Heteroclinic Cycles and Wreath Product Symmetries

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Abstract

We consider the existence and stability of heteroclinic cycles arising by local bifurcation in dynamical systems with wreath product symmetry $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$, where $\mathbf{Z}_2$ acts by ± 1 on $\mathbf{R}$ and $\mathcal{G}$ is a transitive subgroup of the permutation group $\mathbf{S}_N$ (thus $\mathcal{G}$ has degree N). The group Γ acts absolutely irreducibly on $\mathbf{R}^N$. We consider primary (codimension one) bifurcations from an equilibrium to heteroclinic cycles as real eigenvalues pass through zero. We relate the possibility of such cycles to the existence of non-gradient equivariant vector fields of cubic order. Using Hilbert series and the software package MAGMA we show that apart from the cyclic groups $\mathcal{G}$ (already studied by other authors) only five groups $\mathcal{G}$ of degree ≤ 7 are candidates for the existence of heteroclinic cycles. We establish the existence of certain types of heteroclinic cycle in these cases by making use of the concept of a subcycle. We also discuss edge cycles, and a generalisation of heteroclinic cycles which we call a heteroclinic web. We apply our methods to three examples.

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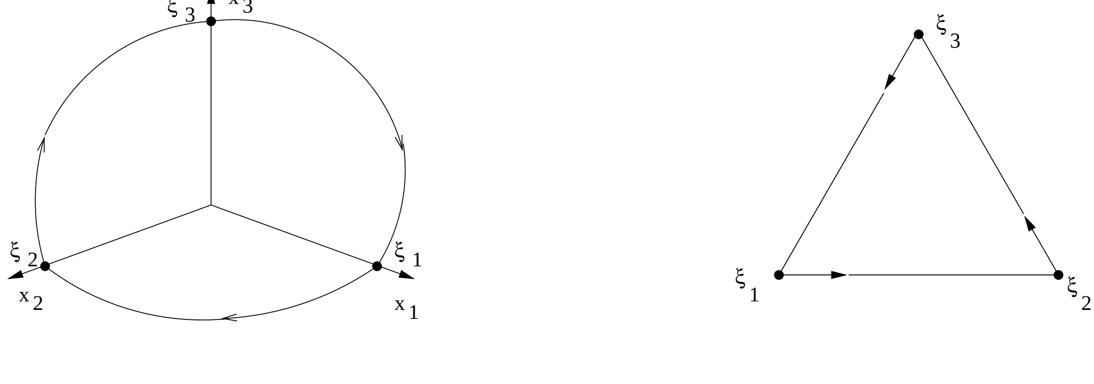
1 Introduction

A nonlinear dynamical system possesses a heteroclinic cycle if it has a series of equilibria that are *connected* in the sense that some trajectory links the unstable manifold of any given equilibrium to the stable manifold of the next (see Arrowsmith and Place [1] or any other advanced textbooks on Dynamical Systems). We refer to such a trajectory as a *(heteroclinic) connection* between the equilibria concerned. A connection limits on one equilibrium in forward time and on the other in backward time.

Heteroclinic cycles are especially common in symmetric dynamical systems: see Guckenheimer and Holmes [20], Krupa and Melbourne [25], Melbourne *et al.* [27]. Moreover, these may be robust in the sense that they persist under perturbations that respect the symmetry. Another issue about heteroclinic cycles is its stability. When for any neighborhood U of a heteroclinic cycle, there is always a smaller neighborhood V such that the trajectories starting in V remain in U for all forward time and are asymptotic to the cycle, then the cycle is said to be *asymptotically stable.* Asymptotically stable heteroclinic cycles are related to the occurrence of intermittency in applications: the system remains near one equilibrium for a significant time, then switches to the next, and so on indefinitely. The best known example of a robust and asymptotically stable heteroclinic cycle in a symmetric dynamical system is the cycle described by Guckenheimer and Holmes [20]. Their system is a simplified version of a model of rotating convection introduced by Busse and Heikes [5]. The system studied by Guckenheimer and Holmes has symmetry $\Gamma = T \oplus \mathbf{Z}_2$, where T is the tetrahedral group consisting of the orientation-preserving transformations of a tetrahedron. Equivalently, Γ is the semidirect product ${\mathbf{Z}_2}^3 \dot{+} \mathbf{Z}_3$, or in wreath product notation $\mathbf{Z}_2 \wr \mathbf{Z}_3$. See Figure 1, where (a) is taken from Krupa [24] and (b) from Field [15].

Although symmetry can force a heteroclinic cycle to be robust, it may also complicate the description of a cycle, because all symmetrically related equilibria should also be taken into account. For example, Figure 1(a) is only one octant of a more complex system of connections between the six equilibria $\pm\xi_1$, $\pm\xi_2$, $\pm\xi_3$. Krupa and Melbourne [25] introduce a precise definition of heteroclinic cycles in symmetric systems: in this definition group orbits of equilibria are connected to other group orbits of equilibria. They also prove a sufficient criterion for asymptotic stability of such a cycle, which we shall make use of in the sequel. See Section 2.3 for a summary of their results. The concept of a heteroclinic cycle has been extended by several authors to allow for more complicated connections between equilibria or other invariant sets. See for example Brannath [4], Kirk and Silber [23], Field and Richardson [16], Field [15] and Ashwin and Chossat [2]. More recently, Ashwin and Field [3] formalized the concept of *heteroclinic network.* This concept covers the known examples of heteroclinic behaviour, but it also includes cycles between invariant sets that are more complicated than equilibria or limit cycles: for example, cycles that connect chaotic sets (*cycling chaos*), and cycles that connect heteroclinic cycles (*cycling cycles*).

In this paper we restrict attention to systems with so-called 'wreath product' symmetry, see Golubitsky *et al.* [18], Dionne *et al.* [12] and Dias and Stewart [8, 9, 10]. To motivate this choice, recall that the symmetry group ${\mathbf{Z}_2}^3 \dot{+} \mathbf{Z}_3$ of the system studied by Guckenheimer and Holmes is one of the simplest examples of a *wreath product* — a semidirect product of a number of copies of a given group by a group of permutations that permutes those copies. In wreath product notation it is $\mathbf{Z}_2 \wr \mathbf{Z}_3$. The wreath product has very well-behaved algebraic



(a) (b)

Figure 1: (a) Heteroclinic cycle in a system with $\mathbf{Z}_2 \wr \mathbf{Z}_3$-symmetry. (b) Schematic picture.

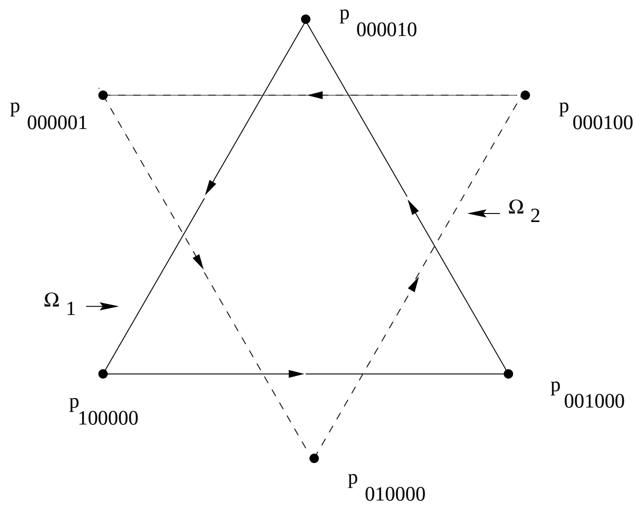


Figure 2: Two coexisting heteroclinic cycles in a system with $\mathbf{Z}_2 \wr \mathbf{Z}_6$-symmetry (schematic picture).

properties, a fact that has persuaded several authors — ourselves included — to study wreath products in the hope of finding new examples of robust and stable heteroclinic cycles. For instance, Field and Swift [17] prove the existence of asymptotically stable heteroclinic cycles in systems with $\mathbf{Z}_2 \wr \mathbf{Z_4}$ symmetry. One of the main references for the study of heteroclinic cycles in wreath product systems is Field [15], where it is shown that heteroclinic cycles in systems with $\mathbf{Z}_2 \wr \mathbf{Z}_N$ symmetry can be asymptotically stable. In particular, Field [15] considers systems with $\mathbf{Z}_2 \wr \mathbf{Z}_N$ symmetry where $N = 5$, 7, 11 or $N = pk$ for $p \geq 3$ and $k \geq 2$. See Figure 2 (from Field [15]) and Field and Richardson [16].

All of the symmetry groups mentioned so far are of the type $\mathbf{Z}_2 \wr \mathbf{Z}_N$, and heteroclinic cycles are common in this particular class of systems. In this paper we study local bifurcations from equilibria to robust heteroclinic cycles in a more general (but still special) class of wreath product systems: those for which the symmetry group has the form $\mathbf{Z}_2 \wr \mathcal{G}$, where $\mathcal{G}$ is a transitive subgroup of the permutation group $\mathbf{S}_N$. In this case, we say that the *degree* of $\mathcal{G}$

is N. This choice is motivated by the occurrence of wreath product symmetry in coupled cell systems, discussed briefly later in this introduction. There the group $\mathbf{Z}_2$ represents the 'internal' symmetry of a cell and $\mathcal{G}$ the 'global' symmetry of the coupled cell network. We select $\mathbf{Z}_2$ because it corresponds to the simplest nontrivial case, in which each cell has a 1-dimensional dynamic. Also, note that the proof of existence of heteroclinic cycles is difficult when the unstable manifolds of the equilibria involved are high-dimensional. Transitivity of $\mathcal{G}$ may be assumed without loss of generality and simplifies the analysis. (Roughly speaking, if $\mathcal{G}$ is intransitive then we can decompose its permutation representation into transitive components and treat each component separately.)

We focus on two main issues: existence of heteroclinic cycles in some appropriate sense, not necessarily that of Krupa and Melbourne [25]; and their asymptotic stability. We show that the choice of the permutation group $\mathcal{G}$ is crucial to the existence and stability of heteroclinic cycles in systems with symmetry $\mathbf{Z}_2 \wr \mathcal{G}$. We consider here dynamics of equivariant systems under groups $\Gamma = {\mathbf{Z}_2}^N \dot{+} \mathcal{G}$, with action defined on $\mathbf{R}^N$, and where $\mathcal{G}$ is a transitive subgroup of the permutation group $\mathbf{S}_N$, that is, $\mathcal{G}$ has degree N. In wreath product notation $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$. Indeed we show that — at least up to degree 7 — for most non-cyclic transitive groups $\mathcal{G}$, all equivariant vector fields have the same local branching pattern as their cubic truncations, and these truncations are *gradient*. This rules out any recurrent behaviour except equilibria, and in particular it rules out heteroclinic cycles arising by local bifurcation.

We do, however, find exactly five non-cyclic transitive groups of degree ≤ 7 with non-gradient dynamics, which are thus sensible candidates for the occurrence of heteroclinic cycles.

The gradient property arises in the following manner. Field [15] has introduced the concept of *k-determinacy* of a group, which roughly speaking means that generically all interesting local bifurcation phenomena, for individual vector fields with one bifurcation parameter, are already present in the truncation of the Taylor series of the vector field to order k. Moreover, Field [15] proves that (for the representations we are considering here) the groups $\mathbf{Z}_2 \wr \mathcal{G}$ are 3-determined. Thus, when seeking heteroclinic cycles, we may restrict attention to $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant vector fields truncated at cubic order. For most transitive groups $\mathcal{G}$ (up to degree 7) we shall prove that all cubic order truncated vector fields of $\mathbf{Z}_2 \wr \mathcal{G}$ are gradient, so local bifurcation to heteroclinic cycles cannot occur for these groups.

For the groups that we identify as possessing non-gradient truncated vector fields, we prove that heteroclinic cycles of the type observed by Guckenheimer and Holmes can occur. However, they are always related to a cyclic subgroup of $\mathcal{G}$. We also predict the occurrence of more complicated 'heteroclinic cycles', but in general these cycles are not of the type defined by Krupa and Melbourne [25] in which equilibria are connected only to equilibria in the same group orbit. Instead, each equilibrium in the 'cycle' can be connected to equilibria that need not be in the same group orbit. We shall call such a configuration a *heteroclinic web*. See Section 6 for an example. Heteroclinic webs pose new questions — for example, their existence is a more difficult issue. The stability criteria of Krupa and Melbourne [25] does not directly apply to them, but it is easily adapted by Melbourne [26] — see Section 6. We discuss these matters in connection with Example 6.2, where $\mathcal{G}$ is the alternating group $\mathbf{A}_4$ of degree 4 and order 12, but in a permutation representation of degree 6.

In closing this introduction, we remark that systems of differential equations with wreath product symmetry have been studied previously, see Golubitsky *et al.* [18], Dionne *et al.* [12] and Dias and Stewart [8, 9, 10], especially in the context of symmetric networks of coupled cells ('oscillators') in which each cell has its own symmetry. Our results are therefore of interest for coupled cell systems. The overall symmetry group Γ of such a system depends on the group of *local* (or *internal*) symmetries $\mathcal{L}$ of an individual cell, and on the *global* group $\mathcal{G}$ of permutations of the cells that preserve the network of couplings. Thus the symmetry of a system of ODEs modelling coupled identical cells is determined by the symmetries of the cells and the symmetries of the couplings between cells. One natural type of coupling gives rise to systems with wreath product symmetry: this is the case where the coupling is independent of internal symmetries. In the case under discussion, $\Gamma = \mathcal{L} \wr \mathcal{G}$. See Golubitsky *et al.* [18] for examples of nonlinear systems with wreath product symmetry.

This paper is organised as follows. In Section 2 we review some basic concepts of equivariant bifurcation theory, wreath products, and heteroclinic cycles. In Section 3 we present a method for determining whether a $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant vector field is gradient. The method is based on Hilbert series for rings of invariants (see for example the survey in Worfolk [30], or Dias and Stewart [11]) and leads to the criterion of Lemma 3.7.

This lemma leads to the main results of the paper, which are as follows. In Theorem 3.9 we apply our method to the groups $\mathbf{Z}_2 \wr \mathcal{G}$, where $\mathcal{G}$ is any transitive permutation group of degree ≤ 7. We find that apart from cyclic groups, only five groups $\mathcal{G}$ are such that $\mathbf{Z}_2 \wr \mathcal{G}$ possesses non-gradient equivariant vector fields of cubic order.

In Section 4 we define *subcycles*: these are heteroclinic cycles associated with invariant subspaces that arise from a block partition determined by $\mathcal{G}$. In Theorem 4.10 we prove that the asymptotic stability of a subcycle (relative to the restricted dynamic on the invariant subspace) implies the existence and (for an open set of parameter values) the asymptotic stability of a heteroclinic cycle for the original problem. We illustrate this result with two examples in Section 4.3.

In Section 5 we characterise the groups $\mathcal{G}$ that permit the existence of cycles of the type occurring in the system studied by Guckenheimer and Holmes. Following [15] we call these *edge cycles*. Our main result in this context is Theorem 5.3.

Finally, in Section 6 we give a typical example of a stability calculation for a heteroclinic web, showing how to exploit the geometry of wreath product representations in connection with criterions for stability like the one developed in Krupa and Melbourne [25].

2 Background

In this section we review some concepts related to equivariant bifurcation theory, and introduce the main results that we require about wreath product systems. We also present the definition of heteroclinic cycles from Krupa and Melbourne [25], and state their criterion for asymptotic stability.

2.1 Equivariant Bifurcation Theory

For a detailed discussion of the basics of equivariant bifurcation theory see Golubitsky *et al.* [19]. We summarise a few key points.

Consider a system of differential equations

$$\dot{x} = F(x, \lambda), \tag{2.1}$$

where $x \in V^N$ with $V \equiv \mathbf{R}^k$, the vector field $F : V^N \times \mathbf{R} \to V^N$ is smooth, and $\lambda \in \mathbf{R}$ is a bifurcation parameter. Recall that F *commutes* with a (compact Lie) group Γ (or F is Γ-*equivariant*) if

$$F(\gamma \cdot x, \lambda) = \gamma \cdot F(x, \lambda)$$

for all $\gamma \in \Gamma$ and $x \in V^N$. Henceforth we assume F to be Γ-equivariant. The group

$$\Sigma_x = \{\gamma \in \Gamma \ : \ \gamma \cdot x = x\} \subseteq \Gamma$$

is the *isotropy subgroup* of $x \in V^N$. The *fixed-point space* of a subgroup $\Sigma \subseteq \Gamma$ is the subspace of V^N defined by

$$\mathrm{Fix}(\Sigma) = \{x \in V^N \ : \ \gamma \cdot x = x, \ \forall\, \gamma \in \Sigma\}.$$

For any Γ-equivariant mapping F and any subgroup $\Sigma \subseteq \Gamma$ we have

$$F(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma).$$

An isotropy subgroup of Γ is *axial* if it has a 1-dimensional fixed-point space. An equilibrium with axial isotropy is called an *axial equilibrium.*

A subspace $W \subseteq V^N$ is *absolutely irreducible* for Γ if the space W is irreducible for Γ and the only matrices commuting with the action of Γ on W are the scalar multiples of the identity.

We assume the following standard hypotheses for (2.1):

Hypotheses 2.1

1. $F(0,0) = 0$ and $(\mathrm{D}_x F)_{0,0} = 0$, that is, $(x, \lambda) = (0,0)$ is a bifurcation point.

2. The space $V^N = \ker(\mathrm{D}_x F)_{0,0}$ is absolutely irreducible for Γ. Thus

   $$(\mathrm{D}_x F)_{0,\lambda} = c(\lambda)\mathrm{Id}_{Nk \times Nk}$$

   and $c(0) = 0$. (This hypothesis is generically true by Golubitsky *et al.* [19] Proposition XIII 3.2.)

3. The stability of the origin changes as λ crosses zero, say $c'(0) > 0$.

4. The action of Γ on V^N is nontrivial and so $\mathrm{Fix}_{V^N}(\Gamma) = \{0\}$. Thus $F(0, \lambda) \equiv 0$.

5. We are interested in the behaviour of F near the bifurcation point. Thus we may assume without loss of generality that $c(\lambda) = \lambda$.

When the above hypotheses hold, steady-state bifurcation from a trivial equilibrium to equilibria with symmetry given by axial subgroups is guaranteed by the Equivariant Branching Lemma of Vanderbauwhede and Cicogna, see Golubitsky *et al.* [19] Theorem XXIII 3.3.

2.2 Wreath Products

Let $\mathcal{L}$ be a closed subgroup of $\mathbf{O}(k)$ acting on V, and let $\mathcal{G}$ be a transitive subgroup of the *symmetric group* $\mathbf{S}_N$ which consists of all permutations of $\{1, ..., N\}$. (Recall that $\mathcal{G}$ is *transitive*, or *acts transitively on* $\Lambda = \{1, ..., N\}$, if it has only one orbit — namely $\mathcal{G}(i) = \Lambda$ for all $i \in \Lambda$. Here, $\mathcal{G}(i) = \{\sigma(i) : \sigma \in \mathcal{G}\}$. Equivalently, $\mathcal{G}$ is transitive if for every pair of points $i, j \in \Lambda$ there exists $\sigma \in \mathcal{G}$ such that $\sigma(i) = j$.)

We consider the action of $\mathcal{L} \wr \mathcal{G}$ on $W = V^N$ given by

$$(l, \sigma).(x_1, \ldots, x_N) = (l_1 \cdot x_{\sigma^{-1}(1)}, \ldots, l_N \cdot x_{\sigma^{-1}(N)}) \tag{2.2}$$

where $l = (l_1, \ldots, l_N) \in \mathcal{L}^N$ and $\sigma \in \mathcal{G}$. A permutation $\sigma \in \mathcal{G}$ acts on $l \in \mathcal{L}^N$ by

$$\sigma(l) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(N)}).$$

It follows that the group multiplication in $\mathcal{L} \wr \mathcal{G}$ is given by

$$(h, \tau)(l, \sigma) = (h\tau(l), \tau\sigma).$$

Throughout this work we study some aspects of the existence and stability of heteroclinic cycles for (2.1) when F commutes with the above action of $\Gamma \equiv \mathbf{Z}_2 \wr \mathcal{G}$. We consider here only the case $V = \mathbf{R}$, where the action of $\mathbf{Z}_2$ on V is multiplication by ± 1. If $V = \mathbf{R}$ and $\mathcal{L} = \mathbf{Z}_2$ then V is absolutely irreducible for $\mathbf{Z}_2$ and V^N is absolutely irreducible for $\mathbf{Z}_2 \wr \mathcal{G}$ (when $\mathcal{G}$ is transitive).

More generally, Dionne *et al.* [12] (Lemma 3.1 and Lemma 3.2) have proved:

Proposition 2.2 *Suppose that $\mathcal{L} \wr \mathcal{G}$ acts on W, such that $\mathcal{L}^N$ acts nontrivially on W and $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$. Then W is absolutely irreducible for $\mathcal{L} \wr \mathcal{G}$ if and only if $W \cong V^N$ where V is an absolutely irreducible representation of $\mathcal{L}$.* ∎

Without loss of generality we may assume that $\mathcal{L}$ acts absolutely irreducibly on V, and $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$. (The justification of this statement involves a possible centre manifold or Liapunov-Schmidt reduction: see Golubitsky *et al.* [19].) We assume these hypotheses from now on unless otherwise stated, but occasionally repeat them for clarity.

We complete this section with a description of the axial subgroups of Γ. A subset $J \subseteq \{1, \ldots, N\}$ is called a *block* if there is a subgroup $\mathcal{H}$ of $\mathcal{G}$ that leaves J invariant and acts transitively on it. As in [12], for a block $J \subseteq \{1, \ldots, N\}$ and an axial subgroup $A \subseteq \mathcal{L}$ acting on V, define the subgroup

$$\Sigma(A, J) \equiv (B_1 \times \cdots \times B_N) \dot{+} \mathcal{G}_J$$

of $\mathcal{L} \wr \mathcal{G}$, where

$$B_j = \begin{cases} A & \text{if } j \in J \\ \mathcal{L} & \text{if } j \notin J \end{cases}$$

and

$$\mathcal{G}_J = \{\sigma \in \mathcal{G} : \sigma(J) = J\}.$$

Lemma 4.1, Lemma 4.2 and Proposition 4.3. of Dionne *et al.* [12] imply:

Theorem 2.3 *An isotropy subgroup Σ of $\mathcal{L} \wr \mathcal{G}$ acting (absolutely irreducibly) on V^N is axial if and only if it is conjugate to an (axial) group of the type $\Sigma(A, J)$ for some block $J \subseteq \{1, \ldots, N\}$ and an axial group A of $\mathcal{L}$ acting on V.* ∎

When $V = \mathbf{R}$ and $\mathcal{L} = \mathbf{Z}_2$, then up to conjugacy the axial groups of $\mathbf{Z}_2 \wr \mathcal{G}$ are the groups $\Sigma(\{\mathbf{1}\}, J)$ for blocks $J \subseteq \{1, \ldots, N\}$ of $\mathcal{G}$.

2.3 Heteroclinic Cycles

We are interested in one-parameter systems (2.1), for which a branch of heteroclinic cycles between equilibria bifurcates (at $\lambda = 0$ say) from the trivial equilibrium, where some of the equilibria involved are those implied by the Equivariant Branching Lemma — that is, they are axial. By Theorem 2.3, such equilibria have isotropy subgroup Σ conjugate to a groups of the form $\Sigma(A, J)$, where $A \subseteq \mathcal{L}$ is axial and J is a block for $\mathcal{G}$, if $\Gamma = \mathcal{L} \wr \mathcal{G}$.

Symmetry forces the existence of flow-invariant subspaces: in particular, fixed-point s-paces. Moreover, if $S \subseteq V^N$ is a flow-invariant set, then so is the set γS for any $\gamma \in \Gamma$. Also, the dynamics on S and γS are similar. We say that two subsets S_1 and S_2 of V^N are *conjugate* if $S_1 = \gamma S_2$ for some $\gamma \in \Gamma$. Following Krupa and Melbourne [25], we can now define a heteroclinic cycle as follows:

Definition 2.4 Suppose that ξ_j, $j = 1, \ldots, m$ are hyperbolic equilibria of (2.1) with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$. Then the set of group orbits of the unstable manifolds

$$\Omega = \{W^u(\gamma\xi_j) : \ j = 1, \ldots, m, \ \gamma \in \Gamma\}$$

forms a *heteroclinic cycle* provided $\dim W^u(\xi_j) \geq 1$ and

$$W^u(\xi_j) \setminus \{\xi_j\} \subseteq \cup_{\gamma \in \Gamma} W^s(\gamma\xi_{j+1}). \tag{2.3}$$

Here ξ_{m+1} denotes ξ_1 and $\setminus$ is set-theoretic difference. In other words, all trajectories that leave ξ_j along its unstable manifold are captured by the stable manifold of some conjugate of ξ_{j+1}.

Definition 2.5

1. A *heteroclinic cycle* Ω is said to be (Liapunov) *stable* if for any neighbourhood U of Ω there exists a neighbourhood $N \subseteq U$ of Ω such that all trajectories starting in N remain in U for all forward time.

2. A *heteroclinic cycle* Ω is said to be *asymptotically stable* if Ω is stable and there exists a neighbourhood N of Ω such that all trajectories starting in N converge to Ω in forward time.

3. If a cycle is not stable, then it is *unstable.*

4. When $m = 1$, the cycle Ω is called a *homoclinic cycle.*

Following Krupa and Melbourne [25], suppose that the cycle Ω satisfies the following hypothesis:

Hypothesis 2.6 For each j, there is a flow-invariant subspace P_j such that $W^u(\xi_j) \subseteq P_j$ and ξ_{j+1} is a sink in P_j.

We define the following quantities related to the eigenvalues of $(\mathrm{D}_x F)_{\xi_j}$:

Definition 2.7

1. $-r_j$ is the maximum real part of the eigenvalues of $(\mathrm{D}_x F)_{\xi_j}$ restricted to $L_j = P_j \cap P_{j-1}$. The corresponding eigenvalue is called the weakest *radial* eigenvalue.

2. $-c_j$ is the maximum real part of the remaining eigenvalues in $P_{j-1} \setminus L_j$. The corresponding eigenvalue is called the weakest *contracting* eigenvalue.

3. $e_j > 0$ is the maximum real part of the eigenvalues of $(\mathrm{D}_x F)_{\xi_j}$. The corresponding eigenvalue is called the strongest *expanding* eigenvalue.

4. $t_j < 0$ is the maximum real part of the eigenvalues of $(\mathrm{D}_x F)_{\xi_j}$ with eigenvectors transverse to $P_{j-1} + P_j$. The corresponding eigenvalue is called the weakest *transverse* eigenvalue. If $P_{j-1} + P_j = V^N$ then by convention $t_j = -\infty$.

We may now state the stability criterion presented by Krupa and Melbourne [25]:

Theorem 2.8 *Suppose that Ω is a heteroclinic cycle of (2.1) satisfying Hypothesis 2.6. Then Ω is asymptotically stable if*

$$\prod_{j=1}^{m} \min(c_j, e_j - t_j) > \prod_{j=1}^{m} e_j. \tag{2.4}$$

$\blacksquare$

Condition (2.4) is sufficient but may not be necessary.

3 $\mathbf{Z}_2 \wr \mathcal{G}$-Equivariant Vector Fields

In this section we consider bifurcation problems with $\mathbf{Z}_2 \wr \mathcal{G}$ symmetry acting on $\mathbf{R}^N$, where $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$. There exists a classification (up to permutation isomorphism) of all transitive permutation groups $\mathcal{G}$ up to degree 22 (that is, $N \leq 22$). We use here the classification up to degree 15 given in Conway *et al.* [7]. It is well known that if F is a gradient vector field, then (2.1) has no heteroclinic cycle, nor any other kind of recurrent behaviour. So one way to determine whether a Γ-equivariant system (2.1) admits heteroclinic cycles is to determine whether the relevant dynamics is gradient. It is highly unusual for *all* Γ-equivariant vector fields to be gradient, yet this does happen when the orbit space of Γ is 1-dimensional, for example $\mathbf{O}(2)$ in its standard representation. However it is not unusual for all Γ-equivariant vector fields of low degree to be gradient. There is a possibility of relating a vector field to some truncated Taylor series, and here a theorem of Field [15] becomes very useful. We are considering local bifurcation from a trivial invariant equilibrium, and so the branching pattern of such bifurcation is fundamental. By *branching pattern* we mean the branches of solutions (classified by their symmetries or isotropy) that generically may be expected to bifurcate, as well as their directions of branching and their asymptotic stabilities. In Theorem 4.29 of [15] Field proves that the groups $\mathbf{Z}_2 \wr \mathcal{G}$, where $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, are 3-*determined*. Roughly speaking, this means that if the Taylor expansion of the bifurcation problem is known to order three, then with certain nondegeneracy conditions involving low-order terms of this expansion (Hypotheses 2.1), the terms of degree greater than three do not affect the branching pattern. When studying local bifurcation to heteroclinic cycles in $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant systems, we may therefore restrict our attention to generic $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant vector fields of degree ≤ 3. In section 3.1 we develop a computational technique for deciding whether all equivariants of degree three are gradient. In Section 3.2 we apply this technique to the groups $\mathbf{Z}_2 \wr \mathcal{G}$ where $\mathcal{G}$ is a transitive permutation group of degree ≤ 7. The methods apply with no extra difficulty to groups of higher degree: we stopped at degree 7 because several interesting questions of existence and stability are already present at that stage.

3.1 Gradient Vector Fields

Let Γ be a compact Lie group acting linearly and orthogonally on $V \cong \mathbf{R}^N$ and assume that $\mathrm{Fix}_V(\Gamma) = \{0\}$. Note that this condition holds if $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ in the representation (2.2). Let $\mathcal{P}_V(\Gamma)$ denote the ring of Γ-invariant polynomials, and let $\mathcal{P}_V^d(\Gamma)$ denote the vector space of homogeneous Γ-invariant polynomials of degree d. Similarly, let $\vec{\mathcal{P}}_V(\Gamma)$ denote the module over $\mathcal{P}_V(\Gamma)$ of all Γ-equivariant polynomial mappings from V to V, and let $\vec{\mathcal{P}}_V^d(\Gamma)$ denote the vector space of homogeneous Γ-equivariant polynomial mappings of degree d.

Definition 3.1 A vector field $f : V \to V$ in $\vec{\mathcal{P}}_V(\Gamma)$ is *gradient* if it is the gradient of a Γ-invariant function $I : V \to \mathbf{R}$.

For $d \geq 0$, define a map ∇ by

$$\nabla : \mathcal{P}_V^{d+1}(\Gamma) \to \vec{\mathcal{P}}_V^d(\Gamma)$$
$$I \mapsto \nabla I = \left(\frac{\partial I}{\partial x_1}, \ldots, \frac{\partial I}{\partial x_N} \right).$$

The function ∇ is $\mathbf{R}$-linear. If $\nabla I = 0$ for some $I \in \mathcal{P}_V^{d+1}(\Gamma)$ then I must be constant. Therefore $\mathcal{P}_V^{d+1}(\Gamma) \cap \ker(\Delta) = \{0\}$ for $d \geq 0$ and so ∇ is injective for $d \geq 0$.

Remark 3.2 If $d \geq 0$ then clearly

$$\dim \mathcal{P}_V^{d+1}(\Gamma) \leq \dim \vec{\mathcal{P}}_V^d(\Gamma).$$

Therefore every $f \in \vec{\mathcal{P}}_V^d(\Gamma)$ is gradient if and only if

$$\dim \mathcal{P}_V^{d+1}(\Gamma) = \dim \vec{\mathcal{P}}_V^d(\Gamma).$$

We now return to the wreath product groups $\mathbf{Z}_2 \wr \mathcal{G}$ where $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$. Let $\mathcal{G}_1$ denote the subgroup of $\mathcal{G}$ fixing 1 (that is, the *stabiliser* of 1 in $\mathcal{G}$).

Lemma 3.3 *Suppose that $\mathcal{G}$ is a subgroup of $\mathbf{S}_N$, $N \geq 1$, and that $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ acts on $V = \mathbf{R}^N$ as in (2.2). Then*

(a) $\mathcal{P}_V(\Gamma) \cong \mathcal{P}_V(\mathcal{G})$.

(b) *If $\mathcal{G}$ acts transitively on $\{1, 2, \ldots, N\}$ then $\vec{\mathcal{P}}_V(\Gamma) \cong \mathcal{P}_V(\mathcal{G}_1)$.*

Here $\cong$ indicates isomorphism of vector spaces.

Proof: We use Lemma 2.30 of Field [15], where it is proved that:

1. A function $q : V \to \mathbf{R}$ is Γ-invariant if and only if

$$q(x_1, \ldots, x_N) = p(x_1^2, \ldots, x_N^2) \tag{3.5}$$

   for some $\mathcal{G}$-invariant $p : V \to \mathbf{R}$.

2. If $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, then $f : V \to V$ is Γ-equivariant if and only if

$$f(x_1, \ldots, x_N) = \Theta_p(x_1, \ldots, x_N) = \begin{pmatrix} p(x_1^2, \ldots, x_N^2)x_1 \\ p(\sigma_2(x_1^2, \ldots, x_N^2))x_2 \\ \vdots \\ p(\sigma_N(x_1^2, \ldots, x_N^2))x_N \end{pmatrix} \tag{3.6}$$

   for a $\mathcal{G}_1$-invariant function $p : V \to \mathbf{R}$ and permutations $\sigma_i \in \mathcal{G}$ such that $\sigma_i^{-1}(1) = i$ for $i = 2, 3, \ldots, N$.

   The map Θ given by $p \mapsto \Theta_p$ is a linear isomorphism between the Γ-equivariants of degree $2d+1$ and the $\mathcal{G}_1$-invariants of degree d. Moreover, Θ is independent of the choice of permutations σ_i: indeed if σ_i and $\tilde{\sigma}_i$ are permutations such that $\sigma_i^{-1}(1) = i$ and $\tilde{\sigma}_i^{-1}(1) = i$ respectively, then $\sigma_i \tilde{\sigma}_i^{-1} \in \mathcal{G}_1$. Thus $\sigma_i = \eta_i \tilde{\sigma}_i$ for some $\eta_i \in \mathcal{G}_1$. Since p is $\mathcal{G}_1$-invariant, $p \circ \sigma_i = p \circ \tilde{\sigma}_i$.

Information about invariant theory of wreath products $\mathcal{L} \wr \mathcal{G}$ can be found in Field [15] and Dias and Stewart [11].

Using Remark 3.2 and Lemma 3.3, we can now characterise $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant gradient vector fields. We suppose throughout that $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ acts on $V = \mathbf{R}^N$, where $\mathbf{Z}_2$ acts on $\mathbf{R}$ by ± 1 and $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$.

Lemma 3.4 *Suppose that $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ acts on $V = \mathbf{R}^N$. For $d \geq 0$, all $f \in \vec{\mathcal{P}}_V^{2d+1}(\Gamma)$ are gradient if and only if*

$$\dim \mathcal{P}_V^d(\mathcal{G}_1) = \dim \mathcal{P}_V^{d+1}(\mathcal{G}).$$

Proof: By Remark 3.2, $f \in \vec{\mathcal{P}}_V^{2d+1}(\Gamma)$ is gradient if and only if

$$\dim\ \mathcal{P}_V^{2d+2}(\Gamma) = \dim\ \vec{\mathcal{P}}_V^{2d+1}(\Gamma).$$

Moreover, by Lemma 3.3, $f : V \to V$ is Γ-equivariant of degree $2d+1$ if and only if $f = \Theta_p$ where p is $\mathcal{G}_1$-invariant of degree d. Thus the Γ-equivariant mappings with polynomial components of degree $2d+1$ are in bijective correspondence with the $\mathcal{G}_1$-invariant polynomials of degree d, so

$$\dim\ \mathcal{P}_V^d(\mathcal{G}_1) = \dim\ \vec{\mathcal{P}}_V^{2d+1}(\Gamma). \tag{3.7}$$

Again Lemma 3.3 implies that $p : V \to \mathbf{R}$ is $\mathcal{G}$-invariant of degree $d+1$ if and only if $q : V \to \mathbf{R}$, defined by $q(x_1, \ldots, x_N) \equiv p(x_1^2, \ldots, x_N^2)$, is $\mathbf{Z}_2 \wr \mathcal{G}$-invariant of degree $2d+2$. Therefore

$$\dim\ \mathcal{P}_V^{2d+2}(\Gamma) = \dim\ \mathcal{P}_V^{d+1}(\mathcal{G}).$$

∎

We focus on the occurrence of branches of heteroclinic cycles between equilibria for (2.1), where F commutes with $\mathbf{Z}_2 \wr \mathcal{G}$ and satisfies Hypotheses 2.1. The calculation of the symmetry-breaking branches of equilibria for (2.1), their symmetries (isotropy subgroups) and stabilities is a central feature of this paper. Field [15] proves that bifurcation problems like (2.1) where $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ are 3-determined.

We therefore restrict attention to the cubic truncation of the Taylor series of the system. Since the action of $\mathbf{Z}_2 \wr \mathcal{G}$ on $\mathbf{R}^N$ is absolutely irreducible, all linear equivariants for $\mathbf{Z}_2 \wr \mathcal{G}$ are scalar multiples of the identity. Moreover, there are no invariants of degree 3, and the invariants of degree 2 are real multiples of $x_1^2 + \cdots + x_N^2$. Thus, in order to determine whether a $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant vector field with polynomial components of degree ≤ 3 is gradient, it suffices to check the $\mathbf{Z}_2 \wr \mathcal{G}$-equivariants with homogeneous components of degree 3.

Lemma 3.4 provides a criterion for determining gradient vector fields when $F \in \vec{\mathcal{P}}_V^{2d+1}(\Gamma)$, for $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$, by comparing the dimensions of $\mathcal{P}_V^d(\mathcal{G}_1)$ and $\mathcal{P}_V^{d+1}(\mathcal{G})$. Information about the structure of these spaces can be encoded in their Hilbert series, which are generating functions for the dimensions of the polynomial invariants of given degree, see Worfolk [30], Dias and Stewart [11]. Moreover, Molien's Theorem (originally proved by Molien in [28]) provides an explicit formula for these series. Before explaining how to exploit such information in the present context, we review Hilbert series for the rings of invariants of compact Lie groups.

3.1.1 Hilbert Series

Suppose that Γ is a compact Lie group acting linearly on $V \cong \mathbf{R}^N$. Let $x_1, x_2, \ldots, x_N$ be coordinates relative to a basis for V. Since the action of $\gamma \in \Gamma$ is given by a matrix M_γ with real entries, we can extend the action of Γ to $\mathbf{C}^N$ by using the same matrices but interpreting their entries as complex numbers. Moreover, the ring of polynomials $\mathbf{R}[x_1, \ldots, x_N]$ in N real variables is naturally included in the ring of complex polynomials $\mathbf{C}[x_1, \ldots, x_N]$, and a basis over $\mathbf{R}$ for the real vector space of Γ-invariant polynomials of degree d is also a basis over $\mathbf{C}$ for the complex vector space of Γ-invariant polynomials of degree d. Thus, the real and complex Hilbert series are the same.

Suppose that V is a N-dimensional vector space over $\mathbf{C}$ and let $x_1, x_2, \ldots, x_N$ denote coordinates relative to a basis for V. Let $\Gamma \subseteq \mathbf{GL}(V)$ be a compact Lie group acting on V. Denote by $\mathcal{P}_V(\Gamma)$ the subalgebra of $\mathbf{C}[x_1, \ldots, x_N]$ formed by the invariant polynomials under Γ. The polynomial ring $\mathbf{C}[x_1, \ldots, x_N]$ is graded:

$$\mathbf{C}[x_1, \ldots, x_N] = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

where R_i is the vector space of all homogeneous polynomials of degree i. Therefore the ring $\mathcal{P}_V(\Gamma)$ is also graded:

$$\mathcal{P}_V(\Gamma) = \mathcal{P}_V^0(\Gamma) \oplus \mathcal{P}_V^1(\Gamma) \oplus \mathcal{P}_V^2(\Gamma) \oplus \cdots$$

where $\mathcal{P}_V^i(\Gamma) = \mathcal{P}_V(\Gamma) \cap R_i$.

Definition 3.5 The *Hilbert series* or *Poincaré series* of the graded algebra $\mathcal{P}_V(\Gamma)$ is the generating function for dim $\mathcal{P}_V^d(\Gamma)$ for $d = 0, 1, \ldots$ More precisely, this generating function is

$$\Phi_\Gamma(t) = \sum_{d=0}^{\infty} \dim\left(\mathcal{P}_V^d(\Gamma)\right) t^d.$$

Molien's Theorem [28], (see also [29]) for a compact Lie group Γ, provides an explicit formula for Φ_Γ; namely,

$$\Phi_\Gamma(t) = \int_\Gamma \frac{1}{\det(\mathbf{1}_\Gamma - M_\gamma t)}$$

where $\int_\Gamma f$ denotes the integral with respect to normalised Haar measure on Γ (see for example Halmos [22]) of a continuous function $f : \Gamma \to \mathbf{R}$. In particular, if Γ is finite, then

$$\Phi_\Gamma(t) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{\det(\mathbf{1}_\Gamma - M_\gamma t)}, \qquad (3.8)$$

where $|\Gamma|$ denotes the order of Γ.

3.1.2 Hilbert Series and Gradient Vector Fields

We can now translate Lemma 3.4 into a statement about Hilbert series:

Lemma 3.6 *All $f \in \vec{\mathcal{P}}_V(\Gamma)$ are gradient if and only if*

$$\Phi_{\mathcal{G}}(t) = t\ \Phi_{\mathcal{G}_1}(t). \tag{3.9}$$

As stated earlier, such a strong property is highly unusual, but if we restrict attention to $f \in \vec{\mathcal{P}}_V^3(\Gamma)$ then condition (3.9) implies the following result (for the definition of 'cycle decomposition' see for example Hall [21]).

Lemma 3.7 *Suppose that $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ acts on $V = \mathbf{R}^N$. Then all $f \in \vec{\mathcal{P}}_V^3(\Gamma)$ are gradient if and only if*

$$\dim\, \mathcal{P}_V^1(\mathcal{G}_1) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \left[\frac{k_1(\sigma)(k_1(\sigma)+1)}{2} + k_2(\sigma) \right] \tag{3.10}$$

where $k_i(\sigma)$ denotes the number of cycles of length i in the cycle decomposition of σ for $i = 1$, 2, ... , N.

Proof: If $\sigma \in \mathcal{G}$ then

$$\det(I_{N\times N} - \sigma t) = \prod_{i=1}^{N} (1-t^i)^{k_i(\sigma)}.$$

Hence (3.8) becomes

$$\Phi_{\mathcal{G}}(t) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \frac{1}{\prod_{i=1}^{N} (1-t^i)^{k_i(\sigma)}}.$$

Therefore,

$$\begin{aligned}
\Phi_{\mathcal{G}}(t) &= \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \left(1 + k_1(\sigma)t + \frac{k_1(\sigma)(k_1(\sigma)+1)}{2} t^2 + \cdots \right) \left(1 + k_2(\sigma)t^2 + \cdots \right) \\
&\qquad \left(1 + k_3(\sigma)t^3 + \cdots \right) \cdots \left(1 + k_N(\sigma)t^N + \cdots \right) \\
&= \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \left[1 + k_1(\sigma)t + \left(\frac{k_1(\sigma)(k_1(\sigma)+1)}{2} + k_2(\sigma) \right) t^2 + \cdots \right] \\
&= 1 + t + \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \left[\frac{k_1(\sigma)(k_1(\sigma)+1)}{2} + k_2(\sigma) \right] t^2 + O(t^3).
\end{aligned}$$

Hence

$$\dim\, \mathcal{P}_V^2(\mathcal{G}) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \left[\frac{k_1(\sigma)(k_1(\sigma)+1)}{2} + k_2(\sigma) \right]$$

and the result follows (using Lemma 3.4). ∎

Remark 3.8 Clearly $\dim\, \mathcal{P}_V^1(\mathcal{G}_1)$ is equal to the number of $\mathcal{G}_1$-orbits on $\{1, ..., N\}$.

3.2 Gradient $\mathbf{Z}_2 \wr \mathcal{G}$-Equivariant Vector Fields for $\mathcal{G}$ of Degree ≤ 7

We now apply Lemma 3.7 to the groups $\mathbf{Z}_2 \wr \mathcal{G}$, where $\mathcal{G}$ is a transitive permutation group of degree $N \leq 7$. The list of transitive groups of degree less than or equal to 7 is given in Table 1 (see Conway *et al.* [7]). The notation is also from Conway *et al.*. Only five non-cyclic groups $\mathcal{G}$ are such that $\mathbf{Z}_2 \wr \mathcal{G}$ has non-gradient cubic equivariant vector fields. Only Γ-equivariant vector fields corresponding to these groups have a chance of yielding heteroclinic cycles by local bifurcation.

Theorem 3.9 *Let $\mathcal{G}$ be a transitive permutation group of degree $N \leq 7$. Let $F : \mathbf{R}^N \to \mathbf{R}^N$ be a polynomial vector field in $\vec{\mathcal{P}}^3_{\mathbf{R}^N}(\mathbf{Z}_2 \wr \mathcal{G})$. Then F is gradient unless (in the notation of Table 1) $\mathcal{G}$ is $\mathbf{C}(N)$ with $3 \leq N \leq 7$, $\mathbf{D}_6(6)$, $\mathbf{A}_4(6)$, $\mathbf{F}_{18}(6)$, $2\mathbf{A}_4(6)$ or $\mathbf{F}_{21}(7)$.*

Proof: By Lemma 3.7 all F in $\vec{\mathcal{P}}^3_{\mathbf{R}^N}(\mathbf{Z}_2 \wr \mathcal{G})$ are gradient if and only equation (3.10) is satisfied. For each group $\mathcal{G}$ listed in Table 1, we use MAGMA [6] to calculate and compare both entries of equation (3.10) which are listed in the columns headed LHS and RHS. Inspection of those columns yields the result. ∎

4 Subcycles in $\mathbf{Z}_2 \wr \mathcal{G}$-Equivariant Systems

Of course the existence of non-gradient dynamics at cubic order does not necessarily imply that local bifurcation to heteroclinic cycles can occur. Both existence and stability remain an issue. In this section we describe a method for proving the existence in $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant systems of a special type of heteroclinic cycle, which we call a subcycle. In this approach, we demonstrate the existence of stable heteroclinic cycles for a restricted vector field, in which the phase space is replaced by an invariant subspace. We also show that the existence of such a *subcycle* can be sufficient to prove the existence of a stable cycle, formed from the union of disjoint (conjugate) subcycles, for the original vector field. In fact, the 'block partition' determined by $\mathcal{G}$ defines suitable invariant subspaces. We begin by introducing the concept of *independent block* for $\mathcal{G}$. Independent blocks induce the decomposition of the phase space into invariant subspaces that make it possible to prove the occurrence of cycles formed from conjugate subcycles.

4.1 Independent Blocks

Let $\mathcal{G}$ be a permutation group acting on $\Lambda = \{1, 2, \ldots, N\}$ and let $J \subseteq \Lambda$.

For $\sigma \in \mathcal{G}$, let

$$\sigma J = \{\sigma(i) : \; i \in J\} \subseteq \Lambda.$$

Recall that the *orbit* of i under $\mathcal{G}$ is

$$\mathcal{G}(i) = \{\sigma(i) : \; \sigma \in \mathcal{G}\}$$

and the *stabiliser* of i in $\mathcal{G}$ is

$$\mathcal{G}_i = \{\sigma \in \mathcal{G} : \; \sigma(i) = i\}.$$

Degree	$\mathcal{G}$	$\lvert\mathcal{G}\rvert$	Generators	LHS	RHS
2	$\mathbf{S}_2$	2	$(1,2)$	2	2
3	$\mathbf{A}_3 = \mathbf{Z}_3$	3	$(1,2,3)$	3	2
	$\mathbf{S}_3$	6	$(1,3),\ (2,3)$	2	2
4	$\mathbf{C}(4) = \mathbf{Z}_4$	4	$(1,2,3,4)$	4	3
	$\mathbf{E}(4)$	4	$(1,4)(2,3),\ (1,2)(3,4)$	4	4
	$\mathbf{D}(4)$	8	$(1,2,3,4),\ (1,3)$	3	3
	$\mathbf{A}_4$	12	$(1,2,4),\ (2,3,4)$	2	2
	$\mathbf{S}_4$	24	$(1,4),\ (2,4),\ (3,4)$	2	2
5	$\mathbf{C}(5) = \mathbf{Z}_5$	5	$(1,2,3,4,5)$	5	3
	$\mathbf{D}(5)$	10	$(1,2,3,4,5),\ (1,4)(2,3)$	3	3
	$\mathbf{F}(5)$	20	$(1,2,3,4,5),\ (1,2,4,3)$	2	2
	$\mathbf{A}_5$	60	$(1,2,5),\ (2,3,5),\ (3,4,5)$	2	2
	$\mathbf{S}_5$	120	$(1,5),\ (2,5),\ (3,5),\ (4,5)$	2	2
6	$\mathbf{C}(6) = \mathbf{Z}_6$	6	$(1,2,3,4,5,6)$	6	4
	$\mathbf{D}_6(6)$	6	$(1,3,5)(2,4,6),\ (1,4)(2,3)(5,6)$	6	5
	$\mathbf{D}(6)$	12	$(1,2,3,4,5,6),\ (1,4)(2,3)(5,6)$	4	4
	$\mathbf{A}_4(6)$	12	$(1,4)(2,5),\ (1,3,5)(2,4,6)$	4	3
	$\mathbf{F}_{18}(6)$	18	$(2,4,6),\ (1,4)(2,5)(3,6)$	4	3
	$2\mathbf{A}_4(6)$	24	$(3,6),\ (1,3,5)(2,4,6)$	4	3
	$\mathbf{S}_4(6d)$	24	$(1,4)(2,5),\ (1,3,5)(2,4,6),\ (1,5)(2,4)$	3	3
	$\mathbf{S}_4(6c)$	24	$(1,4)(2,5),\ (1,3,5)(2,4,6),\ (1,5)(2,4)(3,6)$	3	3
	$\mathbf{F}_{18}(6):2$	36	$(2,4,6),\ (1,5)(2,4),\ (1,4)(2,5)(3,6)$	3	3
	$\mathbf{F}_{36}(6)$	36	$(2,4,6),\ (1,5)(2,4),\ (1,4,5,2)(3,6)$	3	3
	$2\mathbf{S}_4(6)$	48	$(3,6),\ (1,3,5)(2,4,6),\ (1,5)(2,4)$	3	3
	$\mathbf{L}(6)$	60	$(1,2,3,4,6),\ (1,4)(5,6)$	2	2
	$\mathbf{F}_{36}(6):2$	72	$(2,4,6),\ (2,4),\ (1,4)(2,5)(3,6)$	3	3
	$\mathbf{L}(6):2$	120	$(1,2,3,4,6),\ (1,6)(2,3)(4,5)$	2	2
	$\mathbf{A}_6$	360	$(1,2,6),\ (2,3,6),\ (3,4,6),\ (4,5,6)$	2	2
	$\mathbf{S}_6$	720	$(1,6),\ (2,6),\ (3,6),\ (4,6),\ (5,6)$	2	2
7	$\mathbf{C}(7) = \mathbf{Z}_7$	7	$(1,2,3,4,5,6,7)$	7	4
	$\mathbf{D}(7)$	14	$(1,2,3,4,5,6,7),\ (1,6)(2,5)(3,4)$	4	4
	$\mathbf{F}_{21}(7)$	21	$(1,2,3,4,5,6,7),\ (1,2,4)(3,6,5)$	3	2
	$\mathbf{F}_{42}(7)$	42	$(1,2,3,4,5,6,7),\ (1,3,2,6,4,5)$	2	2
	$\mathbf{L}(7)$	168	$(1,2,3,4,5,6,7),\ (1,2)(3,6)$	2	2
	$\mathbf{A}_7$	2520	$(1,2,7),\ (2,3,7),\ (3,4,7),\ (4,5,7),\ (5,6,7)$	2	2
	$\mathbf{S}_7$	5040	$(1,7),\ (2,7),\ (3,7),\ (4,7),\ (5,7),\ (6,7)$	2	2

Table 1: Classification of transitive permutation groups of degree ≤ 7 (up to permutation isomorphism), with degree, order and generators, from Conway *et al.* [7]. Columns headed LHS and RHS refer to the two sides of equation (3.10).

Remark 4.1 Suppose that $\mathcal{G}$ acts transitively on Λ. Then the set of stabilisers $\{\mathcal{G}_i\}_{i\in\Lambda}$ forms a single conjugacy class of subgroups of $\mathcal{G}$. See for example Dixon and Mortimer [13] Corollary 1.4A.

If $\mathcal{G}$ is transitive then the only transitive subgroup of $\mathcal{G}$ containing any $\mathcal{G}_i$ is $\mathcal{G}$ itself.

We now refine the definition of a block from Section 2.2.

Definition 4.2 Let $\mathcal{G}$ act transitively on Λ. Then a nonempty subset J of Λ is an *independent block* for $\mathcal{G}$ if for each $\sigma \in \mathcal{G}$ either $\sigma J = J$ or $\sigma J \cap J = \emptyset$.

For example, in the notation of Table 1, the subset $J = \{1,3,5\}$ of $\Lambda = \{1,2,\ldots,6\}$ is an independent block for $\mathbf{D}_6(6)$ and $\mathbf{F}_{18}(6)$. However, J is not an independent block for $\mathbf{A}_4(6)$.

Suppose that $\mathcal{G}$ is transitive on Λ. Clearly Λ and $\{i\}$, $1 \le i \le N$, are independent blocks. These are the *trivial independent blocks* and any other block is *nontrivial.* The intersection of independent blocks containing a common point is again an independent block.

Proposition 4.3 *Suppose that $\mathcal{G}$ acts transitively on Λ, that $J \subseteq \Lambda$ is an independent block for $\mathcal{G}$, and let $\Sigma_J = \{\sigma J : \ \sigma \in \mathcal{G}\}$. Then the sets in Σ_J form a partition of Λ, and each element of Σ_J is an independent block for $\mathcal{G}$.*

Proof: See Dixon and Mortimer [13] (Section 1.5). ∎

The set Σ_J in Proposition 4.3 is called the *system of independent blocks* containing J.

The *setwise stabiliser* of $J \subseteq \Lambda$ is the group

$$\mathcal{G}_J = \{\sigma \in \mathcal{G} : \ \sigma(J) = J\}.$$

If $\mathcal{G}$ is transitive on Λ and J is an independent block for $\mathcal{G}$ then $\mathcal{G}_J$ is transitive on J.

Remark 4.4 If $\mathcal{G}$ is transitive on Λ and J is an independent block for $\mathcal{G}$ with $i \in J$, then J is the union of some orbits of $\mathcal{G}_i$.

4.2 Subcycles: Existence and Stability

In this section we are interested in flow-invariant subspaces for (2.1) arising as fixed-point spaces of subgroups of $\mathbf{Z}_2 \wr \mathcal{G}$ determined by nontrivial independent blocks J. We show that when (2.1) is restricted to one of these subspaces, it defines a restricted ODE with $\mathbf{Z}_2 \wr \mathcal{G}_J$-symmetry, where $\mathcal{G}_J$ is the setwise stabiliser of J. In particular, for this restricted equivariant problem, heteroclinic cycles can occur and can be asymptotically stable. We call such cycles *subcycles* (Definition 4.7), generalising a concept introduced by Field [15]. We prove the occurrence of heteroclinic cycles for (2.1) formed by the disjoint union of (conjugate) subcycles. The stability of the cycle depends on the linear stability of the equilibria involved in the cycle.

We begin by introducing the relevant flow-invariant subspaces. These are in correspondence with independent blocks, as follows. For any independent block J for $\mathcal{G}$, let

$$V_J = \{x \in V : \ x_i = 0 \text{ whenever } i \notin J\} \tag{4.11}$$

and as defined before

$$\mathcal{G}_J = \{\sigma \in \mathcal{G} : \ \sigma(J) = J\}.$$

We start by proving that the spaces V_J are flow-invariant for F, and that the restriction of F to V_J is equivariant under a wreath product $\mathbf{Z}_2 \wr \mathcal{G}_J$, where $\mathcal{G}_J \subseteq \mathcal{G}$ acts on J by permuting the indices in J transitively.

Lemma 4.5 *Suppose that* $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ *acts on* $V = \mathbf{R}^N$ *as before, where* $\mathcal{G}$ *is a transitive subgroup of* $\mathbf{S}_N$*. Let* $F : V \to V$ *be a* Γ*-equivariant mapping. Let* J *be an independent block for* $\mathcal{G}$ *and let* V_J *be as in (4.11). Then* $F\,|_{V_J}$ *is* $\mathbf{Z}_2 \wr \mathcal{G}_J$*-equivariant.*

Proof: It is easy to show that

$$V_J = \mathrm{Fix}(B_J \dot{+} P_J)$$

where $B_J = B_1 \times \cdots \times B_N$ with $B_i = \mathbf{Z}_2$ if $i \notin J$ and $B_i = \{\mathbf{1}\}$ if $i \in J$, and P_J is the subgroup of $\mathcal{G}$ comprising all permutations that act trivially on J. Since V_J is a fixed-point space, it is flow-invariant, that is, $F(V_J) \subseteq V_J$. Moreover, $\mathcal{G}_J$ is the largest subgroup of $\mathcal{G}$ that leaves J invariant. Since F is $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant, the restriction of F to V_J is $\mathbf{Z}_2 \wr \mathcal{G}_J$-equivariant. (See Golubitsky *et al.* [19] Lemma XIII 10.2.) ∎

Lemma 4.6 *Suppose that* $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ *acts on* $V = \mathbf{R}^N$*, where* $\mathcal{G}$ *is a transitive subgroup of* $\mathbf{S}_N$*, and* F *in (2.1) is a* $\mathbf{Z}_2 \wr \mathcal{G}$*-equivariant vector field. Let* J_1 *be an independent block for* $\mathcal{G}$ *of cardinality* $p \le N$*. Let* $\Sigma_{J_1} = \{J_1, J_2, \ldots, J_q\}$*,* $q = N/p$*, be the system of independent blocks containing* J_1*. Then, the dynamics of (2.1) on the flow-invariant subspaces* V_{J_i}*,* $1 \le i \le q$ *are conjugate.*

Proof: By Lemma 4.5, each V_{J_i} is flow-invariant for F. Since J_1 is an independent block for $\mathcal{G}$ there exist permutations $\sigma_2, \ldots, \sigma_q \in \mathcal{G}$ such that $J_i = \sigma_i(J_1)$ (recall Proposition 4.3), so $V_{J_i} = \sigma_i V_{J_1}$. ∎

Definition 4.7 A *subcycle* Ω_i of (2.1) is a heteroclinic cycle Ω_i of (2.1) restricted to V_{J_i}, where $J_i \neq \{1, \ldots, N\}$ is an independent block for $\mathcal{G}$.

The stability of a heteroclinic cycle depends on various eigenvalues at the equilibria in the cycle. Suppose that V_1 is a flow-invariant subspace for F, and consider equilibria in V_1 that form a subcycle. We divide the relevant eigenvalues into two classes:

- *restricted* eigenvalues, whose eigenvectors lie in V_1
- *supplementary* eigenvalues, whose eigenvectors do not lie in V_1.

We may apply Theorem 2.8, either to the subcycle, in which case we consider only restricted eigenvalues, or to the union of its conjugates in the whole space, in which case we must also take the supplementary eigenvalues into account. Intuitively the restricted eigenvalues govern the stability of the subcycle in V_1, whereas the supplementary eigenvalues determine its stability transverse to V_1.

The main result of this section, Theorem 4.10 below, shows that if (2.1) has an asymptotically stable subcycle Ω_i associated to a space V_{J_i}, then the cycle $\cup_{i=1}^{q}\Omega_i$ is stable in V (for an open subset of parameter values that determine the stability of the cycle transverse to V_{J_i}).

4.2.1 Generic Normal Form

We consider here the third order truncation of the vector field F of (2.1). Note that since the groups $\mathbf{Z}_2 \wr \mathcal{G}$ are 3-determined, the linear stability of the equilibria is the same when considering this truncation (recall beginning of Section 3). As before, we assume that $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ acts on $V = \mathbf{R}^N$, where $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$. (This assumption on $\mathcal{G}$ will not always be stated explicitly.) Suppose that Hypotheses 2.1 on F are valid. By Lemma 3.3, if the vector field F is written as $(F_1,\ F_2,\ \ldots\ ,\ F_N)^\top$, where $\top$ indicates the transpose, then we can assume that

$$F_j(x,\lambda) = x_j[\lambda + a \parallel x \parallel^2 + \sum_{i=2}^{r} a_i \parallel x_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2] \qquad (j = 1, 2, \ldots , N) \tag{4.12}$$

Here $\sigma_1 = \mathbf{1}$ and $\sigma_j \in \mathcal{G}$ (for $2 \leq j \leq N$) are such that $\sigma_j^{-1}(1) = j$, and $\mathcal{O}_1 = \{1\}$, $\mathcal{O}_2, \ldots ,$ $\mathcal{O}_r$ are all the distinct $\mathcal{G}_1$-orbits. Also $\sigma_j^{-1}(\mathcal{O}_i) = \{\sigma_j^{-1}(k) :\ k \in \mathcal{O}_i\}$ and $\parallel x_{\{i_1,\ldots,i_k\}} \parallel^2 = x_{i_1}^2 + \cdots + x_{i_k}^2$.

For any given F, the coefficients $a,\ a_j \in$ (4.12) are specific real numbers. However, we may consider the coefficients to be arbitrary, in which case we obtain what Field [15] calls a *generic normal form* for $\vec{\mathcal{P}}_V^3(\mathbf{Z}_2 \wr \mathcal{G})$: this is a parametrised family $G = (G_1,\ G_2,\ \ldots\ ,\ G_N)^\top$ where

$$G_{j,\alpha}(x,\lambda) = x_j[\lambda + a \parallel x \parallel^2 + \sum_{i=2}^{r} a_i \parallel x_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2] \tag{4.13}$$

and $\alpha = (a, a_2, \ldots , a_r)$. Let $A \cong \mathbf{R}^r$ be the parameter space $A = \{\alpha\}$. For any given $\alpha \in A$ we may specialise (4.13) to define a bifurcation problem $\alpha(x,\lambda)$ and hence a system of ODEs (2.1) with $F = G_\alpha$. We seek, in particular, conditions on $\alpha \in A$ for which (2.1), with this definition on F, has a branch of stable heteroclinic cycles.

Remark 4.8 By Field [15] the group $\mathbf{Z}_2 \wr \mathcal{G}$ is 3-determined. Thus $\mathbf{Z}_2 \wr \mathcal{G}_J$, where J is an independent block for $\mathcal{G}$, is also 3-determined.

Lemma 4.9 *Suppose that $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ acts on $V = \mathbf{R}^N$, where $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, and J is an independent block for $\mathcal{G}$. If $G \in \vec{\mathcal{P}}_V^3(\mathbf{Z}_2 \wr \mathcal{G})$ is a generic normal form then $G\mid_{V_J}$ is a generic normal form in $\vec{\mathcal{P}}_{V_J}^3(\mathbf{Z}_2 \wr \mathcal{G}_J)$ (where the action of $\mathbf{Z}_2 \wr \mathcal{G}_J$ is restricted to V_J.)*

Proof: Since $\mathcal{G}$ is transitive, we may without loss of generality suppose that $1 \in J$. By (3.7) and Remark 3.8, $\dim \vec{\mathcal{P}}_V^3(\mathbf{Z}_2 \wr \mathcal{G})$ is equal to the number of $\mathcal{G}_1$-orbits for the action of

$\mathbf{Z}_2 \wr \mathcal{G}$ on V, and we have a similar relation with $\mathcal{G}_1$ replaced by $(\mathcal{G}_J)_1$ and V by V_J. We prove that the number of $(\mathcal{G}_J)_1$-orbits in J is equal to the number of $\mathcal{G}_1$-orbits in J. Recall that $\mathcal{G}_J$ acts transitively on J. Since J is an independent block for $\mathcal{G}$ (and $\mathcal{G}_J$) and $1 \in J$, it follows that J is a union of orbits for $\mathcal{G}_1$ (Remark 4.4). Thus $\mathcal{G}_1$ leaves J invariant. Therefore $\mathcal{G}_1 \subseteq \mathcal{G}_J$ and $\mathcal{G}_1 \subseteq (\mathcal{G}_J)_1$. But $\mathcal{G}_J \subseteq \mathcal{G}$ implies that $(\mathcal{G}_J)_1 \subseteq \mathcal{G}_1$. Therefore $\mathcal{G}_1 = (\mathcal{G}_J)_1$. ∎

We will show in Theorem 4.10 that for suitable choices of V_J, the division of eigenvalues at the equilibria corresponds to a decomposition $A = A_R \oplus A_S$ where the *restricted parameters* $\alpha \in A_R$ determine the restricted eigenvalues and the *supplementary parameters* $\alpha \in A_S$ determine the supplementary ones. That is, we may construct stable cycles in the whole space by selecting parameter values in A_R that yield a stable subcycle, and then selecting independent parameters in A_S to make the full cycle stable. See Examples 4.11 and 4.12 to clarify this description. Example 6.2 of Section 6 illustrates a typical case where Theorem 4.10 does not apply. The blocks J considered in that example are not independent blocks.

We now state and prove our main result on subcycles.

Theorem 4.10 *Suppose that $\mathcal{G}$ acts transitively on $\{1, 2, \ldots, N\}$, $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ acts on $V = \mathbf{R}^N$, and G is a generic normal form for $\vec{\mathcal{P}}^3_V(\mathbf{Z}_2 \wr \mathcal{G})$ as in (4.13). Let $J \subseteq \{1, 2, \ldots, N\}$ be an independent block for $\mathcal{G}$. Suppose that for some parameter value $\alpha = (\alpha_R, \alpha_S) \in A_R \oplus A_S$ the ODE (2.1) defined by $F = G_\alpha |_{V_J}$ has a branch of subcycles Ω' for $\lambda > 0$ satisfying Hypothesis 2.6 and condition (2.4) of Theorem 2.8. Then:*

1. *Condition (2.4) depends only on α_R.*

2. *For each α_R such that (2.4) holds, there is a nonempty open subset $U \subseteq A_S$ such that $G_{(\alpha_R, \alpha_S)}$ has a branch of asymptotically stable cycles Ω containing Ω' for $\lambda > 0$, whenever $\alpha_S \in U$.*

Proof: Without loss of generality we may assume that $J = \{1, 2, \ldots, p\}$. Let $F = G_\alpha$ for some α. By Lemma 4.5 the space V_J is flow-invariant for F. Moreover, since J is an independent block, if $j \in J$ then $\sigma_j \in \mathcal{G}_J$ and $\sigma_j(\mathcal{O}_i) \subseteq J$ for the orbits $\mathcal{O}_i$ contained in J. Suppose that $\mathcal{O}_1$, $\mathcal{O}_2$, ... , $\mathcal{O}_{r_1}$, with $r_1 \leq r$, are the $\mathcal{G}_1$-orbits inside J. Thus

$$F_j |_{V_J} \equiv 0 \quad \text{if } j \notin J$$

and

$$F_j(x, \lambda) = x_j[\lambda + a \parallel x \parallel^2 + \sum_{i=2}^{r_1} a_i \parallel x_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2] \quad \text{if } j \in J,\ x \in V_J \tag{4.14}$$

so $F(x, \lambda) |_{V_J}$ depends only on the parameters $a, a_2, \ldots, a_{r_1}$. These are the restricted parameters and span the space A_R. The remaining parameters $a_{r_1+1}, \ldots, a_r$ are the supplementary parameters and span A_S. Let Ω' be a subcycle of (2.1) associated with V_J (for $\lambda > 0$ fixed). That is, Ω' is a heteroclinic cycle of (2.1) restricted to V_J. Moreover, suppose that Ω' is asymptotically stable in V_J. Suppose that $\xi_1, \ldots, \xi_m$ are the equilibria involved in the cycle (Definition 2.4) and suppose that condition (2.4) of Theorem 2.8,

$$\prod_{j=1}^{m} \min(c_j, e_j - t_j) > \prod_{j=1}^{m} e_j, \tag{4.15}$$

is satisfied for some region in the space of parameters $\alpha_R = (a, a_2, \ldots, a_{r_1}) \in A_R$. Recall that the quantities c_j, e_j, t_j are as in Definition 2.7, and they depend on the linear stability of ξ_j, hence on the eigenvalues of $(\mathrm{D}_x F)_{\xi_j}$ restricted to V_J. We now prove the existence, for an open set of the parameters $\alpha_S = (a_{r_1+1}, \ldots, a_r) \in A_S$, of a heteroclinic cycle Ω for (2.1) composed of (conjugate) subcycles. Moreover, we prove that this cycle is asymptotically stable.

Let $\mathbf{p}$ be any equilibrium of $F\,|_{V_J}$ belonging to Ω' (for $\lambda > 0$ fixed). Then $\mathbf{p}_i = 0$ if $i \notin J$. The linear stability of $\mathbf{p}$ in V is determined by $(\mathrm{D}_x F)_{\mathbf{p}}$. Since $(\mathrm{D}_x F)_{\mathbf{p}}$ commutes with $\Sigma_{\mathbf{p}}$ and $\{\mathbf{1}\}^p \times \mathbf{Z}_2{}^{s-p} \subseteq \Sigma_{\mathbf{p}}$, we have

$$(\mathrm{D}_x F)_{\mathbf{p}} = \mathrm{Diag}(M_1, M_2)$$

where M_1 determines the stability of $\mathbf{p}$ inside V_J (and so depends on $\lambda, a, a_2, \ldots, a_{r_1}$) and

$$M_2 = \mathrm{Diag}(\lambda_{p+1}, \ldots, \lambda_N).$$

Denote by $\lambda_1, \ldots, \lambda_p$ the eigenvalues of M_1. These are the restricted eigenvalues and $\lambda_{p+1}, \ldots, \lambda_N$ are the supplementary eigenvalues. Thus λ_j for $1 \le j \le p$ are the eigenvalues of $(\mathrm{D}_x F)_{\mathbf{p}}$ restricted to V_J and from (4.14) it follows that they depend only on $\lambda, a, a_2, \ldots, a_{r_1}$. The λ_j for $p+1 \le j \le N$ are the eigenvalues of $(\mathrm{D}_x F)_{\mathbf{p}}$ restricted to $V \setminus V_J$. Moreover,

$$\lambda_j = \left(\frac{\partial F_j}{\partial x_j}\right)\Bigg|_{x=\mathbf{p}} \qquad (j \ge p+1).$$

Choose a fixed value of $j \ge p+1$. By (4.12),

$$\left(\frac{\partial F_j}{\partial x_j}\right) = \lambda + a \parallel x \parallel^2 + \sum_{i=2}^{r} a_i \parallel x_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2 + x_j \left[2 a x_j + \frac{\partial}{\partial x_j}\left(\sum_{i=2}^{r} a_i \parallel x_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2\right)\right]$$

and so

$$\left(\frac{\partial F_j}{\partial x_j}\right)\Bigg|_{x=\mathbf{p}} = \lambda + a \parallel \mathbf{p} \parallel^2 + \sum_{i=2}^{r} a_i \parallel \mathbf{p}_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2 . \tag{4.16}$$

If $\mathbf{p}$ is a non-trivial steady-state equilibrium, we may suppose without loss of generality that $\mathbf{p}_1 \neq 0$, so that

$$\begin{aligned} 0 &= \lambda + a \parallel \mathbf{p} \parallel^2 + \sum_{i=2}^{r} a_i \parallel \mathbf{p}_{\mathcal{O}_i} \parallel^2 \\ &= \lambda + a \parallel \mathbf{p} \parallel^2 + \sum_{i=2}^{r_1} a_i \parallel \mathbf{p}_{\mathcal{O}_i} \parallel^2 \end{aligned}$$

from which

$$\lambda + a \parallel \mathbf{p} \parallel^2 = -\sum_{i=2}^{r_1} a_i \parallel \mathbf{p}_{\mathcal{O}_i} \parallel^2 . \tag{4.17}$$

Substitute (4.17) in (4.16) to get

$$\left(\frac{\partial F_j}{\partial x_j}\right)\Bigg|_{x=\mathbf{p}} = \sum_{i=2}^{r} a_i \parallel \mathbf{p}_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2 - \sum_{i=2}^{r_1} a_i \parallel \mathbf{p}_{\mathcal{O}_i} \parallel^2 .$$

Moreover, since $j \notin J$, then $\sigma_j^{-1}(J) \subseteq \{1, 2, \ldots, N\} \setminus J$ since J is an independent block. Therefore $\parallel \mathbf{p}_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2 = 0$ for $i = 2, \ldots, r_1$, and

$$\lambda_j = \sum_{i=r_1+1}^{r} a_i \parallel \mathbf{p}_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2 - \sum_{i=2}^{r_1} a_i \parallel \mathbf{p}_{\mathcal{O}_i} \parallel^2 \qquad (j \geq p+1) \tag{4.18}$$

We can choose $a_{r_1+1}, a_{r_1+2}, \ldots, a_r$ in another set $U \subseteq A_S$ (indeed it suffices to make them negative enough) so that

$$W^u(\xi_j) \subseteq V_J \tag{4.19}$$

for each ξ_j in the cycle. If $J = J_1$ and if $\Sigma_{J_1} = \{J_1, J_2, \ldots, J_q\}$ $(q = N/p)$ is the corresponding system of independent blocks containing J_1, then by Lemma 4.6 there are disjoint (conjugate) subcycles Ω_i inside V_{J_i} with $\Omega_i = \sigma_i \Omega'$ for some $\sigma_i \in \mathcal{G}$. From (4.19) (recall Definition 2.4) the union $\Omega = \cup_{i=1}^{q} \Omega_i$ forms a heteroclinic cycle for (2.1).

The flow-invariant subspaces P_j and P_{j-1} satisfying Hypothesis 2.6 in V_J also satisfy this hypothesis in V. By Corollary 4.8 of Krupa and Melbourne [25] it follows that Ω is asymptotically stable. That is, choosing parameters $a_{r_1+1}, a_{r_1+2}, \ldots, a_r$ such that (4.19) holds, the stability of the cycle Ω' (and so of Ω) is determined by the stability within V_J. ∎

4.3 Examples of Subcycles

We focus on the groups $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ where $\mathcal{G}$ is $\mathbf{D}_6(6)$ or $\mathbf{F}_{18}(6)$, see Table 2. We illustrate our results for existence and stability of heteroclinic cycles for these two examples. In each case there is an independent block $J = \{1, 3, 5\} \subseteq \{1, \ldots, 6\}$ for $\mathcal{G}$ such that the corresponding Γ-equivariant ODE (2.1), when restricted to V_J, is $\mathbf{Z}_2 \wr \mathbf{Z}_3$-equivariant.

Example 4.11 ($\mathbf{Z}_2 \wr \mathbf{F}_{18}(6)$) Consider (2.1) where $G \in \vec{\mathcal{P}}_{\mathbf{R}^6}(\mathbf{Z}_2 \wr \mathbf{F}_{18}(6))$ and has polynomial components of degree ≤ 3. Lemma 3.3 yields the generic normal form

$$G(x, \lambda) = \lambda x + a \parallel x \parallel^2 x + a_2 g_2(x) + a_3 g_3(x) + a_4 g_4(x),$$

$\mathcal{G}$	$\lvert\mathcal{G}\rvert$	Generators	$\mathcal{G}_1$	$\mathcal{G}_1$-orbits
$\mathbf{D}_6(6)$	6	$(1,3,5)(2,4,6)$ $(1,4)(2,3)(5,6)$	**1**	$\{1\}, \ldots, \{6\}$
$\mathbf{F}_{18}(6)$	18	$(2,4,6)$ $(1,4)(2,5)(3,6)$	$\langle(2,4,6)\rangle$	$\{1\}$, $\{2,4,6\}$, $\{3\}$, $\{5\}$

Table 2: Groups, generators, stabilisers and corresponding orbits for Examples 4.11 and 4.12.

where

$$g_2(x) = \begin{pmatrix} (x_2^2+x_4^2+x_6^2)x_1 \\ (x_5^2+x_1^2+x_3^2)x_2 \\ (x_2^2+x_4^2+x_6^2)x_3 \\ (x_5^2+x_1^2+x_3^2)x_4 \\ (x_2^2+x_4^2+x_6^2)x_5 \\ (x_5^2+x_1^2+x_3^2)x_6 \end{pmatrix} \qquad g_3(x) = \begin{pmatrix} x_3^2x_1 \\ x_4^2x_2 \\ x_5^2x_3 \\ x_6^2x_4 \\ x_1^2x_5 \\ x_2^2x_6 \end{pmatrix} \qquad g_4(x) = \begin{pmatrix} x_5^2x_1 \\ x_6^2x_2 \\ x_1^2x_3 \\ x_2^2x_4 \\ x_3^2x_5 \\ x_4^2x_6 \end{pmatrix}$$

and λ, a, a_2, a_3, a_4 are real (see Table 2 for the $\mathcal{G}_1$-orbits).

We prove that the conditions

$$\begin{gathered} a,\ a_3 < 0,\ a_4 > 0, \\ a_3 + a_4 < 0, \\ a_2 < 0 \end{gathered}$$

on a and a_2, a_3, a_4 are sufficient for the existence and asymptotic stability of a branch of heteroclinic cycles for $\lambda > 0$, each formed by two disjoint subcycles: the first between equilibria of the form $(x,0,0,0,0,0)$, $(0,0,x,0,0,0)$ and $(0,0,0,0,x,0)$, and the second between the equilibria of the form $(0,x,0,0,0,0)$, $(0,0,0,x,0,0)$ and $(0,0,0,0,0,x)$.

If we restrict (2.1) to the flow-invariant space

$$V_1 = \{(x_1,0,x_3,0,x_5,0):\ x_1,x_3,x_5 \in \mathbf{R}\}$$

then we get the $\mathbf{Z}_2 \wr \mathbf{Z}_3$-equivariant ODE

$$\begin{aligned} \dot{x}_1 &= x_1[\lambda + a(x_1^2+x_3^2+x_5^2) + a_3x_3^2 + a_4x_5^2] \\ \dot{x}_3 &= x_3[\lambda + a(x_1^2+x_3^2+x_5^2) + a_3x_5^2 + a_4x_1^2] \\ \dot{x}_5 &= x_5[\lambda + a(x_1^2+x_3^2+x_5^2) + a_3x_1^2 + a_4x_3^2] \end{aligned} \tag{4.20}$$

Note that $J = \{1,3,5\}$ is an independent block for $\mathcal{G}$. The restricted parameters are a, a_3, a_4, and the supplementary parameter is a_2. When $a < 0$, $a_3 < 0$ and $a_4 > 0$, the reduced system (4.20) has a branch of heteroclinic cycles for $\lambda > 0$, between the equilibria of the form $(x,0,0)$, $(0,x,0)$, and $(0,0,x)$, for $x \in \mathbf{R}$ (see for example Proposition 1 of Krupa [24]).

Moreover if $a_3 + a_4 < 0$, then the cycles are asymptotically stable and satisfy condition (2.4). In this case $-r_1 = a$, $-c_1 = a_3$, and $e_1 = a_4$. Each cycle corresponds to a subcycle Ω_1 in $\mathbf{R}^6$ between equilibria of the form $(x, 0, 0, 0, 0, 0)$, $(0, 0, x, 0, 0, 0)$, and $(0, 0, 0, 0, x, 0)$, which have (conjugate) axial isotropy subgroups $\Sigma(\{\mathbf{1}\}, \{1\})$, $\Sigma(\{\mathbf{1}\}, \{3\})$, and $\Sigma(\{\mathbf{1}\}, \{5\})$.

A conjugate subcycle Ω_2 occurs in the flow-invariant space

$$V_2 = \{(0, x_2, 0, x_4, 0, x_6) : \ x_2, x_4, x_6 \in \mathbf{R}\}$$

for the same values of the parameters a, a_3, a_4 (Lemma 4.6). Suppose that $a < 0$, $a_3 < 0$, $a_4 > 0$ and $a_3 + a_4 < 0$. If $a_2 < 0$, then the union of these two subcycles $\Omega = \Omega_1 \cup \Omega_2$ forms a heteroclinic cycle for (2.1). Note that the stability of, for example, an equilibrium of the form $(x, 0, 0, 0, 0, 0)$, is given by eigenvalues of signs determined by a_3, a_4 (with eigenvectors associated with directions in V_1, hence restricted) and a_2 (an eigenvalue with multiplicity 3 and associated with directions in $V \setminus V_1$, that is, supplementary). By Corollary 4.8 of Krupa and Melbourne [25] it follows that Ω is asymptotically stable.

Example 4.12 ($\mathbf{Z}_2 \wr \mathbf{D}_6(6)$) Consider (2.1) where $G \in \vec{\mathcal{P}}_{\mathbf{R}^6}(\mathbf{Z}_2 \wr \mathbf{D}_6(6))$ and has polynomial components of degree ≤ 3. By Lemma 3.3 (see Table 2), the generic normal form is

$$G(x, \lambda) = \lambda x + a \parallel x \parallel^2 x + \sum_{i=2}^{6} a_i g_i(x)$$

where

$$g_2(x) = \begin{pmatrix} x_2^2 x_1 \\ x_1^2 x_2 \\ x_4^2 x_3 \\ x_3^2 x_4 \\ x_6^2 x_5 \\ x_5^2 x_6 \end{pmatrix} \qquad g_3(x) = \begin{pmatrix} x_3^2 x_1 \\ x_6^2 x_2 \\ x_5^2 x_3 \\ x_2^2 x_4 \\ x_1^2 x_5 \\ x_4^2 x_6 \end{pmatrix} \qquad g_4(x) = \begin{pmatrix} x_4^2 x_1 \\ x_5^2 x_2 \\ x_6^2 x_3 \\ x_1^2 x_4 \\ x_2^2 x_5 \\ x_3^2 x_6 \end{pmatrix}$$

$$g_5(x) = \begin{pmatrix} x_5^2 x_1 \\ x_4^2 x_2 \\ x_1^2 x_3 \\ x_6^2 x_4 \\ x_3^2 x_5 \\ x_2^2 x_6 \end{pmatrix} \qquad g_6(x) = \begin{pmatrix} x_6^2 x_1 \\ x_3^2 x_2 \\ x_2^2 x_3 \\ x_5^2 x_4 \\ x_4^2 x_5 \\ x_1^2 x_6 \end{pmatrix}$$

and λ, a, a_i $(i = 2, 3, \ldots, 6)$ are real.

We prove that the conditions

$$\begin{gathered} a, \ a_3 < 0, \ a_5 > 0, \\ a_3 + a_5 < 0, \\ a_2, \ a_4, \ a_6 < 0 \end{gathered}$$

on a and a_i for $i = 2, \ldots, 6$ are sufficient for the existence and asymptotic stability of a branch of heteroclinic cycles for $\lambda > 0$, each formed by two disjoint subcycles: as in the previous example, the first subcycle is between equilibria of the form $(x, 0, 0, 0, 0, 0)$, $(0, 0, x, 0, 0, 0)$ and

$(0,0,0,0,x,0)$, and the second between equilibria of the form $(0,x,0,0,0,0)$, $(0,0,0,x,0,0)$ and $(0,0,0,0,0,x)$.

If we restrict (2.1) to the flow-invariant space V_1 defined in Example 4.11, we get the $\mathbf{Z}_2 \wr \mathbf{Z}_3$-equivariant ODE

$$\begin{aligned}
\dot{x}_1 &= x_1[\lambda + a(x_1^2 + x_3^2 + x_5^2) + a_3 x_3^2 + a_5 x_5^2] \\
\dot{x}_3 &= x_3[\lambda + a(x_1^2 + x_3^2 + x_5^2) + a_3 x_5^2 + a_5 x_1^2] \\
\dot{x}_5 &= x_5[\lambda + a(x_1^2 + x_3^2 + x_5^2) + a_3 x_1^2 + a_5 x_3^2]
\end{aligned}$$

Again $J = \{1,3,5\}$ is an independent block for $\mathcal{G}$. The restricted parameters are a, a_3, a_5. The supplementary parameters are a_2, a_4, a_6. As in Example 4.11, for $a < 0$, $a_3 < 0$, and $a_5 > 0$, there exists a branch of heteroclinic cycles Ω_1 in V_1 (for $\lambda > 0$) corresponding to subcycles in $\mathbf{R}^6$ between equilibria of type $(x,0,0,0,0)$, $(0,0,x,0,0,0)$ and $(0,0,0,0,x,0)$, with axial isotropy subgroups $\Sigma(\{\mathbf{1}\},\{1\})$, $\Sigma(\{\mathbf{1}\},\{3\})$ and $\Sigma(\{\mathbf{1}\},\{5\})$ respectively. Since $-c_1 = a_3$ and $e_1 = a_5$, each cycle for the restricted system is asymptotically stable provided that $a_3 + a_5 < 0$ (again by Theorem 2.8). For the same values of the parameters, there is a conjugate subcycle Ω_2 in the flow-invariant space V_2 as defined in Example 4.11. The union of the two subcycles $\Omega = \Omega_1 \cup \Omega_2$ forms an asymptotically stable heteroclinic cycle for (2.1) provided that the supplementary parameters a_2, a_4, a_6 are chosen negative.

5 Edge Cycles in $\mathbf{Z}_2 \wr \mathcal{G}$-Equivariant Systems

We now turn our attention to heteroclinic cycles in $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant systems called *edge cycles*. We follow the nomenclature of Field [15] and Field and Swift [17]. The truncated normal form of the system that we study is a 1-parameter family of the form $\dot{x} = \lambda x + Q(x)$, where $Q : V \to V$ is a cubic homogeneous polynomial. Transforming to spherical polar coordinates ('blowing up'), we arrive at a vector field on the $(N-1)$-dimensional sphere $\mathbf{S}^{N-1}$. This vector field is called the *phase vector field*, see Field [14]. The phase vector field does not depend on the parameter λ. We may then study the codimension one bifurcations of the phase vector field as the coefficients of Q are varied. Provided Q satisfies a *contraction* condition, Field [14] shows that as λ passes through zero, there is a branch of attracting invariant spheres for the truncated normal form $\lambda x + Q(x)$ (Invariant Sphere Theorem [14]). The heteroclinic cycles that are formed by edges of a 'fundamental domain' under the action of ${\mathbf{Z}_2}^N$ on $\mathbf{S}^{N-1}$ are called *edge cycles*. (See Section 6.1 for more detail on the results stated here and the definition of fundamental domain.)

Definition 5.1 An *edge cycle* is a heteroclinic cycle connecting equilibria with symmetry (conjugate to) $\Sigma(\{\mathbf{1}\},\{i\})$.

The well known example studied by Guckenheimer and Holmes [20] is an edge cycle: the group involved is $\mathbf{Z}_2 \wr \mathbf{Z}_3$.

As before let $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$, where $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, and consider (2.1) where F commutes with Γ and has polynomial components of degree ≤ 3. Given an equilibrium $\mathbf{p} = (x_1, 0, \ldots, 0)$ (for fixed $\lambda > 0$) of (2.1), the linear stability of $\mathbf{p}$ is determined by $(\mathrm{D}_x F)_{x=\mathbf{p}}$. Since this matrix commutes with the isotropy subgroup

$$\Sigma_{\mathbf{p}} = \Sigma(\{\mathbf{1}\},\{1\}) = \{\mathbf{1}\} \times {\mathbf{Z}_2}^{N-1} \dot{+} \mathcal{G}_1$$

of $\mathbf{p}$, it follows that

$$(\mathrm{D}_x F)_{\mathbf{p}} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N).$$

Moreover, the vector $(\lambda_1, \ldots, \lambda_N)$ formed by the eigenvalues commutes with $\mathcal{G}_1$. Thus the eigenvalues λ_i can be divided into r classes in which the value is the same, where r is the number of $\mathcal{G}_1$-orbits; that is, if i, j belong to the same $\mathcal{G}_1$-orbit then $\lambda_i = \lambda_j$. We claim that the converse is also true. From (4.12) and $\lambda_1 = 2ax_1^2$,

$$\lambda_j = \left(\frac{\partial F_j}{\partial x_j}\right)\bigg|_{x=\mathbf{p}} = \sum_{i=2}^{r} a_i \parallel \mathbf{p}_{\sigma_j^{-1}(\mathcal{O}_i)} \parallel^2$$

implies that

$$\lambda_j = a_i x_1^2 \quad \text{if } \sigma_j(1) \in \mathcal{O}_i$$

for $j \geq 2$. Hence $\lambda_k = \lambda_j$ if $\sigma_k(1) \in \mathcal{O}_i$. We assume the generic hypothesis $a_i \neq a_j$ for $i \neq j$. In this case, if $\lambda_k = \lambda_j$ — that is, if $\sigma_j(1)$ and $\sigma_k(1)$ are in the same $\mathcal{G}_1$-orbit — then j and k are also in the same $\mathcal{G}_1$-orbit. (To see this, note that if $\sigma_j(1) = m$ and $\sigma_k(1) = n$ with $m, n \in \mathcal{O}_i$, then $\sigma_k^{-1}\sigma\sigma_j \in \mathcal{G}_1$ for some $\sigma \in \mathcal{G}_1$ such that $\sigma(m) = n$. By the definition of σ_j, σ_k we have $\sigma_k^{-1}\sigma\sigma_j(j) = k$, so j, k are in the same $\mathcal{G}_1$-orbit.)

Therefore $\lambda_k = \lambda_j$ if and only if j, k are in the same $\mathcal{G}_1$-orbit. The eigenvalues are thus of the form $\lambda_j = a_i x_1^2$ with multiplicity given by the size of the $\mathcal{G}_1$-orbit $\mathcal{O}_i$.

Proving the existence of heteroclinic cycles in general is difficult. The more common methods for proving (2.3) are based on the existence of heteroclinic connections between equilibria, usually relying on the existence of a connection in a two-dimensional flow-invariant subspace, where the Poincaré-Bendixson Theorem (see Arrowsmith and Place [1] or any other advanced textbooks on ODEs) can be applied. We therefore choose parameters so that the cycle is as simple as possible. We allow just one orbit of eigenvalues to be positive; namely, only one parameter a_i is chosen to be positive. Hence,

$$W^u(\mathbf{p}) \subseteq H = \{x \in V : \ x_k = 0 \text{ for } \sigma_k(1) \in \{2, \ldots, N\} \setminus \mathcal{O}_i\} \tag{5.21}$$

and

$$\dim W^u(\mathbf{p}) = |\mathcal{O}_i|. \tag{5.22}$$

Denote by ξ_j (for fixed $\lambda > 0$) an equilibrium of (2.1) with isotropy $\Sigma(\{\mathbf{1}\}, \{j\})$.

Lemma 5.2 *Consider (2.1) where $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ and $\mathcal{G}$ is transitive. Assume (5.21) and (5.22), and suppose that there exists a heteroclinic connection between ξ_1 and ξ_j, where $\sigma_j(1) \in \mathcal{O}_i$. A necessary condition for the existence of an edge cycle (involving ξ_1, ξ_j) is that $\mathcal{O}_i$ has cardinality 1.*

Proof: Suppose that $\mathcal{O}_i$ has cardinality $p > 1$. Then the space

$$\{x \in V : \ x_1 \in \mathbf{R}, \ x_j = x_k, \ \text{if } \sigma_j(1), \sigma_k(1) \in \mathcal{O}_i, \ x_k = 0 \text{ otherwise}\}$$

is flow-invariant and contains an axial equilibrium with isotropy $\Sigma(\{\mathbf{1}\}, \mathcal{O}_j)$, where $\mathcal{O}_j$ is the $\mathcal{G}_1$-orbit containing j (and $\mathcal{O}_j$ has cardinality p). Thus the condition $W^u(\xi_1) \subseteq \cup_{\gamma \in \Gamma} W^s(\gamma\xi_j)$ for the existence of the edge cycle cannot be satisfied. ∎

Theorem 5.3 *Assume (2.1) where* $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$ *and* $\mathcal{G}$ *is transitive, and let* ξ_1 *be an equilibrium of (2.1). Suppose that* $\dim W^u(\xi_1) = 1$ *and* e_j *(with* $j \neq 1$*) is an eigenvector associated with the eigenvalue of* $(D_x F)_{\xi_1}$ *with positive real part. Then there is an edge cycle including* ξ_1 *if and only if* $\mathcal{G}_{\{1,j\}}$ *acts trivially on* $\{1, j\}$.

Proof: Suppose that the cycle exists. Since $\dim W^u(\xi_1) = 1$ and e_j is an eigenvector associated with the eigenvalue of $(\mathrm{D}_x F)_{\xi_1}$ with positive real part, the edge cycle includes ξ_1 and ξ_j. If $\mathcal{G}_{\{1,j\}}$ does not act trivially on $\{1, j\}$ then the line $x_1 = x_j$ is flow-invariant for (2.1), so there cannot be a connection between ξ_1 and ξ_j. Thus the condition that $\mathcal{G}_{\{1,j\}}$ acts trivially on $\{1, j\}$ is necessary for the cycle to exist.

We prove now that this condition is sufficient. Since e_j is the unstable direction for ξ_1 (and $\dim W^u(\xi_1)$ is 1-dimensional) the $\mathcal{G}_1$-orbit containing $\sigma_j(1)$ contains only $\sigma_j(1)$. Suppose that $\sigma_j(1) = i$, so that $a_i > 0$ (that is, we suppose that $\mathcal{O}_i = \{i\}$). We use here the notation of (4.12). Consider (2.1) restricted to $\{x \in V : \; x_k = 0 \text{ if } k \neq 1, j\}$ where F is as in (4.12):

$$\begin{aligned} \dot{x}_1 &= x_1[\lambda + a(x_1^2 + x_j^2) + a_p x_j^2] \\ \dot{x}_j &= x_j[\lambda + a(x_1^2 + x_j^2) + a_i x_1^2] \end{aligned} \qquad (5.23)$$

Since $\mathcal{G}_{\{1,j\}}$ acts trivially on $\{1, j\}$ and $\sigma_j(1) = i$, then $i \neq j$ (we know that $\sigma_j(j) = 1$). Thus we may suppose that $j \in \mathcal{O}_p$, where $p \neq i$, so that $a_p < 0$ and $a_i > 0$. Therefore the equilibrium ξ_1 is a saddle and ξ_j is a sink for (5.23). Moreover, since $a_i a_p < 0$ there are no other nontrivial equilibria with isotropy type different from those of ξ_1 and ξ_j. The condition

$$\frac{a_p + a}{a} + \frac{a_i + a}{a} > -2$$

is satisfied provided a is chosen sufficiently negative. By Proposition 2.6 of Melbourne *et al.* [27], all trajectories of (5.23) starting within a circle of radius $O(\lambda)$ stay bounded near the origin. By the Poincaré-Bendixson Theorem there is a connection between ξ_1 and ξ_j (for $\lambda > 0$).

Since $\mathcal{G}$ is transitive, there exists $\sigma \in \mathcal{G}$ with $\sigma(1) = j$. Moreover, since $\mathcal{G}_{\{1,j\}}$ acts trivially on $\{1, j\}$, then $\sigma(j) = k \neq 1$, so $\mathcal{G}_{\{j,\sigma(j)\}}$ also acts trivially on $\{j, \sigma(j)\}$. Since there exists a connection between ξ_1 and ξ_j, there also exists a connection between ξ_j and ξ_k. The cycle includes the equilibria $\xi_{\sigma^i(1)}$ for $0 \leq i \leq q$, where $q \geq 3$ is minimal subject to $\sigma^q(1) = 1$. ∎

Remark 5.4 With the conditions of Theorem 5.3, a sufficient condition for the edge cycle to be asymptotically stable is that $\max_{k \neq i}(a_k) + a_i < 0$, where the a_k for $k \neq i$ are chosen to be negative.

6 Heteroclinic Webs

We now consider the case $\mathcal{G} = \mathbf{A}_4(6)$, which leads to a concept rather more general than that of a heteroclinic cycle, which we shall call a 'heteroclinic web'. For motivation, consider Figure 4, described in detail in Example 6.2 below. This schematic diagram indicates connections between equilibria in a $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant system. However, it does not represent a heteroclinic *cycle* because the unstable manifold of an equilibrium ξ may not be contained in the union of stable manifolds of equilibria ξ' that lie in a single $\mathbf{Z}_2 \wr \mathcal{G}$-orbit. For example the unstable manifold of ξ_6 connects to the stable manifolds of ξ_2, ξ_5 and ξ_{25}, but ξ_{25} is not in the same group orbit as ξ_2 and ξ_5.

This kind of heteroclinic structure seems typical of non-cyclic groups $\mathcal{G}$, and it deserves further study. In this final section we raise some of the issues involved and describe how far we can get using our present methods.

First we formalise the notion of a heteroclinic web.

Definition 6.1 Let Ξ be a set of hyperbolic equilibria of a Γ-equivariant ODE

$$\dot{x} = F(x) \tag{6.24}$$

on $\mathbf{R}^N$. We say that Ξ is a *pre-heteroclinic web* for (6.24) if for all $\xi \in \Xi$

$$W^u(\xi) \setminus \{\xi\} \subseteq \cup_{\zeta \in \Xi} W^s(\zeta). \tag{6.25}$$

In addition to (6.25) we shall require that the connections between equilibria are 'recurrent', to avoid trivial cases (such as, for example, some ξ being a sink to which all unstable manifolds of the remaining ξ' converge). To do this, we introduce a relation $\rightsquigarrow$ on Ξ, as follows:

$$\xi \rightsquigarrow \zeta$$

if

$$(W^u(\xi) \setminus \{\xi\}) \cap W^s(\zeta) \neq \emptyset.$$

That is, some trajectory originating in $W^u(\xi)$ converges to ζ in forward time. The relation $\rightsquigarrow$ gives Ξ the structure of a directed graph, and we shall require that graph to be *orbitally directionally connected* in the following sense:

If $\xi, \zeta \in \Xi$, then there exist $\eta_0, \eta_1, \dots, \eta_k \in \Xi$ and $\gamma \in \Gamma$ such that

$$\begin{aligned} &\eta_0 = \xi \\ &\eta_k = \gamma\zeta \\ &\eta_i \rightsquigarrow \eta_{i+1} \text{ for } i = 0, \dots, k-1. \end{aligned}$$

In this case we say that Ξ forms a *heteroclinic web.*

Orbital directional connectivity of the set Ξ means that the graph of group orbits of equilibria is directionally connected. The case where Γ is trivial possesses independent interest.

The stability condition (2.4) applies to heteroclinic cycles (where Ξ is a closed loop modulo the group action, that is, viewed in orbit space). Similar criteria exist for heteroclinic webs and Γ-heteroclinic webs (Melbourne [26]).

Given $\xi \in \Xi$, condition (6.25) is satisfied: the unstable manifold of the equilibrium ξ is contained in the union of the stable manifolds of equilibria ζ in Ξ that do not lie necessarily in a single Γ-orbit. By Melbourne [26], condition (2.4) has to be verified for each possible

$\mathcal{G}$	$\lvert\mathcal{G}\rvert$	Generators	$\mathcal{G}_1$	$\mathcal{G}_1$-orbits
$\mathbf{A}_4(6)$	12	$(1,4)(2,5)$ $(1,3,5)(2,4,6)$	$<(2,5)(3,6)>$	$\{1\}$, $\{2,5\}$, $\{3,6\}$, $\{4\}$

Table 3: Data for $\mathbf{A}_4(6)$: generators, stabiliser, and corresponding orbits.

path of connections of the equilibrium ξ. For example, suppose we have a heteroclinic web with (group orbits of) equilibria A, B, C, D and A connects to B and C, B connects to D, C connects to A, and D connects to A and C. There are three routes from A back to itself (its group orbit):

$$\begin{array}{lllllllll} A & \to & B & \to & D & \to & A & & \\ A & \to & C & \to & A & & & & \\ A & \to & B & \to & D & \to & C & \to & A. \end{array}$$

Sufficient conditions for the asymptotic stability are:

$$\begin{array}{l} \rho_A \cdot \rho_B \cdot \rho_D > 1 \\ \rho_A \cdot \rho_C > 1 \\ \rho_A \cdot \rho_B \cdot \rho_D \cdot \rho_C > 1, \end{array}$$

where

$$\rho_x = \min(c_j, e_j - t_j)/e_j,$$

if c_j, e_j, t_j are the quantities as defined in Section 2.3 for each equilibrium A, B, C, D. We illustrate this in Example 6.2.

Existence proofs for ($\Gamma-$)heteroclinic webs are likely to be very difficult, because the Poincaré-Bendixson Theorem is usually not applicable. The next example illustrates all of the above observations.

Example 6.2 Consider (2.1) where $G \in \vec{\mathcal{P}}_{\mathbf{R}^6}(\mathbf{Z}_2 \wr \mathbf{A}_4(6))$ and has polynomial components of degree ≤ 3 (see Table 3). Then

$$G(x,\lambda) = \lambda x + a \parallel x \parallel^2 x + \sum_{i=2}^{4} a_i g_i(x)$$

where

$$g_2(x) = \begin{pmatrix} (x_2^2 + x_5^2)x_1 \\ (x_3^2 + x_6^2)x_2 \\ (x_4^2 + x_1^2)x_3 \\ (x_5^2 + x_2^2)x_4 \\ (x_6^2 + x_3^2)x_5 \\ (x_1^2 + x_4^2)x_6 \end{pmatrix} \quad g_3(x) = \begin{pmatrix} (x_3^2 + x_6^2)x_1 \\ (x_1^2 + x_4^2)x_2 \\ (x_5^2 + x_2^2)x_3 \\ (x_3^2 + x_6^2)x_4 \\ (x_1^2 + x_4^2)x_5 \\ (x_5^2 + x_2^2)x_6 \end{pmatrix} \quad g_4(x) = \begin{pmatrix} x_4^2 x_1 \\ x_5^2 x_2 \\ x_6^2 x_3 \\ x_1^2 x_4 \\ x_2^2 x_5 \\ x_3^2 x_6 \end{pmatrix}$$

and λ, a, a_i for $i = 2$, 3 and 4 are real.

We conjecture that the conditions

$$\begin{gathered} a,\ a_3 < 0,\ a_2 > 0, \\ a_4 < 0, \\ 2a_3 - a_4 < 0, \\ a + a_2 < 0, \end{gathered}$$

on the parameters a and a_i for $i = 2,\ 3,\ 4$ are sufficient for the existence of a branch of heteroclinic webs for $\lambda > 0$. The additional conditions

$$\begin{gathered} a_2 + \max(a_3,\ a_4) < 0, \\ (2a_3 - a_4) + \max(-2a_4,\ 2a_2 - a_4) < 0 \end{gathered}$$

are sufficient for its asymptotic stability. We now explain why this is plausible.

The restriction of (2.1) to $V_1 = \{x \in V :\ x_2 = x_4 = x_6 = 0\}$ yields the $\mathbf{Z}_2 \wr \mathbf{Z}_3$-equivariant ODE

$$\begin{aligned} \dot{x}_1 &= x_1[\lambda + a(x_1^2 + x_3^2 + x_5^2) + a_2 x_5^2 + a_3 x_3^2] \\ \dot{x}_3 &= x_3[\lambda + a(x_1^2 + x_3^2 + x_5^2) + a_2 x_1^2 + a_3 x_5^2] \\ \dot{x}_5 &= x_5[\lambda + a(x_1^2 + x_3^2 + x_5^2) + a_2 x_3^2 + a_3 x_1^2] \end{aligned} \tag{6.26}$$

There are parameter values $a,\ a_2,\ a_3$ that guarantee the existence of a branch of heteroclinic cycles in V_1 for $\lambda > 0$ between equilibria of the form $(x, 0, 0, 0, 0, 0)$, $(0, 0, x, 0, 0, 0)$, and $(0, 0, 0, 0, x, 0)$. If we choose $a, a_3 < 0,\ a_2 > 0$ and $a_3 + a_2 < 0$, then each cycle exists and is asymptotically stable (for (6.26) defined on V_1). However, in contrast to the examples given in Section 4.3, the block $J = \{1, 3, 5\}$ is not independent for $\mathcal{G} = \mathbf{A}_4(6)$. Thus we cannot apply Theorem 4.10. Nevertheless, we can proceed as follows.

Consider the linear stability of the equilibrium $\xi_1 = (x, 0, 0, 0, 0, 0)$ (for fixed $\lambda > 0$) of (2.1) (that is determined by the eigenvalues of $(\mathrm{D}_x F)_{\xi_1}$). This matrix commutes with $\Sigma(\{\mathbf{1}\}, \{1\}) = \{\mathbf{1}\} \times {\mathbf{Z}_2}^5 \dot{+} \mathcal{G}_1$, so

$$(\mathrm{D}_x F)_{\xi_1} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$$

where $\lambda_2 = \lambda_5$ and $\lambda_3 = \lambda_6$ since $\mathcal{G}_1 =< (2, 5)(3, 6) >$. Moreover,

$$\begin{aligned} \lambda_1 &= 2ax^2 \\ \lambda_2 &= \lambda_5 = a_3 x^2 \\ \lambda_3 &= \lambda_6 = a_2 x^2 \\ \lambda_4 &= a_4 x^2. \end{aligned}$$

The eigenvalue λ_1 is the radial eigenvalue. We choose $a < 0$. The eigenvalue λ_3 is associated to the expanding direction of ξ_1 in V_1 that gives rise to a connection between ξ_1 and ξ_3. Therefore, if we choose $a, a_3, a_4 < 0$ and $a_2 > 0$, then the unstable manifold of ξ_1 is 2-dimensional. Moreover,

$$W^u(\xi_1) \subseteq H_1 = \{(x_1, 0, x_3, 0, 0, x_6) :\ x_1, x_3, x_6 \in \mathbf{R}\}$$

and H_1 is a flow-invariant space. Consider (2.1) restricted to H_1:

$$\begin{aligned} \dot{x}_1 &= x_1[\lambda + a(x_1^2 + x_3^2 + x_6^2) + a_3(x_3^2 + x_6^2)] \\ \dot{x}_3 &= x_3[\lambda + a(x_1^2 + x_3^2 + x_6^2) + a_2 x_1^2 + a_4 x_6^2] \\ \dot{x}_6 &= x_6[\lambda + a(x_1^2 + x_3^2 + x_6^2) + a_2 x_1^2 + a_4 x_3^2] \end{aligned} \tag{6.27}$$

Then there exist connections between ξ_1 and ξ_3 (in the x_1x_3-plane), and ξ_1 and ξ_6 (in the x_1x_6-plane). Moreover, since $a_2a_3 < 0$ and $a_2a_4 < 0$, there are no nontrivial solutions in the x_1x_3-plane and the x_1x_6-plane except for ξ_1, ξ_3, ξ_6 and their conjugates under the $\mathbf{Z}_2$-symmetries. Consider (6.27) restricted to the x_3x_6-plane:

$$\begin{aligned} \dot{x}_3 &= x_3[\lambda + a(x_3^2 + x_6^2) + a_4x_6^2] \\ \dot{x}_6 &= x_6[\lambda + a(x_3^2 + x_6^2) + a_4x_3^2] \end{aligned} \tag{6.28}$$

The lines $x_3 = x_6$ and $x_3 = -x_6$ are flow-invariant. Denote by ξ_{36} the equilibrium with isotropy $\Sigma(\{\mathbf{1}\}, \{3, 6\})$. The linear stability of ξ_{36} in the x_3x_6-plane is determined by eigenvalues whose signs are those of $-2a_4$ and $4a + 2a_4$, and connections between ξ_3 and ξ_{36}, and ξ_6 and ξ_{36}, exist in this plane, provided that $2a_3 - a_4 < 0$ and a is sufficiently negative such that $3a + a_2 < 0$ (and also $a_3, a_4 < 0, a_2 > 0$). A straightforward calculation shows that, for parameter values satisfying the above conditions, the only nontrivial equilibria for (6.27) are those with isotropy subgroups conjugate to those of ξ_1 (ξ_3 and ξ_6) and ξ_{36}.

6.1 The Invariant Sphere Theorem

To make further progress, we apply the Invariant Sphere Theorem of Field [14] to Example 6.2.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $V = \mathbf{R}^N$, and $\| \cdot \|$ the corresponding Euclidean norm on V. Let $\mathbf{S}^{N-1}$ denote the unit sphere of $\mathbf{R}^N$. Consider a vector field F of the form

$$F(x, \lambda) = \lambda x + Q(x) \tag{6.29}$$

where $Q : V \to V$ is a homogeneous polynomial map of degree $2d+1$ ($d > 0$). Let the *phase vector field* $\mathcal{P}^Q$ of Q be the vector field on $\mathbf{S}^{N-1}$ defined by

$$\mathcal{P}^Q(u) = Q(u) - \langle Q(u), u \rangle u, \quad u \in \mathbf{S}^{N-1}.$$

The vector field Q is *contracting* if

$$\langle Q(x), x \rangle < 0 \quad \text{for all } x \neq 0.$$

Then we recall the Invariant Sphere Theorem:

Theorem 6.3 *Let $Q : V \to V$ be a homogeneous polynomial map of degree $2d + 1$ ($d > 0$) and suppose that Q is contracting. For every $\lambda > 0$ there exists a unique $(N-1)$-dimensional sphere $\mathbf{S}(\lambda) \subseteq \mathbf{R}^N \setminus \{0\}$ which is invariant under the flow of $\dot{x} = F(x, \lambda)$ for F as in (6.29). Moreover:*
(i) $\mathbf{S}(\lambda)$ is embedded as a topological submanifold of $\mathbf{R}^N$, and the bounded component of the complement of $\mathbf{S}(\lambda)$ contains the origin of $\mathbf{R}^N$.
(ii) $\mathbf{S}(\lambda)$ is globally attracting in the sense that every trajectory of $\dot{x} = F(x, \lambda)$ through $x \neq 0$ is forward asymptotic to $\mathbf{S}(\lambda)$.
(iii) The dynamics of $\dot{x} = F(x, \lambda)$ on $\mathbf{S}(\lambda)$ is topologically equivalent to that of $\mathcal{P}^Q$ on $\mathbf{S}^{N-1}$.

Proof. This is Theorem 5.1 of Field [14]. ∎

When Q is contracting, this theorem allows us to reduce the study of the ODE $\dot{x} = F(x,\lambda)$ to the flow of the (λ-independent) vector field $\mathcal{P}^Q$ on $\mathbf{S}^{N-1}$. All dynamical properties of the flow of $\mathcal{P}^Q$ yield corresponding properties for the flow of $\dot{x} = F(x,\lambda)$. Since the groups Γ that we are considering here are 3-determined, properties such as hyperbolicity of zeros, if satisfied for $\mathcal{P}^Q$ and F on an open subset of parameters, will also hold for $f \in \vec{\mathcal{P}}_V(\Gamma)$ whenever the cubic truncation of f coincides with F. If we denote the eigenvalues of $(\mathrm{D}_x F)_x$ by $\lambda_1, \ldots, \lambda_N$ (where x is an equilibrium for $\dot{x} = F(x,\lambda)$), with λ_1 being the radial eigenvalue, then the eigenvalues of the linearization of $\mathcal{P}^Q$ at $u = x/\|x\| \in \mathbf{S}^{N-1}$ are $\lambda_2/\|x\|^2, \ldots, \lambda_N/\|x\|^2$.

We return now to example 6.2. Here

$$Q(x) = a\|x\|^2 x + a_2 g_2(x) + a_3 g_3(x) + a_4 g_4(x)$$

and

$$\begin{aligned}\langle Q(x), x\rangle = \;& a\|x\|^4 + (a_2+a_3)[(x_2^2+x_5^2)(x_1^2+x_4^2) + (x_3^2+x_6^2)(x_2^2+x_5^2) + \\ & (x_4^2+x_1^2)(x_3^2+x_6^2)] + 2a_4(x_1^2x_4^2 + x_2^2x_5^2 + x_3^2x_6^2)]\end{aligned}$$

If $a, a_2+a_3, a_4 < 0$ then Q is contracting. If $V^+ = \{x \in V : \; x_1, \ldots, x_6 \geq 0\}$, then $\Lambda_5 = V^+ \cap \mathbf{S}^5$ is a $\mathcal{G}$-invariant *fundamental domain* for the action of $\Delta_6 = {\mathbf{Z}_2}^6$ on $\mathbf{S}^5$. That is, Λ_5 has the following properties:

- $\cup_{\gamma\in\Delta_6}\gamma\Lambda_5 = \mathbf{S}^5$.
- For all $\gamma \in \Delta_6$, $\gamma(\mathrm{int}(\Lambda_5)) \cap \mathrm{int}(\Lambda_5) \neq \emptyset$ if and only if $\gamma = \mathbf{1}$. (Here int denotes the interior.)
- Λ_5 is $\mathcal{G}$-invariant.

Note that $\mathcal{G}$ acts on Δ_6 by permutation of the six coordinates. Let $H_1^+ = \{x \in H_1 : x_i \geq 0\} \subseteq V^+$ and $\Lambda_2 = H_1^+ \cap \mathbf{S}^6 \subset \Lambda_5$. That is, Λ_2 is a spherical triangle. Moreover H_1 is flow-invariant for G, and the dynamics of G restricted to H_1^+ is equivalent to the dynamics of $\mathcal{P}^Q$ restricted to Λ_2. See Figure 3 for a schematic picture.

If the cycle exists, it necessarily involves the equilibria ξ_1, ξ_3, ξ_5 (recall that G restricted to V_1 is $\mathbf{Z}_2 \wr \mathbf{Z}_3$-equivariant) and ξ_2, ξ_4, ξ_6 (same reason) — but it *also* includes the equilibrium ξ_{36} which is not conjugate to any ξ_i. The linear stability of ξ_{36} is given by eigenvalues with signs determined by $2a_2 - a_4 > 0$ (multiplicity 2), $-2a_4 > 0$, and $2a_3 - a_4 < 0$ (multiplicity 2).

Since ξ_{36} has unstable directions not contained in H_1 we study now (2.1) restricted to the unstable manifold of ξ_{36}, $W^u(\xi_{36})$, that is 3-dimensional. Moreover,

$$W^u(\xi_{36}) \subseteq H_2 = \{(0, x_2, x_3, 0, x_5, x_6) : \; x_2, x_3, x_5, x_6 \in \mathbf{R}\}$$

and H_2 is a flow-invariant space. Consider (2.1) restricted to H_2:

$$\begin{aligned}\dot{x}_2 &= x_2[\lambda + a(x_2^2+x_3^2+x_5^2+x_6^2) + a_2(x_3^2+x_6^2) + a_4x_5^2] \\ \dot{x}_3 &= x_3[\lambda + a(x_2^2+x_3^2+x_5^2+x_6^2) + a_3(x_5^2+x_2^2) + a_4x_6^2] \\ \dot{x}_5 &= x_5[\lambda + a(x_2^2+x_3^2+x_5^2+x_6^2) + a_2(x_3^2+x_6^2) + a_4x_2^2] \\ \dot{x}_6 &= x_6[\lambda + a(x_2^2+x_3^2+x_5^2+x_6^2) + a_3(x_5^2+x_2^2) + a_4x_3^2]\end{aligned} \tag{6.30}$$

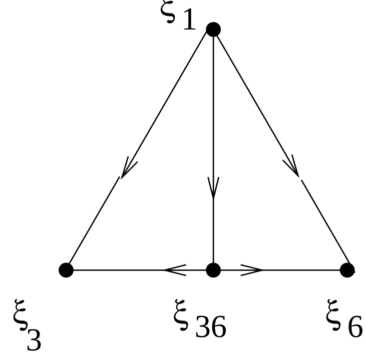


Figure 3: Schematic picture for the dynamics of $\mathcal{P}^Q$ on Λ_2.

Let $H_2^+ = \{x \in H_2 : \ x_i \geq 0\} \subseteq V^+$ and $\Lambda_3 = H_2^+ \cap \mathbf{S}^6 \subset \Lambda_5$, that can be represented as a (regular) tetrahedron. Note that H_2 is flow-invariant for G, and the dynamics of G restricted to H_2^+ is equivalent to the dynamics of $\mathcal{P}^Q$ restricted to Λ_3. Denote by ξ_{25} an equilibrium with isotropy $\Sigma(\{\mathbf{1}\},\{2,5\})$. Connections between ξ_{36} and ξ_2, ξ_5, ξ_{25}, and connection between ξ_3 and ξ_{25} are guaranteed provided $a + a_2 < 0$. Moreover, no equilibria except $\xi_2, \xi_3, \xi_5, \xi_6$ and ξ_{36}, ξ_{25} (and their conjugates by $\mathbf{Z}_2{}^6$) occur for (6.30). See Figure 4 for a schematic picture containing the connections for $\mathcal{P}^Q$ restricted to Λ_3 (and so for (6.30) restricted to H_2^+).

This information is not sufficient to establish the existence of a heteroclinic web, however, because there might in principle exist more complicated attractors (limit cycles, chaotic attractors) that prevent all trajectories that start in $W^u(\xi_{36}) \setminus \{\xi_{36}\}$, say, from limits on other equilibria. In the absence of suitable invariant planes, the Poincaré-Bendixson Theorem no longer assists in ruling out such attractors.

We discuss now the asymptotic stability of the web (supposing that the existence question can be resolved — perhaps by computer-assisted analysis).

The heteroclinic web includes (group orbits of) equilibria $A = \xi_1$ and $B = \xi_{36}$. There are three kinds of routes that we have to consider:

$$A \rightarrow A \tag{6.31}$$
$$B \rightarrow B \tag{6.32}$$
$$A \rightarrow B \rightarrow A. \tag{6.33}$$

For the asymptotic stability, we need

$$a_2 + \max(a_3, a_4) < 0 \tag{6.34}$$

for (6.31) and

$$(2a_3 - a_4) + \max(-2a_4,\ 2a_2 - a_4) < 0 \tag{6.35}$$

for (6.32). The stability for (6.33) is implied by (6.34) and (6.35).

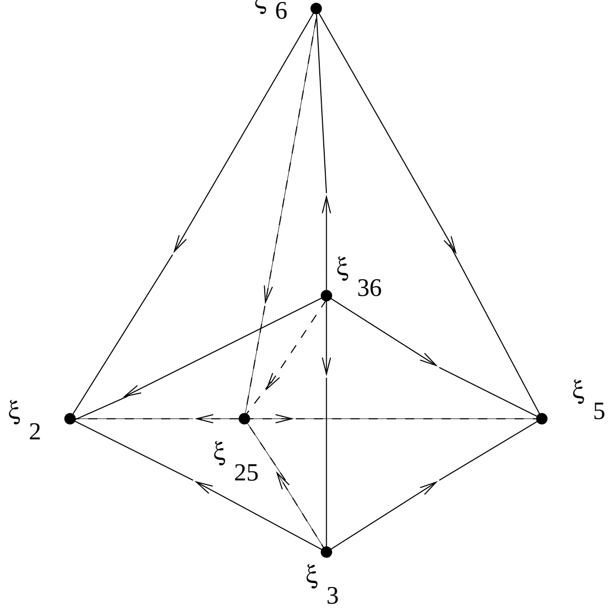


Figure 4: Schematic picture with heteroclinic connections on Λ_3.

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