Hopf bifurcation for wreath products

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Abstract. Systems of ODEs modelling coupled cells with 'wreath product' coupling have been the subject of recent research. For identical cells, such systems can have interesting symmetries. The basic existence theorem for Hopf bifurcation in the symmetric case is the equivariant Hopf theorem, which involves isotropy subgroups with a two-dimensional fixed-point subspace (called **C**-axial). A classification theorem for **C**-axial subgroups in wreath products is presented by Dionne, Golubitsky and Stewart in [8]. However, their classification is incomplete: it omits some **C**-axial subgroups in some cases. We provide a complete classification of the **C**-axial subgroups in wreath products. We also classify the maximal isotropy subgroups for these groups.

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1 Introduction

Over the last few years, there has been increasing interest in the nonlinear dynamics and bifurcations of systems of coupled identical 'cells' - that is, dynamical subsystems such as oscillators: see in particular [8, 9]. It is now well understood that coupled systems of this kind can possess certain types of symmetry. In particular, the individual cells can have a 'local' (or 'internal') group of symmetries $\mathcal{L}$ (a subgroup of $\mathbf{O}(k)$), and the network of couplings can have a 'global' symmetry group $\mathcal{G}$ (a subgroup of the permutation group $\mathbf{S}_N$). Two types of couplings have attracted particular attention, the so-called 'wreath product' case, in which the coupling is invariant under any local symmetry of any cell, and the 'direct product' case when a local symmetry must be applied simultaneously to all cells. The symmetry group of the coupled system is the wreath product $\mathcal{L} \wr \mathcal{G}$, and the direct product $\mathcal{L} \times \mathcal{G}$, respectively. In this paper we will study only the wreath product case.

There are many naturally occurring nonlinear systems with wreath product structure. Golubitsky, Stewart and Dionne [13] list four types of examples, as follows:

1. Coupled arrays of Josephson junctions [17, 1] where the symmetry group is $\mathbf{S}_k \wr \mathbf{S}_N$.

2. Discretizations of PDEs with gauge symmetry, such as the Ginzburg-Landau equation.

3. Molecular dynamics, where to a good approximation coupling between atoms is invariant under symmetries of each individual atom.

4. Heteroclinic cycles, like the Guckenheimer and Holmes [15] example of a structurally stable heteroclinic cycle obtained by abstracting a model for rotating convection developed by Busse and Heikes [2]. Also, with slight modification, the 'instant chaos' scenario of Guckenheimer and Worfolk [16], which involves a subgroup of index two of $\mathbf{Z}_2 \wr \mathbf{Z}_4$.

   More generally, there are now several more recent examples:

5. In [4], Dellnitz *et al.* present numerical evidence that it is possible to replace the equilibria in the heteroclinic cycle presented by Guckenheimer and Holmes (mentioned above) by chaotic sets, once more obtaining cycling behaviour, but now between chaotic sets.

6. Silber and Knobloch [18] study Hopf bifurcation on a square lattice. Their system can be viewed as a wreath product coupling system, and the group here is $\mathbf{O}(2) \wr \mathbf{S}_2$.

7. Callahan and Knobloch [3] study steady-state bifurcations on various cubic lattices. These correspond to various representations of $\mathcal{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$, where $\mathcal{O}$ is the octahedral group, $\mathbf{Z}_2^c$ represents inversion through origin and $\mathbf{T}^3$ is the three-torus of translations. This group is isomorphic to $\mathbf{O}(2) \wr \mathbf{S}_3$.

Our approach applies to the 6-dimensional representation (for the simple cubic lattice), but not to the 8- or 12-dimensional representations (for the face-centred and body-centred cubic lattices).

8. More generally, the Weyl group of type B_n denoted by $W(B_n)$ can be viewed as the wreath product $\mathbf{Z}_2 \wr \mathbf{S}_n$ [11]. This crystallographic group is also called the hyperoctahedral group because it is the symmetry group of the N-dimensional cube. It is the holohedry of a lattice in dimension N. If we extend it by the N-torus $\mathbf{T}^N$, we obtain a compact subgroup of the Euclidean group $\mathbf{E}(N)$ that leaves invariant the space of functions from $\mathbf{R}^N$ to $\mathbf{R}$ that are spatially periodic with respect to this lattice [6, 7]. Again we get a wreath product group: $\mathbf{O}(2) \wr \mathbf{S}_N$.

An appropriate general setting for such questions is the theory of symmetric dynamical systems [12, 14]. In that theory, we study a system of ODEs $\dot{x} = g(x, \lambda)$, for $g : V \times \mathbf{R} \to V$, where V is a finite-dimensional vector space. It turns out that the symmetry of g imposes restrictions on the bifurcations that can occur, and the main aim of the theory is to understand the effect that these restrictions have.

A central part of the theory is the study of bifurcations to periodic solutions in systems commuting with a compact Lie group Γ. Here the main result is the equivariant Hopf theorem [14], which guarantees (with certain nondegeneracy conditions) that for each isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ with a two-dimensional fixed-point subspace (called $\mathbf{C}$-axial) there exists a branch of periodic solutions with that symmetry. This theorem reduces part of the existence problem for Hopf bifurcations to an algebraic problem: the classification of $\mathbf{C}$-axial subgroups.

A classification theorem for the $\mathbf{C}$-axial subgroups in wreath products groups is presented in [8]. However, the proposed classification omits some $\mathbf{C}$-axial subgroups, and is therefore incomplete. In theorem 3.1 of this paper we provide a complete classification theorem for the $\mathbf{C}$-axial subgroups of wreath products groups. The structure of the extra $\mathbf{C}$-axial groups is more complicated than in [8]. However this structure is explicitly described, and depends very clearly on the $\mathbf{C}$-axial subgroups of $\mathcal{L} \times \mathbf{S}^1$ and on the possible blocks that can be obtained from the permutation group $\mathcal{G}$. More precisely, we prove that the $\mathbf{C}$-axial groups, up to conjugacy, of a general $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ are the groups that we denote by $\Sigma = \Sigma(B^\psi, J, \sigma, J_1, p)$ and that we define in section 3. Here B^ψ is $\mathbf{C}$-axial in $\mathcal{L} \times \mathbf{S}^1$ and J is a block. For B^ψ and J chosen, there is a permutation $\sigma \in \mathcal{G}$ that splits J as disjoint union of subsets of the form $\sigma^i(J_1)$ where for some power s' we end up with J_1. Finally, depending on this power and on the twist image $\psi(B)$, there is a cyclic group $\mathbf{Z}_p$ of $\mathbf{S}^1$ on which the structure of Σ depends. The $\mathbf{C}$-axial groups obtained by [8] are those groups Σ with twist image equal to the twist image $\psi(B)$.

In [10] there is the analogous result to the equivariant Hopf theorem, in which Σ can be any maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. We also describe the maximal isotropy subgroups for $\Gamma \times \mathbf{S}^1$ where Γ is a wreath product group. The description of these subgroups is easily obtained from the method used for

$\mathbf{C}$-axial groups. More generally, we find that any submaximal isotropy subgroup can be seen as an intersection of isotropy subgroups that have a structure like the one for $\mathbf{C}$-axial groups where now the isotropy group B^ψ of $\mathcal{L} \times \mathbf{S}^1$ can be any (not necessarily maximal).

Finally, we can conclude that in order to find the isotropy lattice of a general group $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$, we start up by finding the isotropy lattice of $\mathcal{L} \times \mathbf{S}^1$. Once this is obtained, we have to know the block structure of $\mathcal{G}$. Finally, putting together this information, we are able to obtain the complete classification with the groups that we denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$.

The paper is organized as follows. In section 2 we introduce wreath product groups and summarize the mains results obtained in [8] about the linear theory of wreath products. We also present the $\mathbf{C}$-axial groups obtained in [8].

In section 3 we complete the classification of $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$ where Γ is a general wreath product group $\mathcal{L} \wr \mathcal{G}$.

In section 4 we derive the analogous classification for the maximal isotropy subgroups of $\Gamma \times \mathbf{S}^1$.

In section 5 we also prove that isotropy subgroups with an algebraic structure that is a generalization of the one obtained for $\mathbf{C}$-axial groups can be used to describe any isotropy subgroup of $\Gamma \times \mathbf{S}^1$.

Finally in section 6 we illustrate the classification of section 3 for the wreath product group $\mathbf{O}(2) \wr \mathbf{S}_2$, for a representation that is isomorphic to the one used in [18]. The same $\mathbf{C}$-axial subgroups are obtained as those found in [18], but in a much more systematic way. We also obtain the $\mathbf{C}$-axial groups for $\mathbf{O}(2) \wr \mathbf{S}_3$.

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2 Background

We begin this section by introducing the terminology and results of [8] for wreath products. These will be needed in the following sections.

One way of defining a coupled system of ODEs of N identical cells is the following. Let $X_j \in \mathbf{R}^k$ denote the state variables of the jth cell and suppose that $\dot{X}_j = f(X_j)$ governs the internal dynamics of each cell, where f commutes with $\mathcal{L} \subseteq \mathbf{O}(k)$. Let $X = (X_1, \ldots, X_N) \in (\mathbf{R}^k)^N$ be the state variables for the entire N-cell system. A system of ODEs

$$\frac{dX}{dt} = F(X) \tag{1}$$

is a system of coupled cells if

$$F_j(X) = f(X_j) + h_j(X)$$

where h_j governs the coupling between cells.

We say that (1) has symmetry the wreath product $\mathcal{L} \wr \mathcal{G}$, where $\mathcal{G}$ is a subgroup of the permutation group $\mathbf{S}_N$, if the symmetry group that commutes with F is the group generated by the groups $\mathcal{L}^N$ and $\mathcal{G}$. This is, for example, the case when h_j is invariant under $\mathcal{L}^{N-1}$ acting on $(X_1, X_2, \dots, X_{j-1}, X_{j+1}, \dots, X_N)$, and equivariant under $\mathcal{L}$ acting on X_j.

As in [8] we may assume that
(H_T) $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$.
This assumption does not lead to a loss of generality, because if the action of $\mathcal{G}$ is intransitive, we can consider the group orbits of cells under $\mathcal{G}$, which reduces the discussion to a finite list of cases in each of which the condition (H_T) holds.

2.1 Group structure of the wreath product

Let $V = \mathbf{R}^k$ and let V^N be the state space of the system (1). The action of $\mathcal{L} \wr \mathcal{G}$ on V^N is given by

$$(l, \sigma).(X_1, \dots, X_N) = (l_1 X_{\sigma^{-1}(1)}, \dots, l_N X_{\sigma^{-1}(N)})$$

where $l = (l_1, \dots, l_N) \in \mathcal{L}^N$, $\sigma \in \mathcal{G}$ and $(X_1, \dots, X_N) \in V^N$.

The permutations act on $l \in \mathcal{L}^N$ by

$$\sigma(l) = (l_{\sigma^{-1}(1)}, \dots, l_{\sigma^{-1}(N)})$$

and it follows that the group product in $\mathcal{L} \wr \mathcal{G}$ is given by

$$(h, \tau)(l, \sigma) = (h\tau(l), \tau\sigma).$$

2.2 The linear theory

Throughout, let $\Gamma = \mathcal{L} \wr \mathcal{G}$. By [14], when considering Hopf bifurcation we may assume the generic hypothesis
(H_H) Γ acts Γ-simply on the centre subspace.
It is important to understand how Γ decomposes the state space into irreducible subspaces. We are interested in studying bifurcations with combined local and global symmetries. Let $W \subseteq V^N$ be a Γ-irreducible subspace. This subspace is invariant under $\mathcal{L}^N$. If $\mathcal{L}^N$ acts trivially on W, then the local symmetries will have no effect on a bifurcation supported by this representation and we end up with a bifurcation problem with symmetry $\mathcal{G}$ which is not the aim of this work.

Therefore, as in [8], it is assumed
$(H_{\mathcal{L}})$ $\mathcal{L}^N$ acts nontrivially on W.

It is proved in [8] that Γ acts absolutely irreducible on V^N if and only if $\mathcal{L}$ acts absolutely irreducibly on V. For Hopf bifurcation points, the centre subspace is generically Γ-simple. That is, it has the form $W \oplus W$ where W is absolutely Γ-irreducible or is a nonabsolutely Γ-irreducible subspace. Assuming (H_T) and $(H_{\mathcal{L}})$, lemmas 3.2 and 3.1 of [8] imply that the centre subspace is either $(U \oplus U)^N$ where U is absolutely $\mathcal{L}$-irreducible, or it is U^N where U is nonabsolutely $\mathcal{L}$-irreducible. That is, U is $\mathcal{L}$-simple. See [8] (end of section 3).

2.3 C-axial subgroups

For a compact Lie group Γ acting on a vector space V, the *isotropy subgroup* of $v \in V$ is the group

$$\Sigma_v = \{\gamma \in \Gamma : \gamma \cdot v = v\}.$$

Recall that points in V that are in the same Γ-orbit have conjugate isotropy subgroups.

An isotropy subgroup Σ of Γ is *maximal* if there does not exist an isotropy subgroup Δ of Γ satisfying $\Sigma \subset \Delta \subset \Gamma$.

As mentioned before, we say that a subgroup $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-*axial* if it is an isotropy subgroup having a two-dimensional fixed-point subspace (over $\mathbf{R}$).

In the classification of $\mathbf{C}$-axial subgroups of wreath products groups $\mathcal{L} \wr \mathcal{G}$, the structure of these subgroups is determined by the possible blocks that are derived from the permutation group $\mathcal{G}$. We define: a subset of indices $J \subseteq \{1, \ldots, N\}$ is a *block* if there exists a subgroup $\mathcal{H}$ of $\mathcal{G}$ that leaves J invariant and acts transitively on J.

To each block J we can associate the permutation subgroup

$$Q_J = \{\sigma \in \mathcal{G} : \sigma(J) = J\}$$

which acts transitively on J.

Assume (H_H) and consider the natural action of $\mathbf{S}^1$ on the centre subspace obtained by giving a complex structure to this space as in [14]. Consider a block J and let B^ψ a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$ where $\psi : B \to \mathbf{S}^1$ is an homomorphism and

$$B^\psi = \{(b, \psi(b)) : b \in B\}.$$

Following [14] we call the group B^ψ a *twisted* subgroup of $\mathcal{L} \times \mathbf{S}^1$. The image $\psi(B)$ is a closed subgroup of $\mathbf{S}^1$. The closed subgroups of $\mathbf{S}^1$ are $\mathbf{1}$, $\mathbf{Z}_n$ ($n = 2, 3, 4, \ldots$) and $\mathbf{S}^1$. We say that B^ψ is of *finite* twist type if the image $\psi(B)$ is not $\mathbf{S}^1$.

Consider

$$\Sigma(B^\psi, J) = (\mathbf{1}^N, Q_J, 0) + ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) + ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$$

where $+$ indicates 'group generated by' as in [8]. Here it is assumed that $J = \{1, \ldots, s\}$ and the subgroup $\hat{B}$ is defined by

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

It is proved in [8] that $\Sigma(B^{\psi}, J)$ is a $\mathbf{C}$-axial subgroup of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$. It is also claimed that every $\mathbf{C}$-axial subgroup is of this type. However, we show in the next section that *not* all the $\mathbf{C}$-axial subgroups of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ are of this type.

3 Classification of C-axial subgroups

Our aim in this section is to give a complete description up to conjugacy of all the $\mathbf{C}$-axial subgroups of the groups of the type $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$. We begin by describing subgroups that we denote by $\Sigma(B^{\psi}, J, \sigma, J_1, p)$. We will show that these subgroups are $\mathbf{C}$-axial and that any $\mathbf{C}$-axial subgroup is conjugate to one of these.

As in the previous section we assume the generic hypothesis (H_H), so that we can write the centre subspace as V^N, where either $\mathcal{L}$ acts nonabsolutely irreducibly on V, or $V = U \oplus U$ and $\mathcal{L}$ acts absolutely irreducibly on U. That is, the space V is $\mathcal{L}$-simple. If $\mathcal{L}$ acts trivially on V, then the action of Γ on V^N is reduced to the action of $\mathcal{G}$ on V^N and in this case, Hopf bifurcation with symmetry Γ is reduced to that with symmetry $\mathcal{G}$. We are therefore interested in the cases where $\mathcal{L}$ does not act trivially on V. We note that for these actions, $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$.

From now on consider $\Gamma = \mathcal{L} \wr \mathcal{G}$ where we are considering the action of $\mathcal{L} \wr \mathcal{G}$ on the Γ-simple space V^N is as defined in the previous section.

The group $\Sigma(B^{\psi}, J, \sigma, J_1, p)$

Consider a block $J \subseteq \{1, \ldots, N\}$ and let Q_J be the subgroup of $\mathcal{G}$ that leaves J invariant. Suppose

$$J = \{1, \ldots, s\}.$$

Let J_1 be a subset of J such that for some permutation $\sigma \in Q_J$

$$J = J_1 \,\dot{\cup}\, \sigma(J_1) \,\dot{\cup}\, \cdots \,\dot{\cup}\, \sigma^{s'-1}(J_1)$$

where $\dot{\cup}$ is disjoint union and

$$\sigma^{s'}(J_1) = J_1.$$

In particular it follows that $|J| = s'|J_1|$.

Choose notation so that

$$J_{i+1} = \sigma^i(J_1), \quad i = 1, \ldots, s' - 1$$

and let

$$Q_{J,J_1} = \{\tau \in Q_J : \tau(J_i) = J_i,\ i = 1, \ldots, s'\}.$$

Suppose Q_{J,J_1} acts transitively on J_1. This implies that Q_{J,J_1} acts transitively on all J_i.

Note that by definition of block the group Q_J acts transitively on J. Therefore $\sigma = 1$ and $J_1 = J$ are under those conditions.

Define

$$Q_{J,J_k} = \{\tau \in Q_J : \tau(J_j) = \sigma^{k-1}(J_j),\ j = 1, \ldots, s^{'}\},$$

for $k = 2, \ldots, s^{'}$. That is, each permutation in Q_{J,J_k} interchanges the subsets J_i of J in the same way as σ^{k-1}.

Let B^ψ be a $\mathbf{C}$-axial subgroup of $\mathcal{L}\times\mathbf{S}^1$ of finite twist type $\mathbf{Z}_r$, and let $\hat{B}$ be the subgroup of B^s defined by

$$\hat{B} = \{(b_1, \cdots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Let $\mathbf{Z}_p =< \xi_p >$ be a cyclic subgroup of $\mathbf{S}^1$ such that

$$s^{'} = \min_{i>0} \{\xi_p^i \in \mathbf{Z}_r\}.$$

Call $\xi_{r'} = \xi_p^{s^{'}}$. It follows that $\mathbf{Z}_p = \mathbf{Z}_{s' r'}$ where $\mathbf{Z}_{r'} \subseteq \mathbf{Z}_r$.

Define B_k the subgroup of B^s by

$$B_k = \left\{ \quad (b_1, \ldots, b_s) \in B^s : \psi(b_j) = \quad \begin{cases} \xi_{r'}, & \text{if } j \in J_1 \cup \cdots \cup J_{k-1}, \\ 0, & \text{if } j \in J_k \cup \cdots \cup J_{s'} \end{cases} \quad \right\}$$

Finally denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$ the subgroup of $\Gamma \times \mathbf{S}^1$ generated by the following groups:

$$\Sigma(B^\psi, J, \sigma, J_1, p) = ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) \ + \ ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \ +$$

$$+ \ (\mathbf{1}^N, Q_{J,J_1}, 0) \ + \bigcup_{k=2,\ldots,s'} ((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1}).$$

Note that this group depends on the block J, the permutation σ (and so on J_1). Also the group Q_{J,J_1} has to act transitively on J_1. Finally, it depends on B^ψ (a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$) and on the cyclic subgroup $\mathbf{Z}_p$ of $\mathbf{S}^1$ (where some divisor $r^{'}$ of r divides p).

We state our main theorem:

Theorem 3.1 *An isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-axial if and only if it is conjugate to a ($\mathbf{C}$-axial) group of the type $\Sigma(B^\psi, J, \sigma, J_1, p)$, for some $\mathbf{C}$-axial group B^ψ of $\mathcal{L} \times \mathbf{S}^1$, a block J, a permutation σ of $\mathcal{G}$, a subset J_1 of J, and a nonnegative integer p.*

The rest of this section is dedicated to the proof of this theorem. First we show in proposition 3.2 that the groups $\Sigma(B^\psi, J, \sigma, J_1, p)$ defined above are $\mathbf{C}$-axial. Basically, using algebraic calculations, we are able to describe the fixed-point subspaces of these groups. Then, in proposition 3.6, we show that every $\mathbf{C}$-axial Σ of $\Gamma \times \mathbf{S}^1$ is conjugate to some group of this type. For that we need to prove first two lemmas. In lemma 3.4 we prove that once we choose an element w fixed by Σ, the nonzero components have indices corresponding to a block and the projection of Σ on the group $\mathcal{G}$ is a permutation group acting transitively on that block. In lemma 3.5 we show that if we choose a nonzero component of w, then the corresponding isotropy subgroup (now of $\mathcal{L} \times \mathbf{S}^1$) is also $\mathbf{C}$-axial. Finally, in proposition 3.6, using these two lemmas, we manipulate the vector w and conclude that, up to conjugacy, we can assume that w (a representative point for the isotropy subgroup Σ) belongs to the fixed-point subspace of one of those groups $\Sigma(B^\psi, J, \sigma, J_1, p)$.

Proposition 3.2 *With the above notation $\Sigma(B^\psi, J, \sigma, J_1, p)$ is a $\mathbf{C}$-axial subgroup of $\Gamma \times \mathbf{S}^1$.*

Proof. Let $\Sigma = \Sigma(B^\psi, J, \sigma, J_1, p)$ and $w = (w_1, \ldots, w_N) \in V^N$ be fixed by Σ. Since $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ fixes w and $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$, we have $w_{s+1} = \cdots = w_N = 0$. Thus

$$w = (w_1, \ldots, w_s, 0, \ldots, 0).$$

Suppose

$$J_1 = \{1, \ldots, \frac{s}{s'}\},\ J_2 = \{\frac{s}{s'} + 1, \ldots, 2\frac{s}{s'}\},\ \ldots,\ J_{s'} = \{(s' - 1)\frac{s}{s'} + 1, \ldots, s\}.$$

Since $(\mathbf{1}^N, Q_{J,J_1}, 0)$ fixes w and Q_{J,J_1} is transitive on each part J_i, the components w_i corresponding to each J_i are equal. Denote by $< w_i >$ the vector with s/s' components equal to w_i and $\xi = \xi_p$. Since $((B_k, \mathbf{1}^{N-s}), \sigma^{k-1}, \xi^{k-1})$ for $k = 2, \ldots, s'$ fixes w, it follows that

$$w = (< w_1 >, < \xi . w_1 >, \ldots, < \xi^{s'-1} \cdot w_1 >, 0, \ldots, 0).$$

Since w is fixed by $((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$ we have $w_1 \in \mathrm{Fix}_V(\mathrm{B}^\psi)$. As B^ψ is $\mathbf{C}$-axial, we see that $\mathrm{Fix}_{V^N}(\Sigma)$ is two-dimensional.

To complete the proof we show that Σ is the isotropy subgroup of w. Let Σ_w be the isotropy subgroup of

$$w = (< w_1 >, < \xi \cdot w_1 >, \ldots, < \xi^{s'-1} \cdot w_1 >, 0, \ldots, 0).$$

From the previous discussion and straightforward calculations $\Sigma \subseteq \Sigma_w$. To verify the reverse inclusion we show that if $(l, \tau, \theta) \in \Sigma_w$ then $(l, \tau, \theta) \in \Sigma$. As $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ fixes w we have that $((l_1, \ldots, l_s, 1, \ldots, 1), \tau, \theta)$ fixes w.

If $\tau \in Q_{J,J_1}$, as $(\mathbf{1}^N, Q_{J,J_1}, 0)$ fixes w, then $\gamma = ((l_1, \ldots, l_s, 1, \ldots, 1), 1, \theta)$ fixes w. But for $\gamma \cdot w = w$, then $(l_j, \theta) \in \mathcal{L} \times \mathbf{S}^1$ fixes w_1. Since B^ψ is the isotropy subgroup of w_1, it follows that $(l_j, \theta) \in B^\psi$ and $\theta = \psi(l_j)$. Thus $\gamma \in ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$ and $(l, \tau, \theta) \in \Sigma$.

If $\theta \in \mathbf{Z}_r$, then it follows immediately that $\tau \in Q_{J,J_1}$ and we have the previous case.

If $\theta \notin \mathbf{Z}_r$ and as $((l_1,\ldots,l_s,1,\ldots,1),\tau,\theta)$ fixes w, then $\theta = \xi^i \xi_r^j$ for some $i < s^{'}$ and $j \in \{0,\ldots,r-1\}$. As we have $((b,1,\ldots,1),1,\xi_r^{-j}) \in ((\hat{B},\mathbf{1}^{N-s}),\mathbf{1},\psi)$ for some $b \in \hat{B}$, then

$$\gamma = ((b_1 l_1,\ldots,b_s l_s,1,\ldots,1),\tau,\xi^i)$$

fixes w. Suppose that $J_1 = \tau^{-1}(J_{j+1})$. From $\gamma \cdot w = w$ and since $i < s^{'}$, then $j = i$ and $\tau \in Q_{J,J_{i+1}}$. There is $\gamma^{'} = (((b^{'})^{-1},1,\ldots,1),\tau,\xi^i) \in ((B_{i+1},\mathbf{1}^{N-s}),Q_{J,J_{i+1}},\xi^i)$. Moreover, we can choose $b^{'} \in B_{i+1}$ such that $b^{'}_j = 1$ for $j \in J_{i+1} \cup \ldots \cup J_{s^{'}}$. Take $(\gamma^{'})^{-1} = ((\tau^{-1}(b^{'}),1^{N-s}),\tau^{-1},\xi^{-i})$. Now

$$(\gamma^{'})^{-1}\gamma = ((\tau^{-1}(b^{'}bl),1^{N-s}),1,0)$$

fixes w. It follows that $\tau^{-1}(b^{'}bl) \in \hat{B}$ with $\psi((b^{'}bl)_{\tau(i)}) = 0$ and $l \in B^s$ since b and $b^{'}$ are in B^s. Therefore $\psi(l_l) = \xi_r^j \xi_{r^{'}}$ if $l \in J_1 \cup \ldots \cup J_i$ and $\psi(l_l) = \xi_r^j$ if $l \in J_{i+1} \cup \ldots \cup J_{s^{'}}$. It follows that $\gamma \in ((B_{i+1},\mathbf{1}^{N-s}),Q_{J,J_{i+1}},\xi^i)$ and $((l_1,\ldots,l_s,1,\ldots,1),\tau,\theta) \in ((B_{i+1},\mathbf{1}^{N-s}),Q_{J,J_{i+1}},\xi^i) + ((\hat{B},\mathbf{1}^{N-s}),\mathbf{1},\psi)$. $\square$

Some of the groups $\Sigma(B^\psi,J,\sigma,J_1,p)$ are the same as the groups $\Sigma(B^\psi,J)$ found by Dionne *et al.* [8]. Using the notation of the previous section we have:

Corollary 3.3 *With the conditions of proposition 3.2, and if*

$$\mathbf{Z}_p \subseteq \mathbf{Z}_r,$$

then $\Sigma(B^\psi,J,\sigma,J_1,p)$ is conjugate to $\Sigma(B^\psi,J)$ and is of the same twist type as B^ψ.

Proof. Note that if $\mathbf{Z}_p \subseteq \mathbf{Z}_r$ then Σ is conjugate to $\Sigma_{(w_1,\ldots,w_1,0,\ldots,0)}$ since any element $(l_i,\theta_i)w_1$ in V (with $w_1 \in \mathrm{Fix}_V(\mathrm{B}^\psi)$) is in the $\mathcal{L}$-orbit of w_1. $\square$

However, if $\mathbf{Z}_p \not\subseteq \mathbf{Z}_r$, then there are new possibilities. We show that all $\mathbf{C}$-axial subgroups of finite twist type of $\Gamma \times \mathbf{S}^1$ are conjugate to subgroups of the form $\Sigma(B^\psi,J,\sigma,J_1,p)$.

Let $\Pi_{\mathcal{G}} : \Gamma \times \mathbf{S}^1 \to \mathcal{G}$ be projection and let

$$V_J = \{(w_1,\ldots,w_N) \in V^N : w_j = 0 \text{ for } j \notin J\}.$$

Lemma 3.4 *Let Σ be a $\mathbf{C}$-axial subgroup of $\Gamma \times \mathbf{S}^1$. Then $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on some block J, and $\mathrm{Fix}_{V^N}(\Sigma) \subseteq V_J$.*

Proof. Let w be a nonzero vector of $\mathrm{Fix}_{V^N}(\Sigma)$ and let J be the set of the indices $j \in \{1,\ldots,N\}$ such that $w_j \neq 0$. We will show that $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Since $(l_j,\theta).w_{\sigma^{-1}(j)} = 0$ if and only if $w_{\sigma^{-1}(j)} = 0$, we have $\Pi_{\mathcal{G}}(\Sigma)J \subseteq J$. Suppose that there exist two disjoint subsets J_1 and J_2 of J such that $\Pi_{\mathcal{G}}(\Sigma)J_i \subseteq J_i$ for $i = 1,2$. If so, construct two vectors $y_1,\ y_2 \in V^N$, where

$$y_1^j = \begin{cases} w_j & \text{if } j \in J_1, \\ 0 & \text{if } j \notin J_1 \end{cases} \qquad y_2^j = \begin{cases} w_j & \text{if } j \in J_2, \\ 0 & \text{if } j \notin J_2 \end{cases}$$

These vectors in $\mathrm{Fix}_{V^N}(\Sigma)$ are linearly independent. Moreover, we can split

$$\mathrm{Fix}_{V^N}(\Sigma) = \mathrm{V}_1 \oplus \mathrm{V}_2,$$

if we take

$$V_i = \{y_i \in \mathrm{Fix}_{V^N}(\Sigma) : y_i^j = 0 \text{ if } j \notin J_i\}$$

for $i = 1, 2$. Therefore

$$\Sigma_w \subset \Sigma_{y_1} \subset \Gamma \times \mathbf{S}^1$$

and $\Sigma_w = \Sigma$ is not maximal. This contradicts the fact that Σ is $\mathbf{C}$-axial, so $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J, and J is a block. $\square$

The projection $\Pi_{\mathcal{G}}(\Sigma)$ as a subgroup of $\mathcal{G}$, must decompose the set $\{1, \ldots, N\}$ into a union of blocks. Using the above lemma, we get that if Σ is $\mathbf{C}$-axial, then a vector w fixed by Σ is supported on precisely one of these blocks. That is, only the components corresponding to one of the blocks are nonzero and $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on that block. This restriction implies information on the vectors w fixed by Σ.

We know that all proper isotropy subgroups of $\Gamma \times \mathbf{S}^1$ are twisted subgroups (see for example [14]). Because $\Gamma \times \mathbf{S}^1$ acts Γ-simply on V^N, also $\mathcal{L} \times \mathbf{S}^1$ acts $\mathcal{L}$-simple on V and so if $\Sigma_{w_1} \subset \mathcal{L} \times \mathbf{S}^1$ is the isotropy subgroup of $w_1 \in V$ then it is a twisted subgroup of $\mathcal{L} \times \mathbf{S}^1$.

Lemma 3.5 *Let $\Sigma = H^\theta \subset \Gamma \times \mathbf{S}^1$ be a twisted $\mathbf{C}$-axial subgroup. Let w be a nonzero vector fixed by Σ with $w_1 \neq 0$. Let $\Sigma_{w_1} = B^\psi$ be the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$. Then Σ_{w_1} is $\mathbf{C}$-axial and $\psi(B) \subseteq \theta(H)$.*

Proof. For w a nonzero vector of $\mathrm{Fix}_{V^N}(\Sigma)$, let J be the set of indices $j \in \{1, \ldots, N\}$ such that $w_j \neq 0$. As we are assuming $w_1 \neq 0$ we have $1 \in J$. By lemma 3.4 we know that $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Therefore for all $i \in J \setminus \{1\}$ there exists a permutation $\sigma_i \in \Pi_{\mathcal{G}}(\Sigma)$ such that $\sigma_i^{-1}(i) = 1$. Moreover, since

$$(l^i, \sigma_i, \theta_i) \cdot w = w$$

for some (l^i, θ_i) it follows that

$$w_i = (l_i^i, \theta_i) \cdot w_1.$$

Therefore the vector w has the form

$$w = (w_1, (l_2^2, \theta_2) \cdot w_1, \ldots, (l_s^s, \theta_s) \cdot w_1, 0, \ldots, 0)$$

and Σ is conjugate to the isotropy subgroup of

$$w^{'} = (w_1, \theta_2 \cdot w_1, \ldots, \theta_s \cdot w_1, 0, \ldots, 0).$$

Let $\Sigma = \Sigma_{w'}$ and define

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Note that $\hat{B}$ is a subgroup of B^s because ψ is a group homomorphism. Now

$$((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \subset \Sigma$$

and so $\psi(B) \subseteq \theta(H)$.

It remains to show that B^ψ is $\mathbf{C}$-axial. If B^ψ fixes an element w_2 that is not a multiple of w_1, then Σ fixes $w = (w_2, \theta_2 \cdot w_2, \ldots, \theta_s \cdot w_2)$ and then Σ is not $\mathbf{C}$-axial, a contradiction. $\square$

Remark. Let $\Sigma_{w_1} = B^\psi \subset \mathcal{L} \times \mathbf{S}^1$ be the isotropy subgroup of w_1, and let w be a nonzero vector fixed by $\Sigma = H^\theta \subset \Gamma \times \mathbf{S}^1$ as in lemma 3.5. If Σ_{w_1} is of twist type $\mathbf{S}^1$ then Σ also has twist type $\mathbf{S}^1$. In this case Σ is conjugate to $\Sigma(B^\psi, J)$. Moreover, we shall see later that if Σ is of twist type $\mathbf{S}^1$ then Σ_{w_1} is also of twist type $\mathbf{S}^1$.

We prove now our main result. Using lemmas 3.4 and 3.5, we show that all $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$ are conjugate to groups of the form $\Sigma(B^\psi, J, \sigma, J_1, p)$.

Proposition 3.6 *Let $\Sigma = H^\theta \subset \Gamma \times \mathbf{S}^1$ be a twisted $\mathbf{C}$-axial subgroup of finite twist type. If $\Sigma = \Sigma_w$ with $w_1 \neq 0$, assume that the isotropy subgroup $B^\psi = \Sigma_{w_1}$ of w_1 is of twist type $\mathbf{Z}_r$.*

Then Σ is conjugate to $\Sigma(B^\psi, J, \sigma, J_1, p)$ for a block J, a permutation $\sigma \in \Pi_{\mathcal{G}}(\Sigma)$, a subset J_1 of J, and a nonnegative integer p.

Proof. Let $w \neq 0$ be a vector fixed by Σ. We know that $\Pi_{\mathcal{G}}(\Sigma)$ decomposes $\{1, \ldots, N\}$ into a union of blocks. From lemma 3.4, since Σ is $\mathbf{C}$-axial, w is supported on precisely one of these blocks J. To simplify notation, assume that the block $J = \{1, \ldots, s\}$ where $s \leq N$, and let

$$w = (w_1, \ldots, w_s, 0 \ldots, 0).$$

Then the group $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ fixes w.

Construct a partition $J = J_1 \cup \cdots \cup J_q$ of the block J, by putting two indices l and m in the same part if w_l and w_m lie on the same $\mathcal{L}$-orbit. Conjugate w so that all w_i in the same part J_i are equal.

If all the components lie on the same $\mathcal{L}$-orbit, then Σ is conjugate to

$$\Sigma_{(w_1, \ldots, w_1, 0, \ldots, 0)}$$

which is of the type described in [8], that is, Σ is conjugate to $\Sigma(B^\psi, J)$. In our notation it is $\Sigma(B^\psi, J, 1, J, r)$.

Suppose now that $J = J_1 \cup \ldots \cup J_q$ with $q > 1$. From lemma 3.4, the group $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Suppose without loss of generality that $1 \in J_1$, and choose $i_2 \in J_2$. Then there exists $(l^2, \sigma_2, \theta_2) \in H^\theta$ with $\theta_2 \neq 0$ such that $\sigma_2(1) = i_2$. So

$$w_{i_2} = (l^2_{i_2}, \theta_2).w_1.$$

Similarly, for $j = 3, \ldots, q$, we can choose $i_j \in J_j$ and find $(l^j, \sigma_j, \theta_j) \in H^\theta$ with $\theta_j \neq 0$ such that $\sigma_j(1) = i_j$. Now suppose that $\theta(H) = \mathbf{Z}_l$. By lemma 3.5 we have $\mathbf{Z}_r \subseteq \mathbf{Z}_l$. So $\theta_2, \ldots, \theta_q \in \mathbf{Z}_l \setminus \mathbf{Z}_r$. Note that if, for example, $\theta_2 \in \mathbf{Z}_r$, then w_{i_2} would belong to the same $\mathcal{L}$-orbit as w_1: since $\theta_2 = \psi(b_2)$ for some $b_2 \in B$, then

$$w_{i_2} = (l_{i_2}^2, \theta_2) \cdot w_1 = (l_{i_2}^2 b_2^{-1}, 0)(b_2, \theta_2) \cdot w_1 = l_{i_2}^2 b_2^{-1} \cdot w_1.$$

Moreover, if $l = kr$ for some positive integer k, then up to conjugacy we can always choose

$$\theta_i \in \left\{ \frac{2\pi}{l}, \cdots, (k-1)\frac{2\pi}{l} \right\}.$$

Associated with each choice of the θ_i we have a permutation $\sigma_i \in \Pi_{\mathcal{G}}(\Sigma)$ and an element $\theta_i = \theta(l^i, \sigma_i)$ for some $(l^i, \sigma_i) \in H$. In fact $\sigma_i^{-1}(J_i) = J_1$ for $i = 2, \ldots, q$, so all the $J_i s$ have the same size t where $tq = s$.

For simplicity, take $J_1 = \{1, \ldots, t\}, \ldots, J_q = \{(q-1)t+1, \ldots, s\}$. Conjugate w to have the form

$$w = (< w_1 >, < \theta_2 \cdot w_1 >, \ldots, < \theta_q \cdot w_1 >, 0, \ldots, 0).$$

We know from lemma 3.5 that

$$((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \subseteq \Sigma.$$

Since $\theta_2, \ldots, \theta_q \in \mathbf{Z}_l$, the subgroup $< \theta_2, \ldots, \theta_q >$ generated by these elements is a cyclic group, say $\mathbf{Z}_p$, and there is a generator $\xi_p \in \mathbf{Z}_l \setminus \mathbf{Z}_r$. For some (l, σ) we have $(l, \sigma, \xi_p) \in \Sigma$, so that $(l, \sigma, \xi_p)^i \in \Sigma$ for all i.

Let

$$s^{'} = \min_{i>0} \{\xi_p^i \in \mathbf{Z}_r\}.$$

Consider J_1. Then exists J_2, such that $J_1 \cap J_2 = \emptyset$ and $\sigma(J_1) = J_2$. The reason is that $(l, \sigma, \xi_p) \in \Sigma$. Also $w_i = (l_i, \xi_p) \cdot w_1$ for all $i \in J_2$. Since $(l, \sigma, \xi_p)^2 = (l\sigma(l), \sigma^2, \xi_p^2) \in \Sigma$, there exists J_3 such that $\sigma^2(J_1) = J_3$ and $J_3 \cap (J_1 \cup J_2) = \emptyset$. Again

$$w_i = (l_i \sigma(l_i), \xi_p^2) \cdot w_1$$

for $i \in J_3$. We can do the same for each $i \leq s^{'}$, so we eventually have

$$J_1 \cup \cdots \cup J_{s^{'}} \subseteq J.$$

Suppose that there is another sub-block $J_{s^{'}+1}$. Then for $i \in J_{s^{'}+1}$ we have w_i in the $\mathcal{L}$-orbit of one of the sub-blocks J_i for $i \leq s^{'}$, which is not the case. For simplicity, we take $J_2, \ldots, J_{s^{'}}$ as before. Therefore we can take

$$w = (< w_1 >, < \xi_p \cdot w_1 >, \ldots, < \xi_p^{s^{'}-1} \cdot w_1 >, 0, \ldots, 0).$$

Now define

$$Q_{J,J_k} = \{\tau \in \Pi_{\mathcal{G}}(\Sigma) : \tau(J_j) = \sigma^{k-1}(J_j),\ j = 1, \ldots, s^{'}\}$$

for $k = 1, \ldots, s^{'}$ (where $\sigma^0 = 1$) and take B_k as defined before proposition 3.2. Then $((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1}) \subseteq \Sigma$ and $((\hat{B}, \mathbf{1}^{N-s}), Q_{J,J_1}, \psi) \subseteq \Sigma$.

Let $(l, \tau, \theta) \in \Sigma$. Since $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) \subseteq \Sigma$, then

$$\gamma = ((l_1, \ldots, l_s, 1^{N-s}), \tau, \theta) \in \Sigma.$$

If $\theta \in \mathbf{Z}_r$, then from $\gamma \cdot w = w$ we must have $\tau \in Q_{J,J_1}$ and so $\theta = \psi(l_1) = \cdots = \psi(l_s)$. Therefore $\gamma \in ((\hat{B}, \mathbf{1}^{N-s}), Q_{J,J_1}, \psi) \subseteq \Sigma$.

If $\theta \notin \mathbf{Z}_r$, then from $\gamma \cdot w = w$, we have $\theta = \xi_p^i \xi_r^j$ for some $i < s^{'}$ and some $j \in \mathbf{Z}_0^+$. As $((b, 1, \ldots, 1), 1, \xi_r^{-j}) \in ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$ for some $b \in \hat{B}$, it follows that $\gamma^{'} = ((b_1 l_1, \ldots, b_s l_s, 1, \ldots, 1), \tau, \xi_p^i)$ fixes w. Now we use the end of the proof of proposition 3.2 and conclude that $\tau \in Q_{J,J_{i+1}}$ and $\gamma \in ((B_{i+1}, \mathbf{1}^{N-s}), Q_{J,J_{i+1}}, \xi_p^i) + ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$. $\square$

Corollary 3.7 *Let $H^\theta = \Sigma \subset \Gamma \times \mathbf{S}^1$ be a $\mathbf{C}$-axial subgroup of twist type $\mathbf{S}^1$. If $\Sigma = \Sigma_w$ with $w_1 \neq 0$, let Σ_{w_1} be the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$. Then Σ_{w_1} is $\mathbf{C}$-axial of twist type $\mathbf{S}^1$.*

Proof. From lemma 3.5 the group Σ_{w_1} is $\mathbf{C}$-axial. Suppose Σ_{w_1} is of finite twist type, say $\mathbf{Z}_r$. As in the proof of proposition 3.6 we can conjugate w to

$$w = (< w_1 >, < \theta_2 \cdot w_1 >, \ldots, < \theta_q \cdot w_1 >, 0, \ldots, 0)$$

for some $\theta_2, \ldots, \theta_q \in \theta(H) \setminus \mathbf{Z}_r$. Moreover $< \theta_2, \ldots, \theta_q >$ is a finite subgroup of $\theta(H) = \mathbf{S}^1$. Let $(l, \tau, \theta) \in \Sigma_w$. If $\theta \in \theta(H) \setminus \mathbf{Z}_r$, then from $(l, \tau, \theta) \cdot w = w$ it follows that $\theta\theta_i\theta_j^{-1} \in \mathbf{Z}_r$ for some i and j in $\{2, \ldots, q\}$ and so $\theta = \xi\xi^{'}$ for some $\xi \in < \theta_2, \ldots, \theta_q >$ and $\xi^{'} \in \mathbf{Z}_r$. But $\theta(H) = \mathbf{S}^1$. So Σ_{w_1} has to be of twist type $\mathbf{S}^1$. $\square$

Corollary 3.8 *Let $\Sigma_w \subset \Gamma \times \mathbf{S}^1$ where $w_1 \neq 0$, and let Σ_{w_1}, the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$, be $\mathbf{C}$-axial. Then Σ is of twist type $\mathbf{S}^1$ if and only if Σ_{w_1} is of twist type $\mathbf{S}^1$.*

Proof. This follows from the last remark and from the corollary 3.7. $\square$

Corollary 3.9 *Let $\Sigma_w \subset \Gamma \times \mathbf{S}^1$ be $\mathbf{C}$-axial and let $B^\psi = \Sigma_{w_1}$ where $w_1 \neq 0$. Then Σ_w is conjugate to $\Sigma(B^\psi, J)$ for some block J if and only if Σ_w and Σ_{w_1} are of the same twist type.*

Proof. Let $\Sigma_w = H^\theta$ and Σ_{w_1} have the same twist type. By lemma 3.4 the subgroup $\Pi_{\mathcal{G}}(\Sigma)$ of $\mathcal{G}$ acts transitively on some block J. Suppose $J = \{1, \ldots, s\}$. Up to conjugacy we need only consider the isotropy subgroups of elements of the form

$$w = (< w_1 >, 0, \ldots, 0),$$

where the vector $< w_1 >$ has s components equal to w_1, since $\theta(H) = \psi(B)$. Note that any element $(l_i, \theta_i) \cdot w_1 \in V$ (with $\theta_i \in \theta(H)$ and $w_1 \in \mathrm{Fix}_V(\mathrm{B}^\psi)$) is in the $\mathcal{L}$-orbit of w_1. Therefore Σ is conjugate to $\Sigma(B^\psi, J)$. Now, if $H^\theta = \Sigma$ is conjugate to $\Sigma(B^\psi, J)$ for some block J, then $\theta(H) = \psi(B)$ from the structure of $\Sigma(B^\psi, J)$. $\square$

4 Maximal isotropy subgroups

We describe now all the maximal isotropy subgroups of the group $\Gamma \times \mathbf{S}^1$ where, as usual, $\Gamma = \mathcal{L} \wr \mathcal{G}$. Generalizing the classification for the $\mathbf{C}$-axial subgroups, we prove that these groups are conjugate to subgroups of the form $\Sigma(B^\psi, J, \sigma, J_1, p)$ where now B^ψ is any maximal isotropy subgroup of $\mathcal{L} \times \mathbf{S}^1$. As before denote by $\Pi_{\mathcal{G}}(\Sigma)$ the projection of Σ on the permutation group $\mathcal{G}$ and for a block J let

$$V_J = \{(w_1, \ldots, w_N) \in V^N : w_j = 0 \text{ for } j \notin J\}.$$

Lemma 4.1 *Let Σ be a maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Then $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on some block J and $\mathrm{Fix}_{V^N}(\Sigma) \subseteq V_J$.*

Proof. Let w be a nonzero vector of $\mathrm{Fix}_{V^N}(\Sigma)$ and let J be the set of indices $j \in \{1, \ldots, N\}$ such that $w_j \neq 0$. We show that $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Since $(l_j, \theta).w_{\sigma^{-1}(j)} = 0$ if and only if $w_{\sigma^{-1}(j)} = 0$, we have $\Pi_{\mathcal{G}}(\Sigma)J \subseteq J$. Suppose that there are two disjoint subsets J_1 and J_2 of J such that $\Pi_{\mathcal{G}}(\Sigma)J_i \subseteq J_i$ for $i = 1, 2$. Let

$$V_1 = \{y_1 \in \mathrm{Fix}_{V^N}(\Sigma) : y_1^j = 0 \text{ if } j \notin J_1\}.$$

Therefore

$$\Sigma_w \subset \Sigma_{y_1} \subset \Gamma \times \mathbf{S}^1$$

and $\Sigma_w = \Sigma$ is not maximal, a contradiction. Thus $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J and J is a block. $\square$

Lemma 4.2 *Let $\Sigma_w = H^\theta$ be a twisted maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Suppose that $w_1 \neq 0$. Let $\Sigma_{w_1} = B^\psi$ be the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$. Then*
(a) $\dim \mathrm{Fix}_V(\Sigma_{w_1}) \leq \dim \mathrm{Fix}_{V^N}(\Sigma_w)$;
(b) Σ_{w_1} is maximal in $\mathcal{L} \times \mathbf{S}^1$ and $\psi(B) \subseteq \theta(H)$.

Proof. The vector w is nonzero and is fixed by Σ_w. Let J be the set of indices $j \in \{1, \ldots, N\}$ such that $w_j \neq 0$. By lemma 4.1 the group $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on some block J. Therefore we can conjugate Σ_w to the isotropy subgroup of the vector

$$w = (w_1, \theta_2 \cdot w_1, \ldots, \theta_s \cdot w_1, 0, \ldots, 0)$$

if we take $J = \{1, \ldots, s\}$.

It follows that if there are p linearly independent vectors in $\mathrm{Fix}_V(\Sigma_{w_1})$, then we can construct p vectors in V^N that are linearly independent and are in $\mathrm{Fix}_{V^N}(\Sigma_w)$, so we have proved (a).

Now define

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Then

$$((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \subset \Sigma$$

and so $\psi(B) \subseteq \theta(H)$.

Suppose that Σ_{w_1} is not maximal in $\mathcal{L} \times \mathbf{S}^1$. This means that there exists an isotropy subgroup $\Sigma_{v_1} = C^\phi$ of $\mathcal{L} \times \mathbf{S}^1$ such that

$$\Sigma_{w_1} \subset \Sigma_{v_1} \subset \mathcal{L} \times \mathbf{S}^1.$$

Let

$$v = (v_1, \theta_2 \cdot v_1, \ldots, \theta_s \cdot v_1, 0, \ldots, 0)$$

and consider the isotropy subgroup Σ_v of v in $\Gamma \times \mathbf{S}^1$. Then

$$((\hat{C}, \mathbf{1}^{N-s}), \mathbf{1}, \phi) \subset \Sigma_v$$

and

$$((\hat{C}, \mathbf{1}^{N-s}), \mathbf{1}, \phi) \not\subset \Sigma_w.$$

Moreover,

$$\Sigma_w \subset \Sigma_v \subset \Gamma \times \mathbf{S}^1$$

and so Σ_w is not maximal, a contradiction. Therefore Σ_{w_1} is maximal in $\mathcal{L} \times \mathbf{S}^1$, as required. $\square$

Corollary 4.3 *Let Σ_w be a maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Suppose that $w_1 \neq 0$ and let $\Sigma_{w_1} = B^\psi$ be the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$. Then Σ_w is conjugate to $\Sigma(B^\psi, J, \sigma, J_1, p)$ for some block $J \subseteq \{1, \ldots, N\}$, a permutation σ in $\mathcal{G}$, a subset J_1 of J, a positive integer p and B^ψ is maximal in $\mathcal{L} \times \mathbf{S}^1$.*

Proof. The group B^ψ is maximal by lemma 4.2. The rest follows as in the proof of proposition 3.6 (using lemmas 4.1 and 4.2) if Σ_w is of finite twist type, or as in corollaries 3.7 and 3.9 if Σ_w is of twist type $\mathbf{S}^1$. $\square$

Remarks.

(a) By proposition 3.2 with B^ψ maximal, it follows that every group $\Sigma = \Sigma(B^\psi, J, \sigma, J_1, p)$ is a maximal isotropy subgroup and

$$\dim \mathrm{Fix}_{V^N}(\Sigma) = \dim \mathrm{Fix}_V(B^\psi).$$

(b) From (a), assuming the conditions of corollary 4.3 we have

$$\dim \mathrm{Fix}_{V^N}(\Sigma_w) = \dim \mathrm{Fix}_V(\Sigma_{w_1}).$$

In particular,

$$\dim \mathrm{Fix}_{V^N}(\Sigma_w) \leq \dim V.$$

5 Isotropy subgroups

We show now that we can describe a general isotropy subgroup of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ using groups with a structure similar to that obtained for the maximal isotropy subgroups.

Once again consider V^N, a $\mathcal{L} \wr \mathcal{G}$-simple space. Let $\Gamma = \mathcal{L} \wr \mathcal{G}$ and take a partition of $\{1, \ldots, N\}$ in p blocks

$$\{1, \ldots, N\} = J_1 \cup \cdots \cup J_p.$$

Let $\Sigma_1, \ldots, \Sigma_p$ be isotropy subgroups in $\Gamma \times \mathbf{S}^1$ of the type

$$\Sigma(B_i^{\psi_i}, J_i, \sigma_i, J_1^i, p_i) \tag{2}$$

for $B_i^{\psi_i}$ an isotropy subgroup in $\mathcal{L} \times \mathbf{S}^1$, a part J_1^i of J_i, a permutation σ_i in $\mathcal{G}$ and a nonnegative integer p_i. Note that, if for example $J_1 = \{1, \ldots, q\}$, then a vector w fixed by Σ_1 is of the form

$$w = (w_{J_1}, 0, \ldots, 0), \tag{3}$$

where if we assume $J_1^1 = \{1, \ldots, t\}$, $\sigma_1(J_1^1) = \{t+1, \ldots, 2t\}, \ldots$, then

$$\begin{aligned}
&w_{J_1} = (< w_1 >, < \xi_{p_1} \cdot w_1 >, \ldots, < \xi_{p_1}^{s_1^{'}-1} \cdot w_1 >),\\
&s_1^{'} = \min_{i>0} \{\xi_{p_1}^i \in \psi_1(B_1)\},\\
&w_1 \in \mathrm{Fix}_V(B_1^{\psi_1})
\end{aligned}$$

and so $q = s_1^{'} t$.

Let

$$\Sigma = \cap_{i=1}^p \Sigma_i.$$

Proposition 5.1 *Σ is an isotropy subgroup of $\Gamma \times \mathbf{S}^1$ acting on V^N and every isotropy subgroup of $\Gamma \times \mathbf{S}^1$ is conjugate to such a Σ.*

Proof. Let W_i be a (nonzero) vector fixed by Σ_i. This means that

$$W_i = (0, \ldots, 0, w_{J_i}, 0, \ldots, 0)$$

where w_{J_i} denotes the components of W_i corresponding to the block J_i and we are assuming for simplicity that the blocks J_i have consecutive indices. Suppose that the first components of each w_{J_i} are in distinct $\mathcal{L} \times \mathbf{S}^1$ orbits.

Let

$$w = W_1 + \cdots + W_p.$$

By construction Σ fixes w, i.e., $\Sigma \subseteq \Sigma_w$.

Let now $(l, \sigma, \theta) \in \Sigma_w$. As we are assuming that the first components of each w_{J_i} are in distinct $\mathcal{L} \times \mathbf{S}^1$ orbits, we must have

$$\Pi_{\mathcal{G}}(\Sigma_w) J_i = J_i$$

and so, from $(l,\sigma,\theta)\cdot w = w$ we get $(l,\sigma,\theta)\cdot W_i = W_i$, for $i = 1,\dots,p$. Thus $\Sigma_w \subseteq \Sigma$.

Let Σ_w be any isotropy subgroup of $\Gamma\times\mathbf{S}^1$. Consider the projection $\Pi_{\mathcal{G}}(\Sigma_w) = G$. Since G is a subgroup of the permutation group $\mathbf{S}_N$, there is a partition $J_1\cup\cdots\cup J_p$ of the set $\{1,\dots,N\}$ such that each J_i is a G-orbit. That is, each J_i is a block and the components corresponding to each J_i are all null or all nonzero. Consider Σ_i the isotropy subgroup in $\Gamma\times\mathbf{S}^1$ of the vector

$$W_i = (0,\dots,0,w_{J_i},0,\dots,0).$$

Again we are assuming J_i formed by consecutives indices. Since G acts transitively on J_i, then $\forall i_1, i_2 \in J_i$ we can find some $(l,\sigma,\theta)\in\Sigma_w$ such that

$$w_{i_1} = (l_i,\theta)\cdot w_{i_2}.$$

Thus, if for some i_1 we have $w_{i_1} = 0$, then all the components with indices in J_i are null. If it is not the case, then up to conjugacy, we can suppose that each w_{J_i} has a form like in (3) (see proposition 3.6). Thus each Σ_i is of the type (2) and $GJ_i = J_i$.

It is now straightforward to prove that Σ_w is the intersection of the isotropy subgroups Σ_i with $i = 1,\dots,p$. □

6 Examples

We now illustrate our results on the groups $\mathbf{O}(2)\wr\mathbf{S}_2$ and $\mathbf{O}(2)\wr\mathbf{S}_3$.

6.1 Group action of $(\mathbf{O}(2)\wr\mathbf{S}_2)\times\mathbf{S}^1$

Let $\Gamma = \mathbf{O}(2)\wr\mathbf{S}_2$ and let $V = \mathbf{C}\oplus\mathbf{C}$. Consider the following action of $\mathbf{O}(2)\times\mathbf{S}^1$ on V:

$$\begin{array}{ll} \theta(z_1,z_2) = (e^{i\theta}z_1, e^{i\theta}z_2) & (\theta\in\mathbf{S}^1) \\ \kappa(z_1,z_2) = (z_2,z_1), & (\kappa = \text{flip in } \mathbf{O}(2)) \\ \psi(z_1,z_2) = (e^{-i\psi}z_1, e^{i\psi}z_2) & (\psi\in\mathbf{SO}(2)) \end{array}$$

Here $\mathbf{S}_2$ is the group of the identity and the transposition (12) that interchanges the indices 1 and 2. Also the group multiplication in $\Gamma\times\mathbf{S}^1$ is given by

$$(h,\tau,\theta_1)(l,\sigma,\theta_2) = (h\tau(l),\tau\sigma,\theta_1\theta_2)$$

and the action of $\Gamma\times\mathbf{S}^1$ on V^2 is given by:

$$((l_1,l_2),\sigma,\theta)w = ((l_1,\theta)w_{\sigma^{-1}(1)},(l_2,\theta)w_{\sigma^{-1}(2)}),$$

for $(l_1,l_2)\in\mathbf{O}(2)^2$, $\sigma\in\mathbf{S}_2$ and $\theta\in\mathbf{S}^1$ (with $w = (w_1,\ w_2)\in V^2$).

Note that as V is $\mathbf{O}(2)$-simple, also V^2 is Γ-simple by the results stated in section 2.

6.2 C-axial groups for $(\mathbf{O}(2) \wr \mathbf{S}_2) \times \mathbf{S}^1$

Using the method of section 3, we first have to calculate the $\mathbf{C}$-axial subgroups of $\mathbf{O}(2) \times \mathbf{S}^1$. By [14] for example (proposition XVII 1.1.) we have (up to conjugacy) two types of $\mathbf{C}$-axial subgroups. See table A.

Table A.

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(a,0),\ a>0$	$\widetilde{\mathbf{SO}}(2) = \{(\theta,\theta)\}$	$\{(z_1,0)\}$
$(a,a),\ a>0$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c = \{(1,0),(\kappa,0),(\pi,\pi),(\kappa\pi,\pi)\}$	$\{(z_1,z_1)\}$

Usually periodic solutions with symmetry $\widetilde{\mathbf{SO}}(2)$ are called *rotating waves* and those with symmetry $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ are *standing waves.*

We can now compute the isotropy subgroups for $\Gamma \times \mathbf{S}^1$:

Proposition 6.1 *There are five conjugacy classes of* $\mathbf{C}$*-axial subgroups for* $\Gamma \times \mathbf{S}^1$ *with the above action on* V^2*. They are listed, together with their orbit representatives and fixed-point subspaces, in table B.*

In table B we use the notation of the sections 2 and 3. Also a denotes a real positive number.

Table B.

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(a,0,0,0)$	$\Sigma_1 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1\})$	$\{(z_1,0,0,0)\}$
$(a,a,0,0)$	$\Sigma_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1\})$	$\{(z_1,z_1,0,0)\}$
$(a,0,a,0)$	$\Sigma_3 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1,2\})$	$\{(z_1,0,z_1,0)\}$
(a,a,a,a)	$\Sigma_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\})$	$\{(z_1,z_1,z_1,z_1)\}$
(a,a,ia,ia)	$\Sigma_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\},(12),\{1\},4)$	$\{(z_1,z_1,iz_1,iz_1)\}$

Proof of proposition 6.1. Up to conjugacy we need consider only the blocks

$$J = \{1\}, \quad J = \{1,2\}$$

and by lemmas 3.4 and 3.5 of section 3 we need to look for **C**-axial subgroups Σ_w with

$$\begin{array}{l} w = (w_1, 0) \\ w = (w_1, \theta_2 w_1) \end{array} ,$$

where $\theta_2 \in \mathbf{S}^1$ and $w_1 \in V$ is such that

$$\dim \mathrm{Fix}_V \ (\Sigma_{w_1}) = 2.$$

Here Σ_{w_1} is the isotropy subgroup of w_1 in $O(2) \times \mathbf{S}^1$.

For Σ_{w_1}, up to conjugacy, we have two choices: $\widetilde{\mathbf{SO}}(2)$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$. As the first one is of twist type $\mathbf{S}^1$, if $w_1 \in \mathrm{Fix}_V(\widetilde{\mathbf{SO}}(2))$ we can assume (up to conjugacy) that w has equal nonzero components and we get Σ_1 and Σ_3: for example, as $(\theta_2, \theta_2) \in \widetilde{\mathbf{SO}}(2)$, then $w = (w_1, \theta_2 w_1) = (w_1, (\theta_2^{-1}\theta_2, \theta_2) w_1) = ((1, \theta_2^{-1}), 1, 0)(w_1, w_1)$ and so the isotropy subgroup of $(w_1, \theta_2 w_1)$ is conjugate to the isotropy subgroup of $w = (w_1, w_1)$. See also corollaries 3.8 and 3.9.

Let now $w_1 \in \mathrm{Fix}_V \ (\mathbf{Z}_2 \oplus \mathbf{Z}_2^c)$. Note that $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ is of twist type $\mathbf{Z}_2$. It follows that $\Sigma_{(w_1,0)}$ is Σ_2 and if $\theta_2 \in \mathbf{Z}_2$, then $\Sigma_{(w_1, \theta_2 w_1)}$ is conjugate to Σ_4 by the same reason as before. Now using proposition 3.6 we end up just with Σ_5. Note that, once we fixe $s^{'} = 2$, the possibilities for p are 2 and 4. But for $p = 2$ we must have $\theta_2 \in \mathbf{Z}_2$. $\square$

Remark. This action of $\Gamma \times \mathbf{S}^1$ on V^2 is isomorphic to the action of $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$ presented in [18]. Moreover, we obtain **C**-axial subgroups $\Sigma_1, \ldots, \Sigma_5$ that are in precise correspondence with the **C**-axial subgroups obtained in [18].

6.3 C-axial groups for $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$

A more detailed discussion of dynamics with this symmetry group can be found in [5]. Here we limit our discussion to classifying the **C**-axial subgroups.

Consider $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_3$ and $V = \mathbf{C} \oplus \mathbf{C}$. We use the same action of $\mathbf{O}(2) \times \mathbf{S}^1$ on V as in section 6.1 and $\mathbf{S}_3$ represents the group of permutations of the set $\{1, 2, 3\}$. The group multiplication in $\Gamma \times \mathbf{S}^1$ and the action of $\Gamma \times \mathbf{S}^1$ on V^3 are the natural extensions of the ones used in section 6.1. We have:

Proposition 6.2 *There are eight conjugacy classes of* **C***-axial subgroups for the group* $\Gamma \times \mathbf{S}^1$ *with the above action on* V^3*. They are listed, together with their orbit representatives and fixed-point subspaces, in table C.*

In table C we denote $\xi = \frac{2\pi}{3} \in \mathbf{S}^1$.

Table C.

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(a,0,0,0,0,0)$	$\Sigma_1 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1\})$	$\{(z_1,0,0,0,0,0)\}$
$(a,a,0,0,0,0)$	$\Sigma_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1\})$	$\{(z_1,z_1,0,0,0,0)\}$
$(a,0,a,0,0,0)$	$\Sigma_3 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1,2\})$	$\{(z_1,0,z_1,0,0,0)\}$
$(a,a,a,a,0,0)$	$\Sigma_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\})$	$\{(z_1,z_1,z_1,z_1,0,0)\}$
$(a,a,ia,ia,0,0)$	$\Sigma_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\},(12),\{1\},4)$	$\{(z_1,z_1,iz_1,iz_1,0,0)\}$
$(a,0,a,0,a,0)$	$\Sigma_6 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1,2,3\})$	$\{(z_1,0,z_1,0,z_1,0)\}$
(a,a,a,a,a,a)	$\Sigma_7 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2,3\})$	$\{(z_1,z_1,z_1,z_1,z_1,z_1)\}$
$(a,a,\xi a,\xi a,\xi^2 a,\xi^2 a)$	$\Sigma_8 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2,3\},(123),\{1\},3)$	$\{(z_1,z_1,\xi z_1,\xi z_1,\xi^2 z_1,\xi^2 z_1)\}$

Proof of proposition 6.2. We can follow the proof of proposition 6.1 where now we consider $w = (w_1, 0, 0)$ or $w = (w_1, \theta_2 w_1, 0)$ and we obtain the groups $\Sigma_1, \ldots, \Sigma_5$. In addition, we have to consider the block

$$J = \{1, 2, 3\},$$

and again, by lemmas 3.4 and 3.5 we need to look for $\mathbf{C}$-axial groups Σ_w with

$$w = (w_1, \theta_2 w_1, \theta_3 w_1),$$

where $\theta_2, \theta_3 \in \mathbf{S}^1$ and w_1 is a vector of V fixed by a $\mathbf{C}$-axial group of $\mathbf{O}(2) \times \mathbf{S}^1$. If w_1 is fixed by $\widetilde{\mathbf{SO}}(2)$, then we obtain a group of the type Σ_6. Suppose now that w_1 is fixed by $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$. If θ_2 and θ_3 are in $\mathbf{Z}_2$, then we can conjugate Σ_w to Σ_7. If some θ_2 or θ_3 is not in $\mathbf{Z}_2$, then the only possibility for $w = (w_1, \theta_2 w_1, \theta_3 w_1)$ to be fixed by a $\mathbf{C}$-axial subgroup (up to conjugacy) is if $\theta_2 = \xi$ and $\theta_3 = \xi^2$ and we have Σ_8 (see proposition 3.6 and note that now with $s' = 3$ we only need to consider $p = 3$ or $p = 6$ since the corresponding isotropy subgroups are conjugate). $\square$

Remark. Considering now $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_n$ acting on V^n, we see that $\mathbf{C}$-axial groups of $(\mathbf{O}(2) \wr \mathbf{S}_{n-1}) \times \mathbf{S}^1$ are, with appropriate adjustment, included in those of $\Gamma \times \mathbf{S}^1$.

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