}\right) = \frac{b}{a}$$

In both cases

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the $(a_1, b_1, (0,0,1))$-basis. Moreover

$$\left(\widetilde{\mathbf{O}(2)} \oplus {\mathbf{Z}_2}^c\right) \cap \mathbf{O}(2)^- = \mathbf{D}_2^-$$

where

$$\mathbf{D}_2^- = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

in the $((a,b,0), (-b,a,0), (0,0,1))$ basis. □

Theorem 4.2 *Suppose we have an orbit of axisymmetric periodic solutions with* $\mathbf{O}(2)^-$ *symmetry (entry* 1 *of Table 3 for l odd) to the sphere problem. On restriction we obtain a unique orbit of solutions of the hemisphere problem with spatial symmetry group* ${\mathbf{Z}_2}^-$ *(inside* $\mathbf{O}(2)^-$*) and spatio-temporal symmetry* $(\mathbf{D}_2^-)^\theta$ *(inside* $\mathbf{O}(2)^- \times \mathbf{S}^1$*) where* θ *has twist type* $\mathbf{Z}_2$.

Proof We determine now the subgroups of $\mathbf{O}(3)$ conjugate to $\mathbf{O}(2)^-$ (when l is odd) that contain τ. We denote by $\widetilde{\mathbf{O}(2)^-}$ an element in the conjugacy class of $\mathbf{O}(2)^-$ in $\mathbf{O}(3)$. Now $\widetilde{\mathbf{O}(2)^-} = \widetilde{\mathbf{SO}(2)} \cup \widetilde{\sigma}\widetilde{\mathbf{SO}(2)}$, where $\widetilde{\mathbf{SO}(2)}$ contains all rotations of the plane orthogonal to the axis of symmetry that fix this axis. Here we can take $\widetilde{\sigma}$ any reflection with fixed point set containing the axis of symmetry of $\widetilde{\mathbf{SO}(2)}$. Let $\Sigma = \widetilde{\mathbf{O}(2)^-}$ be such that $\tau \in \Sigma$. Following the same lines as in the proof of Theorem 4.1, we also have to consider cases **(a)** and **(b)** according the axis of rotation of $\widetilde{\mathbf{SO}(2)}$ is the x_3-axis or perpendicular to the x_3-axis, respectively. In case **(a)** $\tau \notin \widetilde{\mathbf{O}(2)^-}$ so these solutions do not occur when l is odd. In case **(b)** we have $\tau \in \widetilde{\mathbf{O}(2)^-}$ and

$$(\widetilde{\mathbf{O}(2)^-}) \cap \mathbf{O}(2)^- = \mathbf{Z}_2^-$$

To see this, note that the elements of Σ are of one of the following two types:

- $\widetilde{R_\theta} \in \widetilde{\mathbf{SO}(2)} \subset \mathbf{SO}(3)$,
- $\widetilde{\sigma}\widetilde{R_\theta} \in \mathbf{O}(3)$.

Following cases **(b.1)** and **(b.4)** of the proof of Theorem 4.1, we obtain

$$(\widetilde{\mathbf{O}(2)^-}) \cap \mathbf{O}(2)^- = {\mathbf{Z}_2}^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

in the $((a,b,0),(-b,a,0),(0,0,1))$ basis, where $(a,b,0)$ is a nonzero vector in the axis of rotation of $\widetilde{\mathbf{SO}(2)}$.

For $H^\theta \cap (\mathbf{O}(2)^- \times \mathbf{S}^1)$, when l is odd, we need to compute $(\widetilde{\mathbf{O}(2)} \oplus {\mathbf{Z}_2}^c) \cap \mathbf{O}(2)^-$. The above computations, done for even l, are also valid for odd l, so we have

$$(\widetilde{\mathbf{O}(2)} \oplus {\mathbf{Z}_2}^c) \cap \mathbf{O}(2)^- = \mathbf{D}_2^-$$

Note that the axis of symmetry of $\widetilde{\mathbf{SO}(2)}$ is perpendicular to x_3 (case **(b)** in the computations for even l in the proof of Theorem 4.1). □

5 Solutions with Finite Spatial Symmetry

We consider now periodic solutions with twisted symmetry groups that have finite spatial symmetry groups for the sphere problem that restrict to solutions of the hemisphere problem.

We recall that a group G is the disjoint union of subgroups $G_i,\ i \in I$, if $G = \cup_{i\in I} G_i$ and for all $i,j \in I,\ i \neq j,\ G_i \cap G_j$ is the identity element of G. We write $G = \dot{\cup}_{i\in I} G_i$.

Lemma 5.1 *The subgroups* $\mathbf{I}$, $\mathbf{O}$, $\mathbf{T}$ *and* $\mathbf{O}^-$ *of* $\mathbf{O}(3)$ *have the following disjoint union decompositions:*

$$
\begin{aligned}
\mathbf{I} &= \dot\cup^6\, \mathbf{Z}_5\, \dot\cup^{10}\, \mathbf{Z}_3\, \dot\cup^{15}\, \mathbf{Z}_2 \\
\mathbf{O} &= \dot\cup^3\, \mathbf{Z}_4\, \dot\cup^4\, \mathbf{Z}_3\, \dot\cup^6\, \mathbf{Z}_2 \\
\mathbf{T} &= \dot\cup^4\, \mathbf{Z}_3\, \dot\cup^3\, \mathbf{Z}_2 \\
\mathbf{O}^- &= \dot\cup^3\, \mathbf{Z}_4^-\, \dot\cup^4\, \mathbf{Z}_3\, \dot\cup^6\, \mathbf{Z}_2^-
\end{aligned}
$$

Here $\dot\cup^k \mathbf{Z}_l$ *(resp.* $\dot\cup^k \mathbf{Z}_l^-$*) denotes a disjoint decomposition of* k *copies of subgroups all conjugate in* $\mathbf{SO}(3)$ *(resp.* $\mathbf{O}(3)$*) to* $\mathbf{Z}_l$ *(resp.* $\mathbf{Z}_l^-$*).*

Proof See [11] or [9, pp. 105, 123]. □

Theorem 5.2 *Suppose that we have an* $\mathbf{O}(3) \times \mathbf{S}^1$*-orbit of periodic solutions to the sphere problem with finite group of spatial symmetries given in Table 2 (entries* $2-7$*). On restriction, we obtain the following* $\mathbf{O}(2)^- \times \mathbf{S}^1$ *orbits of solutions to the hemisphere problem:*

(a) Spatial group $K = \mathbf{Z}_k \oplus \mathbf{Z}_2{}^c$
For each even k *(and* $2 \le k \le l$*) there is one orbit of solutions with spatial symmetry group* $\mathbf{Z}_k$*, and spatio-temporal symmetry* $(\mathbf{SO}(2))\theta$ *where* $\theta(\psi) = k\psi$ *for* $\psi \in \mathbf{SO}(2)$*.*

(b) Spatial group $K = \mathbf{I} \oplus \mathbf{Z}_2{}^c$ *(***Icosahedral solutions***)*
There are 15 orbits of solutions with (spatial) symmetry group $\mathbf{Z}_2{}^-$*.*

(c) Spatial group $K = \mathbf{O} \oplus \mathbf{Z}_2{}^c$ *(***Octahedral solutions***)*
There are 3 orbits of solutions with (spatial) symmetry group $\mathbf{D}_4^-$*.*
There are 6 orbits of solutions with (spatial) symmetry group $\mathbf{D}_2^-$*.*

(d) Spatial group $K = \mathbf{T} \oplus \mathbf{Z}_2{}^c$ *(***Tetrahedral solutions***)*
There are 3 orbits of solutions with spatial symmetry group $\mathbf{D}_2^-$*, and spatio-temporal symmetry* $(\mathbf{D}_4^-)^\theta$ *where* θ *has twist type* $\mathbf{Z}_2$*.*

(e) Spatial group $K = \mathbf{D}_2 \oplus \mathbf{Z}_2{}^c$ *(***Dihedral solutions***)*
There are 3 orbits of solutions with spatial symmetry group $\mathbf{D}_2^-$*.*

(f) Spatial group $K = \mathbf{D}_{n/2} \oplus \mathbf{Z}_2{}^c$ *(***Dihedral solutions***)*
For each odd $n/2$ *(and* $l < n \le 2l$*) there are* $n/2$ *orbits of solutions. The solutions have spatial symmetry group* $\mathbf{Z}_2$*, and spatio-temporal symmetry* $(\mathbf{D}_2^-)^\theta$ *where* θ *has twist type* $\mathbf{Z}_2$*.*

For each even $n/2$ (and $l < n \leq 2l$) there are $n/2+1$ orbits of solutions. One orbit consists of solutions with spatial symmetry group $\mathbf{D}_{n/2}^-$, and spatio-temporal symmetry $(\mathbf{D}_n^-)^\theta$ where θ has twist type $\mathbf{Z}_2$. The other $n/2$ orbits consist of solutions with (spatial) symmetry group $\mathbf{D}_2^-$.

Proof The groups K are of the type $\Delta \oplus \mathbf{Z}_2^c$ where $\Delta \subset \mathbf{SO}(3)$. By Remark 3.3, reflections conjugate to τ (Lemma 3.4) are found by composing the involutions in Δ with $-I_{\mathbf{R}^3}$.

(a) Suppose $K = \mathbf{Z}_k \oplus \mathbf{Z}_2{}^c$. When k is odd, K does not contain involutions. When k is even $R_\pi \in \mathbf{SO}(2)$ is the only involution of $\mathbf{Z}_k$. Moreover, $\tau = -R_\pi \in K$. Thus by Lemma 3.4 for each even k there is one orbit of solutions.

Let k be even. For the spatial intersection, $K \cap \mathbf{O}(2)^-$, where $K = \mathbf{Z}_k \oplus \mathbf{Z}_2{}^c$, note that $\mathbf{O}(2)^- = \mathbf{SO}(2) \cup (-\kappa)\mathbf{SO}(2)$ and $\mathbf{Z}_k \oplus \mathbf{Z}_2{}^c = \mathbf{Z}_k \cup (-I)\mathbf{Z}_k$. The intersection $\mathbf{SO}(2) \cap \mathbf{Z}_k$ is $\mathbf{Z}_k$ and the intersection $[(-\kappa)\mathbf{SO}(2)] \cap [(-I)\mathbf{Z}_k]$ is empty because every element of $(-\kappa)\mathbf{SO}(2)$ fixes the third coordinate and no element of $(-I)\mathbf{Z}_k$ does so. We conclude that the intersection $(\mathbf{Z}_k \oplus \mathbf{Z}_2{}^c) \cap \mathbf{O}(2)^-$ is $\mathbf{Z}_k$.

In order to calculate $H^\theta \cap (\mathbf{O}(2)^- \times \mathbf{S}^1)$ where $H = \mathbf{SO}(2) \oplus \mathbf{Z}_2{}^c$, we calculate now $(\mathbf{SO}(2) \oplus \mathbf{Z}_2{}^c) \cap \mathbf{O}(2)^-$ which is $\mathbf{SO}(2)$. Therefore we have that $H^\theta \cap (\mathbf{O}(2)^- \times \mathbf{S}^1) = (\mathbf{SO}(2))^\theta$ where $\theta(\psi) = k\psi$, for $\psi \in \mathbf{SO}(2)$.

(b) By Lemma 5.1, in $\mathbf{I}$ there are 15 involutions in $\mathbf{Z}_2$. Composing these involutions with $-I$ it follows that $\mathbf{I} \oplus \mathbf{Z}_2^c$ has 15 distinct reflections (recall Remark 3.3). Moreover, by Lemma 3.4, these reflections are conjugate to τ.

Geometrically, these 15 (orbits) of solutions to the hemisphere problem correspond to the 15 different ways we may slice (an object with the same symmetry as) the icosahedron (through the corresponding 15 planes of symmetry). The solutions for the hemisphere problem in this case have spatial $\mathbf{Z}_2^-$-symmetry.

(c) By Lemma 5.1, in $\mathbf{O}$ there are three involutions in $\mathbf{Z}_4$ and six in $\mathbf{Z}_2$. Composing these involutions with $-I$ it follows that $\mathbf{O} \oplus \mathbf{Z}_2^c$ has 9 distinct reflections.

Geometrically, the nine (orbits) of periodic solutions to the hemisphere problem that we obtain, correspond to the nine different ways we may slice (an object with the same symmetry of) the cube as we describe now. Three of them correspond to slice the cube through the three planes of symmetry parallel to, and halfway between, two opposite faces of the cube. The solutions for the hemisphere problem in this case have spatial symmetry $\mathbf{D}_4^-$. The other six (orbits) of solutions correspond to slice the cube through the planes of symmetry containing opposite edges of the cube. The solutions for the hemisphere problem in this case have spatial $\mathbf{D}_2^-$-symmetry.

(d) By Lemma 5.1, in $\mathbf{T}$ there are three involutions in $\mathbf{Z}_2$. Composing these involutions with $-I$ it follows that $\mathbf{T} \oplus {\mathbf{Z}_2}^c$ has three distinct reflections conjugate to τ.

For the intersection $K \cap \mathbf{O}(2)^-$, note that the group $\mathbf{T} \oplus {\mathbf{Z}_2}^c$ can be realized as $\mathbf{Z}_3 \dot{+} {\mathbf{Z}_2}^3$. The elements of $\mathbf{Z}_3 \dot{+} {\mathbf{Z}_2}^3$ that fix the third coordinate are two groups isomorphic to $\mathbf{Z}_2$, forming the group $\mathbf{D}_2^- =< R_\pi, -\kappa >$.

For the intersection $H^\theta \cap (\mathbf{O}(2)^- \times \mathbf{S}^1)$, note that the group $H = \mathbf{O} \oplus {\mathbf{Z}_2}^c$ can be realized as $\mathbf{S}_3 \dot{+} {\mathbf{Z}_2}^3$. Moreover, $H \cap \mathbf{O}(2)^-$ is the group $\mathbf{D}_4^- =< R_{\pi/2}, -\kappa >$. Thus $H^\theta \cap (\mathbf{O}(2)^- \times \mathbf{S}^1) = (\mathbf{D}_4^-)^\theta$ and θ has twist type $\mathbf{Z}_2$.

(e) We have that $\mathbf{D}_2 =< R_\pi, \kappa >$ and so $\mathbf{D}_2 \oplus {\mathbf{Z}_2}^c$ contains three reflections, $\tau = -R_\pi$, $-\kappa$ and $-\kappa R_\pi$, which are conjugate to τ by Lemma 3.4.

The intersection $K \cap \mathbf{O}(2)^-$ where K is (conjugate to) $\mathbf{D}_2 \oplus {\mathbf{Z}_2}^c$ is (conjugate to) $\mathbf{D}_2^-$ (generated by R_π and $-\kappa$).

From the previous item we have that $H^\theta \cap (\mathbf{O}(2)^- \times \mathbf{S}^1) = (\mathbf{D}_2^-)^\theta$, and θ has trivial twist type.

(f) The $n/2$ elements in $\mathbf{D}_{n/2} \setminus \mathbf{Z}_{n/2}$ are involutions in $\mathbf{D}_{n/2}$. When $n/2$ is even, we also have that $R_\pi \in \mathbf{D}_{n/2}$ and so $-R_\pi = \tau \in \mathbf{D}_{n/2} \oplus {\mathbf{Z}_2}^c$. We conclude that when $n/2$ is even there are $n/2+1$ orbits of solutions containing τ for the sphere problem. When $n/2$ is odd there are $n/2$ orbits of solutions for the sphere problem.

Let $n/2$ be even. For the intersection $K \cap \mathbf{O}(2)^-$ where $K = \mathbf{D}_{n/2} \oplus \mathbf{Z}_2^c$, note that

$$\mathbf{D}_{n/2} \oplus {\mathbf{Z}_2}^c = \mathbf{Z}_{n/2} \cup \kappa\mathbf{Z}_{n/2} \cup (-I)\mathbf{Z}_{n/2} \cup (-\kappa)\mathbf{Z}_{n/2}.$$

The elements of K that fix the x_3-axis belong to $\mathbf{Z}_{n/2} \cup (-\kappa)\mathbf{Z}_{n/2}$. This is the group $\mathbf{D}_{n/2}^-$ (generated by $R_{4\pi/n}$ and $-\kappa$) and it is contained in $\mathbf{O}(2)^-$. It follows then that $K \cap \mathbf{O}(2)^-$ is $\mathbf{D}_{n/2}^-$. Now, let $H = \mathbf{D}_n \oplus {\mathbf{Z}_2}^c$. The elements of H that fix the x_3-axis form the group $\mathbf{D}_n^-$ (generated by $R_{2\pi/n}$ and $-\kappa$) which is contained in $\mathbf{O}(2)^-$. It follows then that the intersection $H^\theta \cap (\mathbf{O}(2)^- \times \mathbf{S}^1)$ is $(\mathbf{D}_n^-)^\theta$ and θ has twist type $\mathbf{Z}_2$.

For the remaining $n/2$ orbits of solutions when $n/2$ is even, from the proof of Theorem 4.1 (b) it follows that the solutions for the hemisphere problem have spatial symmetry $\mathbf{D}_2^-$ and trivial twist type. That is, we have $K \cap \mathbf{O}(2)^- = H \cap \mathbf{O}(2)^- = \mathbf{D}_2^-$.

Let $n/2$ be odd. In this case, again following the proof of Theorem 4.1 (b), we have that $K \cap \mathbf{O}(2)^- = \mathbf{Z}_2$ and $H \cap \mathbf{O}(2)^- = \mathbf{D}_2^-$. Thus the solutions for the hemisphere problem have spatial symmetry $\mathbf{Z}_2$ and twist type $\mathbf{Z}_2$. $\square$

Remark 5.3 Consider the natural representation of $\mathbf{O}(3)$ on V_l. Then the group $\mathbf{O} \oplus \mathbf{Z}_2{}^c$ is a maximal isotropy subgroup of $\mathbf{O}(3)$ for the even values of l greater or equal to 4. Also the group $\mathbf{I} \oplus \mathbf{Z}_2{}^c$ is a maximal isotropy subgroup of $\mathbf{O}(3)$ for the following values of l: $6, 10, 12$ and all other even numbers greater or equal to 16. See [9, Theorem XIII 9.8]. Also the normalizer of $\mathbf{O} \oplus \mathbf{Z}_2{}^c$ in $\mathbf{O}(3)$ is $\mathbf{O} \oplus \mathbf{Z}_2{}^c$. Similarly, the normalizer of $\mathbf{I} \oplus \mathbf{Z}_2{}^c$ coincides with $\mathbf{I} \oplus \mathbf{Z}_2{}^c$. It follows then that we can use the above facts to show that $\mathbf{O} \oplus \mathbf{Z}_2{}^c$ and $\mathbf{I} \oplus \mathbf{Z}_2{}^c$ are maximal isotropy subgroups of $\mathbf{O}(3) \times \mathbf{S}^1$ of trivial twist type (for the corresponding action on $V_l \oplus V_l$). For the above values of l that are not listed in the entries 3 and 4 of Table 2, we can use Fiedler's result [4] (see [5] or [9, Theorem XVI 4.5]) to conclude the generic existence of branches of periodic solutions with symmetry given by the isotropy subgroups in the conjugacy classes of $\mathbf{O} \oplus \mathbf{Z}_2{}^c$ and $\mathbf{I} \oplus \mathbf{Z}_2{}^c$, in Hopf bifurcation problems with spherical symmetry posed on $V_l \oplus V_l$. It follows then that the results of Theorem 5.2 (b)-(c) are also valid for those values of l. $\diamond$

Lemma 5.4 *The periodic solutions to $\mathbf{O}(3)$-equivariant Hopf bifurcation problems with isotropy conjugate to the twisted groups with spatial group $\mathbf{I}$ or $\mathbf{O}$ or $\mathbf{D}_n$ (see Table 3, entries $3, 4, 6$ and 7) cannot restrict to solutions of the Neumann problem on the hemisphere.*

Proof The groups $\mathbf{I}$, $\mathbf{O}$, $\mathbf{D}_2$ and $\mathbf{D}_n$ contain no orientation-reversing elements. Therefore, they do not contain reflections. $\square$

Theorem 5.5 *Suppose that we have an $\mathbf{O}(3) \times \mathbf{S}^1$-orbit of periodic solutions to the sphere problem with finite group of spatial symmetries given in Table 3 (entries $2, 5$). On restriction, we obtain the following $\mathbf{O}(2)^- \times \mathbf{S}^1$ orbits of solutions to the hemisphere problem:*

(a) Spatial group $K = \mathbf{Z}_{2k}^-$
For each odd k (and $1 \leq k \leq l$) there is one orbit of solutions with spatial symmetry group $\mathbf{Z}_k$, and spatio-temporal symmetry $(\mathbf{SO}(2))\theta$ where $\theta(\psi) = k\psi$ for $\psi \in \mathbf{SO}(2)$.

*(b) Spatial group $K = \mathbf{O}^-$ (**Octahedral solutions**)*
There are 6 orbits of solutions with spatial symmetry group $\mathbf{Z}_2^-$, and spatio-temporal symmetry $(\mathbf{D}_2^-)^\theta$ where θ has twist type $\mathbf{Z}_2$.

Proof

(a) Note that the group $\mathbf{Z}_{2k}^-$ is generated by $-R_{\pi/k}$. If k is even then the involution of $\mathbf{Z}_{2k}^-$ is $(-R_{\pi/k})^k = R_\pi$ which is not a reflection. If k is odd then $(-R_{\pi/k})^k = -R_\pi = \tau \in \mathbf{Z}_{2k}^-$.

Let k be odd. The elements of $\mathbf{Z}_{2k}^-$ with determinant one form the group $\mathbf{Z}_k$, which is contained in $\mathbf{O}(2)^-$. Moreover, the elements in $\mathbf{Z}_{2k}^- \setminus \mathbf{Z}_k$ do not fix the third coordinate and so do not belong to $\mathbf{O}(2)^-$. Thus $\mathbf{Z}_{2k}^- \ \cap\ \mathbf{O}(2)^- = \mathbf{Z}_k$. Moreover, $(\mathbf{SO}(2) \oplus {\mathbf{Z}_2}^c) \ \cap\ \mathbf{O}(2)^- = \mathbf{SO}(2)$.

(b) Recall the decomposition of $\mathbf{O}^-$ in Lemma 5.1. The reflections in $\mathbf{O}^-$ are the order two elements of $\mathbf{Z}_2^-$. Thus by Lemma 3.4 there are six orbits of solutions.

The proof of the intersections follows closely the proof of item (c) of Theorem 5.2.

□

6 Figures

Following Field *et al.* [6], given $l \geq 2$, we identify the spherical harmonics of degree l (in V_l) with the deformations of a sphere in the following way: since a spherical harmonic is a real-valued function on the sphere, we can picture it by deforming the sphere in the radial direction by an amount equal to the value of that spherical harmonic.

In this section we show periodic one-parameter families of deformations of the sphere, the parameter being time, to illustrate the symmetries of the periodic solutions predicted by Theorems 4.2, 5.2, and 5.5 for the sphere problem and for their restriction to the hemisphere.

In Figure 3 (a) we assume $l = 3$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbf{O}(2)^-$, twisted symmetry $(\mathbf{O}(2) \oplus \mathbf{Z}_2^c)^\theta$ (and twist type $\mathbf{Z}_2$). Thus the deformations *maintain an* $\mathbf{O}(2)^-$*-symmetric shape* and oscillate with twist type $\mathbf{Z}_2$. In Theorem 4.2 we show that this solution may be sliced in one way (up to $\mathbf{O}(2)^-$-symmetry) to obtain a solution to the equation posed on the hemisphere, having spatial symmetry $\mathbf{Z}_2^-$ and spatio-temporal symmetry $(\mathbf{D}_2^-)^\theta$ (having twist type $\mathbf{Z}_2$). We show that in Figure 3 (b).

In Figure 4 (a) we assume $l = 3$ and we picture a rotating wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbf{Z}_6^-$, twisted symmetry $\mathbf{SO}(2)^\theta$ (and twist type $\mathbf{S}^1$). Thus the deformations maintain a $\mathbf{Z}_6^-$-symmetric *constant shape* and rotate about the x_3-axis. In Figure 4 (b) we picture the restriction to hemisphere with $\mathbf{Z}_3$-symmetric constant shape and rotating about the x_3-axis. Recall Theorem 5.5 (a) for $l = 3$ and $k = 3$.

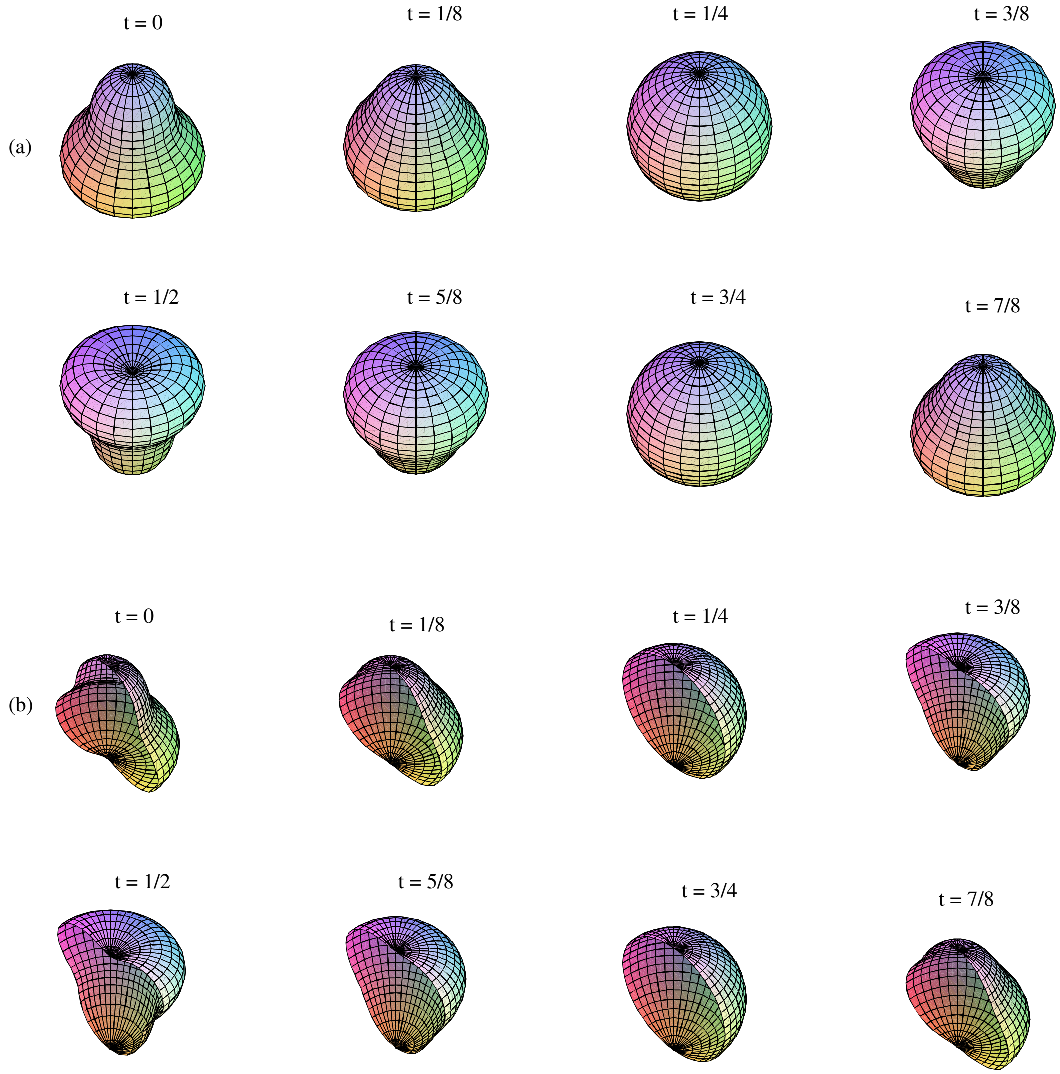

Figure 3: (a) Oscillations of a sphere deformed by spherical harmonics of order $l = 3$: spatial symmetry $\mathbf{O}(2)^-$ and twisted symmetry $(\mathbf{O}(2) \oplus \mathbf{Z}_2^c)^\theta$ (twist type $\mathbf{Z}_2$). (b) Restriction to hemisphere: spatial symmetry $\mathbf{Z}_2^-$, twisted symmetry $(\mathbf{D}(2)^-)^\theta$, twist type $\mathbf{Z}_2$.

In Figure 5 (a) we assume $l = 6$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbf{T} \oplus \mathbf{Z}_2^c$, twisted symmetry $(\mathbf{O} \oplus \mathbf{Z}_2^c)^\theta$. Thus the deformations are $\mathbf{T} \oplus \mathbf{Z}_2^c$-symmetric and oscillate with twist type $\mathbf{Z}_2$. In Theorem 5.2 (d) we show that this solution may be sliced in

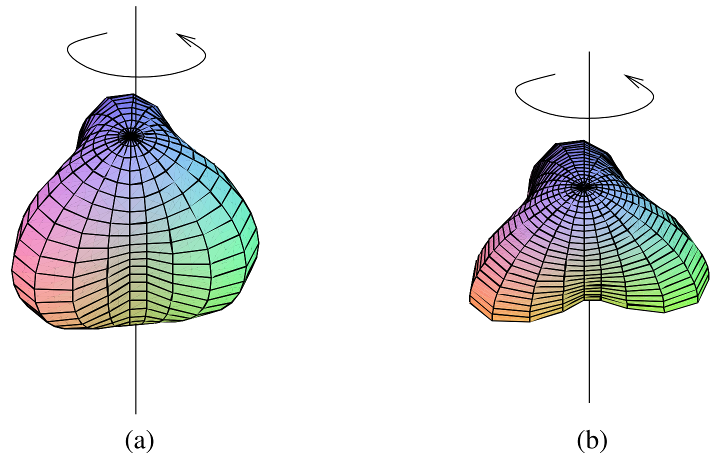

Figure 4: (a) Oscillations of a sphere deformed by spherical harmonics of order $l = 3$: spatial symmetry $\mathbf{Z}_6^-$ and twisted symmetry $(\mathbf{SO}(2))^\theta$. (b) Restriction to hemisphere: spatial symmetry $\mathbf{Z}_3$ and twisted symmetry $(\mathbf{SO}(2))^\theta$.

three ways to obtain solutions to the equation posed on the hemisphere, all of them have spatial symmetry $\mathbf{D}_2^-$ and spatio-temporal symmetry $(\mathbf{D}_4^-)^\theta$ (having twist type $\mathbf{Z}_2$). We show that in Figure 5 (b-d).

In Figure 1 (Section 1) we assume $l = 6$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbf{D}_2 \oplus \mathbf{Z}_2^c$, twisted symmetry $(\mathbf{T} \oplus \mathbf{Z}_2^c)^\theta$. Thus the deformations are $\mathbf{D}_2 \oplus \mathbf{Z}_2^c$-symmetric and oscillate with twist type $\mathbf{Z}_3$. In Theorem 5.2 (e) we show that this solution may be sliced in three ways to obtain solutions to the equation posed on the hemisphere, all of them have spatial symmetry $\mathbf{D}_2^-$ and trivial twist type. We show that in Figures 2 (Section 1), 6, 7.

In Figure 8 (a) we assume $l = 2$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbf{D}_2 \oplus \mathbf{Z}_2^c$, twisted symmetry $(\mathbf{D}_4 \oplus \mathbf{Z}_2^c)^\theta$. Thus the deformations are $\mathbf{D}_2 \oplus \mathbf{Z}_2^c$-symmetric and oscillate with twist type $\mathbf{Z}_2$. In Theorem 5.2 (f) for $l = 2$ and $n = 4$, we show that this solution may be sliced in three ways to obtain solutions to the equation posed on the hemisphere, all of them have spatial symmetry $\mathbf{D}_2^-$. One it has twisted symmetry $(\mathbf{D}_4^-)^\theta$ (twist type $\mathbf{Z}_2$), and the other two have trivial twist type. We show that in Figure 8 (b-d).

7 Smoothness of Extended Solutions

In Section 3 we state that solutions of (3.2) on H that satisfy the boundary condition (3.3) can be extended to solutions of (3.2) on S by defining u on the lower hemisphere

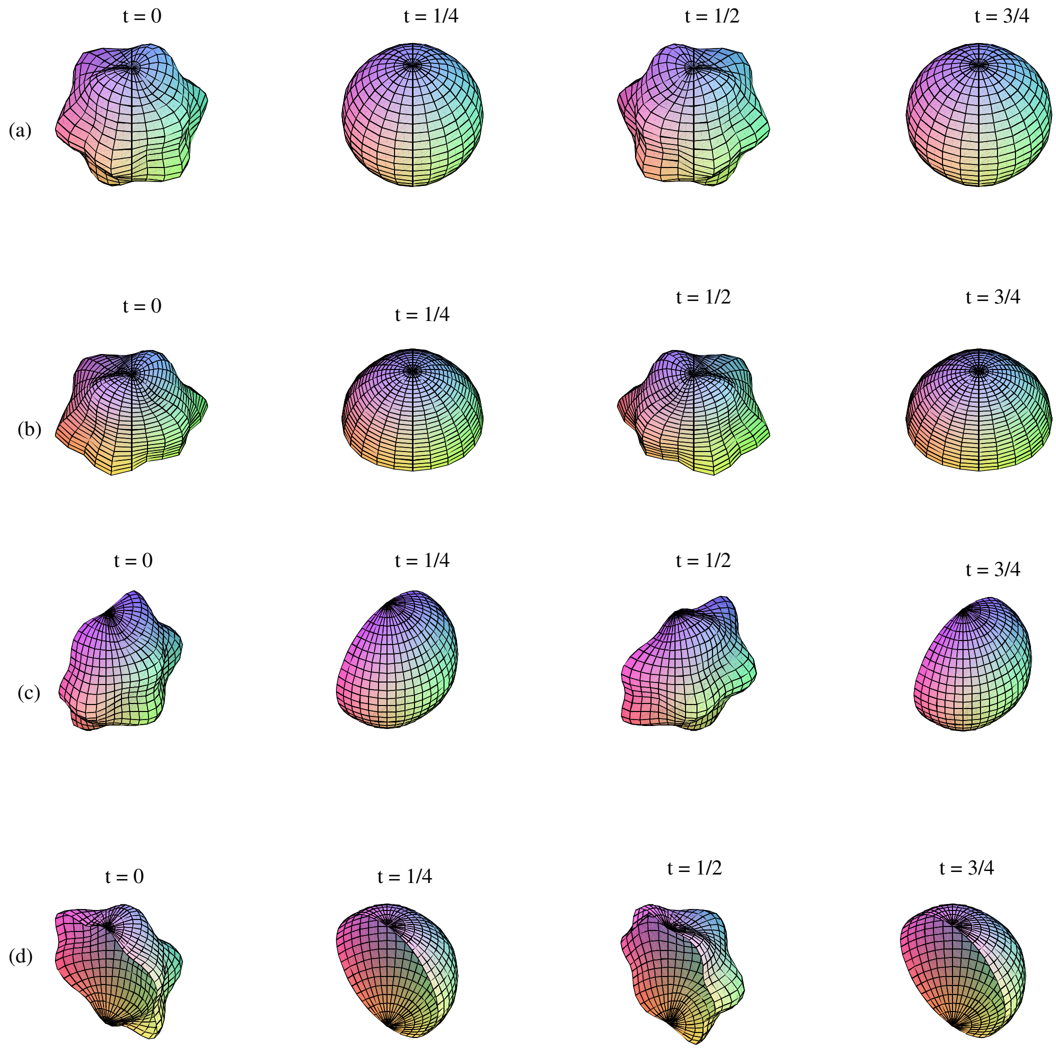

Figure 5: (a) Oscillations of a sphere deformed by spherical harmonics of order $l = 6$: spatial symmetry $\mathbf{T} \oplus \mathbf{Z}_2^c$ and spatio-temporal symmetry $(\mathbf{O} \oplus \mathbf{Z}_2^c)^\theta$. (b-d) (Three orbits) Restriction to hemisphere: spatial symmetry $\mathbf{D}_2^-$, spatio-temporal symmetry $(\mathbf{D}_4^-)^\theta$, twist type $\mathbf{Z}_2$.

by the reflection $\tau : S \to S$ across ∂H. Field *et al.* [6, Theorem 5.18] prove the regularity of the steady-state extended solutions along ∂H obtained by this method. A similar result is valid for periodic solutions of (3.2). Before stating this result, we briefly describe the abstract setting assumed by [6] and where the results hold.

Let M be a smooth, compact, connected, Riemannian n-dimensional manifold

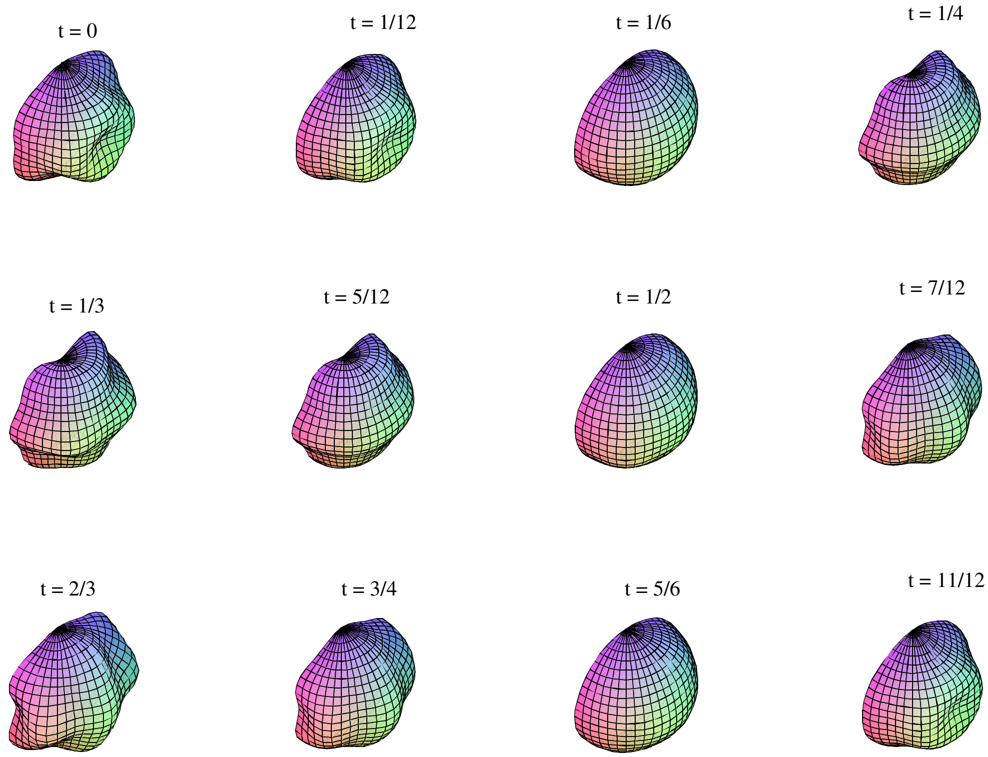

Figure 6: (Orbit 2) Restriction of solution of Figure 1 to hemisphere with spatial symmetry $\mathbf{D}_2^-$.

without boundary. Suppose K is a finite group of transformations of M generated by an admissible set $\mathcal{R}$ of p reflections. Thus K is isomorphic to $\mathbf{Z}_2^p$ and it may be assumed a group of isometries of M. Take N to be a connected component of $M_K = \{x \in M : (k \in K \wedge kx = x) \Rightarrow k = I_M\}$, where I_M denotes the identity map of M, and note that every isometry on N (for the Riemannian structure induced from M) extends uniquely to an isometry on M. Denote the boundary of N by ∂N, the group of isometries of M by $ISO(M)$, and the space of smooth real-valued functions on M by $C^\infty(M)$. Recall that the natural action of $ISO(M)$ on $C^\infty(M)$ is defined by $u \to g(u)$, where for $u \in C^\infty(M)$ and $g \in ISO(M)$ we have $g(u)(x) = u(g^{-1}x)$. Finally, consider $\mathcal{P}$ to be a semi-linear elliptic operator on $C^\infty(M)$ defined by

$$\mathcal{P}(u) = \Delta u + f(u), \tag{7.5}$$

where Δ is the Laplace operator associated to the Riemannian structure on M and

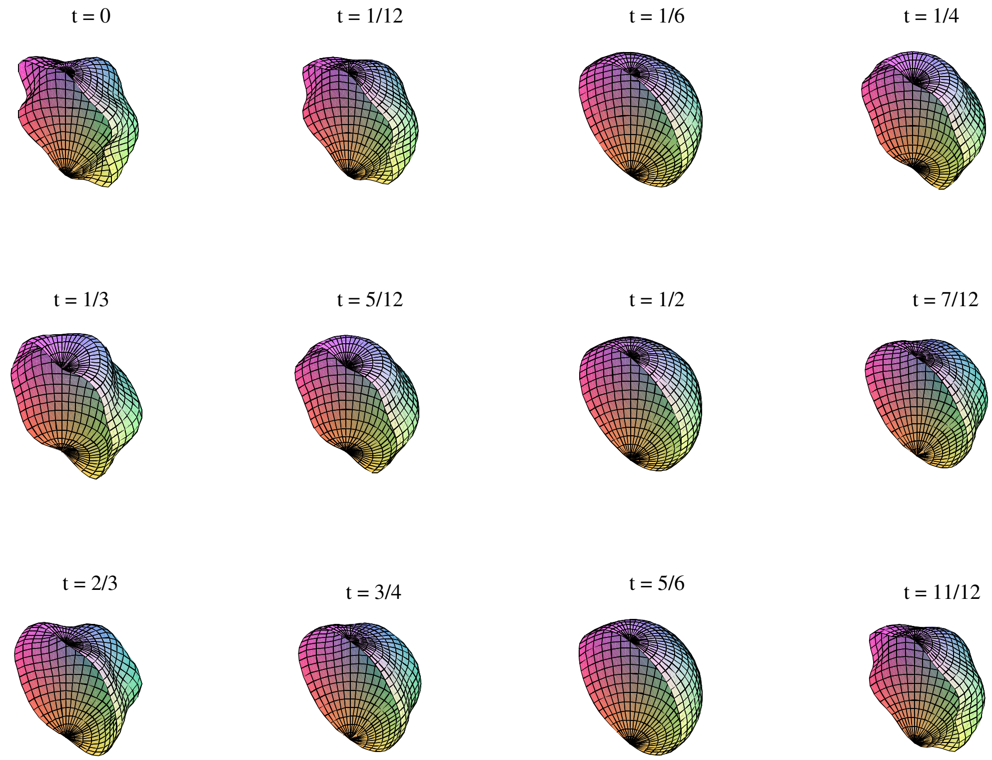

Figure 7: (Orbit 3) Restriction of solution of Figure 1 to hemisphere: spatial symmetry $\mathbf{D}_2^-$.

$f : \mathbf{R} \to \mathbf{R}$ is smooth. Note that the operator $\mathcal{P}$ on $C^\infty(M)$ defined by (7.5) satisfies

$$\mathcal{P}(g(u)) = g(\mathcal{P}(u))$$

for all $u \in C^\infty(M)$ and $g \in ISO(M)$ ($\mathcal{P}$ is $ISO(M)$-*invariant*).

Let $C^1(M)$ (respectively, $C^1(\overline{N})$) denote the space of C^1 real-valued functions on M (respectively, $\overline{N}$), and $C([0,T], C^1(M))$ the space of continuous mappings $u :\ [0,T] \to C^1(M)$.

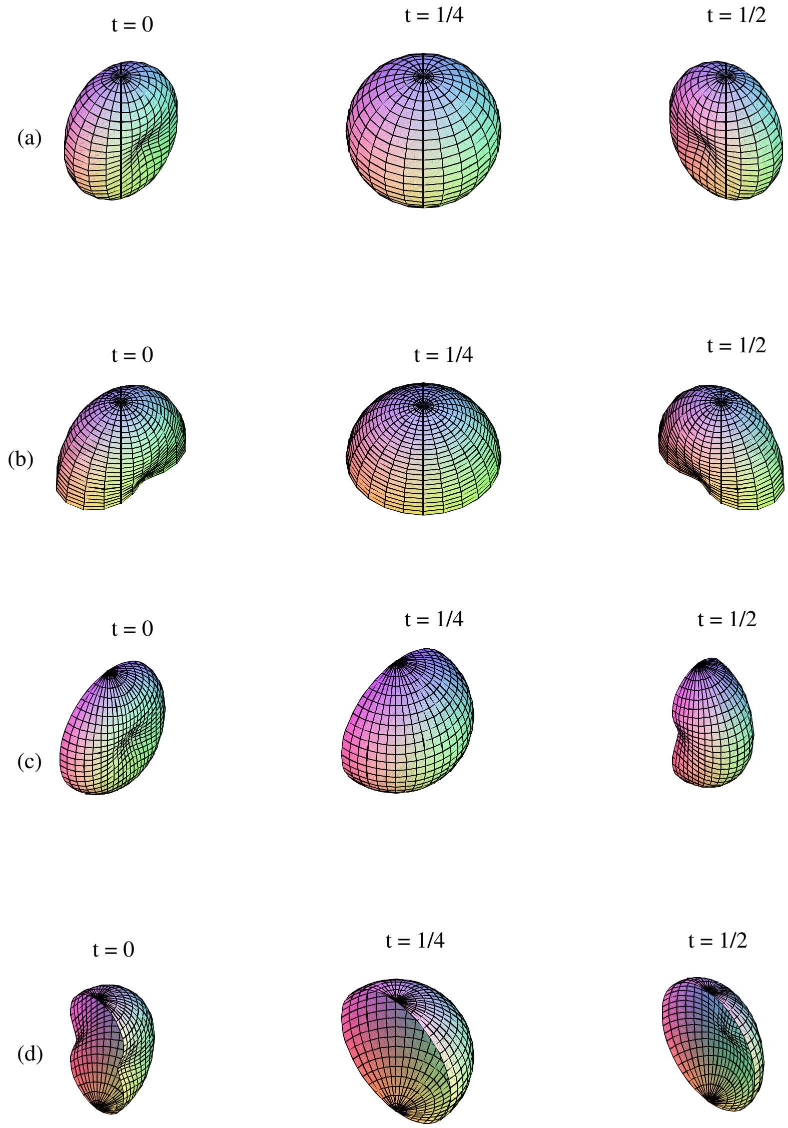

Figure 8: (a) Oscillations of a sphere deformed by spherical harmonics of order $l = 2$: spatial symmetry $\mathbf{D}_2 \oplus \mathbf{Z}_2^c$ and spatio-temporal symmetry $(\mathbf{D}_4 \oplus \mathbf{Z}_2^c)^\theta$ (twist type $\mathbf{Z}_2$). (b-d) (Three orbits) Restriction to hemisphere: spatial symmetry $\mathbf{D}_2^-$; (b) has spatio-temporal symmetry $(\mathbf{D}_4^-)^\theta$ and so twist type $\mathbf{Z}_2$; (c) and (d) have trivial twist type.

Proposition 7.1 [14] *Consider the equation*

$$\frac{\partial u}{\partial t} = \Delta u + F(x, u, \nabla u, t), \quad u(x, 0) = g \tag{7.6}$$

for $u(x,t)$ a function on $M \times [0,T]$, where M is a compact manifold without boundary, and F is C^∞ in its arguments. Given $g \in C^1(M)$, the above equation has, for some $T > 0$, a unique solution

$$u \in C([0,T], C^1(M)) \ \cap \ C^\infty(M \times]0,T]). \tag{7.7}$$

Proof See Taylor [14, Chapter 15, Proposition 1.2]. □

Given $f : M \to M$, let $\mathrm{Fix}(f) = \{x \in M : \ f(x) = x\}$ denote the *fixed-point set* of f.

We recall that a solution u such that $u(x,0) \in C^1(\overline{N})$ of

$$\frac{\partial u}{\partial t} = \mathcal{P}(u)$$

posed on N satisfies Neumann boundary conditions (NBC) on N if for every $\tau \in \mathcal{R}$ and all $x \in \partial N \cap \mathrm{Fix}(\tau)$, we have

$$\frac{\partial u}{\partial n}(x) = 0, \tag{7.8}$$

where n is the normal direction to $\mathrm{Fix}(\tau)$ at x.

A solution u (satisfying (7.7)) of the equation (7.6) will be called *smooth*.

We can now state the extension theorem:

Theorem 7.2 *Let $\mathcal{P}$ be the K-invariant operator defined by (7.5). Then the following hold:*

1. *Every smooth K-invariant solution u of*

$$\frac{\partial u}{\partial t} = \mathcal{P}(u) \tag{7.9}$$

   on M restricts to a smooth solution of the Neumann problem for (7.9) on N.

2. *Let u be a solution to the Neumann problem with $u(x,0) \in C^1(\overline{N})$ for (7.9) on $\overline{N}$. Then:*

   (a) u is smooth.

   (b) u extends uniquely to a smooth K-invariant solution of (7.9) on M.

Proof The proof follows the same lines as [6, Theorem 5.18], where now we consider the parabolic equation (7.9). In proving item (b) we use Proposition 7.1 above instead of [6, Lemma 5.15]. □

Acknowledgements

We thank Michael Field and Míriam Manoel for helpful discussions, and Ian Melbourne for pointing out Remark 5.3. SMCA thanks Universidade Portucalense for granting leave. The research of SMCA was supported by Centro de Matemática da Universidade do Porto.

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