Linear Equivalence and ODE-equivalence for Coupled Cell Networks

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26 October 2004

Abstract

Coupled cell systems are systems of ODEs, defined by 'admissible' vector fields, associated with a network whose nodes represent variables and whose edges specify couplings between nodes. It is known that non-isomorphic networks can correspond to the same space of admissible vector fields. Such networks are said to be 'ODE-equivalent'. We prove that two networks are ODE-equivalent if and only if they determine the same space of *linear* vector fields; moreover, the variable associated with each node may be assumed 1-dimensional for that purpose. We briefly discuss the combinatorics of the resulting linear algebra problem.

AMS classification scheme numbers: 37C10 20L05

1 Introduction

Networks of nonlinear dynamical systems have become the topic of considerable attention recently, mainly because a wide variety of physical and biological systems can naturally be modelled by such networks, see Wang [17], Stewart [15]. In particular, there is considerable interest in networks of neurons (biological ones, not 'neural nets'), genetic regulatory networks, cellular metabolic networks, and food webs in ecosystems.

The theoretical understanding of such systems is also under intensive development. Of course, every (finite) network of dynamical systems can be considered as a single dynamical

§CMUP is supported by Fundação para a Ciência e a Tecnologia through Programa Operacional Ciência, Tecnologia e Inovação (POCTI) and Programa Operacional Sociedade da Informação (POSI) of Quadro Comunitário de Apoio III (2000-2006) with European union fundings (FEDER) and national fundings.

system, and every dynamical system is trivially a network with only one node and no edges, so it might seem that networks offer no gain in generality. However, networks possess additional structure, namely, canonical observables—the dynamical behaviour of the individual nodes [10]. These observables can be compared, revealing such features as synchrony or phase-relations, and it is precisely these features that are important in many applications. Any theoretical treatment of network dynamics must therefore take this additional structure into account, so conventional dynamical systems theory must be modified to preserve that structure. The topology (or 'architecture') of the network imposes constraints on the dynamics, with the result that many new phenomena become 'generic' for a given architecture. These include multiple eigenvalues unrelated to symmetries, nilpotent linearizations, and the coexistence of large numbers of equilibria which bifurcate together but typically have different stabilities, Golubitsky *et al.* [8].

A network (or graph) is a schematic representation of a set of dynamical systems (that is, ordinary differential equations or ODEs) that are coupled together. The nodes of the graph ('cells' of the network) represent the individual dynamical systems, and the directed edges ('arrows') represent couplings. Associated with each network G is a class of differential equations on P, which correspond to 'admissible' vector fields on P. These are the ODEs that are compatible with the network topology and the choice of phase spaces for the cells (that is, the variables of the system).

The growing emphasis on network dynamics in applied science, especially biology, makes it important to understand how network topology contrains the associated dynamics. The combinatorial complexity is huge: for instance, there are precisely 13505116827457 connected networks with six identical nodes and six identical arrows entering each node, Aldosray and Stewart [2]. However, some general principles are emerging, in particular the notion of a balanced equivalence relation and its quotient network, Stewart *et al.* [16], Golubitsky *et al.* [12]. This concept classifies 'robust' patterns of synchrony, 'rigid' patterns of synchronous equilibria, and (conjecturally but plausibly) 'rigid' patterns of phase-locking in periodic states. Moreover, progress has been made in understanding steady and Hopf bifurcation from synchronous states to asynchronous ones, Golubitsky *et al.* [12, 9]. An alternative 'synthetic' approach to robust patterns of synchrony, exploiting combinatorial properties of the network, can be found in Field [7].

It has also become clear that topologically distinct networks have related dynamics. In particular it was observed in Golubitsky *et al.* [12] that topologically distinct coupled cell networks can give rise to the same space of admissible vector fields (for a suitable choice of cell phase spaces), a phenomenon known as 'ODE-equivalence'. This phenomenon raises the prospect of simplifying the network structure while preserving some (in this case, all) of its dynamical features, and it is important to understand how far this technique can be developed. In this paper we make a start on this question by determining necessary and sufficient conditions for two networks to be ODE-equivalent.

We prove two main theorems, each of which represents a substantial simplification of the problem. The first (Theorem 7.1 below) reduces the problem of ODE-equivalence to 'linear equivalence', where two networks (with suitably identified phase spaces) are linearly equivalent if they determine the same space of *linear* admissible vector fields. The second (a simple but useful corollary) is that when deciding linear equivalence, it can without loss of generality be assumed that each cell phase space is 1-dimensional (Corollary 7.9).

These theorems classify ODE-equivalence in terms of a linear algebra invariant. However, we also show that the combinatorial issues here are very complicated. More precisely, the characterization of linearly equivalent networks reduces to a combinatorial condition in linear algebra. In a sense, this condition completely solves the problem of linear equivalence, hence of ODE-equivalence. However, the relation between network topology and the linear algebra condition is deceptively simple; in particular, there seems to be no straightforward combinatorial condition on the two networks that determines linear equivalence, other than a suitably 'encoded' form of the linear algebra condition. This topic will be the subject of future work by Aguiar and Dias [1].

In order to make the discussion precise, we must specify the relation between the network and the associated class of dynamical systems. Here we work in the context of 'coupled cell systems', which provides a convenient formal framework. In this formulation, introduced by Stewart *et al.* [16] and extended into a technically more convenient form by Golubitsky *et al.* [12], both arrows and cells are labelled to indicate various 'types' of dynamical behaviour. To each cell c is associated a choice of 'cell phase space' P_c, which we will assume is a finite-dimensional vector space $\mathbf{R}^k$ over $\mathbf{R}$, where k may depend on c. (More generally, it could be a finite-dimensional smooth manifold, but we do not consider this generalization here.) The overall phase space P of the coupled cell system is the direct product of the spaces P_c.

Associated with each network G is a class of differential equations on P, which correspond to 'admissible' vector fields on P. These are the ODEs that are compatible with the network topology and the choice of cell phase spaces. The admissible vector fields can be characterised in terms of an algebraic structure known as the 'symmetry groupoid' of the network. A groupoid is similar to a group, except that product of two elements may not always be defined. The symmetry groupoid $\mathcal{B}_G$ consists of all 'input isomorphisms' between pairs of cells c, d—that is, type-preserving bijections between the set of arrows entering cell c and the corresponding set for cell d. The admissible vector fields then turn out to be precisely those that are equivariant under a natural action of the groupoid $\mathcal{B}_G$ on P, in a sense that generalizes the usual notion of equivariance under the action of a group [10, 11].

The groupoid formalism may seem a little strange at first, but it helps to organise the theory and relate it to the existing group-theoretic viewpoint in a natural and useful way. A more concrete combinatorial approach could perhaps be used, just as a group could be replaced by a set of suitable permutations. We do not wish to debate the relative virtues of these philosophies here, except to note that several key theorems (such as the characterisation of rigid patterns of equilibria in terms of balanced equivalence relations in [12]) have been motivated and proved within the groupoid framework, whereas it is unclear how to derive them using other methods.

Sections 2, 3, 4 of the paper provide formal definitions for, and basic properties of, coupled cell networks, the associated symmetry groupoid, and admissible vector fields. Section 5 defines ODE-equivalence. Section 6 discusses linear equivalence, including a typical example that shows how the network topology encodes a linear algebra condition. Section 7 proves the main theorem that ODE-equivalence is the same as linear equivalence, and deduces as a corollary that linear equivalence does not depend on the choice of cell phase spaces (provided their dimensions are at least 1), so that when deciding linear (hence ODE) equivalence, all cells may be assumed to have 1-dimensional phase spaces. The proofs are closely related to the methods developed in Dias and Stewart [5] to solve the lifting problem for admissible vector fields in the single-arrow formalism. Finally Section 8 provides a brief discussion of the combinatorial issues associated with linear equivalence.

2 Coupled Cell Networks

A coupled cell network can be represented schematically by a directed graph (see for example Figures 1, 2, 3 below) whose nodes correspond to cells and whose edges represent couplings. We employ the following definition, introduced by Golubitsky*et al.* [12], which permits multiple arrows and self-coupling. This formulation has several technical advantages over the more restricted version described in [16]. For example, quotient networks naturally lead to multiple arrows, and Neumann boundary conditions in the sense of Epstein and Golubitsky [6] naturally lead to self-coupling ([12] section 1). More significantly, admissible vector fields always lift from quotient networks ([12] Theorem 5.2). This is not the case in the single-arrow formalism of [16], where necessary and sufficient conditions for lifting have been derived by Dias and Stewart [5].

Definition 2.1 [12] In the *multiarrow formalism*, a *coupled cell network* G consists of:

(a) A finite set $\mathcal{C} = \{1, \dots, n\}$ of *nodes* (or *cells*).

(b) An equivalence relation $\sim_C$ on the nodes in $\mathcal{C}$.
The *type* or *cell label* of cell c is the $\sim_C$-equivalence class $[c]_C$ of c.

(c) A finite set $\mathcal{E}$ of *edges* or *arrows*.

(d) An equivalence relation $\sim_E$ on the edges in $\mathcal{E}$.
The *type* or *coupling label* of edge e is the $\sim_E$-equivalence class $[e]_E$ of e.

(e) Two maps $\mathcal{H} : \ \mathcal{E} \to \mathcal{C}$ and $\mathcal{T} : \ \mathcal{E} \to \mathcal{C}$.
For $e \in \mathcal{E}$ we call $\mathcal{H}(e)$ the *head* of e and $\mathcal{T}(e)$ the *tail* of e.

(f) Equivalent edges have equivalent tails and heads. That is, if $e_1, e_2 \in \mathcal{E}$ and $e_1 \sim_E e_2$, then
$$\mathcal{H}(e_1) \sim_C \mathcal{H}(e_2) \qquad \mathcal{T}(e_1) \sim_C \mathcal{T}(e_2)$$

We write $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. ◇

Observe that in this definition of coupled cell network, self-coupling is permitted since $\mathcal{H}(e) = \mathcal{T}(e)$ for $e \in \mathcal{E}$ is permitted. Also multiarrows are permitted since we can have $\mathcal{H}(e_1) = \mathcal{H}(e_2)$ and $\mathcal{T}(e_1) = \mathcal{T}(e_2)$ for $e_1 \neq e_2$.

Indeed, two cells may be coupled by several distinct types of arrows, each with its own multiplicity. There seems to be no good reason to rule out such a possibility, which is a natural consequence of the formal set-up, has greater generality, and reflects kinds of coupling that might arise in some applications.

With hindsight, the formalism of [16] can be viewed as a special case of the multiarrow formalism in which we require all arrows to be single and do not permit self-coupling. The main results of this paper are valid in either formalism, with very similar proofs. For definiteness we work in the multiarrow formalism, because of the aforementioned technical advantages, at the price of minor complications in the definitions.

3 Symmetry Groupoid of a Coupled Cell Network

Given a graph $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ as in Definition 2.1, we can define the 'symmetry groupoid' $\mathcal{B}_G$ of G. This definition centres upon the notion of 'input set'.

Definition 3.1 Following [**12**], let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network.

(a) Given $c \in \mathcal{C}$, then the *input set* of c is

$$I(c) = \{e \in \mathcal{E} : \ \mathcal{H}(e) = c\}$$

An element of $I(c)$ is called an *input edge* or *input arrow* of c.

(b) The relation $\sim_I$ of *input-equivalence* on $\mathcal{C}$ is defined by $c \sim_I d$ if and only if there exists a bijection

$$\beta : \ I(c) \to I(d)$$

that preserves edge type. That is, for every input edge $i \in I(c)$

$$i \sim_E \beta(i)$$

Any such β is called an *input isomorphism* from cell c to cell d. We denote the set of all input isomorphisms from cell c to cell d by $B(c, d)$, and define

$$\mathcal{B}_G = \dot{\bigcup_{c,d \ \in \ \mathcal{C}}} B(c, d)$$

where $\dot{\cup}$ indicates disjoint union. A natural product operation can be defined on $\mathcal{B}_G$ as follows: elements $\beta_2 \in B(c, d)$ and $\beta_1 \in B(a, b)$ can be multiplied only when $b = c$, and in this case $\beta_2\beta_1 \in B(a, d)$ is the usual composition of functions. Now $\mathcal{B}_G$ is a groupoid whose objects are the nodes of G, and the $\mathcal{B}_G$-morphisms are the elements of the sets $B(c, d)$, with the product operation between the morphisms as defined above. Some references on groupoids are Brandt [3], Brown [4] and Higgins [13]. Following [12, 16] we call $\mathcal{B}_G$ the *symmetry groupoid of the network* G. For any $c \in \mathcal{C}$, the subset $B(c, c)$ is a group, the *vertex group* corresponding to c.

◇

Structure of $B(c,d)$

Let $B(c,d) \subseteq \mathcal{B}_G$. We can specify the structure of the set $B(c,d)$ in terms of the structure of G. We distinguish three cases:

1. If $c \not\sim_I d$ then $B(c,d) = \emptyset$.

2. If $c = d$ we can define an equivalence relation $\equiv_c$ on $\mathcal{T}(I(c))$ by

$$\mathcal{T}(j_1) \equiv_c \mathcal{T}(j_2) \iff j_1 \sim_E j_2 \tag{3.1}$$

   where $j_1, j_2 \in I(c)$. If $K_1, K_2, \ldots, K_{r(c)}$ are the $\equiv_c$-equivalence classes (on $\mathcal{T}(I(c))$), then

$$B(c,c) = \mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r(c)}} \tag{3.2}$$

   where each $\mathbf{S}_{K_i}$ comprises all permutations of the set K_i. If we extend by the identity on $\mathcal{T}(I(c)) \setminus K_i$, then there is a natural embedding of $\mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r(c)}}$, and hence of $B(c,c)$, in the group $\mathbf{S}_{n(c)}$, where $n(c) = |\mathcal{T}(I(c))|$.

3. If $c \neq d$ and $c \sim_I d$ (so that $B(c,d) \neq \emptyset$), then for *any* $\beta \in B(c,d)$ we have

$$B(c,d) = \beta B(c,c) = B(d,d)\beta$$

For proofs of the above facts see [16], end of Section 3.

4 Admissible Vector Fields

We make now precise the connection between coupled cell systems and coupled cell networks. Essentially, the network is a schematic diagram (graph), whereas the system is a set of ODEs whose couplings correspond to the edges of the network. To obtain these ODEs we must associate variables x_c with cells c, that is, we must choose a phase space for each cell.

By a *coupled cell system* we mean a network of dynamical systems coupled together, where we use a labelled directed graph G (that is, a coupled cell network in the sense of Definition 2.1), whose nodes correspond to *cells*, and whose edges represent *couplings*. The term 'coupling' here is used in the sense that the output of certain cells affects the time-evolution of other cells.

Again, we follow the treatment of Stewart *et al.* [16] and Golubitsky *et al.* [12]. Consider a coupled cell network $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$, with symmetry groupoid $\mathcal{B}_G$. We wish to define a class $\mathcal{F}_G^P$ of 'admissible' vector fields corresponding to G. This class consists of all vector fields that are 'compatible' with the labelled graph structure, and it depends on a choice of 'total phase space' P.

To each cell $c \in \mathcal{C}$ we associate a *cell phase space* P_c, which for simplicity we assume is a nonzero finite-dimensional real vector space.

If c, d are in the same $\sim_C$-equivalence class, then we insist that $P_c = P_d$, and we identify these spaces canonically. The *total phase space* is

$$P = \prod_{c \in \mathcal{C}} P_c$$

with coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

on P. If $\mathcal{D} = (d_1, \ldots, d_s)$ is any finite ordered subset of s cells in $\mathcal{C}$ we define

$$P_{\mathcal{D}} = P_{d_1} \times \cdots \times P_{d_s}$$

and we write

$$x_{\mathcal{D}} = (x_{d_1}, \ldots, x_{d_s})$$

where $x_{d_i} \in P_{d_i}$. Note that the same cell can appear more than once in $\mathcal{D}$. (This condition must be permitted because of the multiarrow formalism.)

Given $c \in \mathcal{C}$, denote by $\mathcal{T}(I(c))$ the ordered set of cells $(\mathcal{T}(i_1), \ldots, \mathcal{T}(i_s))$ where the arrows i_k run through $I(c)$. Suppose that $c \sim_I d$ and consider the ordered sets $\mathcal{D}_1 = \mathcal{T}(I(c)), \mathcal{D}_2 = \mathcal{T}(I(d))$ of $\mathcal{C}$. Let $\beta \in B(c,d)$. Then β is a bijection between $I(c)$ and $I(d)$. Moreover for all $i \in I(c)$ we have $i \sim_E \beta(i)$, and so $\mathcal{T}(i) \sim_C \mathcal{T}(\beta(i))$. We can define the *pullback map*

$$\beta^* : \; P_{\mathcal{D}_2} \to P_{\mathcal{D}_1}$$

by

$$\big(\beta^*(z)\big)_{\mathcal{T}(j)} = z_{\mathcal{T}(\beta(j))}$$

for all $\mathcal{T}(j) \in \mathcal{D}_1$ and $z \in P_{\mathcal{D}_2}$. If $\mathcal{T}\left(I(c)\right) = (\mathcal{T}(i_1), \ldots, \mathcal{T}(i_s))$ then $x_{\mathcal{T}(I(c))} = \big(x_{\mathcal{T}(i_1)}, \ldots, x_{\mathcal{T}(i_s)}\big)$ and $\beta^*\left(x_{\mathcal{T}(I(d))}\right) = \big(x_{\mathcal{T}(\beta(i_1))}, \ldots, x_{\mathcal{T}(\beta(i_s))}\big)$.

We use pullback maps to relate different components of a vector field associated with a given coupled cell network.

For a given cell c the *internal phase space* is P_c and the *coupling phase space* is

$$P_{\mathcal{T}(I(c))} = P_{\mathcal{T}(i_1)} \times \cdots \times P_{\mathcal{T}(i_s)}$$

where as before $\mathcal{T}(I(c))$ denotes the ordered set of cells $(\mathcal{T}(i_1), \ldots, \mathcal{T}(i_s))$.

Definition 4.1 [12] Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network with symmetry groupoid $\mathcal{B}_G$. For a given choice of the P_c, a (smooth) vector field $f : \; P \to P$ is $\mathcal{B}_G$-*equivariant* or G-*admissible* if:

(a) *Domain Condition*: For any $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variable x_c and the coupling phase space variables $x_{\mathcal{T}(I(c))}$. That is, there exists a (smooth) function $\hat{f}_c : \; P_c \times P_{\mathcal{T}(I(c))} \to P_c$ such that

$$f_c(x) = \hat{f}_c\big(x_c, x_{\mathcal{T}(I(c))}\big)$$

(b) *Pullback Condition*: For all $c, d \in \mathcal{C}$ and $\beta \in B(c,d)$

$$\hat{f}_d\big(x_d, x_{\mathcal{T}(I(d))}\big) = \hat{f}_c\big(x_d, \beta^*\left(x_{\mathcal{T}(I(d))}\right)\big)$$

for all $x \in P$.

◇

Theorem 4.2 *Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network and $\mathcal{B}_G$ the corresponding symmetry groupoid. A vector field $f: P \to P$ for a given choice of the P_c is $\mathcal{B}_G$-equivariant if and only if for each connected component $\mathcal{Q}$ of $\mathcal{B}_G$ (that is, each $\sim_I$-equivalence class)*

(a) $\hat{f}_c$ is $B(c,c)$-invariant for some $c \in \mathcal{Q}$.

(b) For $d \in \mathcal{Q}$ such that $d \neq c$, given (any) $\beta \in B(c,d)$, we have

$$f_d\big(x_d, x_{\mathcal{T}(I(d))}\big) = \hat{f}_c\big(x_d, \beta^*\left(x_{\mathcal{T}(I(d))}\right)\big)$$

Proof This is a generalization of Lemma [16] 4.5 and is proved the same way. □

Now we introduce notation for the space of G-admissible vector fields on P:

Definition 4.3 Let G be a coupled cell network. For a given choice of the P_c, define $\mathcal{F}_G^P$ to consist of all smooth G-admissible vector fields on P. Clearly $\mathcal{F}_G^P$ is a vector space over **R**. Like all function spaces, it can be equipped with a variety of topologies, but here only the vector space structure is relevant. Let $\mathcal{P}_G^P$ be the subspace of $\mathcal{F}_G^P$ consisting of the G-admissible polynomial vector fields on P, and let $\mathcal{L}_G^P$ be the subspace of $\mathcal{P}_G^P$ consisting of the G-admissible linear vector fields on P. ◇

The space of $\mathcal{B}_G$-equivariant maps has a natural decomposition according to the 'connected components' of the groupoid $\mathcal{B}_G$, and this decomposition is inherited by the polynomial and linear vector fields:

Definition 4.4 Let $\mathcal{Q} \subseteq \mathcal{C}$ be an $\sim_I$-equivalence class. Define

$$\begin{aligned}
\mathcal{F}_G^P(\mathcal{Q}) &= \left\{f \in \mathcal{F}_G^P: \; f_s(x) = 0, \; \forall s \notin \mathcal{Q}\right\} \\
\mathcal{P}_G^P(\mathcal{Q}) &= \left\{f \in \mathcal{P}_G^P: \; f_s(x) = 0, \; \forall s \notin \mathcal{Q}\right\} \\
\mathcal{L}_G^P(\mathcal{Q}) &= \left\{f \in \mathcal{L}_G^P: \; f_s(x) = 0, \; \forall s \notin \mathcal{Q}\right\}
\end{aligned}$$

We say that vector fields in $\mathcal{F}_G^P(\mathcal{Q})$, $\mathcal{P}_G^P(\mathcal{Q})$, and $\mathcal{L}_G^P(\mathcal{Q})$ are *supported on* $\mathcal{Q}$. ◇

Remark 4.5 From the above theorem there are direct sum decompositions

$$\mathcal{F}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{F}_G^P(\mathcal{Q}) \qquad \mathcal{P}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{P}_G^P(\mathcal{Q}) \qquad \mathcal{L}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{L}_G^P(\mathcal{Q})$$

where $\mathcal{Q}$ runs over the $\sim_I$-equivalence classes of G. ◇

For detailed proofs see [16], end of Section 4, especially Proposition 4.6.

5 ODE-equivalence

As pointed out by Golubitsky *et al.* [12], in the class of coupled cell networks that permits self-coupling and multiarrows, it is possible for two different coupled cell networks G_1 and G_2 to generate the same space of admissible vector fields. Figure 1 shows a simple example, taken from Golubitsky *et al.* [12]. In G_1 both cells have the same cell type, and similarly for G_2. Suppose that the phase space for all four cells is $\mathbf{R}^k$ and identify these spaces canonically. Then the total phase space for both G_1 and G_2 is $\mathbf{R}^k \times \mathbf{R}^k$.

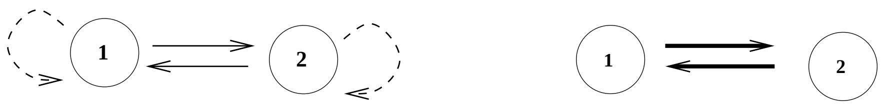

Figure 1: Two coupled cell networks G_1 (on the left) and G_2 (on the right) that generate the same space of admissible vector fields.

The admissible vector fields for G_1 have the form

$$H(x_1, x_2) = (h(x_1, x_1, x_2), h(x_2, x_2, x_1))$$

where $h : \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$ is any smooth function, and the admissible vector fields for G_2 have the form

$$F(x_1, x_2) = (f(x_1, x_2), f(x_2, x_1))$$

where $f : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$ is any smooth function. It is now easy to see that the set $\{H\}$ of all H is the same as the set $\{F\}$ of all F. Namely, given f we can set $h(x, y, z) = f(x, z)$, so that $\{H\} \subseteq \{F\}$. Given h we can set $f(a, b) = h(a, a, b)$ so that $\{F\} \subseteq \{H\}$. Therefore the spaces $\mathcal{F}_{G_1}^{P_1}$ and $\mathcal{F}_{G_2}^{P_2}$ are the same.

For a less trivial example, see Figure 2 of Section 6. Note that the above comparison of admissible vector fields involves identifying cells in the two networks, a step that we formalise in general in terms of a bijection between the two sets of cells.

In the next definition, given a coupled cell network G_i and a choice of total phase space P_i for G_i, we denote by $P_{i,c}$ the cell phase space corresponding to cell c of $\mathcal{C}_i$.

Definition 5.1 Two coupled cell networks G_1 and G_2 are γ-*ODE-equivalent* if:

1. There is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence, such that:

2. If we choose cell phase spaces $P_c \neq 0$ for G_1, and define the corresponding choice of cell phase spaces for G_2 by

$$P_{2,\gamma(c)} = P_{1,c}$$

so that the corresponding total phase spaces are

$$P_1 = \prod_{c \in \mathcal{C}_1} P_{1,c} \qquad P_2 = \prod_{c \in \mathcal{C}_1} P_{1,\gamma(c)}$$

then:

3. The condition
$$\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2} \tag{5.3}$$
is satisfied.

Two coupled cell networks G_1 and G_2 are *ODE-equivalent* if they are γ-ODE-equivalent for some bijection γ. ◇

Remarks 5.2 (a) The cells of G_2 can be renumbered so that $\gamma = \text{id}$. In this case, we omit explicit reference to γ.

(b) It is shown in Section 7 below that if (5.3) holds for some choice of nonzero cell phase spaces P_c, then it holds for all such choices. We postpone proving this fact until we have looked at a typical example, which makes the result obvious.

It follows that ODE-equivalence of two networks depends only on their architecture, and not on the particular choice of cell phase spaces. Note, however, the appearance of the bijection γ that associates cells in the two networks (and must preserve cell-equivalence), and the requirement that $P_{\gamma(c)} = P_c$.

◇

Isomorphic networks (in the usual graph-theoretic sense) are always ODE-equivalent. As pointed out by Golubitsky *et al.* [12], ODE-equivalent networks are not necessarily isomorphic (see for instance Figure 1). The aim of this paper is to describe necessary and sufficient conditions for two coupled cell networks to be ODE-equivalent. In the next section we define the notion of 'linear equivalence' between two networks. We show in Section 7 that two coupled cell networks are ODE-equivalent if and only if they are linearly equivalent.

6 Linear Equivalence

In this section we define the notion of 'linear equivalence' (Definition 6.2 below). We start with an example to illustrate the ideas involved, and, in particular, the effect of multiple arrows.

Figure 2: Coupled cell networks G_1 (left) and G_2 (right). The number k attached to the right of each edge symbolizes k edges of that type.

Example 6.1 Consider the coupled cell networks G_1 and G_2 of Figure 2. Here all cells are cell-equivalent in each graph, and the $\sim_I$-equivalence classes of both graphs are:

$$\mathcal{Q}_1 = \{1, 2, 3\},\ \mathcal{Q}_2 = \{4\}$$

The identity function on $\{1, 2, 3, 4\} = \mathcal{C}_1 = \mathcal{C}_2$ preserves cell-equivalence and input-equivalence.

First, choose all cell phase spaces to be $P_c = \mathbf{R}$. We now describe the linear admissible vector fields for both graphs, that is, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ of linear groupoid-equivariant maps. Let Y_c denote coordinates on the phase space of cell c, for $c = 1, \ldots, 4$, in both graphs. Any linear G_1-admissible vector field $F = (f_1, f_2, f_3, f_4) : \mathbf{R}^4 \to \mathbf{R}^4$ has the form:

$$\begin{aligned} f_1(Y_1) &= aY_1 \\ f_2(Y_2) &= aY_2 \\ f_3(Y_3) &= aY_3 \\ f_4(Y_4, Y_1, Y_2, Y_3) &= bY_4 + c(5Y_1 + Y_3) + d(2Y_1 + Y_2 + Y_3) \end{aligned}$$

where a, b, c, d are real constants, and any linear G_2-admissible vector field $G = (g_1, g_2, g_3, g_4) : \mathbf{R}^4 \to \mathbf{R}^4$ has the form:

$$\begin{aligned} g_1(Y_1) &= eY_1 \\ g_2(Y_2) &= eY_2 \\ g_3(Y_3) &= eY_3 \\ g_4(Y_4, Y_1, Y_2, Y_3) &= hY_4 + j(5Y_1 + Y_3) + l(5Y_2 + 3Y_3) \end{aligned}$$

where e, h, j, l are real constants. Now recall Definition 4.1, and use the notation $\mathbf{R}\{z_1, \ldots, z_m\}$ for the real vector space spanned by $z_1, \ldots, z_m$. It is clear that

$$\mathbf{R}\{Y_4,\ 5Y_1 + Y_3,\ 2Y_1 + Y_2 + Y_3\} = \mathbf{R}\{Y_4,\ 5Y_1 + Y_3,\ 5Y_2 + 3Y_3\} \tag{6.4}$$

since $5Y_2+3Y_3 = 5(2Y_1+Y_2+Y_3)-2(5Y_1+Y_3)$ and $2Y_1+Y_2+Y_3 = \frac{2}{5}(5Y_1+Y_3)+\frac{1}{5}(5Y_2+3Y_3)$. Therefore the space $\mathcal{L}^P_{G_1}$ of linear G_1-admissible vector fields on $\mathbf{R}^4$ equals the space $\mathcal{L}^P_{G_2}$ of linear G_2-admissible vector fields on $\mathbf{R}^4$. We prove in Theorem 7.1 that this is a necessary and sufficient condition for the graphs G_1 and G_2 to be ODE-equivalent. $\diamond$

If we let $P_c = \mathbf{R}^k$ for $k > 1$ the identical calculation can be carried over, with the only change being that the Y_j now represent arbitrary vectors in $\mathbf{R}^k$. However, condition (6.4) can be interpreted as the condition that the rows of the 3×4 matrices

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 5 & 0 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 \end{bmatrix}$$

should span the same subspaces of $\mathbf{R}^4$. The entries in these matrices are determined by the corresponding network topology, so this condition does not depend on the size of k. This fact generalises, see Corollary 7.9 below. (It is also easy to give an independent proof, along the lines of the above example.)

Definition 6.2 Two coupled cell networks G_1 and G_2 are *γ-linearly equivalent* if:

1. There is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence, such that:

2. If we choose cell phase spaces $P_c \neq 0$ for G_1, and define the corresponding choice of cell phase spaces for G_2 by
$$P_{2,\gamma(c)} = P_{1,c}$$
so that the corresponding total phase spaces are
$$P_1 = \prod_{c \in \mathcal{C}_1} P_{1,c} \qquad P_2 = \prod_{c \in \mathcal{C}_1} P_{1,\gamma(c)}$$
then:

3. The condition
$$\mathcal{L}_{G_1}^{P_1} \;=\; \mathcal{L}_{G_2}^{P_2} \tag{6.5}$$
is satisfied.

Two coupled cell networks G_1 and G_2 are *linearly equivalent* if they are γ-linearly equivalent for some γ. ◇

We show in the next section that this definition is independent of the dimensions of the P_c. Again, we may renumber to make γ the identity.

7 Linear Equivalence and ODE-equivalence

We now come to the main theorem of this paper, which reduces ODE-equivalence to linear equivalence, and its corollary, that the cell phase spaces may be assumed 1-dimensional in that context. Recall Definition 5.1 of ODE-equivalence and Definition 6.2 of linear equivalence of coupled cell networks. Our main result is:

Theorem 7.1 *Let $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ be a bijection that preserves cell-equivalence and input-equivalence. Then two coupled cell networks G_1 and G_2 are γ-ODE-equivalent if and only if they are γ-linearly equivalent.*

Proof The proof is divided in two steps. We prove in Proposition 7.2 below that given two coupled cell networks G_1 and G_2 and a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserving cell-equivalence and input-equivalence, together with a choice of total phase space P_1 for G_1 and P_2 for G_2 according to Definition 5.1, then $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$ if and only if $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$. The rest of the proof consists in proving in Proposition 7.3 below that $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$ if and only if G_1 and G_2 are γ-linearly equivalent for some bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserving cell-equivalence and input-equivalence. As a corollary, we deduce that $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$ if and only if $\mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2}$, and that in this context we may without loss of generality assume that all cell phase spaces are 1-dimensional. □

In the rest of the section we state and prove Propositions 7.2 and 7.3.

Proposition 7.2 *Let G_1 and G_2 be two coupled cell networks such that there is bijection $\gamma: \mathcal{C}_1 \to \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence. Consider a choice of total phase space $P_1 = \prod_{c\in\mathcal{C}_1} P_{1,c}$ for G_1, and let $P_2 = \prod_{c\in\mathcal{C}_1} P_{1,\gamma(c)}$ be the corresponding phase space for G_2. Then $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$ if and only if $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$.*

Proof Trivially, if $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$ then $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$. Suppose now that $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$. By Theorem 4.2, every smooth equivariant vector field $f \in \mathcal{F}_{G_i}^{P_i}$ is determined uniquely by its components f_c where c runs through a set of representatives for the connected components (that is, the $\sim_I$-equivalence classes) of the groupoid $\mathcal{B}_{G_i}$. Note that since $\gamma: \mathcal{C}_1 \to \mathcal{C}_2$ is a bijection that preserves input-equivalence, if $\mathcal{Q}$ is a connected component of the groupoid $\mathcal{B}_{G_1}$ then $\gamma(\mathcal{Q})$ is a connected component of $\mathcal{B}_{G_2}$, and if $\mathcal{R}$ is a set of representatives for the connected components of $\mathcal{B}_{G_1}$ then $\gamma(\mathcal{R})$ is a set of representatives for the connected components of $\mathcal{B}_{G_2}$. The only constraints on f_c are that it depends only on $x_c, x_{\mathcal{T}(I(c))}$ and is invariant under the vertex group $B(c,c)$. Thus every smooth equivariant vector field f is determined uniquely by a finite set of $B(c,c)$-invariant functions, where c runs through a set of representatives for the connected components of the groupoid. Moreover, if $d \sim_I c$ then f_d is related to f_c by a pullback map β^* for $\beta \in B(c,d)$. Pullbacks permute variables, hence preserve smoothness (and also map polynomials to polynomials).

Schwarz [14] proves that in general for any compact Lie group Γ with an orthogonal action on $\mathbf{R}^n$, if the algebra of Γ-invariant polynomials is generated by $\rho_1, \ldots, \rho_k$ (and by Hilbert's basis theorem such a finite basis always exist), then any Γ-invariant C^∞-function of n variables is a C^∞-function of the generators $\rho_1, \ldots, \rho_k$. Since $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$, the vector space of polynomial $B_1(c,c)$-invariants (where $B_1(c,c) \subseteq \mathcal{B}_{G_1}$) coincides with the vector space of polynomial $B_2(\gamma(c),\gamma(c))$-invariants (where $B_2(\gamma(c),\gamma(c)) \subseteq \mathcal{B}_{G_2}$). In particular, the two spaces share a set of invariant polynomial generators. Thus, given an $\sim_I$-equivalence class $\mathcal{Q} \subseteq \mathcal{C}_1$, the equality

$$\mathcal{P}_{G_1}^{P_1}(\mathcal{Q}) = \mathcal{P}_{G_2}^{P_2}(\gamma(\mathcal{Q}))$$

implies that

$$\mathcal{F}_{G_1}^{P_1}(\mathcal{Q}) = \mathcal{F}_{G_2}^{P_2}(\gamma(\mathcal{Q}))$$

Now Theorem 4.2 implies that $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$. □

Proposition 7.3 *Assume the conditions of Proposition 7.2. Then*

$$\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$$

if and only if G_1 and G_2 are γ-linearly equivalent.

Before we prove Proposition 7.3, we state and prove two lemmas that explore the structure of the symmetry groupoids of coupled cell networks.

Lemma 7.4 *Consider $V_1^{d_1}, \ldots, V_s^{d_s}$ where each V_i is a nonzero finite-dimensional vector space of dimension k_i, and denote coordinates on $V_i^{d_i}$ by $x_i = (x_{i,1}, \ldots, x_{i,d_i})$. Let*

$$\Gamma = \mathbf{S}_{d_1} \times \cdots \times \mathbf{S}_{d_s}$$

and

$$V = V_1^{d_1} \oplus \cdots \oplus V_s^{d_s}$$

Define a Γ*-action on* V *by: if* $\sigma \in \mathbf{S}_{d_i}$*, then*

$$\sigma \cdot x = (x_1, \ldots, x_{i-1}, \sigma \cdot x_i, x_{i+1}, \ldots, x_s)$$

where

$$\sigma \cdot x_i = (x_{i,\sigma^{-1}(1)}, \ldots, x_{i,\sigma^{-1}(d_i)})$$

Then any real Γ*-invariant polynomial is a sum of polynomials of the form*

$$q_1(x_1)q_2(x_2)\cdots q_s(x_s)$$

where for $j = 1, \ldots, s$*, each* $q_j(x_j)$ *is* $\mathbf{S}_{d_j}$*-invariant.*

Proof The idea of the proof is simple but the notation is complicated. Essentially, we use the fact that any invariant can be obtained as a linear combination of symmetrized monomials, so the proof reduces to computations with monomials.

In detail, recall that $p : V \to \mathbf{R}$ is Γ-invariant if and only if

$$p(\sigma \cdot x) = p(x) \quad \forall \sigma \in \Gamma, \ x \in V$$

This condition holds if and only if $p : V \to \mathbf{R}$ is $\mathbf{S}_{d_i}$-invariant, where $\mathbf{S}_{d_i}$ acts nontrivially only on $V_i^{d_i}$.

Denote by $\mathbf{Z}_0^+$ the set of nonnegative integers. Monomials in x_1 have the form

$$x_{1,1}^{I_1} \cdots x_{1,d_1}^{I_{d_1}}$$

where $I_1, \ldots, I_{d_1} \in (\mathbf{Z}_0^+)^{k_1}$, and each $x_{1,j}^{I_j}$ is a monomial in the k_1 components of $x_{1,j}$.

Let $p : V \to \mathbf{R}$ be a Γ-invariant polynomial, and write it as linear combination of monomials in x_1 with coefficients in $\mathbf{R}[x_2, \ldots, x_s]$. Suppose that $p(x)$ contains a term that is a scalar multiple of

$$x_{1,1}^{I_1} \cdots x_{1,d_1}^{I_{d_1}} \, q(x_2, \ldots, x_s)$$

Since p is $\mathbf{S}_{d_1}$-invariant and $\mathbf{S}_{d_1}$ acts trivially on $x_2, \ldots, x_s$, then $p(x)$ must also contain

$$x_{1,\sigma(1)}^{I_1} \cdots x_{1,\sigma(d_1)}^{I_{d_1}} \, q(x_2, \ldots, x_s)$$

for all $\sigma \in \mathbf{S}_{d_1}$. It follows that $p(x)$ contains a scalar multiple of

$$\left(\sum_{\sigma \in \mathbf{S}_{d_1}} x_{1,\sigma(1)}^{I_1} \cdots x_{1,\sigma(d_1)}^{I_{d_1}} \right) q(x_2, \ldots, x_s) = q_1(x_1) \cdot q(x_2, \ldots, x_s)$$

where $q_1(x_1) = \sum_{\sigma \in \mathbf{S}_{d_1}} x_{1,\sigma(1)}^{I_1} \cdots x_{1,\sigma(d_1)}^{I_{d_1}}$. Now we repeat the same argument for $q(x_2, \ldots, x_s)$ inductively. $\square$

Remark 7.5 This proof can be presented in a more abstract way: inductively, consider the polynomial $\mathbf{S}_{d_{j+1}}$-invariants over the ring of polynomial invariants for the subgroup $\mathbf{S}_{d_1} \times \cdots \times \mathbf{S}_{d_j} \times \mathbf{1} \times \cdots \times \mathbf{1}$. $\diamondsuit$

Lemma 7.6 *Let V be a nonzero finite-dimensional real vector space of dimension d, and denote coordinates on V^t by $y = (y_1, \ldots, y_t)$. Let $\Gamma = \mathbf{S}_t$ and consider the action of Γ on V^t defined by:*

$$\sigma \cdot y = \big(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(t)}\big) \quad (\sigma \in \mathbf{S}_t,\ y \in V)$$

Then the ring of the Γ-invariant polynomials from V^t to $\mathbf{R}$ is generated by the set of all Γ-invariant polynomials of the form

$$y_1^I + \cdots + y_t^I$$

where $I \in (\mathbf{Z}_0^+)^d$ and each y_i^I is a monomial in the d components of y_i.

Proof Choose coordinates $(y_1, \ldots, y_t)$ on V^t, where $y_i = (y_{i,1}, \ldots, y_{i,d})$. Thus, if $\sigma \in \mathbf{S}_t$,

$$\sigma \cdot (y_1, \ldots, y_t) = \big(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(t)}\big)$$

where

$$y_{\sigma^{-1}(i)} = \big(y_{\sigma^{-1}(i),1}, \ldots, y_{\sigma^{-1}(i),d}\big)$$

A real polynomial $\mathbf{S}_t$-invariant on V^t is a linear combination of $\mathbf{S}_t$-invariants of the form

$$\sum_{\sigma \in \mathbf{S}_t} y_{\sigma(1)}^{I_1} \cdots y_{\sigma(t)}^{I_t} \tag{7.6}$$

where $I_i \in (\mathbf{Z}_0^+)^d$.

To continue the proof we need some terminology. Say that a scalar multiple of a polynomial (7.6) is of *type* m, where $1 \leq m \leq t$, if only m sets of indices, without loss of generality, $I_1, \ldots, I_m$, are non-zero. That is, $I_{m+1} = \cdots = I_t = (0, \ldots, 0)$, and $I_j \neq (0, \ldots, 0)$ for $j = 1, \ldots, m$.

Now observe that if $m = 1$, then given *any* $I_1 \in (\mathbf{Z}_0^+)^d$, an expression (7.6) of type 1 has the form $\sum_{\sigma \in \mathbf{S}_t} y_{\sigma(1)}^{I_1}$ which is a scalar multiple of the type 1 polynomial

$$p_{I_1}(y) = y_1^{I_1} + \cdots + y_t^{I_1}$$

The proof of the lemma is carried out by induction on the type m of the Γ-invariant polynomial. Suppose that any polynomial of the form (7.6) of type less than or equal to m is a polynomial in polynomials of type 1. We prove that the same holds for polynomials of type $m + 1$. Consider

$$p_{I_1, \ldots, I_{m+1}}(y) = \sum_{\sigma \in \mathbf{S}_t} y_{\sigma(1)}^{I_1} y_{\sigma(2)}^{I_2} \cdots y_{\sigma(m+1)}^{I_{m+1}}$$

Take the $\mathbf{S}_t$-invariant polynomial

$$p(y) = p_{I_1}(y) \cdots p_{I_{m+1}}(y)$$

where

$$p_{I_i}(y) = y_1^{I_i} + \cdots + y_t^{I_i}$$

Note that $p_{I_j}(y)$ for all j is a Γ-invariant polynomial of type 1. Moreover

$$p(y) = p_{I_1,\ldots,I_{m+1}}(y) + \sum_i \beta_i r_i(y)$$

where each $\beta_i \in \mathbf{R}$ and each $r_i(y)$ is an $\mathbf{S}_t$-invariant of the form (7.6) of type less than or equal to m. By hypothesis $r_i(y)$ is Γ-invariant and can be written as a polynomial in Γ-invariant polynomials of type 1. Thus

$$p_{I_1,\ldots,I_{m+1}}(y) = p(y) - \sum_i \beta_i r_i(y)$$

and so $p_{I_1,\ldots,I_{m+1}}(y)$ is a Γ-invariant polynomial that can be written as a polynomial of Γ-invariant polynomials of type 1. □

Again, this proof can be presented more abstractly, along the lines of Remark 7.5.

We introduce some notation that we use in the proof of Proposition 7.3. Consider two coupled cell networks $G_i = (\mathcal{C}_i, \mathcal{E}_i, \sim_{C_i}, \sim_{E_i})$ for $i = 1, 2$ such that there is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserving cell-equivalence and input-equivalence. Given a connected component $\mathcal{Q}$ of $\mathcal{B}_{G_1}$ and $c \in \mathcal{Q}$, consider

$$\mathcal{T}(I_1(c)) = K_1 \,\dot\cup \cdots \dot\cup\, K_{r_1(c)}$$

where $K_1, \ldots, K_{r_1(c)}$ are the $\equiv_c$-equivalence classes (on $\mathcal{T}(I_1(c))$) and $n_1(c)$ is the cardinality of $\mathcal{T}(I_1(c))$. (See (3.1) for the definition of the relation $\equiv_c$.) Consider

$$\mathcal{T}(I_2(\gamma(c))) = L_1 \,\dot\cup \cdots \dot\cup\, L_{r_2(\gamma(c))}$$

where $L_1, \ldots, L_{r_2(\gamma(c))}$ are the $\equiv_{\gamma(c)}$-equivalence classes on $\mathcal{T}(I_2(\gamma(c)))$, and $n_2(\gamma(c))$ is the cardinality of $\mathcal{T}(I_2(\gamma(c)))$.

We use the notation $\mathbf{R}[z_1, \ldots, z_m]$ for the polynomial ring in indeterminates $z_1, \ldots, z_m$ over $\mathbf{R}$, and $\mathbf{R}\{z_1, \ldots, z_m\}$ for the real vector space spanned by $z_1, \ldots, z_m$. Let

$$R_1 = \mathbf{R}[Y_c, Y_{i_1}, \ldots, Y_{i_{n_1(c)}}]$$

be the real vector space of polynomials in the indeterminates $Y_c, Y_{i_1}, \ldots, Y_{i_{n_1(c)}}$, where $c, i_1, \ldots, i_{n_1(c)} \in \mathcal{C}_1$.

Remark 7.7 We avoid notational complications here if we permit repetition of the indeterminates (that is, we allow $z_i = z_j$ when $i \neq j$), and interpret the resulting ring of polynomials to be the same as the ring obtained when any repeated indeterminates are replaced by the corresponding single indeterminate. Again, this convention arises from the multiarrow formalism. It amounts to performing calculations in the polynomial ring $\mathbf{R}[z_1, \ldots, z_m]$ where the z_j are independent indeterminates, and then applying a ring homomorphism to identify various z_i. ◇

Let

$$R_2 = \mathbf{R}[Y_c, Y_{\gamma^{-1}(j_1)}, \ldots, Y_{\gamma^{-1}(j_{n_2(\gamma(c))})}]$$

be the real vector space of polynomials in the indeterminates $Y_c, Y_{\gamma^{-1}(j_1)}, \ldots, Y_{\gamma^{-1}(j_{n_2(\gamma(c))})}$, where $\gamma(c), j_1, \ldots, j_{n2(\gamma(c))} \in \mathcal{C}_2$.

Consider the subspace S_1^c of R_1 defined by

$$S_1^c = \mathbf{R}\left\{ Y_c, \sum_{i \in K_1} Y_i,\ \ldots,\ \sum_{i \in K_{r_1(c)}} Y_i \right\} \tag{7.7}$$

Thus S_1^c contains the linear polynomials of R_1 that are $B_1(c,c)$-invariant. Similarly, let

$$S_2^{\gamma(c)} = \mathbf{R}\left\{ Y_c, \sum_{i \in L_1} Y_{\gamma^{-1}(i)},\ \ldots,\ \sum_{i \in L_{r_2(\gamma(c))}} Y_{\gamma^{-1}(i)} \right\} \subseteq R_2 \tag{7.8}$$

be formed by the linear polynomials of R_2 that are $B_2(\gamma(c), \gamma(c))$-invariant.

Example 7.8 Recall the coupled cell networks G_1 and G_2 of Figure 2. For both networks, all cells are cell-equivalent and the $\sim_I$-equivalence classes are $\mathcal{Q}_1 = \{1,2,3\}$ and $\mathcal{Q}_2 = \{4\}$. Thus the identity function γ on $\{1,2,3,4\} = \mathcal{C}_1 = \mathcal{C}_2$ preserves cell-equivalence and input-equivalence. Consider $\mathcal{Q}_2$ and recall (7.7) and (7.8) where now $c = 4 = \gamma(4)$. Then

$$S_1^4 = \mathbf{R}\{Y_4,\ 5Y_1 + Y_3,\ 2Y_1 + Y_2 + Y_3\}$$

and

$$S_2^{\gamma(4)} = \mathbf{R}\{Y_4,\ 5Y_1 + Y_3,\ 5Y_2 + 3Y_3\}$$

As noted earlier, $S_1^4 = S_2^{\gamma(4)}$ and so $\mathcal{L}^P_{G_1}(\mathcal{Q}_2) = \mathcal{L}^P_{G_2}(\mathcal{Q}_2)$ when all cell phase spaces are taken to be $\mathbf{R}$. (It is also true for any choice of phase space P compatible with cell-equivalence.) Trivially, $\mathcal{L}^P_{G_1}(\mathcal{Q}_1) = \mathcal{L}^P_{G_2}(\mathcal{Q}_1)$. It follows from Theorem 4.2 that $\mathcal{L}^P_{G_1} = \mathcal{L}^P_{G_2}$ and G_1, G_2 are γ-linearly equivalent. We show in Proposition 7.3 that this is necessary and sufficient for G_1 and G_2 to be ODE-equivalent, and as a corollary, $\mathcal{L}^P_{G_1} = \mathcal{L}^P_{G_2}$ for *any* choice of P compatible with cell-equivalence.

Proof of Proposition 7.3 Let $G_1,\ G_2$ be two coupled cell networks and let $\gamma:\ \mathcal{C}_1 \to \mathcal{C}_2$ be a bijection that preserves cell-equivalence and input-equivalence. Renumber the cells in $\mathcal{C}_2$ so that the bijection γ is the identity. Thus $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ and $P_1 = P_2 = P$.

Trivially if $\mathcal{P}^P_{G_1} = \mathcal{P}^P_{G_2}$ then for each connected component $\mathcal{Q}$ of G_1 (and G_2) we have $\mathcal{L}^P_{G_1}(\mathcal{Q}) = \mathcal{L}^P_{G_2}(\mathcal{Q})$. In particular, this is true when all cell phase spaces are taken to be $\mathbf{R}$. We now prove the converse. Suppose that for each connected component $\mathcal{Q}$ of G_1 (and G_2) we have

$$\mathcal{L}^P_{G_1}(\mathcal{Q}) = \mathcal{L}^P_{G_2}(\mathcal{Q})$$

Note that in particular this implies that

$$S_1^c = S_2^c$$

for all $c \in \mathcal{Q}$. Let f be any admissible polynomial vector field in $\mathcal{P}^P_{G_1}$. By Theorem 4.2, for any component $\mathcal{Q}$ of $\mathcal{B}_{G_1}$ and $c \in \mathcal{Q}$, the property of $\mathcal{B}_{G_1}$-equivariance of f on the component $\widehat{f}_c : \; P_c \times P_{\mathcal{T}(I_1(c))} \to P_c$ is equivalent to $B_1(c,c)$-invariance of $\widehat{f}_c$. Moreover, $B_1(c,c)$-invariance of $\widehat{f}_c$ is equivalent to $B_1(c,c)$-invariance of each real component of $\widehat{f}_c$. Note that if $d \sim_I c$ then f_d is related to f_c by a pullback map β^* for $\beta \in B(c,d)$. We choose the same coordinate system for P_1 and P_2, and then prove that if $S_1^c = S_2^c$ then the ring of the real polynomial $B_1(c,c)$-invariants is the same as the ring of the real polynomial $B_2(c,c)$-invariants. Moreover, as $\mathcal{L}^P_{G_1} = \mathcal{L}^P_{G_2}$, it follows that $\mathcal{P}^P_{G_1}(\mathcal{Q}) = \mathcal{P}^P_{G_2}(\mathcal{Q})$. Again by Theorem 4.2 we get $\mathcal{P}^P_{G_1} = \mathcal{P}^P_{G_2}$.

Given a $B(c,c)$-invariant polynomial function f_c, we denote by $[f_c]_Q$ the $\mathcal{B}_{G_1}$-equivariant vector field supported on $\mathcal{Q}$ with that f_c component.

By Lemmas 7.4 and 7.6 the ring of real polynomial $B_1(c,c)$-invariants is generated as a ring by the real polynomial $B_1(c,c)$-invariants of type 1. Similarly, the ring of real polynomial $B_2(c,c)$-invariants is generated as a ring by the real polynomial $B_2(c,c)$-invariants of type 1. We show that these rings are the same. Now it is enough to prove that if $S_1^c = S_2^c$, then any type 1 real polynomial $B_1(c,c)$-invariant is a real polynomial $B_2(c,c)$-invariant. (Or equivalently, that any type 1 real polynomial $B_2(c,c)$-invariant is a real polynomial $B_1(c,c)$-invariant.)

As before, $\mathcal{T}\,(I_1(c)) = K_1 \,\dot\cup \cdots \dot\cup\, K_{r_1(c)}$ where $K_1, \ldots, K_{r_1(c)}$ are the $\equiv_c$-equivalence classes on $\mathcal{T}\,(I_1(c))$. Thus

$$B_1(c,c) = \mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r_1(c)}}$$

Similarly, $\mathcal{T}\,(I_2(c)) = L_1 \,\dot\cup \cdots \dot\cup\, L_{r_2(c)}$ where $L_1, \ldots, L_{r_2(c)}$ are the $\equiv_c$-equivalence classes on $\mathcal{T}\,(I_2(c))$, and

$$B_2(c,c) = \mathbf{S}_{L_1} \times \cdots \times \mathbf{S}_{L_{r_2(c)}}$$

By Lemma 7.4 any real $B_1(c,c)$-invariant polynomial is a product of real $\mathbf{S}_{K_i}$-invariant polynomials. Set $K_i = \{1, \ldots, t\}$, so that $\mathbf{S}_t = \mathbf{S}_{K_i}$ and let $V^t = P_{K_i}$ where $V = P_l$ (for any $l \in K_i$) is a non-zero finite-dimensional real vector space. Suppose that V has dimension d, and denote coordinates on V^t by $y = (y_1, \ldots, y_t)$, where each $y_i = (y_{i,1}, \ldots, y_{i,d})$. Thus, if $\sigma \in \mathbf{S}_t$ then

$$\sigma \cdot (y_1, \ldots, y_t) = \left(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(t)}\right)$$

where

$$y_{\sigma^{-1}(i)} = \left(y_{\sigma^{-1}(i),1}, \ldots, y_{\sigma^{-1}(i),d}\right)$$

Given $I \in (\mathbf{Z}_0^+)^d$, the polynomial

$$p_I(y) = y_1^I + \cdots + y_t^I$$

is $\mathbf{S}_t$-invariant (and so $B_1(c,c)$-invariant) and

$$Y_1 + \cdots + Y_t \in S_1^c$$

where S_1^c is as defined in (7.7). By hypothesis, $S_1^c = S_2^c$, where S_2^c is defined in (7.8). Thus there exist real coefficients $\alpha_0, \alpha_{j_1}, \alpha_{j_2}, \ldots$, such that

$$Y_1 + \cdots + Y_t = \alpha_0 p_0(Y) + \alpha_{j_1} p_{j_1}(Y) + \alpha_{j_2} p_{j_2}(Y) + \cdots \tag{7.9}$$

where $p_0(Y) = Y_c$ and $p_{j_p}(Y) = \sum_{i \in L_{j_p}} Y_i$ for $p \geq 1$. Here $L_{j_1}, L_{j_2}, \ldots$ denote $\equiv_c$-equivalence classes on $\mathcal{T}(I_2(c))$. Note also that since $\mathcal{L}^P_{G_1}(\mathcal{Q}) = \mathcal{L}^P_{G_2}(\mathcal{Q})$, it follows that

$$[Y_1 + \cdots + Y_t]_Q = \alpha_0[Y_c]_Q + \alpha_{j_1}[p_{j_1}(Y)]_Q + \alpha_{j_2}[p_{j_2}(Y)]_Q + \cdots \tag{7.10}$$

We claim that the $p_{j_p}(Y)$ that appear in (7.9) can be chosen to depend only on Y_k, where the cell phase space $P_k = P_1$. To see this note that for all $m \in \{1, \ldots, t\} = K_i \subseteq \mathcal{T}(I_1(c))$ we know that $1 = \mathcal{T}(e_1)$ and $m = \mathcal{T}(e_m)$ for arrows $e_1,\ e_m$ where $e_1 \sim_{E_1} e_m$. Thus $1 \sim_{C_1} m$ and so $P_1 = P_m$. Also, cells in the same $\equiv_c$-equivalence class are cell-equivalent tails. Thus if some $p_{j_p}(Y)$ (with $\alpha_{j_p} \neq 0$) in (7.9) depends on Y_l, Y_k such that $P_l = P_1$, then as $k \sim_{C_2} l$ since $k = \mathcal{T}(e_k)$ and $l = \mathcal{T}(e_l)$ where $e_l \sim_{E_2} e_k$ and $l, k \in L_{j_p} \subseteq \mathcal{T}(I_2(c))$, we have that $P_k = P_l = P_1$. This proves the claim.

Set $Y_j = y^I_j$ for all j. Substituting all of this into equation (7.9) we get

$$p_I(y) = Y_1 + \cdots + Y_t = q_I(y)$$

where

$$q_I(y) = \alpha_0 y^I_c + \alpha_{j_1}\left(\sum_{i \in L_{j_1}} y^I_i\right) + \alpha_{j_2}\left(\sum_{i \in L_{j_2}} y^I_i\right) + \cdots$$

is a $B_2(c,c)$-invariant. Moreover, from (7.10) it follows that

$$[q_I(y)]_Q = \alpha_0[y^I_c]_Q + \alpha_{j_1}\left[\left(\sum_{i \in L_{j_1}} y^I_i\right)\right]_Q + \alpha_{j_2}\left[\left(\sum_{i \in L_{j_2}} y^I_i\right)\right]_Q + \cdots$$

is $\mathcal{B}_{G_2}$-equivariant and supported on $\mathcal{Q}$. □

Corollary 7.9 *The following conditions on two networks G_1, G_2 are equivalent:*

(a) *G_1 and G_2 are γ-linearly equivalent.*

(b) *With the identification $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ are equal for all P.*

(c) *With the identification $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ are equal when all cell phase spaces are taken to be* **R**.

Proof Condition (a) implies γ-ODE-equivalence for any choice of P, and γ-ODE-equivalence implies (b) by restricting to linear vector fields. Then (c) is special case of (b). Finally, (c) clearly implies (a). □

8 Combinatorics of Linear Equivalence

Given two coupled cell networks, G_1 and G_2, Theorem 7.1 implies that in order to verify their ODE-equivalence, we need only check their linear equivalence for some bijection between the corresponding sets of cells $\mathcal{C}_i$, preserving cell-equivalence and input-equivalence. A problem concerning the topology of networks as directed graphs has now become a linear algebra problem of finding when two vector spaces of linear polynomial vector fields are equal. Standard methods of linear algebra can solve this problem efficiently in any given case.

It might seem likely that linear equivalence is simpler to work with than ODE-equivalence, in the sense that linear equivalence can be read off easily from the topologies two networks concerned, modulo elementary linear algebra. But it is not clear whether the above linear algebra problem can be simplified significantly by exploiting the network topology (say by defining some kind of 'normal form' for the network, with a topological procedure that reduces any given network to normal form) in a way that is not a trivial encoding of the corresponding linear algebra computation. To illustrate the combinatorial complexities that may arise when determining linear equivalence, we generalize Example 6.1.

Recall Example 7.8, corresponding to the coupled cell networks of Figure 2. The two networks are ODE-equivalent since they are linearly equivalent, taking γ to be the identity function on $\mathcal{C}_1 = \mathcal{C}_2 = \{1, 2, 3, 4\}$. Other examples of coupled cell networks can easily be constructed that are also linearly equivalent to G_1 and G_2, in the following way. Note that taking

$$V = \{\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 : \ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbf{R}\}$$

then

$$S_1^4 = S_2^{\gamma(4)} = \{\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 \in V : \ \lambda_1 + 3\lambda_2 = 5\lambda_3\}$$

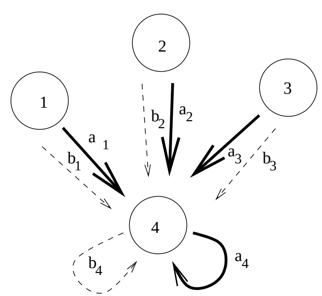

Figure 3: A coupled cell network with four identical cells, two edge-equivalence classes. The symbols a_i and b_i attached to the right of each edge symbolizes the number of edges of that type. Thus a_i, b_i denote nonnegative integers.

Now consider Figure 3. Any coupled cell network with four identical cells and two edge-equivalence classes, as in Figure 3, is γ-linearly equivalent to G_1 of Figure 2 provided that

$$\mathbf{R}\{Y_4, \ a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4, \ b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y_4\} =$$

$$\{\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 \in V : \ \lambda_1 + 3\lambda_2 = 5\lambda_3\}$$

Here a_i, b_i are nonnegative integers and γ is the identity on $\{1, 2, 3, 4\}$.

Other questions that we can pose include the following. Given an ODE-equivalence class of coupled cell networks, is there a canonical 'normal form' — perhaps a graph, or a set of graphs, such that the number of edges is minimal among all the graphs of that ODE-class? Moreover, given a graph G, when can we find a subgraph that is ODE-equivalent to G? These questions are addressed by Aguiar and Dias [1].

Acknowledgements

We thank Marty Golubitsky and Manuela Aguiar for helpful discussions. APSD thanks Departamento de Matemática Pura da Faculdade de Ciências da Universidade do Porto for granting leave, and the Mathematics Institute of the University of Warwick, where this work was partially carried out, for hospitality. The research of APSD was supported by Centro de Matemática da Universidade do Porto. That of INS was partially supported by a grant from the Engineering and Physical Sciences Research Council of the UK, and by NSF grant DMS-0244529.

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Figure 1 caption:

Two coupled cell networks G_1 (on the left) and G_2 (on the right) that generate the same space of admissible vector fields.

Figure 2 caption:

Coupled cell networks G_1 (left) and G_2 (right). The number k attached to the right of each edge symbolizes k edges of that type.

Figure 3 caption:

A coupled cell network with four identical cells, two edge-equivalence classes. The symbols a_i and b_i attached to the right of each edge symbolizes the number of edges of that type. Thus a_i, b_i denote nonnegative integers.