Bifurcations with wreath product symmetry

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Declaration

I declare that, to the best of my knowledge and unless where otherwise stated, all the work presented in this thesis is original and was done under the supervision of Professor Ian Stewart.

A paper with the contents of chapter 3 of this thesis has been accepted for publication by Nonlinearity.

Summary

The aim of this thesis is to make clear how patterns formed through steady-state and Hopf bifurcations in wreath product systems depend both on the internal and global symmetries. We continue the work of Dionne *et al.* developed in [10]. We provide a complete classification theorem for the $\mathbf{C}$-axial subgroups of wreath product groups. Although the structure of these groups can be complicated, the dependence on the internal and global symmetries is always evident. We also provide a description of the isotropy lattice of wreath product groups (extended by the phase-shift symmetries $\mathbf{S}^1$) keeping in mind the distinction between both of these kinds of symmetries. We use the classification theorems for the axial and the $\mathbf{C}$-axial subgroups of wreath product groups to calculate the stability of equilibria in steady-state bifurcation, and of periodic solutions in Hopf bifurcation. We apply these results to study Hopf bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_3$ symmetry. This group is the compact subgroup of the Euclidean group $\mathbf{E}(3)$ that leaves invariant the space of functions from $\mathbf{R}^3$ to $\mathbf{R}$ that are spatially periodic with respect to a cubic lattice. We also study steady-state bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_N$ symmetry.

To my parents
and
Isabel.

Chapter 1

Introduction

Over the last few years, there has been increasing interest in the nonlinear dynamics and bifurcations of systems of coupled identical 'cells' - that is, dynamical subsystems such as oscillators: see in particular [10, 11]. It is now well understood that coupled systems of this kind can possess certain types of symmetry. In particular the individual cells can have a 'local' (or 'internal') group of symmetries $\mathcal{L}$ (a subgroup of $\mathbf{O}(k)$), and the network of couplings can have a 'global' symmetry group $\mathcal{G}$ (a subgroup of the permutation group $\mathbf{S}_N$). Two types of couplings have attracted particular attention, the so-called 'wreath product' case, in which the coupling is invariant under any local symmetry of any cell, and the 'direct product' case when a local symmetry must be applied simultaneously to all cells. The symmetry group of the coupled system is the wreath product $\mathcal{L} \wr \mathcal{G}$, and the direct product $\mathcal{L} \times \mathcal{G}$, respectively. In this work we will study only the wreath product case.

There are many naturally occurring nonlinear systems with wreath product structure. Golubitsky, Stewart and Dionne [19] list four types of examples, as follows:

1. Coupled arrays of Josephson junctions [25, 2] where the symmetry group is $\mathbf{S}_k \wr \mathbf{S}_N$.

2. Discretizations of PDEs with gauge symmetry, such as the Ginzburg-Landau equation.

3. Molecular dynamics, where to a good approximation coupling between atoms is invariant under symmetries of each individual atom.

4. Heteroclinic cycles, like the Guckenheimer and Holmes [23] example of a structurally stable heteroclinic cycle obtained by abstracting a model for rotating convection developed by Busse and Heikes [5]. Also, with slight modification, the 'instant chaos' scenario of Guckenheimer and Worfolk [24], which involves a subgroup of index two of $\mathbf{Z}_2 \wr \mathbf{Z}_4$.

   More generally, there are now several more recent examples:

5. In [7], Dellnitz *et al.* present numerical evidence that it is possible to replace the equilibria in the heteroclinic cycle presented by Guckenheimer and Holmes (mentioned above) by chaotic sets, once more obtaining cycling behaviour, but now between chaotic sets.

6. Silber and Knobloch [34] study Hopf bifurcation on a square lattice. Their system can be viewed as a wreath product coupling system, and the group here is $\mathbf{O}(2) \wr \mathbf{S}_2$.

7. Callahan and Knobloch [6] study steady-state bifurcations on various cubic lattices. These correspond to various representations of $(\mathcal{O} \oplus \mathbf{Z}_2^c) \dot{+} \mathbf{T}^3$, where $\mathcal{O}$ is the octahedral group, $\mathbf{Z}_2^c$ represents inversion through the origin and $\mathbf{T}^3$ is the three-torus of translations. This group is isomorphic to $\mathbf{O}(2) \wr \mathbf{S}_3$. Our approach applies to the 6-dimensional representation (for the simple cubic lattice), but not to the 8- and 12-dimensional representations (for the face-centred and body-centred cubic lattices).

8. More generally, the Weyl group of type B_n denoted by $W(B_n)$ [4] can be viewed as the wreath product $\mathbf{Z}_2 \wr \mathbf{S}_n$ [14]. This crystallographic group is also called the hyperoctahedral group because it is the symmetry group of the N-dimensional cube. It is the holohedry of a lattice in dimension N. If we extend it by the N-torus $\mathbf{T}^N$, we obtain a compact subgroup of the Euclidean group $\mathbf{E}(N)$, that leaves invariant the space of functions from $\mathbf{R}^N$ to $\mathbf{R}$ that are spatially periodic with respect to this lattice [8, 9]. Again we get a wreath product group: $\mathbf{O}(2) \wr \mathbf{S}_N$.

An appropriate general setting for such questions is the theory of symmetric dynamical systems [17, 20]. In that theory, we study a system of ODEs $\dot{x} = g(x, \lambda)$, for $g : V \times \mathbf{R} \to V$, where V is a finite-dimensional vector space. It turns out that the symmetry of g imposes restrictions on the bifurcations that can occur, and the main aim of the theory is to understand the effect that these restrictions have.

The basic existence theorem for steady-state bifurcation is the equivariant branching lemma [20]. Generically, branches of steady-states with symmetries fixing one-dimensional fixed-point subspaces bifurcate from a steady-state with the full symmetry for an ODE equivariant under this group.

A central part of the theory is the study of bifurcations to periodic solutions in systems commuting with a compact Lie group Γ. Here the main result is the equivariant Hopf theorem [20], which guarantees (with certain nondegeneracy conditions) that for each isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ with a two-dimensional fixed-point subspace (called $\mathbf{C}$-axial) there exists a branch of periodic solutions with that symmetry. This theorem reduces part of the existence problem for Hopf bifurcations to an algebraic problem: the classification of $\mathbf{C}$-axial subgroups.

In [10] a theory is developed of how patterns formed through steady-state and Hopf bifurcations in wreath product systems depend both on $\mathcal{L}$ and $\mathcal{G}$.

A classification theorem for the $\mathbf{C}$-axial subgroups in wreath product groups is presented in [10]. However, the proposed classification omits some $\mathbf{C}$-axial subgroups, and is therefore incomplete. We provide a complete classification theorem for the $\mathbf{C}$-axial subgroups of wreath product groups (theorem 3.2.1). The structure of the extra $\mathbf{C}$-axial groups is more complicated than in [10]. However this structure is explicitly described, and depends very clearly on the $\mathbf{C}$-axial subgroups of $\mathcal{L} \times \mathbf{S}^1$ and on the possible blocks that can be obtained from the permutation group $\mathcal{G}$. More precisely, we prove that the $\mathbf{C}$-axial groups, up to conjugacy, of a general $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ are the groups that we denote by $\Sigma = \Sigma(B^\psi, J, \sigma, J_1, p)$ and that we define in chapter 3. Here B^ψ is $\mathbf{C}$-axial in $\mathcal{L} \times \mathbf{S}^1$ and J is a block. For B^ψ and J chosen, there is a permutation $\sigma \in \mathcal{G}$ that splits J as disjoint union of subsets of the form $\sigma^i(J_1)$ where for some power $s^{'}$ we end up with J_1. Finally, depending on this power and on the twist image $\psi(B)$, there is a cyclic group $\mathbf{Z}_p$ of $\mathbf{S}^1$ on which the structure of Σ depends. The $\mathbf{C}$-axial groups obtained by [10] are those groups Σ with twist image equal to the twist image $\psi(B)$.

In [13] there is the analogous result to the equivariant Hopf theorem, in which Σ can be any maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. We also describe the maximal isotropy subgroups for $\Gamma \times \mathbf{S}^1$ where Γ is a wreath product group. The description of these subgroups is easily obtained from the method used for $\mathbf{C}$-axial groups. More generally, we find that any submaximal isotropy subgroup can be seen as an intersection of isotropy subgroups that have a structure like the one for $\mathbf{C}$-axial groups where now the isotropy group B^ψ of $\mathcal{L} \times \mathbf{S}^1$ can be any (not necessarily maximal).

Finally, we conclude that in order to find the isotropy lattice of a general group $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$, we start by finding the isotropy lattice of $\mathcal{L} \times \mathbf{S}^1$. Once this is obtained, we have to know the block structure of $\mathcal{G}$. Finally, putting together this information, we are able to obtain the complete classification with the groups that we denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$.

The work is organized as follows. In chapter 2 we present the background needed for the present work.

In chapter 3 we present our main work. We begin by introducing wreath product groups and summarizing the mains results obtained in [10] about the linear theory of wreath products. We also present the axial and the $\mathbf{C}$-axial groups obtained in [10]. Our main result is the complete classification of the $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$ where Γ is a general wreath product group $\mathcal{L} \wr \mathcal{G}$. Moreover, we derive the analogous classification for the maximal isotropy subgroups of $\Gamma \times \mathbf{S}^1$. Finally, we prove that an algebraic structure that is a generalization of the one obtained for $\mathbf{C}$-axial groups can be used to describe any isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Thus in this work we describe the complete isotropy lattice of a general group $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$, using the isotropy lattice of $\mathcal{L} \times \mathbf{S}^1$ and the block structure originated by $\mathcal{G}$.

In chapter 4 we explore the natural association between coupled cell systems and wreath product systems, showing that some information on the dynamics of coupled cell systems can be derived from the dynamics of the one-cell system. We focus here on the equivariant branching lemma and on the equivariant Hopf theorem. Moreover, we use the classification obtained in chapter 3 to reduce the form of the derivative of the vector field calculated at steady-state equilibria with maximal isotropy in $\mathcal{L} \wr \mathcal{G}$ (for $\mathcal{L} \wr \mathcal{G}$-absolutely irreducible representations) or in $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ (for $\mathcal{L} \wr \mathcal{G}$-simple representations).

In chapter 5 we apply the theory thus developed to the symmetry group $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_3$. We study Hopf bifurcation with symmetry Γ. We derive the general form of a vector field commuting with $\Gamma \times \mathbf{S}^1$ up to sixth degree and we find the possible branches of periodic solutions with maximal symmetry by classifying the maximal isotropy subgroups of $\Gamma \times \mathbf{S}^1$. Also we determine the conditions on the coefficients of the third order truncation of the vector field that determine (in the generic case) the stability of these solutions. Moreover, we prove that the same conditions determine the stability of these solutions, even if only the Taylor series of degree three commutes with $\Gamma \times \mathbf{S}^1$ but the original vector field commutes only with Γ. We also find branches of periodic solutions with submaximal isotropy, but now we can no longer guarantee the existence of these solutions for the original vector field. At this stage, we used the isotropy lattice of $\Gamma \times \mathbf{S}^1$. We remark that the Hopf bifurcation problem on a square lattice reduced to the Hopf bifurcation with symmetry $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ (on a eight-dimensional vector space) studied by [34] is a subproblem of our problem, i.e., there is an isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ with eight-dimensional fixed-point subspace such that the elements of the normalizer of Σ that act nontrivially on $\mathrm{Fix}(\Sigma)$ form the group $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$. We also explore some possibilities for heteroclinic cycles between periodic solutions with maximal symmetry.

Finally, in chapter 6 we study generic bifurcation with symmetry $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_N$ on $V = \mathbf{C}^N$, where V is a Γ-absolutely irreducible representation. Our results are strictly related to the generic bifurcation with symmetry $\mathbf{Z}_2 \wr \mathbf{S}_N$ (that is the Weyl group $W(B_N)$). We derive the general form of a commuting vector field with Γ, and we obtain that generically there are only N branches of steady-state equilibriums (with maximal symmetry) that are guaranteed by the equivariant branching lemma. The stability of these equilibriums is determined by the Taylor series of degree three of the vector field. Moreover, only equilibria corresponding to one of two of the N branches can be stable, and then only if all branches are supercritical.

Chapter 2

Background

When we study a system of ODEs

$$\dot{v} = g(v, \lambda), \tag{2.1}$$

with $v \in V$, $\lambda \in \mathbf{R}$ where V is a finite-dimensional real vector space and λ is the bifurcation parameter, and g is a symmetric function, we know that the symmetry of the problem imposes restrictions on the type and the way that solutions can bifurcate from an invariant steady-state equilibrium.

There are techniques that simplify the analysis of symmetric bifurcation problems and these techniques exploit the symmetries of the problem (which in often aspects complicate the study). Two of these are invariant theory and restriction to fixed-point subspaces. We begin by presenting a few results concerning representation of compact Lie groups. We also state the equivariant branching lemma, a fundamental result in steady-state bifurcations. It involves linear subspaces of V that are invariant under the vector field g and that are called fixed-point subspaces of subgroups Σ of the symmetry group being studied. Generically, there are branches of steady-state solutions corresponding to each one-dimensional fixed-point subspace. These solutions are in those spaces and therefore are solutions of (2.1) restricted to the corresponding fixed-point subspace.

When the derivative $(dg)_{0,0}$ has purely imaginary eigenvalues, then under an additional hypothesis of nondegeneracy this condition implies the occurrence of a branch of periodic solutions. We introduce here the equivariant Hopf theorem and again this result states that we can expect bifurcating branches of periodic solutions. These correspond to solutions of (2.1) restricted to two-dimensional fixed-point subspaces of groups that are related now with symmetries that involve the original symmetry group of the problem and an extra group of symmetries called phase-shift symmetries. These extra symmetries can be related to the circle group $\mathbf{S}^1$.

2.1 Group theory

We start by stating some useful results concerning invariant theory that are related to the representation of compact Lie groups on finite-dimensional real vector spaces.

Let Γ be a compact Lie group acting linearly on V where V is a finite-dimensional real vector space. Throughout this work, we consider linear Lie groups, that is, closed subgroups of $GL(n)$ where $GL(n)$ denotes the group of all invertible linear transformations of the vector space $\mathbf{R}^n$ into itself. We note that in [3], it is proved that every compact Lie group is topologically isomorphic to a linear Lie group. We say closed subgroup of $GL(n)$ in the sense that it is a closed subset (and subgroup) of $GL(n)$ where $GL(n)$ is seen as an open subset of the space of all $n \times n$ matrices that can be identified with $\mathbf{R}^{n^2}$.

It turns out that a representation of Γ on V can always be decomposed into a direct sum of simpler representations that are called irreducible. Before we state this result formally we define: a subspace $W \subseteq V$ is Γ-*invariant* if $\gamma \cdot w \in W$ for all $w \in W$ and $\gamma \in \Gamma$. If in addition, the only Γ-invariant subspaces of W are $\{0\}$ and W, then the representation of Γ on W is said to be *irreducible* and W is called Γ-*irreducible*.

Let V and W be n-dimensional vector spaces and assume that the Lie group Γ acts both on V and W. The actions are said to be *isomorphic*, or the spaces V and W are Γ-*isomorphic*, if there exists a (linear) isomorphism $A : V \to W$ such that $A(\gamma \cdot v) = \gamma \cdot A(v)$, for all $v \in V$ and $\gamma \in \Gamma$, i.e., we get the same group of matrices if we identify the spaces V and W (via the linear isomorphism).

Lemma 2.1.1 *Let Γ be a compact Lie group acting on V, let $A : V \to V$ be a linear mapping that commutes with Γ, and let $W \subseteq V$ be a Γ-irreducible subspace. Then $A(W)$ is Γ-invariant, and either $A(W) = \{0\}$ or the representations of Γ on W and $A(W)$ are isomorphic.*

Proof See [20] lemma XII 3.4. □

Theorem 2.1.2 *Let Γ be a compact Lie group acting on V.*
(a) Up to Γ-isomorphism there are a finite number of distinct Γ-irreducible subspaces of V. Call these $U_1, \ldots, U_t$.
(b) Define W_k to be the sum of all Γ-irreducible subspaces W of V such that W is Γ-isomorphic to U_k. Then

$$V = W_1 \oplus \cdots \oplus W_t.$$

Proof See [20] theorem XII 2.5. □

The subspaces W_k are called the *isotypic components* of V, of type U_k, for the action of Γ. We say that a mapping $g : V \to V$ is Γ-*equivariant* or *commutes with* Γ if

$$g(\gamma \cdot v) = \gamma \cdot g(v)$$

for all $\gamma \in \Gamma$ and $v \in V$.

A special kind of commuting mappings are the linear ones.

Definition 2.1.3 *A representation of a group Γ on a vector space V is absolutely irreducible if the only linear mappings on V that commute with Γ are the scalar multiples of the identity.*

We see later how we can use the decomposition of theorem 2.1.2 and the theorem below to compute the linearized stability of steady-state solutions.

Theorem 2.1.4 *Let Γ be a compact Lie group acting on the vector space V. Decompose V into isotypic components*

$$V = W_1 \oplus \cdots \oplus W_s.$$

Let $A : V \to V$ be a linear mapping commuting with Γ. Then

$$A(W_k) \subseteq W_k$$

for $k = 1, \ldots, s$.

Proof See [20] theorem XII 3.5. □

The symmetry of a mapping imposes restrictions on its form.

We say that a real-valued function $f : V \to \mathbf{R}$ is *invariant* under Γ if

$$f(\gamma \cdot v) = f(v)$$

for all $\gamma \in \Gamma$ and $v \in V$.

There are results that permit the description of the $\mathbf{C}^\infty$ functions that are equivariant by Γ.

Denote by $\mathcal{P}(\Gamma)$ ($\mathcal{E}(\Gamma)$) the ring of polynomials ($\mathbf{C}^\infty$ germs) from V to $\mathbf{R}$ invariant under Γ.

When there is a *finite* subset of invariant polynomials such that *every* invariant polynomial may be written as a polynomial function of them, this set is said to *generate* or to form a *Hilbert basis* for $\mathcal{P}(\Gamma)$.

Theorem 2.1.5 (Hilbert-Weyl theorem) *Let Γ be a compact Lie group acting on V. Then there exists a finite Hilbert basis for $\mathcal{P}(\Gamma)$.*

Proof See [20] theorem XII 4.2. □

Theorem 2.1.6 (Schwarz) *Let Γ be a compact Lie group acting on V. Let $\mu_1, \ldots, \mu_s$ be a Hilbert basis for $\mathcal{P}(\Gamma)$. Let $f \in \mathcal{E}(\Gamma)$. Then there exists a smooth germ $h \in \mathcal{E}_s$ such that*

$$f(v) = h(\mu_1(v), \ldots, \mu_s(v)).$$

Here $\mathcal{E}_s$ is the ring of $\mathbf{C}^\infty$ germs $\mathbf{R}^s \to \mathbf{R}$.

Proof See [32]. □

Denote now by $\vec{\mathcal{P}}(\Gamma)$ the space of Γ-equivariant polynomial mappings from V into V and $\vec{\mathcal{E}}(\Gamma)$ the space of Γ-equivariant germs (at the origin) $\mathbf{C}^\infty$ from V to V.

As we have that if $f \in \mathcal{P}(\Gamma)$ and $g \in \vec{\mathcal{P}}(\Gamma)$, then $f \cdot g \in \vec{\mathcal{P}}(\Gamma)$, the space $\vec{\mathcal{P}}(\Gamma)$ is a module over the ring $\mathcal{P}(\Gamma)$. Similarly, the space $\vec{\mathcal{E}}(\Gamma)$ is a module over the ring $\mathcal{E}(\Gamma)$.

Let $g_1, \ldots, g_r$ be Γ-equivariant polynomials such that every $g \in \vec{\mathcal{P}}(\Gamma)$ ($\vec{\mathcal{E}}(\Gamma)$) can be written as

$$g = f_1 g_1 + \cdots + f_r g_r$$

for $f_j \in \mathcal{P}(\Gamma)$ $(\mathcal{E}(\Gamma))$. Then $g_1, \ldots, g_r$ are said to *generate* $\vec{\mathcal{P}}(\Gamma)$ $(\vec{\mathcal{E}}(\Gamma))$ over $\mathcal{P}(\Gamma)$ $(\mathcal{E}(\Gamma))$. If the relation

$$f_1 g_1 + \cdots + f_r g_r \equiv 0$$

implies that

$$f_1 \equiv \cdots \equiv f_r \equiv 0,$$

then we say that $g_1, \ldots, g_r$ *freely generate* $\vec{\mathcal{P}}(\Gamma)$ $(\vec{\mathcal{E}}(\Gamma))$ and $\vec{\mathcal{P}}(\Gamma)$ $(\vec{\mathcal{E}}(\Gamma))$ is called a *free* module over $\mathcal{P}(\Gamma)$ $(\mathcal{E}(\Gamma))$.

Theorem 2.1.7 *Let Γ be a compact Lie group acting on V. Then there exists a finite set of Γ-equivarint polynomials $g_1, \ldots, g_r$ that generates the module $\vec{\mathcal{P}}(\Gamma)$ over the ring $\mathcal{P}(\Gamma)$.*

Proof See [20] theorem XII 5.2. $\square$

Theorem 2.1.8 (Poénaru) *Let Γ be a compact Lie group acting on V and let $g_1, \ldots, g_r$ generate the module $\vec{\mathcal{P}}(\Gamma)$ over the ring $\mathcal{P}(\Gamma)$. Then $g_1, \ldots, g_r$ generate the module $\vec{\mathcal{E}}(\Gamma)$ over the ring $\mathcal{E}(\Gamma)$.*

Proof See [30]. $\square$

2.2 Symmetry-breaking in steady-state bifurcation

We concentrate now on the structure of bifurcations of steady-state solutions of systems of ODEs

$$\dot{v} = g(v, \lambda),$$

where $g : V \times \mathbf{R} \to V$ commutes with the action of a compact Lie group Γ on V and $\lambda \in \mathbf{R}$ is the bifurcation parameter. A steady-state solution v satisfies

$$g(v, \lambda) = 0$$

and since g commutes with Γ, if v is a solution, then $\gamma \cdot v$ is also a solution, for $\gamma \in \Gamma$.

We define

$$\Gamma v = \{\gamma \cdot v : \gamma \in \Gamma\},$$

the *orbit* of v under Γ, and

$$\Sigma_v = \{\gamma \in \Gamma : \gamma \cdot v = v\},$$

the *isotropy subgroup* of $v \in V$ in Γ.

Recall that points in V that are in the same Γ-orbit have conjugate isotropy subgroups.

Define the *fixed-point subspace* of $\Sigma \subseteq \Gamma$ by

$$\text{Fix}\ (\Sigma) = \{v \in \text{V} : \gamma \cdot v = v,\ \forall \gamma \in \Sigma\}.$$

This is always a linear subspace invariant under g, that is,

$$g(\text{Fix}(\Sigma)) \subseteq \text{Fix}(\Sigma).$$

This follows from the fact that if $\sigma \in \Sigma$ and $v \in \text{Fix}(\Sigma)$, then $g(v) = g(\sigma v)$; and as g commutes with Σ, then $g(\sigma v) = \sigma g(v)$ and so $g(v)$ is also fixed by Σ. Note that this result holds even when g is nonlinear. One consequence is that when we look for solutions with a specific isotropy subgroup Σ, we can restrict g to $\text{Fix}(\Sigma)$ and then solve the equation on this space.

Another consequence is the existence of trivial zeros of Γ-equivariant mappings g. Suppose that $\text{Fix}(\Gamma) = \{0\}$. Then it follows that $g(0, \lambda) = 0$ for all $\lambda \in \mathbf{R}$.

The larger is the orbit Γv for $v \in V$, the smaller is the isotropy subgroup Σ_v:

Proposition 2.2.1 *Let Γ be a compact Lie group acting on V. Then*
(a) If $|\Gamma| < \infty$, then $|\Gamma| = |\Sigma_v||\Gamma v|$.
(b) $\dim\ \Gamma = \dim\ \Sigma_v + \dim\ \Gamma v$.

Proof See [20] proposition XIII 1.2. □

An important class of isotropy subgroups are called *maximal.*

Definition 2.2.2 *Let Γ be a compact Lie group acting on V. An isotropy subgroup $\Sigma \subseteq \Gamma$ is maximal if there does not exist an isotropy subgroup Δ of Γ satisfying $\Sigma \subset \Delta \subset \Gamma$.*

One class of maximal isotropy subgroups of a Lie group Γ are those with one-dimensional fixed-point subspace: these are called *axial*. These subgroups are important because (generically) they lead to solutions for bifurcation problems with symmetry Γ.

Definition 2.2.3 *Let Γ be a Lie group acting on a vector space V. A bifurcation problem with symmetry Γ is a germ $g \in \vec{\mathcal{E}}_{v,\lambda}(\Gamma)$ satisfying $g(0,0) = 0$ and $(dg)_{0,0} = 0$.*

Here we use $g \in \vec{\mathcal{E}}_{v,\lambda}(\Gamma)$ to denote a germ (based at the origin $(v, \lambda) = (0,0)$) of a Γ-equivariant mapping, which we also denote by g by abuse of notation. By equivariance, we mean

$$g(\gamma \cdot v, \lambda) = \gamma \cdot g(v, \lambda)$$

for all $\gamma \in \Gamma,\ v \in V$ and $\lambda \in \mathbf{R}$.

Note that in the definition we assume $(dg)_{0,0} = 0$. We know that by applying a Liapunov-Schmidt reduction with symmetries we can always reduce g such that $(dg)_{0,0}$ vanishes.

Consider a property $\mathcal{P}$ defined by a finite number of derivatives of a germ g evaluated at the origin. As in [20], when we state that a set $\mathcal{S}$ of germs is *generic* for the property $\mathcal{P}$, we mean that there exists a finite number of inequalities $\mathcal{Q}$ involving a finite number of derivatives of g at the origin (and not contradicting any of the equalities in $\mathcal{P}$), such that $g \in \mathcal{S}$ if and only if g has property $\mathcal{P}$ and satisfies the inequalities in $\mathcal{Q}$.

Proposition 2.2.4 *Let $G : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}^N$ be a one-parameter family of Γ-equivariant mappings with $G(0,0) = 0$. Let $V = \ker\ (dG)_{0,0}$. Then generically the action of Γ on V is absolutely irreducible.*

Proof See [20] proposition XIII 3.2. □

If g commutes with Γ, then from $g(\gamma \cdot v, \lambda) = \gamma \cdot g(v, \lambda)$ we get

$$(dg)_{\gamma v,\lambda} \cdot \gamma = \gamma \cdot (dg)_{v,\lambda}.$$

In particular, for $v = 0$ it follows that $(dg)_{0,\lambda}$ commutes with Γ. If the action is absolutely irreducible, then $(dg)_{0,\lambda}$ is a multiple of the identity, i.e., there exists $C(\lambda)$ such that $(dg)_{0,\lambda} = C(\lambda) Id$.

Theorem 2.2.5 (Equivariant branching lemma) *Let Γ be a Lie group acting absolutely irreducibly on V and let $g \in \vec{\mathcal{E}}_{v,\lambda}(\Gamma)$ be a Γ-equivariant bifurcation problem. Let $(dg)_{0,\lambda} = C(\lambda)Id$ and suppose that $C'(0) \neq 0$. Let Σ be an isotropy subgroup satisfying*

$$\dim\ \mathrm{Fix}(\Sigma) = 1.$$

Then there exists a unique smooth solution branch to $g = 0$ such that the isotropy subgroup of each solution is Σ.

Proof See [20] theorem XIII 3.3. □

We note that since $\mathrm{Fix_V}(\Gamma) = \{0\}$, the assumption that the space V is absolutely irreducible for Γ and g commutes with Γ implies that $g(0, \lambda) = 0$, so $v = 0$ is an equilibrium for all $\lambda \in \mathbf{R}$. In the conditions of the equivariant branching lemma, since it is assumed that $C'(0) \neq 0$, there is an exchange of stability of this trivial equilibrium (for λ near 0). We say that the bifurcating solution branch is *subcritical* if the branch occurs for parameter values of λ where the trivial equilibrium is stable and *supercritical* otherwise. Usually we assume that $v = 0$ is stable for $\lambda < 0$ and so subcritical branches occur for $\lambda < 0$ and supercritical branches for $\lambda > 0$.

Suppose now that v_0 is an equilibrium solution of (2.1) where g commutes with Γ and let $\Sigma = \Sigma_{v_0}$ be the isotropy subgroup of v_0. The solution v_0 is asymptotically stable if every trajectory $v(t)$ of (2.1) which begins near v_0 stays near v_0 for all $t > 0$, and also $\lim_{t\to\infty} v(t) = v_0$. It follows that if $\dim\ \Sigma < \dim\ \Gamma$, then v_0 cannot be asymptotically stable (essentially because of proposition 2.2.1). However v_0 can be *neutrally stable*, i.e., every trajectory $v(t)$ of the ODE which begins near v_0 stays near v_0 for all time $t > 0$.

In fact in this case, we have

$$T_{v_0}\Gamma v_0 \subseteq \ker\ (dg)_{v_0}$$

where $T_{v_0}\Gamma v_0$ denotes the tangent space of Γv_0 at v_0. To see this, let $y(t) = \gamma(t) \cdot v_0$ be a smooth curve in the orbit Γv_0 with $\gamma(t)$ a smooth curve in Γ such that $\gamma(0) = 1$. Then $(\frac{d\gamma}{dt})(0) \cdot v_0$ is an eigenvector of $(dg)_{v_0}$ with eigenvalue zero and we have a method for calculating null vectors of $(dg)_{v_0}$.

We say that the equilibrium v_0 is *orbitally stable* if v_0 is neutrally stable and if whenever $v(t)$ is a trajectory beginning near v_0, then $\lim_{t\to\infty} v(t)$ exists and lies in Γv_0.

Also we say that v_0 is *linearly orbitally* stable if the eigenvalues of $(dg)_{v_0}$ other than those arising from $T_{v_0}\Gamma v_0$ have negative real part. That is, the dimension of the null space is dim $\Gamma -$ dim Σ_{v_0} and the eigenvalues of $(dg)_{v_0}$ not forced to be zero by the group action have negative real part.

Theorem 2.2.6 *Linear orbital stability implies (asymptotic) orbital stability.*

Proof See [20] theorem XIII 4.3. □

As we saw before, from $g(\gamma \cdot v, \lambda) = \gamma \cdot g(v, \lambda),\ \forall \gamma \in \Gamma$ we get that

$$(dg)_{\gamma v,\lambda} \cdot \gamma = \gamma \cdot (dg)_{v,\lambda}.$$

In particular if $\Sigma \subseteq \Gamma$ is the isotropy subgroup of v it follows that $(dg)_{v,\lambda}$ commutes with Σ.

Also we know by theorem 2.1.2 that we can decompose V into isotypic components for Σ, say $V = W_1 \oplus \cdots \oplus W_k$. Now if we use theorem 2.1.4 we have

$$(dg)_{v,\lambda}(W_i) \subseteq W_i$$

and we can always take $W_1 = \mathrm{Fix}(\Sigma)$. Therefore from Σ_v we can get information about the derivative $(dg)_{v,\lambda}$. For example we can calculate the eigenvalues restricting the derivative to each W_i.

2.3 Symmetry-breaking in Hopf bifurcation

We say that an ODE

$$\dot{v} = g(v, \lambda), \quad g(0,0) = 0, \tag{2.2}$$

with g smooth undergoes a Hopf bifurcation at $\lambda = 0$ if $(dg)_{0,0}$ has a purely imaginary eigenvalue. Under an additional hypothesis of nondegeneracy a branch of periodic solutions bifurcates from $\lambda = 0$.

When g commutes with a symmetry group Γ, this symmetry imposes restrictions on the imaginary eigenspace.

Definition 2.3.1 *A representation W of Γ is Γ-simple if either*
(a) $W \cong V \oplus V$ where V is absolutely irreducible for Γ, or
(b) W is non-absolutely irreducible for Γ.

Proposition 2.3.2 *Let $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ be a Γ-equivariant bifurcation problem. Suppose that $(dg)_{0,0}$ has purely imaginary eigenvalues $\mp iw$. Let G_{iw} be the corresponding real generalized eigenspace of $(dg)_{0,0}$. Then generically G_{iw} is Γ-simple. Moreover, $G_{iw} = E_{iw}$.*

Proof See [20] proposition XVI 1.4. $\square$

In fact, under the conditions of the previous proposition and if $\mathbf{R}^n$ is Γ-simple, then we may assume

$$(dg)_{0,0} = \begin{bmatrix} 0 & -Id_m \\ Id_m & 0 \end{bmatrix} = J$$

where $m = n/2$. This is because:

Lemma 2.3.3 *Assume that $\mathbf{R}^n$ is Γ-simple, the mapping g is Γ-equivariant and $(dg)_{0,0}$ has i as an eigenvalue. Then*
(a) The eigenvalues of $(dg)_{0,\lambda}$ consist of a complex conjugate pair $\sigma(\lambda)\mp i\rho(\lambda)$ each with multiplicity m. Moreover, σ and ρ are smooth functions of λ.
(b) There is an invertible linear map $S : \mathbf{R}^n \to \mathbf{R}^n$ commuting with Γ such that

$$(dg)_{0,0} = SJS^{-1}.$$

Proof See [20] lemma XVI 1.5. $\square$

Again as in steady-state bifurcation, one method to find periodic solutions to (2.2) is to look for solutions with a specific symmetry.

Identify the circle $\mathbf{S}^1$ with $\mathbf{R}/2\pi\mathbf{Z}$ and suppose that $v(t)$ is 2π-periodic in t. A symmetry of the periodic solution $v(t)$ is an element $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ such that

$$\gamma \cdot v(t) = v(t - \theta).$$

If we introduce the action of $\Gamma \times \mathbf{S}^1$ on the space $\mathcal{C}_{2\pi}$ of 2π-periodic functions $\mathbf{R} \to \mathbf{R}^n$, defined by

$$(\gamma, \theta) \cdot v = \gamma \cdot v(t + \theta),$$

then the symmetry group of $v(t)$ is the isotropy subgroup of v with respect to this action.

So if we assume (2.2) where g commutes with Γ and $(dg)_{0,0} = L$ has purely imaginary eigenvalues, we can apply a Liapunov-Schmidt reduction preserving symmetries that will induce a different action of $\mathbf{S}^1$ on a finite-dimensional space, which can be identified with the exponential of $L\mid_{E_i}$ acting on the imaginary eigenspace E_i of L. The reduced function of g will commute with $\Gamma \times \mathbf{S}^1$.

Basically the equivariant Hopf theorem states that for each isotropy subgroup of $\Gamma \times \mathbf{S}^1$ with two-dimensional fixed-point subspace there exists a unique branch of periodic solutions with that symmetry (with a nondegeneracy crossing condition of the eigenvalues):

Theorem 2.3.4 (Equivariant Hopf theorem) *Consider the system of ODEs*

$$\frac{dv}{dt} = g(v, \lambda) \tag{2.3}$$

where $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ *is smooth and commutes with a compact Lie group* Γ.

Assume the generic hypothesis that $\mathbf{R}^n$ *is* Γ*-simple and* $(dg)_{0,0}$ *has* i *as eigenvalue, i.e., choose coordinates so that* $(dg)_{0,0} = J$*, where* $m = n/2$*. Using lemma 2.3.3 we know that the eigenvalues of* $(dg)_{0,\lambda}$ *are* $\sigma(\lambda) \mp i\rho(\lambda)$*, each of multiplicity* m*. Therefore* $\sigma(0) = 0$ *and* $\rho(0) = 1$*.*

Assume now that

$$\sigma'(0) \neq 0,$$

that is, the eigenvalues of $(dg)_{0,\lambda}$ *cross the imaginary axis with nonzero speed.*

Let $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ *be an isotropy subgroup such that*

$$\dim\ \mathrm{Fix}(\Sigma) = 2.$$

Then there exists a unique branch of small-amplitude periodic solutions to (2.3) with period near 2π *having* Σ *as their group of symmetries.*

Proof See [20] theorem XVI 4.1. □

The basic idea in the equivariant Hopf theorem is that small-amplitude periodic solutions of (2.3) of period near 2π correspond to zeros of a reduced equation $\phi(v, \lambda, \tau) = 0$ where τ is the period-perturbing parameter. To find periodic solutions of (2.3) with symmetries Σ is equivalent to find zeros of the reduced equation with isotropy Σ and they correspond to the zeros of the reduced equation restricted to Fix (Σ).

The main tool for calculating the stabilities of the periodic solutions (including those guaranteed by the equivariant Hopf theorem) presented in [20] (and using the ideas of [12]) is to use a Birkhoff normal form of g: by a suitable coordinate change, up to any given order k, the vector field g can be made to commute with Γ and $\mathbf{S}^1$ (in the Hopf case). This result is the equivariant version of the Poincaré-Birkhoff normal form theorem. Let $\vec{\mathcal{P}}_k(\Gamma)$ be the space of the Γ-equivariant homogeneous polynomial mappings of degree k on $\mathbf{R}^n$ and define the linear map $\mathrm{ad}_L : \vec{\mathcal{P}}_k(\Gamma) \to \vec{\mathcal{P}}_k(\Gamma)$ by

$$\mathrm{ad}_L(P_k)(y) = LP_k(y) - (dP_k)_y Ly.$$

Theorem 2.3.5 *Let g be Γ-equivariant and $L = (dg)_0$. Choose a value of k. Then there exists a Γ-equivariant change of coordinates of degree k such that in the new coordinates the system (2.2) has the form*

$$\dot{y} = Ly + g_2(y) + \cdots + g_k(y) + h, \tag{2.4}$$

where $g_j \in \mathcal{G}_j$, h is of order $k+1$ and

$$\vec{\mathcal{P}}_j(\Gamma) = \mathcal{G}_j \oplus ad_L(\vec{\mathcal{P}}_j(\Gamma)).$$

Proof See [20] theorem XVI 5.8. □

In [12] it is proved that there exists a canonical choice for the complement $\mathcal{G}_j$ in which the elements of $\mathcal{G}_j$ commute with a one-parameter group S of mappings defined in terms of the linear part L of g. In the Hopf case, where the derivative $L = J$, the action of S may be interpreted as the symmetries induced by phase-shifts $\mathbf{S}^1$. That is, it is possible to choose a complement to $\mathrm{ad}_L(\vec{\mathcal{P}}_j(\Gamma))$ where the elements commute with $\mathbf{S}^1$ (besides Γ):

$$\vec{\mathcal{P}}_j(\Gamma) = \vec{\mathcal{P}}_j(\Gamma \times \mathbf{S}^1) \oplus \mathrm{ad}_L(\vec{\mathcal{P}}_j(\Gamma))$$

(see [20] theorem XVI 5.9). Therefore, when we suppose g in (2.3) is in Birkhoff normal form we suppose that it was made this choice in the complements $\mathrm{ad}_L(\vec{\mathcal{P}}_j(\Gamma))$ and so g commutes with $\Gamma \times \mathbf{S}^1$. This hypothesis is especially important when we wish to calculate the stability of the periodic solutions.

Theorem 2.3.6 *Suppose that the vector field g in (2.3) is in Birkhoff normal form. Then it is possible to perform a Liapunov-Schmidt reduction on (2.3) such that the reduced equation ϕ has the form*

$$\phi(v, \lambda, \tau) = g(v, \lambda) - (1 + \tau)Jv$$

where τ is the period-scaling parameter.

Proof See [20] theorem XVI 10.1 or [36, 38]. □

Corollary 2.3.7 *Suppose that the vector field g in (2.3) is in Birkhoff normal form and that $\phi(v, \lambda, \tau)$ is the mapping obtained by using the Liapunov-Schmidt procedure. Let (v_0, λ_0, τ_0) be a solution to $\phi = 0$, and let $v(t)$ be the corresponding periodic solution of (2.3). Then $v(t)$ is orbitally stable if the $n - d_\Sigma$ (where $d_\Sigma = \dim \Gamma + 1 - \dim \Sigma$)eigenvalues of $(d\phi)_{(v_0, \lambda_0, \tau_0)}$ which are not forced to zero by the group action have negative real parts.*

Proof See [20] corollary XVI 10.2. □

Thus the assumption of Birkhoff normal form implies that we can apply the standard Hopf theorem to $\dot{v} = g(v, \lambda)$ restricted to $\text{Fix}(\Sigma) \times \mathbf{R}$. In this case, exchange of stability happens, so that if the trivial steady-state solution is stable subcritically, then a subcritical branch of periodic solutions with isotropy subgroup Σ is unstable. Supercritical branches may be stable or unstable depending on the signs of the eigenvalues of the real parts of the eigenvalues on the complement of $\text{Fix}(\Sigma)$.

Call the system (as in theorem 2.3.5)

$$\dot{y} = Ly + g_2(y) + \cdots + g_k(y)$$

the (kth order) *truncated Birkhoff normal form.*

The dynamics of the truncated Birkhoff normal form are related to, but not identical with, the local dynamics of the system (2.2) around the equilibrium $v = 0$. In fact it is still an unsolved problem to know how much of the dynamics is preserved for the kth order truncated normal form.

On the other hand, in general it is not possible to find a single change of coordinates that puts g into normal form to all orders. And if it is, then there is the problem of the flat 'tail'.

The results of theorem 2.3.6 and corollary 2.3.7 hold when g is in Birkhoff normal form. So when discussing the stability of the solutions found using the equivariant Hopf theorem we suppose that the kth order truncation of g commutes also with $\mathbf{S}^1$ and we use these results. Thus we are ignoring terms of high order that do not commute necessarily with $\mathbf{S}^1$ and that can change the dynamics (and so the stability of these periodic solutions that exist even for the nontruncated system by the equivariant Hopf theorem).

However, in some cases the stability results for the periodic solutions can hold even when g is of the form

$$\tilde{g}(v, \lambda) + o(||v||^k),$$

where $\tilde{g}$ commutes with $\Gamma \times \mathbf{S}^1$ but $o||v||^k$ commutes only with Γ, provided k is large enough. We use $h(v) = o(||v||^k)$ to mean that $\frac{h(v)}{||v||^k} \to 0$ as $||v|| \to 0$.

Suppose that $\dim \mathrm{Fix}(\Sigma) = 2$. Then Σ has *p-determined stability* if all eigenvalues of $(d\tilde{g})_{(v_0,\lambda_0)} - (1+\tau_0)J$, other than those forced to zero by Σ, have the form

$$\mu_j = \alpha_j a^{m_j} + o(a^{m_j})$$

on a periodic solution $v(s)$ of

$$\dot{v} = \tilde{g}(v, \lambda) \tag{2.5}$$

such that $||v(s)|| = a$, where α_j is a $\mathbf{C}$-valued function of the Taylor coefficients of terms of degree lower or equal p in $\tilde{g}$. We expect that the real parts of the α_j to be generically nonzero: these are the nondegeneracy conditions on the Taylor coefficients of $\tilde{g}$ at the origin that are obtained when computing stabilities along the branches. In this case, we say that $\tilde{g}$ is *nondegenerate for* Σ.

Theorem 2.3.8 *Suppose that the hypothesis of theorem 2.3.4 hold, and the isotropy subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ has p-determined stability. Let $k \geq p$ and assume that $g(v,\lambda) = \tilde{g}(v,\lambda) + o(||v||^k)$ where $\tilde{g}$ commutes with $\Gamma \times \mathbf{S}^1$ and is nondegenerate for Σ. Then for λ sufficiently near 0, the stabilities of a periodic solution of $\dot{v} = g(v,\lambda)$ with isotropy Σ are given by the same expressions in the coefficients of g as those that define the stability of a solution of the truncated Birkhoff normal form $\dot{v} = \tilde{g}(v,\lambda)$ with isotropy subgroup Σ.*

Proof See [20] theorem XVI 11.2. □

By theorem 2.3.5 there always exists a polynomial change putting g in the form $\tilde{g}(v, \lambda) + o(||v||^k)$. Thus if the p-determined stability condition holds, theorem 2.3.8 completes the stability analysis for g.

Chapter 3

Hopf bifurcation for wreath products

3.1 Background

We begin by introducing the terminology and results of [10] for wreath products. These will be needed in the following sections.

One way of defining a coupled system of ODEs of N identical cells is the following. Let $X_j \in \mathbf{R}^k$ denote the state variables of the jth cell and suppose that $\dot{X}_j = f(X_j)$ governs the internal dynamics of each cell, where f commutes with $\mathcal{L} \subseteq \mathbf{O}(k)$. Let $X = (X_1, \ldots, X_N) \in (\mathbf{R}^k)^N$ be the state variables for the entire N-cell system. A system of ODEs

$$\frac{dX}{dt} = F(X) \tag{3.1}$$

is a system of coupled cells if

$$F_j(X) = f(X_j) + h_j(X)$$

where h_j governs the coupling between cells.

We say that (3.1) has symmetry the *wreath product* $\mathcal{L} \wr \mathcal{G}$, where $\mathcal{G}$ is a subgroup of the permutation group $\mathbf{S}_N$, if the symmetry group that commutes with F is the group generated by the groups $\mathcal{L}^N$ and $\mathcal{G}$. This is, for example, the case when h_j is invariant under $\mathcal{L}^{N-1}$ acting on $(X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_N)$, and equivariant under $\mathcal{L}$ acting on X_j.

If F commutes not with $\mathcal{L}^N$ but instead with the diagonal subgroup of $\mathcal{L}^N$, that is, the element $l \in \mathcal{L}$ is a symmetry of (3.1) only if it acts simultaneously on each cell, then the total group of symmetries is the *direct product* $\mathcal{L} \times \mathcal{G}$. This case is analysed in [11].

In our work we concentrate on the wreath product case, studied in [10].

As in [10] we may assume that

(H_T) $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$.

This assumption does not lead to a loss of generality, because if the action of $\mathcal{G}$ is intransitive, we can consider the group orbits of cells under $\mathcal{G}$, which reduces the discussion to a finite list of cases in each of which the condition (H_T) holds.

3.1.1 Group structure of the wreath product

Let $V = \mathbf{R}^k$ and let V^N be the state space of the system (3.1). The action of $\mathcal{L} \wr \mathcal{G}$ on V^N is given by

$$(l, \sigma).(X_1, \ldots, X_N) = (l_1 X_{\sigma^{-1}(1)}, \ldots, l_N X_{\sigma^{-1}(N)})$$

where $l = (l_1, \ldots, l_N) \in \mathcal{L}^N$, $\sigma \in \mathcal{G}$ and $(X_1, \ldots, X_N) \in V^N$.

The permutations act on $l \in \mathcal{L}^N$ by

$$\sigma(l) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(N)})$$

and it follows that the group multiplication in $\mathcal{L} \wr \mathcal{G}$ is given by

$$(h, \tau)(l, \sigma) = (h\tau(l), \tau\sigma).$$

For general information on wreath products see for example [33] pg 215.

3.1.2 The linear theory

Throughout, let $\Gamma = \mathcal{L} \wr \mathcal{G}$. By [20] (proposition 2.3.2), when considering Hopf bifurcation we may assume the generic hypothesis

(H_H) Γ acts Γ-simply on the centre subspace.

It is important to understand how Γ decomposes the state space into irreducible subspaces. We are interested in studying bifurcations with combined local and global symmetries. Let $W \subseteq V^N$ be a Γ-irreducible subspace. This subspace is invariant under $\mathcal{L}^N$. If $\mathcal{L}^N$ acts trivially on W, then the local symmetries will have no effect on a bifurcation supported by this representation and we end up with a bifurcation problem with symmetry $\mathcal{G}$ which is

not the aim of this work. Therefore, as in [10], it is assumed
$(H_{\mathcal{L}})$ $\mathcal{L}^N$ acts nontrivially on W.

It is proved in [10] that Γ acts absolutely irreducible on V^N if and only if $\mathcal{L}$ acts absolutely irreducibly on V. For Hopf bifurcation points, the centre subspace is generically Γ-simple. That is, it has the form $W \oplus W$ where W is absolutely Γ-irreducible or is a nonabsolutely Γ-irreducible subspace. Assuming (H_T) and $(H_{\mathcal{L}})$, lemmas 3.2 and 3.1 of [10] imply that the centre subspace is either $(U \oplus U)^N$ where U is absolutely $\mathcal{L}$-irreducible, or it is U^N where U is nonabsolutely $\mathcal{L}$-irreducible. That is, U is $\mathcal{L}$-simple. See [10] (end of section 3).

3.1.3 C-axial subgroups

As mentioned before, we say that a subgroup $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-*axial* if it is an isotropy subgroup having a two-dimensional fixed-point subspace (over $\mathbf{R}$).

In the classification of $\mathbf{C}$-axial subgroups of wreath products groups $\mathcal{L} \wr \mathcal{G}$, the structure of these subgroups is determined by the possible blocks that are derived from the permutation group $\mathcal{G}$. We define: a subset of indices $J \subseteq \{1, \ldots, N\}$ is a *block* if there exists a subgroup $\mathcal{H}$ of $\mathcal{G}$ that leaves J invariant and acts transitively on J.

To each block J we can associate the permutation subgroup

$$Q_J = \{\sigma \in \mathcal{G} : \sigma(J) = J\}$$

which acts transitively on J.

Assume (H_H) and consider the natural action of $\mathbf{S}^1$ on the centre subspace obtained by giving a complex structure to this space as in [20]. Consider a block J and let B^ψ be a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$ (acting on V) where $\psi : B \to \mathbf{S}^1$ is an homomorphism and

$$B^\psi = \{(b, \psi(b)) : b \in B\}.$$

Following [20] we call the group B^ψ a *twisted* subgroup of $\mathcal{L} \times \mathbf{S}^1$. The image $\psi(B)$ is a closed subgroup of $\mathbf{S}^1$. The closed subgroups of $\mathbf{S}^1$ are $\mathbf{1}$, $\mathbf{Z}_n$ $(n = 2, 3, 4, \ldots)$ and $\mathbf{S}^1$. We say that B^ψ is of *finite* twist type if the image $\psi(B)$ is not $\mathbf{S}^1$.

Consider

$$\Sigma(B^\psi, J) = (\mathbf{1}^N, Q_J, 0) + ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) + ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$$

where + indicates 'group generated by' as in [10]. Here it is assumed that $J = \{1, \ldots, s\}$ and the subgroup $\hat{B}$ is defined by

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

It is proved in [10] that $\Sigma(B^\psi, J)$ is a $\mathbf{C}$-axial subgroup of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$. It is also claimed that every $\mathbf{C}$-axial subgroup is of this type. However, we show in the next section that *not* all the $\mathbf{C}$-axial subgroups of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ are of this type.

3.2 Classification of C-axial subgroups

Our aim in this section is to give a complete description up to conjugacy of all the $\mathbf{C}$-axial subgroups of the groups of the type $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$. We begin by describing subgroups that we denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$. We will show that these subgroups are $\mathbf{C}$-axial and that any $\mathbf{C}$-axial subgroup is conjugate to one of these.

As in the previous section we assume the generic hypothesis (H_H), so that we can write the centre subspace as V^N, where either $\mathcal{L}$ acts nonabsolutely irreducibly on V, or $V = U \oplus U$ and $\mathcal{L}$ acts absolutely irreducibly on U. That is, the space V is $\mathcal{L}$-simple. If $\mathcal{L}$ acts trivially on V, then the action of Γ on V^N is reduced to the action of $\mathcal{G}$ on V^N and in this case, Hopf bifurcation with symmetry Γ is reduced to that with symmetry $\mathcal{G}$. We are therefore interested in the cases where $\mathcal{L}$ does not act trivially on V. We note that for these actions, $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$.

From now on consider $\Gamma = \mathcal{L} \wr \mathcal{G}$ where we are considering the action of $\mathcal{L} \wr \mathcal{G}$ on the Γ-simple space V^N is as defined in the previous section.

The group $\Sigma(B^\psi, J, \sigma, J_1, p)$

Consider a block $J \subseteq \{1, \ldots, N\}$ and let Q_J be the subgroup of $\mathcal{G}$ that leaves J invariant. Suppose

$$J = \{1, \ldots, s\}.$$

Let J_1 be a subset of J such that for some permutation $\sigma \in Q_J$

$$J = J_1 \,\dot\cup\, \sigma(J_1) \,\dot\cup\, \cdots \,\dot\cup\, \sigma^{s'-1}(J_1)$$

where $\dot{\cup}$ is disjoint union and

$$\sigma^{s'}(J_1) = J_1.$$

In particular it follows that $|J| = s'|J_1|$.

Choose notation so that

$$J_{i+1} = \sigma^i(J_1), \quad i = 1, \ldots, s' - 1$$

and let

$$Q_{J,J_1} = \{\tau \in Q_J : \tau(J_i) = J_i, \ i = 1, \ldots, s'\}.$$

Suppose Q_{J,J_1} acts transitively on J_1. This implies that Q_{J,J_1} acts transitively on all J_i.

Note that by definition of block the group Q_J acts transitively on J. Therefore $\sigma = 1$ and $J_1 = J$ are under those conditions.

Define

$$Q_{J,J_k} = \{\tau \in Q_J : \tau(J_j) = \sigma^{k-1}(J_j), \ j = 1, \ldots, s'\},$$

for $k = 2, \ldots, s'$. That is, each permutation in Q_{J,J_k} interchanges the subsets J_i of J in the same way as σ^{k-1}.

Let B^ψ be a $\mathbf{C}$-axial subgroup of $\mathcal{L}\times\mathbf{S}^1$ of finite twist type $\mathbf{Z}_r$, and let $\hat{B}$ be the subgroup of B^s defined by

$$\hat{B} = \{(b_1, \cdots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Let $\mathbf{Z}_p =< \xi_p >$ be a cyclic subgroup of $\mathbf{S}^1$ such that

$$s' = \min_{i>0} \{\xi_p^i \in \mathbf{Z}_r\}.$$

Call $\xi_{r'} = \xi_p^{s'}$. It follows that $\mathbf{Z}_p = \mathbf{Z}_{s'r'}$ where $\mathbf{Z}_{r'} \subseteq \mathbf{Z}_r$.

Define B_k the subgroup of B^s by

$$B_k = \left\{ \quad (b_1, \ldots, b_s) \in B^s : \psi(b_j) = \quad \begin{cases} \xi_{r'}, & \text{if } j \in J_1 \cup \cdots \cup J_{k-1}, \\ 0, & \text{if } j \in J_k \cup \cdots \cup J_{s'} \end{cases} \quad \right\}$$

Finally denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$ the subgroup of $\Gamma \times \mathbf{S}^1$ generated by the following groups:

$$\Sigma(B^\psi, J, \sigma, J_1, p) = ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) \ + \ ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \ +$$

$$+ \ (\mathbf{1}^N, Q_{J,J_1}, 0) \ + \bigcup_{k=2,\ldots,s'} ((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1}).$$

Note that this group depends on the block J, the permutation σ (and so on J_1). Also the group Q_{J,J_1} has to act transitively on J_1. Finally, it depends on B^ψ (a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$) and on the cyclic subgroup $\mathbf{Z}_p$ of $\mathbf{S}^1$ (where some divisor r' of r divides p).

We state our main theorem:

Theorem 3.2.1 *An isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-axial if and only if it is conjugate to a ($\mathbf{C}$-axial) group of the type $\Sigma(B^\psi, J, \sigma, J_1, p)$, for some $\mathbf{C}$-axial group B^ψ of $\mathcal{L} \times \mathbf{S}^1$, a block J, a permutation σ of $\mathcal{G}$, a subset J_1 of J, and a nonnegative integer p.*

The rest of this section is dedicated to the proof of this theorem. First we show in proposition 3.2.2 that the groups $\Sigma(B^\psi, J, \sigma, J_1, p)$ defined above are $\mathbf{C}$-axial. Basically, using algebraic calculations, we are able to describe the fixed-point subspaces of these groups. Then, in proposition 3.2.7, we show that every $\mathbf{C}$-axial Σ of $\Gamma \times \mathbf{S}^1$ is conjugate to some group of this type. For that we need to prove first two lemmas. In lemma 3.2.4 we prove that once we choose an element w fixed by Σ, the nonzero components have indices corresponding to a block and the projection of Σ on the group $\mathcal{G}$ is a permutation group acting transitively on that block. In lemma 3.2.5 we show that if we choose a nonzero component of w, then the corresponding isotropy subgroup (now of $\mathcal{L} \times \mathbf{S}^1$) is also $\mathbf{C}$-axial. Finally, in proposition 3.2.7, using these two lemmas, we manipulate the vector w and conclude that, up to conjugacy, we can assume that w (a representative point for the isotropy subgroup Σ) belongs to the fixed-point subspace of one of those groups $\Sigma(B^\psi, J, \sigma, J_1, p)$.

Proposition 3.2.2 *With the above notation $\Sigma(B^\psi, J, \sigma, J_1, p)$ is a $\mathbf{C}$-axial subgroup of $\Gamma \times \mathbf{S}^1$.*

Proof. Let $\Sigma = \Sigma(B^\psi, J, \sigma, J_1, p)$ and $w = (w_1, \ldots, w_N) \in V^N$ be fixed by Σ. Since $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ fixes w and $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$, we have $w_{s+1} = \cdots = w_N = 0$. Thus

$$w = (w_1, \ldots, w_s, 0, \ldots, 0).$$

Suppose

$$J_1 = \{1, \ldots, \frac{s}{s'}\}, \ J_2 = \{\frac{s}{s'} + 1, \ldots, 2\frac{s}{s'}\}, \ \ldots, \ J_{s'} = \{(s' - 1)\frac{s}{s'} + 1, \ldots, s\}.$$

Since $(\mathbf{1}^N, Q_{J,J_1}, 0)$ fixes w and Q_{J,J_1} is transitive on each part J_i, the components w_i corresponding to each J_i are equal. Denote by $< w_i >$ the vector with s/s' components equal to w_i and $\xi = \xi_p$. Since $((B_k, \mathbf{1}^{N-s}), \sigma^{k-1}, \xi^{k-1})$ for $k = 2, \ldots, s'$ fixes w, it follows that

$$w = (< w_1 >, < \xi.w_1 >, \ldots, < \xi^{s'-1} \cdot w_1 >, 0, \ldots, 0).$$

Since w is fixed by $((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$ we have $w_1 \in \mathrm{Fix}_V(\mathrm{B}^\psi)$. As B^ψ is $\mathbf{C}$-axial, we see that $\mathrm{Fix}_{V^N}(\Sigma)$ is two-dimensional.

To complete the proof we show that Σ is the isotropy subgroup of w. Let Σ_w be the isotropy subgroup of

$$w = (< w_1 >, < \xi \cdot w_1 >, \ldots, < \xi^{s'-1} \cdot w_1 >, 0, \ldots, 0).$$

From the previous discussion and straightforward calculations $\Sigma \subseteq \Sigma_w$. To verify the reverse inclusion we show that if $(l, \tau, \theta) \in \Sigma_w$ then $(l, \tau, \theta) \in \Sigma$. As $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ fixes w we have that $((l_1, \ldots, l_s, 1, \ldots, 1), \tau, \theta)$ fixes w.

If $\tau \in Q_{J,J_1}$, as $(\mathbf{1}^N, Q_{J,J_1}, 0)$ fixes w, then $\gamma = ((l_1, \ldots, l_s, 1, \ldots, 1), 1, \theta)$ fixes w. But for $\gamma \cdot w = w$, then $(l_j, \theta) \in \mathcal{L} \times \mathbf{S}^1$ fixes w_1. Since B^ψ is the isotropy subgroup of w_1, it follows that $(l_j, \theta) \in B^\psi$ and $\theta = \psi(l_j)$. Thus $\gamma \in ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$ and $(l, \tau, \theta) \in \Sigma$.

If $\theta \in \mathbf{Z}_r$, then it follows immediately that $\tau \in Q_{J,J_1}$ and we have the previous case.

If $\theta \notin \mathbf{Z}_r$ and as $((l_1, \ldots, l_s, 1, \ldots, 1), \tau, \theta)$ fixes w, then $\theta = \xi^i \xi_r^j$ for some $i < s'$ and $j \in \{0, \ldots, r-1\}$. As we have $((b, 1, \ldots, 1), 1, \xi_r^{-j}) \in ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$ for some $b \in \hat{B}$, then

$$\gamma = ((b_1 l_1, \ldots, b_s l_s, 1, \ldots, 1), \tau, \xi^i)$$

fixes w. Suppose that $J_1 = \tau^{-1}(J_{j+1})$. From $\gamma \cdot w = w$ and since $i < s'$, then $j = i$ and $\tau \in Q_{J,J_{i+1}}$. There is $\gamma' = (((b')^{-1}, 1, \ldots, 1), \tau, \xi^i) \in ((B_{i+1}, \mathbf{1}^{N-s}), Q_{J,J_{i+1}}, \xi^i)$. Moreover, we can choose $b' \in B_{i+1}$ such that $b'_j = 1$ for $j \in J_{i+1} \cup \ldots \cup J_{s'}$. Take $(\gamma')^{-1} = ((\tau^{-1}(b'), 1^{N-s}), \tau^{-1}, \xi^{-i})$. Now

$$(\gamma')^{-1}\gamma = ((\tau^{-1}(b'bl), 1^{N-s}), 1, 0)$$

fixes w. It follows that $\tau^{-1}(b'bl) \in \hat{B}$ with $\psi((b'bl)_{\tau(i)}) = 0$ and $l \in B^s$ since b and b' are in B^s. Therefore $\psi(l_l) = \xi_r^j \xi_{r'}$ if $l \in J_1 \cup \ldots \cup J_i$ and $\psi(l_l) = \xi_r^j$ if $l \in J_{i+1} \cup \ldots \cup J_{s'}$. It follows that $\gamma \in ((B_{i+1}, \mathbf{1}^{N-s}), Q_{J,J_{i+1}}, \xi^i)$

and $((l_1,\ldots,l_s,1,\ldots,1),\tau,\theta)\in((B_{i+1},\mathbf{1}^{N-s}),Q_{J,J_{i+1}},\xi^i)+((\hat{B},\mathbf{1}^{N-s}),\mathbf{1},\psi)$.
□

Some of the groups $\Sigma(B^\psi,J,\sigma,J_1,p)$ are the same as the groups $\Sigma(B^\psi,J)$ found by Dionne *et al.* [10]. Using the notation of the previous section we have:

Corollary 3.2.3 *With the conditions of proposition 3.2.2, and if*

$$\mathbf{Z}_p\subseteq\mathbf{Z}_r,$$

then $\Sigma(B^\psi,J,\sigma,J_1,p)$ is conjugate to $\Sigma(B^\psi,J)$ and is of the same twist type as B^ψ.

Proof. Note that if $\mathbf{Z}_p\subseteq\mathbf{Z}_r$, then Σ is conjugate to $\Sigma_{(w_1,\ldots,w_1,0,\ldots,0)}$ since any element $(l_i,\theta_i)\cdot w_1$ in V (with $w_1\in\mathrm{Fix}_V(\mathrm{B}^\psi)$) is in the $\mathcal{L}$-orbit of w_1. □

However, if $\mathbf{Z}_p\not\subseteq\mathbf{Z}_r$, then there are new possibilities. We show that all $\mathbf{C}$-axial subgroups of finite twist type of $\Gamma\times\mathbf{S}^1$ are conjugate to subgroups of the form $\Sigma(B^\psi,J,\sigma,J_1,p)$.

Let $\Pi_{\mathcal{G}}:\Gamma\times\mathbf{S}^1\to\mathcal{G}$ be projection and let

$$V_J=\{(w_1,\ldots,w_N)\in V^N: w_j=0\text{ for }j\notin J\}.$$

Lemma 3.2.4 *Let Σ be a $\mathbf{C}$-axial subgroup of $\Gamma\times\mathbf{S}^1$. Then $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on some block J, and $\mathrm{Fix}_{V^N}(\Sigma)\subseteq V_J$.*

Proof. Let w be a nonzero vector of $\mathrm{Fix}_{V^N}(\Sigma)$ and let J be the set of the indices $j\in\{1,\ldots,N\}$ such that $w_j\neq0$. We will show that $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Since $(l_j,\theta).w_{\sigma^{-1}(j)}=0$ if and only if $w_{\sigma^{-1}(j)}=0$, we have $\Pi_{\mathcal{G}}(\Sigma)J\subseteq J$. Suppose that there exist two disjoint subsets J_1 and J_2 of J such that $\Pi_{\mathcal{G}}(\Sigma)J_i\subseteq J_i$ for $i=1,2$. If so, construct two vectors $y_1,\ y_2\in V^N$, where

$$y_1^j=\begin{cases}w_j & \text{if } j\in J_1,\\ 0 & \text{if } j\notin J_1\end{cases}\qquad y_2^j=\begin{cases}w_j & \text{if } j\in J_2,\\ 0 & \text{if } j\notin J_2.\end{cases}$$

These vectors in $\mathrm{Fix}_{V^N}(\Sigma)$ are linearly independent. Moreover, we can split

$$\mathrm{Fix}_{V^N}(\Sigma)=\mathrm{V}_1\oplus\mathrm{V}_2,$$

if we take

$$V_i = \{y_i \in \mathrm{Fix}_{V^N}(\Sigma) : \mathrm{y}_i^j = 0 \text{ if } j \notin J_i\}$$

for $i = 1, 2$. Therefore

$$\Sigma_w \subset \Sigma_{y_1} \subset \Gamma \times \mathbf{S}^1$$

and $\Sigma_w = \Sigma$ is not maximal. This contradicts the fact that Σ is $\mathbf{C}$-axial, so $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J, and J is a block. $\square$

The projection $\Pi_{\mathcal{G}}(\Sigma)$ as a subgroup of $\mathcal{G}$, must decompose the set of indices $\{1, \ldots, N\}$ into a union of blocks. Using the above lemma, we get that if Σ is $\mathbf{C}$-axial, then a vector w fixed by Σ is supported on precisely one of these blocks. That is, only the components corresponding to one of the blocks are nonzero and $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on that block. This restriction implies information on the vectors w fixed by Σ.

We know that all proper isotropy subgroups of $\Gamma \times \mathbf{S}^1$ are twisted subgroups (see for example [20]). Because $\Gamma \times \mathbf{S}^1$ acts Γ-simply on V^N, also $\mathcal{L} \times \mathbf{S}^1$ acts $\mathcal{L}$-simple on V and so if $\Sigma_{w_1} \subset \mathcal{L} \times \mathbf{S}^1$ is the isotropy subgroup of $w_1 \in V$, then it is a twisted subgroup of $\mathcal{L} \times \mathbf{S}^1$.

Lemma 3.2.5 *Let $\Sigma = H^\theta \subset \Gamma \times \mathbf{S}^1$ be a twisted $\mathbf{C}$-axial subgroup. Let w be a nonzero vector fixed by Σ with $w_1 \neq 0$. Let $\Sigma_{w_1} = B^\psi$ be the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$. Then Σ_{w_1} is $\mathbf{C}$-axial and $\psi(B) \subseteq \theta(H)$.*

Proof. For w a nonzero vector of $\mathrm{Fix}_{V^N}(\Sigma)$, let J be the set of indices $j \in \{1, \ldots, N\}$ such that $w_j \neq 0$. As we are assuming $w_1 \neq 0$ we have $1 \in J$. By lemma 3.2.4 we know that $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Therefore for all $i \in J \setminus \{1\}$ there exists a permutation $\sigma_i \in \Pi_{\mathcal{G}}(\Sigma)$ such that $\sigma_i^{-1}(i) = 1$. Moreover, since

$$(l^i, \sigma_i, \theta_i) \cdot w = w$$

for some (l^i, θ_i), it follows that

$$w_i = (l_i^i, \theta_i) \cdot w_1.$$

Therefore the vector w has the form

$$w = (w_1, (l_2^2, \theta_2) \cdot w_1, \ldots, (l_s^s, \theta_s) \cdot w_1, 0, \ldots, 0)$$

and Σ is conjugate to the isotropy subgroup of

$$w' = (w_1, \theta_2 \cdot w_1, \ldots, \theta_s \cdot w_1, 0, \ldots, 0).$$

Let $\Sigma = \Sigma_{w'}$ and define

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Note that $\hat{B}$ is a subgroup of B^s because ψ is a group homomorphism. Now

$$((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \subset \Sigma$$

and so $\psi(B) \subseteq \theta(H)$.

It remains to show that B^ψ is $\mathbf{C}$-axial. If B^ψ fixes an element w_2 that is not a multiple of w_1, then Σ fixes $w = (w_2, \theta_2 \cdot w_2, \ldots, \theta_s \cdot w_2)$ and then Σ is not $\mathbf{C}$-axial, a contradiction. $\Box$

Remark 3.2.6 *Let $\Sigma_{w_1} = B^\psi \subset \mathcal{L} \times \mathbf{S}^1$ be the isotropy subgroup of w_1, and let w be a nonzero vector fixed by $\Sigma = H^\theta \subset \Gamma \times \mathbf{S}^1$ as in lemma 3.2.5. If Σ_{w_1} is of twist type $\mathbf{S}^1$, then Σ also has twist type $\mathbf{S}^1$. In this case Σ is conjugate to $\Sigma(B^\psi, J)$. Moreover, we shall see later that if Σ is of twist type $\mathbf{S}^1$, then Σ_{w_1} is also of twist type $\mathbf{S}^1$.*

We prove now our main result. Using lemmas 3.2.4 and 3.2.5, we show that all $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$ are conjugate to groups of the form $\Sigma(B^\psi, J, \sigma, J_1, p)$.

Proposition 3.2.7 *Let $\Sigma = H^\theta \subset \Gamma \times \mathbf{S}^1$ be a twisted $\mathbf{C}$-axial subgroup of finite twist type. If $\Sigma = \Sigma_w$ with $w_1 \neq 0$, assume that the isotropy subgroup $B^\psi = \Sigma_{w_1}$ of w_1 is of twist type $\mathbf{Z}_r$.*

Then Σ is conjugate to $\Sigma(B^\psi, J, \sigma, J_1, p)$ for a block J, a permutation $\sigma \in \Pi_{\mathcal{G}}(\Sigma)$, a subset J_1 of J, and a nonnegative integer p.

Proof. Let $w \neq 0$ be a vector fixed by Σ. We know that $\Pi_{\mathcal{G}}(\Sigma)$ decomposes $\{1, \ldots, N\}$ into a union of blocks. From lemma 3.2.4, since Σ is $\mathbf{C}$-axial, w is supported on precisely one of these blocks J. To simplify notation, assume that the block $J = \{1, \ldots, s\}$ where $s \leq N$, and let

$$w = (w_1, \ldots, w_s, 0 \ldots, 0).$$

Then the group $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0)$ fixes w.

Construct a partition $J = J_1 \cup \cdots \cup J_q$ of the block J, by putting two indices l and m in the same part if w_l and w_m lie on the same $\mathcal{L}$-orbit. Conjugate w so that all w_i in the same part J_i are equal.

If all the components lie on the same $\mathcal{L}$-orbit, then Σ is conjugate to

$$\Sigma_{(w_1,\ldots,w_1,0,\ldots,0)}$$

which is of the type described in [10], that is, the group Σ is conjugate to $\Sigma(B^{\psi}, J)$. In our notation it is $\Sigma(B^{\psi}, J, 1, J, r)$.

Suppose now that $J = J_1 \cup \ldots \cup J_q$ with $q > 1$. From lemma 3.2.4, the group $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Suppose without loss of generality that $1 \in J_1$, and choose $i_2 \in J_2$. Then there exists $(l^2, \sigma_2, \theta_2) \in H^{\theta}$ with $\theta_2 \neq 0$ such that $\sigma_2(1) = i_2$. So

$$w_{i_2} = (l^2_{i_2}, \theta_2).w_1.$$

Similarly, for $j = 3, \ldots, q$, we can choose $i_j \in J_j$ and find $(l^j, \sigma_j, \theta_j) \in H^{\theta}$ with $\theta_j \neq 0$ such that $\sigma_j(1) = i_j$. Now suppose that $\theta(H) = \mathbf{Z}_l$. By lemma 3.2.5 we have $\mathbf{Z}_r \subseteq \mathbf{Z}_l$. So $\theta_2, \ldots, \theta_q \in \mathbf{Z}_l \setminus \mathbf{Z}_r$. Note that if, for example, $\theta_2 \in \mathbf{Z}_r$, then w_{i_2} would belong to the same $\mathcal{L}$-orbit as w_1: since $\theta_2 = \psi(b_2)$ for some $b_2 \in B$, then

$$w_{i_2} = (l^2_{i_2}, \theta_2) \cdot w_1 = (l^2_{i_2} b_2^{-1}, 0)(b_2, \theta_2) \cdot w_1 = l^2_{i_2} b_2^{-1} \cdot w_1.$$

Moreover, if $l = kr$ for some positive integer k, then up to conjugacy we can always choose

$$\theta_i \in \left\{ \frac{2\pi}{l}, \cdots, (k-1)\frac{2\pi}{l} \right\}.$$

Associated with each choice of the θ_i we have a permutation $\sigma_i \in \Pi_{\mathcal{G}}(\Sigma)$ and an element $\theta_i = \theta(l^i, \sigma_i)$ for some $(l^i, \sigma_i) \in H$. In fact $\sigma_i^{-1}(J_i) = J_1$ for $i = 2, \ldots, q$, so all the $J_i s$ have the same size t where $tq = s$.

For simplicity, take $J_1 = \{1, \ldots, t\}, \ldots, J_q = \{(q-1)t + 1, \ldots, s\}$. Conjugate w to have the form

$$w = (< w_1 >, < \theta_2 \cdot w_1 >, \ldots, < \theta_q \cdot w_1 >, 0, \ldots, 0).$$

We know from lemma 3.2.5 that

$$((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \subseteq \Sigma.$$

Since $\theta_2, \ldots, \theta_q \in \mathbf{Z}_l$, the subgroup $< \theta_2, \ldots, \theta_q >$ generated by these elements is a cyclic group, say $\mathbf{Z}_p$, and there is a generator $\xi_p \in \mathbf{Z}_l \setminus \mathbf{Z}_r$. For some (l, σ) we have $(l, \sigma, \xi_p) \in \Sigma$, so that $(l, \sigma, \xi_p)^i \in \Sigma$ for all i.

Let

$$s' = \min_{i>0} \{\xi_p^i \in \mathbf{Z}_r\}.$$

Consider J_1. Then exists J_2, such that $J_1 \cap J_2 = \emptyset$ and $\sigma(J_1) = J_2$. The reason is that $(l, \sigma, \xi_p) \in \Sigma$. Also $w_i = (l_i, \xi_p) \cdot w_1$ for all $i \in J_2$. Since $(l, \sigma, \xi_p)^2 = (l\sigma(l), \sigma^2, \xi_p^2) \in \Sigma$, there exists J_3 such that $\sigma^2(J_1) = J_3$ and $J_3 \cap (J_1 \cup J_2) = \emptyset$. Again

$$w_i = (l_i\sigma(l_i), \xi_p^2) \cdot w_1$$

for $i \in J_3$. We can do the same for each $i \leq s^{'}$, so we eventually have

$$J_1 \cup \cdots \cup J_{s'} \subseteq J.$$

Suppose that there is another sub-block $J_{s'+1}$. Then for $i \in J_{s'+1}$ we have w_i in the $\mathcal{L}$-orbit of one of the sub-blocks J_i for $i \leq s^{'}$, which is not the case. For simplicity, we take $J_2, \ldots, J_{s'}$ as before. Therefore we can take

$$w = (< w_1 >, < \xi_p \cdot w_1 >, \ldots, < \xi_p^{s^{'}-1} \cdot w_1 >, 0, \ldots, 0).$$

Now define

$$Q_{J,J_k} = \{\tau \in \Pi_{\mathcal{G}}(\Sigma) : \tau(J_j) = \sigma^{k-1}(J_j),\ j = 1, \ldots, s^{'}\}$$

for $k = 1, \ldots, s^{'}$ (where $\sigma^0 = 1$) and take B_k as defined before proposition 3.2.2. Then $((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1}) \subseteq \Sigma$ and $((\hat{B}, \mathbf{1}^{N-s}), Q_{J,J_1}, \psi) \subseteq \Sigma$.

Let $(l, \tau, \theta) \in \Sigma$. Since $((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) \subseteq \Sigma$, then

$$\gamma = ((l_1, \ldots, l_s, 1^{N-s}), \tau, \theta) \in \Sigma.$$

If $\theta \in \mathbf{Z}_r$, then from $\gamma \cdot w = w$ we must have $\tau \in Q_{J,J_1}$ and so $\theta = \psi(l_1) = \cdots = \psi(l_s)$. Therefore $\gamma \in ((\hat{B}, \mathbf{1}^{N-s}), Q_{J,J_1}, \psi) \subseteq \Sigma$.

If $\theta \notin \mathbf{Z}_r$, then from $\gamma \cdot w = w$, we have $\theta = \xi_p^i \xi_r^j$ for some $i < s^{'}$ and some $j \in \mathbf{Z}_0^+$. As $((b, 1, \ldots, 1), 1, \xi_r^{-j}) \in ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$ for some $b \in \hat{B}$, it follows that $\gamma^{'} = ((b_1 l_1, \ldots, b_s l_s, 1, \ldots, 1), \tau, \xi_p^i)$ fixes w. Now we use the end of the proof of proposition 3.2.2 and conclude that $\tau \in Q_{J,J_{i+1}}$ and $\gamma \in ((B_{i+1}, \mathbf{1}^{N-s}), Q_{J,J_{i+1}}, \xi_p^i) + ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$. □

Corollary 3.2.8 *Let* $H^\theta = \Sigma \subset \Gamma \times \mathbf{S}^1$ *be a* $\mathbf{C}$*-axial subgroup of twist type* $\mathbf{S}^1$*. If* $\Sigma = \Sigma_w$ *with* $w_1 \neq 0$*, let* Σ_{w_1} *be the isotropy subgroup of* w_1 *in* $\mathcal{L} \times \mathbf{S}^1$*. Then* Σ_{w_1} *is* $\mathbf{C}$*-axial of twist type* $\mathbf{S}^1$*.*

Proof. From lemma 3.2.5 the group Σ_{w_1} is $\mathbf{C}$-axial. Suppose Σ_{w_1} is of finite twist type, say $\mathbf{Z}_r$. As in the proof of proposition 3.2.7 we can conjugate w to

$$w = (< w_1 >, < \theta_2 \cdot w_1 >, \ldots, < \theta_q \cdot w_1 >, 0, \ldots, 0)$$

for some $\theta_2, \ldots, \theta_q \in \theta(H) \setminus \mathbf{Z}_r$. Moreover $< \theta_2, \ldots, \theta_q >$ is a finite subgroup of $\theta(H) = \mathbf{S}^1$. Let $(l, \tau, \theta) \in \Sigma_w$. If $\theta \in \theta(H) \setminus \mathbf{Z}_r$, then from $(l, \tau, \theta) \cdot w = w$, it follows that $\theta\theta_i\theta_j^{-1} \in \mathbf{Z}_r$ for some i and j in $\{2, \ldots, q\}$, and so $\theta = \xi\xi'$ for some $\xi \in < \theta_2, \ldots, \theta_q >$ and $\xi' \in \mathbf{Z}_r$. But $\theta(H) = \mathbf{S}^1$. So Σ_{w_1} has to be of twist type $\mathbf{S}^1$. $\square$

Corollary 3.2.9 *Let $\Sigma_w \subset \Gamma \times \mathbf{S}^1$ where $w_1 \neq 0$, and let Σ_{w_1}, the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$, be $\mathbf{C}$-axial. Then Σ is of twist type $\mathbf{S}^1$ if and only if Σ_{w_1} is of twist type $\mathbf{S}^1$.*

Proof. This follows from the remark 3.2.6 and from the corollary 3.2.8. $\square$

Corollary 3.2.10 *Let $\Sigma_w \subset \Gamma \times \mathbf{S}^1$ be $\mathbf{C}$-axial and let $B^\psi = \Sigma_{w_1}$ where $w_1 \neq 0$. Then Σ_w is conjugate to $\Sigma(B^\psi, J)$ for some block J if and only if Σ_w and Σ_{w_1} are of the same twist type.*

Proof. Let $\Sigma_w = H^\theta$ and Σ_{w_1} have the same twist type. By lemma 3.2.5 we have $\theta(H) = \psi(B)$ and by lemma 3.2.4 the subgroup $\Pi_{\mathcal{G}}(\Sigma)$ of $\mathcal{G}$ acts transitively on some block J. Suppose $J = \{1, \ldots, s\}$. Up to conjugacy we need only consider the isotropy subgroups of elements of the form

$$w = (< w_1 >, 0, \ldots, 0),$$

where the vector $< w_1 >$ has s components equal to w_1, since $\theta(H) = \psi(B)$. Note that any element $(l_i, \theta_i) \cdot w_1 \in V$ (with $\theta_i \in \theta(H)$ and $w_1 \in \mathrm{Fix}_V(\mathrm{B}^\psi)$) is in the $\mathcal{L}$-orbit of w_1. Therefore Σ is conjugate to $\Sigma(B^\psi, J)$. Now, if $H^\theta = \Sigma$ is conjugate to $\Sigma(B^\psi, J)$ for some block J, then $\theta(H) = \psi(B)$ from the structure of $\Sigma(B^\psi, J)$. $\square$

3.3 Maximal isotropy subgroups

We describe now all the maximal isotropy subgroups of the group $\Gamma \times \mathbf{S}^1$ where, as usual, $\Gamma = \mathcal{L} \wr \mathcal{G}$. Generalizing the classification for the $\mathbf{C}$-axial subgroups, we prove that these groups are conjugate to subgroups of the form $\Sigma(B^\psi, J, \sigma, J_1, p)$ where now B^ψ is any maximal isotropy subgroup of $\mathcal{L} \times \mathbf{S}^1$. As before denote by $\Pi_{\mathcal{G}}(\Sigma)$ the projection of Σ on the permutation group $\mathcal{G}$ and for a block J let

$$V_J = \{(w_1, \ldots, w_N) \in V^N : w_j = 0 \text{ for } j \notin J\}.$$

Lemma 3.3.1 *Let Σ be a maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Then $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on some block J and $\mathrm{Fix}_{V^N}(\Sigma) \subseteq V_J$.*

Proof. Let w be a nonzero vector of $\mathrm{Fix}_{V^N}(\Sigma)$ and let J be the set of indices $j \in \{1, \ldots, N\}$ such that $w_j \neq 0$. We show that $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J. Since $(l_j, \theta).w_{\sigma^{-1}(j)} = 0$ if and only if $w_{\sigma^{-1}(j)} = 0$, we have $\Pi_{\mathcal{G}}(\Sigma)J \subseteq J$. Suppose that there are two disjoint subsets J_1 and J_2 of J such that $\Pi_{\mathcal{G}}(\Sigma)J_i \subseteq J_i$ for $i = 1, 2$. Let

$$V_1 = \{y_1 \in \mathrm{Fix}_{V^N}(\Sigma) : y_1^j = 0 \text{ if } j \notin J_1\}.$$

Therefore

$$\Sigma_w \subset \Sigma_{y_1} \subset \Gamma \times \mathbf{S}^1$$

and $\Sigma_w = \Sigma$ is not maximal, a contradiction. Thus $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on J and J is a block. $\square$

Lemma 3.3.2 *Let $\Sigma_w = H^\theta$ be a twisted maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Suppose that $w_1 \neq 0$. Let $\Sigma_{w_1} = B^\psi$ be the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$. Then*
(a) $\dim \mathrm{Fix}_V(\Sigma_{w_1}) \leq \dim \mathrm{Fix}_{V^N}(\Sigma_w)$;
(b) Σ_{w_1} is maximal in $\mathcal{L} \times \mathbf{S}^1$ and $\psi(B) \subseteq \theta(H)$.

Proof. The vector w is nonzero and is fixed by Σ_w. Let J be the set of indices $j \in \{1, \ldots, N\}$ such that $w_j \neq 0$. By lemma 3.3.1 the group $\Pi_{\mathcal{G}}(\Sigma)$ acts transitively on some block J. Therefore we can conjugate Σ_w to the isotropy subgroup of the vector

$$w = (w_1, \theta_2 \cdot w_1, \ldots, \theta_s \cdot w_1, 0, \ldots, 0),$$

if we take $J = \{1, \ldots, s\}$.

It follows that if there are p linearly independent vectors in $\mathrm{Fix}_V(\Sigma_{w_1})$, then we can construct p vectors in V^N that are linearly independent and are in $\mathrm{Fix}_{V^N}(\Sigma_w)$, so we have proved (a).

Now define

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Then

$$((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \subset \Sigma$$

and so $\psi(B) \subseteq \theta(H)$.

Suppose that Σ_{w_1} is not maximal in $\mathcal{L} \times \mathbf{S}^1$. This means that there exists an isotropy subgroup $\Sigma_{v_1} = C^\phi$ of $\mathcal{L} \times \mathbf{S}^1$ such that

$$\Sigma_{w_1} \subset \Sigma_{v_1} \subset \mathcal{L} \times \mathbf{S}^1.$$

Let

$$v = (v_1, \theta_2 \cdot v_1, \ldots, \theta_s \cdot v_1, 0, \ldots, 0)$$

and consider the isotropy subgroup Σ_v of v in $\Gamma \times \mathbf{S}^1$. Then

$$((\hat{C}, \mathbf{1}^{N-s}), \mathbf{1}, \phi) \subset \Sigma_v$$

and

$$((\hat{C}, \mathbf{1}^{N-s}), \mathbf{1}, \phi) \not\subset \Sigma_w.$$

Moreover,

$$\Sigma_w \subset \Sigma_v \subset \Gamma \times \mathbf{S}^1$$

and so Σ_w is not maximal, a contradiction. Therefore Σ_{w_1} is maximal in $\mathcal{L} \times \mathbf{S}^1$, as required. □

Corollary 3.3.3 *Let Σ_w be a maximal isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Suppose that $w_1 \neq 0$ and let $\Sigma_{w_1} = B^\psi$ be the isotropy subgroup of w_1 in $\mathcal{L} \times \mathbf{S}^1$. Then Σ_w is conjugate to $\Sigma(B^\psi, J, \sigma, J_1, p)$ for some block $J \subseteq \{1, \ldots, N\}$, a permutation σ in $\mathcal{G}$, a subset J_1 of J, a positive integer p and B^ψ is maximal in $\mathcal{L} \times \mathbf{S}^1$.*

Proof. The group B^ψ is maximal by lemma 3.3.2. The rest follows as in the proof of proposition 3.2.7 (using lemmas 3.3.1 and 3.3.2) if Σ_w is of finite twist type, or as in corollaries 3.2.8 and 3.2.10 if Σ_w is of twist type $\mathbf{S}^1$. □

Remark 3.3.4 *(a) By proposition 3.2.2 with B^ψ maximal, it follows that every group $\Sigma = \Sigma(B^\psi, J, \sigma, J_1, p)$ is a maximal isotropy subgroup and*

$$\dim \mathrm{Fix}_{V^N}(\Sigma) = \dim \mathrm{Fix}_V(B^\psi).$$

(b) From (a), assuming the conditions of corollary 3.3.3 we have

$$\dim \mathrm{Fix}_{V^N}(\Sigma_w) = \dim \mathrm{Fix}_V(\Sigma_{w_1}).$$

In particular,

$$\dim \mathrm{Fix}_{V^N}(\Sigma_w) \leq \dim V.$$

3.4 Isotropy subgroups

We show now that we can describe a general isotropy subgroup of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ using groups with a structure similar to that obtained for the maximal isotropy subgroups.

Once again consider V^N, a $\mathcal{L} \wr \mathcal{G}$-simple space. Let $\Gamma = \mathcal{L} \wr \mathcal{G}$ and take a partition of $\{1, \ldots, N\}$ in p blocks

$$\{1, \ldots, N\} = J_1 \cup \cdots \cup J_p.$$

Let $\Sigma_1, \ldots, \Sigma_p$ be isotropy subgroups in $\Gamma \times \mathbf{S}^1$ of the type

$$\Sigma(B_i^{\psi_i}, J_i, \sigma_i, J_1^i, p_i) \tag{3.2}$$

for $B_i^{\psi_i}$ an isotropy subgroup in $\mathcal{L} \times \mathbf{S}^1$, a part J_1^i of J_i, a permutation σ_i in $\mathcal{G}$ and a nonnegative integer p_i. Note that, if for example $J_1 = \{1, \ldots, q\}$, then a vector w fixed by Σ_1 is of the form

$$w = (w_{J_1}, 0, \ldots, 0), \tag{3.3}$$

where if we assume $J_1^1 = \{1, \ldots, t\}$, $\sigma_1(J_1^1) = \{t+1, \ldots, 2t\}, \ldots$, then

$$\begin{array}{l} w_{J_1} = (< w_1 >, < \xi_{p_1} \cdot w_1 >, \ldots, < \xi_{p_1}^{s_1^{'}-1} \cdot w_1 >), \\ s_1^{'} = \min_{i>0} \{\xi_{p_1}^i \in \psi_1(B_1)\}, \\ w_1 \in \mathrm{Fix}_V(B_1^{\psi_1}) \end{array}$$

and so $q = s_1^{'} t$.

Let

$$\Sigma = \bigcap_{i=1}^{p} \Sigma_i.$$

Theorem 3.4.1 *Σ is an isotropy subgroup of $\Gamma \times \mathbf{S}^1$ acting on V^N and every isotropy subgroup of $\Gamma \times \mathbf{S}^1$ is conjugate to such a Σ.*

Proof. Let W_i be a (nonzero) vector fixed by Σ_i. This means that

$$W_i = (0, \ldots, 0, w_{J_i}, 0, \ldots, 0),$$

where w_{J_i} denotes the components of W_i corresponding to the block J_i and we are assuming for simplicity that the blocks J_i have consecutives indices. Suppose that the first components of each w_{J_i} are in distinct $\mathcal{L} \times \mathbf{S}^1$ orbits.

Let

$$w = W_1 + \cdots + W_p.$$

By construction Σ fixes w, i.e., $\Sigma \subseteq \Sigma_w$.

Let now $(l, \sigma, \theta) \in \Sigma_w$. As we are assuming that the first components of each w_{J_i} are in distinct $\mathcal{L} \times \mathbf{S}^1$ orbits, we must have

$$\Pi_{\mathcal{G}}(\Sigma_w) J_i = J_i$$

and so, from $(l, \sigma, \theta) \cdot w = w$ we get $(l, \sigma, \theta) \cdot W_i = W_i$, for $i = 1, \ldots, p$. Thus $\Sigma_w \subseteq \Sigma$.

Let Σ_w be any isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Consider the projection $\Pi_{\mathcal{G}}(\Sigma_w) = G$. Since G is a subgroup of the permutation group $\mathbf{S}_N$, there is a partition $J_1 \cup \cdots \cup J_p$ of the set $\{1, \ldots, N\}$ such that each J_i is a G-orbit. That is, each J_i is a block and the components corresponding to each J_i are all null or all nonzero. Consider Σ_i the isotropy subgroup in $\Gamma \times \mathbf{S}^1$ of the vector

$$W_i = (0, \ldots, 0, w_{J_i}, 0, \ldots, 0).$$

Again we are assuming J_i formed by consecutives indices. Since G acts transitively on J_i, then $\forall\ i_1, i_2 \in J_i$ we can find some $(l, \sigma, \theta) \in \Sigma_w$ such that

$$w_{i_1} = (l_{i_1}, \theta) \cdot w_{i_2}.$$

Thus, if for some i_1 we have $w_{i_1} = 0$, then all the components with indices in J_i are null. If it is not the case, then up to conjugacy, we can suppose that each w_{J_i} has a form like in (3.3) (see proposition 3.2.7). Thus each Σ_i is of the type (3.2) and $GJ_i = J_i$.

It is now straightforward to prove that Σ_w is the intersection of the isotropy subgroups Σ_i with $i = 1, \ldots, p$. $\square$

Chapter 4

Symmetry-breaking for wreath product systems

Keeping in mind the association between wreath product systems and coupled cell networks, we now describe symmetry-breaking for wreath product systems, where we can naturally recognize the equations governing the dynamics of one individual cell and the coupling that produces the full coupled system. That is, we suppose in general that the dynamics of one cell are known, and we wish to use this information to study the global behaviour of the entire system.

We begin by clarifying how the coupling function and individual cell dynamics are naturally associated with systems of ODEs involving vector fields that commute with symmetries of wreath product groups.

We show that when we apply the equivariant branching lemma and the equivariant Hopf theorem, then we can reduce the conditions on the entire system to a set of conditions on the one-cell system.

The structure of the axial groups obtained in [10] and the **C**-axial groups obtained in [10] as amended in chapter 3 is then used to describe the general form of the matrices commuting with these groups.

4.1 Coupled cell systems and wreath product groups

As defined before, a system of ODEs $\dot{X} = F(X)$ is a system of coupled cells if $F_j(X) = f(X_j) + h_j(X)$, where h_j governs the coupling between cells and

f governs the dynamics of one individual cell (we are assuming the cells are identical). We note that there is no loss of generality in the association of identical coupled cells systems with systems that are symmetric under wreath product groups. In fact, let $F = (F_1, \ldots, F_N)$ be a vector field defined on a vector space V^N and commuting with $\mathcal{L} \wr \mathcal{G}$, where the action of the group $\mathcal{L} \wr \mathcal{G}$ on V^N is as defined in section 3.1 for $\mathcal{L} \subseteq \mathbf{O}(k)$ (acting on V) and $\mathcal{G}$ a transitive group of $\mathbf{S}_N$. Because F commutes with $\mathcal{G}$ and $\mathcal{G}$ is transitive on $\{1, \ldots, N\}$, there are permutations σ_j for $j = 2, \ldots, N$ such that $\sigma_j^{-1}(1) = j$ and so

$$F_j(X) = F_1(\sigma_j(X)) = F_1(X_j, X_{\sigma_j^{-1}(2)}, \ldots, X_{\sigma_j^{-1}(N)}).$$

There is no loss of generality in assuming we can distinguish in F_1 one part that depends only on X_1 and so we can write F_1 as

$$F_1(X) = f(X_1) + h_1(X)$$

for some mappings $f : V \to V$ and $h_1 : V^N \to V$, and we get

$$F_j(X) = f(X_j) + h_1(\sigma_j(X)),$$

for $j = 2, \ldots, N$. In general, the mapping f is not unique.

Now if we denote by $H(X) = (h_1(X), h_1(\sigma_2(X)), \ldots, h_1(\sigma_N(X)))$, then F commutes with $\mathcal{L} \wr \mathcal{G}$ if and only if f commutes with $\mathcal{L}$, the vector field H commutes with $\mathcal{G}$ and h_1 is equivariant in X_1 and invariant in $X_2, \ldots, X_N$, both under $\mathcal{L}$.

It is often common to assume that the effect of coupling on the jth cell is the summing of the effects of all the cells that are coupled to jth cell, that is, if $H(X) = (h_1(X), \ldots, h_N(X))$, then

$$h_j(X) = \sum_{i=1}^{N} C(i,j) h_{ij}(X_i, X_j),$$

where $C(i,j) = 1$ if cell i is coupled to cell j and zero otherwise. If the coupling is identical, then $h_{ij} = h$, for all i and j. In this case, if $C = (C(i,j))$, then H commutes with $\mathcal{L} \wr \mathcal{G}$ if and only if $\sigma C \sigma^{-1} = C$, $\forall\, \sigma \in \mathcal{G}$ and $h(y_1, y_2)$ is invariant in y_1 and equivariant in y_2 under $\mathcal{L}$.

4.2 Equivariant branching lemma and equivariant Hopf theorem in wreath product systems

The aim of the work presented here and in [10] is to make clear how patterns formed through steady-state and Hopf bifurcations in wreath product systems depend both on the internal and global symmetries. Steady-state bifurcation to equilibria with symmetry given by axial subgroups is guaranteed by the equivariant branching lemma, and Hopf bifurcation to periodic solutions with symmetry given by **C**-axial subgroups can be justified by the equivariant Hopf theorem.

The axial and **C**-axial groups, up to conjugacy, of groups $\mathcal{L} \wr \mathcal{G}$ are described in chapter 3 and [10] and it is clear how the structure of these groups depends both on $\mathcal{L}$ and $\mathcal{G}$.

Let $\dot{X} = F(X, \lambda)$ be a system of ODEs, where $X \in V^N$, the bifurcation parameter is $\lambda \in \mathbf{R}$ and F commutes with $\Gamma = \mathcal{L} \wr \mathcal{G}$. Suppose that $F_j(X, \lambda) = f(X_j, \lambda) + h_j(X)$. We are interested in relating the conditions of the equivariant branching lemma and the equivariant Hopf theorem between the coupled cell system $\dot{X} = F(X, \lambda)$ and the one-cell system $\dot{x} = f(x, \lambda)$. For the equivariant branching lemma we have:

Proposition 4.2.1 *Let $\dot{x} = f(x, \lambda)$, $x \in V$, $\lambda \in \mathbf{R}$ be a system of ODEs where f commutes with $\mathcal{L} \subseteq \mathbf{O}(k)$ and $\mathcal{L}$ acts nontrivially and absolutely irreducibly on V. Suppose that $(x, \lambda) = (0, 0)$ is a bifurcation point, i.e., $f(0, 0) = 0$ and $(df)_{0,0} = 0$. Let $A = \Sigma_{x_0} \subseteq \mathcal{L}$ be axial in V and if $(df)_{0,\lambda} = C(\lambda) Id_V$, then $C'(0) \neq 0$. Suppose that $f(x_0, \lambda_0) = 0$, i.e., the point $(x, \lambda) = (x_0, \lambda_0)$ belongs to the branch of steady solutions of $\dot{x} = f(x, \lambda)$ that bifurcate from $(0, 0)$ (with symmetry A) guaranteed by the equivariant branching lemma.*

Consider now $\dot{X} = F(X, \lambda)$, $X = (X_1, \ldots, X_N) \in V^N$, $\lambda \in \mathbf{R}$, where F commutes with $\mathcal{L} \wr \mathcal{G}$, the group $\mathcal{G} \subseteq \mathbf{S}_N$ is transitive and $F_j(X, \lambda) = f(X_j, \lambda) + h_j(X)$ for $h_j : V^N \to V$. Suppose that $J = \{1, \ldots, s\}$ is a block, $h_1(x_0, \ldots, x_0, 0, \ldots, 0) = 0$ and $\left(\frac{\partial h_1}{\partial X_1}\right)_0 = 0$. Then we have the conditions of the equivariant branching lemma satisfied for F, i.e., the vector $(x_0, \ldots, x_0, 0, \ldots, 0, \lambda_0)$ belongs to the branch of equilibria of $\dot{X} = F(X, \lambda)$ bifurcating from $(X, \lambda) = (0, 0)$ with symmetry $\Sigma(A, J) = A^s \times \mathcal{L}^{N-s} \dot{+} Q_J$.

Proof. Since $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$, and as $f(\mathrm{Fix}_V(\mathcal{L})) \subseteq \mathrm{Fix}_V(\mathcal{L})$, then $f(0,\lambda) = 0,\ \forall\ \lambda \in \mathbf{R}$. Similarly, since $\mathrm{Fix}_{V^N}(\mathcal{L} \wr \mathcal{G}) = \{0\}$, we have $F(0,\lambda) = 0,\ \forall\ \lambda \in \mathbf{R}$. Because $(df)_{0,0} = 0$ and $(df)_{0,\lambda} = C(\lambda)Id_V$, then $C(0) = 0$. Thus $C(0) = 0$ and $C^{'}(0) \neq 0$.

The space V^N is a $\mathcal{L}\wr\mathcal{G}$-absolutely irreducible and $(dF)_{0,\lambda}$ commutes with this group. Thus $(dF)_{0,\lambda}$ is a scalar multiple of Id_{V^N}. As

$$\left(\frac{\partial F_1}{\partial X_1}\right)_{0,\lambda} = (df)_{0,\lambda} + \left(\frac{\partial h_1}{\partial X_1}\right)_0 = (df)_{0,\lambda} = C(\lambda)Id_V,$$

it follows that

$$(dF)_{0,\lambda} = C(\lambda)Id_{V^N}.$$

From [10], the group $\Sigma(A,J)$ is axial and $\Sigma(A,J) = \Sigma_{(x_0,\ldots,x_0,0,\ldots,0)}$. Since Q_J is transitive on J, we can choose $\sigma_j \in Q_J$, for $j = 2,\ldots,s$, such that $\sigma_j^{-1}(1) = j$ and $h_j(X) = h_1(\sigma_j(X))$. Thus

$$\begin{aligned} F_j(x_0,\ldots,x_0,0,\ldots,0,\lambda_0) &= f(x_0,\lambda_0) + h_1(\sigma_j(x_0,\ldots,x_0,0,\ldots,0)) \\ &= h_1(x_0,\ldots,x_0,0,\ldots,0) = 0, \end{aligned}$$

since $f(x_0,\lambda_0) = 0$ and $h_1(\sigma_j(x_0,\ldots,x_0,0,\ldots,0)) = h_1(x_0,\ldots,x_0,0,\ldots,0) = 0$, for $j \leq s$. For $j = s+1,\ldots,N$, consider $\sigma_j \in \mathcal{G}$ such that $\sigma_j^{-1}(1) = j$. As $h_1(X_1,\ldots,X_N)$ is $\mathcal{L}$-equivariant in X_1 and $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$, then

$$h_1(0,X_2,\ldots,X_N) = 0, \quad \forall\ X_2,\ldots,X_N \in V,$$

and so

$$h_j(x_0,\ldots,x_0,0,\ldots,0) = h_1(\sigma_j(x_0,\ldots,x_0,0,\ldots,0)) = 0,$$

and

$$F_j(x_0,\ldots,x_0,0,\ldots,0,\lambda_0) = f(0,\lambda_0) = 0$$

for $j = s+1,\ldots,N$. Thus $F(x_0,\ldots,x_0,0,\ldots,0,\lambda_0) = 0$ and $(x_0,\ldots,x_0,0,\ldots,0,\lambda_0)$ belongs to the unique branch of solutions $\dot{X} = F(X,\lambda)$ with symmetry $\Sigma(A,J)$ guaranteed by the equivariant branching lemma. $\square$

Remark 4.2.2 *In proposition 4.2.1 we assume that $J = \{1,\ldots,s\}$ is a block and $h_1(x_0,\ldots,x_0,0,\ldots,0) = 0$ to make easier exposition. The result is still true if instead we suppose that the block J is $\{i_1,\ldots,i_s\}$ and, for example $h_{i_1}(X^*) = 0$ where $X^*_i = x_0$ if $i \in J$ and zero otherwise. For this case (X^*,λ_0) belongs to the branch of equilibria with symmetry $\Sigma(A,J)$ bifurcating from the origin.*

Proposition 4.2.3 *Let* $\dot{x} = f(x, \lambda),\ x \in V,\ \lambda \in \mathbf{R}$ *be a system of ODEs where* f *commutes with* $\mathcal{L} \subseteq \mathbf{O}(k)$ *and* $\mathcal{L}$ *acts nontrivially and* $\mathcal{L}$*-simply on* V*. Suppose that* $(df)_{0,0} = J$*, where*

$$J = \begin{bmatrix} 0 & -Id_m \\ Id_m & 0 \end{bmatrix},$$

and $m = \dim V/2$ *where* $k = \dim V$*. Call the eigenvalues of* $(df)_{0,\lambda}$ *by* $\sigma(\lambda) \mp i\rho(\lambda)$ *(and so* $\sigma(0) = 0$ *and* $\rho(0) = 1$*) and suppose that* $\sigma'(0) \neq 0$*. Let* B^{ψ} *be a* $\mathbf{C}$*-axial group of* $\mathcal{L} \times \mathbf{S}^1$*.*

Let $\dot{X} = F(X, \lambda),\ X = (X_1, \ldots, X_N) \in V^N,\ \lambda \in \mathbf{R}$ *be a coupled cell system where* $F_j(X, \lambda) = f(X_j, \lambda) + h_j(X)$ *for* $j = 1, \ldots, N$ *and* F *commutes with* $\mathcal{L} \wr \mathcal{G}$ *for a transitive group* $\mathcal{G}$ *of* $\mathbf{S}_N$*. Let* $\Sigma = \Sigma(B^{\psi}, J, \sigma, J_1, p)$ *be a* $\mathbf{C}$*-axial group of* $\Gamma \times \mathbf{S}^1$*. Suppose that* h_1 *satisfies* $\left(\frac{\partial h_1}{\partial X_1}\right)_0 = 0$*. Then the conditions of the equivariant Hopf theorem are satisfied for* F *and thus there exists a unique branch of small-amplitude periodic solutions of* $\dot{X} = F(X, \lambda)$ *with period near* 2π*, having* Σ *as their group of symmetries.*

Proof. Note that since V is $\mathcal{L}$-simple, then V^N is $\mathcal{L} \wr \mathcal{G}$-simple. Also $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$ and $\mathrm{Fix}_{V^N}(\mathcal{L} \wr \mathcal{G}) = \{0\}$. As h_1 is invariant under $\mathcal{L}$ in $X_2, \ldots, X_N$, then $\left(\frac{\partial F_1}{\partial X_i}\right)_0 = 0$ for $i \geq 2$: from $h_1(l \cdot X_1, X_2, \ldots, X_N) = l \cdot h_1(X),\ \forall\, l \in \mathcal{L}$, it follows that

$$\left(\frac{\partial h_1}{\partial X_j}\right)_0 = l \cdot \left(\frac{\partial h_1}{\partial X_j}\right)_0,\ \forall\, l \in \mathcal{L}$$

and so $\left(\frac{\partial h_1}{\partial X_j}\right)_0 = 0$ for $j \geq 2$. Note that, if $M \in \mathbf{M}_{k\times k}(\mathbf{R})$ and $l \cdot M = M,\ \forall\, l \in \mathcal{L}$, then from $l \cdot Mv = Mv,\ \forall\, l \in \mathcal{L},\ v \in V$, we get $M = 0$. In general, $\left(\frac{\partial F_j}{\partial X_i}\right)_0 = 0$ if $i \neq j$. Since $\left(\frac{\partial h_1}{\partial X_1}\right)_0 = 0$ by hypothesis and so $\left(\frac{\partial h_j}{\partial X_j}\right)_0 = 0$, then

$$\left(\frac{\partial F_j}{\partial X_j}\right)_{0,0} = (df)_{0,0}$$

for $j = 1, \ldots, N$. Thus

$$(dF)_{0,0} = \mathrm{Diag}(J, \ldots, J),$$

(and the eigenvalues of this matrix are $\mp i$) and by lemma 2.3.3 the eigenvalues of $(\partial F)_{0,\lambda}$ are complex conjugates $\sigma^*(\lambda) \mp i\rho^*(\lambda)$ (where $\sigma^*(0) = 0$ and

$\rho^*(0) = 1$). As

$$(dF)_{0,\lambda} = \mathrm{Diag}((\mathrm{d}f)_{0,\lambda}, \ldots, (\mathrm{d}f)_{0,\lambda}),$$

it follows that $\sigma^*(\lambda) = \sigma(\lambda)$, $\rho^*(\lambda) = \rho(\lambda)$ and $(\sigma^*)'(0) \neq 0$. $\square$

Remark 4.2.4 *We note that with the conditions of proposition 4.2.3 and if the vector fields f and h_1 commute with $\mathcal{L} \times \mathbf{S}^1$ (to be more precise h_1 is $\mathcal{L} \times \mathbf{S}^1$ invariant in $X_2, \ldots, X_N$ and equivariant by the same group in X_1), and so they are in Birkhoff normal form, then the vector field F is also in Birkhoff normal form.*

4.3 Isotropy restrictions

As we saw in section 2.1, if Γ is a Lie group acting on V and $g \in \vec{\mathcal{E}}_x(\Gamma)$, then $(dg)_x$ commutes with $\Sigma = \Sigma_x \subset \Gamma$, where Σ_x is the isotropy subgroup of $x \in V$. And if we decompose V into isotypic components for the action of Σ, say $V = W_1 \oplus \cdots \oplus W_k$, then $(dg)_x$ is invariant for each W_j. We are interested now in the case where $\Gamma = \mathcal{L} \wr \mathcal{G}$ and Σ is a maximal isotropy subgroup of Γ (or $\Gamma \times \mathbf{S}^1$).

4.3.1 Axial groups

We wish to use the structure of the axial subgroups of a general wreath product group $\mathcal{L} \wr \mathcal{G}$ obtained [10] to describe the general form of the commuting matrices for these groups. We show that this form depends both on the groups $\mathcal{L}$ and $\mathcal{G}$, where each one imposes constraints in a systematic way.

By lemmas 3.1 and 3.2 of [10], if we assume that $\mathcal{L} \wr \mathcal{G}$ acts absolutely irreducibly on W where $\mathcal{L}^N$ acts nontrivially and if $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, then we can write $W = V^N$, where $\mathcal{L}$ acts absolutely irreducibly on V. In [10], it is proved that for each block $J \subseteq \{1, \ldots, N\}$ and each axial subgroup $A \subset \mathcal{L}$ acting on V, the subgroup $\Sigma(A, J) \subset \mathcal{L} \wr \mathcal{G}$ defined by

$$\Sigma(A, J) = (B_1 \times \cdots \times B_N) \dot{+} Q_J,$$

where

$$B_j = \begin{cases} A & \text{if } j \in J, \\ \mathcal{L} & \text{if } j \notin J, \end{cases}$$

and

$$Q_J = \{\sigma \in \mathcal{G} : \sigma(J) = J\},$$

is an axial subgroup and any axial subgroup of $\mathcal{L} \wr \mathcal{G}$ is conjugate to one of $\Sigma(A, J)$ for some axial $A \subset \mathcal{L}$ and some block $J \subseteq \{1, \ldots, N\}$.

Theorem 4.3.1 *Assume that $\mathcal{L} \subseteq \mathbf{O}(k)$ is acting nontrivially and absolutely irreducibly on $V = \mathbf{R}^k$. Let $\mathcal{G}$ be a transitive subgroup of $\mathbf{S}_N$. Let $G \in \mathbf{M}_{Nk \times Nk}(\mathbf{R})$. Let $\Sigma(A, J)$ be an axial subgroup of $\mathcal{L} \wr \mathcal{G}$ as defined above for an axial subgroup A of $\mathcal{L}$ and for some block $J \subseteq \{1, \ldots, N\}$. Suppose $J = \{1, \ldots, s\}$ and that G commutes with $\Sigma(A, J)$. Then there exists a basis of V^N such that*

$$G = \mathrm{Diag}(G_1, G_2),$$

with

$$G_1 = \mathrm{Diag}(C, C_1, \ldots, C_1)$$

and

$$G_2 = \mathrm{Diag}(\lambda_{s+1} Id_{k \times k}, \ldots, \lambda_N Id_{k \times k}),$$

where $c_{i,j}, \ldots, \lambda_i$ are real, the matrix $C_1 \in \mathbf{M}_{(k-1)\times(k-1)}(\mathbf{R})$, and the matrices

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,s} \\ c_{2,1} & c_{1,1} & \cdots & c_{2,s} \\ \cdots & \cdots & \cdots & \cdots \\ c_{s,1} & c_{s,2} & \cdots & c_{1,1} \end{bmatrix}$$

and G_2 commute with Q_J. More precisely, C_1 commutes with $Q_J \mid_J$ and G_2 with $Q_J \mid_{\{s+1,\ldots,N\}}$.

Before we prove this theorem, we start by proving a lemma.

Lemma 4.3.2 *Suppose that a group H acts on a space $U_1 \oplus U_2$ where each U_j is H-invariant. Suppose that U_1 has no H-irreducible component that is H-isomorphic to an H-irreducible component of U_2. Let P be a linear transformation on $U_1 \oplus U_2$ that commutes with H. Then $P = \mathrm{Diag}(\mathrm{P}_1, \mathrm{P}_2)$ where P_1 is a linear transformation of U_1 and P_2 is a linear transformation of U_2. Moreover, both P_j commute with H.*

Proof Commuting matrices correspond to H-module homomorphisms. Write

$$P = \begin{pmatrix} P_1 & Q_1 \\ Q_2 & P_2 \end{pmatrix}$$

with respect to $U_1 \oplus U_2$. Then Q_1 defines an H-module homomorphism $U_2 \to U_1$, and Q_2 defines one from $U_1 \to U_2$. Since U_1, U_2 have no common irreducible component, then $Q_1 = 0$ and $Q_2 = 0$ (see lemma 2.1.1). So

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

as claimed. $\Box$

Proof of theorem 4.3.1 Since we are assuming $J = \{1, \ldots, s\}$, we have

$$\Sigma = \Sigma(A, J) = (A^s \times \mathcal{L}^{N-s}) \dot{+} Q_J.$$

Since V is an $\mathcal{L}$-irreducible space and $\mathcal{L}$ acts nontrivially on V, we have $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$. Thus if $M \in \mathbf{M}_{k\times k}(\mathbf{R})$ and $l \cdot M = M$, $\forall\, l \in \mathcal{L}$ or $M \cdot l = M$, $\forall\, l \in \mathcal{L}$, then $M = 0_{k\times k}$. To show this suppose $l \cdot M = M$, $\forall\, l \in \mathcal{L}$. Let $v \in V$. Then

$$l \cdot Mv = Mv, \ \forall\, l \in \mathcal{L}.$$

Thus $Mv \in \mathrm{Fix}_V(\mathcal{L})$ and so $Mv = 0$, $\forall\, v \in V$. Therefore $M = 0$. Now suppose $M \cdot l = M$, $\forall\, l \in \mathcal{L}$. Then

$$l^\top \cdot M^\top = M^\top, \ \forall\, l \in \mathcal{L}.$$

So by same argument, $M^\top = 0$. Note that the $\mathcal{L}$-action is orthogonal and so $l^\top = l^{-1}$, $\forall\, l \in \mathcal{L}$.

Since G commutes with $(\mathbf{1}^s, \mathcal{L}^{N-s})$, by lemma 4.3.2,

$$G = \mathrm{Diag}(G_1, G_2),$$

where $G_1 \in \mathbf{M}_{sk\times sk}$ and $G_2 \in \mathbf{M}_{(N-s)k\times(N-s)k}$. Note that the group $(\mathbf{1}^s, \mathcal{L}^{N-s})$ is acting on $V^s \times V^{N-s}$, where the action on $V^s \times \{0\}^{N-s}$ is trivial and on $\{0\}^s \times V^{N-s}$ is by $\mathcal{L}^{N-s}$. The latter, being a diagonal action of $\mathcal{L}$, has no trivial component. Again, since $\mathcal{L}$ acts absolutely irreducibly on V and $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$, using lemma 4.3.2, it follows that

$$G_2 = \mathrm{Diag}(\lambda_{s+1} Id_{k\times k}, \ldots, \lambda_N Id_{k\times k})$$

for some constants $\lambda_{s+1}, \ldots, \lambda_N \in \mathbf{R}$. Note that for each $j > s$, the spaces

$$\begin{array}{c}(0, \ldots, 0; 0, \quad \ldots, 0, V, 0, \ldots, \quad 0) \\ \uparrow \\ j\text{th place}\end{array}$$

are $(\mathbf{1}^s, \mathcal{L}^{N-s})$-irreducible and non $(\mathbf{1}^s, \mathcal{L}^{N-s})$-isomorphic.

Since G commutes with $(A^s, \mathbf{1}^{N-s})$, the matrix G_1 commutes with A^s. Let $G_1 = (g_{i,j})$, where $g_{i,j} \in \mathbf{M}_{k\times k}(\mathbf{R})$, for $i, j = 1, \ldots, s$. Then

$$g_{i,i} \cdot a = a \cdot g_{i,i}, \quad \forall\ a \in A, \quad (i = 1, \ldots, s),$$

and

$$g_{i,j} \cdot a = a \cdot g_{i,j} = g_{i,j}, \quad \forall\ a \in A, \quad (i \neq j,\ i, j = 1, \ldots, s).$$

Since G commutes with Q_J and Q_J is transitive on J, then

$$g_{1,1} = \cdots = g_{s,s}.$$

Since A is axial, it has a one-dimensional fixed-point subspace. We can choose a basis $\{b_1, \ldots, b_k\}$ for V such that $\mathrm{Fix}_V(\mathrm{A}) =< b_1 >$. Suppose that $V = W_1 \oplus \ldots \oplus W_p$, where W_i for $i = 1, \ldots, p$ are the isotypic components of V for the action of A, and let $W_1 = \mathrm{Fix}_V(\mathrm{A})$. Since $g_{1,1}$ commutes with A, there exists $c_{1,1} \in \mathbf{R}$ and a matrix $C_1 \in \mathbf{M}_{(k-1)\times(k-1)}(\mathbf{R})$ such that $g_{1,1}$ is written with respect to this basis $\{b_1, \ldots, b_k\}$ as

$$g_{1,1} = \mathrm{Diag}(c_{1,1}, C_1)$$

(this follows from theorem 2.1.4). Since $g_{i,j}$ for $i \neq j$ also commutes with A, we can use the same argument and write

$$g_{i,j} = \mathrm{Diag}(c_{i,j}, C_{i,j})$$

for constants $c_{i,j} \in \mathbf{R}$ and matrices $C_{i,j} \in \mathbf{M}_{(k-1)\times(k-1)}$. Since $a \cdot g_{i,j} = g_{i,j}$, $\forall\ a \in A$ and $i \neq j$, then $C_{i,j} = 0_{(k-1)\times(k-1)}$ because $\mathrm{Fix}_{W_2\oplus\cdots\oplus W_p}(\mathrm{A}) = \{0\}$ (recall the observation at the beginning of the proof). Thus

$$g_{i,j} = \mathrm{Diag}(c_{i,j}, 0)$$

and so

$$G_1 = \begin{bmatrix} c_{1,1} & & c_{1,2} & & \dots & c_{1,s} & \\ & C_1 & & 0 & \dots & & 0 \\ c_{2,1} & & c_{1,1} & & \dots & c_{2,s} & \\ & 0 & & C_1 & \dots & & 0 \\ \dots & & \dots & & \dots & \dots & \\ c_{s,1} & & c_{s,2} & & \dots & c_{1,1} & \\ & 0 & & 0 & \dots & & C_1 \end{bmatrix}$$

where the blank entries are zero matrices. Now if we change the order of the basis the result follows. □

Corollary 4.3.3 *With the conditions of theorem 4.2.1 the derivative* $(dF)_{(x_0,\dots,x_0,0,\dots,0,\lambda_0)}$ *viewed as a matrix in* $(\mathbf{R}^k)^N$ *if* $V = \mathbf{R}^k$, *is of the form presented in theorem 4.3.1 for some basis of* V^N.

Proof It is straightforward. □

Remark 4.3.4 *(a) Since* C *commutes with* Q_J *and* Q_J *acts transitively on* $\{1,\dots,s\}$, *this symmetry will strongly restrict the form of* C. *In particular, if* C *is symmetric, then it is diagonalizable. However, the matrix* C *is not always diagonalizable. For example, if we take* $Q_J = \mathbf{Z}_4$, *then the matrix*

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

commutes with $\mathbf{Z}_4$ *(since* $\mathbf{Z}_4$ *is an abelian group) and it is not diagonalizable.*

(b) From theorem 4.3.1, it follows that the problem of finding the eigenvalues of a matrix that commutes with $\Sigma(A,J)$ *is reduced to the problem of finding the eigenvalues of the two matrices* $C \in \mathbf{M}_{s\times s}(\mathbf{R})$ *and* $C_1 \in \mathbf{M}_{(k-1)(k-1)}(\mathbf{R})$. *Moreover, each eigenvalue of* C_1 *will have multiplicity at least* s.

Corollary 4.3.5 *With the conditions of theorem 4.3.1, if* Q_J *acts transitively on* $\{s{+}1,\dots,N\}$, *then* $\lambda_{s+1} = \cdots = \lambda_N$ *and so* $G_2 = \lambda_{s+1} Id_{(N-s)k\times(N-s)k}$.

Proof. This follows since G_2 commutes with $Q_J\mid_{\{s+1,\ldots,N\}}$. $\square$

Remark 4.3.6 *Since V is $\mathcal{L}$-absolutely irreducible and if Q_J acts transitively on $\{s+1,\ldots,N\}$, the space V^{N-s} is $\mathcal{L}\wr Q_J$-absolutely irreducible. Thus any linear mapping commuting with $\mathcal{L}\wr Q_J$ (considering the action of this group on V^{N-s}) must be a scalar multiple of $Id_{V^{N-s}}$. Applying this to the matrix G_2 in the previous corollary, we would obtain the same result.*

In general, if $\mathcal{G}\subseteq \mathbf{S}_N$ and $\{1,\ldots,N\} = J\cup J^{'}$, where J is a block, then Q_J acts transitively on $J^{'}$ if and only if $J^{'}$ is a block: since Q_J leaves J invariant, then it leaves $J^{'}$ invariant. So if Q_J acts transitively on $J^{'}$, then $J^{'}$ is a block. On the other hand, if J and $J^{'}$ are both blocks, as Q_J leaves $J^{'}$ invariant then $Q_J\subseteq Q_{J'}$, and as $Q_{J'}$ leaves J invariant, then $Q_{J'}\subseteq Q_J$. Thus $Q_J = Q_{J'}$ and Q_J acts transitively on $J^{'}$.

Examples (a) Let $\mathcal{G} = \mathbf{Z}_3 \subset \mathbf{S}_3$. Then $J = \{1\}$ is a block, where $Q_J = \{1\}$ and Q_J does not act transitively on $\{2,3\}$.

(b) Let $\mathcal{G} = \mathbf{Z}_4 \subset \mathbf{S}_4$. Then $\mathcal{G}$ is generated by the permutation (1234) and $J = \{1,3\}$ is a block. In this case the group $Q_J = \{1,\sigma^2\}$ acts transitively on $\{2,4\}$ and $\{2,4\}$ is also a block.

Corollary 4.3.7 *With the conditions of theorem 4.3.1, if $\mathcal{G} = \mathbf{S}_N$, then $G_2 = \lambda_{s+1} Id_{(N-s)k\times(N-s)k}$ and $c_{i,j} = c_{j,i}$ for all $i,j = 1,\ldots,s$, i.e.,*

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,2} \\ c_{1,2} & c_{1,1} & \cdots & c_{1,2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1,2} & c_{1,2} & \cdots & c_{1,1} \end{bmatrix}.$$

Proof. For all $1 \le s \le N$, the set $J = \{1,\ldots,s\}$ is a block and $Q_J = \mathbf{S}_s \times \mathbf{S}_{N-s}$. For G_2, as Q_J is transitive on $\{s+1,\ldots,N\}$ it holds corollary 4.3.5. Since C commutes with $Q_J\mid_J = \mathbf{S}_s$ and this group is generated by $\{(1,k),\ k = 2,\ldots,s\}$, the result follows. $\square$

4.3.2 C-axial groups

Let V be a $\mathcal{L}$-simple space where $\mathcal{L}$ acts nontrivially and $\mathcal{G}$ is a transitive group of $\mathbf{S}_N$. Again, by [10], the space V^N is $\mathcal{L}\wr\mathcal{G}$-simple and the subgroups of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ of the form $\Sigma(B^\psi, J)$ (defined in section 3.1.3) are $\mathbf{C}$-axial. Here, the group $B^\psi \subset \mathcal{L} \times \mathbf{S}^1$ is $\mathbf{C}$-axial in V and $J \subseteq \{1, \ldots N\}$ is a block. As we did in theorem 4.3.1, we can use the structure of these groups to describe the general form of the commuting matrices with these groups.

Theorem 4.3.8 *Assume that $V = \mathbf{R}^k$ is a $\mathcal{L}$-simple space where $\mathcal{L} \subseteq \mathbf{O}(k)$ acts nontrivially on V. Let $\mathcal{G}$ be a transitive subgroup of $\mathbf{S}_N$. Let $G \in \mathbf{M}_{Nk \times Nk}(\mathbf{R})$ and $\Sigma(B^\psi, J)$ be a $\mathbf{C}$-axial subgroup of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ where $B^\psi \subset \mathcal{L} \times \mathbf{S}^1$ is $\mathbf{C}$-axial (in V) and J is a block of $\{1, \ldots, N\}$. Suppose that $J = \{1, \ldots, s\}$ and G commutes with $\Sigma(B^\psi, J)$. Then there exists a basis of V^N such that*

$$G = \mathrm{Diag}(G_1, G_2),$$

with

$$G_1 = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,s} \\ g_{2,1} & g_{1,1} & \cdots & g_{2,s} \\ \cdots & \cdots & \cdots & \cdots \\ g_{s,1} & g_{s,2} & \cdots & g_{1,1} \end{bmatrix},$$

for $g_{1,1} \in \mathbf{M}_{k \times k}(\mathbf{R})$ that commutes with B^ψ, and the matrices $g_{i,j} \in \mathbf{M}_{k \times k}(\mathbf{R})$ for $i \neq j$ also commute with B^ψ. The matrix

$$G_2 = \mathrm{Diag}(g_{s+1}, \ldots, g_N),$$

for matrices $g_{s+1}, \ldots, g_N \in \mathbf{M}_{k \times k}(\mathbf{R})$ commuting with $\mathcal{L} \times \psi(B)$. Moreover, as $\dim \mathrm{Fix}_V(\mathrm{B}^\psi) = 2$, for some basis B of V^N the matrix

$$g_{1,1} = \mathrm{Diag}(c_{1,1}, C_1)$$

where $c_{1,1} \in \mathbf{M}_{2 \times 2}(\mathbf{R})$ and $C_1 \in \mathbf{M}_{(k-2) \times (k-2)}(\mathbf{R})$, and for $i \neq j$ we have

$$g_{i,j} = \mathrm{Diag}(c_{i,j}, C_{i,j})$$

for $c_{i,j} \in \mathbf{M}_{2 \times 2}(\mathbf{R})$ and $C_{i,j} \in \mathbf{M}_{(k-2) \times (k-2)}(\mathbf{R})$. Also the matrices G_1 and G_2 commute with Q_J.

Proof. The proof follows as in the proof of theorem 4.3.1 where now $\dim \mathrm{Fix}_V(\mathrm{B}^\psi) = 2$ and V is $\mathcal{L}$-simple. $\square$

Remark 4.3.9 *(a) From theorem 4.2.3, if* $\Sigma = \Sigma(B^\psi, J)$ *and* F *commutes also with* $\mathbf{S}^1$*, it follows that we can use the above theorem to deduce the form of the derivative of the reduced vector field obtained by a Liapunov-Schmidt reduction (theorem 2.3.6) at the zero corresponding to the periodic solution with symmetry* Σ*.*

(b) With the conditions of theorem 4.3.8 but now with Σ *of type* $\Sigma = \Sigma(B^\psi, J, \sigma, J_1, p)$*, the matrix* G *is also* $G = \mathrm{Diag}(G_1, G_2)$*, but now the matrices* $g_{i,i}$ *are not equal. This is because of the more complicated structure of* Σ*: now* G_1 *and* G_2 *commute only with* Q_{J_1} *(and not* Q_J*); in addition,* G *commutes with*

$$\bigcup_{k=2,\ldots,s'} ((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1})$$

(see section 3.2). The rest holds.

(c) If $\psi(B) = \mathbf{S}^1$*, then with the conditions of the previous theorem, the matrices* $g_{s+1}, \ldots, g_N$ *commute with* $\mathcal{L} \times \mathbf{S}^1$ *and so are of 'complex type' by [20] lemma XVI 3.4.*

Chapter 5

Hopf bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_3$ symmetry

Suppose we have a system of ODEs

$$\dot{v} = f(v, \lambda), \tag{5.1}$$

where $v \in V$, $\lambda \in \mathbf{R}$ is a bifurcation parameter and $f : V \times \mathbf{R} \to V$ is a smooth (C^∞) mapping commuting with the action of a compact Lie group Γ on V. Also suppose that $f(0,0) = 0$ and $(df)_{0,0}$ has purely imaginary eigenvalues (nonresonant) $\pm i$ (after rescaling time if necessary).

We are interested in bifurcation to periodic solutions. The basic existence theorem is the equivariant Hopf theorem which states that, under appropriate conditions there will be a symmetry-breaking branch of periodic solutions with symmetry involving (isotropy) subgroups Σ with a two-dimensional fixed-point subspace (the $\mathbf{C}$-axial subgroups). That is, there are branches of periodic solutions corresponding to each $\mathbf{C}$-axial subgroup of $\Gamma \times \mathbf{S}^1$.

At points of Hopf bifurcation in Γ-equivariant systems, generically the centre subspace is Γ-simple (see proposition 2.3.2) and there is a complex structure on this space with a natural action of the circle group $\mathbf{S}^1$.

Using [20], periodic solutions to (5.1) of period near 2π are in one-to-one correspondence with zeros of a reduced bifurcation equation

$$g(y, \lambda, \tau) = 0 \tag{5.2}$$

where if Y is the $\pm i$ real eigenspace of $(df)_{0,0}$, then $g : Y \times \mathbf{R} \times \mathbf{R} \to Y$ is C^∞ and $\Gamma \times \mathbf{S}^1$-equivariant. These are zeros of the restricted system of equations $g\ |_{\mathrm{Fix}(\Sigma)} = 0$, and since the fixed-point subspaces are invariant for g, then we are actually seeking the zero set of $g : \mathrm{Fix}(\Sigma) \times \mathbf{R} \times \mathbf{R} \to \mathrm{Fix}(\Sigma)$. In particular, if

$$\dim\ \mathrm{Fix}\ (\Sigma) = 2$$

then we end up having only two equations to solve. Moreover, when we assume that f commutes also with $\mathbf{S}^1$ we can apply theorem 2.3.6 and corollary 2.3.7: we get an explicit form of g and the stability of a periodic solution is determined by the eigenvalues of the derivative of g at the corresponding zero. Also these stability results can hold even when f does not commute with $\mathbf{S}^1$ provided the conditions of theorem 2.3.8 are satisfied.

For Hopf bifurcation with $\Gamma = \mathcal{L} \wr \mathcal{G}$ symmetry, and the generic hypothesis that Γ acts Γ-simply on the center subspace, we know by lemmas 3.2 and 3.1 of [10] that it is equivalent to assume that we can write the centre subspace as U^N, where the space U is $\mathcal{L}$-simple. Therefore, when studying Hopf bifurcation with symmetry a wreath product $\mathcal{L} \wr \mathcal{G}$, we start by considering $\mathcal{L}$ acting on a $\mathcal{L}$-simple subspace.

Let $\mathcal{O} \oplus \mathbf{Z}_2^c$ be the symmetry group of the centred cube on $\mathbf{R}^3$, where $\mathcal{O}$ denotes the group of all orientation-preserving symmetries of the cube and $\mathbf{Z}_2^c$ represents inversion through the origin. We note that we can consider this group as the wreath product $\mathbf{Z}_2 \wr \mathbf{S}_3$ (see also [14]). Also the group $(\mathcal{O} \oplus \mathbf{Z}_2^c) \dot{+} \mathbf{T}^3$ (where $\mathbf{T}^3$ represents the three-torus translations) is the largest compact subgroup of the symmetries of the Euclidean group $\mathbf{E}(3)$ that leaves invariant the space of the spatially periodic functions $u : \mathbf{R}^3 \to \mathbf{R}$ with respect to a primitive cubic lattice [8]. This group is the wreath product $\mathbf{O}(2) \wr \mathbf{S}_3$. We explain below the interest of this group in terms of physical problems. In this chapter we study Hopf bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_3$ symmetry. We begin to choose the standard action of $\mathbf{O}(2) \times \mathbf{S}^1$ on $\mathbf{C} \oplus \mathbf{C}$. Using the results of chapter 3 we find eight $\mathbf{C}$-axial subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ (that are maximal isotropy subgroups). We also obtain the complete isotropy lattice of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

We find the general form of a commuting mapping with $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$, i.e., we derive the generators for the invariants and equivariants for the action of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ up to degree six.

By the equivariant Hopf theorem there exists for each one of the maximal groups a distinct branch of periodic solutions, provided that the eigenvalues cross the imaginary axis with nonzero speed. We analyze the branching directions and the stability of these solutions in terms of the Taylor expansion of the vector field and we prove that these depend only on the coefficients of degree lower or equal three of the Taylor expansion of f around the origin. Moreover, we show that the $\mathbf{C}$-axial subgroups have 3-determined stability and so we can use theorem 2.3.8 to justify that the same stability results hold even when the original vector field f commutes only with $\mathbf{O}(2) \wr \mathbf{S}_3$.

We show that branches of periodic solutions with submaximal isotropy type can also bifurcate generically and we explore some possibilities for heteroclinic cycles between periodic solutions.

5.1 Physical motivation

Throughout this chapter we shall be studying ODEs with certain symmetries, especially Hopf bifurcation in such systems. We digress to indicate how such systems may arise from PDEs of a kind that is common in mathematical physics and elsewhere.

Systems of partial differential equations arising in many areas of applied science are often posed on $\mathbf{R}^3$ and in many cases have Euclidean symmetry $\mathbf{E}(3)$ (group of all symmetries in the space that preserves distances, i.e., translations, rotations and reflections). As a parameter is varied, time-periodic, spatially periodic solutions often arise by a bifurcation from an invariant equilibrium. Suppose that we restrict attention to solutions u that are triply spatially periodic with respect to a primitive cubic lattice, i.e.,

$$u(X) = u(X + n_1\omega_1 + n_2\omega_2 + n_3\omega_3),$$

for any integers $n_1,\ n_2,\ n_3$ and three fixed vectors that generate a primitive cubic lattice [28]: here we choose $\omega_1 = (1,0,0)$, $\omega_2 = (0,1,0)$ and $\omega_3 = (0,0,1)$. We suppose that for all the parameter values there is a trivial solution (which is therefore an $\mathbf{E}(3)$-invariant equilibrium) and that this equilibrium loses stability at some value of the parameter for which the linearization of the equation has conjugate purely imaginary eigenvalues.

To be more precise, we can write such a system of PDEs in the form

$$\frac{d}{dt}u = F(u, \lambda)$$

with $F : \mathcal{X} \times \mathbf{R} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are function spaces, the bifurcation parameter is $\lambda \in \mathbf{R}$, and $u : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$. We assume that $F(0, \lambda) = 0$ for all values of λ, and we look for time-periodic solutions by Hopf bifurcation from $u = 0$ at $\lambda = 0$ that respect the condition

$$u(X + l, t) = u(X, t)$$

for all $l \in \mathcal{L}$, where

$$\mathcal{L} = \{n_1\omega_1 + n_2\omega_2 + n_3\omega_3,\ n_1, n_2, n_3 \in \mathbf{Z}\}.$$

These solutions are in general called *planforms*. Moreover, we assume that F is $\mathbf{E}(3)$-equivariant, where the action of $\mathbf{E}(3)$ on functions u that we consider here is defined by

$$g \cdot u(X, t) = u(g^{-1}X, t)$$

for all $g \in \mathbf{E}(3)$, and so

$$F : \mathcal{X}_{\mathcal{L}} \times \mathbf{R} \to \mathcal{Y}_{\mathcal{L}}, \tag{5.3}$$

where $\mathcal{X}_{\mathcal{L}}$ and $\mathcal{Y}_{\mathcal{L}}$ represent the spaces of all $\mathcal{L}$-periodic functions in $\mathcal{X}$ and $\mathcal{Y}$ respectively.

There is a natural compact group of symmetries acting on the space of the $\mathcal{L}$-periodic solutions, which is derived from the action of the group $\mathbf{E}(3)$ on the spaces $\mathcal{X}$ and $\mathcal{Y}$. This group is

$$\Gamma = H \dot{+} \mathbf{T}^3$$

where $H = \mathcal{O} \oplus \mathbf{Z}_2^c$ is the holohedry of the lattice $\mathcal{L}$, i.e., the largest subgroup of $\mathbf{O}(3)$ that leaves $\mathcal{L}$ invariant. More details can be found for example in [8] or [9]. The group Γ is the largest group that can be constructed from $\mathbf{E}(3) = \mathbf{O}(3) \dot{+} \mathbf{R}^3$ that acts on $\mathcal{X}_{\mathcal{L}}$ (and leaves it invariant) and F in (5.3) is Γ-equivariant. Thus we can suppose that the real generalized eigenspace $G_{\mp i}\ (dF)_{0,0}$ for the eigenvalues $\mp i$ is finite-dimensional (for F in (5.3)) and we demand that

$$V = G_{\mp i}\ (dF)_{0,0} \neq \{0\}$$

for bifurcation to occur. We scale time so that the (only) purely imaginary eigenvalues of $(dF)_{0,0}$ are $\mp i$.

We consider here the case when the symmetry forces the dimension of V to be twelve, i.e., when the real generalized eigenspace associated with $\mp i$ is of the form $U \oplus U$ where U is Γ-absolutely irreducible (and six-dimensional). The space V can be identified with the six-dimensional complex vector space spanned by the six travelling waves

$$e^{i2\pi(t - \omega_j \cdot X)}, \qquad j = 1, \ldots, 6,$$

where $\omega_4 = -\omega_1,\ \omega_5 = -\omega_2$ and $\omega_6 = -\omega_3$. Here, we are assuming that the functions in $\mathcal{X}_{\mathcal{L}}$ are regular enough to have Fourier expansions in terms of these travelling waves. The six vectors ω_i for $i = 1, \ldots, 6$ are called *wave vectors* and were selected at the midpoint of each of the six surfaces of a cube. Explicitly, we can identify $(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{C}^6$ with the function $u : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ defined by

$$u(X,t) = \mathrm{Re}\{\sum_{j=1}^{3} z_{2j-1} e^{i2\pi(t-\omega_j \cdot X)} + \sum_{j=1}^{3} z_{2j} e^{i2\pi(t+\omega_j \cdot X)}\}. \tag{5.4}$$

The center manifold theorem reduces the original system to a system of differential equations on V [26], and it is this system that it is studied in this chapter.

Hopf bifurcations leading to time-periodic spatially periodic solutions in Euclidean equivariant systems have been considered before. In [31], Roberts *et al.* considered the six-dimensional irreducible representation of $\mathbf{D}_6 \dot{+} \mathbf{T}^2$, which is associated with an hexagonal lattice. In [34], Silber and Knobloch considered the four-dimensional irreducible representation of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ associated with a square lattice. In [35], Silber *et al.* study Hopf bifurcation for the rhombic lattice, where the group is $\mathbf{D}_2 \dot{+} \mathbf{T}^2$.

5.2 Isotropy subgroups

5.2.1 Group action of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$

Throughout let $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_3$ and $V = \mathbf{C} \oplus \mathbf{C}$, which is abstracted from the physical situation described in section 5.3 below. Consider the following action of $\mathbf{O}(2) \times \mathbf{S}^1$ on V [16]:

$$\begin{array}{ll} \theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2), & (\theta \in \mathbf{S}^1) \\ \kappa(z_1, z_2) = (z_2, z_1), & (\kappa = \text{flip in } \mathbf{O}(2)) \\ \psi(z_1, z_2) = (e^{-i\psi} z_1, e^{i\psi} z_2) & (\psi \in \mathbf{SO}(2)). \end{array}$$

Here $\mathbf{S}_3$ is the group of the permutations of the set $\{1, 2, 3\}$. Also the group multiplication in $\Gamma \times \mathbf{S}^1$ is given by

$$(h, \tau, \theta_1)(l, \sigma, \theta_2) = (h\tau(l), \tau\sigma, \theta_1\theta_2)$$

and the action of $\Gamma \times \mathbf{S}^1$ on V^3 is given by:

$$((l_1, l_2, l_3), \sigma, \theta)w = ((l_1, \theta)w_{\sigma^{-1}(1)}, (l_2, \theta)w_{\sigma^{-1}(2)}, (l_3, \theta)w_{\sigma^{-1}(3)}),$$

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(a,0),\ a>0$	$\widetilde{\mathbf{SO}}(2)=\{(\theta,\theta)\}$	$\{(z_1,0)\}$
$(a,a),\ a>0$	$\mathbf{Z}_2\oplus\mathbf{Z}_2^c=\{(1,0),(\kappa,0),(\pi,\pi),(\kappa\pi,\pi)\}$	$\{(z_1,z_1)\}$

Table 5.1: Orbit representatives, isotropy subgroups and fixed-point subspaces for the standard action of $\mathbf{O}(2)\times\mathbf{S}^1$ on $\mathbf{C}\oplus\mathbf{C}$.

for $(l_1,l_2,l_3)\in\mathbf{O}(2)^3,\ \sigma\in\mathbf{S}_3$ and $\theta\in\mathbf{S}^1$ (with $w=(w_1,\ w_2,\ w_3)\in V^3$).

Note that as V is $\mathbf{O}(2)$-simple, also V^3 is Γ-simple by the results stated in section 3.1.

5.2.2 C-axial groups of $(\mathbf{O}(2)\wr\mathbf{S}_3)\times\mathbf{S}^1$

Using the method of section 3.2, we first have to calculate the $\mathbf{C}$-axial subgroups of $\mathbf{O}(2)\times\mathbf{S}^1$. By [20] for example (proposition XVII 1.1.) we have (up to conjugacy) two types of $\mathbf{C}$-axial subgroups. See table 5.1.

Usually periodic solutions with symmetry $\widetilde{\mathbf{SO}}(2)$ are called *rotating waves* and those with symmetry $\mathbf{Z}_2\oplus\mathbf{Z}_2^c$ are *standing waves*.
We can now compute the $\mathbf{C}$-axial groups of $\Gamma\times\mathbf{S}^1$.

Theorem 5.2.1 *There are eight conjugacy classes of* $\mathbf{C}$*-axial subgroups of* $\Gamma\times\mathbf{S}^1$ *with the above action on* V^3*. They are listed, together with their orbit representatives and fixed-point subspaces, in table 5.2. In table 5.3 we present the generators of these groups. We use the notation of chapter 3. Also* a *denotes a real positive number,* $\xi=\frac{2\pi}{3}\in\mathbf{S}^1,\ z_i\in\mathbf{C}$*, for* $i=1,\ldots,6$*, and* $\kappa_1=(\kappa,1,1),\ \kappa_2=(1,\kappa,1),\ \kappa_3=(1,1,\kappa)\in\mathbf{O}(2)^3$.

Proof of theorem 5.2.1. Up to conjugacy we need to consider only the blocks

$$J=\{1\},\quad J=\{1,2\},\quad J=\{1,2,3\},$$

and by lemmas 3.2.4 and 3.2.5 of section 3.2 we need to look for $\mathbf{C}$-axial subgroups Σ_w with

$$\begin{aligned}w&=(w_1,0,0),\\ w&=(w_1,\theta_2 w_1,0),\\ w&=(w_1,\theta_2 w_1,\theta_3 w_1),\end{aligned}$$

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(a,0,0,0,0,0)$	$\Sigma_1 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1\})$	$\{(z_1,0,0,0,0,0)\}$
$(a,a,0,0,0,0)$	$\Sigma_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1\})$	$\{(z_1,z_1,0,0,0,0)\}$
$(a,0,a,0,0,0)$	$\Sigma_3 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1,2\})$	$\{(z_1,0,z_1,0,0,0)\}$
$(a,a,a,a,0,0)$	$\Sigma_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\})$	$\{(z_1,z_1,z_1,z_1,0,0)\}$
$(a,a,ia,ia,0,0)$	$\Sigma_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\},(12),\{1\},4)$	$\{(z_1,z_1,iz_1,iz_1,0,0)\}$
$(a,0,a,0,a,0)$	$\Sigma_6 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1,2,3\})$	$\{(z_1,0,z_1,0,z_1,0)\}$
(a,a,a,a,a,a)	$\Sigma_7 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2,3\})$	$\{(z_1,z_1,z_1,z_1,z_1,z_1)\}$
$(a,a,\xi a,\xi a,\xi^2 a,\xi^2 a)$	$\Sigma_8 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2,3\},(123),\{1\},3)$	$\{(z_1,z_1,\xi z_1,\xi z_1,\xi^2 z_1,\xi^2 z_1)\}$

Table 5.2: Orbit representatives, $\mathbf{C}$-axial groups and fixed-point subspaces of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

where $\theta_2,\ \theta_3 \in \mathbf{S}^1$ and $w_1 \in V$ is such that

$$\dim \mathrm{Fix}_V\ (\Sigma_{w_1}) = 2.$$

Here Σ_{w_1} is the isotropy subgroup of w_1 in $\mathbf{O}(2) \times \mathbf{S}^1$.

For Σ_{w_1}, up to conjugacy, we have two choices: $\widetilde{\mathbf{SO}}(2)$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ As the first one is of twist type $\mathbf{S}^1$, if $w_1 \in \mathrm{Fix}_V(\widetilde{\mathbf{SO}}(2))$, then we can assume (up to conjugacy) that w has equal nonzero components and we get $\Sigma_1,\ \Sigma_3$ and Σ_6: for example, as $(\theta_2, \theta_2) \in \widetilde{\mathbf{SO}}(2)$, then $w = (w_1, \theta_2 w_1, 0) = (w_1, (\theta_2^{-1}\theta_2, \theta_2)w_1, 0) = ((1, \theta_2^{-1}, 1), 1, 0)(w_1, w_1, 0)$ and so the isotropy subgroup of $(w_1, \theta_2 w_1, 0)$ is conjugate to the isotropy subgroup of $w = (w_1, w_1, 0)$. See also corollaries 3.2.9 and 3.2.10.

Let now $w_1 \in \mathrm{Fix}_V\ (\mathbf{Z}_2 \oplus \mathbf{Z}_2^c)$. Note that $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ is of twist type $\mathbf{Z}_2$. It follows that $\Sigma_{(w_1,0,0)}$ is Σ_2 and if $\theta_2 \in \mathbf{Z}_2$, then $\Sigma_{(w_1,\theta_2 w_1,0)}$ is conjugate to Σ_4 by the same reason as before. Now using proposition 3.2.7 we need to consider only Σ_5 for the case where $\theta_2 \notin \mathbf{Z}_2$ and $w = (w_1, \theta_2 w_1, 0)$. Note that, once we fixe $s' = 2$, the possibilities for p are 2 and 4. But for $p = 2$ we must have $\theta_2 \in \mathbf{Z}_2$.

Similarly, if θ_2 and θ_3 are in $\mathbf{Z}_2$, then we can conjugate $\Sigma_{(w_1,\theta_2 w_1,\theta_3 w_1)}$ to Σ_7. If some θ_2 or θ_3 is not in $\mathbf{Z}_2$, then the only possibility for $w = (w_1, \theta_2 w_1, \theta_3 w_1)$ to be fixed by a $\mathbf{C}$-axial subgroup (up to conjugacy) is if $\theta_2 = \xi$ and $\theta_3 = \xi^2$ and we have Σ_8 (see proposition 3.2.7 and note that now with $s' = 3$ we only need to consider $p = 3$ or $p = 6$ since the corresponding isotropy subgroups are conjugate). $\square$

Remark 5.2.2 *If $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_N$ is acting on V^N, the $\mathbf{C}$-axial groups of $(\mathbf{O}(2) \wr \mathbf{S}_{N-1}) \times \mathbf{S}^1$ acting on V^{N-1} are, with appropriate adjustment, included in those of $\Gamma \times \mathbf{S}^1$. As an example, it is the case of the group $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$ presented in [34] that can be seen as the group $(\mathbf{O}(2) \wr \mathbf{S}_2) \times \mathbf{S}^1$. Note that the $\mathbf{C}$-axial subgroups $\Sigma_1, \ldots, \Sigma_5$ are in precise correspondence with the $\mathbf{C}$-axial subgroups obtained in [34].*

5.2.3 Isotropy lattice of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$

The isotropy lattice of $\mathbf{O}(2) \times \mathbf{S}^1$ is given in figure 5.1. As stated in chapter 2, if we consider the action of $\mathbf{O}(2) \times \mathbf{S}^1$ on the space of $\mathcal{C}_{2\pi}$ of 2π-periodic functions $\mathbf{R} \to \mathbf{R}^4$ be defined by $(\gamma, \theta) \cdot v(t) = \gamma \cdot v(t + \theta)$, for $\gamma \in \mathbf{O}(2)$ and $\theta \in \mathbf{S}^1$, then we can speak in the symmetry group of $v(t)$ as the isotropy

Isotropy subgroup	Generators
$\Sigma_1 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1\})$	$\mathbf{1} \times \mathbf{O}(2)^2,\ \{((\theta, 1, 1), 1, \theta)\},\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Sigma_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\})$	$\mathbf{1} \times \mathbf{O}(2)^2,\ \kappa_1,\ ((\pi, 1, 1), 1, \pi),\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Sigma_3 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1, 2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \{((\theta, \theta, 1), 1, \theta)\},\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Sigma_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ ((\pi, \pi, 1), 1, \pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Sigma_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\}, (12), \{1\}, 4)$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ \kappa_2,\ ((\pi, \pi, 1), 1, \pi),\ ((\pi, 1, 1), (12), \frac{\pi}{2})$
$\Sigma_6 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1, 2, 3\})$	$\{((\theta, \theta, \theta), 1, \theta)\},\ \mathbf{S}_3$
$\Sigma_7 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2, 3\})$	$\kappa_1,\ ((\pi, \pi, \pi), 1, \pi),\ \mathbf{S}_3$
$\Sigma_8 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2, 3\}, (123), \{1\}, 3)$	$\kappa_1,\ \kappa_2,\ \kappa_3,\ ((\pi, \pi, \pi), 1, \pi),\ (1^3, (123), \xi)$

Table 5.3: Generators of the $\mathbf{C}$-axial groups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

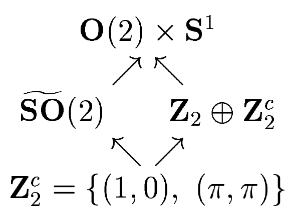


Figure 5.1: Isotropy lattice of $\mathbf{O}(2) \times \mathbf{S}^1$.

subgroup of v with respect to this action. As (π, π) acts trivially on V, every periodic solution satisfies $\mathrm{R}_\pi \cdot v(t) = v(t+\pi)$, that is the effect of a spatial rotation of $v(t)$ through angle π is the same as shifting the phase of $v(t)$ by half a period. The rotating waves also satisfy $\mathrm{R}_\theta \cdot v(t) = v(t+\theta)$, for all θ and the standing waves satisfy $\kappa \cdot v(t) = v(t)$.

Theorem 5.2.3 *The complete isotropy lattice of* $\Gamma \times \mathbf{S}^1$ *up to conjugacy is given by the groups listed in table 5.2 together with the isotropy subgroups in table 5.4 (these are the groups with fixed-point subspaces with dimension higher than two). The groups* Δ_i *have four-dimensional fixed-point subspaces,* $\mathrm{Fix}(\Pi_i)$ *are six-dimensional,* $\mathrm{Fix}(\Lambda_i)$ *are eight-dimensional,* $\mathrm{Fix}(\Phi)$ *is ten-dimensional. Finally the group* T *acts trivially on the all space* V^3*. In table 5.5 we present the generators of the subgroups of table 5.4. We use the notation of chapter 3 and* ξ *denotes* $\frac{2\pi}{3} \in \mathbf{S}^1$*.*

Proof of theorem 5.2.3. We can apply theorem 3.4.1. Let $\Sigma = \Sigma_w$ be an isotropy subgroup of $\Gamma \times \mathbf{S}^1$, where $w \in V^3$ is nonzero. The subgroup Σ is not $\Gamma \times \mathbf{S}^1$ since $\mathrm{Fix}_{\mathrm{V}^3}(\Gamma \times \mathbf{S}^1) = \{0\}$. The possible partitions of the set of indices $\{1,2,3\}$ in blocks that we need to consider are:

$$\{1,2,3\} = \{1\} \cup \{2\} \cup \{3\},$$

$$\{1,2,3\} = \{1,2\} \cup \{3\},$$

$$\{1,2,3\} = \{1,2,3\}.$$

Suppose that $w = (w_1, w_2, w_3)$ where $w_1,\ w_2,\ w_3 \in V$. Using the proof of theorem 3.4.1, we have to look for isotropy subgroups of elements of the kind

$(a.1)$ $(w_1, 0, 0)$,

$(b.1)$ $(w_1, \theta_2 w_1, 0)$, $(b.2)$ $(w_1, w_2, 0)$,

$(c.1)$ $(w_1, \theta_2 w_1, \theta_3 w_1)$, $(c.2)$ $(w_1, \theta_2 w_1, w_2)$, $(c.3)$ (w_1, w_2, w_3),

where Σ_{w_i} for $i = 1,2,3$ are isotropy subgroups of $\mathbf{O}(2) \times \mathbf{S}^1$ (and distinct from $\mathbf{O}(2) \times \mathbf{S}^1$, i.e., the vectors $w_1,\ w_2$ and w_3 are nonnull) and so

$$\Sigma_{w_1},\ \Sigma_{w_2},\ \Sigma_{w_3} \in \{\widetilde{\mathbf{SO}}(2),\ \mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\ \mathbf{Z}_2^c\}.$$

Isotropy subgroup	Fixed-point subspace
$\Delta_1 = \Sigma(\mathbf{Z}_2^c, \{1\})$	$\{(z_1, z_2, 0, 0, 0, 0)\}$
$\Delta_2 = \Sigma(\mathbf{Z}_2^c, \{1, 2\})$	$\{(z_1, z_2, z_1, z_2, 0, 0)\}$
$\Delta_3 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}, (12), \{1\}, 4)$	$\{(z_1, z_2, iz_1, iz_2, 0, 0)\}$
$\Delta_4 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{2\})$	$\{(z_1, 0, z_2, 0, 0, 0)\}$
$\Delta_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\})$	$\{(z_1, z_1, z_2, z_2, 0, 0)\}$
$\Delta_6 = \Sigma(\mathbf{Z}_2^c, \{1, 2, 3\})$	$\{(z_1, z_2, z_1, z_2, z_1, z_2)\}$
$\Delta_7 = \Sigma(\mathbf{Z}_2^c, \{1, 2, 3\}, (123), \{1\}, 3)$	$\{(z_1, z_2, \xi z_1, \xi z_2, \xi^2 z_1, \xi^2 z_2)\}$
$\Delta_8 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1, 2\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, 0, z_1, 0, z_2, 0)\}$
$\Delta_9 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_1, z_1, z_2, z_2)\}$
$\Delta_{10} = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\}, (12), \{1\}, 4) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, z_1, iz_1, iz_1, z_2, 0)\}$
$\Pi_1 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\{(z_1, z_1, z_2, z_3, 0, 0)\}$
$\Pi_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_2, z_2, z_3, z_3)\}$
$\Pi_3 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{2\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, 0, z_2, 0, z_3, 0)\}$
$\Pi_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2, 3\})$	$\{(z_1, z_1, z_2, z_3, z_2, z_3)\}$
$\Pi_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_1, z_1, z_2, z_3)\}$
$\Pi_6 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}, (12), \{1\}, 4) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, z_2, iz_1, iz_2, z_3, 0)\}$

Table 5.4: Fixed-point subspaces corresponding to the submaximal isotropy subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

Isotropy subgroup	Generators
$\Delta_1 = \Sigma(\mathbf{Z}_2^c, \{1\})$	$\mathbf{1} \times \mathbf{O}(2)^2,\ ((\pi,1,1),1,\pi),\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Delta_2 = \Sigma(\mathbf{Z}_2^c, \{1,2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ ((\pi,\pi,1),1,\pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Delta_3 = \Sigma(\mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$	$\mathbf{1}^2 \times \mathbf{O}(2),\ ((\pi,\pi,1),1,\pi),$ $((\pi,1,1),(12),\frac{\pi}{2})$
$\Delta_4 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \{(\theta,\theta,1),1,\theta)\}$
$\Delta_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ \kappa_2,\ ((\pi,\pi,1),1,\pi)$
$\Delta_6 = \Sigma(\mathbf{Z}_2^c, \{1,2,3\})$	$((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_3$
$\Delta_7 = \Sigma(\mathbf{Z}_2^c, \{1,2,3\}, (123), \{1\}, 3)$	$((\pi,\pi,\pi),1,\pi),\ (1^3,(123),\xi)$
$\Delta_8 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1,2\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(\theta,\theta,\theta),1,\theta)\},\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Delta_9 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ \kappa_3,\ ((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Delta_{10} = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$ $\cap\ \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\kappa_1,\ \kappa_2,\ ((\pi,\pi,\pi),1,\pi),$ $((\pi,1,\frac{\pi}{2}),(12),\frac{\pi}{2})$
$\Pi_1 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ ((\pi,\pi,1),1,\pi)$
$\Pi_2 = \bigcap_{i=1,2,3} \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{i\})$	$\kappa_1,\ \kappa_2,\ \kappa_3,\ ((\pi,\pi,\pi),1,\pi)$
$\Pi_3 = \bigcap_{i=1,2,3} \Sigma(\widetilde{\mathbf{SO}}(2), \{i\})$	$\{((\theta,\theta,\theta),1,\theta)\}$
$\Pi_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2,3\})$	$\kappa_1,\ ((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Pi_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ ((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Pi_6 = \Sigma(\mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$((\pi,\pi,\pi),1,\pi),\ ((\pi,1,\frac{\pi}{2}),(12),\frac{\pi}{2})$

Table 5.5: Generators of the submaximal isotropy subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

Isotropy subgroup	Fixed-point subspace
$\Lambda_1 = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\{(z_1, z_2, z_3, z_4, 0, 0)\}$
$\Lambda_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_2, z_2, z_3, z_4)\}$
$\Lambda_3 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_2, z_1, z_2, z_3, z_4)\}$
$\Phi = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_2, z_3, z_4, z_5)\}$
$T = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_2, z_3, z_4, z_5, z_6)\}$

Table 5.4: Continuation.

Isotropy subgroup	Generators
$\Lambda_1 = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ ((\pi, \pi, 1), 1, \pi)$
$\Lambda_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ \kappa_2,\ ((\pi, \pi, \pi), 1, \pi)$
$\Lambda_3 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$((\pi, \pi, \pi), 1, \pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Phi = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ ((\pi, \pi, \pi), 1, \pi)$
$T = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$((\pi, \pi, \pi), 1, \pi)$

Table 5.5: Continuation.

Also $\theta_2,\ \theta_3 \in \mathbf{S}^1$, and for each case we have:

$(a.1)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{1\}) = \{1\}$,

$(b.1)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{1, 2\}) = \{1, 2\}$, $(b.2)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{i\}) = \{i\},\ i = 1, 2, 3,$

$(c.2)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{1, 2\}) = \{1, 2\}$, $(c.3)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{i\}) = \{i\},\ i = 1, 2, 3,$

i.e., the elements $w_1,\ w_2,\ w_3 \in V \setminus \{0\}$ are in distinct $\mathbf{O}(2) \times \mathbf{S}^1$ orbits.

Case $(a.1)$

We get Σ_1 and Σ_2 (as we have seen in theorem 5.2.1) and Δ_1.

Case $(b.1)$

As in theorem 5.2.1, we get $\Sigma_3,\ \Sigma_4$ and Σ_5 if we consider Σ_{w_1} to be maximal. See now the case $\Sigma_{w_1} = \mathbf{Z}_2^c$. As $\mathbf{Z}_2^c = \{(1,0),\ (\pi,\pi)\} \subset \mathbf{O}(2) \times \mathbf{S}^1$ is of type $\mathbf{Z}_2$, we only need to consider up to conjugacy, the groups $\Sigma(\mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$ and $\Sigma(\mathbf{Z}_2^c, \{1,2\})$, i.e., we obtain Δ_2 and Δ_3.

Case $(b.2)$

By theorem 3.4.1 we have

$$\Sigma_{(w_1,w_2,0)} = \Sigma_{(w_1,0,0)} \cap \Sigma_{(0,w_2,0)}.$$

The isotropy subgroup $\Sigma_{(w_1,0,0)}$ is of type $\Sigma(A,\{1\})$, where

$$A \in \{\widetilde{\mathbf{SO}}(2),\ \mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\ \mathbf{Z}_2^c\}.$$

Similarly, the isotropy group $\Sigma_{(0,w_2,0)}$ is of type $\Sigma(B,\{2\})$, where B belongs to the same set as A. Considering the possible intersections, we get $\Delta_4,\ \Delta_5,\ \Pi_1$ and Λ_1.

Case $(c.1)$

This case is similar to case $(b.1)$: we obtain the maximal $\Sigma_6,\ \Sigma_7$ and Σ_8 (as before) and again, because $\mathbf{Z}_2^c$ is of twist type $\mathbf{Z}_2$, we consider only $\Sigma(\mathbf{Z}_2^c, \{1,2,3\}, (123), \{1\}, 3)$ and $\Sigma(\mathbf{Z}_2^c, \{1,2,3\})$, i.e., the groups Δ_7 and Δ_6 respectively.

Case $(c.2)$

Now

$$\Sigma_{(w_1,\theta_2 w_1,w_2)} = \Sigma_{(w_1,\theta_2 w_1,0)} \cap \Sigma_{(0,0,w_2)}.$$

Up to conjugacy, the group $\Sigma_{(w_1,\theta_2 w_1,0)}$ is of type $\Sigma(A,\{1,2\}$ with

$$A \in \{\widetilde{\mathbf{SO}}(2),\ \mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\ \mathbf{Z}_2^c\}$$

or

$$\Sigma_{(w_1,\theta_2 w_1,0)} \in \{\Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4),\ \Sigma(\mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)\}.$$

We note that if $\Sigma_{w_1} = \widetilde{\mathbf{SO}}(2)$, then as it is of twist type $\mathbf{S}^1$, the group $\Sigma_{(w_1,\theta_2 w_1,0)}$ is always conjugate to $\Sigma_{(w_1,w_1,0)}$ and so it is included in the first possibility.

The group $\Sigma_{(0,0,w_2)}$ can be $\Sigma(B, \{3\})$ where again B is maximal or $\mathbf{Z}_2^c$. We obtain, up to conjugacy, the groups $\Delta_8,\ \Delta_9,\ \Delta_{10}$ and $\Pi_4,\ \Pi_5,\ \Pi_6$ and Λ_3.

Case $(c.3)$
We have the intersection

$$\Sigma_{(w_1,w_2,w_3)} = \Sigma_{(w_1,0,0)} \cap \Sigma_{(0,w_2,0)} \cap \Sigma_{(0,0,w_3)},$$

where the groups $\Sigma_{(w_1,0,0)},\ \Sigma_{(0,w_2,0)}$ and $\Sigma_{(0,0,w_3)}$ can be $\Sigma(A_1, \{1\}),\ \Sigma(A_2, \{2\})$ and $\Sigma(A_3, \{3\})$ respectively, with A_i maximal or $\mathbf{Z}_2^c$ (in $\mathbf{O}(2) \times \mathbf{S}^1$). We obtain $\Pi_2,\ \Pi_3,\ \Lambda_2,\ \Phi$ and T. $\square$

Figure 5.2 contains the isotropy lattice of $\Gamma \times \mathbf{S}^1$ where each entry represents the entire conjugacy class of that group. We use $H \to K$ to mean that $g^{-1}Hg \subseteq K$ for some $g \in \Gamma \times \mathbf{S}^1$ where $H,\ K$ are subgroups of $\Gamma \times \mathbf{S}^1$. In terms of fixed-point subspaces, this implies that $g(\mathrm{Fix(K)}) \subseteq \mathrm{Fix(H)}$. For example, $((1, \pi/2, 1), 1, 0)(\mathrm{Fix}(\Delta_8)) \subseteq \mathrm{Fix}(\Pi_6)$ and so we have the arrow $\Pi_6 \to \Delta_8$.

5.3 Planforms

We now show some diagrams of planforms that are time-periodic and spatially periodic with respect to the primitive cubic lattice considered in section 5.1. We consider here functions $u(X, t)$ with $X \in \mathbf{R}^3$ corresponding to the maximal solutions with symmetry $\Sigma_1, \ldots, \Sigma_8$ obtained in theorem 5.2.1.

At a Hopf bifurcation point of a PDE, subject to the usual nondegeneracy conditions on the bifurcation, the branch of bifurcating solutions is tangent to a branch of solutions of the linearized PDE at the bifurcation point (in any appropriate Banach space of functions). By continuity it follows that sufficiently near the bifurcation point, the periodic solutions along the Hopf branch are arbitrarily closely approximated by solutions of the linearized PDE. This is also the case in symmetric Hopf bifurcation, when solution branches are obtained by passing to a Liapunov-Schmidt reduced equation or an equivariant center manifold reduction and restricting to a fixed-point subspace. In other words, we can approximate the relevant periodic solutions of the nonlinear PDE by a suitable superposition of linearized eigenfunctions. For similar reasons, planforms of periodic solutions of the

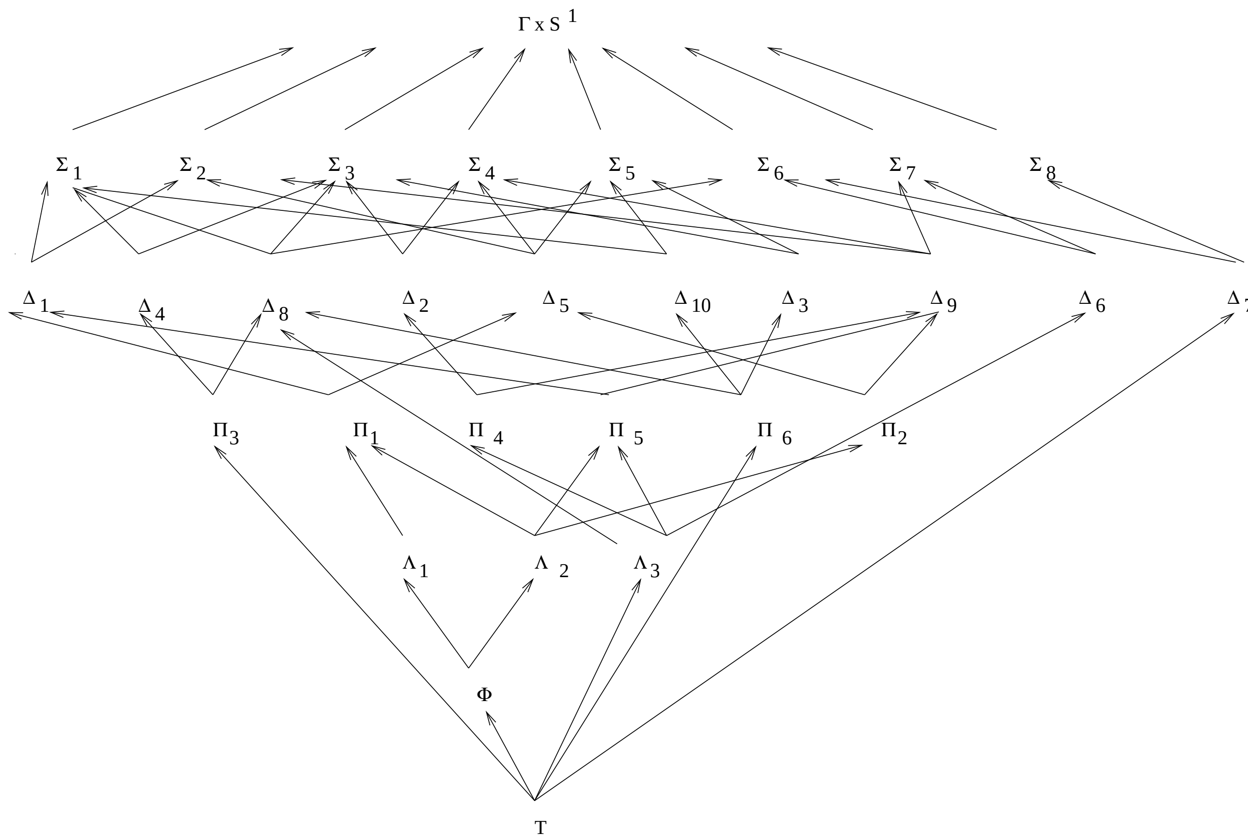


Figure 5.2: Isotropy lattice of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

nonlinear PDE, near an equivariant Hopf bifurcation point, are well approximated by planforms of suitable linearized eigenfunctions. This observation is often exploited in the literature, and we shall continue that practice here.

The Fourier sum (5.4) is plotted at discrete times for representative elements $(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{C}^6$ of the fixed-point subspaces listed in table 5.2. We identify $(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{C}^6$ with the function $u : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ defined by

$$u(X,t) = \mathrm{Re}\{z_1 e^{i2\pi(t-X_1)} + z_3 e^{i2\pi(t-X_2)} + z_5 e^{i2\pi(t-X_3)} + z_2 e^{i2\pi(t+X_1)} + z_4 e^{i2\pi(t+X_2)} + z_6 e^{i2\pi(t+X_3)}\}. \tag{5.5}$$

Note that, using this identification, the action of $\mathbf{SO}(2)^3$ on $(\mathbf{C} \oplus \mathbf{C})^3$ can be described in terms of translations: identify (ψ_1, ψ_2, ψ_3) with the translation $u(X,t) \to u(X+d,t)$, where $d \cdot \omega_i = \psi_i/2\pi$, for $i = 1,2,3$. Also $\theta \in \mathbf{S}^1$ is identified with the phase-shift in time, i.e., $u(X,t) \to u(x, t+\theta/2\pi)$. The elements of $\mathbf{S}_3$ permute the vectors ω_i and the flips $\kappa_i \in \mathbf{O}(2)^3$ act as $\kappa_i \omega_i = -\omega_i$, for $i = 1,2,3$.

The groups Σ_1, Σ_3 and Σ_6 are of twist type $\mathbf{S}_1$. The corresponding planforms move at constant velocity but remain the same. The groups Σ_2, Σ_4 and Σ_7 are of twist type $\mathbf{Z}_2$. The corresponding planforms also remain the same for all the time. Their symmetries are only spatial symmetries (except $((\pi,1,1),1,\pi)$ in Σ_2, $((\pi,\pi,1),1,\pi)$ in Σ_4 and $((\pi,\pi,\pi),1,\pi)$ in Σ_7). The groups Σ_5 and Σ_8 have non-trivial spatial and spatial-temporal symmetries. Thus their planforms change with the time.

In each case, we consider the spatial domain containing four unit cells (cubes) and the density plot of the planform $u(X,t)$ is projected in the planar domain (X_1, X_2) (or (X_1, X_3) when indicated), thus for four unit squares (we consider $X_1, X_2 \in [0,2]$). The grey scale indicates the magnitude of $u(X,t)$, with white denoting maximum and black minimum. The planforms corresponding to the symmetries Σ_i for $i = 1, \ldots, 5$ are two-dimensional planforms in the sense that they don't depend on the third spatial variable X_3. These are the planforms obtained by Silber and Knobloch [34] and we adopt the same names as the ones considered there. For a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_2 we take $(z_1, z_2, z_3, z_4, z_5, z_6) = (a, a, 0, 0, 0, 0)$ with $a = 1$. For a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_4, we take $(z_1, z_2, z_3, z_4, z_5, z_6) = (a, a, a, a, 0, 0)$ with $a = 1$. These are called *standing rolls* and *standing squares* planforms respectively [34]. We show in figure 5.3 the plot and in figure 5.4 the density plot of these planforms $u(X,t)$ for the times $t = 0$ and $t = 1/2$. For a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_7, we consider $(z_1,z_2,z_3,z_4,z_5,z_6)=(a,a,a,a,a,a)$ with $a=1$. We call a *standing cubes* planform. Figure 5.4 also contains the densityplot of $u(X_1,X_2,X_3,0)$ for $X_2=0,\ 1/4,\ 1/2,\ 3/4$ and $X_3=0,\ 1/4,\ 1/2,\ 3/4$ (equal because of the $\mathbf{S}_3$-symmetry), and of $u(X_1,X_2,X_3,1/2)$ for $X_3=0,\ 1/4,\ 1/2,\ 3/4$. If $0\le t<1/4$, then $U(X,t)$ has maximum $6\cos(2\pi t)$ at the points (n_1,n_2,n_3) with $n_i\in\mathbf{Z}$ and minimum $-6\cos(2\pi t)$ at $(1/2+n_1,1/2+n_2,1/2+n_3)$. If $1/4<t\le 1/2$, then $U(X,t)$ has maximum $-6\cos(2\pi t)$ at the points $(1/2+n_1,1/2+n_2,1/2+n_3)$ with $n_i\in\mathbf{Z}$ and minimum $6\cos(2\pi t)$ at (n_1,n_2,n_3).

Consider now a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_1. We take $(z_1,z_2,z_3,z_4,z_5,z_6)=(a,0,0,0,0,0)$ with $a=1$. For a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_3, we consider $(z_1,z_2,z_3,z_4,z_5,z_6)=(a,0,a,0,0,0)$ with $a=1$. As in [34], these planforms are called *travelling rolls* and *travelling squares* planforms, respectively. We show in figure 5.5 the plot and in figure 5.6 the density plot of these planforms $u(X,t)$ for the times $t=0$ and $t=1/2$. For a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_6, we consider $(z_1,z_2,z_3,z_4,z_5,z_6)=(a,0,a,0,a,0)$ with $a=1$. We call this a *travelling cubes* planform. Figure 5.6 also contains the densityplot of these planforms $u(X_1,X_2,X_3,0)$ for $X_2,X_3=0,\ 1/4,\ 1/2,\ 3/4$ (equal because of the $\mathbf{S}_3$ symmetry) and $u(X_1,X_2,X_3,1/2)$ for $X_3=0,\ 1/4,\ 1/2,\ 3/4$. In this case, the maximum of $u(X,t_0)$ (for $0\le t_0<1/2$ is always 3 at points $(t_0+n_1,t_0+n_2,t_0+n_3)$ and the minimum is -3 at $(1/2+t_0+n_1,1/2+t_0+n_2,1/2+t_0+n_3)$ for integers n_1,n_2,n_3.

For a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_5, we take $(z_1,z_2,z_3,z_4,z_5,z_6)=(a,a,ia,ia,0,0)$ with $a=1$. As in [34] we call an *alternating rolls* planform. We show in figure 5.7 the density plot of the planform $u(X,t)$ for the times $t=i/8,\ i=0,\dots,7$.

For a planform $u(X,t)$ corresponding to a periodic solution with symmetry Σ_8, let $(z_1,z_2,z_3,z_4,z_5,z_6)=(a,a,e^{2\pi/3}a,e^{2\pi/3}a,e^{4\pi/3}a,e^{4\pi/3}a)$ with $a=1$. We call an *alternating cubes* planform. Figure 5.7 also contains the plot of $u(X,t)$ for $X_3=0$ and $t=0$ and the density plot of $u(X,t)$ for $X_3=0,\ 1/3,\ 1/2,\ 2/3$ and $t=0$. Figure 5.8 contains the plot of $u(X,t)$ for $X_2=0,\ t=1/12$ and $X_3=0,\ t=1/12$. Also contains the density plot of $u(X,t)$ for $X_2=0,\ 1/3,\ 1/2,\ 2/3,\ t=1/12$ (projection on the plane (X_1,X_3)) and $X_3=0,\ 1/3,\ 1/2,\ 2/3,\ t=1/12$ (projection on the plane (X_1,X_2)). Figure 5.9 contains (for $X_3=0,\ 1/3,\ 1/2,\ 2/3$) the density plot of this planform $u(X,t)$ for the times $t = i/12,\ i = 2,\dots,5$. Figures 5.10 and 5.11 contain (for $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$) the density plot of this planform $u(X,t)$ for the times $t = i/12,\ i = 6,\dots,11$. Note the symmetry $(\pi,\pi,\pi),1,\pi)$ of these planforms (comparing with figures 5.7, ..., 5.9). For $0 \le t < 1/12$, the maximum of $u(X,t)$ is at points $(n_1, 1/2+n_2, 1/2+n_3)$ and the minimum at $(1/2+n_1, n_2, n_3)$ for integers n_i. For $1/12 < t < 1/4$, the maximum of $u(X,t)$ is at the points $(n_1, 1/2+n_2, n_3)$ and the minimum at $(1/2+n_1, n_2, 1/2+n_3)$. For $1/4 < t < 5/12$, the maximum of $u(X,t)$ is at the points $(1/2+n_1, 1/2+n_2, n_3)$ and the minimum at $(n_1, n_2, 1/2+n_3)$. For $5/12 < t < 1/2$, the maximum of $u(X,t)$ is at the points $(1/2+n_1, n_2, 1/2+n_3)$ and the minimum at $(n_1, 1/2+n_2, n_3)$. For $u(X, 1/12)$, the maximum is at any point $(n_1, 1/2+n_2, X_3)$, where n_1, n_2 are integers, and the minimum at $(1/2+n_1, n_2, X_3)$. For $t = 1/4$, for the points $(X_1, 1/2+n_1, n_2)$ the function $u(X, 1/4)$ has the maximum and minimum for $(X_1, n_1, 1/2+n_2)$. Finally, for $t = 5/12$, the maximum is for points $(1/2+n_1, X_2, n_2)$ and the minimum for $(n_1, X_2, 1/2+n_2)$.

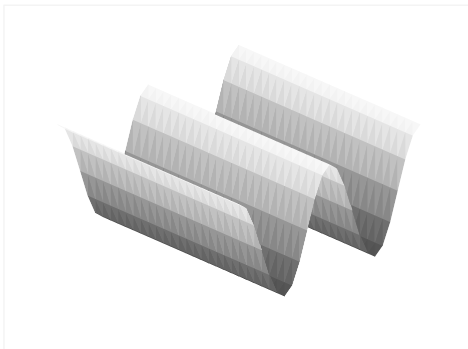
5.4 Invariant theory for $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$

In order to determine the direction of branching and the stability of the bifurcating branches of periodic solutions, we must compute the general form of an $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-equivariant bifurcation problem, i.e., we need to find the invariants and the equivariants by the group $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ for the action considered in section 5.2. We calculate the general form of a vector field $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-equivariant with polynomial components of degree less than seven. However we will see in the sections 5.5 and 5.6 that the third order terms of the Taylor expansion around the bifurcation point of a general commuting vector field detemine the branching directions and the stabilities of the solutions corresponding to the bifurcating branches found in this work.

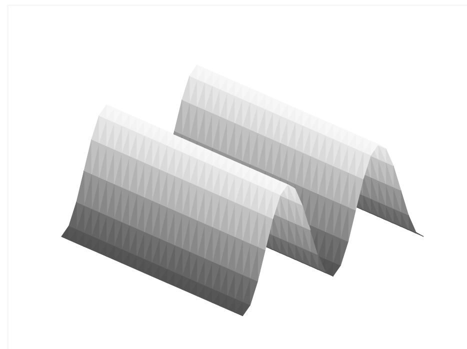
Proposition 5.4.1 *Up to the 6th order*
(a) For every polynomial function $f : (\mathbf{C} \oplus \mathbf{C})^3 \to \mathbf{R}$ that is invariant by $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$, there exists a polynomial function P such that

$$f(z) = P(I_1, \dots, I_9),$$

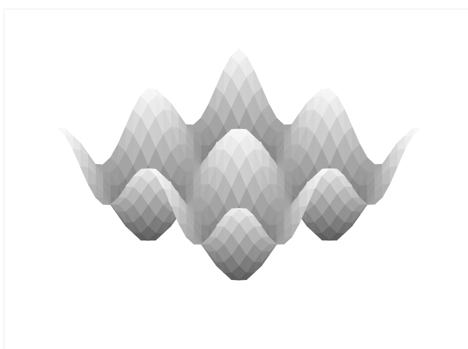
where



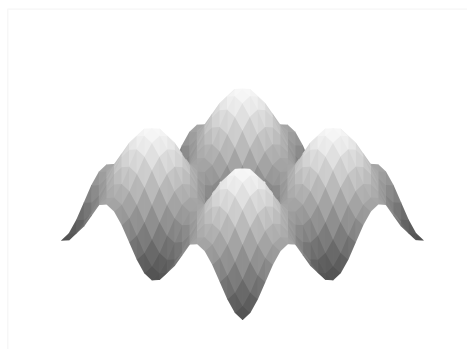
(a) Standing rolls: $t = 0$.



(b) Standing rolls: $t = 1/2$.

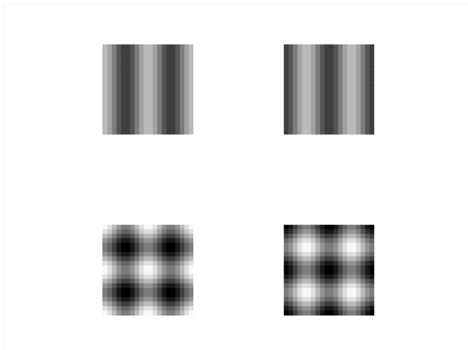


(c) Standing squares: $t = 0$.

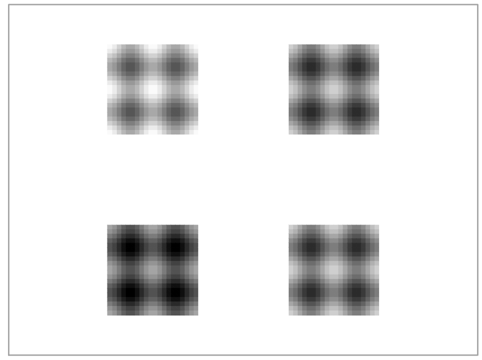


(d) Standing squares: $t = 1/2$.

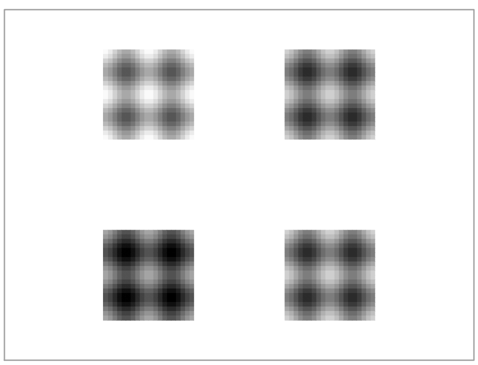
Figure 5.3: Standing rolls and standing squares.



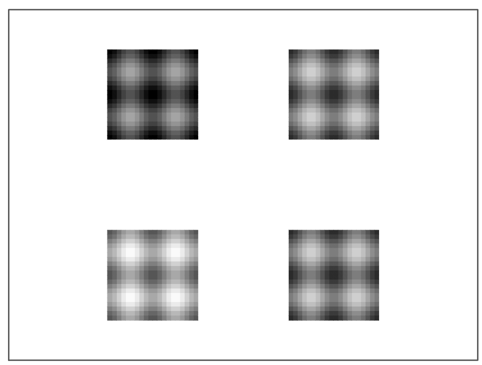
(a) Standing rolls on the top and standing squares above, both for $t = 0$ and $t = 1/2$.



(b) Standing cubes: $t = 0$ and $X_2 = 0,\ 1/4,\ 1/2,\ 3/4$.

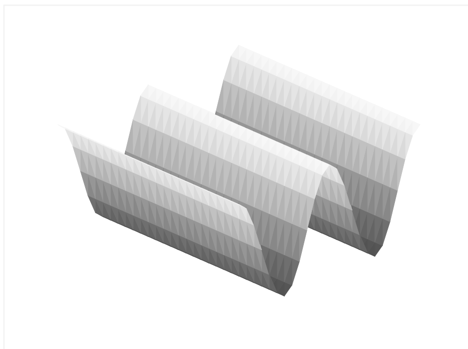


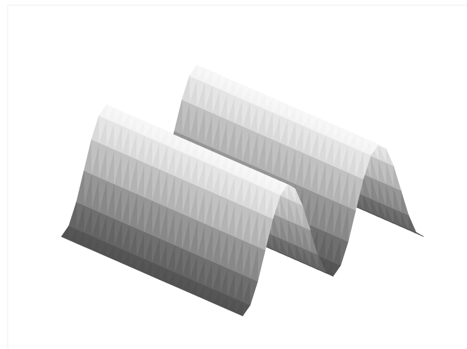
(c) Standing cubes: $t = 0$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.



(d) Standing cubes: $t = 1/2$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.

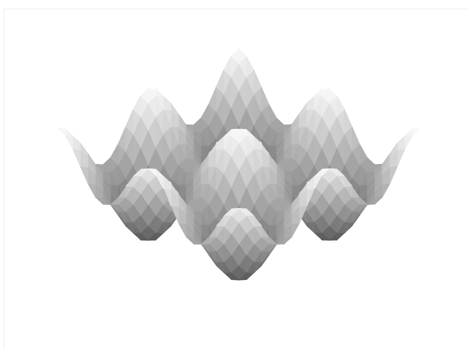
Figure 5.4: Standing rolls, standing squares and standing cubes.

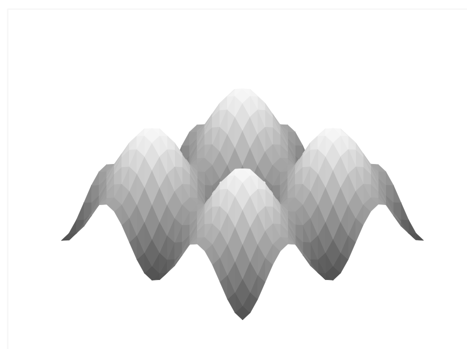




(a) Travelling rolls: $t = 0$.

(b) Travelling rolls: $t = 1/2$.

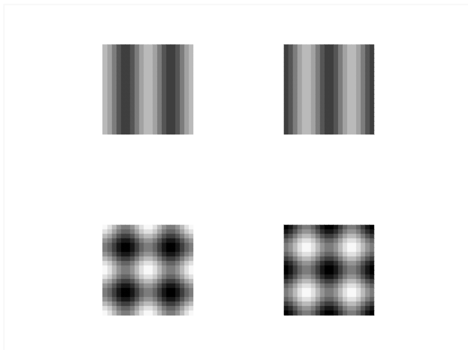




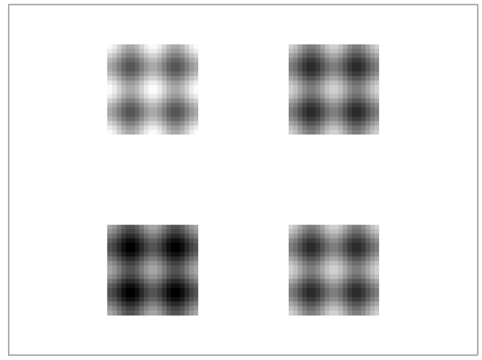
(c) Travelling squares: $t = 0$.

(d) Travelling squares: $t = 1/2$.

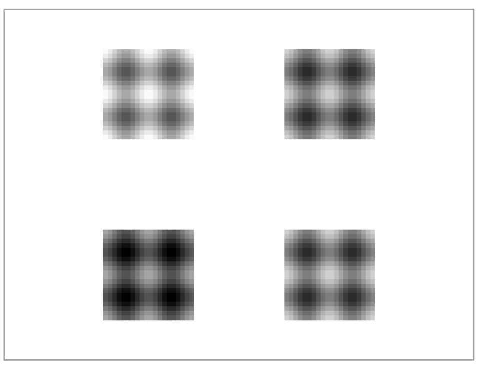
Figure 5.5: Travelling rolls and travelling squares.



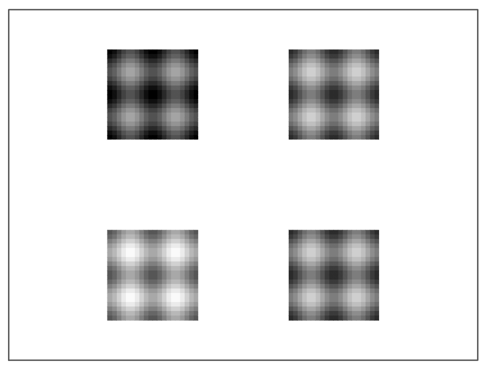
(a) Travelling rolls on the top and travelling squares above, both for $t = 0$ and $t = 1/2$.



(b) Travelling cubes: $t = 0$ and $X_2 = 0,\ 1/4,\ 1/2,\ 3/4$.

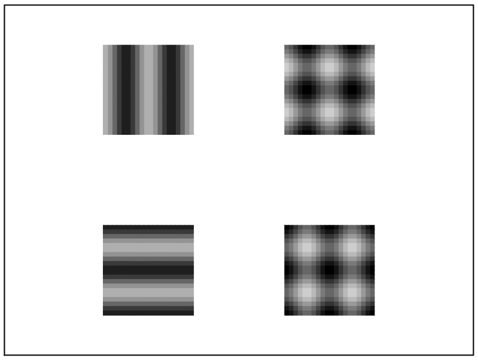


(c) Travelling cubes: $t = 0$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.

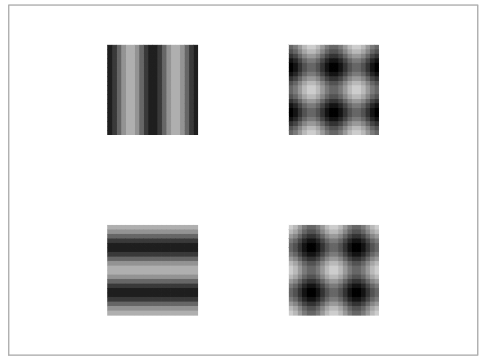


(d) Travelling cubes: $t = 1/2$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.

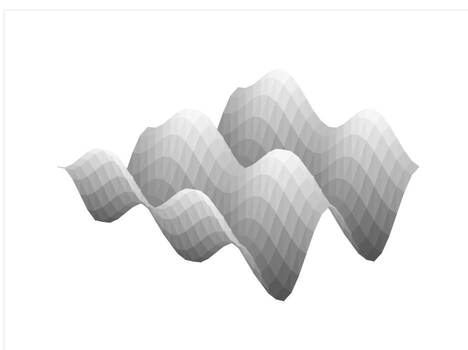
Figure 5.6: Travelling rolls, travelling squares and travelling cubes.



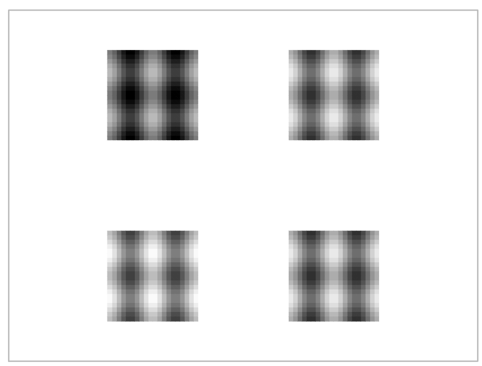
(a) Alternating rolls: $t = 0,\ 1/8,\ 1/4,\ 3/8$.



(b) Alternating rolls: $t = 1/2,\ 5/8,\ 3/4,\ 7/8$.

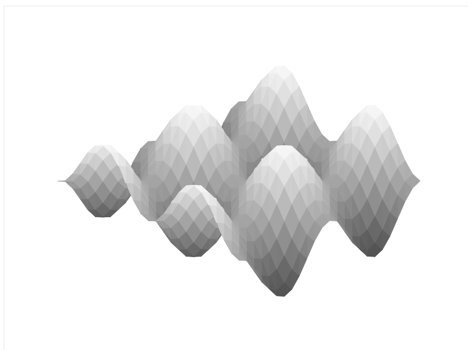


(c) Alternating cubes: $t = 0$ and $X_3 = 0$.

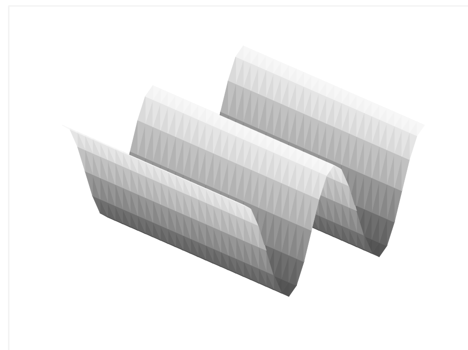


(d) Alternating cubes: $t = 0$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.

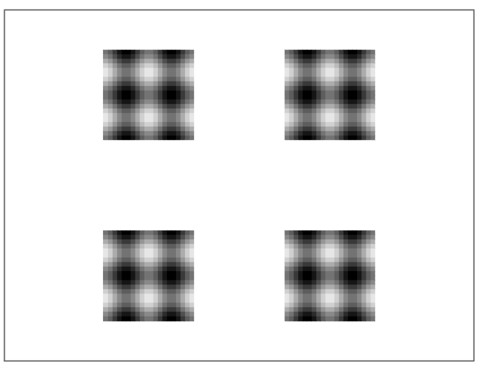
Figure 5.7: Alternating rolls and alternating cubes.



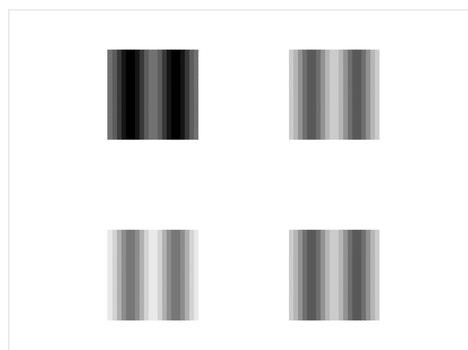
(a) $t = 1/12$ and $X_3 = 0$.



(b) $t = 1/12$ and $X_2 = 0$.

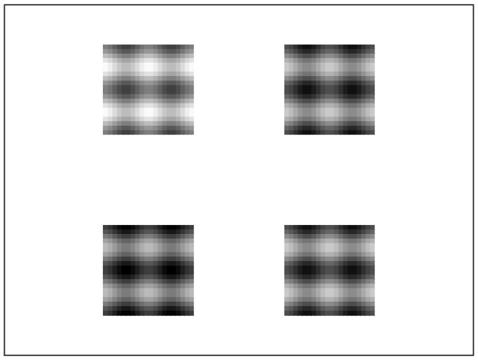


(c) $t = 1/12$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.

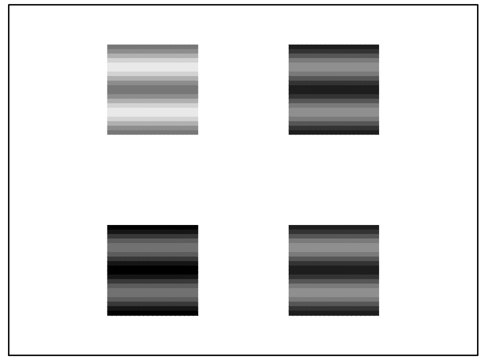


(d) $t = 1/12$ and $X_2 = 0,\ 1/3,\ 1/2,\ 2/3$.

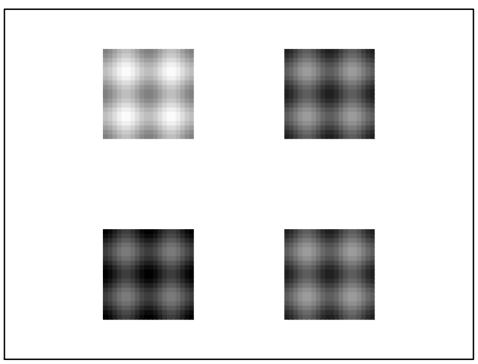
Figure 5.8: Alternating cubes.



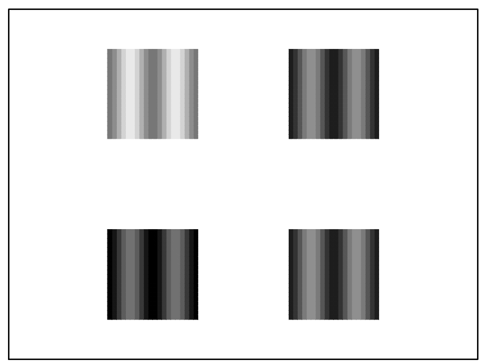
(a) $t = \frac{1}{6}$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.



(b) $t = \frac{1}{4}$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.

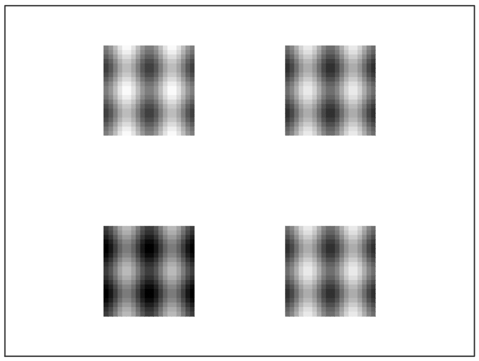


(c) $t = \frac{1}{3}$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$. Same pattern as for $t = 0$ and $X_1 = 0,\ 1/3,\ 1/2,\ 2/3$.

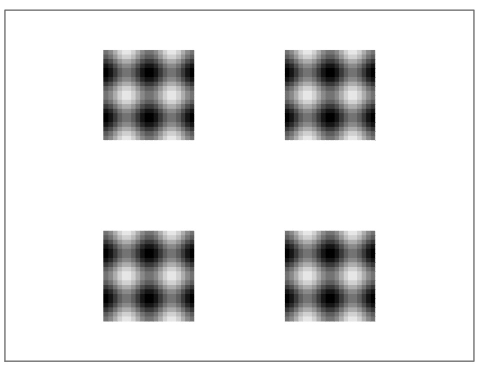


(d) $t = \frac{5}{12}$ and $X_3 = 0\,,1/3,\ 1/2,\ 2/3$.

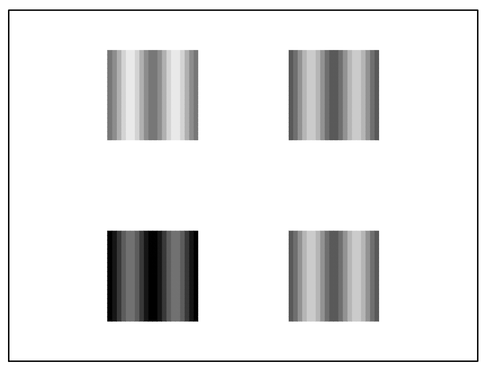
Figure 5.9: Alternating cubes.



(a) $t = \frac{1}{2}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6.$

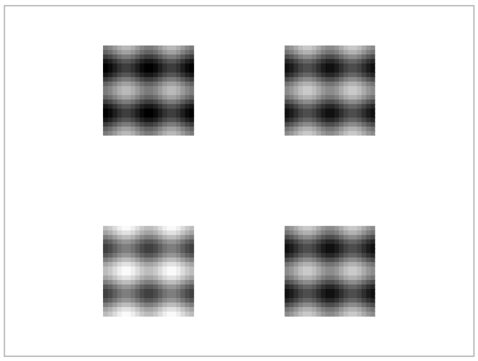


(b) $t = \frac{7}{12}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6.$

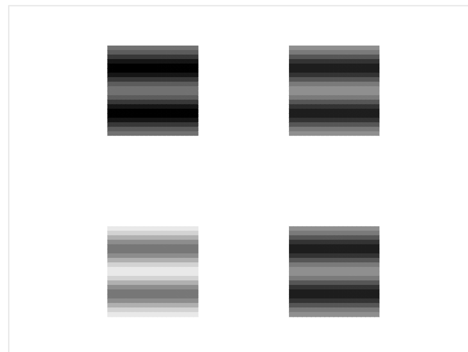


(c) $t = \frac{7}{12}$ and $X_2 = 0,\ 1/6,\ 1/2,\ 5/6.$

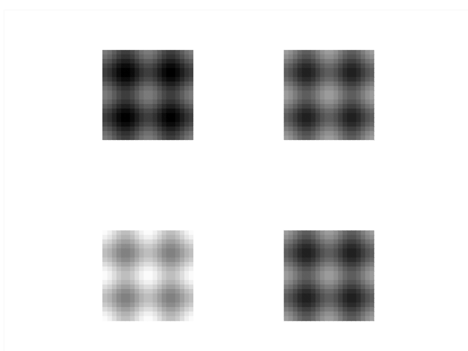
Figure 5.10: Alternating cubes.



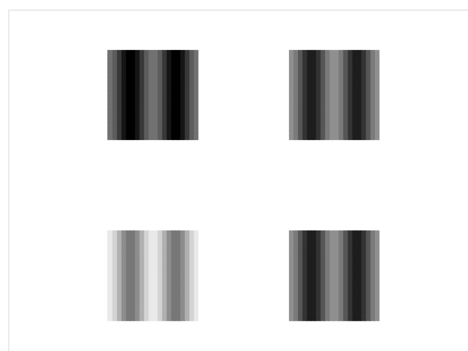
(a) $t = \frac{2}{3}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.



(b) $t = \frac{3}{4}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.



(c) $t = \frac{5}{6}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.



(d) $t = \frac{11}{12}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.

Figure 5.11: Alternating cubes.

$$I_1 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2$$

$$I_2 = (|z_1|^2 + |z_2|^2)^2 + (|z_3|^2 + |z_4|^2)^2 + (|z_5|^2 + |z_6|^2)^2$$

$$I_3 = |z_1|^2|z_2|^2 + |z_3|^2|z_4|^2 + |z_5|^2|z_6|^2$$

$$I_4 = Re(z_1z_2\overline{z_3z_4} + z_3z_4\overline{z_5z_6} + z_5z_6\overline{z_1z_2})$$

$$I_5 = (|z_1|^2 + |z_2|^2)^3 + (|z_3|^2 + |z_4|^2)^3 + (|z_5|^2 + |z_6|^2)^3$$

$$I_6 = (|z_1|^2+|z_2|^2)|z_1|^2|z_2|^2+(|z_3|^2+|z_4|^2)|z_3|^2|z_4|^2+(|z_5|^2+|z_6|^2)|z_5|^2|z_6|^2$$

$$I_7 = (|z_5|^2 + |z_6|^2)Re(z_1z_2\overline{z_3z_4}) + (|z_1|^2 + |z_2|^2)Re(z_3z_4\overline{z_5z_6}) + (|z_3|^2 + |z_4|^2)Re(z_5z_6\overline{z_1z_2})$$

$$I_8 = (|z_1|^2 + |z_2|^2)Re(z_1z_2\overline{z_3z_4} + z_5z_6\overline{z_1z_2}) + (|z_3|^2 + |z_4|^2)Re(z_3z_4\overline{z_5z_6} + z_1z_2\overline{z_3z_4}) + (|z_5|^2 + |z_6|^2)Re(z_3z_4\overline{z_5z_6} + z_5z_6\overline{z_1z_2})$$

$$I_9 = (|z_1|^2 + |z_2|^2)Im(z_1z_2\overline{z_3z_4} - z_5z_6\overline{z_1z_2}) + (|z_3|^2 + |z_4|^2)Im(z_3z_4\overline{z_5z_6} - z_1z_2\overline{z_3z_4}) + (|z_5|^2 + |z_6|^2)Im(z_5z_6\overline{z_1z_2} - z_3z_4\overline{z_5z_6})$$

(b) Every function g with polynomial components from $(\mathbf{C}\oplus\mathbf{C})^3$ to $(\mathbf{C}\oplus\mathbf{C})^3$ that is $(\mathbf{O}(2)\wr\mathbf{S}_3)\times\mathbf{S}^1$-equivariant can be written as:

$$g(z) = \sum_{j=1}^{11}(p_j + iq_j)E_j,$$

where $p_j,\ q_j$ are $(\mathbf{O}(2)\wr\mathbf{S}_3)\times\mathbf{S}^1$-invariant germs of degree lower or equal six,

$$E_j = \begin{bmatrix} f_j(z_1,z_2,z_3,z_4,z_5,z_6) \\ f_j(z_2,z_1,z_3,z_4,z_5,z_6) \\ f_j(z_3,z_4,z_1,z_2,z_5,z_6) \\ f_j(z_4,z_3,z_1,z_2,z_5,z_6) \\ f_j(z_5,z_6,z_3,z_4,z_1,z_2) \\ f_j(z_6,z_5,z_3,z_4,z_1,z_2) \end{bmatrix}$$

and

$$
\begin{array}{ll}
f_1 = z_1 & f_2 = z_1|z_1|^2 \\
f_3 = z_1|z_2|^2 & f_4 = \overline{z_2}(z_3 z_4 + z_5 z_6) \\
f_5 = z_1|z_1|^4 & f_6 = z_1|z_2|^4 \\
f_7 = z_1|z_1|^2|z_2|^2 & f_8 = z_1 Re(z_3 z_4 \overline{z_5 z_6}) \\
f_9 = z_2 z_1^2(\overline{z_3 z_4} + \overline{z_5 z_6}) & f_{10} = \overline{z_2}|z_2|^2(z_3 z_4 + z_5 z_6) \\
f_{11} = \overline{z_2}[z_3 z_4(|z_3|^2 + |z_4|^2) + z_5 z_6(|z_5|^2 + |z_6|^2)] &
\end{array}
$$

In order to prove these two propositions, we begin by proving first two lemmas involving only the invariance under $\mathbf{SO}(2)^N \times \mathbf{S}^1$ and $\mathbf{O}(2)^N \times \mathbf{S}^1$.

Lemma 5.4.2 *Every polynomial germ $f : (\mathbf{C} \oplus \mathbf{C})^N \to \mathbf{C}$ invariant under $\mathbf{O}(2)^N \times \mathbf{S}^1$ has the form*

$$f(z) = P(u_1, \ldots, v_1, \ldots, c_{12}, \ldots, \overline{c_{12}}, \ldots),$$

where

$$
\begin{array}{ll}
u_i = |z_{2i-1}|^2 |z_{2i}|^2, & i = 1, \ldots, N \\
v_i = |z_{2i-1}|^2 + |z_{2i}|^2, & i = 1, \ldots, N \\
c_{ij} = z_{2i-1} z_{2i} \overline{z_{2j-1} z_{2j}}, & 1 \le i < j \le N.
\end{array}
$$

Proof By Schwarz's theorem we may assume that f is polynomial. Let $f : \mathbf{C}^{2N} \to \mathbf{C}$ be written as

$$f(z) = \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta,$$

where $a_{\alpha\beta} \in \mathbf{C},\ \alpha,\ \beta \in (\mathbf{Z}_0^+)^{2N}$ and $z \in \mathbf{C}^{2N}$. Here we use multi-indices. In order f to be $\mathbf{SO}(2)^N \times \mathbf{S}^1$ invariant, i.e.,

$$f((\varphi_1, \ldots, \varphi_N, \theta)z) = f(z),$$

for all $(\varphi_1, \ldots, \varphi_N, \theta) \in \mathbf{SO}(2)^N \times \mathbf{S}^1$, we have for each $\alpha,\ \beta \in (\mathbf{Z}_0^+)^{2N}$

$$a_{\alpha\beta} = 0 \vee \theta(|\alpha| - |\beta|) + [\varphi_1(\alpha_2 - \beta_2 + \beta_1 - \alpha_1) + \cdots + \varphi_N(\alpha_{2N} - \beta_{2N} + \beta_{2N-1} - \alpha_{2N-1})] = 0.$$

Therefore, if $a_{\alpha\beta} \neq 0$, then

$$|\alpha| = |\beta| \qquad \wedge \qquad \alpha_{2i} - \beta_{2i} = \alpha_{2i-1} - \beta_{2i-1},\ i = 1, \ldots, N,$$

which is equivalent to have

$$\sum_{i=1}^{N} \beta_{2i-1} = \sum_{i=1}^{N} \alpha_{2i-1} \quad \wedge \quad \alpha_{2i} - \beta_{2i} = \alpha_{2i-1} - \beta_{2i-1},\ i = 1, \ldots, N. \qquad (5.6)$$

In addition, we have

$$f(z) = f(z_2, z_1, z_3, z_4, \ldots) = \cdots = f(z_1, z_2, \ldots, z_{2N}, z_{2N-1}).$$

So f will be the sum of monomials like

$$(z_1^{\alpha_1} z_2^{\alpha_2} \overline{z}_1^{\beta_1} \overline{z}_2^{\beta_2} + z_1^{\alpha_2} z_2^{\alpha_1} \overline{z}_1^{\beta_2} \overline{z}_2^{\beta_1}) \cdots (z_{2N-1}^{\alpha_{2N-1}} z_{2N}^{\alpha_{2N}} \overline{z}_{2N-1}^{\beta_{2N-1}} \overline{z}_{2N}^{\beta_{2N}} + z_{2N-1}^{\alpha_{2N}} z_{2N}^{\alpha_{2N-1}} \overline{z}_{2N-1}^{\beta_{2N}} \overline{z}_{2N}^{\beta_{2N-1}})$$

with $\alpha_{2i-1} \leq \alpha_{2i}$ and $\beta_{2i-1} \leq \beta_{2i}$ for $i = 1, \ldots, N$. Note that here we used the fact that if $\alpha_{2i} - \alpha_{2i-1} \geq 0$, then also $\beta_{2i} - \beta_{2i-1} \geq 0$. Moreover, if (α, β) satisfies (5.6), then for example $(\alpha_2, \alpha_1, \ldots, \beta_2, \beta_1, \ldots)$ also satisfies (5.6). Now, if we factor out the largest powers of $z_{2i-1}z_{2i}$ and $\overline{z_{2i-1}z_{2i}}$ and use (5.6), we will have polynomials terms like

$$(z_1 z_2)^{\alpha_1} (\overline{z_1 z_2})^{\beta_1} \ldots (z_{2N-1} z_{2N})^{\alpha_{2N-1}} (\overline{z_{2N-1} z_{2N}})^{\beta_{2N-1}}$$

with

$$\sum_{i=1}^{N} \alpha_{2i-1} = \sum_{i=1}^{N} \beta_{2i-1}$$

and

$$(|z_{2i-1}|^2)^{k_i} + (|z_{2i}|^2)^{k_i}.$$

By pairing the $z_{2i-1}z_{2i}$ with $\overline{z_{2j-1}z_{2j}}$ we can always write the first one as monomial in the u_i's and c_{ij}'s and the second one as a polynomial in the u_i's and v_i's. □

Note that by [20] lemma XVI 9.2., if we have the $\mathbf{C}$-valued invariants in z, $\overline{z}$, then we take the real and imaginary parts of these and we have the $\mathbf{R}$-valued invariants.

Lemma 5.4.3 *Every polynomial germ $f : (\mathbf{C} \oplus \mathbf{C})^N \to \mathbf{C}$ invariant under $\mathbf{SO}(2)^N \times \mathbf{S}^1$ has the following form*

$$f(z) = P(v_1, \ldots, c_{12}, \ldots),$$

where

$$\begin{aligned} v_i &= |z_i|^2, & i &= 1, \ldots, 2N, \\ c_{ij} &= z_{2i-1} z_{2i} \overline{z_{2j-1} z_{2j}}, & 1 &\leq i < j \leq N. \end{aligned}$$

Proof From the proof of lemma 5.4.2 we know that

$$f(z) = \sum a_{\alpha\beta} z^{\alpha} \overline{z}^{\beta},$$

where $a_{\alpha\beta} \in \mathbf{C},\ \alpha,\ \beta \in (\mathbf{Z}_0^+)^{2N}$ and each (α, β) must satisfy

$$\begin{array}{l} \sum \alpha_{2i} = \sum \beta_{2i} \quad \wedge \quad \sum \alpha_{2i-1} = \sum \beta_{2i-1} \\ \alpha_{2i} - \beta_{2i} = \alpha_{2i-1} - \beta_{2i-1}, \quad i = 1, \ldots, N. \end{array}$$

For $\alpha,\ \beta \in (\mathbf{Z}_0^+)^{2N}$ in those conditions, we can factor out

$$z_{2i-1}^{\alpha_{2i-1}} z_{2i}^{\alpha_{2i}} \overline{z}_{2i-1}^{\beta_{2i-1}} \overline{z}_{2i}^{\beta_{2i}}$$

as

$$\left(z_{2i-1}\overline{z}_{2i-1}\right)^{\beta_{2i-1}} \left(z_{2i}\overline{z}_{2i}\right)^{\beta_{2i}} \left(z_{2i-1}z_{2i}\right)^{\alpha_{2i}-\beta_{2i}}$$

if $\alpha_{2i} \geq \beta_{2i}$ (and so $\alpha_{2i-1} \geq \beta_{2i-1}$), or as

$$\left(z_{2i-1}\overline{z}_{2i-1}\right)^{\alpha_{2i-1}} \left(z_{2i}\overline{z}_{2i}\right)^{\alpha_{2i}} \left(\overline{z_{2i-1}z_{2i}}\right)^{\beta_{2i}-\alpha_{2i}}$$

if $\alpha_{2i} \leq \beta_{2i}$ (and so $\alpha_{2i-1} \leq \beta_{2i-1}$). Also we know that

$$\sum_{\{i:\alpha_{2i} \geq \beta_{2i}\}} (\alpha_{2i} - \beta_{2i}) = \sum_{\{i:\alpha_{2i} < \beta_{2i}\}} (\beta_{2i} - \alpha_{2i})$$

Therefore we only need the v_i's and the c_{ij}'s. $\square$

Remark 5.4.4 *(a) Using a similar result as [21] theorem 4, we have that if* $I_1, \ldots, I_r$ *generate the* $\mathbf{C}$*-valued invariants by* $\mathbf{SO}(2)^N \times \mathbf{S}^1$*, then the equivariants are generated by the mappings*

$$\textit{row } j \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial I_g}{\partial \overline{z}_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $1 \leq j \leq 2N$ *and* $1 \leq g \leq r$*. This follows from the fact that if* $g = (g_1, \ldots, g_{2N})$ *is* $\mathbf{SO}(2)^N \times \mathbf{S}^1$*-equivariant, then* $\overline{z}_j g_j$ *is* $\mathbf{SO}(2)^N \times \mathbf{S}^1$*-invariant (for* $j = 1, \ldots, 2N$*).*

(b) If I is an invariant polynomial under $(\mathbf{O}(2) \wr \mathbf{S}_N) \times \mathbf{S}^1$ over $\mathbf{C}$, then

$$g = (\frac{\partial I}{\partial \overline{z}_1}, \ldots, \frac{\partial I}{\partial \overline{z}_{2N}})$$

commutes with $(\mathbf{O}(2) \wr \mathbf{S}_N) \times \mathbf{S}^1$. The polynomial I is invariant under $(\mathbf{Z}_2^\kappa)^N \times \mathbf{S}^1$ if we denote $\mathbf{Z}_2^\kappa = \{1, \kappa\} \subset \mathbf{O}(2)$. Let $g_1(z) = \frac{\partial I}{\partial \overline{z}_1} |_z$. Then $g(z) = (g_1(z), g_1(z_2, z_1, \ldots), g_1(z_3, z_4, z_1, z_2, \ldots), \ldots)$ and g_1 is invariant under $\mathbf{S}_{N-1}$ and $(\mathbf{Z}_2^\kappa)^{N-1}$ in the last $N-1$ variables. It follows that g commutes with $\mathbf{S}_N$ and $(\mathbf{Z}_2^\kappa)^N$. But I is $\mathbf{SO}(2)^N \times \mathbf{S}^1$ invariant. So $(g_1, 0, \ldots, 0)$ is $\mathbf{SO}(2)^N \times \mathbf{S}^1$-equivariant. It follows that g commutes with $(\mathbf{O}(2) \wr \mathbf{S}_N) \times \mathbf{S}^1$.

Proof of proposition 5.4.1

(a) First f is $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-invariant if and only if it is invariant by $\mathbf{O}(2)^3 \times \mathbf{S}^1$ and $\mathbf{S}_3$. By lemma 5.4.2, we have the invariants for $\mathbf{O}(2)^3 \times \mathbf{S}^1$. We have to find the polynomials

$$p(u_1, \ldots, v_1, \ldots, c_{12}, \ldots, \overline{c}_{12}, \ldots)$$

that are $\mathbf{S}_3$-invariants. Moreover, we wish to find the homogeneous polynomials up to degree six. We end with three problems: to find the homogeneous invariants of degree two, four and six.

For the degree two we look for the polynomials p of degree one in $v_1,\ v_2,\ v_3$ that are $\mathbf{S}_3$-invariants. As $v_1,\ v_2,\ v_3$ are algebrically independents, we only need to consider the linear combination $v_1 + v_2 + v_3$ and we have I_1.

For the degree four we have two possibilities: polynomials of degree two in $v_1,\ v_2,\ v_3$ and the polynomials of degree one in $u_1,\ u_2,\ u_3$ and $c_{12},\ c_{13},\ c_{23}$ (and their conjugates). For the first case it is enough to take $v_1^2 + v_2^2 + v_3^2$ and we have I_2. For the second case, we can have a polynomial just involving $u_1,\ u_2,\ u_3$ and we take I_3, or a polynomial involving the c_{ij}'s and their conjugates and we get I_4.

By last, for the degree six, we can have polynomials in the v_i's of the degree three (we can choose I_5), polynomials of degree two in the u_i's and the v_i's or in the v_i's and $c_{ij}, \overline{c}_{ij}$'s. We obtain $I_6, \ldots, I_9$.

(b) Let $g : (\mathbf{C} \oplus \mathbf{C})^3 \to (\mathbf{C} \oplus \mathbf{C})^3$ be $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-equivariant. Using lemma 5.4.3 we have a general mapping g equivariant by $\mathbf{SO}(2)^3 \times \mathbf{S}^1$. Therefore we have to look for such g that is also equivariant by the flips in $\mathbf{O}(2)^3$ and by the permutations in $\mathbf{S}_3$. Remember that $\mathbf{S}_3$ is generated for example by the transpositions (12) and (13). If we have

$$g = (g_1, g_2, g_3, g_4, g_5, g_6),$$

imposing the equivariance by the flips and the transpositions we must have

$$\begin{aligned} g_2(z) &= g_1(z_2, z_1, z_3, z_4, z_5, z_6) \\ g_3(z) &= g_1(z_3, z_4, z_1, z_2, z_5, z_6) \\ g_4(z) &= g_1(z_4, z_3, z_1, z_2, z_5, z_6) \\ g_5(z) &= g_1(z_5, z_6, z_3, z_4, z_1, z_2) \\ g_6(z) &= g_1(z_6, z_5, z_3, z_4, z_1, z_2) \end{aligned}$$

and

$$g_1(z) = g_1(z_1, z_2, z_5, z_6, z_3, z_4) = g_1(z_1, z_2, z_4, z_3, z_5, z_6) = g_1(z_1, z_2, z_3, z_4, z_6, z_5).$$

From lemma 5.4.3 and the remark 5.4.4,

$$g_1(z) = z_1 p_1 + \overline{z_2} z_3 z_4 p_2 + \overline{z_2} z_5 z_6 p_3,$$

where the p_i are polynomials in $v_i,\ c_{ij}$ and $\overline{c_{ij}}$. For degree one we only need to consider $g_1(z) = az_1$ for $a \in \mathbf{C}$. We get f_1.

For the degree three,

$$g_1(z) = z_1 p_1(v_1, \ldots) + \overline{z_2} z_3 z_4 b_1 + \overline{z_2} z_5 z_6 b_2$$

for constants $b_1,\ b_2\ \in \mathbf{C}$ and we get $f_2,\ f_3,\ f_4$.

By last, for the degree five,

$$g_1(z) = z_1 p_1(v_1, \ldots) + z_1 p_2(c_{12}, \ldots, \overline{c_{12}}, \ldots) + \overline{z_2} z_3 z_4 p_3(v_1, \ldots) + \overline{z_2} z_5 z_6 p_4(v_1, \ldots)$$

and we get $f_5, \ldots, f_{11}$. □

5.5 Branching equations and stability of the periodic solutions with maximal isotropy

From theorem 5.2.1 we have (up to conjugacy) the $\mathbf{C}$-axial subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$. Therefore we can use the equivariant Hopf theorem to prove the existence of periodic solutions with these symmetries for a bifurcation problem with symmetry $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_3$.

Proposition 5.5.1 *Consider the system of ODEs*

$$\dot{z} = f(z, \lambda), \tag{5.7}$$

where
(i) $f : V^3 \times \mathbf{R} \to V^3$ *is smooth, commuting with* Γ *and*
(ii) $(df)_{0,\lambda}$ *has eigenvalues* $\sigma(\lambda) \mp i\rho(\lambda)$ *with* $\sigma(0) = 0,\ \rho(0) = 1$ *and* $\sigma'(0) \neq 0$.

Then for each isotropy subgroup of $\Gamma \times \mathbf{S}^1$ *conjugate to one of the groups in table 5.2, there is a unique branch of periodic solutions of (5.7) bifurcating from* $(0, 0)$ *with period near* 2π *and with symmetry that isotropy subgroup.*

Proof As $\mathrm{Fix}_{V^3}(\Gamma) = \{0\}$, from (i) it follows that

$$f(0, \lambda) = 0$$

and so $(0, \lambda)$ is an equilibrium for all λ that changes stability when λ crosses zero since $\sigma'(0) \neq 0$.

Again we note that as V is $\mathbf{O}(2)$-simple, the space V^3 is Γ-simple. After assuming (i), the assumption (ii) imposes the eigenvalue crossing condition of the equivariant Hopf theorem (and scales the eigenvalues of $(df)_{0,0}$ to $\mp i$). Recall lemma 2.3.3. From theorem 5.2.1 we have the $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$. □

Our aim in this section is to determine the conditions on the coefficients of the lowest order terms of f that are necessary for each of the different types of bifurcating solutions (with maximal symmetry) to be stable. That is, we wish to calculate the stability of these bifurcating periodic solutions in terms of the coefficients of the lowest degree terms in the Taylor series expansion of a general vector field commuting with Γ.

We note that as it was stated in chapter 2, the theory of Birkhoff normal form asserts that for any positive integer k the Taylor series of degree k of a vector field commuting with Γ can be made to commute with the $\mathbf{S}^1$-action by a change of coordinates in V^3 that commutes with the action of Γ. We can assume that the degree k Taylor series also commutes with $\mathbf{S}^1$. We point out that this does not imply that the original vector field commutes with $\mathbf{S}^1$.

In this work we assume that the Taylor series of degree three of f commutes also with $\mathbf{S}^1$. We show that the $\mathbf{C}$-axial groups are 3-determined and that the stability of the bifurcating periodic solutions of (5.7) with maximal symmetry depends only on the coefficients of this Taylor series of degree three.

Let f be as in (5.7). If we suppose that the Taylor series of degree three of f around $z=0$ commutes also with $\mathbf{S}^1$, then by proposition 5.4.1 we can write $f=(f_1,\dots,f_6)$, where

$$f_1(z,\lambda)=[\mu(\lambda)+a|z|^2+b|z_1|^2+c|z_2|^2]z_1+d\overline{z}_2(z_3z_4+z_5z_6)+\text{ terms of degree } \geq 5 \tag{5.8}$$

and

$$\begin{aligned}
f_2(z,\lambda) &= f_1(z_2,z_1,z_3,z_4,z_5,z_6),\\
f_3(z,\lambda) &= f_1(z_3,z_4,z_1,z_2,z_5,z_6),\\
f_4(z,\lambda) &= f_1(z_4,z_3,z_1,z_2,z_5,z_6),\\
f_5(z,\lambda) &= f_1(z_5,z_6,z_3,z_4,z_1,z_2),\\
f_6(z,\lambda) &= f_1(z_6,z_5,z_3,z_4,z_1,z_2).
\end{aligned}$$

Also, the coefficients $a,\ b,\ c,\ d$ are complex smooth functions of λ and as in proposition 5.5.1

$$\mu(0)=i \quad \wedge \quad \mathrm{Re}(\mu^{'}(0))\neq 0.$$

Suppose that

$$\mathrm{Re}(\mu'(0))>0.$$

Rescaling λ if necessary we can suppose that

$$\mathrm{Re}(\mu(\lambda))=\lambda+\text{ higher order terms in }\lambda.$$

Thus the trivial solution of (5.7) is stable for λ negative and and unstable for λ positive (near zero).

We show now that the coefficients $a,\ b,\ c$ and d determine (generically) the directions of branching and the stability of the periodic solutions guaranteed by the equivariant Hopf theorem. However we note that the periodic solutions whose existence is guaranteed by this theorem are not necessarily the only periodic solutions bifurcating from $(0,0)$. We seek in the next section periodic solutions with submaximal isotropy.

The truncated normal form of degree three obtained in [31] to study Hopf bifurcation on a hexagonal lattice is equivariant by the group $\mathbf{D}_6\dot{+}\mathbf{T}^2$. As it was pointed out there, that truncated vector field has an extra symmetry. We remark that it is equivariant by a group that is a subgroup of $\Gamma\times\mathbf{S}^1$.

Throughout, subscripts r and I on the coefficients a,b,c and d refer to real and imaginary parts.

Isotropy subgroup	Branching equation
Σ_1	$\nu + (a+b)\vert z\vert^2 + \cdots = 0$
Σ_2	$\nu + (2a+b+c)\vert z\vert^2 + \cdots = 0$
Σ_3	$\nu + (2a+b)\vert z\vert^2 + \cdots = 0$
Σ_4	$\nu + (4a+b+c+d)\vert z\vert^2 + \cdots = 0$
Σ_5	$\nu + (4a+b+c-d)\vert z\vert^2 \cdots = 0$
Σ_6	$\nu + (3a+b)\vert z\vert^2 + \cdots = 0$
Σ_7	$\nu + (6a+b+c+2d)\vert z\vert^2 + \cdots = 0$
Σ_8	$\nu + (6a+b+c-d)\vert z\vert^2 + \cdots = 0$

Table 5.6: Branching equations for $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ Hopf bifurcation.

Theorem 5.5.2 *Consider the system (5.7) where f is as in (5.8). For each symmetry group Σ_i listed in table 5.2, let $s_0, \ldots, s_r$ be the functions of a, b, c and d listed in table 5.8 evaluated at $\lambda = 0$.*

(1) For each Σ_i the corresponding branch of periodic solutions is supercritical if $s_0 < 0$ and subcritical if $s_0 > 0$. Tables 5.6 and 5.7 list the branching equations.

(2) For each Σ_i, if $s_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $s_j < 0$ for all j, then the branch of periodic solutions is stable near $\lambda = 0$ and $z = 0$.

Proof of theorem 5.5.2 By the equivariant Hopf theorem we know that we can apply a Liapunov-Schmidt reduction to a map $g(z, \lambda, \tau)$ from $V^3 \times \mathbf{R} \times \mathbf{R}$ to V^3 (note that V^3 is the $\pm i$- real eigenspace of $(df)_{0,0}$) commuting with $\Gamma \times \mathbf{S}^1$ whose zeros are in one-to-one correspondence with periodic solutions of (5.7) of period $\frac{2\pi}{1+\tau}$ and where τ corresponds to the period-perturbing parameter.

Isotropy subgroup	Branching equation
Σ_1	$\lambda = -(a_r + b_r)\|z\|^2 + \cdots$
Σ_2	$\lambda = -(2a_r + b_r + c_r)\|z\|^2 + \cdots$
Σ_3	$\lambda = -(2a_r + b_r)\|z\|^2 + \cdots$
Σ_4	$\lambda = -(4a_r + b_r + c_r + d_r)\|z\|^2 + \cdots$
Σ_5	$\lambda = -(4a_r + b_r + c_r - d_r)\|z\|^2 + \cdots$
Σ_6	$\lambda = -(3a_r + b_r)\|z\|^2 + \cdots$
Σ_7	$\lambda = -(6a_r + b_r + c_r + 2d_r)\|z\|^2 + \cdots$
Σ_8	$\lambda = -(6a_r + b_r + c_r - d_r)\|z\|^2 + \cdots$

Table 5.7: Branches of periodic solutions for $\mathbf{O}(2) \wr \mathbf{S}_3$ Hopf bifurcation.

Symmetry of the solution	s_0	$s_1, \ldots, s_r$
Σ_1	$a_r + b_r$	$-b_r + c_r$ $-b_r$
Σ_2	$2a_r + b_r + c_r$	$b_r - c_r$ $-b_r - c_r$ $-(\lvert b+c\rvert^2 - \lvert d\rvert^2)$
Σ_3	$2a_r + b_r$	b_r $-b_r + c_r + d_r$ $-b_r + c_r - d_r$ $-b_r$
Σ_4	$4a_r + b_r + c_r + d_r$	$b_r - c_r - d_r$ $-b_r - c_r - d_r$ $-[\lvert b+c\rvert^2 - 3\lvert d\rvert^2 + 2\mathrm{Re}((b+c)\overline{d})]$ $b_r + c_r - 3d_r$ $-[\lvert d\rvert^2 - \mathrm{Re}((b+c)\overline{d})]$
Σ_5	$4a_r + b_r + c_r - d_r$	$b_r - c_r + d_r$ $b_r + c_r + 3d_r$ $-[\lvert d\rvert^2 + \mathrm{Re}((b+c)\overline{d})]$ $-b_r - c_r + d_r$
Σ_6	$3a_r + b_r$	b_r $-b_r + c_r + 2d_r$ $-b_r + c_r - d_r$
Σ_7	$6a_r + b_r + c_r + 2d_r$	$b_r - c_r - 2d_r$ $b_r + c_r - 4d_r$ $-[\lvert d\rvert^2 - \mathrm{Re}((b+c)\overline{d})]$
Σ_8	$6a_r + b_r + c_r - d_r$	$b_r - c_r + d_r$ $b_r + c_r + 2d_r \mp \mathrm{Re}(\delta_2 + i\delta_2')^{1/2}$ where $\delta_2 = (b_r + c_r + 2d_r)^2 - 9d_I^2 +$ $6\lvert d\rvert^2 - 6\mathrm{Re}((b+c)\overline{d})$ $\delta_2' = -6d_r(2d_I + b_I + c_I)$

Table 5.8: Stability for $\mathbf{O}(2) \wr \mathbf{S}_3$ Hopf bifurcation.

Moreover, if we assume that f also commutes with $\mathbf{S}^1$ then by theorem 2.3.6, the Liapunov-Schmidt reduction function g of (5.7) has the explicit form

$$g(z, \lambda, \tau) = f(z, \lambda) - (1 + \tau) iz \tag{5.9}$$

(and so the branching equations for f are those for g since the τ-dependence enters in a simple way in those equations) and the asymptotic stability of a bifurcating solution of (5.7) is equivalent to the linearized stability of the corresponding solution of

$$g(z, \lambda, \tau) = 0. \tag{5.10}$$

By corollary 2.3.7, if $z(t)$ is a periodic solution of (5.7) with $\lambda = \lambda_0$ and $\tau = \tau_0$, and (z_0, λ_0, τ_0) is the corresponding solution of (5.10), then there is a correspondence between the Floquet multipliers of $z(t)$ and the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ such that a multiplier lies inside (respectively outside) the unit circle if and only if the corresponding eigenvalue has negative (respectively positive) real part. So to determine the stability of each type of bifurcating periodic orbit we can calculate the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ (to the lowest order in z). As g commutes with $\Gamma \times \mathbf{S}^1$, it maps $\mathrm{Fix}(\Sigma)$ into itself, and for each of the conjugacy classes Σ_i in table 5.2, we have a distinct branch of periodic solutions of (5.7) that are in correspondence with the zeros of g with isotropy Σ_i. These zeros are found by solving $g\mid_{\mathrm{Fix}(\Sigma_i)} = 0$ (and $\mathrm{Fix}(\Sigma_i)$ is two-dimensional). Note that to find the zeros of g, it suffices to look at representative points on $\Gamma \times \mathbf{S}^1$ orbits.

We do the following: we assume that the initial vector field f commutes also with $\mathbf{S}^1$ and so we can apply the results stated above to determine the stability. We show that the stability and the branching equations for the periodic solutions with symmetry the groups in table 5.2 are completely determined by the Taylor series of f of degree three. Moreover, the groups are 3-determined. Then we use theorem 2.3.8 to argue that the same conditions determine the stability and the direction of the branches of the periodic solutions with maximal symmetry of (5.7) even if f does not commute with $\mathbf{S}^1$. Note that by theorem 2.3.5 we can always choose a coordinates change such that the third order truncated normal form of f commutes with $\Gamma \times \mathbf{S}^1$. Thus there is no loss of generality in assuming that f has the form (5.8).

If we consider the Taylor series of f around $z = 0$ as in (5.8) we may write g as $(g_1, g_2, g_3, g_4, g_5, g_6)$, where

$$g_1(z, \lambda, \tau) = [\nu + a|z|^2 + b|z_1|^2 + c|z_2|^2] z_1 + d\bar{z}_2(z_3 z_4 + z_5 z_6) + \cdots,$$

where

$$\begin{aligned}
g_2(z,\lambda) &= g_1(z_2,z_1,z_3,z_4,z_5,z_6),\\
g_3(z,\lambda) &= g_1(z_3,z_4,z_1,z_2,z_5,z_6),\\
g_4(z,\lambda) &= g_1(z_4,z_3,z_1,z_2,z_5,z_6),\\
g_5(z,\lambda) &= g_1(z_5,z_6,z_3,z_4,z_1,z_2),\\
g_6(z,\lambda) &= g_1(z_6,z_5,z_3,z_4,z_1,z_2),
\end{aligned}$$

and $\nu(\lambda,\tau)=\mu(\lambda)-(1+\tau)i$.

When restricted to a two-dimensional fixed-point subspace $\mathrm{Fix}(\Sigma)$, equation (5.10) gives a single complex scalar equation (table 5.6) and hence two real ones. Use the imaginary equation (involving $\mathrm{Im}(\nu(\lambda,\tau))$) to solve for τ. Now the other equation we can solve for z as a function of λ by the implicit function theorem.

To determine for each values of the coefficients of the Taylor expansion of g (around zero), and so f, the periodic solutions predicted by the equivariant Hopf theorem bifurcate supercritically or subcritically, we have to solve the real part of (5.10) restricted to $\mathrm{Fix}(\Sigma)$ and we get table 5.7.

To determine the stability of each type of the bifurcating periodic orbit, we calculate the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ (to the lowest order in z) in terms of the coefficients of the Taylor expansion of g.

First we find conditions on the matrix $(dg)_{(z_0,\lambda_0,\tau_0)}$ which must be satisfied as a consequence of the fact that it commutes with Σ. We can use table 5.3 and theorem 4.3.8.

Second we use these conditions to find a basis for V^3 with respect to which the matrix is block diagonal with 2×2 blocks. Another way of doing this could be by decomposing V^3 into subspaces, each of each is invariant under a different representation of the isotropy subgroup Σ_{z_0}. That is, we could form the isotypic decomposition $V^3 = W_1 \oplus \cdots \oplus W_k$ where each isotypic component W_i may be further decomposed into subspaces each of which transforms according to the ith irreducible representation of Σ_{z_0} (each W_i would be the sum of all Σ_{z_0}-irreducible subspaces of V^3 that are Σ_{z_0}-isomorphic). We can always take $W_1 = \mathrm{Fix}(\Sigma_{z_0})$ so that W_1 is the sum of all subspaces of V^3 on which Σ_{z_0} acts trivially. Note that the orbit of z_0 under the action of $\mathbf{S}^1$ is in W_1, so that there is a zero eigenvalue of $(dg)_{(z_0,\lambda_0,\tau_0)}$ restricted to W_1 for each solution z_0. Thus for the solutions with $\mathbf{C}$-axial symmetry, the nonzero eigenvalue of $(dg)_{(z_0,\lambda_0,\tau_0)}$ restricted to $\mathrm{Fix}(\Sigma_{z_0})$ is given by the trace of $(dg)_{(z_0,\lambda_0,\tau_0)}\,|_{\mathrm{Fix}(\Sigma_{z_0})}$ and it is associated with the eigenvector in the plane of the limit cycle and transverse to it. This eigenvalue depends on the direction of the bifurcating branch.

Finally, we calculate the entries in the matrix in terms of the Taylor coefficients and obtain the signs of the real parts of the eigenvalues.

Note that as the group action forces some of the Floquet multipliers to be equal to one, it also forces the corresponding eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to be equal to zero. The eigenvectors of these eigenvalues at the point z_0 are the tangent vectors to the orbit of $\mathbf{SO}(2)^3 \times \mathbf{S}^1$ through z_0. In fact, as pointed out before, if the solution z_0 has symmetry Σ_{z_0}, then the group orbit has the dimension of $(\Gamma \times \mathbf{S}^1)/\Sigma_{z_0}$ and so the number of zero eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ forced by the group action is

$$d_{\Sigma_{z_0}} = 4 - \dim(\Sigma_{z_0}),$$

since $\dim \Gamma \times \mathbf{S}^1 = 4$. Thus if the group Σ_{z_0} is discrete, then there are four zero eigenvalues: three associated with the $\mathbf{O}(2)^3$ symmetry and one associated with the $\mathbf{S}^1$ phase-shift symmetry. This is the case for Σ_7 and Σ_8. We have $d_{\Sigma_i} = 3$, for $i = 4, 5, 6$, also $d_{\Sigma_i} = 2$ for $i = 2, 3$ and finally $d_{\Sigma_1} = 1$.

We take co-ordinate functions on V^3:

$$z_1, \overline{z_1}, \ldots, z_6, \overline{z_6}.$$

These correspond to a basis B for V^3, the elements of which we will denote by:

$$l_1, \overline{l_1}, \ldots, l_6, \overline{l_6}.$$

An $\mathbf{R}$-linear mapping on $\mathbf{C}$ has the form

$$w \longmapsto \alpha w + \beta \overline{w},$$

where α and β are complex and the matrix of this mapping in these co-ordinates $M = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix}$ has

$$\mathrm{Tr}(M) = 2\mathrm{Re}(\alpha) \ \wedge \ \mathrm{Det}(M) = |\alpha|^2 - |\beta|^2.$$

The eigenvalues of this matrix are

$$\frac{\mathrm{tr}(M)}{2} \mp \sqrt{\left(\frac{\mathrm{Tr}(M)}{2}\right)^2 - \mathrm{Det}(M)}.$$

If one eigenvalue is zero, then $\mathrm{Det}(M) = 0$ and the sign of the other eigenvalue (if it is not zero) is given by the sign of the real part of α. If there are no zero eigenvalues, then the eigenvalues have negative real part if and only if the determinant is positive and the trace is negative.

(Σ_1)

The fixed-point subspace is $z_2 = \cdots = z_6 = 0$ and $z_1 = z$. Using the equation (5.10) after dividing by z we have

$$\nu(\lambda) + (a+b)|z|^2 + \cdots = 0$$

where $+\cdots$ denotes terms of higher order in z and $\overline{z}$, and taking the real part of this equation, we obtain

$$\lambda = -(a_r + b_r)|z|^2 + \cdots$$

where the functions $a_r,\ b_r$ are evaluated at $\lambda = 0$ and $+\cdots$ indicates higher order terms in $z,\ \overline{z}$ and λ.

It follows that if $a_r + b_r < 0$, then the branch bifurcates supercritically.

Throughout we denote by (z_0, λ_0, τ_0) a zero of (5.9) with $z_0 \in \mathrm{Fix}(\Sigma_i)$ and we wish to calculate $(dg)_{(z_0,\lambda_0,\tau_0)}$.

Let $\Sigma = \Sigma_1$ be the isotropy subgroup of $z_0 = (z, 0, 0, 0, 0, 0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$\mathrm{diag}(A, B, C, C, C, C),$$

where $A,\ B$ and C are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1' \\ \overline{a}_1' & \overline{a}_1 \end{bmatrix},\quad B = \begin{bmatrix} b_1 & 0 \\ 0 & \overline{b}_1 \end{bmatrix},\quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix}$$

and

$$a_1 = \tfrac{\partial g_1}{\partial z_1},\quad a_1' = \tfrac{\partial g_1}{\partial \overline{z}_1},\quad b_1 = \tfrac{\partial g_2}{\partial z_2},\quad c_1 = \tfrac{\partial g_3}{\partial z_3},$$

calculated at (z_0, λ_0, τ_0).

The tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector $(iz, 0, 0, 0, 0, 0)$. Thus the matrix A has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(a_1) = 2(a_r + b_r)|z|^2 + \cdots,$$

whose sign is determined by

$$a_r + b_r$$

if it is assumed nonzero (where $a_r + b_r$ is calculated at zero).

The eigenvalues of B are the conjugate complex numbers

$$b_1,\ \overline{b}_1$$

and

$$b_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,z,0,0,0,0,\lambda_0,\tau_0)} = (-b+c)|z|^2 + \cdots$$

Thus the sign of $\mathrm{Re}(b_1)$ is determined by the sign of

$$-b_r + c_r.$$

The matrix C has eigenvalues

$$c_1,\ \overline{c}_1$$

four times each one and as

$$c_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,z,0,0,0,\lambda_0,\tau_0)} = -b|z|^2 + \cdots,$$

then $\mathrm{Re}(c_1)$ is determined by

$$-b_r.$$

(Σ_2)

Let $\Sigma = \Sigma_2$ be the isotropy subgroup of $z_0 = (z, z, 0, 0, 0, 0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & B & & & & \\ B & A & & & & \\ & & C & D & & \\ & & D & C & & \\ & & & & C & D \\ & & & & D & C \end{bmatrix},$$

where $A,\ B,\ C$ and D are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1' \\ \overline{a}_1' & \overline{a}_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_1' \\ \overline{b}_1' & \overline{b}_1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & d_1' \\ \overline{d}_1' & 0 \end{bmatrix}$$

and

$$a_1 = \frac{\partial g_1}{\partial z_1}, \quad a_1' = \frac{\partial g_1}{\partial \bar{z}_1}, \quad b_1 = \frac{\partial g_1}{\partial z_2}, \quad b_1' = \frac{\partial g_1}{\partial \bar{z}_2}$$
$$c_1 = \frac{\partial g_3}{\partial z_3}, \quad d_1' = \frac{\partial g_3}{\partial \bar{z}_4},$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis B':

$$l_1 + l_2, \ c.c., \ l_1 - l_2, \ c.c., \ l_3 + l_4, \ c.c.,$$
$$l_3 - l_4, \ c.c., \ l_5 + l_6, \ c.c., \ l_5 - l_6, \ c.c.,$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A + B, \ A - B, \ C + D, \ C - D, \ C + D, \ C - D).$$

That is, the matrices are similar:

$$\mathrm{diag}(A{+}B, \ A{-}B, \ C{+}D, \ C{-}D, \ C{+}D, \ C{-}D) = S \begin{bmatrix} A & B & & & & \\ B & A & & & & \\ & & C & D & & \\ & & D & C & & \\ & & & & C & D \\ & & & & D & C \end{bmatrix} S^{-1},$$

where

$$S = S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} Id & Id & & & & \\ Id & -Id & & & & \\ & & Id & Id & & \\ & & Id & -Id & & \\ & & & & Id & Id \\ & & & & Id & -Id \end{bmatrix}.$$

The null vectors can be chosen to be

$$(iz, iz, 0, 0, 0, 0),$$
$$(-iz, iz, 0, 0, 0, 0).$$

For example, for the second one note that

$$\frac{d}{dt}(e^{-it}z, e^{it}z, 0, 0, 0, 0) \mid_{t=0} = (-iz, iz, 0, 0, 0, 0).$$

Zero is an eigenvalue of $A + B$ and $A - B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1 + b_1), \ 2\mathrm{Re}(a_1 - b_1),$$

and since

$$\begin{aligned} a_1 + b_1 &= (2a+b+c)|z|^2 + \cdots \\ a_1 - b_1 &= (b-c)|z|^2 + \cdots, \end{aligned}$$

their signs depend on

$$2a_r + b_r + c_r, \ \ b_r - c_r.$$

For $C+D$ and $C-D$, the eigenvalues are

$$\mathrm{Re}(c_1) \mp \sqrt{|d_1^{'}|^2 - \mathrm{Im}\ ^2(c_1)}$$

each four times and these eigenvalues have negative real part if and only if

$$\mathrm{Tr}(C+D) = \mathrm{Tr}(C-D) < 0 \ \wedge \ \mathrm{Det}(C+D) = \mathrm{Det}(C-D) > 0.$$

As

$$\begin{aligned} c_1 &= \left(\tfrac{\partial g_1}{\partial z_1}\right)_{(0,0,z,z,0,0,\lambda_0,\tau_0)} = -(b+c)|z|^2 + \cdots \\ d_1^{'} &= \left(\tfrac{\partial g_1}{\partial \overline{z}_2}\right)_{(0,0,z,z,0,0,\lambda_0,\tau_0)} = d|z|^2 + \cdots, \end{aligned}$$

we have

$$\mathrm{Tr}(C+D) = 2\mathrm{Re}(c_1) = -2(b_r + c_r)|z|^2 + \cdots$$

and

$$\mathrm{Det}(C+D) = |c_1|^2 - |d_1^{'}|^2 = (|b+c|^2 - |d|^2)|z|^4 + \cdots$$

(Σ_3)

Let $\Sigma = \Sigma_3$ be the isotropy subgroup of $z_0 = (z,0,z,0,0,0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & & B & & & \\ & C & & D & & \\ B & & A & & & \\ & D & & C & & \\ & & & & E & \\ & & & & & E \end{bmatrix},$$

where $A,\ B,\ C,\ D$ and E are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{bmatrix},$$
$$E = \begin{bmatrix} e_1 & 0 \\ 0 & \overline{e}_1 \end{bmatrix}$$

and

$$
\begin{array}{llll}
a_1 = \frac{\partial g_1}{\partial z_1}, & a_1^{'} = \frac{\partial g_1}{\partial \overline{z}_1}, & b_1 = \frac{\partial g_1}{\partial z_3}, & b_1^{'} = \frac{\partial g_1}{\partial \overline{z}_3}, \\
c_1 = \frac{\partial g_2}{\partial z_2}, & d_1 = \frac{\partial g_2}{\partial z_4}, & e_1 = \frac{\partial g_5}{\partial z_5}, &
\end{array}
$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis $B^{'}$:

$$
\begin{array}{l}
l_1 + l_3,\ c.c.,\ l_2 + l_4,\ c.c.,\ l_1 - l_3,\ c.c., \\
l_2 - l_4,\ c.c.,\ l_5,\ c.c.,\ l_6,\ c.c.,
\end{array}
$$

the matrix becomes block diagonal with 2×2 blocks:

$$
\mathrm{diag}(A + B,\ C + D,\ A - B,\ C - D,\ E,\ E).
$$

The null vectors can be

$$
\begin{array}{l}
(iz, 0, iz, 0, 0, 0) \\
(iz, 0, -iz, 0, 0, 0)
\end{array}
$$

and zero is an eigenvalue (two times) of $A + B$ and $A - B$. The others eigenvalues of these matrices are

$$
2\mathrm{Re}(a_1 + b_1),\ 2\mathrm{Re}(a_1 - b_1),
$$

where

$$
\begin{array}{l}
a_1 = (a + b)|z|^2 + \cdots \\
b_1 = a|z|^2 + \cdots
\end{array}
$$

and so their signs depend on

$$
2a_r + b_r,\ b_r.
$$

For $C + D$ and $C - D$, the eigenvalues are

$$
c_1 \mp d_1,\ c.c.
$$

and for E are

$$
e_1,\ \overline{e}_1,
$$

each occurring two times. Since

$$
\begin{array}{lll}
c_1 & = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,z,z,0,0,0,\lambda_0,\tau_0)} & = (-b + c)|z|^2 + \cdots \\
d_1 & = \left(\frac{\partial g_1}{\partial z_4}\right)_{(0,z,z,0,0,0,\lambda_0,\tau_0)} & = d|z|^2 + \cdots \\
e_1 & = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,z,0,z,0,\lambda_0,\tau_0)} & = -b|z|^2 + \cdots,
\end{array}
$$

their real parts are

$$\begin{aligned}
&\mathrm{Re}(c_1+d_1) = (-b_r+c_r+d_r)|z|^2+\cdots\\
&\mathrm{Re}(c_1-d_1) = (-b_r+c_r-d_r)|z|^2+\cdots\\
&\mathrm{Re}(e_1) = -b_r|z|^2+\cdots
\end{aligned}$$

(Σ_4)

Let $\Sigma=\Sigma_4$ be the isotropy subgroup of $z_0=(z,z,z,z,0,0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & B & C & C & & \\ B & A & C & C & & \\ C & C & A & B & & \\ C & C & B & A & & \\ & & & & D & E \\ & & & & E & D \end{bmatrix},$$

where $A,\ B,\ C,\ D$ and E are the 2×2 matrices:

$$A=\begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix},\quad B=\begin{bmatrix} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{bmatrix},\quad C=\begin{bmatrix} c_1 & c_1^{'} \\ \overline{c}_1^{'} & \overline{c}_1 \end{bmatrix},\quad D=\begin{bmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{bmatrix},$$
$$E=\begin{bmatrix} 0 & e_1^{'} \\ \overline{e}_1^{'} & 0 \end{bmatrix}$$

and

$$\begin{aligned}
&a_1=\tfrac{\partial g_1}{\partial z_1},\quad a_1^{'}=\tfrac{\partial g_1}{\partial \overline{z}_1},\quad b_1=\tfrac{\partial g_1}{\partial z_2},\quad b_1^{'}=\tfrac{\partial g_1}{\partial \overline{z}_2},\\
&c_1=\tfrac{\partial g_1}{\partial z_3},\quad c_1^{'}=\tfrac{\partial g_1}{\partial \overline{z}_3},\quad d_1=\tfrac{\partial g_5}{\partial z_5},\quad e_1^{'}=\tfrac{\partial g_5}{\partial \overline{z}_6},
\end{aligned}$$

calculated at (z_0,λ_0,τ_0).

With respect to the new basis $B^{'}$:

$$\begin{aligned}
&l_1-l_2,\ c.c.,\ l_3-l_4,\ c.c.,\ l_1+l_2+l_3+l_4,\ c.c.,\\
&l_1+l_2-l_3-l_4,\ c.c.,\ l_5+l_6,\ c.c.,\ l_5-l_6,\ c.c.,
\end{aligned}$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A-B,\ A-B,\ A+B+2C,\ A+B-2C, D+E,\ D-E).$$

The null vectors are

$$\begin{aligned}
&(iz,iz,iz,iz,0,0),\\
&(-iz,iz,0,0,0,0),\\
&(0,0,-iz,iz,0,0)
\end{aligned}$$

and zero is an eigenvalue $A+B+2C$ and $A-B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1-b_1),\ 2\mathrm{Re}(a_1+b_1+2c_1)$$

and as

$$\begin{aligned} a_1 &= (a+b-d)|z|^2+\cdots \\ b_1 &= (a+c)|z|^2+\cdots \\ c_1 &= (a+d)|z|^2+\cdots, \end{aligned}$$

then

$$\begin{aligned} \mathrm{Re}(a_1-b_1) &= (b_r-c_r-d_r)|z|^2+\cdots \\ \mathrm{Re}(a_1+b_1+2c_1) &= (4a_r+b_r+c_r+d_r)|z|^2+\cdots \end{aligned}$$

The matrix $A+B-2C$ has eigenvalues complex conjugates whose real part is negative if and only if

$$\mathrm{Tr}(A+B-2C)<0\ \wedge\ \mathrm{Det}(A+B-2C)>0.$$

And we have

$$\begin{aligned} \mathrm{Tr}(A+B-2C) &= 2\mathrm{Re}(a_1+b_1-2c_1)=2(b_r+c_r-3d_r)|z|^2+\cdots \\ \mathrm{Det}(A+B-2C) &= |a_1+b_1-2c_1|^2-|a_1^{'}+b_1^{'}-2c_1^{'}|^2=8[|d|^2-\mathrm{Re}((b+c)\overline{d})]|z|^4+\cdots, \end{aligned}$$

since

$$\begin{aligned} a_1^{'} &= (a+b)z^2+\cdots \\ b_1^{'} &= (a+c+d)z^2+\cdots \\ c_1^{'} &= az^2+\cdots \end{aligned}$$

Finally, the eigenvalues of $D+E$ and $D-E$ are

$$\mathrm{Re}(d_1)\mp\sqrt{|e_1^{'}|^2-\mathrm{Im}^2(d_1)},$$

which have negative real part if and only if

$$\mathrm{Tr}(D+E)=2\mathrm{Re}(d_1)<0\ \wedge\ \mathrm{Det}(D+E)=|d_1|^2-|e_1^{'}|^2>0.$$

Since

$$\begin{aligned} d_1 &= \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,z,z,z,z,\lambda_0,\tau_0)}=-(b+c+d)|z|^2+\cdots \\ e_1^{'} &= \left(\frac{\partial g_1}{\partial \overline{z}_2}\right)_{(0,0,z,z,z,z,\lambda_0,\tau_0)}=2dz^2+\cdots, \end{aligned}$$

we have

$$\begin{aligned} \mathrm{Re}(d_1) &= -(b_r+c_r+d_r)|z|^2+\cdots \\ |d_1|^2-|e_1^{'}|^2 &= [|b+c|^2-3|d|^2+2\mathrm{Re}((b+c)\overline{d})]|z|^4+\cdots \end{aligned}$$

(Σ_5)

Let $\Sigma = \Sigma_5$ be the isotropy subgroup of $z_0 = (z, z, iz, iz, 0, 0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & B & C' & C' & & \\ B & A & C' & C' & & \\ C & C & A' & B' & & \\ C & C & B' & A' & & \\ & & & & D & \\ & & & & & D \end{bmatrix},$$

where $A,\ B,\ C,\ D,\ A',\ B',\ C'$ and D are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1' \\ \overline{a}_1' & \overline{a}_1 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_1' \\ \overline{b}_1' & \overline{b}_1 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 & c_1' \\ \overline{c}_1' & \overline{c}_1 \end{bmatrix}, \qquad D = \begin{bmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{bmatrix},$$
$$A' = \begin{bmatrix} a_1 & -a_1' \\ -\overline{a}_1' & \overline{a}_1 \end{bmatrix}, \quad B' = \begin{bmatrix} b_1 & -b_1' \\ -\overline{b}_1' & \overline{b}_1 \end{bmatrix}, \quad C' = \begin{bmatrix} -c_1 & c_1' \\ \overline{c}_1' & -\overline{c}_1 \end{bmatrix}$$

and

$$a_1 = \tfrac{\partial g_1}{\partial z_1}, \quad a_1' = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad b_1 = \tfrac{\partial g_1}{\partial z_2}, \quad b_1' = \tfrac{\partial g_1}{\partial \overline{z}_2},$$
$$-c_1 = \tfrac{\partial g_1}{\partial z_3}, \quad c_1' = \tfrac{\partial g_1}{\partial \overline{z}_3}, \quad d_1 = \tfrac{\partial g_5}{\partial z_5},$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis B':

$$l_1 - l_2,\ c.c.,\ l_3 - l_4,\ c.c.,\ l_1 + l_2 + i(l_3 + l_4),\ c.c.,$$
$$l_1 + l_2 - i(l_3 + l_4),\ c.c.,\ l_5,\ c.c.,\ l_6,\ c.c.,$$

the matrix becomes block diagonal with 2×2 blocks:

$$\operatorname{diag}(A - B,\ A' - B',\ M,\ N, D,\ D),$$

where

$$M = \begin{bmatrix} a_1 + b_1 - 2ic_1 & a_1' + b_1' - 2ic_1' \\ \overline{a}_1' + \overline{b}_1' + 2i\overline{c}_1' & \overline{a}_1 + \overline{b}_1 + 2i\overline{c}_1 \end{bmatrix}$$

and

$$N = \begin{bmatrix} a_1 + b_1 + 2ic_1 & a_1' + b_1' + 2ic_1' \\ \overline{a}_1' + \overline{b}_1' - 2i\overline{c}_1' & \overline{a}_1 + \overline{b}_1 - 2i\overline{c}_1 \end{bmatrix}.$$

The null vectors are:

$$\begin{array}{l}(iz, iz, -z, -z, 0, 0),\\ (-iz, iz, 0, 0, 0, 0),\\ (0, 0, z, -z, 0, 0)\end{array}$$

and zero is an eigenvalue of M, $A - B$ and $A' - B'$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1 - b_1),\ 2\mathrm{Re}(a_1 + b_1 - 2ic_1),$$

and since

$$\begin{array}{l}a_1 = (a + b + d)|z|^2 + \cdots\\ b_1 = (a + c)|z|^2 + \cdots\\ c_1 = (a - d)i|z|^2 + \cdots,\end{array}$$

then

$$\begin{array}{l}\mathrm{Re}(a_1 - b_1) = (b_r - c_r + d_r)|z|^2 + \cdots\\ \mathrm{Re}(a_1 + b_1 - 2ic_1) = (4a_r + b_r + c_r - d_r)|z|^2 + \cdots\end{array}$$

The matrix N has eigenvalues complex conjugates whose real part is negative if and only if

$$\mathrm{Tr}(N) < 0 \ \wedge \ \mathrm{Det}(N) > 0.$$

Since

$$\begin{array}{l}a_1' = (a + b)z^2 + \cdots\\ b_1' = (a + c - d)z^2 + \cdots\\ c_1' = aiz^2 + \cdots,\end{array}$$

then

$$\begin{array}{l}\mathrm{Tr}(N) = 2\mathrm{Re}(a_1 + b_1 + 2ic_1) = 2(b_r + c_r + 3d_r)|z|^2 + \cdots\\ \mathrm{Det}(N) = |a_1 + b_1 + 2ic_1|^2 - |a_1' + b_1' + 2ic_1'|^2 = 8[|d|^2 + \mathrm{Re}((b + c)\overline{d})]|z|^4 + \cdots\end{array}$$

Finally, the matrix D has the eigenvalues

$$d_1,\ \overline{d}_1$$

and

$$d_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,iz,iz,z,z,\lambda_0,\tau_0)} = (-b - c + d)|z|^2 + \cdots$$

Thus

$$\mathrm{Re}(d_1) = (-b_r - c_r + d_r)|z|^2 + \cdots$$

(Σ_6)

Let $\Sigma = \Sigma_6$ be the isotropy subgroup of $z_0 = (z, 0, z, 0, z, 0)$. With respect to the basis B, any matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & & B & & B & \\ & C & & D & & D \\ B & & A & & B & \\ & D & & C & & D \\ B & & B & & A & \\ & D & & D & & C \end{bmatrix},$$

where $A,\ B,\ C$ and D are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix},\quad B = \begin{bmatrix} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{bmatrix},\quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix},\quad D = \begin{bmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{bmatrix}$$

and

$$\begin{array}{l} a_1 = \frac{\partial g_1}{\partial z_1},\quad a_1^{'} = \frac{\partial g_1}{\partial \overline{z}_1},\quad b_1 = \frac{\partial g_1}{\partial z_3},\quad b_1^{'} = \frac{\partial g_1}{\partial \overline{z}_3}, \\ c_1 = \frac{\partial g_2}{\partial z_2},\quad d_1 = \frac{\partial g_2}{\partial z_4}, \end{array}$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis $B^{'}$:

$$\begin{array}{l} l_1 + l_3 + l_5,\ c.c.,\ l_2 + l_4 + l_6,\ c.c.,\ l_1 - 2l_3 + l_5,\ c.c., \\ l_2 - 2l_4 + l_6,\ c.c.,\ l_1 - l_3,\ c.c.,\ l_2 - l_4,\ c.c., \end{array}$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A + 2B,\ C + 2D,\ A - B,\ C - D, A - B,\ C - D).$$

The null vectors can be

$$\begin{array}{l} (iz, 0, iz, 0, iz, 0), \\ (iz, 0, -iz, 0, 0, 0), \\ (iz, 0, -2iz, 0, iz, 0) \end{array}$$

and zero is an eigenvalue (three times) of $A + 2B$ and $A - B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1 + 2b_1),\ 2\mathrm{Re}(a_1 - b_1),$$

and since

$$a_1 = (a+b)|z|^2 + \cdots$$
$$b_1 = a|z|^2 + \cdots,$$

then

$$\mathrm{Re}(a_1 + 2b_1) = (3a_r + b_r)|z|^2 + \cdots$$
$$\mathrm{Re}(a_1 - b_1) = b_r|z|^2 + \cdots$$

The matrices $C+2D$ and $C-D$ have eigenvalues complex conjugates

$$c_1 + 2d_1,\ c.c.,\ c_1 - d_1,\ c.c.$$

As

$$c_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,z,z,0,z,0,\lambda_0,\tau_0)} = (-b+c)|z|^2 + \cdots$$
$$d_1 = \left(\frac{\partial g_1}{\partial z_4}\right)_{(0,z,z,0,z,0,\lambda_0,\tau_0)} = d|z|^2 + \cdots,$$

then

$$\mathrm{Re}(c_1 + 2d_1) = (-b_r + c_r + 2d_r)|z|^2 + \cdots$$
$$\mathrm{Re}(c_1 - d_1) = (-b_r + c_r - d_r)|z|^2 + \cdots$$

(Σ_7)

Let $\Sigma = \Sigma_7$ be the isotropy subgroup of $z_0 = (z,z,z,z,z,z)$. With respect to the basis B, any matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & B & C & C & C & C \\ B & A & C & C & C & C \\ C & C & A & B & C & C \\ C & C & B & A & C & C \\ C & C & C & C & A & B \\ C & C & C & C & B & A \end{bmatrix},$$

where $A,\ B$ and C are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix},\quad B = \begin{bmatrix} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{bmatrix},\quad C = \begin{bmatrix} c_1 & c_1^{'} \\ \overline{c}_1^{'} & \overline{c}_1 \end{bmatrix}$$

and

$$a_1 = \frac{\partial g_1}{\partial z_1},\quad a_1^{'} = \frac{\partial g_1}{\partial \overline{z}_1},\quad b_1 = \frac{\partial g_1}{\partial z_2},\quad b_1^{'} = \frac{\partial g_1}{\partial \overline{z}_2},$$
$$c_1 = \frac{\partial g_1}{\partial z_3},\quad c_1^{'} = \frac{\partial g_1}{\partial \overline{z}_3},$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis B':

$$\begin{aligned}&l_1+l_2+l_3+l_4+l_5+l_6,\ c.c.,\ l_1+l_2-(l_5+l_6),\ c.c.,\\ &l_1-l_2+l_3-l_4,\ c.c.,\ l_3-l_4+l_5-l_6,\ c.c.,\ l_1-l_2+l_5-l_6,\ c.c.,\\ &l_1+l_2-(l_3+l_4),\ c.c.,\end{aligned}$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A+B+4C,\ A+B-2C,\ A-B,\ A-B,\ A-B,\ A+B-2C).$$

The null vectors are:

$$\begin{aligned}&(iz,iz,iz,iz,iz,iz),\\ &(-iz,iz,-iz,iz,0,0),\\ &(0,0,-iz,iz,-iz,iz),\\ &(-iz,iz,0,0,-iz,iz)\end{aligned}$$

and zero is an eigenvalue of $A+B+4C$ and $A-B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1+b_1+4c_1),\ 2\mathrm{Re}(a_1-b_1),$$

where

$$\begin{aligned}a_1&=(a+b-2d)|z|^2+\cdots\\ b_1&=(a+c)|z|^2+\cdots\\ c_1&=(a+d)|z|^2+\cdots\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Re}(a_1+b_1+4c_1)&=(6a_r+b_r+c_r+2d_r)|z|^2+\cdots\\ \mathrm{Re}(a_1-b_1)&=(b_r-c_r-2d_r)|z|^2+\cdots\end{aligned}$$

The matrix $A+B-2C$ has eigenvalues complex conjugates that have negative real part if and only if

$$\begin{aligned}&\mathrm{Tr}(A+B-2C)=2\mathrm{Re}(a_1+b_1-2c_1)<0\\ &\mathrm{Det}(A+B-2C)=|a_1+b_1-2c_1|^2-|a_1'+b_1'-2c_1'|^2>0.\end{aligned}$$

Since

$$\begin{aligned}a_1'&=(a+b)z^2+\cdots\\ b_1'&=(a+c+2d)z^2+\cdots\\ c_1'&=az^2+\cdots,\end{aligned}$$

then

$$\mathrm{Re}(a_1+b_1-2c_1) = (b_r+c_r-4d_r)|z|^2+\cdots$$
$$|a_1+b_1-2c_1|^2-|a_1'+b_1'-2c_1'|^2 = 12[|d|^2-\mathrm{Re}((b+c)\overline{d})]|z|^4+\cdots$$

(Σ_8)

Let $\Sigma=\Sigma_8$ be the isotropy subgroup of $z_0=(z,z,\xi z,\xi z,\xi^2 z,\xi^2 z)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A_1 & A_2 & B_2^{\xi} & B_2^{\xi} & B_1^{\xi^2} & B_1^{\xi^2} \\ A_2 & A_1 & B_2^{\xi} & B_2^{\xi} & B_1^{\xi^2} & B_1^{\xi^2} \\ B_1 & B_1 & A_1^{\xi} & A_2^{\xi} & B_2^{\xi^2} & B_2^{\xi^2} \\ B_1 & B_1 & A_2^{\xi} & A_1^{\xi} & B_2^{\xi^2} & B_2^{\xi^2} \\ B_2 & B_2 & B_1^{\xi} & B_1^{\xi} & A_1^{\xi^2} & A_2^{\xi^2} \\ B_2 & B_2 & B_1^{\xi} & B_1^{\xi} & A_2^{\xi^2} & A_1^{\xi^2} \end{bmatrix},$$

where $A_1,\ A_2,\ B_1,\ B_2,\ A_1^{\xi},\ A_1^{\xi^2},\ldots,$ are the 2×2 matrices:

$$A_1=\begin{bmatrix} a_1 & a_1' \\ \overline{a_1}' & \overline{a_1}\end{bmatrix},\quad A_2=\begin{bmatrix} b_1 & b_1' \\ \overline{b_1}' & \overline{b_1}\end{bmatrix},\qquad B_1=\begin{bmatrix} c_1 & c_1' \\ \overline{c_1}' & \overline{c_1}\end{bmatrix}$$
$$B_2=\begin{bmatrix} d_1 & d_1' \\ \overline{d}_1' & \overline{d}_1\end{bmatrix},\quad A_1^{\xi}=\begin{bmatrix} a_1 & \xi^2 a_1' \\ \xi\overline{a}_1' & \overline{a}_1\end{bmatrix},\quad A_1^{\xi^2}=\begin{bmatrix} a_1 & \xi a_1' \\ \xi^2\overline{a}_1' & \overline{a}_1\end{bmatrix}$$

With respect to the new basis B':

$$l_1+l_2+\xi(l_3+l_4)+\xi^2(l_5+l_6),\ c.c.,\ l_1-l_2,\ c.c.,\ l_3-l_4,\ c.c.,$$
$$l_5-l_6,\ \ c.c.,\ l_1+l_2+l_3+l_4+l_5+l_6,\ \overline{l}_1+\overline{l}_2+\xi(\overline{l}_3+\overline{l}_4)+\xi^2(\overline{l}_5+\overline{l}_6),$$
$$l_1+l_2+\xi^2(l_3+l_4)+\xi(l_5+l_6),\ \overline{l}_1+\overline{l}_2+\overline{l}_3+\overline{l}_4+\overline{l}_5+\overline{l}_6,$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(M_1,\ A_1-A_2,\ A_1^{\xi}-A_2^{\xi},\ A_1^{\xi^2}-A_2^{\xi^2},\ M_2,\ M_3),$$

where

$$M_1 = \begin{bmatrix} a_1 + b_1 + 2\xi d_1 + 2\xi^2 c_1 & a_1^{'} + b_1^{'} + 2\xi d_1^{'} + 2\xi^2 c_1^{'} \\ \overline{a}_1^{'} + \overline{b}_1^{'} + 2\xi^2 \overline{d}_1^{'} + 2\xi \overline{c}_1^{'} & \overline{a}_1 + \overline{b}_1 + 2\xi^2 \overline{d}_1 + 2\xi \overline{c_1} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} a_1 + b_1 + 2c_1 + 2d_1 & a_1^{'} + b_1^{'} + 2c_1^{'} + 2d_1^{'} \\ \overline{a}_1^{'} + \overline{b}_1^{'} + 2\xi \overline{d}_1^{'} + 2\xi^2 \overline{c}_1^{'} & \overline{a}_1 + \overline{b}_1 + 2\xi \overline{d}_1 + 2\xi^2 \overline{c}_1 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} a_1 + b_1 + 2\xi c_1 + 2\xi^2 d_1 & a_1^{'} + b_1^{'} + 2\xi c_1^{'} + 2\xi^2 d_1^{'} \\ \overline{a}_1^{'} + \overline{b}_1^{'} + 2\overline{c}_1^{'} + 2\overline{d}_1^{'} & \overline{a}_1 + \overline{b}_1 + 2\overline{d}_1 + 2\overline{c}_1 \end{bmatrix}$$

and

$$\begin{array}{llll} a_1 = \frac{\partial g_1}{\partial z_1}, & a_1^{'} = \frac{\partial g_1}{\partial \overline{z}_1}, & b_1 = \frac{\partial g_1}{\partial z_2}, & b_1^{'} = \frac{\partial g_1}{\partial \overline{z}_2}, \\ c_1 = \frac{\partial g_1}{\partial z_5}, & \xi c_1^{'} = \frac{\partial g_1}{\partial \overline{z}_5}, & d_1 = \frac{\partial g_1}{\partial z_3}, & \xi^2 d_1^{'} = \frac{\partial g_1}{\partial \overline{z}_3}, \end{array}$$

calculated at (z_0, λ_0, τ_0).

The null eigenvectors can be

$$\begin{array}{l} (iz, iz, i\xi z, i\xi z, i\xi^2 z, i\xi^2 z), \\ (-iz, iz, 0, 0, 0, 0), \\ (0, 0, -i\xi z, i\xi z, 0, 0), \\ (0, 0, 0, 0, -i\xi^2 z, i\xi^2 z), \end{array}$$

and so zero is an eigenvalue of the matrices M_1, $A_1 - A_2$, $A_1^{\xi} - A_2^{\xi}$ and $A_1^{\xi^2} - A_2^{\xi^2}$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1 + b_1 + 2\xi d_1 + 2\xi^2 c_1),\ 2\mathrm{Re}(a_1 - b_1),$$

and since

$$\begin{array}{l} a_1 = (a + b + d)|z|^2 + \cdots \\ b_1 = (a + c)|z|^2 + \cdots \\ d_1 = (\xi^2 a + \xi d)|z|^2 + \cdots \\ c_1 = (\xi a + \xi^2 d)|z|^2 + \cdots, \end{array}$$

it follows that

$$\begin{array}{l} \mathrm{Re}(a_1 + b_1 + 2\xi d_1 + 2\xi^2 c_1) = (6a_r + b_r + c_r - d_r)|z|^2 + \cdots \\ \mathrm{Re}(a_1 - b_1) = (b_r - c_r + d_r)|z|^2 + \cdots \end{array}$$

The eigenvalues of M_i for $i = 2,\ 3$ are

$$T_i \mp \sqrt{T_i^2 - D_i}$$

where $T_i = \frac{\mathrm{Tr}(M_i)}{2}$ and $D_i = \mathrm{Det} M_i$, and they have negative real part if and only if

$$\mathrm{Re}(T_i \mp \sqrt{T_i^2 - D_i}) < 0.$$

Note that

$$\begin{aligned}
T_2 &= \mathrm{Re}(a_1) + \mathrm{Re}(b_1) + (c_1 + \xi^2 \overline{c}_1) + (d_1 + \xi \overline{d}_1) \\
T_3 &= \mathrm{Re}(a) + \mathrm{Re}(b) + (\overline{c} + \xi c) + (\overline{d} + \xi^2 d) \\
D_2 &= (a_1 + b_1 + 2c_1 + 2d_1)(\overline{a}_1 + \overline{b}_1 + 2\xi^2 \overline{c}_1 + 2\xi \overline{d}_1) \\
&\quad -(a_1' + b_1' + 2c_1' + 2d_1')(\overline{a}_1' + \overline{b}_1' + 2\xi^2 \overline{c}_1' + 2\xi \overline{d}_1') \\
D_3 &= (a_1 + b_1 + 2\xi c_1 + 2\xi^2 d_1)(\overline{a}_1 + \overline{b}_1 + 2\overline{c}_1 + 2\overline{d}_1) \\
&\quad -(a_1' + b_1' + 2\xi c_1' + 2\xi^2 d_1')(\overline{a}_1' + \overline{b}_1' + 2\overline{c}_1' + 2\overline{d_1'})
\end{aligned}$$

and so $T_3 = \overline{T}_2$ and $D_3 = \overline{D}_2$. The eigenvalues of M_3 are the conjugates of the eigenvalues of the matrix M_2. Since

$$\begin{aligned}
a_1' &= (a+b)z^2 + \cdots \\
b_1' &= (a+c-d)z^2 + \cdots \\
d_1' &= \xi^2 a z^2 + \cdots \\
c_1' &= \xi a z^2 + \cdots,
\end{aligned}$$

the expressions for T_2 and D_2 are

$$\begin{aligned}
T_2 &= (b_r + c_r + 2d_r - i3d_I)|z|^2 + \cdots \\
D_2 &= 6[(b+c)\overline{d} - |d|^2]|z|^4 + \cdots,
\end{aligned}$$

and it follows the expressions in the table 5.8. □

5.6 Periodic solutions

In the previous section we considered the possible branches of periodic solutions with maximal isotropy that could generically bifurcate for the system (5.7). We wish now to look for possible branches of periodic solutions that can bifurcate with submaximal isotropy.

As was stated, when f is supposed to commute also with $\mathbf{S}^1$, then the problem of finding periodic solutions of $\dot{z} = f(z, \lambda)$ can be transformed to the problem of finding the zeros of $\dot{z} = g(z, \lambda, \tau)$ where $g = f - (1+\tau)iz$.

Also the stability of these zeros gives the stability of the corresponding solutions. As before we consider the third order truncation of f commuting with $\Gamma \times \mathbf{S}^1$. However, we point out that for the branches of periodic solutions with submaximal isotropy that are found here, we can no longer guarantee that they exist for (5.7) if f commutes only with Γ (even with the third order Taylor series commuting with $\mathbf{S}^1$). These solution branches are guaranteed only for the third order truncation, with which we work from now on. Throughout, consider the truncation of f as in (5.8) of degree three and the respective reduced vector field $g = f - (1+\tau)iz$ of the same degree. Thus let $g = (g_1, g_2, g_3, g_4, g_5, g_6)$ with

$$g_1(z, \lambda, \tau) = [\nu + a|z|^2 + b|z_1|^2 + c|z_2|^2]z_1 + d\overline{z}_2(z_3 z_4 + z_5 z_6),$$

where $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in V^3$, the coefficients $a,\ b,\ c,\ d$ are complex and depend on λ. Also, $\nu = \mu(\lambda) - (1+\tau)i$, where $\mu(0) = i$ and λ is scaled such that $\mathrm{Re}(\mu) = \lambda + o(\lambda^2))$. Thus the trivial solution is stable for λ negative and unstable for λ positive (near zero) and the nondegeneracy condition of the equivariant Hopf theorem is satisfied.

Recall that since g is $\Gamma \times \mathbf{S}^1$ equivariant, each fixed-point subspace is invariant under the dynamics, i.e., $g(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma)$, where Σ is an isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Recall tables 5.2 and 5.4, and figure 5.2.

We analyse symmetric solutions by analysing the zeros of the vector field g restricted to each fixed-point subspace, that is, we study

$$\dot{v} = g\ |_{\mathrm{Fix}(\Sigma)}\ (v), \qquad v \in \mathrm{Fix}(\Sigma).$$

$\mathrm{Fix}(\Sigma_i)$, $i = 1, \ldots, 8$.

For each Σ_i for $i = 1, \ldots, 8$, the equivariant Hopf theorem guarantees the existence of a branch of small-amplitude periodic solutions with symmetry Σ_i bifurcating from the trivial solution at $\lambda = 0$. For each two-dimensional $\mathrm{Fix}(\Sigma_i)$, the dynamics of $\dot{z}_1 = g\ |_{\mathrm{Fix}(\Sigma_i)}\ (z_1)$ are governed by

$$\dot{z}_1 = \nu z_1 + A|z_1|^2 z_1, \quad z_1 \in \mathbf{C}. \tag{5.11}$$

Note that the assumption of Birkhoff normal form implies that we can apply the standard Hopf theorem to the problem restricted to the two-dimensional fixed-point subspace (i.e., the vector field restricted to the fixed-point subspace commutes with $\mathbf{S}^1$). Thus as we are assuming that $\mathrm{Re}(\mu'(0)) > 0$,

Symmetry	Name (abbreviation)	A_r (stability)
Σ_1	Travelling rolls (TR)	$a_r + b_r$
Σ_2	Standing rolls (SR)	$2a_r + b_r + c_r$
Σ_3	Travelling squares (TS)	$2a_r + b_r$
Σ_4	Standing squares (SS)	$4a_r + b_r + c_r + d_r$
Σ_5	Alternating rolls (AR)	$4a_r + b_r + c_r - d_r$
Σ_6	Travelling cubes (TC)	$3a_r + b_r$
Σ_7	Standing cubes (SC)	$6a_r + b_r + c_r + 2d_r$
Σ_8	Alternating cubes (AC)	$6a_r + b_r + c_r - d_r$

Table 5.9: Stability of the periodic solution of the Hopf equation for each of the $\mathbf{C}$-axial groups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

the trivial solution is stable for λ negative and the stability of the periodic orbit in the fixed-point subspace is determined by the real part A_r of A. If $A_r < 0$, then the Hopf bifurcation is supercritical and the solution is stable to small perturbations within $\mathrm{Fix}(\Sigma_i)$. If $A_r > 0$, then the Hopf bifurcation is subcritical and the solution is unstable. See table 5.9.

Consider now the dynamics of $\dot{z} = g(z, \lambda)$ restricted to the four-dimensional fixed-point subspaces Δ_i (see table 5.4).

$\mathrm{Fix}(\Delta_5)$

When we restrict g to $\mathrm{Fix}(\Delta_5)$ we obtain the system:

$$\begin{aligned} \dot{z}_1 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_1|^2]z_1 + C\overline{z}_1 z_2^2, \\ \dot{z}_2 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_2|^2]z_2 + C\overline{z}_2 z_1^2, \end{aligned} \tag{5.12}$$

where $A = 2a$, $B = b + c$ and $C = d$. This is the normal form for the generic Hopf bifurcation problem with symmetry D_4 studied in [20, 18, 37].

The nontrivial solutions in the space $\mathrm{Fix}(\Delta_5)$ with maximal isotropy are: the (SR) solutions with symmetry Σ_2 corresponding to the zeros of (5.12) of type $z_2 = 0$, the (SS) solutions with symmetry Σ_4 corresponding to zeros satisfying $z_1 = z_2$ and the (AR) solutions with symmetry Σ_5, where the zeros satisfy $z_2 = iz_1$. Their stability properties in this subspace are summarized in table 5.10. The nondegeneracy conditions that determine the stability are

$$A_r + B_r,\ 2A_r + B_r \mp C_r \neq 0,$$

$$|B|^2 - |C|^2,\ |C|^2 \mp \mathrm{Re}(B\overline{C}) \neq 0.$$

Also $B_r = 0$ is a degenerate condition when $|B|^2 > |C|^2$ (in this case the eigenvalues $\mu_1,\ \mu_2$ are conjugate purely imaginary numbers). Similar, the conditions $B_r = \pm 3C_r$ are degenerate when $|C|^2 \mp \mathrm{Re}(B\overline{C}) > 0$. In table 5.10 we also show the corresponding conditions on the coefficients of g that were obtained in the full stability analysis in section 5.5 and that determine the stability of the three solutions in the space $\mathrm{Fix}(\Delta_5)$. As (5.12) is equivariant under the transformation (parameter symmetry [37])

$$(z_1, z_2; C) \to (z_1, iz_2; -C), \tag{5.13}$$

the stability conditions for the (AR) solution are obtained from those for the (SS) solution by letting $C \to -C$ (see table 5.10). In each case, the first two eigenvalues are in the radial and phase directions, respectively, and $\mu_1,\ \mu_2$ are in directions transverse to the plane containing the periodic orbit.

By [37], in addition to these periodic solutions, there can be a fourth branch of periodic solutions to (5.12) with $|z_1| \neq |z_2|$ and $z_1 z_2 \neq 0$. This solution branch exists if

$$|\mathrm{Re}((b+c)\overline{d})| < |d|^2 < |b+c|^2,$$

and the solutions are generically unstable.

In figure 5.12 we reproduce the possible bifurcation diagrams for this problem for the branches of solutions with maximal isotropy, obtained by [20], assuming that the expressions of table 5.10 are nonzero. The bifurcation diagrams are plotted for $C_r > 0$. The parameter symmetry (5.13) allows to infer the bifurcation diagrams for $C_r < 0$ (the same diagrams with (SS) and (AR) labels interchanged). There are regions of the parameter space (A, B, C) (and so of (a, b, c, d)) where a stable supercritical branch and one

Solution	Eigenvalues	Expressions from the full analysis for stability
$\Sigma_2\ (SR)$ $(z_1, z_2) = (z_1, 0)$	$2(A_r + B_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -2B_r\lvert z_1\rvert^2$ $\mu_1\mu_2 = (\lvert B\rvert^2 - \lvert C\rvert^2)\lvert z_1\rvert^4$	$2a_r + b_r + c_r$ $-b_r - c_r$ $-(\lvert b+c\rvert^2 - \lvert d\rvert^2)$
$\Sigma_4\ (SS)$ $(z_1, z_2) = (z_1, z_1)$	$2(2A_r + B_r + C_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = 2(B_r - 3C_r)\lvert z_1\rvert^2$ $\mu_1\mu_2 = 8[\lvert C\rvert^2 - \mathrm{Re}(B\overline{C})]\lvert z_1\rvert^4$	$4a_r + b_r + c_r + d_r$ $b_r + c_r - 3d_r$ $-[\lvert d\rvert^2 - \mathrm{Re}((b+c)\overline{d})]$
$\Sigma_5\ (AR)$ $(z_1, z_2) = (z_1, iz_1)$	$2(2A_r + B_r - C_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = 2(B_r + 3C_r)\lvert z_1\rvert^2$ $\mu_1\mu_2 = 8[\lvert C\rvert^2 + \mathrm{Re}(B\overline{C})]\lvert z_1\rvert^4$	$4a_r + b_r + c_r - d_r$ $b_r + c_r + 3d_r$ $-[\lvert d\rvert^2 + \mathrm{Re}((b+c)\overline{d})]$

Table 5.10: Stability of the periodic solutions with maximal isotropy in $\mathbf{D}_4$-Hopf bifurcation.

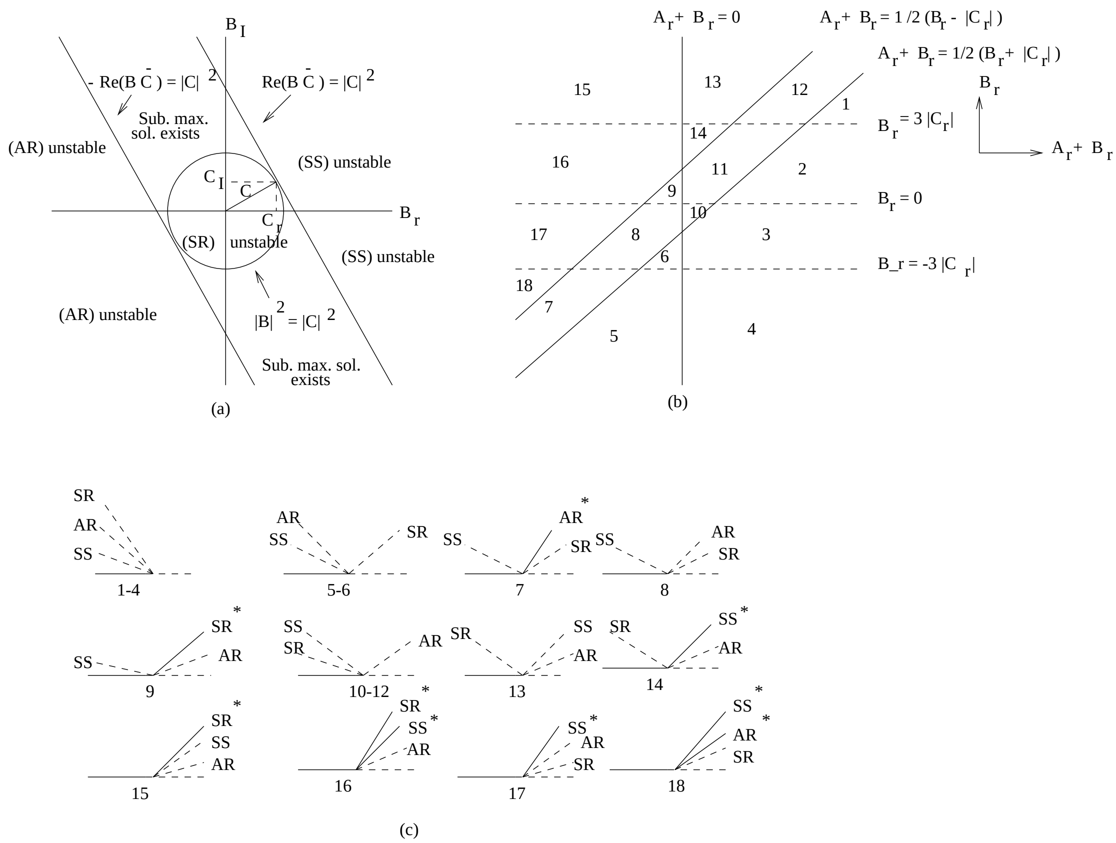


Figure 5.12: Bifurcation diagram for the nondegenerate Hopf bifurcation with $\mathbf{D}_4$ symmetry. (a) The complex coefficient C is plotted in the complex B plane. (c) Broken bifurcation curves indicate unstable solutions. An asterisk on solution indicates that it is possible for the solution to be stable and it depends on where B lies in (a). The diagrams are plotted for $C_r > 0$.

subcritical branch bifurcate from the origin. If two solutions branches bifurcate subcritically, then all the three solutions are unstable. Also, there are regions where all of the three solutions branches are supercritical and two of them are stable. However, it is not possible to have the three branches stable. Another possible bifurcation diagram arises when all branches are supercritical and unstable. Note that the stability of the Σ_4 and Σ_5 branches implies the instability of the Σ_2 branch: if $b_r + c_r + 3d_r$ and $b_r + c_r - 3d_r$ are both negative then $b_r + c_r$ is also negative.

$\mathrm{Fix}(\Delta_i)$, $i = 1, 2, 3, 4, 6, 7$.

If we restrict the initial vector field g to each $\mathrm{Fix}(\Delta_i)$ for $i = 1, 2, 3, 4, 6, 7$, then we obtain the normal form for the generic Hopf bifurcation problem with $\mathbf{O}(2)$ symmetry (studied by [20]):

$$\begin{aligned} \dot{z}_1 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_1|^2]z_1, \\ \dot{z}_2 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_2|^2]z_2. \end{aligned} \tag{5.14}$$

See table 5.11 with the coefficients A and B in terms of the coefficients of g. The two nontrivial periodic solutions of (5.14) satisfy the conditions $z_2 = 0$ and $z_1 = z_2$, the first ones we call by TW (travelling waves) and the second ones SW (standing waves).

Note that for the space $\mathrm{Fix}(\Delta_3)$, the TW solution has symmetry a conjugate subgroup of Σ_3 since $(z_1, 0, iz_1, 0, 0, 0)$ and $(z_1, 0, z_1, 0, 0, 0)$ are in the same orbit by the group $\Gamma \times \mathbf{S}^1$. The same happens for the TW solution contained in $\mathrm{Fix}(\Delta_7)$ with isotropy conjugate to Σ_6. We recall that solutions in the same orbit by the group $\Gamma \times \mathbf{S}^1$ have the same stability. The stability properties of the solutions in these subspaces are given in table 5.12. As (5.14) is a special case of (5.12) (take $C = 0$), this table is just obtained from table 5.10 with $C = 0$. In this case, the nondegeneracy conditions that determine the stability are

$$A_r + B_r,\ 2A_r + B_r,\ B_r \neq 0.$$

We also show the conditions on the coefficients of g obtained in the full stability analysis for the stability in each of the subspaces $\mathrm{Fix}(\Delta_i)$ for $i = 1, 2, 3, 4, 6, 7$.

Note that the SW and TW solutions of table 5.12 cannot be both stable. In fact, one of the two branches of solutions is stable only if both branches of the bifurcating solutions are supercritical, in which case, generically, precisely

Δ_i	$\mathrm{Fix}(\Delta_i)$	TW	SW	A	B
Δ_1	$\{(z_1, z_2, 0, 0, 0, 0)\}$	Σ_1 (TR)	Σ_2 (SR)	$a + c$	$b - c$
Δ_2	$\{(z_1, z_2, z_1, z_2, 0, 0)\}$	Σ_3 (TS)	Σ_4 (SS)	$2a + c + d$	$b - c - d$
Δ_3	$\{(z_1, z_2, iz_1, iz_2, 0, 0)\}$	Σ_3 (TS)	Σ_5 (AR)	$2a + c - d$	$b - c + d$
Δ_4	$\{(z_1, 0, z_2, 0, 0, 0)\}$	Σ_1 (TR)	Σ_3 (TS)	a	b
Δ_6	$\{(z_1, z_2, z_1, z_2, z_1, z_2)\}$	Σ_6 (TC)	Σ_7 (SC)	$3a + c + 2d$	$b - c - 2d$
Δ_7	$\{(z_1, z_2, \xi z_1, \xi z_2, \xi^2 z_1, \xi^2 z_2)\}$	Σ_6 (TC)	Σ_8 (AC)	$3a + c - d$	$b - c + d$

Table 5.11: Correspondence between solutions in $\mathbf{O}(2)$ Hopf bifurcation and solutions of $\dot{z} = f(z, \lambda)$ restricted to the appropriate fixed-point subspace.

one branch is stable. Thus, for example, the Σ_3 solution cannot be stable if any of the Σ_1, Σ_4 or Σ_5 solutions is stable. In figure 5.13 we reproduce the bifurcation diagram for the nondegenerate Hopf bifurcation with $\mathbf{O}(2)$ symmetry [20].

$\mathrm{Fix}(\Delta_8)$

Proposition 5.6.1 *Consider the equations for f restricted to* $\mathrm{Fix}(\Delta_8)$. *Generically (specifically, if $b_r \neq 0$), there are only branches of periodic solutions with symmetry Σ_1, Σ_3 and Σ_6. Table 5.13 contains the list of the eigenvalues depending on the coefficients of f that generically determine the stability of these solutions in the space* $\mathrm{Fix}(\Delta_8)$, *i.e., if $a_r + b_r$, $2a_r + b_r$, $3a_r + b_r$ and b_r are assumed nonzero.*

Proof Consider g restricted to $\mathrm{Fix}(\Delta_8)$. The equations are governed by

$$\begin{aligned} \dot{z}_1 &= [\nu + a(2|z_1|^2 + |z_2|^2) + b|z_1|^2]z_1, \\ \dot{z}_2 &= [\nu + a(2|z_1|^2 + |z_2|^2) + b|z_2|^2]z_2, \end{aligned} \tag{5.15}$$

for $(z_1, z_2) \in \mathbf{C}^2$. We have by the equivariant Hopf theorem periodic solutions with symmetry Σ_1, Σ_3 and Σ_6. We show now that these are the only periodic solutions. Let $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ in (5.15). Since these equations

Space Solution	Eigenvalues	Solution	Eigenvalues
(TW) $(z_1, z_2) = (z_1, 0)$	$2(A_r + B_r)\|z_1\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -2B_r\|z_1\|^2,$ $\mu_1\mu_2 = \|B\|^2\|z_1\|^4;$	(SW) $(z_1, z_2) = (z_1, z_1)$	$2(2A_r + B_r)\|z_1\|^2,\ 0;$ $2B_r\|z_1\|^2,\ 0;$
$\mathrm{Fix}(\Delta_1)$ $\Sigma_1\ (TR)$	$a_r + b_r$ $-b_r + c_r$	$\Sigma_2\ (SR)$	$2a_r + b_r + c_r$ $b_r - c_r$
$\mathrm{Fix}(\Delta_2)$ $\Sigma_3\ (TS)$	$2a_r + b_r$ $-b_r + c_r + d_r$	$\Sigma_4\ (SS)$	$4a_r + b_r + c_r + d_r$ $b_r - c_r - d_r$
$\mathrm{Fix}(\Delta_3)$ $\Sigma_3\ (TS)$	$2a_r + b_r$ $-b_r + c_r - d_r$	$\Sigma_5\ (AR)$	$4a_r + b_r + c_r - d_r$ $b_r - c_r + d_r$
$\mathrm{Fix}(\Delta_4)$ $\Sigma_1\ (TR)$	$a_r + b_r$ $-b_r$	$\Sigma_3\ (TS)$	$2a_r + b_r$ b_r
$\mathrm{Fix}(\Delta_6)$ $\Sigma_6\ (TC)$	$3a_r + b_r$ $-b_r + c_r + 2d_r$	$\Sigma_7\ (SC)$	$6a_r + b_r + c_r + 2d_r$ $b_r - c_r - 2d_r$
$\mathrm{Fix}(\Delta_7)$ $\Sigma_6\ (TC)$	$3a_r + b_r$ $-b_r + c_r - d_r$	$\Sigma_8\ (AC)$	$6a_r + b_r + c_r - d_r$ $b_r - c_r + d_r$

Table 5.12: Stability of the periodic solutions in $\mathbf{O}(2)$ Hopf bifurcation.

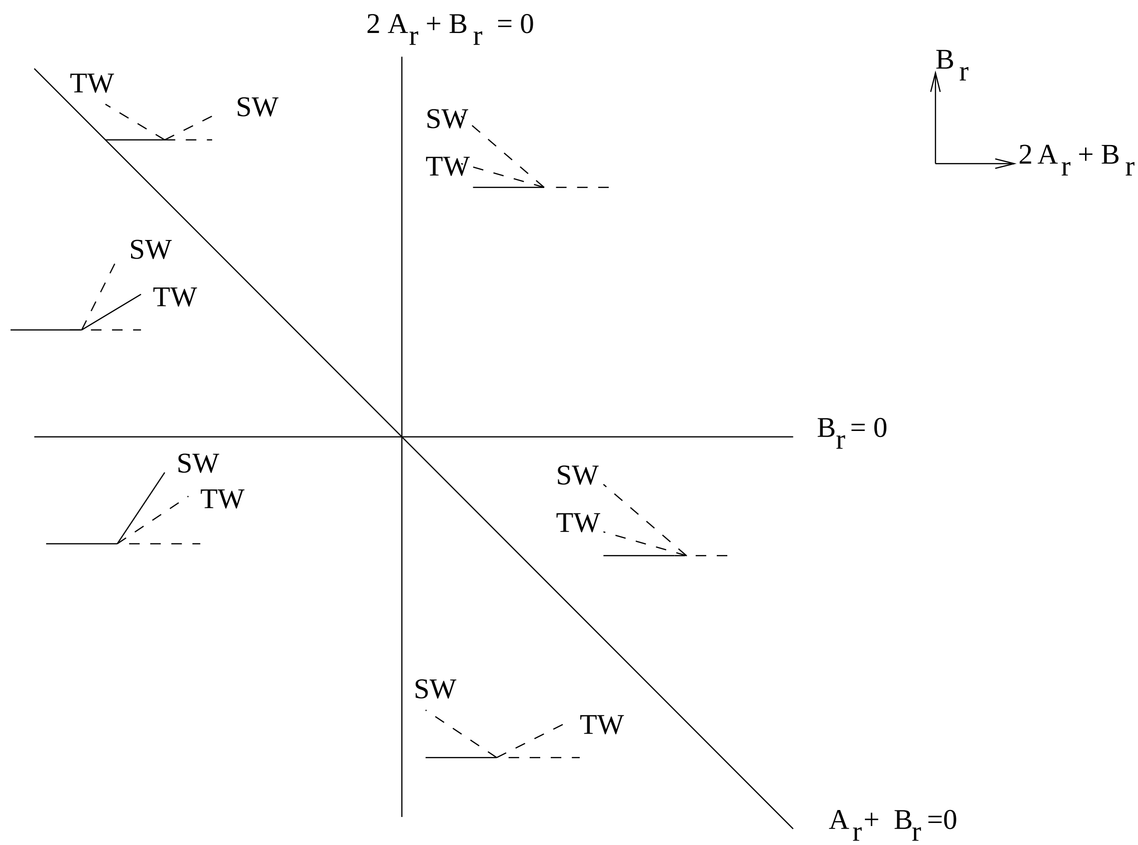


Figure 5.13: Bifurcation diagram for the nondegenerate Hopf bifurcation with $\mathbf{O}(2)$ symmetry. Broken bifurcation curves indicate unstable solutions and unbroken curves indicate stable solutions.

commute with $\mathbf{SO}(2) \times \mathbf{S}^1$, the (r_1, r_2)-equations decouple from the phases (ϕ_1, ϕ_2). We obtain the following equations:

$$\begin{aligned} \dot{r}_1 &= [\lambda + (2a_r + b_r)r_1^2 + a_r r_2^2]r_1, \\ \dot{r}_2 &= [\lambda + (a_r + b_r)r_2^2 + 2a_r r_1^2]r_2. \end{aligned} \tag{5.16}$$

Here we are taking $\mathrm{Re}(\nu) = \lambda$. Note that we are studying bifurcations near $\lambda = 0$, and so there is no loss of generality in this assumption.

The equilibria of this new system correspond to the zeros of (5.15) (and so to periodic solutions of the original problem). The periodic solutions with symmetry Σ_1 correspond to the fixed-points of (5.16) of the type $(r_1, r_2) = (0, r_2)$, the (TS) solutions to the fixed-points of the type $(r_1, r_2) = (r_1, 0)$

Solution	Eigenvalues	Solution	Eigenvalues
Σ_1 (TR) $(z_1, z_2) = (0, z_2)$	$2(a_r + b_r)\lvert z_2\rvert^2,\ 0;$ $-2b_r\lvert z_2\rvert^2,\ 0;$	Σ_3 (TS) $(z_1, z_2) = (z_1, 0)$	$2(2a_r + b_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -2b_r\lvert z_1\rvert^2,$ $\mu_1\mu_2 = \lvert b\rvert^2\lvert z_1\rvert^4;$
Σ_6 (TC) $(z_1, z_2) = (z_1, z_1)$	$2(3a_r + b_r)\lvert z_1\rvert^2,\ 0;$ $2b_r\lvert z_1\rvert^2, 0.$		

Table 5.13: Stability of the periodic solutions of $\dot{z} = f(z, \lambda)$ restricted to the space $\mathrm{Fix}(\Delta_8)$.

and the (TC) solutions to the zeros of the type $(r_1, r_2) = (r_1, r_1)$. If $b_r \neq 0$, then there are no fixed-points besides the corresponding to these maximal solutions. We show in table 5.13 the stability of these solutions in the space $\mathrm{Fix}(\Delta_8)$. □

$\mathrm{Fix}(\Delta_{10})$

Proposition 5.6.2 *Consider the equations for f restricted to* $\mathrm{Fix}(\Delta_{10})$. *Generically (see remark 5.6.3 for details), there are only branches of periodic solutions with symmetry Σ_1 and Σ_5. In table 5.14 we list the eigenvalues depending on the coefficients of f that determine (in the nondegenerate case) the stability of these solutions in the space* $\mathrm{Fix}(\Delta_{10})$, *i.e., if $4a_r + b_r + c_r - d_r,\ -b_r - c_r + d_r,\ b_r$ and $a_r + b_r$ are assumed nonzero.*

Proof Consider g restricted to $\mathrm{Fix}(\Delta_{10})$. The equations are governed by

$$\begin{aligned} \dot{z}_1 &= [\nu + a(4|z_1|^2 + |z_2|^2) + (b + c - d)|z_1|^2]z_1, \\ \dot{z}_2 &= [\nu + a(4|z_1|^2 + |z_2|^2) + b|z_2|^2]z_2, \end{aligned} \tag{5.17}$$

for $(z_1, z_2) \in \mathbf{C}^2$. We note that by the equivariant Hopf theorem there are periodic solutions with symmetry Σ_5 in the space $z_2 = 0$ and with conjugate symmetry to Σ_1 in the space $z_1 = 0$.

We wish to know now if there are any zeros of (5.17) with $z_1 z_2 \neq 0$. Let $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ in (5.17) and note that these equations commute with $\mathbf{SO}(2) \times \mathbf{S}^1$. Zeros of (5.17) with $z_1 z_2 \neq 0$ satisfy

$$\begin{aligned} \nu + (4a + b + c - d)r_1^2 + ar_2^2 = 0, \\ \nu + (a + b)r_2^2 + 4ar_1^2 = 0, \end{aligned} \tag{5.18}$$

with $r_1 r_2 \neq 0$, and so

$$(4a + b + c - d)r_1^2 + ar_2^2 = (a + b)r_2^2 + 4ar_1^2,$$

i.e.,

$$(b + c - d)r_1^2 = br_2^2. \tag{5.19}$$

If $r_1 = r_2$, then $(c - d)r_1^2 = 0$ and so, if $c - d \neq 0$, then there are no solutions for this case. If $r_1 \neq r_2$ (and $r_1 r_2 \neq 0$), then the real and imaginary parts of (5.19) give two equations for $(r_1/r_2)^2$, that is,

$$\begin{aligned} (b_r + c_r - d_r)r_1^2 = b_r r_2^2 \\ (b_I + c_I - d_I)r_1^2 = b_I r_2^2 \end{aligned}.$$

Thus, provided

$$(b_r + c_r - d_r)b_I \neq (b_I + c_I - d_I)b_r,$$

then there are no solutions in this case.

We show in table 5.14 the stability of the maximal solutions in the space $\mathrm{Fix}(\Delta_{10})$ depending on the coefficients of f (if assumed nonzero). □

Remark 5.6.3 *From the above proof, the nondegeneracy conditions (referred in the word "generically") in proposition 5.6.2 that guarantee that only branches of periodic solutions with maximal isotropy can bifurcate at $\lambda = 0$ for the equations for f restricted to* $\mathrm{Fix}(\Delta_{10})$ *are*

$$c - d \neq 0, \ (b_r + c_r - d_r)b_I \neq (b_I + c_I - d_I)b_r.$$

In the degenerate case when

$$(b_r + c_r - d_r)b_I = (b_I + c_I - d_I)b_r$$

and

$$b_r(b_r + c_r - d_r) > 0,$$

these solutions correspond to zeros of the radial equations obtained from (5.17):

$$\begin{aligned}\dot{r}_1 &= [\lambda + (4a_r + b_r + c_r - d_r)r_1^2 + a_r r_2^2]r_1,\\ \dot{r}_2 &= [\lambda + (a_r + b_r)r_2^2 + 4a_r r_1^2]r_2.\end{aligned} \tag{5.20}$$

Note that the periodic solutions with symmetry Σ_5 correspond to zeros of (5.20) of the type $r_2 = 0$. The periodic solutions with symmetry conjugate to Σ_1 correspond to the zeros of (5.20) of the type $r_1 = 0$.

The stability of the submaximal periodic solutions (for this degenerate case) in the space $\mathrm{Fix}(\Delta_{10})$, can be derived from the stability of the corresponding zeros of (5.20) which have type (r_1, r_2) with $r_1 r_2 \neq 0$. That is, the asymptotic stability of an equilibrium in the radial equations corresponds to the (orbitally) asymptotic stability of the corresponding solution in the original system when restricted to $\mathrm{Fix}(\Delta_{10})$.

The branch equation is

$$\lambda = -\left[\frac{(a_r + b_r)(b_r + c_r - d_r) + 4a_r b_r}{b_r + c_r - d_r}\right] r_2^2$$

and the eigenvalues are μ_1 and μ_2 such that

$$\mu_1 + \mu_2 = \frac{(4a_r + b_r + c_r - d_r)b_r + (a_r + b_r)(b_r + c_r - d_r)}{b_r + c_r - d_r} 2r_2^2$$

and

$$\mu_1\mu_2 = 4\left[(a_r + b_r)(b_r + c_r - d_r) + 4a_r b_r\right](r_1 r_2)^2.$$

Consider first the case when $b_r + c_r - d_r$ and b_r are both positive. If the branch bifurcates supercritically then the solutions are unstable. If it bifurcates subcritically then the solutions can be stable or unstable. When $b_r + c_r - d_r$ and b_r are both negative, then subcritical bifurcation is unstable and supercritical can be stable or unstable.

$\mathrm{Fix}(\Delta_9)$

Proposition 5.6.4 *Consider the equations for f restricted to* $\mathrm{Fix}(\Delta_9)$. *Then there are branches of periodic solutions with symmetry Σ_4, Σ_2 and Σ_7. We list in table 5.15 the eigenvalues depending on the coefficients of f that generically (see remark 5.6.5) determine the stability of these solutions in the space* $\mathrm{Fix}(\Delta_9)$.

Solution	Eigenvalues
Σ_5 (AR) $(z_1,z_2)=(z_1,0)$	$2(4a_r+b_r+c_r-d_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1+\mu_2=2(-b_r-c_r+d_r)\lvert z_1\rvert^2,$ $\mu_1\mu_2=\lvert -b-c+d\rvert^2\lvert z_1\rvert^4;$
Σ_1 (TR) $(z_1,z_2)=(0,z_2)$	$2(a_r+b_r)\lvert z_2\rvert^2,\ 0;$ $-2b_r\lvert z_2\rvert^2,\ 0;$

Table 5.14: Stability of the periodic solutions of $\dot{z}=f(z,\lambda)$ restricted to $\mathrm{Fix}(\Delta_{10})$.

Generically, there can be branches of periodic solutions with submaximal isotropy. These correspond to zeros of g *restricted to* $\mathrm{Fix}(\Delta_9)$ *of the type* (z_1,z_2) *with* $|z_1|\neq|z_2|$ *and* $z_1z_2\neq 0$. *The number of the branches depends on the real zeros in* $]0,\pi[$ *of the following equation in* $\phi=arg(z_2\overline{z}_1)$*:*

$$Im(d\overline{f})(1-\cos(2\phi))+3Re(d\overline{f})\sin(2\phi)+|d|^2\sin(2\phi)(1-4\cos(2\phi))=0 \quad (5.21)$$

with $f=b+c$*, that satisfy the conditions*

$$\begin{array}{l} f_r+d_r-2Re(\overline{d}e^{i2\phi})\neq 0,\\ f_I+d_I+2Im(\overline{d}e^{i2\phi})\neq 0\end{array}$$

and

$$\frac{f_r-Re(de^{i2\phi})}{f_r+d_r-2Re(\overline{d}e^{i2\phi})}>0,\neq 1,$$

for an open set of the parameter space (b,c,d)*.*

Proof The equations $g\mid_{\mathrm{Fix}(\Delta_9)}=0$ are:

$$\begin{array}{l}[\nu+2a(2|z_1|^2+|z_2|^2)+(b+c+d)|z_1|^2]\,z_1+d\overline{z}_1z_2^2=0,\\ [\nu+2a(2|z_1|^2+|z_2|^2)+(b+c)|z_2|^2]\,z_2+2d\overline{z}_2z_1^2=0,\end{array} \quad (5.22)$$

with $(z_1,z_2)\in\mathbf{C}^2$. The existence of periodic solutions with symmetry Σ_2, Σ_4 and Σ_7 is guaranteed by the equivariant Hopf theorem. These solutions correspond to the zeros of this system of the type $z_1=0$ (and $z_2\neq 0$), $z_2=0$

Solution	Eigenvalues
Σ_2 (SR) $(z_1, z_2) = (0, z_2)$	$2(2a_r + b_r + c_r)\|z_2\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -2(b_r + c_r)\|z_2\|^2,$ $\mu_1\mu_2 = (\|b + c\|^2 - \|d\|^2)\|z_2\|^4;$
Σ_4 (SS) $(z_1, z_2) = (z_1, 0)$	$2(4a_r + b_r + c_r + d_r)\|z_1\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -(b_r + c_r + d_r)\|z_1\|^2,$ $\mu_1\mu_2 = [\|b + c\|^2 - 3\|d\|^2 + 2\text{Re}((b + c)\overline{d})]\|z_1\|^4;$
Σ_7 (SC) $(z_1, z_2) = (z_1, z_1)$	$2(6a_r + b_r + c_r + 2d_r)\|z_1\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = 2(b_r + c_r - 4d_r)\|z_1\|^2,$ $\mu_1\mu_2 = 12[\|d\|^2 - \text{Re}((b + c)\overline{d})]\|z_1\|^4;$

Table 5.15: Stability of the periodic solutions of $\dot{z} = f(z, \lambda)$ restricted to $\text{Fix}(\Delta_9)$.

(and $z_1 \neq 0$) and $z_1 = z_2 \neq 0$ respectively. The stability of these solutions in the space $\mathrm{Fix}(\Delta_9)$ is given in table 5.15.

In order to find submaximal solutions for the equations for f restricted to $\mathrm{Fix}(\Delta_9)$, we have to look for zeros (z_1, z_2) of the equations (5.22) such that $z_1 \neq z_2$ and $z_1 z_2 \neq 0$. Let $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ in (5.22) and denote by $\phi = \phi_2 - \phi_1 = \arg(z_2 \overline{z_1})$. Zeros of (5.22) (with $z_1 z_2 \neq 0$) satisfy

$$\begin{aligned} &\nu + (4a + b + c + d) r_1^2 + 2a r_2^2 + d e^{i2\phi} r_2^2 = 0, \\ &\nu + (2a + b + c) r_2^2 + 4a r_1^2 + 2d e^{-i2\phi} r_1^2 = 0 \end{aligned}$$

and so, if we denote by $f = b + c$, we get

$$f(r_1^2 - r_2^2) + d r_1^2 + d(e^{i2\phi} r_2^2 - 2e^{-i2\phi} r_1^2) = 0. \tag{5.23}$$

The real and imaginary parts of (5.23) correspond to the equations

$$\begin{aligned} r_1^2(f_r + d_r - 2d_r \cos(2\phi) - 2d_I \sin(2\phi)) + r_2^2(-f_r + d_r \cos(2\phi) - d_I \sin(2\phi)) = 0, \\ r_1^2(f_I + d_I - 2d_I \cos(2\phi) + 2d_r \sin(2\phi)) + r_2^2(-f_I + d_I \cos(2\phi) + d_r \sin(2\phi)) = 0. \end{aligned}$$

These equations may be solved for $(r_1/r_2)^2$ obtaining

$$\left(\frac{r_1}{r_2}\right)^2 = \frac{f_r - d_r \cos(2\phi) + d_I \sin(2\phi)}{f_r + d_r - 2d_r \cos(2\phi) - 2d_I \sin(2\phi)}, \tag{5.24}$$

provided

$$f_r + d_r - 2\mathrm{Re}(\overline{d} e^{i2\phi}) \neq 0$$

and one equation for $\cos(2\phi)$:

$$\mathrm{Im}(d\overline{f})(1 - \cos(2\phi)) + 3\mathrm{Re}(d\overline{f}) \sin(2\phi) + |d|^2 \sin(2\phi)(1 - 4\cos(2\phi)) = 0, \tag{5.25}$$

provided

$$f_I + d_I + 2\mathrm{Im}(\overline{d} e^{i2\phi}) \neq 0.$$

Note that the expression for $(r_1/r_2)^2$ in (5.24) has to be positive and different from 1. This last because if $r_1^2 = r_2^2$ in (5.23), then ϕ satisfies $\cos(2\phi) = 1$ and $\sin(2\phi) = 0$ and we obtain a maximal solution. □

Remark 5.6.5 *The nondegeneracy conditions (referred to in the word "generically") that determine the stability of the solutions with maximal isotropy for the equations for f restricted to* $\mathrm{Fix}(\Delta_9)$ *(proposition 5.6.4) are:*

$$2a_r + b_r + c_r,\ 4a_r + b_r + c_r + d_r,\ 6a_r + b_r + c_r + 2d_r \neq 0,$$

$$D_1 = |b+c|^2 - |d|^2 \neq 0, \quad D_2 = [|b+c|^2 - 3|d|^2 + 2Re((b+c)\overline{d})] \neq 0,$$

$$D_3 = |d|^2 - Re((b+c)\overline{d})] \neq 0$$

and

$$b_r + c_r \neq 0 \text{ if } D_1 > 0,$$

$$b_r + c_r + d_r \neq 0 \text{ if } D_2 > 0,$$

$$b_r + c_r - 4d_r \neq 0 \text{ if } D_3 > 0.$$

We can find regions of the parameter space (b, c, d) where there can coexist two distinct branches of submaximal solutions, one branch and no branch of submaximal solutions for the equations for f restricted to $\mathrm{Fix}(\Delta_9)$. For example, fixing $b = c = 0.5 + i,\ d_r = 1$ and varying d_I we can find these three cases. The author did not find cases where there were three branches of periodic solutions with submaximal isotropy (in the space $\mathrm{Fix}(\Delta_9)$). Note that solving (5.25) for $\sin(2\phi)$ and substituting in $\sin^2(2\phi) + \cos^2(2\phi) = 1$, we get an equation in ϕ of the type $\cos^3(2\phi) + A\cos^2(2\phi) + B\cos(2\phi) + C = 0$ with real coefficients $A,\ B$ and C.

Remark 5.6.6 *With the conditions of proposition 5.6.4, for parameter values for which there are branches of periodic solutions for f restricted to* $\mathrm{Fix}(\Delta_9)$ *with submaximal isotropy, the branching equation is*

$$\nu = -(4a + b + c + d)r_1^2 - 2ar_2^2 - de^{i2\phi}r_2^2,$$

where $r_1 = |z_1|$ and $r_2 = |z_2|$ (and $z = (z_1, z_1, z_1, z_1, z_2, z_2)$ in $\mathrm{Fix}(\Delta_9)$*) satisfy*

$$\left(\frac{r_1}{r_2}\right)^2 = \frac{f_r - d_r\cos(2\phi) + d_I\sin(2\phi)}{f_r + d_r - 2d_r\cos(2\phi) - 2d_I\sin(2\phi)},$$

for $\phi = arg(z_2\overline{z}_1)$ solution of the equation

$$Im(d\overline{f})(1 - \cos(2\phi)) + 3Re(d\overline{f})\sin(2\phi) + |d|^2\sin(2\phi)(1 - 4\cos(2\phi)) = 0.$$

$\mathrm{Fix}(\Pi_3)$

Proposition 5.6.7 *Consider the equations for f restricted to the space* $\mathrm{Fix}(\Pi_3)$*. Then generically (specifically, if $b \neq 0$), only branches of periodic solutions with symmetry (conjugate to) Σ_1, Σ_3 and Σ_6 can bifurcate at the origin.*

Proof If we take coordinates $(z_1, 0, z_2, 0, z_3, 0) = (r_1 e^{i\phi_1}, 0, r_2 e^{i\phi_2}, 0, r_3 e^{i\phi_3}, 0)$ in $\mathrm{Fix}(\Pi_3)$, then from $g\,|_{\mathrm{Fix}(\Pi_3)} = 0$, we get

$$\begin{array}{l} (\nu + ar^2 + br_1^2) r_1 = 0 \\ (\nu + ar^2 + br_2^2) r_2 = 0 \\ (\nu + ar^2 + br_3^2) r_3 = 0, \end{array} \tag{5.26}$$

where $r^2 = r_1^2 + r_2^2 + r_3^2$. Generically, i.e., if $b \neq 0$, then the only nontrivial zeros of (5.26) are the corresponding to the zeros with symmetry Σ_1, Σ_3 and Σ_6. □

$\mathrm{Fix}(\Pi_6)$

Proposition 5.6.8 *Consider the equations for f restricted to the space* $\mathrm{Fix}(\Pi_6)$*. Then generically (see remark 5.6.9 for details), only branches of periodic solutions with symmetry (conjugate to) Σ_1, Σ_3, Σ_5 and Σ_6 can bifurcate at the origin.*

Proof Take $(z_1, z_2, iz_1, iz_2, z_3, 0) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, ir_1 e^{i\phi_1}, ir_2 e^{i\phi_2}, r_3 e^{i\phi_3}, 0)$ in $\mathrm{Fix}(\Pi_6)$. From $g\,|_{\mathrm{Fix}(\Pi_6)} = 0$, we obtain

$$\begin{array}{l} [\nu + ar^2 + br_1^2 + (c-d) r_2^2]\, r_1 = 0 \\ [\nu + ar^2 + br_2^2 + (c-d) r_1^2]\, r_2 = 0 \\ [\nu + ar^2 + br_3^2]\, r_3 = 0, \end{array} \tag{5.27}$$

where $r^2 = 2r_1^2 + 2r_2^2 + r_3^2$. If some r_i is zero, then we obtain zeros of $g\,|_{\mathrm{Fix}(\Pi_6)} = 0$ corresponding to periodic solutions with symmetry conjugate to Σ_1, Σ_3, Σ_5 and Σ_6. If $r_i \neq 0$, for $i = 1, 2, 3$, then from (5.27) we get

$$br_1^2 + (c-d) r_2^2 = br_2^2 + (c-d) r_1^2 = br_3^2.$$

From the first equality, we obtain

$$(b - c + d)(r_1^2 - r_2^2) = 0$$

and so, generically, assuming $b - c + d \neq 0$, we have that $r_1^2 = r_2^2$. From the second equality, we get

$$(b + c - d)r_1^2 = br_3^2$$

and so, two equations (real and imaginary) for $(r_1/r_3)^2$. □

Remark 5.6.9 *From the above proof, the nondegeneracy conditions in proposition 5.6.8 are*

$$b - c + d \neq 0,\ (b_r + c_r - d_r)b_I \neq (b_I + c_I - d_I)b_r.$$

$\mathrm{Fix}(\Lambda_1)$

When we restrict f to the space $\mathrm{Fix}(\Lambda_1)$, the restricted ODEs are equivariant under the group $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$ and we can apply the work done by Silber and Knobloch in [34]. We summarize their results in the next section. The main point at this stage is that generically, the only branches of periodic solutions that can bifurcate at the origin correspond to solutions with symmetry (conjugate to) $\Sigma_1, \ldots, \Sigma_5$ or with submaximal isotropy obtained by Swift [37] (in $\mathrm{Fix}(\Delta_5)$).

$\mathrm{Fix}(\Lambda_3)$

Theorem 5.6.10 *Consider the equations for f restricted to the space* $\mathrm{Fix}(\Lambda_3)$. *Then there are branches of periodic solutions with symmetry Σ_1, Σ_2, Σ_3, Σ_4, Σ_6 and Σ_7.*

Generically (see remark 5.6.11 for details), there can be branches of periodic solutions with submaximal symmetry and these correspond to the ones obtained in proposition 5.6.4.

Proof The equations $g\ |_{\mathrm{Fix}(\Lambda_3)} = 0$ are:

$$\begin{array}{l} [\nu + a|z|^2 + (a+b)|z_1|^2 + (a+c+d)|z_2|^2]\, z_1 + d\overline{z}_2 z_3 z_4 = 0 \\ [\nu + a|z|^2 + (a+b)|z_2|^2 + (a+c+d)|z_1|^2]\, z_2 + d\overline{z}_1 z_3 z_4 = 0 \\ [\nu + 2a|z|^2 + (-a+b)|z_3|^2 + (-a+c)|z_4|^2]\, z_3 + 2d\overline{z}_4 z_1 z_2 = 0 \\ [\nu + 2a|z|^2 + (-a+b)|z_4|^2 + (-a+c)|z_3|^2]\, z_4 + 2d\overline{z}_3 z_1 z_2 = 0, \end{array} \tag{5.28}$$

where $|z|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2$. The only cases, up to conjugacy, that we need to consider are:

$$\begin{array}{l}
(1)\ z_1 \neq 0\ \wedge\ z_2 = z_3 = z_4 = 0 \\
(2)\ z_3 \neq 0\ \wedge\ z_1 = z_2 = z_4 = 0 \\
(3)\ z_1 \neq 0\ \wedge z_3 \neq 0\ \wedge z_2 = z_4 = 0 \\
(4)\ z_1 \neq 0\ \wedge z_2 \neq 0\ \wedge\ z_3 = z_4 = 0 \\
(5)\ z_3 \neq 0\ \wedge z_4 \neq 0\ \wedge\ z_1 = z_2 = 0 \\
(6)\ z_1 z_2 \overline{z}_3 \overline{z}_4 \neq 0.
\end{array}$$

For the cases (1)-(5) we obtain the solutions with symmetry Σ_i for $i = 1, 2, 3, 4, 6$ (apply proposition 5.6.1 and recall that f restricted to $\mathrm{Fix}(\Delta_2)$ and $\mathrm{Fix}(\Delta_1)$ is $\mathbf{O}(2)$-symmetric). Consider now the case (6). Zeros of (5.28) satisfy

$$G = \begin{pmatrix} [\nu + ar^2 + (a+b)r_1^2 + (a+c+d)r_2^2]\, r_1 + dr_2 r_3 r_4 e^{i2\psi} \\ [\nu + ar^2 + (a+b)r_2^2 + (a+c+d)r_1^2]\, r_2 + dr_1 r_3 r_4 e^{i2\psi} \\ [\nu + 2ar^2 + (-a+b)r_3^2 + (-a+c)r_4^2]\, r_3 + 2dr_1 r_2 r_4 e^{-i2\psi} \\ [\nu + 2ar^2 + (-a+b)r_4^2 + (-a+c)r_3^2]\, r_4 + 2dr_1 r_2 r_3 e^{-i2\psi} \end{pmatrix} = 0 \quad (5.29)$$

where $r_i = |z_i|$ for $i = 1, \ldots, 4$ and $r^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2$. Also, $2\psi = \arg(\overline{z}_1 \overline{z}_2 z_3 z_4)$. Note that in this case $r_i \neq 0$ for $i = 1, \ldots, 4$. As

$$\begin{aligned}
r_1 G_1 - r_2 G_2 &= [\nu + ar^2 + (a+b)(r_1^2 + r_2^2)](r_1^2 - r_2^2) = 0 \\
r_3 G_3 - r_4 G_4 &= [\nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2)](r_3^2 - r_4^2) = 0,
\end{aligned}$$

then case (6) is subdivided into four cases:

$$\begin{array}{l}
(6.a)\ r_1^2 = r_2^2\ \wedge\ r_3^2 = r_4^2 \\
(6.b)\ r_1^2 = r_2^2\ \wedge\ \nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0 \\
(6.c)\ r_3^2 = r_4^2\ \wedge\ \nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0 \\
(6.d)\ \nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0\ \wedge\ \nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0.
\end{array}$$

Consider first the case $(6.a)$. If $r_1 = r_2 = r_3 = r_4$, then from (5.29) we get that $\cos(2\psi) = 1$ and $\sin(2\psi) = 0$ and we have the solution with symmetry Σ_7. Now, if $r_1 = r_2 \neq r_3 = r_4$ in (5.29), we get

$$\begin{aligned}
&\nu + (4a+b+c+d)r_1^2 + 2ar_3^2 + dr_3^2 e^{i2\psi} = 0 \\
&\nu + (2a+b+c)r_3^2 + 4ar_1^2 + 2dr_1^2 e^{-i2\psi} = 0
\end{aligned}$$

and so we have the submaximal solutions obtained in proposition 5.6.4. Note that, an element in $\mathrm{Fix}(\Lambda_3)$ such that $r_1 = r_2 \neq r_3 = r_4$ and $r_1 r_3 \neq 0$ is conjugate to an element in $\mathrm{Fix}(\Delta_9)$. That is, the solutions obtained from the above equations have symmetry (conjugate to) Δ_9.

For the case $(6.b)$: we can use $\nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0$ in $G_3 = 0$ of (5.29) and we get

$$r_3 r_4 e^{i2\psi} = \frac{2d}{b-c} r_1 r_2 \tag{5.30}$$

assuming $b - c \neq 0$. We use the conditions $\nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0$ and $r_1 = r_2$ in $G_1 = 0$ of (5.29), and substituting $r_3 r_4 e^{i2\psi}$ as in (5.30), we obtain

$$\left(b + c + d + \frac{2d^2}{b-c}\right) r_1^2 = b(r_3^2 + r_4^2). \tag{5.31}$$

The real and imaginary parts of this equation give two equations for the ratio $r_1^2/(r_3^2 + r_4^2)$. Thus generically, there are no solutions in case $(6.b)$.

The case $(6.c)$ is similar. We can use $\nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0$ in $G_1 = 0$ of (5.29) and we get

$$r_1 r_2 e^{-i2\psi} = \frac{d}{b-c-d} r_3 r_4, \tag{5.32}$$

assuming $b - c - d \neq 0$. Again, in $G_3 = 0$ of (5.29) we use $r_3 = r_4$ and $\nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0$. Also substituting $r_1 r_2 e^{-i2\psi}$ as in (5.32), we obtain

$$\left(b + c + \frac{2d^2}{b-c-d}\right) r_3^2 = b(r_1^2 + r_2^2). \tag{5.33}$$

The real and imaginary parts of this equation give two equations for the ratio $r_3^2/(r_1^2 + r_2^2)$ and so, generically, there are no solutions in case $(6.c)$.

Finally, case $(6.d)$. Using the condition $\nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0$ in $G_1 = 0$ of (5.29), we get

$$(-b + c + d) r_1 r_2 + d r_3 r_4 e^{i2\psi} = 0, \tag{5.34}$$

and in $G_3 = 0$ the condition $\nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0$, we obtain

$$2d r_1 r_2 + (c - b) r_3 r_4 e^{i2\psi} = 0. \tag{5.35}$$

Since, we are supposing $r_1 r_2 r_3 r_4 \neq 0$, the conditions (5.34) and (5.35) imply that

$$(b - c - d)(b - c) = 2d^2. \tag{5.36}$$

Thus, generically there are no solutions in case $(6.d)$. □

Remark 5.6.11 *From the above proof, the precise nondegeneracy conditions in theorem 5.6.10 required are*

$$b \neq 0,$$

$$b - c \neq 0, \quad Re\left(b + c + d + \tfrac{2d^2}{b-c}\right) b_I \neq Im\left(b + c + d + \tfrac{2d^2}{b-c}\right) b_r$$

$$b - c - d \neq 0, \quad Re\left(b + c + \tfrac{2d^2}{b-c-d}\right) b_I \neq Im\left(b + c + \tfrac{2d^2}{b-c-d}\right) b_r$$

$$(b - c - d)(b - c) \neq 2d^2.$$

In figure 5.14 we list some interesting bifurcation diagrams. We include only the branches of solutions with maximal symmetry. Observe that it is possible to have two branches of stable solutions with two subcritical branches. Also, for some parameter values, all the branches can bifurcate supercritically, either being all unstable or having just one or two stable branches. We note that the solutions with symmetry Σ_4, Σ_5 can not be both stable: from $b_r + c_r + 3d_r,\ -b_r - c_r + d_r < 0$ we get $d_r < 0$ and from $b_r + c_r - 3d_r,\ b_r + c_r + 3d_r < 0$ we have $b_r + c_r < 0$ and so $-b_r - c_r - d_r > 0$. This contrasts with the D_4-Hopf bifurcation problem. See figure 5.12 where (SS) and (AR) stand for the solutions with symmetry Σ_4 and Σ_5 respectively. Similarly, it can not happen that the first branch bifurcates subcritically with the second being stable, again as it happens for some parameter values in D_4-Hopf problem.

The solutions with symmetry Σ_3 are always unstable. Also the solutions with symmetry Σ_6 and Σ_8 cannot both be stable. The same happens for the solutions with symmetry Σ_1 and Σ_6.

5.7 The square lattice problem

The square lattice problem studied in [34] is reduced to the study of the dynamics of a vector field commuting with $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ on V^2. We point out here that this group is the wreath product group $\mathbf{O}(2) \wr \mathbf{S}_2$. Our problem reduced to fixed-point subspace $\mathrm{Fix}(\Lambda_1)$ is equivariant by this group.

The correspondence between both notations is the following: the $\mathbf{C}^4$ coordinates are $(v_1, w_1, v_2, w_2) = (z_1, z_2, z_3, z_4)$; for the group $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ they denote $\mathbf{T}^2 = \mathbf{SO}(2)^2$ and the elements for $\mathbf{D}_4$ are $\sigma_V = (1, \kappa, 1) \in \mathbf{O}(2)^3$ and

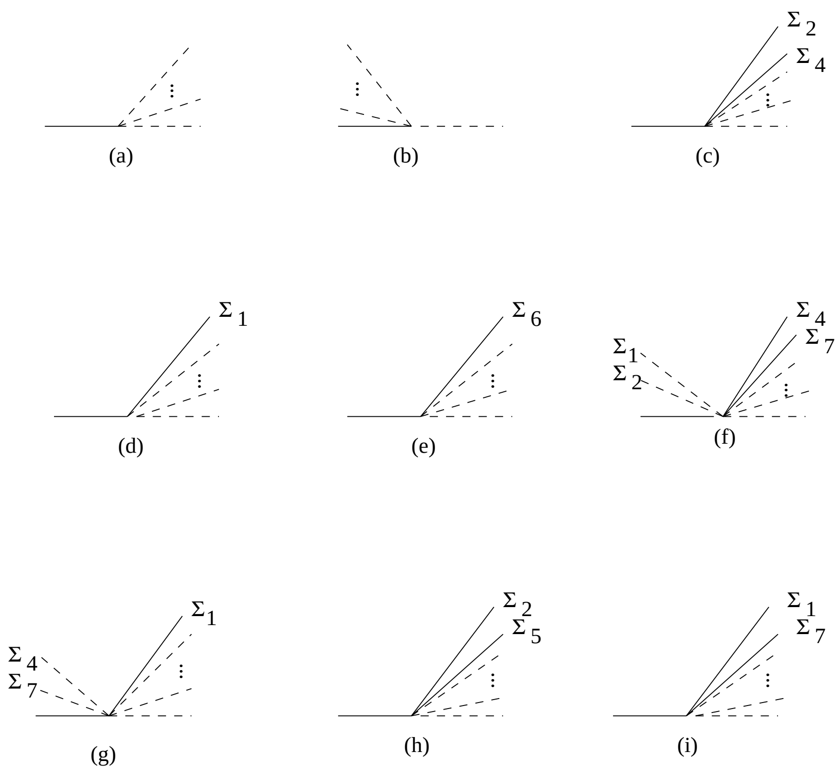


Figure 5.14: Possible bifurcation diagrams for the nondegenerate Hopf bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_3$ symmetry. Broken (unbroken) bifurcation curves indicate unstable (stable) solutions.

$r_{\pi/2} \circ \sigma_V = (12) \in \mathbf{S}_2 \subset \mathbf{S}_3$. Consider $\dot{z} = f \mid_{\mathrm{Fix}(\Lambda_1)}$ (where we take the third order truncation of f as in (5.8)), i.e.,

$$\begin{array}{l}
\dot{z}_1 = [\mu + a(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) + b|z_1|^2 + c|z_2|^2]z_1 + d\overline{z}_2 z_3 z_4 \\
\dot{z}_2 = [\mu + a(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) + b|z_2|^2 + c|z_1|^2]z_2 + d\overline{z}_1 z_3 z_4 \\
\dot{z}_3 = [\mu + a(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) + b|z_3|^2 + c|z_4|^2]z_3 + d\overline{z}_4 z_1 z_2 \\
\dot{z}_4 = [\mu + a(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) + b|z_4|^2 + c|z_3|^2]z_4 + d\overline{z}_3 z_1 z_2.
\end{array} \tag{5.37}$$

The group $\Lambda_1 = \Sigma(\mathbb{Z}_4^c\{1\}) \cap \Sigma(\mathbb{Z}_4^c\{2\})$ is generated by $\mathbf{1}^2 \times \mathbf{O}(2)$. The normalizer $N(\Lambda_1)$ maps $\mathrm{Fix}(\Lambda_1)$ to itself and $f \mid_{\mathrm{Fix}(\Lambda_1)}$ is $N(\Lambda_1)$-equivariant where only the elements of $N(\Lambda_1)/\Lambda_1$ act non-trivially on $\mathrm{Fix}(\Lambda_1)$. As

$$N(\Lambda_1)/\Lambda_1 \cong (\mathbf{O}(2) \wr \mathbf{S}_2) \times \mathbf{S}^1 = (\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1,$$

we obtain for $f \mid_{\mathrm{Fix}(\Lambda_1)}$ the same form as the vector field studied in [34].

In [34], Silber and Knobloch study extended systems invariant under the Euclidean group $\mathbf{E}(2)$, when a spatially uniform quiescent state loses stability to waves of wave number $k \neq 0$ and frequency $\omega \neq 0$. They restrict the space of solutions to those that are periodic on a square lattice (thus periodic boundary conditions are imposed in two orthogonal horizontal directions), and then they observe that the centre manifold theorem allows a reduction of the infinite-dimensional problem to a bifurcation problem on $\mathbf{C}^4$ with symmetry $\mathbf{D}_4 \dot{+} \mathbf{T}^2$. The relation between the isotropy lattice of this group and $\Gamma \times \mathbf{S}^1$ is the following: the maximal isotropy subgroups are related to Σ_i for $i = 1, \ldots, 5$. The groups with four-dimensional fixed-point subspace have correspondence to Δ_i for $i = 1, \ldots, 5$ and the group with six-dimensional fixed-point subspace corresponds to Π_1.

In [34] it is proved that there are five branches of periodic solutions with maximal isotropy that can bifurcate from the origin, considering a generic vector field commuting with $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$ and truncated at the third order. These solutions correspond to the solutions for our problem with symmetry $\Sigma_1, \ldots, \Sigma_5$. We observe that the stability for these solutions in the space $\mathrm{Fix}(\Lambda_1)$ is determined by the derivative (dg) (where g is the reduced vector field) calculated at the corresponding zero of g and restricted to the fixed-point subspace of Λ_1. This was done in section 5.5 and table 5.16 contains the conditions on the coefficients of f that determine the stability (this table is contained in the table 5.8 that lists the stability conditions for the all space V^3). These conditions agree with the results of [34].

Symmetry of the solution	Eigenvalues
Σ_1 (TR)	$a_r + b_r$ $-b_r + c_r$ $-b_r$
Σ_2 (SR)	$2a_r + b_r + c_r$ $b_r - c_r$ $-b_r - c_r$ $-(\|b+c\|^2 - \|d\|^2)$
Σ_3 (TS)	$2a_r + b_r$ b_r $-b_r + c_r + d_r$ $-b_r + c_r - d_r$
Σ_4 (SS)	$4a_r + b_r + c_r + d_r$ $b_r + c_r - 3d_r$ $-[\|d\|^2 - \mathrm{Re}((b+c)\overline{d})]$ $b_r - c_r - d_r$
Σ_5 (AR)	$4a_r + b_r + c_r - d_r$ $b_r + c_r + 3d_r$ $-[\|d\|^2 + \mathrm{Re}((b+c)\overline{d})]$ $b_r - c_r + d_r$

Table 5.16: Stability of the periodic solutions of the square lattice problem.

We note that the stability of the solutions with symmetry Σ_1 and Σ_2 is the same in the space $\mathrm{Fix}(\Lambda_1)$ as for the all space V^3. Note that $\mathbf{S}_1 \times \mathbf{S}_2$ is contained in the groups Σ_1 and Σ_2: this justifies the repetition in the case of the derivative for the (TR) solution of the matrix

$$\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$$

and in the case of the (SR) solution of the matrix

$$\begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

obtained in the proof of proposition 5.5.2. However, the solutions with symmetry Σ_3 in $\mathrm{Fix}(\Lambda_1)$ can be stable, but in the full space the branch is generically unstable: there are eigenvalues in the transverse direction to $\mathrm{Fix}(\Lambda_1)$ wit real parts with sign determined by $-b_r$.

In $\mathrm{Fix}(\Lambda_1)$ the stability for (AR) can be inferred from the stability for (SS) letting $d \mapsto -d$ because of the parameter symmetry [37], that is, the map $g\mid_{\mathrm{Fix}(\Lambda_1)}$ is equivariant under the transformation

$$(z_1, z_2, z_3, z_4; d) \mapsto (z_1, z_2, iz_3, iz_4; -d).$$

However, in the all space, the stability for the (SS) solution is also determined by the eigenvalues such that the sum has sign determined by $-b_r - c_r - d_r$ and product is determined by $|b+c|^2 - 3|d|^2 + 2\mathrm{Re}((b+c)\overline{d})$. For the (AR) solution, similar case happens but now the extra eigenvalues are complex conjugate (or equal and real) with real part determined $-b_r - c_r + d_r$.

In [34], Silber and Knobloch also show that an unstable branch of periodic solutions with submaximal symmetry can generically bifurcate from the origin. This solution branch corresponds to the one obtained in the restricted problem to the space $\mathrm{Fix}(\Delta_5)$ which has $\mathbf{D}_4$ symmetry and that it was studied in [37].

All the possible bifurcation diagrams (when considering the expressions in table 5.16 nonzero) are described in [34]. They show that it is possible to have the five branches to bifurcate supercritically with two of them being stable. The Σ_4 solution can be stable even with two supercritical branches. For some parameter region all five branches bifurcate supercritically with none being stable. They also explore the possibility of a primary bifurcation to a structurally stable heteroclinic cycle. We return to this matter in the next section.

5.8 Heteroclinic cycles

In addition to the periodic solutions described in the previous sections, there can exist quasiperiodic and heteroclinic cycle solutions to

$$\dot{z} = f(z, \lambda). \tag{5.38}$$

These solutions exist in open regions of the coefficient space (a, b, c, d) and bifurcate directly from the trivial solution.

The existence of quasiperiodic solutions follows from the fact that the vector field f restricted to the subspace $\mathrm{Fix}(\Delta_5)$ has $\mathbf{D}_4 \times \mathbf{S}^1$ symmetry. Swift [37] proves that there exists a branch of quasiperiodic solutions to this bifurcation problem.

We explore now the possibility of a heteroclinic cycle involving the periodic solutions (TR), (SR) and (AR), i.e., the heteroclinic cycle would consist in three periodic solutions together with the heteroclinic orbits connecting them. We exhibit the relevant part of the isotropy lattice of $\Gamma \times \mathbf{S}^1$:

$$\begin{array}{ccccccc}
\Sigma_1 & & \Sigma_2 & & \Sigma_5 & & \Sigma_1 \\
& \nwarrow \nearrow & & \nwarrow \nearrow & & \nwarrow \nearrow & \\
& \Delta_1 & & \Delta_5 & & \Delta_{10} &
\end{array}$$

Thus we are interested in the cycle $(TR) \to (SR) \to (AR) \to (TR)$. We need to show the existence of heteroclinic orbits $(TR) \to (SR)$ (in $\mathrm{Fix}(\Delta_1)$), $(SR) \to (AR)$ (in $\mathrm{Fix}(\Delta_5)$) and $(AR) \to (TR)$ (in $\mathrm{Fix}(\Delta_{10})$). We begin with the heteroclinic connections $(TR) \to (SR)$ and $(AR) \to (TR)$.

Consider the dynamics of g restricted to $\mathrm{Fix}(\Delta_1)$. As we saw in the previous section, we obtain the normal form for a generic Hopf bifurcation problem with symmetry $\mathbf{O}(2) \times \mathbf{S}^1$ (5.14), where $A = a + c$ and $B = b - c$, and if we use the coordinates $(z_1, z_2, z_3, z_4, z_5, z_6) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, 0, 0, 0, 0)$ in $\mathrm{Fix}(\Delta_1)$, then the (r_1, r_2)-equations decouple from the phases (ϕ_1, ϕ_2) and are given by

$$\begin{aligned}
\dot{r}_1 &= [\lambda + (a_r + b_r) r_1^2 + (a_r + c_r) r_2^2] r_1 \\
\dot{r}_2 &= [\lambda + (a_r + b_r) r_2^2 + (a_r + c_r) r_1^2] r_2.
\end{aligned} \tag{5.39}$$

The (TR) solutions correspond to zeros of (5.39) of the type $\xi_1 = (r_1, 0)$ (where $r_1 \neq 0$) and the (SR) solutions to zeros of the type $\xi_2 = (r_1, r_1)$ with $r_1 \neq 0$. See table 5.12 for stability. Suppose that both of the branches of these solutions bifurcate supercritically, i.e.,

$$a_r + b_r < 0 \ \wedge \ 2a_r + b_r + c_r < 0.$$

Using [20] lemma XVII 4.1, a zero of the amplitude equations (5.39) is asymptotically stable if and only if the corresponding solution is (orbitally) asymptotically stable. In this case the sign of the nonradial eigenvalue for the zeros corresponding to (TR) and (SR) is determined by $c_r - b_r$: if we assume

$$b_r - c_r < 0,$$

then a zero corresponding to a (TR) solution is a saddle and the one corresponding to (SR) is a sink.

That there is a saddle-sink connection $(TR) \to (SR)$ follows easily: the equations (5.39) admit no equilibria other than those corresponding to (TR) and (SR) solutions in the region $r_1 > 0,\ r_2 \geq 0$; the unstable manifold $W^u(\xi_1)$ remains within $O(\sqrt{\lambda})$ from the origin, by [23] or [27] proposition 2.6; the Poincaré-Bendixson theorem (see for example [22]) implies the existence of the heteroclinic connection $\xi_1 \to \xi_2$.

The saddle-sink connection $(AR) \to (TR)$ is proved in a similar way: consider the dynamics of g restricted to $\mathrm{Fix}(\Delta_{10})$. As we saw in the previous section we obtain the equations (5.17) and considering the coordinates $(z_1, z_2, z_3, z_4, z_5, z_6) = (r_1 e^{i\phi_1}, r_1 e^{i\phi_1}, i r_1 e^{i\phi_1}, i r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, 0)$ for $\mathrm{Fix}(\Delta_{10})$, the (r_1, r_2)-equations decouple from the phase equations and we get

$$\begin{aligned} \dot{r}_1 &= [\lambda + (4a_r + b_r + c_r - d_r) r_1^2 + a_r r_2^2] r_1 \\ \dot{r}_2 &= [\lambda + (a_r + b_r) r_2^2 + 4 a_r r_1^2] r_2. \end{aligned} \tag{5.40}$$

The only nontrivial zeros of this system are the corresponding to the (TR) and (AR) solutions provided that $b_r + c_r - d_r$ and $-b_r$ have the same sign. Recall proposition (5.6.2) from the previous section. See also table 5.14 for stability and branching directions. Consider that both of the branches of solutions corresponding to (TR) and (AR) bifurcate supercritically, that is,

$$a_r + b_r < 0 \ \wedge \ 4a_r + b_r + c_r - d_r < 0.$$

The zeros of (5.40) of type $\xi_1 = (r_1, 0)$, with $r_1 \neq 0$, correspond to solutions with symmetry Σ_5 and the zeros of type $\xi_2 = (0, r_2)$, with $r_2 \neq 0$, correspond to the solutions with symmetry Σ_1. The signs of the eigenvalues for ξ_1 are determined by the radial eigenvalue $4a_r + b_r + c_r - d_r$ and $-b_r - c_r + d_r$ and the signs of the eigenvalues for ξ_2 are determined by $a_r + b_r$ and $-b_r$. In order to ξ_1 to be a saddle and ξ_2 to be a sink, we consider

$$-b_r - c_r + d_r > 0 \ \wedge \ -b_r < 0.$$

Again we use [20] or [27] to prove that the unstable manifold of ξ_1 remains bounded near the origin. Note that the condition of [27] proposition 2.6

$$C = \frac{a_r}{a_r + b_r} + \frac{4a_r}{4a_r + b_r + c_r - d_r} > -2$$

is naturally satisfied since that from $a_r + b_r < 0$ and $-b_r < 0$ it follows that $a_r < 0$. Using the Poincaré-Bendixson theorem we proved the existence of the heteroclinic connection $\xi_1 \to \xi_2$. We recall the conditions on the coefficients of f that we are supposing for the saddle-sink connections $(TR) \to (SR)$ and $(AR) \to (TR)$:

$$\begin{array}{lll} a_r + b_r < 0, & 2a_r + b_r + c_r < 0, & 4a_r + b_r + c_r - d_r < 0, \\ b_r - c_r < 0, & b_r + c_r - d_r < 0, & -b_r < 0 \end{array}$$

and we note that these conditions imply that $a_r < 0$ and $b_r,\ c_r,\ d_r > 0$.

Finally, we have to prove that there is an heteroclinic connection between the (SR) solution and the (AR) solution (for parameter values for which there are the connections $(TR) \to (SR)$ and $(AR) \to (TR)$). As we saw previously, the dynamics of f restricted to the subspace $\mathrm{Fix}(\Delta_5)$ are given by the normal form of a Hopf bifurcation problem with symmetry $\mathbf{D}_4 \times \mathbf{S}^1$ studied by [20, 18, 37]: equations (5.12) with $A = 2a,\ B = b + c$ and $C = d$. By [37] there are periodic solutions besides (SR), (SS) and (AR) (in our problem) provided that

$$|\mathrm{Re}((b + c)\overline{d})| < |d|^2 < |b + c|^2.$$

Swift [37] projects the dynamics on the space $\mathrm{Fix}(\Delta_5)$ onto the surface of the unit sphere, obtaining a system in polar coordinates (r, θ, ϕ) not depending on the total phase $\psi = \arg(z_1\overline{z}_2)$. Since the corresponding $(\dot{\theta}, \dot{\phi})$-equations depend on r only through an overall multiplicative factor which can be removed by introducing a new time τ such that $\dot{\tau} = r,\ \tau(0) = 0$, Swift proves that when solving the associated spherical system on the sphere S^2, there is a straight correspondence between the fixed-points and limit cycles of this associated spherical system and the original system (5.12). More details about this associated spherical system appear in appendix A. Recall the stability for the solutions (SR), (SS), (AR) in the space $\mathrm{Fix}(\Delta_5)$ given in the previous section in table 5.10. Again we are considering supercritical bifurcations and the eigenvalues $\mu_1^*,\ \mu_2^*$ for the corresponding fixed-points of this reduced spherical system have products and sums with signs that depend on the same expressions as the eigenvalues $\mu_1,\ \mu_2$ obtained in table 5.10. For the fixed-point corresponding to (SR), say ξ_1^* the sum depends on $-b_r - c_r$ and the product on $|b+c|^2 - |d|^2$, for the one corresponding to (SS), that we call ξ_2^*, the sum depends on $b_r + c_r - 3d_r$ and the product on $|d|^2 - \mathrm{Re}((b+c)\overline{d})$. Finally for (AR), call ξ_3^*, the sum depends on $b_r + c_r + 3d_r$ and the product on $|d|^2 + \mathrm{Re}((b+c)\overline{d})$.

In order ξ_1^* to be a saddle, since the sum of the eigenvalues depends on $-b_r - c_r$ that is always negative in the parameter region in which we wish to work, the best we can have is when the product of the eigenvalues is negative, that is, when we suppose that

$$|b+c|^2 - |d|^2 < 0.$$

Note that with this condition, in this parameter region there are no submaximal solutions for (5.12). Now ξ_3^* has eigenvalues with sum determined by $b_r + c_r + 3d_r$ that is always positive in the parameter set we are working (since $b_r,\ c_r,\ d_r > 0$). Thus only if

$$|d|^2 + \mathrm{Re}((b+c)\overline{d}) < 0,$$

then this equilibrium has one negative eigenvalue and one positive. Thus it is a saddle. Generically a heteroclinic connection between these two fixed-points cannot happen. Heteroclinic connections saddle-saddle can happen just as a codimension one phenomenon. So the heteroclinic cycle does not occur generically.

Other possibilities were explored that lead to a similar conclusion. For example the possible cycle $(TR) \to (SR) \to (SS) \to (TS) \to (TR)$ also cannot happen generically. Recall tables 5.12 and 5.10 for stability and branching directions. Again we consider supercritical bifurcations and for the connection $(TR) \to (SR)$ to happen in $\mathrm{Fix}(\Delta_1)$, we impose $c_r - b_r > 0$, for the connection $(SS) \to (TS)$ in $\mathrm{Fix}(\Delta_2)$, we have $b_r - c_r - d_r > 0$ and for the $(TS) \to (TR)$ in $\mathrm{Fix}(\Delta_4)$, we require $b_r > 0$. Again, with these conditions, the best possibility for the $(SR) \to (SS)$ to happen (in $\mathrm{Fix}(\Delta_5)$) is the case when the (SR) and (SS) solutions are both saddles (for the spherical system inclusive). Similar conclusion we arrived for the possibilities of the cycles $(TR) \to (SR) \to (AR) \to (TS) \to (TR)$ or $(TR) \to (AR) \to (SR) \to (TR)$ and $(TR) \to (TS) \to (SS) \to (SR) \to (TR)$.

Another example: the cycle $(TR) \to (TS) \to (AR) \to (TR)$ also cannot happen since that for $(TR) \to (TS)$ in the space $\mathrm{Fix}(\Delta_4)$ we need $-b_r > 0$ and for the $(AR) \to (TR)$ in $\mathrm{Fix}(\Delta_{10})$ we need $-b_r < 0$.

In [34], Silber and Knobloch explore the possibility for the heteroclinic cycle involving the solutions (TS), (SS), (AR). The appropriate part of the isotropy lattice of $\Gamma \times \mathbf{S}^1$ is:

$$
\begin{array}{ccccccc}
\Sigma_3 & & \Sigma_4 & & \Sigma_5 & & \Sigma_3 \\
 & \nwarrow\ \nearrow & & \nwarrow\ \nearrow & & \nwarrow\ \nearrow & \\
 & \Delta_2 & & \Delta_5 & & \Delta_3 &
\end{array}
$$

Thus the cycle is formed by the orbits denoted by $(TS) \to (SS) \to (AR) \to (TS)$. They need to show the heteroclinic orbits $(TS) \to (SS)$ (in $\mathrm{Fix}(\Delta_2)$), $(SS) \to (AR)$ (in $\mathrm{Fix}(\Delta_5)$) and $(AR) \to (TS)$ (in $\mathrm{Fix}(\Delta_3)$). The heteroclinic connections $(TS) \to (SS)$ and $(AR) \to (TS)$ can occur by [27]. Note that the dynamics of f restricted to the spaces $\mathrm{Fix}(\Delta_2)$ and $\mathrm{Fix}(\Delta_3)$ are $\mathbf{O}(2)$-symmetric. Recall table 5.12. The conditions needed for the saddle-sink connections to happen are:

$$
\begin{gathered}
2a_r + b_r < 0, \quad 4a_r + b_r + c_r + d_r < 0, \quad 4a_r + b_r + c_r - d_r < 0, \\
-b_r + c_r + d_r > 0, \quad b_r - c_r + d_r > 0.
\end{gathered}
$$

The heteroclinic connection between the (SS) solution and the (AR) solution (for parameter values for which there are the connections $(TS) \to (SS)$ and $(AR) \to (TS)$) is conjectured based on the results of [37]. As we saw before, the dynamics of f restricted to the subspace $\mathrm{Fix}(\Delta_5)$ are given by the normal form of a Hopf bifurcation problem with symmetry $\mathbf{D}_4 \times \mathbf{S}^1$: equations (5.12) with $A = 2a$, $B = b + c$ and $C = d$. The possibility for the connection $(SS) \to (AR)$ is translated to the possibilty of a saddle-sink orbit of the spherical reduced system in the sphere. In this reduced system the stability for the fixed-point corresponding to the (SS) solution is determined by eigenvalues with trace with sign determined by $b_r + c_r - 3d_r$ and determinant $|d|^2 - \mathrm{Re}((b+c)\overline{d})$ (table 5.10). For the fixed-point corresponding to (AR) the eigenvalues have trace with sign determined by $b_r + c_r + 3d_r$ and determinant $|d|^2 + \mathrm{Re}((b+c)\overline{d})$. It is possible to choose values of a, b, c, d such that the first is a saddle and the second is a sink. And using the work of [37] they conjecture that there is a saddle-sink connection in the sphere.

We now explore the dynamics in the space $\mathrm{Fix}(\Delta_8)$.

Proposition 5.8.1 *Consider the dynamics of f reduced to the space* $\mathrm{Fix}(\Delta_8)$. *Provided that*

$$
\begin{gathered}
a_r + b_r < 0, \quad 2a_r + b_r < 0, \quad 3a_r + b_r < 0, \\
b_r < 0, \quad (2a_r + b_r)^2 + (a_r + b_r)(4a_r + b_r) > 0,
\end{gathered}
$$

there exists a bifurcation at $\lambda = 0$ to a set of heteroclinic saddle-sink connections between periodic solutions with symmetry (and conjugate to) Σ_1, Σ_6 and Σ_3. We illustrate part of the orbit by $(TR) \to (TC) \leftarrow (TS)$. These connections are structurally stable.

Proof Restricting g to the space $\mathrm{Fix}(\Delta_8)$ we obtain the equations (5.15) (previous section) and using the change of coordinates in the space $\mathrm{Fix}(\Delta_8)$ as $(z_1, z_2, z_3, z_4, z_5, z_6) = (r_1 e^{i\phi_1}, 0, r_1 e^{i\phi_1}, 0, r_2 e^{i\phi_2}, 0)$, we obtain the (r_1, r_2) equations (5.16). Recall table 5.13 for stability and branching directions. Consider supercritical bifurcations, thus $a_r + b_r,\ 2a_r + b_r,\ 3a_r + b_r < 0$. Consider the region $r_1, r_2 \geq 0$. Denote the only nontrivial fixed-points of (5.16) (for λ fixed) in this region by $\xi_1 = (r_1, 0)$, $\xi_2 = (0, r_2)$ and $\xi_3 = (r_3, r_3)$. These correspond to the periodic solutions of the initial problem with symmetry Σ_3, Σ_1 and Σ_6 respectively. We can apply [27] proposition 2.6 to the two regions $0 \leq r_1 \leq r_2$ and $0 \leq r_2 \leq r_1$ to guarantee that the unstable manifolds (in the positive quadrant) of ξ_1 and ξ_2 are bounded provided the condition

$$\frac{a_r}{a_r + b_r} + \frac{2a_r}{2a_r + b_r} > -2$$

holds. Thus we have the saddle-sink connections $\xi_1 \to \xi_3$ and $\xi_2 \to \xi_3$ by the Poincaré-Bendixson theorem. Applying the same ideas to the other quadrants we have connections between the equilibriums that are conjugate to ξ_i. We note that these saddle-sink connections in the plane are structurally stable. $\square$

Hopf bifurcation with symmetry the wreath product group $\mathbf{Z}_2 \wr \mathbf{S}_3$ was studied in [39] where $\mathbf{Z}_2$ acts nonabsolutely irreducibly on $\mathbf{C}$ by multiplication by -1. Thus $\mathbf{Z}_2 \wr \mathbf{S}_3$ acts also nonabsolutely irreducibly on $\mathbf{C}^3$. When f is reduced to the space $\mathrm{Fix}(\Pi_3)$ we obtain a vector field that is equivariant by $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$. Therefore the type of bifurcation obtained in the previous proposition also occurs for this one. In this case the space $\mathrm{Fix}(\Delta_8)$ corresponds to the space $\mathrm{Fix}(\mathbf{S}_2 \times \mathbf{S}_1)$ contained in $\mathbf{C}^3$.

5.9 Review

We proved that, generically, there are eight branches of periodic solutions with maximal isotropy that bifurcate from the trivial solution at Hopf bifurcation with symmetry $\mathbf{O}(2) \wr \mathbf{S}_3$. The stability for these solutions depends only on the terms of degree three of the vector field (and the truncated vector field can be supposed to commute with $\mathbf{S}^1$). Solutions with submaximal isotropy can also exist for an open region of the parameter space. We are able to prove the existence of these branches only for the Birkhoff normal form of the original vector field of degree three. The existence of one branch of periodic solutions with submaximal isotropy is justified by [37] since for a four-dimensional fixed-point subspace the truncated vector field commutes with $\mathbf{D}_4 \times \mathbf{S}^1$. Swift [37] proves that these solutions are generically unstable. Regions of the parameter space were found for which one or two new branches of solutions with submaximal isotropy can bifurcate from the trivial solution. We did not discuss the stability of these solutions: the equations defining these solutions are complicated.

Although we did not discuss the full $\mathbf{O}(2) \wr \mathbf{S}_3$-equivariant Hopf problem, we can use the classification obtained by [34], propositions 5.6.7, 5.6.8 and theorem 5.6.10 to have a partial classification of the dynamics of the $\mathbf{O}(2) \wr \mathbf{S}_3$-equivariant Hopf problem.

Chapter 6

Steady-state bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_N$ symmetry

Our aim now is to find the branching patterns in steady-state bifurcations for an N-cell system coupled (identically) in a simplex, where each cell has an internal $\mathbf{O}(2)$ symmetry. The wreath product group now in question is $\mathbf{O}(2) \wr \mathbf{S}_N$, where we suppose that the dynamics of an individual cell happens on $\mathbf{C}$ and where we consider the standard action of $\mathbf{O}(2)$ on the complex plane.

We begin by showing how the invariant theory for $\mathbf{O}(2) \wr \mathcal{G}$ is related to the theory for $\mathbf{O}(2)$ and $\mathcal{G}$, where $\mathcal{G}$ is any subgroup of $\mathbf{S}_N$, and we apply these results to the group $\mathbf{O}(2) \wr \mathbf{S}_N$.

Then we introduce the general equations corresponding to an N-cell system coupled in a simplex ('all-to-all' coupling). We find the branching patterns in steady-state bifurcations by looking for solutions corresponding to symmetries having one-dimensional fixed-point subspaces. These subgroups are described using [10]. We give conditions on the derivative that guarantee the existence of those branches. Up to conjugacy, we consider just N kinds of symmetries with one-dimensional fixed-point subspaces. Generically the equivariant branching lemma guarantees the existence of solutions with those symmetries. For these solutions, the conditions on the coefficients, in the general form of the function associated with the system, in the least degenerate case, which determine the direction of the criticality and the stability of those solutions, involve just terms of order one and three. Moreover, the eigenvalues (associated with the derivative calculated at each of those solutions) are real, and only solutions of one of two kinds of branches can be orbitally stable depending on the value of the coefficients.

Finally, we use results of [14], to show that generically the only (local) solution branches for the $\mathbf{O}(2) \wr \mathbf{S}_N$ bifurcation problem are those obtained using the equivariant branching lemma.

6.1 Invariant theory for $\mathbf{O}(2) \wr \mathcal{G}$

Consider the group $\mathbf{O}(2)$ generated by $\mathbf{S}^1$ together with the flip κ, acting on $\mathbf{C}$ by

$$\begin{array}{ll} \theta z = e^{i\theta} z \quad \text{if } \theta \in \mathbf{S}^1, & \\ \kappa z = \overline{z}. & \end{array} \qquad (z \in \mathbf{C})$$

Let $\mathcal{G}$ be a subgroup of $\mathbf{S}_N$ and W a finite-dimensional vector space. Define the action of $\mathcal{G}$ on W^N by

$$\sigma(x) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(N)}),$$

if $\sigma \in \mathcal{G}$ and $x = (x_1, x_2, \ldots, x_N) \in W^N$. That is, each element $\sigma \in \mathcal{G}$ permutes the set of indices $\{1, \ldots, N\}$.

From the structure of $\mathcal{P}(\mathbf{O}(2))$ and $\mathcal{P}(\mathcal{G})$ where $\mathcal{G}$ is a subgroup of $\mathbf{S}_N$, we can describe the structure of $\mathcal{P}(\Gamma)$ if Γ is the wreath product group $\mathbf{O}(2) \wr \mathcal{G}$.

Theorem 6.1.1 *Let $V = \mathbf{C}$. Let $\mathcal{G}$ be a subgroup of $\mathbf{S}_N$ and Γ the wreath product $\mathbf{O}(2) \wr \mathcal{G}$. Consider the action of Γ on V^N and the action of $\mathcal{G}$ on $\mathbf{R}^N$ as defined above.*

Then $\mu_1, \mu_2, \ldots, \mu_s$ form a Hilbert basis for $\mathcal{P}(\mathcal{G})$ if and only if for $f \in \mathcal{E}(\Gamma)$ there exists a smooth germ $h : \mathbf{R}^s \to \mathbf{R}$ such that

$$f(z) = h(\mu_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N), \ldots, \mu_s(z_1\overline{z}_1, \ldots, z_N\overline{z}_N))$$

for all $z = (z_1, \ldots, z_N) \in V^N$.

Remark 6.1.2 *From the definiton of $\mathcal{L} \wr \mathcal{G}$, it follows that a germ p belongs to $\mathcal{E}(\Gamma)$ if and only if p is invariant under $\mathcal{G}$ and under $\mathbf{O}(2)^N$, both acting on V^N, where the action of $\mathbf{O}(2)^N$ on V^N is defined by*

$$l(z) = (l_1 z_1, \ldots, l_N z_N),$$

for $l = (l_1, \ldots, l_N) \in \mathbf{O}(2)^N$ and $z = (z_1, \ldots, z_N) \in V^N$.

Similarly, a germ g belongs to $\vec{\mathcal{E}}(\Gamma)$ if and only if g is $\mathbf{O}(2)^N$-equivariant and $\mathcal{G}$-equivariant.

In order to prove theorem 6.1.1 we use first (lemma 6.1.3 below) the invariance under $\mathbf{O}(2)^N$.

Lemma 6.1.3 *Let $V = \mathbf{C}$ and $p : V^N \to \mathbf{R}$ be a polynomial function. Then p is invariant under $\mathbf{O}(2)^N$ if and only if there exists a polynomial function $h : \mathbf{R}^N \to \mathbf{R}$ such that*

$$p(z) = h(z_1\overline{z}_1, \ldots, z_N\overline{z}_N),$$

for all $z = (z_1, \ldots, z_N) \in V^N$.

Proof Let $p : V^N \to \mathbf{R}$ be a polynomial function invariant under $\mathbf{O}(2)^N$. We can write p as a polynomial in the 'real' coordinates $z_1, \overline{z}_1, \ldots, z_N, \overline{z}_N$ on V^N in the form

$$p(z) = \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta,$$

where $a_{\alpha\beta} \in \mathbf{C}$ and $\alpha,\ \beta \in (\mathbf{Z}_0^+)^N$ (using multi-indices).

As p is real-valued, $p = \overline{p}$, so the coefficients $a_{\alpha\beta}$ satisfy

$$\overline{a}_{\alpha\beta} = a_{\beta\alpha}. \tag{6.1}$$

Since

$$p(e^{i\theta_1} z_1, \ldots, e^{i\theta_N} z_N) = p(z)$$

for all $(\theta_1, \ldots, \theta_N) \in (\mathbf{S}^1)^N$, we must have

$$\sum a_{\alpha\beta} e^{i[\theta_1(\alpha_1-\beta_1)+\cdots+\theta_N(\alpha_N-\beta_N)]} z^\alpha \overline{z}^\beta = p(z).$$

This is an equality between polynomials, and therefore holds if and only if they have identical coefficients. That is, each $a_{\alpha\beta}$ must satisfy

$$a_{\alpha\beta} = e^{i[\theta_1(\alpha_1-\beta_1)+\cdots+\theta_N(\alpha_N-\beta_N)]} a_{\alpha\beta}.$$

This equality holds for all $(\theta_1, \ldots, \theta_N) \in (\mathbf{S}^1)^N$ if and only if

$$a_{\alpha\beta} = 0 \ \vee\ (\alpha_1 = \beta_1 \wedge \cdots \wedge \alpha_N = \beta_N).$$

So

$$p(z) = \sum a_{\alpha\alpha} (z_1\overline{z}_1)^{\alpha_1} \ldots (z_N\overline{z}_N)^{\alpha_N},$$

where by (6.1) the coefficients $a_{\alpha\alpha}$ must be real. As p is invariant under $\mathbf{O}(2)^N$, we can take

$$h(x) = \sum a_{\alpha\alpha} x^\alpha. \quad \square$$

Proof of theorem 6.1.1 Since $\mathcal{G}$ is a subgroup of $\mathbf{S}_N$, it is a compact Lie group acting on $\mathbf{R}^N$, and so there exists a finite Hilbert basis $\mu_1, \mu_2, \ldots, \mu_s$ for the ring $\mathcal{P}(\mathcal{G})$.

Using a result of Schwarz [32] it is sufficient to show that

$$\mu_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N), \ldots, \mu_s(z_1\overline{z}_1, \ldots, z_N\overline{z}_N)$$

generate the ring $\mathcal{P}(\Gamma)$.

From lemma 6.1.3, we have that $p : V^N \to \mathbf{R}$ belongs to $\mathcal{P}(\mathbf{O}(2)^N)$ if and only if there exists $h : \mathbf{R}^N \to \mathbf{R}$ such that

$$p(z) = h(z_1\overline{z}_1, \ldots, z_N\overline{z}_N),$$

for all $z = (z_1, \ldots, z_N) \in V^N$. If we denote $x_i = z_i\overline{z}_i \in \mathbf{R}$ for $i = 1, \ldots, N$, then p is $\mathcal{G}$-invariant if and only if h is $\mathcal{G}$-invariant. From the remark 6.1.2 the proof is complete. $\square$

Similar ideas are implicit in the calculation of $\vec{\mathcal{E}}(\mathbf{O}(2) \wr \mathcal{G})$.

Theorem 6.1.4 *Let $V = \mathbf{C}$. Let $\mathcal{G}$ be a subgroup of $\mathbf{S}_N$ and Γ the wreath product $\mathbf{O}(2) \wr \mathcal{G}$. Consider the action of Γ on V^N and the action of $\mathcal{G}$ on $\mathbf{R}^N$ as defined before.*

Then $g_1, g_2, \ldots, g_r$ generate the module $\vec{\mathcal{E}}(\mathcal{G})$ over the ring $\mathcal{E}(\mathcal{G})$ if and only if

$$g_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) * z, \ldots, g_r(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) * z$$

generate the module $\vec{\mathcal{E}}(\Gamma)$ over the ring $\mathcal{E}(\Gamma)$, where $z = (z_1, \ldots, z_N) \in V^N$ and the product $*$ between two vectors is used to denote the vector obtained multiplying corresponding components of the two vectors.

In order to prove theorem 6.1.4 we use (lemma 6.1.5 below) the equivariance under $\mathbf{O}(2)^N$.

Lemma 6.1.5 *Let $V = \mathbf{C}$ and $q : V^N \to V^N$ be a mapping with polynomial components. Then q is $\mathbf{O}(2)^N$-equivariant if and only if there are polynomial functions $h_i : \mathbf{R}^N \to \mathbf{R}$ for $i = 1, \ldots, N$ such that*

$$q(z) = (h_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N)z_1, \ldots, h_N(z_1\overline{z}_1, \ldots, z_N\overline{z}_N)z_N)$$

for all $z = (z_1, \ldots, z_N) \in V^N$.

Proof Suppose $q = (q_1, \ldots, q_N)$ with $q_i : V^N \to V$ for $i = 1, \ldots, N$. As q is $\mathbf{O}(2)^N$-equivariant, we must have

$$q(e^{i\theta_1} z_1, \ldots, e^{i\theta_N} z_N) = (e^{i\theta_1} q_1(z), \ldots, e^{i\theta_N} q_N(z))),$$

for all $(\theta_1, \ldots, \theta_N) \in (\mathbf{S}^1)^N$. So for $i = 1, \ldots, N$

$$q_i(z) = e^{-i\theta_i} q_i(e^{i\theta_1} z_1, \ldots, e^{i\theta_N} z_N).$$

In the coordinates $z_1, \overline{z}_1, \ldots, z_N, \overline{z}_N$ and using multi-indices, suppose

$$q_1(z) = \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta.$$

Then

$$q_1(z) = \sum a_{\alpha\beta} e^{i[\theta_1(\alpha_1 - \beta_1 - 1) + \theta_2(\alpha_2 - \beta_2) + \cdots + \theta_N(\alpha_N - \beta_N)]} z^\alpha \overline{z}^\beta.$$

This equality is satisfied for all $(\theta_1, \ldots, \theta_N) \in (\mathbf{S}^1)^N$ if and only if

$$a_{\alpha\beta} = 0 \ \vee \ (\alpha_1 = \beta_1 + 1 \wedge \alpha_2 = \beta_2 \wedge \cdots \wedge \alpha_N = \beta_N).$$

So

$$q_1(z) = \sum a_{\beta_1+1\beta_1\beta_2\beta_2\cdots\beta_N\beta_N} (z_1\overline{z}_1)^{\beta_1} \cdots (z_N\overline{z}_N)^{\beta_N} z_1.$$

Let

$$h_1(x) = \sum \mathrm{Re}(a_{\beta_1+1\beta_1\beta_2\beta_2\cdots\beta_N\beta_N}) x^\beta$$

and

$$t_1(x) = \sum \mathrm{Im}(a_{\beta_1+1\beta_1\beta_2\beta_2\cdots\beta_N\beta_N}) x^\beta.$$

Then

$$q_1(z) = h_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) z_1 + t_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) i z_1.$$

We also must have

$$q(\kappa z_1, \ldots, z_N) = (\kappa q_1(z), q_2(z), \ldots, q_N(z))$$

and so $q_1(\overline{z}_1, z_2, \ldots, z_N) = \overline{q_1(z)}$. This implies that

$$t_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) = 0.$$

Similarly we prove the existence of the other h_i such that

$$q_i(z) = h_i(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) z_i$$

and

$$q(z) = (h_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N)z_1, \ldots, h_N(z_1\overline{z}_1, \ldots, z_N\overline{z}_N)z_N)$$

is $\mathbf{O}(2)^N$-equivariant. $\square$

Proof of theorem 6.1.4 Since $\mathcal{G}$ is a compact Lie group acting on $\mathbf{R}^N$, there exists a finite set of $\mathcal{G}$-equivariant polynomials $g_1, g_2, \ldots, g_r$ that generates the module $\vec{\mathcal{P}}(\mathcal{G})$.

Using a result of Poénaru [30], it is enough to prove that the set

$$g_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) * z, \ldots, g_r(z_1\overline{z}_1, \ldots, z_N\overline{z}_N) * z$$

generates the module $\vec{\mathcal{P}}(\Gamma)$.

From lemma 6.1.5 we have $q : V^N \to V^N$ in $\vec{\mathcal{P}}(\mathbf{O}(2)^N)$ if and only if there are polynomial functions $h_i : \mathbf{R}^N \to \mathbf{R}$ for $i = 1, \ldots, N$ such that

$$q(z) = (h_1(z_1\overline{z}_1, \ldots, z_N\overline{z}_N)z_1, \ldots, h_N(z_1\overline{z}_1, \ldots, z_N\overline{z}_N)z_N).$$

Now we have just to note that q is $\mathcal{G}$-equivariant if and only if $(h_1, \ldots, h_N)$ is $\mathcal{G}$-equivariant. From the remark 6.1.2, it follows the result. $\square$

Remark 6.1.6

In theorem 6.1.1 we proved that if $\{\mu_i,\ i = 1, \ldots, s\}$ generates the ring $\mathcal{P}(\mathcal{G})$ where $\mathcal{G} \subseteq \mathbf{S}_N$, then $\{\mu_i(|z_1|^2, \ldots, |z_N|^2),\ i = 1, \ldots, s\}$ generates $\mathcal{P}(\mathbf{O}(2) \wr \mathcal{G})$. The representation of $\mathbf{O}(2)$ on $\mathbf{C}$ is radial. In general, a representaion of $\mathcal{L}$ on V is called *radial* if the module $\vec{\mathcal{P}}(\mathcal{L})$ over the ring $\mathcal{P}(\mathcal{L})$ is free with basis the identity map of V. This implies that $\mathcal{P}(\mathcal{L})$ is generated by the polynomial $||v||^2$ and V is $\mathcal{L}$-absolutely irreducible [14].

In [14] it is proved that in general if the representation of $\mathcal{L}$ on V is radial, then the rings $\mathcal{P}(\mathcal{L} \wr T)$, $\mathcal{P}(\mathbf{Z}_2 \wr T)$ and $\mathcal{P}(T)$ are isomorphic, if T is a transitive group of $\mathbf{S}_N$ and $\mathbf{Z}_2$ acts on $\mathbf{R}$ as multiplication by ∓ 1. Here the group $\mathcal{L} \wr T$ acts on V^N, and the groups $\mathbf{Z}_2 \wr T$ and T act on $\mathbf{R}^N$.

In fact, if we denote coordinates on $\mathbf{R}^N$ by $(x_1, \ldots, x_N)$ and on V^N by $(X_1, \ldots, X_N)$, then $p(x_1, \ldots, x_N)$ is T-invariant if and only if $p(x_1^2, \ldots, x_N^2)$ is $\mathbf{Z}_2 \wr T$-invariant if and only if $p(|X_1|^2, \ldots, |X_N|^2)$ is $\mathcal{L} \wr T$-invariant. Similar ideas are implicit for the equivariants. See [14] proposition 2.35 (pg 22). This relation can be used to relate the dynamics between equivariant systems for these groups. Later on we discuss this point.

6.2 Invariant theory for $\mathbf{O}(2) \wr \mathbf{S}_N$

Using the results of section 6.1, in order to obtain a general form for a bifurcation problem with $\mathbf{O}(2) \wr \mathbf{S}_N$ symmetry, we only need to know the general form of a germ $f \in \vec{\mathcal{E}}(\mathbf{S}_N)$, where $\mathbf{S}_N$ is acting on $\mathbf{R}^N$. Here $\mathbf{S}_N$ is the set of all permutations of the set of indices $\{1, \ldots, N\}$.

Corollary 6.2.1 *Let $f : \mathbf{C}^N \to \mathbf{R}$ be a smooth germ. Then f is invariant under $\mathbf{O}(2) \wr \mathbf{S}_N$ if and only if there exists a smooth germ $h : \mathbf{R}^N \to \mathbf{R}$ such that*

$$f(z) = h(\mu_1, \ldots, \mu_N)$$

for all $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$, where

$$\mu_k = (z_1\overline{z}_1)^k + \cdots + (z_N\overline{z}_N)^k$$

for $k = 1, \ldots, n$.

Proof Using theorem 6.1.1 we only need to know a Hilbert basis for $\mathcal{P}(\mathbf{S}_N)$. The functions

$$\mu_k(x_1, \ldots, x_N) = x_1^k + \cdots + x_N^k,$$

for $k = 1, \ldots, N$ form a Hilbert basis for $\mathcal{P}(\mathbf{S}_N)$. This follows from the main theorem for symmetric polynomials and from the Newton's formulae for sums of powers which enable the elementary symmetric polynomials to be expressed as polynomials in μ_k, for $k = 1, \ldots, N$ (with rational coefficients). See for example [1]. In fact, $\mathcal{P}(\mathbf{S}_N)$ is a polynomial ring and the set $\{\mu_k,\ k = 1, \ldots n\}$ is a Hilbert basis without relations. □

Corollary 6.2.2 *The module of the equivariants maps under $\mathbf{O}(2) \wr \mathbf{S}_N$ is generated over $\mathcal{E}(\mathbf{O}(2) \wr \mathbf{S}_N)$ by*

$$\begin{pmatrix} (z_1\overline{z}_1)^k z_1 \\ (z_2\overline{z}_2)^k z_2 \\ \vdots \\ (z_N\overline{z}_N)^k z_N \end{pmatrix},\ k = 0, \ldots, N-1.$$

Proof The result follows from theorem 6.1.4 and from the fact that

$$\begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_N^k \end{pmatrix}, \ k = 0, \ldots, N-1$$

generate the module $\vec{\mathcal{E}}(\mathbf{S}_N)$ over $\mathcal{E}(\mathbf{S}_N)$ (see appendix B). $\square$

6.3 Branching and stability

Consider the system of ordinary differential equations

$$\dot{z} = g(z, \lambda), \tag{6.2}$$

where $z \in \mathbf{C}^N$, $\lambda \in \mathbf{R}$ and $g : \mathbf{C}^N \times \mathbf{R} \to \mathbf{C}^N$ is a smooth germ at 0 commuting with $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_N$. Suppose also that $n \geq 3$.

From section 6.2 we know that there are polynomials $p_0, \ \ldots, \ p_{N-1}$ from $\mathbf{R}^N \times \mathbf{R}$ to $\mathbf{R}$ such that

$$g(z, \lambda) = \sum_{k=0}^{N-1} p_k(\mu_1, \ldots, \mu_N, \lambda) \begin{pmatrix} (z_1\overline{z}_1)^k z_1 \\ (z_2\overline{z}_2)^k z_2 \\ \vdots \\ (z_N\overline{z}_N)^k z_N \end{pmatrix}, \tag{6.3}$$

where

$$\mu_j = (z_1\overline{z}_1)^j + \cdots + (z_N\overline{z}_N)^j$$

for $j = 1, \ldots, N$ and $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$.

In order the equation $g(z, \lambda) = 0$ to be a bifurcation problem with Γ symmetry, we must have $g(0) = 0$ and $(dg)_{0,0} = 0$. That is, we have to impose

$$p_0(0) = 0 \tag{6.4}$$

and so the kernel of the linearized equations is the entire space $\mathbf{C}^N$. We note that if $(dg)_{0,0}$ were nonzero, then we could use the Liapunov-Schmidt reduction with symmetries to reduce g to the case where the Jacobian vanishes and this would also change the representation of Γ, and the space, which is not the aim of this work.

We have

$$\Sigma_{(0,\ldots,0)} = \mathbf{O}(2) \wr \mathbf{S}_N, \tag{6.5}$$

and the action of Γ is absolutely irreducible on $\mathbf{C}^N$ where $\mathbf{O}(2)^N$ acts nontrivially. Thus $z = 0$ is a solution for (6.2) for all $\lambda \in \mathbf{R}$. Moreover, if we impose the generic hypothesis (of the equivariant branching lemma)

$$p_{0,\lambda}(0) > 0, \tag{6.6}$$

then $z = 0$ is asymptotically stable for λ negative (and near zero) and it loses stability at $\lambda = 0$. Our aim now is to find the solutions $z \neq 0$ that can appear from this loss of stability of $z = 0$ at $\lambda = 0$. Such solutions often have isotropy subgroups Σ smaller than Γ. Generically, the equivariant branching lemma guarantees the existence of solutions with symmetries axial subgroups. We find that the only (nontrivial) branches of solutions that bifurcate at $\lambda = 0$ are those corresponding to the axial groups of Γ.

We begin to find the isotropy subgroups of Γ with one-dimensional fixed-point subspace (the axial groups) using [10].

Recall that by lemmas 3.1 and 3.2 of [10], if we assume $\mathcal{L} \wr \mathcal{G}$ acting absolutely irreducibly on W where $\mathcal{L}^N$ acts nontrivially and if $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, then we can write $W = U^N$ where $\mathcal{L}$ acts absolutely irreducibly on U. In [10], it is proved that for each block $J \subseteq \{1, \ldots, N\}$ and each axial subgroup $A \subseteq \mathcal{L}$ acting on U, the subgroup $\Sigma(A, J) \subseteq \mathcal{L} \wr \mathcal{G}$ is an axial subgroup and any axial subgroup of $\mathcal{L} \wr \mathcal{G}$ is conjugate to one of $\Sigma(A, J)$ for some axial $A \subseteq \mathcal{L}$ and some block J. Recall section 4.3.1.

Proposition 6.3.1 *There are N conjugacy classes of axial subgroups for the group $\mathbf{O}(2) \wr \mathbf{S}_N$ with the action defined before. They are*

$$\Sigma(\mathbf{Z}_2^\kappa, \{1, \ldots, j\}) = (\mathbf{Z}_2^\kappa)^j \times \mathbf{O}(2)^{N-j} \dot{+} \mathbf{S}_j \times \mathbf{S}_{N-j}$$

for $j = 1, \ldots, N$, and where $\mathbf{Z}_2^\kappa = \{1, \kappa\} \subset \mathbf{O}(2)$.

Proof As the only axial subgroup of $\mathbf{O}(2)$, up to conjugacy, is $\mathbf{Z}_2^\kappa = \{1, \kappa\}$, the axial subgroups of Γ, up to conjugacy, are the groups $\Sigma(\mathbf{Z}_2^\kappa, J)$, for each subset of indices $J \subseteq \{1, \ldots, N\}$, where as defined above

$$\Sigma(\mathbf{Z}_2^\kappa, J) = (B_1 \times \cdots \times B_N) \dot{+} Q_J,$$

with

$$B_j = \begin{cases} \mathbf{Z}_2^\kappa & \text{if } j \in J \\ \mathbf{O}(2) & \text{if } j \notin J \end{cases}$$

and

$$Q_J = \{\sigma \in \mathbf{S}_N \ : \ \sigma(J) = J\}.$$

That is, $\Sigma(\mathbf{Z}_2^\kappa, J)$ is a direct product of a number of copies of $\mathbf{Z}_2^\kappa$, one in each cell $j \in J$, and copies of $\mathbf{O}(2)$ in each remaining cell, all extended by Q_J. Note that for any subset of indices J of $\{1, \ldots, N\}$ there is a subgroup $\mathcal{H}$ of $\mathbf{S}_N$ that acts transitively on J and so J is a block.

Let $J = \{i_1, \ldots, i_p\}$, with $p \geq 1$. Let

$$Q_J = (\mathbf{S}_p \times \mathbf{S}_{N-p})_J,$$

be the set of permutations that leave the block invariant, i.e., the group of permutations of $\mathbf{S}_N$ such that the indices of the block J are permuted by an element of $\mathbf{S}_p$ (where $\mathbf{S}_p$ is the group of permutations of the indices J) and the indices of the complement of J in $\{1, \ldots, N\}$, say J', are permuted by an element of $\mathbf{S}_{N-p}$ (here $\mathbf{S}_{N-p}$ is the group of permutations of the indices in J'). The group Q_J acts transitively on J.

Let $x = (x_1, \ldots, x_N) \in \mathrm{Fix}(\Sigma(\mathbf{Z}_2^\kappa, \mathrm{J}))$. As $x_j \in \mathrm{Fix}_{\mathbf{C}}(\mathbf{O}(2))$ if $j \in J'$, then $x_j = 0$ (call such a kind of cell *quiescent* and the others *active*). Since Q_J acts transitively on J, all the active x_j are equal for $j \in J$. Thus any state with isotropy subgroup $\Sigma(\mathbf{Z}_2^\kappa, J)$ corresponds to quiescent cells for $j \in J'$ and identical active cells for $j \in J$.

Moreover, we do not have to consider all the possible blocks of $\{1, \ldots, N\}$. If we have two blocks, say J_1 and J_2, with the same number of indices, the corresponding $\Sigma(\mathbf{Z}_2^\kappa, J_1)$ and $\Sigma(\mathbf{Z}_2^\kappa, J_2)$ will be conjugate: these will be isotropy subgroups of elements in the same orbit under Γ (just use a permutation that takes one block to the other).

Therefore, if we call

$$J_j = \{1, \ldots, j\},$$

then up to conjugacy the axial subgroups of $\mathbf{O}(2) \wr \mathbf{S}_N$ are $\Sigma(\mathbf{Z}_2^\kappa, J_j)$ for the blocks J_j with $j = 1, \ldots, N$. $\square$

Proposition 6.3.2 *Consider (6.2) where g is as in (6.3). Assume conditions (6.4) and (6.6). Thus the trivial equilibrium is stable subcritically and unstable supercritically (for λ near 0).*

Then for each $\Sigma_j = \Sigma(\mathbf{Z}_2^\kappa, J_j)$ with $j = 1, \ldots, N$, there exists locally a unique branch of solutions for (6.2) with that symmetry.
(a) The Σ_j branch is supercritical (subcritical) if $j p_{0,\mu_1}(0) + p_1(0) < 0$ (> 0)

for $j = 1, \ldots, N$.
(b) The Σ_1 *branch is stable if* $p_{0,\mu_1}(0) + p_1(0) < 0$ *and* $p_1(0) > 0$.
(c) The Σ_j *branch for* $1 < j < N$ *is always unstable if we assume* $p_1(0) \neq 0$.
(d) The Σ_N *branch is stable if* $N p_{0,\mu_1}(0) + p_1(0) < 0$ *and* $p_1(0) < 0$.

Proof With the conditions (6.4) and (6.6), the equivariant branching lemma states that for each $\Sigma(\mathbf{Z}_2^k, J_j)$ there exists locally a unique branch of solutions with the symmetry of that group. From now on we call Σ_j the group $\Sigma(\mathbf{Z}_2^k, J_j)$.

We have

$$\mathrm{Fix}(\Sigma_j) = \{(x, \ldots, \underset{\substack{\uparrow \\ j-\ \text{position}}}{x}, \ 0, \ldots, 0) \ : \ x \in \mathbf{R}\}.$$

Restricting g to each fixed-point subspace $\mathrm{Fix}(\Sigma_j)$ and solving the equations $g = 0$, we get

$$p_0(jx^2, jx^4, \ldots, jx^{2N}, \lambda) + x^2 p_1(jx^2, jx^4, \ldots, jx^{2N}, \lambda) + \cdots +$$
$$+ x^{2(N-1)} p_{N-1}(jx^2, jx^4, \ldots, jx^{2N}, \lambda) = 0,$$

for $x > 0$.

The lowest lower terms in the equation for Σ_j solutions are

$$x^2[j p_{0,\mu_1}(0) + p_1(0)] + \lambda p_{0,\lambda}(0) + \cdots,$$

and the Σ_j is supercritical (subcritical) when

$$[j p_{0,\mu_1}(0) + p_1(0)] \cdot p_{0,\lambda}(0) < 0 \ (> 0).$$

If we assume

$$j p_{0,\mu_1}(0) + p_1(0) \neq 0,$$

for $j = 1, \ldots, N$, the directions of the branchings are determined (since we are assuming (6.6)).

We now discuss stabilities. Using the coordinates $z_1, \overline{z}_1, \ldots, z_N, \overline{z}_N$ for g, and denoting the components of g by g_i, for $i = 1, \ldots, N$, the derivative of g is

$$(dg) \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} =$$

$$\begin{bmatrix} g_{1,z_1}w_1 + g_{1,\overline{z}_1}\overline{w}_1 + \cdots + g_{1,z_N}w_N + g_{1,\overline{z}_N}\overline{w}_N \\ \cdots \\ g_{N,z_1}w_1 + g_{N,\overline{z}_1}\overline{w}_1 + \cdots + g_{N,z_N}w_N + g_{N,\overline{z}_N}\overline{w}_N \end{bmatrix}$$

Along the trivial solution $z_1 = \cdots = z_N = 0$ we have

$$(dg)_{0,\lambda}\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} = p_0(0,\lambda)\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} =$$

$$(\lambda p_{0,\lambda}(0) + \cdots)\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}.$$

So $(dg)_{0,\lambda}$ is a multiple of the identity, with the repeated eigenvalue $p_0(0,\lambda)$ whose sign is determined by $\lambda p_{0,\lambda}(0)$ (since $p_0(0) = 0$).

Now we will consider separately the Σ_1, Σ_j for $1 < j < N$ and Σ_N solutions. The reason is that for these three cases we can decompose $\mathbf{C}^N$ in invariant subspaces for the derivative calculated at representative points of the corresponding fixed-points subspaces in slightly different way.

Consider the Σ_1 solution $(x, 0, \ldots, 0) \in \mathrm{Fix}(\Sigma_1)$. Remember that $\Sigma_1 = \mathbf{Z}_2^\kappa \times \mathbf{O}(2)^N \dot{+} \mathbf{S}_1 \times \mathbf{S}_{N-1}$ and that $(dg)_{(x,0,\ldots,0)}$ commutes with Σ_1. Using theorem 4.3.1, corollary 4.3.5, the branch equation and the symmetry of g, we have

$$(dg)_{(x,0,\ldots,0,\lambda)}\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} g_{1,z_1}(w_1 + \overline{w}_1) \\ p_0(x^2, \ldots, x^{2n}, \lambda)w_2 \\ \vdots \\ p_0(x^2, \ldots, x^{2n}, \lambda)w_N \end{bmatrix}$$

with g_{1,z_1} evaluated at $(x, 0, \ldots, 0, \lambda)$. In detail: from theorem 4.3.1 the derivative $(dg)_{(x,0,\ldots,0,\lambda)}$ is a diagonal matrix $\mathrm{Diag}(G_1, G_2)$. For G_2 we can use corollary 4.3.5 and conclude that it is a scalar of the identity on $\mathbf{C}^{N-1}$. For G_1 we can use the branching equation and the symmetry of g to get that $g_{1,z_1} = g_{1,\overline{z}_1}$.

We can decompose $\mathbf{C}^N$ in three invariant subspaces for $(dg)_{(x,0,\ldots,0,\lambda)}$, say V_0, V_1 and V_2. See table 6.1. Let V_0 be the kernel of the derivative which is one-dimensional as it was expected since the orbit of $(x, 0, \ldots, 0)$ under Γ is a one-dimensional manifold. The fixed-point subspace is V_1 and V_2 is the complement of the sum of these two subspaces. Considering $(dg)_{(x,0,\ldots,0,\lambda)}\,|_{V_1}$ we find that

$$2g_{1,z_1}$$

is a eigenvalue. Using the branch equation, the sign is determined by

$$p_{0,\mu_1}(0)+p_1(0).$$

Now with $(dg)_{(x,0,\ldots,0,\lambda)}\,|_{V_2}$ we see that

$$p_0(x^2,\ldots,x^{2n},\lambda)$$

is an eigenvalue whose sign is determined by

$$-p_1(0)$$

if it is assumed nonzero.

For Σ_j with $1<j<N$ supose that $(x,\ldots,x,0,\ldots,0,\lambda)$ is a solution with $(x,\ldots,x,0,\ldots,0)\in \mathrm{Fix}(\Sigma_j)$. Calculate

$$(dg)_{(x,\ldots,x,0,\ldots,0,\lambda)}\begin{bmatrix} w_1\\ \vdots\\ w_j\\ w_{j+1}\\ \vdots\\ w_N\end{bmatrix}=\begin{bmatrix} g_{1,z_1}(w_1+\overline{w}_1)+g_{1,z_2}(w_2+\overline{w}_2+\cdots+w_j+\overline{w}_j)\\ \vdots\\ g_{1,z_1}(w_j+\overline{w}_j)+g_{1,z_2}(w_1+\overline{w}_1+\ldots+w_{j-1}+\overline{w}_{j-1})\\ p_0(jx^2,\cdots,jx^{2n},\lambda)w_{j+1}\\ \vdots\\ p_0(jx^2,\cdots,jx^{2n},\lambda)w_N\end{bmatrix},$$

with g_{1,z_1} and g_{1,z_2} evaluated at $(x,\ldots,x,0,\ldots,0,\lambda)$. Again, we can use theorem 4.3.1, corollaries 4.3.5 and 4.3.7 (as $\Sigma_j=(\mathbf{Z}_2^\kappa)^j\times\mathbf{O}(2)^{N-j}\dot{+}\mathbf{S}_j\times\mathbf{S}_{N-j}$), the symmetry of g and the branching equation.

We can decompose $\mathbf{C}^N$ into four invariant subspaces for $(dg)_{(x,\ldots,x,0,\ldots,0,\lambda)}$, call V_0, V_1, V_2 and V_3 (see table 6.1). The kernel of the derivative is V_0 and is j-dimensional. The fixed-point subspace is V_1. Considering $(dg)_{(x,\ldots,x,0,\ldots,0,\lambda)}\,|_{V_1}$ we have

$$2(g_{1,z_1}+(j-1)g_{1,z_2})$$

is an eigenvalue whose sign is determined by

$$jp_{0,\mu_1}(0)+p_1(0).$$

Isotropy subgroup	Orbit representative	Invariant subspaces
$\Sigma(\mathbf{Z}_2^\kappa, \{1\})$	$(x, 0, \ldots, 0)$ $(x \in \mathbf{R})$	$V_0 = \{(iy, 0, \ldots, 0),\ y \in \mathbf{R}\}$ $V_1 = \{(y, 0, \ldots, 0),\ y \in \mathbf{R}\}$ $V_2 = \{(0, z_2, \ldots, z_N),\ z_2, \ldots, z_N \in \mathbf{C}\}$
$\Sigma(\mathbf{Z}_2^\kappa, \{1, \ldots, j\})$, $1 < j < N$	$(x, \ldots, x, 0, \ldots, 0)$ $(x \in \mathbf{R})$	$V_0 = \{(iy_1, \ldots, iy_j, 0, \ldots, 0),\ y_1, \ldots, y_j \in \mathbf{R}\}$ $V_1 = \{(y, \ldots, y, 0, \ldots, 0),\ y \in \mathbf{R}\}$ $V_2 = \{(0, \ldots, 0, z_{j+1}, \ldots, z_N),$ $z_{j+1}, \ldots, z_N \in \mathbf{R}\}$ $V_3 = \mathcal{L}(\{(1-j, 1, \ldots, 1, 0, \ldots, 0), \ldots,$ $(1, \ldots, 1-j, 1, 0, \ldots, 0)\})$
$\Sigma(\mathbf{Z}_2^\kappa, \{1, \ldots, N\})$	$(x, \ldots, x)$ $(x \in \mathbf{R})$	$V_0 = \{(iy_1, \ldots, iy_N),\ y_1, \ldots, y_N \in \mathbf{R}\}$ $V_1 = \{(y, \ldots, y),\ y \in \mathbf{R}\}$ $V_3 = \mathcal{L}(\{(1-N, 1, \ldots, 1), \ldots,$ $(1, \ldots, 1-N, 1)\})$

Table 6.1: Decomposition of $\mathbf{C}^N$ into invariant subspaces for $\Sigma(\mathbf{Z}_2^\kappa, \{1, \ldots, j\})$.

For $(dg)_{(x,\ldots,x,0,\ldots,0,\lambda)}\ |_{V_2}$ we see that

$$p_0(jx^2,\ldots,jx^{2n},\lambda)$$

is an eigenvalue and the sign is determined by

$$-p_1(0).$$

From the form of the derivative we can take more information than just decompose $\mathbf{C}^N$ in invariant subspaces for dg. In fact, the $j-1$ vectors chosen to generate V_3 in table 6.1 are the eigenvectors corresponding to a single eigenvalue

$$2(g_{1,z_1}-g_{1,z_2}),$$

and again using the branch equation we see that the sign is determined by

$$p_1(0)$$

if it is assumed nonzero.

Finally, for the Σ_N branch, consider $(x,\ldots,x,\lambda)$ a solution with $(x,\ldots,x)\in \mathrm{Fix}(\Sigma_N)$. Then

$$(dg)_{(x,\ldots,x,\lambda)}\begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} g_{1,z_1}(w_1+\overline{w}_1)+g_{1,z_2}(w_2+\overline{w}_2+\cdots+w_N+\overline{w}_N) \\ \vdots \\ g_{1,z_1}(w_N+\overline{w}_N)+g_{1,z_2}(w_1+\overline{w}_1+\cdots+w_{n-1}+\overline{w}_{n-1}) \end{bmatrix},$$

with g_{1,z_1} and g_{1,z_2} evaluated at $(x,\ldots,x,\lambda)$. We used theorem 4.3.1, corollary 4.3.7, the branch equation and the symmetry of g. In this case it holds $g_{1,z_1}=g_{1,\overline{z}_1}$ and $g_{1,z_2}=g_{1,\overline{z}_2}$ when evaluated at the solution.

We can decompose $\mathbf{C}^N$ into three invariant subspaces for $(dg)_{(x,\ldots,x,\lambda)}$, and the only difference from the previous branches is that we no longer have the invariant subspace corresponding to V_2.

The kernel is V_0. From $(dg)_{(x,\ldots,x,\lambda)}\ |_{V_1}$ we get the eigenvalue

$$2(g_{1,z_1}+(N-1)g_{1,z_2}),$$

whose sign is determined by

$$Np_{0,\mu_1}(0)+p_1(0).$$

Isotropy subgroup	Branching equation	Signs of the eigenvalues
$\mathbf{O}(2) \wr \mathbf{S}_N$	$z_1 = \cdots = z_N = 0$	$p_{0,\lambda}\lambda$ (2N times)
$\Sigma(\mathbf{Z}_2^\kappa, \{1\})$	$\lambda = -\frac{p_{0,\mu_1}+p_1}{p_{0,\lambda}}x^2 + \cdots$	$p_{0,\mu_1} + p_1$(one time) $-p_1(2(N-1)$ times)
$\Sigma(\mathbf{Z}_2^\kappa, \{1, \ldots, j\})$, $1 < j < N$	$\lambda = -\frac{jp_{0,\mu_1}+p_1}{p_{0,\lambda}}x^2 + \cdots$	$jp_{0,\mu_1} + p_1$(one time) $-p_1(2(N-j)$ times) $p_1(j-1$ times)
$\Sigma(\mathbf{Z}_2^\kappa, \{1, \ldots, N\})$	$\lambda = -\frac{Np_{0,\mu_1}+p_1}{p_{0,\lambda}}x^2 + \cdots$	$Np_{0,\mu_1} + p_1$(one time) $p_1(N-1$ times)

Table 6.2: Branches and stability of solutions of generic $\mathbf{O}(2) \wr \mathbf{S}_N$ -equivariant bifurcation problem.

The set of the $N-1$ vectors that generate V_3 are the eigenvectors of the eigenvalue

$$2(g_{1,z_1} - g_{1,z_2})$$

whose sign is determined by

$$p_1(0).$$

The bifurcation diagram of $g = 0$ corresponding to branches of solutions with symmetries with one-dimensional fixed-point subspaces (guaranteed by the equivariant branching lemmma) is determined if we assume

$$p_0(0) = 0, \ p_{0,\lambda}(0) \neq 0$$

$$p_1(0) \neq -jp_{0,\mu_1}(0), \ j = 1, \ldots, N$$

$$p_1(0) \neq 0.$$

See table 6.2 (we are denoting p_{0,μ_1}, p_1 and $p_{0,\lambda}$ the values of the corresponding functions at the origin) and figure 6.1. $\square$

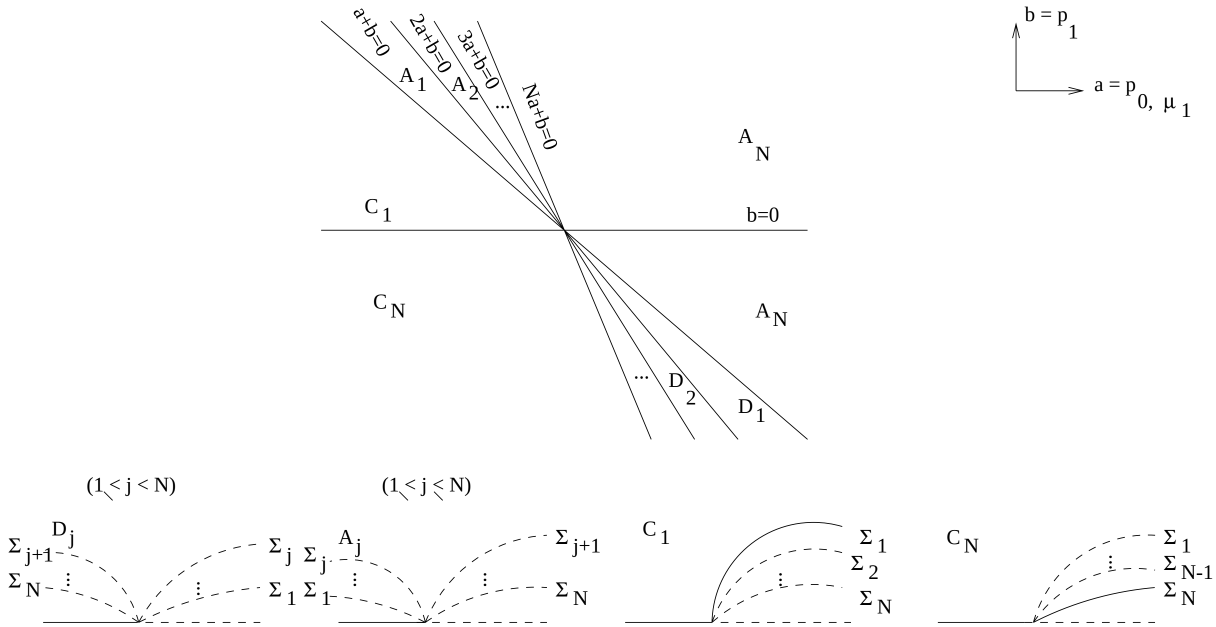


Figure 6.1: Bifurcation diagram for nondegenerate steady-state bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_N$ symmetry. Broken (unbroken) bifurcation curves indicate unstable (stable) solutions.

Remarks

(a) In [14], proposition 2.36 (pg 22) it is proved that if the representation of $\mathcal{L}$ on V is radial and $\mathcal{G}$ is a transitive subgroup of $\mathcal{S}_N$, then the dynamics on V^N for $\mathcal{L} \wr \mathcal{G}$-equivariant maps is equivalent to the dynamics of $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant maps on $\mathbf{R}^N$. Moreover, this result follows from the strict restriction between the invariant theory for $\mathcal{L} \wr \mathcal{G}$ and for $\mathbf{Z}_2 \wr \mathcal{G}$ (and that we referred already in remark 6.1.6).

We are interested in steady-state bifurcation with $\mathbf{O}(2) \wr \mathbf{S}_N$ symmetry, i.e., we compute the branches of steady solutions that bifurcate in a generic $\mathbf{O}(2) \wr \mathbf{S}_N$ bifurcation problem. With the same aim it has been studied the group $\mathbf{Z}_2 \wr \mathbf{S}_N$ acting on $\mathbf{R}^N$ (with $\mathbf{Z}_2$ acting on $\mathbf{R}$ by multiplication by ∓ 1): see for example [40] or [14]. In fact, in [14, 15], they prove that in the generic case, up to conjugacy, there are only N bifurcating branches of nontrivial equilibriums with symmetries corresponding to the N axial (conjugacy classes of) subgroups of $\mathbf{Z}_2 \wr \mathbf{S}_N$: the MISC holds for the group $\mathbf{Z}_2 \wr \mathbf{S}_N$ (that is the Weyl group of type B_N); each nontrivial branch of solutions of $g = 0$ where g is equivariant by the group $\mathbf{Z}_2 \wr \mathbf{S}_N$ and $(x, \lambda) = (0, 0)$ is a bifurcating point, corresponds to a branch of solutions with maximal isotropy subgroup. Moreover, they prove that the third order truncation of the generic vector field g determines the stability and the directions of these branches. This bifurcation problem is said to be 3-determined. In fact, our problem with symmetry $\mathbf{O}(2) \wr \mathbf{S}_N$ is also 3-determined since the action of $\mathbf{O}(2)$ on V is radial. The branches obtained by us are in correspondence to the branches to the $\mathbf{Z}_2 \wr \mathbf{S}_N$ problem and provided that $p_1(0) \neq 0$ these are the only branches that bifurcate at $\lambda = 0$. Moreover, of the N possible branches of solutions (up to conjugacy) only two can be generically stable near the bifurcation point, and these can not be both stable.

(b) As it was stated in chapter 1, the group $\mathbf{Z}_2 \wr \mathbf{S}_N$ is the symmetry group of the N-dimensional cube and the holohedry of a primitive cubic lattice in dimension N. If this group is extended by the N-torus $\mathbf{T}^N$, then we obtain the wreath product group $\mathbf{O}(2) \wr \mathbf{S}_N$. This group leaves invariant the space of functions from $\mathbf{R}^N$ to $\mathbf{R}$ that are spatially periodic with respect to this lattice. As in chapter 5, section 5.1, the problem that we have studied in this chapter can been thought as the problem that would have been obtained by a center manifold reduction from a PDE when looking for spatially periodic solutions with respect to a primitive cubic lattice in dimension N, i.e., when studying steady-state symmetry-breaking on a primitive cubic lattice. See [6] for $N = 3$.

Appendix A

D_4-Hopf bifurcation problem

Golubitsky and Stewart [18] find that the cubic truncation of the most general equivariant system under $\mathbf{D}_4 \times \mathbf{S}^1$ is sufficient to compute the stability and branching direction of the three periodic solutions with maximal isotropy denoted by SR, SS, AR. The Birkhoff normal form, truncated to third order is

$$\begin{aligned} \dot{z}_1 &= [\lambda + i\omega + A(|z_1|^2 + |z_2|^2) + B|z_1|^2]z_1 + C\overline{z}_1 z_2^2 \\ \dot{z}_2 &= [\lambda + i\omega + A(|z_1|^2 + |z_2|^2) + B|z_2|^2]z_2 + C\overline{z}_2 z_1^2 \end{aligned} \tag{A.1}$$

where A, B, C are complex and $\omega(0) = 1$. The three cubic terms make the analysis of (A.1) complicate. However Swift [37] uses the fact that all nonlinear terms are of the same degree to reduce the dynamics to an ODE onto the 2-sphere.

Using the parameters $\alpha_u = R_u + iI_u,\ \alpha_v = R_v + iI_v$ and $\alpha_w = R_w + iI_w$, where $R_u,\ I_u, \ldots$ are real and where $A = \alpha_u + \alpha_v - \alpha_w,\ B = -\alpha_u - \alpha_v + 2\alpha_w$ and $C = \alpha_u - \alpha_v$, the system (A.1) becomes

$$\begin{aligned} \dot{z}_1 &= \mu z_1 + \alpha_u z_2(z_1\overline{z}_2 + \overline{z}_1 z_2) + \alpha_v z_2(z_1\overline{z}_2 - \overline{z}_1 z_2) + \alpha_w z_1(|z_1|^2 - |z_2|^2) \\ \dot{z}_2 &= \mu z_2 + \alpha_u z_1(z_2\overline{z}_1 + \overline{z}_2 z_1) + \alpha_v z_1(z_2\overline{z}_1 - \overline{z}_2 z_1) + \alpha_w z_2(|z_2|^2 - |z_1|^2). \end{aligned} \tag{A.2}$$

This system is then reduced to a system in $\mathbf{R}^3$ with coordinates (u, v, w): in order to eliminate the average phase (derived from the $\mathbf{S}^1$ symmetry) $\psi = \arg(z_1\overline{z}_2)$ from the normal form, Swift uses the following coordinate transformation

$$\begin{aligned} u + iv &= r\sin(\theta)e^{i\phi} = 2z_1\overline{z}_2 \\ w &= r\cos(\theta) = |z_1|^2 - |z_2|^2 \\ e^{i\psi} &= \tfrac{z_1 z_2}{|z_1 z_2|}, \end{aligned}$$

where (r, θ, ϕ) are the usual spherical polar coordinates on $\mathbf{R}^3$ and this is a two-to-one transformation from (z_1, z_2) to (u, v, w, ψ):

$$\begin{aligned} z_1 &= r^{1/2} \cos(\theta/2) e^{i(\phi+\psi)/2} \\ z_2 &= r^{1/2} \sin(\theta/2) e^{i(-\phi+\psi)/2}. \end{aligned}$$

The system (A.2) now becomes

$$\begin{aligned} \tfrac{1}{2}\dot{u} &= u(\lambda + R_u r) + (I_v - I_w)vw \\ \tfrac{1}{2}\dot{v} &= v(\lambda + R_v r) + (I_w - I_u)wu \\ \tfrac{1}{2}\dot{w} &= w(\lambda + R_w r) + (I_u - I_v)uv \\ \tfrac{1}{2}\dot{\psi} &= \omega + o(r) \end{aligned} \tag{A.3}$$

where $r = (u^2 + v^2 + w^2)^{1/2}$. As the first three equations do not contain ψ, Swift ignores ψ and consider the reduced system in $\mathbf{R}^3$ defined by the first three equations of (A.3): fixed-points of this $\mathbf{R}^3$ system correspond to periodic orbits in the original four-dimensional system and the stability is calculated from the reduced system alone: the Floquet exponents of a periodic solution in the four-dimensional system are 0 and the eigenvalues of the corresponding fixed-point in the reduced system. The correspondence between the three maximal solutions and the fixed-points is the following: the fixed-point of the type $v = w = 0$ corresponds to a solution with $z_1 = z_2$ (called 'u' solution), the fixed-point of type $u = w = 0$ corresponds to a solution with $z_2 = iz_1$ (called 'v' solution) and finally a solution with $z_2 = 0$ corresponds to a fixed-point with $v = w = 0$ (and called 'w' solution).

The $\mathbf{R}^3$ system is then put into spherical polar coordinates:

$$\begin{aligned} \dot{r} &= r\{2\lambda + r[2A_r + B_r(1 + \cos^2(\theta)) + C_r \sin^2(\theta)\cos(2\phi)]\} \\ \dot{\theta} &= r\sin(\theta)[\cos(\theta)(-B_r + C_r\cos(2\phi)) - C_I \sin(2\phi)] \\ \dot{\phi} &= r[\cos(\theta)(B_I - C_I\cos(2\phi)) - C_r\sin(2\phi)], \end{aligned} \tag{A.4}$$

where the radius r occurs in the equations for $\dot{\theta}$ and $\dot{\phi}$ as a multiplicative factor. Swift proves that the projection of the trajectories of this system onto the 2-sphere, i.e., the dynamics of

$$\begin{aligned} \dot{\theta} &= \sin(\theta)[\cos(\theta)(-B_r + C_r\cos(2\phi)) - C_I\sin(2\phi)] \\ \dot{\phi} &= \cos(\theta)(B_I - C_I\cos(2\phi)) - C_r\sin(2\phi), \end{aligned} \tag{A.5}$$

called *the associated spherical system*, when translated back to (A.4), the factor of r only speeds up or slows down the motion. Each fixed-point (limit cycle) of (A.5) corresponds to a unique fixed-point (limit cycle) of (A.4), provided a condition on the expression of the r-equation in (A.4) is satisfied. The eigenvalues (Floquet exponents) also can be translated proving the above statement. See the associated spherical lemma in [37].

The periodic solutions of (A.1) with submaximal symmetry correspond to fixed-points of the $\mathbf{R}^3$ system (the first three equations of (A.3)) with no symmetry. These solutions exist precisely when the parameters satisfy $|B|^2 > |C|^2 > |\mathrm{Re}(B\overline{C})|$. And in this case Swift proves that all of the symmetric solutions are either sources or sinks in the spherical system. When there is no non-symmetric solutions, then one of the symmetric solutions is a saddle in the spherical system. Swift conjectured that for the associated spherical system (A.5), with B and C fixed at nondegenerate values, up to time reversal, this system falls into one of four topological equivalence classes that are distinguished by the existence or non-existence of non-symmetric solutions and limit cycles. One of these classes presents dynamics where an heteroclinic connection between the 'u' solution (saddle) and the 'v' solution (sink) is possible for the spherical system. And he also conjectured that it is possible that the same type of saddle-sink connection happens for an open set of the parameters where now the solutions involved are 'w' and 'u'. If this is true, we can expect that for an open set of the parameter values this heteroclinic saddle-sink orbit occurs for the initial system (A.1).

Appendix B

The module $\vec{\mathcal{E}}(\mathbf{S}_N)$ over the ring $\mathcal{E}(\mathbf{S}_N)$

The following result is presumably well known, but we provide a simple self-contained proof.

Lemma The module $\vec{\mathcal{E}}(\mathbf{S}_N)$ is generated over $\mathcal{E}(\mathbf{S}_N)$ by

$$\begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_N^k \end{pmatrix}, \quad k = 0, \ldots, N-1.$$

Proof Applying [30], we restrict attention to polynomials. Let $g = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix} \in \vec{\mathcal{P}}(\mathbf{S}_N)$. It is enough to have

$$g(\sigma x) = \sigma g(x),$$

for $\sigma \in \{(1k),\ k = 2, \ldots, N\}$ (and $x = (x_1, \ldots, x_N)$) since this set generates the group $\mathbf{S}_N$ and where by $(1k)$ we denote the permutation which interchanges the integers 1, k and leaves the rest of the $N-2$ integers unchanged. It follows that

$$g(x) = \begin{pmatrix} g_1(x_1, x_2, \ldots, x_N) \\ g_1(x_2, x_1, \ldots, x_N) \\ \cdots \\ g_1(x_N, x_2, \ldots, x_1) \end{pmatrix}$$

where $g_1(x_1, x_2, x_3, \ldots, x_N) = g_1(x_1, x_3, x_2, \cdots, x_N) = \cdots = g_1(x_1, x_N, x_3, \ldots, x_2)$. From this, g_1 must be invariant under $\mathbf{S}_{N-1}$ in the last $N-1$ indeterminates. So we can write g_1 as a polynomial of x_1 and the elements of a Hilbert basis of $\mathcal{P}(\mathbf{S}_{N-1})$, for example

$$g_1(x) = p(x_1, x_2 + \cdots + x_N, x_2^2 + \cdots + x_N^2, \ldots, x_2^{N-1} + \cdots + x_N^{N-1}).$$

So

$$g(x) =$$

$$\begin{pmatrix} p(x_1, x_2 + x_3 + \cdots + x_N, \ldots, x_2^{N-1} + x_3^{N-1} + \cdots + x_N^{N-1}) \\ p(x_2, x_1 + x_3 + \cdots + x_N, \ldots, x_1^{N-1} + x_3^{N-1} + \cdots + x_N^{N-1}) \\ \cdots \\ p(x_N, x_1 + x_2 + \cdots + x_{N-1}, \ldots, x_1^{N-1} + x_2^{N-1} + \cdots + x_{N-1}^{N-1}) \end{pmatrix} =$$

$$\sum_{\alpha_1 \ldots \alpha_N} a_{\alpha_1 \ldots \alpha_N} \begin{pmatrix} x_1^{\alpha_1}(x_2 + x_3 + \cdots + x_N)^{\alpha_2} \cdots (x_2^{N-1} + x_3^{N-1} + \cdots + x_N^{N-1})^{\alpha_N} \\ x_2^{\alpha_1}(x_1 + x_3 + \cdots + x_N)^{\alpha_2} \cdots (x_1^{N-1} + x_3^{N-1} + \cdots + x_N^{N-1})^{\alpha_N} \\ \cdots \\ x_N^{\alpha_1}(x_1 + x_2 + \cdots + x_{N-1})^{\alpha_2} \ldots (x_1^{N-1} + x_2^{N-1} + \cdots + x_{N-1}^{N-1})^{\alpha_N} \end{pmatrix}$$

Since

$$(X_1+X_2+\cdots+X_N)^\alpha - (X_2+\cdots+X_N)^\alpha = [X_1+(X_2+\cdots+X_N)]^\alpha - (X_2+\cdots+X_N)^\alpha$$

$$= X_1^\alpha + \sum_{\substack{i_1 + i_2 = \alpha; \\ i_1 \neq 0, \alpha}} \begin{pmatrix} \alpha \\ i_1, i_2 \end{pmatrix} X_1^{i_1}(X_2 + \cdots + X_N)^{i_2},$$

we have

$$\begin{pmatrix} (X_2 + X_3 + \cdots + X_N)^\alpha \\ (X_1 + X_3 + \cdots + X_N)^\alpha \\ \cdots \\ (X_1 + X_2 + \cdots + X_{N-1})^\alpha \end{pmatrix} = (X_1 + \cdots + X_N)^\alpha \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ \vdots \\ X_N^\alpha \end{pmatrix}$$

$$- \sum_{\substack{i_1 + i_2 = \alpha; \\ i_1 \neq 0, \alpha}} \begin{pmatrix} \alpha \\ i_1, i_2 \end{pmatrix} \begin{pmatrix} X_1^{i_1}(X_2 + \cdots + X_N)^{i_2} \\ X_2^{i_1}(X_1 + \cdots + X_N)^{i_2} \\ \cdots \\ X_N^{i_1}(X_1 + \cdots + X_{N-1})^{i_2} \end{pmatrix}$$

This means that for each $\alpha_1, \ldots, \alpha_N$ we can write

$$\begin{pmatrix} x_1^{\alpha_1}(x_2 + x_3 \cdots + x_N)^{\alpha_2} \cdots (x_2^{N-1} + x_3^{N-1} + \cdots + x_N^{N-1})^{\alpha_N} \\ x_2^{\alpha_1}(x_1 + x_3 + \cdots + x_N)^{\alpha_2} \cdots (x_1^{N-1} + x_3^{N-1} + \cdots + x_N^{N-1})^{\alpha_N} \\ \cdots \\ x_N^{\alpha_1}(x_1 + x_2 + \cdots + x_{N-1})^{\alpha_2} \cdots (x_1^{N-1} + x_2^{N-1} + \cdots + x_{N-1}^{N-1})^{\alpha_N} \end{pmatrix}$$

as a sum of terms like

$$(x_1^k + \cdots + x_N^k)^\alpha \begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_N^i \end{pmatrix}$$

for integers $k > 0$ and $\alpha, i \geq 0$.

Now, as we know that

$$\phi(x) = (x - x_1) \cdots (x - x_N) = x^N - \sigma_1 x^{N-1} + \cdots + (-1)^N \sigma_N,$$

where $\sigma_1, \ldots, \sigma_N$ are the elementary symmetric polynomials, it follows that

$$x_i^N - \sigma_1 x_i^{N-1} + \cdots + (-1)^N \sigma_N = 0,$$

for $i = 1, \ldots, N$, and so

$$\begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_N^k \end{pmatrix} = \sigma_1 \begin{pmatrix} x_1^{k-1} \\ x_2^{k-1} \\ \vdots \\ x_N^{k-1} \end{pmatrix} - \cdots + (-1)^{N+1} \sigma_N \begin{pmatrix} x_1^{k-N} \\ x_2^{k-N} \\ \vdots \\ x_N^{k-N} \end{pmatrix}, \ k \geq N.$$

That is, the $\mathbf{S}_N$-equivariants functions

$$\begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_N^k \end{pmatrix}$$

for $k = 0, \ldots, N-1$, generate $\vec{\mathcal{E}}(\mathbf{S}_N)$ over $\mathcal{E}(\mathbf{S}_N)$. $\square$

Appendix C

Planforms with Maple

Maple is a symbolic mathematics program. Its facilities include three-dimensional graphic plottings. For more detailed information see for example [29]. We use Maple to plot three-dimensional graphics and plane grayscale plots. We include one program producing figure 5.7 (c) and (d).

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