Hopf bifurcation on a primitive cubic lattice

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Abstract. We study Hopf bifurcation for differential equations defined on the space of functions on $\mathbf{R}^3$ which are triply periodic with respect to a primitive cubic lattice. The center manifold theorem reduces the problem to a system of ordinary differential equations on the space $(\mathbf{C} \oplus \mathbf{C})^3$ and symmetric by the group $(\mathcal{O} \oplus \mathbf{Z}_2^c)\dot{+}\mathbf{T}^3$. We abstract this group as the wreath product group $\mathbf{O}(2) \wr \mathbf{S}_3$. We use the wreath product group theory to find (up to conjugacy) all branches of periodic solutions with maximal isotropy. The stability of these solutions is calculated. Branches of periodic solutions with submaximal isotropy can also exist. Some possibilties of bifurcations to heteroclinic cycles are explored.

1 Introduction

Systems of partial differential equations with Euclidean symmetry can arise when abstracting applications in the physical sciences. In many doubly diffusive convection systems, which have Euclidean symmetry, the heat-conduction solution loses stability via Hopf bifurcation. This occurs, for example, in magnetoconvection [2], thermosolutal convection [30], and binary fluid convection [17]. Many pattern-forming systems produce approximately spatially periodic patterns. We are interested in patterns arising in a system with Euclidean symmetry from a Hopf bifurcation, that is, we look for time-periodic, spatially periodic solutions arising by a bifurcation from an invariant equilibrium (as a parameter is varied). Previous studies of Hopf bifurcation in Euclidean equivariant systems, leading to doubly periodic patterns in a plane, explored periodicity with respect to an hexagonal lattice [25], a square lattice [27] and a rhombic lattice [28]. The results obtained by Roberts *et al.* [25] were used to investigate Hopf bifurcation in the two-layer Bénard problem [23], in the Bénard problem for a viscoelastic liquid [24], and in thermosolutal convection [22]. The analysis obtained by Silber and Knobloch [27] was useful to clarify the numerical results obtained in a three-dimensional compressible magnetoconvection in a square box with periodic boundary conditions in the horizontal directions [19]. In [28], Silber *et al.* work on the oscillatory instability of spatially anisotropic two-dimensional hydrodynamic systems. Little similar work has been done for patterns in three dimensions. There are a large number of bifurcation problems that can be studied in three dimensions (see for example [1]), but it is not always clear if there is a particular physical situation for which the results can be applied. However, numerical simulation of model equations, such as the Brusselator model, in three dimensions revealed the presence of spatially periodic patterns with non-trivial three-dimensional structure [3].

In this paper we focus attention on patterns that are periodic on a primitive cubic lattice in three dimensions and form by Hopf bifurcation. Hopf bifurcation in a three-dimensional lattice seems not to have been investigated before. Consider a system of partial differential equations posed on $\mathbf{R}^3$ with Euclidean symmetry $\mathbf{E}(3)$ (group of all symmetries in the space that preserve distances, i.e., translations, rotations and reflections). As a parameter is varied, time-periodic, spatially periodic solutions can arise by a bifurcation from an invariant equilibrium. We restrict attention to solutions that are triply spatially periodic with respect to a primitive cubic lattice. There is a natural compact group of symmetries acting on this space, which is derived from the action of the group $\mathbf{E}(3)$ on the initial space of functions for the PDE. This group is $\Gamma = H \dot{+} \mathbf{T}^3$, where $H = \mathcal{O} \oplus \mathbf{Z}_2^c$ is the holohedry of the lattice and $\mathbf{T}^3$ represents the three-torus of translations. The center manifold theorem [18] reduces the original system to a system of differential equations posed on a finite-dimensional vector space and commuting with an induced action of Γ on this space. We consider here the case when the symmetry forces the dimension of the space to be twelve, i.e., when it has the form $W \oplus W$ where W is Γ-absolutely irreducible (and six-dimensional). In this case, we can view this group in the context of the wreath product construction in group theory, that is, the group Γ is the wreath product group $\mathbf{O}(2) \wr \mathbf{S}_3$ (this remark is explained in more detail in the next section). Using the theory developed by Golubitsky *et al* [13], we can give a complex structure to the space $W \oplus W$, define a natural action of the circle group $\mathbf{S}^1$ on this space and use the equivariant Hopf theorem to find branches of periodic solutions with symmetry involving (maximal isotropy) subgroups with a two-dimensional fixed-point subspace (the $\mathbf{C}$-axial subgroups) of the group $\Gamma \times \mathbf{S}^1$. Each of these periodic solutions has the corresponding Σ as its group of "spatio-temporal" symmetries.

In this paper, we prove that generically, up to conjugacy, there are eight branches of periodic solutions with maximal isotropy that bifurcate from the trivial solution at generic Hopf bifurcation with symmetry $\mathbf{O}(2) \wr \mathbf{S}_3$. Moreover, the stability for these solutions depends only on the terms of degree three of the vector field (and the truncated vector field can be supposed to commute with $\mathbf{S}^1$). Solutions with submaximal isotropy can also exist for an open region of the parameter space. We are able to prove the existence of these branches only for the Birkhoff normal form of the original vector field of degree three. The existence of one branch of periodic solutions with submaximal isotropy that are generically unstable is justified by [29]. Regions of the parameter space were found for which one or two new branches of solutions with submaximal isotropy can bifurcate from the trivial solution. We can use the classification obtained by the work developed in this paper and the results of [27] to have a partial classification of the dynamics of the generic $\mathbf{O}(2) \wr \mathbf{S}_3$-equivariant Hopf problem.

The paper is organized as follows. In the next section, we explain the necessary assumptions that allow the formulation of the generic Hopf bifurcation problem on a primitive cubic lattice as a Hopf bifurcation problem with symmetry $\mathbf{O}(2) \wr \mathbf{S}_3$ posed on a twelve-dimensional centre manifold. In section 3, we state the results of Dias [4] and Dionne *et al.* [8] with respect to the classification of the isotropy lattice of $\Gamma \times \mathbf{S}^1$, if Γ is a general wreath product group $\mathcal{L} \wr \mathcal{G}$. Using these results, we find (in theorem 3.1) eight conjugacy classes of $\mathbf{C}$-axial subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ (that are maximal isotropy subgroups). Here we use the identification of $W \oplus W$ with $(\mathbf{C} \oplus \mathbf{C})^3$ and we consider the induced action of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ on this space. We also obtain in theorem 3.3 the complete isotropy lattice of this group. In section 4 we find the most general form of a commuting mapping with $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ of degree three (proposition 4.1). In section 5 we use the equivariant Hopf theorem to prove the existence for a generic Hopf problem with symmetry $\mathbf{O}(2) \wr \mathbf{S}_3$ of branches of periodic solutions with symmetry each one of the maximal isotropy groups, provided that some generic condition on the vector field is satisfied. In theorem 5.2 we analyze the branching directions and the stability of these solutions in terms of the Taylor expansion of the vector field commuting with $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$. We prove that, generically, these depend only on the coefficients of degree lower or equal three of the Taylor expansion of the vector field around the origin. Moreover, we show that the $\mathbf{C}$-axial subgroups have three-determined stability and so we can use the theory developed by Golubitsky *et al.* [13] to justify that the same stability results hold even when the original vector field commutes only with $\mathbf{O}(2) \wr \mathbf{S}_3$. In section 6 we show that branches of periodic solutions with submaximal isotropy type can also bifurcate generically. The Hopf bifurcation problem on a square lattice reduced to the Hopf bifurcation with symmetry $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ (on a eight-dimensional vector space) studied by [27] is included in our problem: specifically, there is an isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ with eight-dimensional fixed-point subspace such that the elements of the normalizer of Σ that act nontrivially on $\mathrm{Fix}(\Sigma)$ form the group $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$. Finally, in section 7, we explore some possibilities for heteroclinic cycles between periodic solutions.

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2 Problem formulation

Consider a system of partial differential equations posed on $\mathbf{R}^3$ with Euclidean symmetry $\mathbf{E}(3)$. As a parameter is varied, time-periodic, spatially periodic solutions can arise by a bifurcation from an invariant equilibrium. We restrict attention to solutions u that are triply spatially periodic with respect to a primitive cubic lattice, i.e.,

$$u(X) = u(X + n_1\omega_1 + n_2\omega_2 + n_3\omega_3),$$

for any integers n_1, n_2, n_3 and three fixed vectors that generate a primitive cubic lattice [21]: we choose $\omega_1 = (1,0,0)$, $\omega_2 = (0,1,0)$ and $\omega_3 = (0,0,1)$. We suppose that for all the parameter values there is a trivial solution (which is therefore an $\mathbf{E}(3)$-invariant equilibrium) and that this equilibrium loses stability at some value of the parameter for which the linearization of the equation has conjugate purely imaginary eigenvalues. We can write such a system of PDEs in the form

$$\frac{d}{dt}u = F(u, \lambda)$$

with $F : \mathcal{X} \times \mathbf{R} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are suitable function spaces, the bifurcation parameter is $\lambda \in \mathbf{R}$, and $u(X,t)$ is a vector-valued function of $X \in \mathbf{R}^3$ and time t. We assume that $F(0,\lambda) = 0$ for all values of λ, and we look for time-periodic solutions by Hopf bifurcation from $u = 0$ at $\lambda = 0$ that respect the condition

$$u(X + l, t) = u(X, t)$$

for all $l \in \mathcal{L}$, where

$$\mathcal{L} = \{n_1\omega_1 + n_2\omega_2 + n_3\omega_3,\ n_1, n_2, n_3 \in \mathbf{Z}\}.$$

These solutions are in general called *planforms*. Moreover, we assume that F is $\mathbf{E}(3)$-equivariant, where the action of $\mathbf{E}(3)$ on functions u that we consider here is defined by

$$g \cdot u(X,t) = u(g^{-1}X,t)$$

for all $g \in \mathbf{E}(3)$, and so

$$F : \mathcal{X}_{\mathcal{L}} \times \mathbf{R} \to \mathcal{Y}_{\mathcal{L}}, \tag{1}$$

where $\mathcal{X}_{\mathcal{L}}$ and $\mathcal{Y}_{\mathcal{L}}$ represent the spaces of all $\mathcal{L}$-periodic functions in $\mathcal{X}$ and $\mathcal{Y}$ respectively.

There is a natural compact group of symmetries acting on the space of the $\mathcal{L}$-periodic solutions, which is derived from the action of the group $\mathbf{E}(3)$ on the spaces $\mathcal{X}$ and $\mathcal{Y}$. This group is

$$\Gamma = H \dot{+} \mathbf{T}^3$$

where $H = \mathcal{O} \oplus \mathbf{Z}_2^c$ is the holohedry of the lattice $\mathcal{L}$, i.e., the largest subgroup of $\mathbf{O}(3)$ that leaves $\mathcal{L}$ invariant [6, 7], and $\mathbf{T}^3$ represents the three-torus translations. The group Γ is the largest group that can be constructed from $\mathbf{E}(3) = \mathbf{O}(3) \dot{+} \mathbf{R}^3$ that acts on $\mathcal{X}_{\mathcal{L}}$ (and leaves it invariant) and F in (1) is Γ-equivariant. The group $\mathcal{O} \oplus \mathbf{Z}_2^c$ is the symmetry group of the centred cube on $\mathbf{R}^3$, where $\mathcal{O}$ denotes the group of all orientation-preserving symmetries of the cube and $\mathbf{Z}_2^c$ represents inversion through the origin. It is instructive to view this group in the context of the wreath product construction in general group theory.

Following [8, 12], if $\mathcal{L} \subseteq \mathbf{O}(k)$ and $\mathcal{G}$ is a subgroup of the permutation group $\mathbf{S}_N$, we denote the group $\mathcal{L}^N \dot{+} \mathcal{G}$ (group generated by the groups $\mathcal{L}^N$ and $\mathcal{G}$) by $\mathcal{L} \wr \mathcal{G}$, where $\wr$ stands for *wreath product*. These kind of groups are focused when studying nonlinear dynamics and bifurcations of systems of coupled identical 'cells' (dynamical subsystems such as oscillators) that possess certain types of symmetries. That is, when the individual cells have a 'local' (or 'internal') group of symmetries $\mathcal{L}$, and the network of couplings have a 'global' symmetry group $\mathcal{G}$. The wreath product group $\mathcal{L} \wr \mathcal{G}$ is the total symmetry group of the coupled system when the coupling is invariant under any local symmetry of any cell. More details can be found in [8, 12]. We remark that we can abstract $\mathcal{O} \oplus \mathbf{Z}_2^c$ as the wreath product group $\mathbf{Z}_2 \wr \mathbf{S}_3$ (see also [9]) and so $(\mathcal{O} \oplus \mathbf{Z}_2^c) \dot{+} \mathbf{T}^3$ is the wreath product

$$\Gamma = \mathbf{O}(2) \wr \mathbf{S}_3.$$

For F in (1), the space $\ker(dF)_{0,0}$ is finite-dimensional and we demand that

$$U = \ker\ (dF)_{0,0} \neq \{0\}$$

for bifurcation to occur. We scale time so that the (only) purely imaginary eigenvalues of $(dF)_{0,0}$ are $\mp i$.

We consider here the case when the symmetry forces the dimension of U to be twelve, i.e., when the real generalized eigenspace associated with $\mp i$ is of the form $W \oplus W$ where W is Γ-absolutely irreducible (and six-dimensional).

The space U can be identified with the six-dimensional complex vector space spanned by the six travelling waves

$$e^{i2\pi(t-\omega_j\cdot X)}, \qquad j=1,\ldots,6, \tag{2}$$

where $\omega_4 = -\omega_1$, $\omega_5 = -\omega_2$ and $\omega_6 = -\omega_3$. Here, we are assuming that the functions in $\mathcal{X}_\mathcal{L}$ are regular enough to have Fourier expansions in terms of these travelling waves. The six wave vectors ω_i for $i=1,\ldots,6$ were selected at the midpoint of each of the six surfaces of a cube. Explicitly, we can identify $(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{C}^6$ with the function $u : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ defined by

$$u(X,t) = \mathrm{Re}\left\{\sum_{j=1}^{3} z_{2j-1}e^{i2\pi(t-\omega_j\cdot X)} + \sum_{j=1}^{3} z_{2j}e^{i2\pi(t+\omega_j\cdot X)}\right\}. \tag{3}$$

The center manifold theorem [18] reduces the original system to a system of differential equations

$$\dot{v} = f(v,\lambda) \tag{4}$$

on U, and it is this system that it is studied in this paper. The equivariance of F with respect to $\mathbf{E}(3)$ implies that this reduced system of differential equations can be chosen to be equivariant with respect to the induced action on U (using the above identification) of the group Γ of $\mathbf{E}(3)$ that preserves U. In the next section we discuss in more detail this action.

As stated above, previous studies of Hopf bifurcation leading to time-periodic spatially periodic solutions in Euclidean equivariant systems have been considered before. In [25], Roberts *et al.* considered the six-dimensional irreducible representation of $\mathbf{D}_6\dot{+}\mathbf{T}^2$, which is associated with an hexagonal lattice. In [27], Silber and Knobloch considered the four-dimensional irreducible representation of $\mathbf{D}_4\dot{+}\mathbf{T}^2$ associated with a square lattice. In [28], Silber *et al.* study Hopf bifurcation for the rhombic lattice, where the group is $\mathbf{D}_2\dot{+}\mathbf{T}^2$.

In [13], Golubitsky *et al.* show that at points of Hopf bifurcation in Γ-equivariant systems, generically the centre subspace U is Γ-simple (that is, $U \cong W \oplus W$ where W is absolutely irreducible for Γ or U is non-absolutely irreducible for Γ) and there is a complex structure on this space with a natural action of the circle group $\mathbf{S}^1$: this action is defined by "phase-shifting" the waves $e^{i2\pi(t-\omega_j\cdot X)}$. The equivariant Hopf theorem [13] states that, under appropriate conditions there will be a symmetry-breaking branch of periodic solutions with symmetry involving isotropy subgroups of $\Gamma \times \mathbf{S}^1$ with a two-dimensional fixed-point subspace (the $\mathbf{C}$-axial subgroups). Moreover, periodic solutions of the original ODE (4) of period near 2π are in one-to-one correspondence with zeros of a reduced bifurcation equation

$$g(v,\lambda,\tau) = 0, \tag{5}$$

where if U is the $\mp i$ real eigenspace of $(df)_{0,0}$, then $g : U \times \mathbf{R} \times \mathbf{R} \to U$ is C^∞ and commutes with $\Gamma \times \mathbf{S}^1$. These are zeros of the restricted system of equations $g\mid_{\mathrm{Fix}(\Sigma)} = 0$, and recall that the fixed-point subspaces are invariant for g. Moreover, when we assume that f also commutes with $\mathbf{S}^1$, applying results of [13], we can get an explicit form of g and the stability of a periodic solution is determined by the eigenvalues of the derivative of g at the corresponding zero. Also, these stability results can hold even when f does not commute with $\mathbf{S}^1$, provided some conditons on the eigenvalues are satisfied. In section 4 we apply these results.

3 Isotropy groups

Our aim now is to classify the possible symmetries that periodic solutions can have. For that we need to describe the isotropy lattice of $\Gamma \times \mathbf{S}^1$, where $\Gamma = (\mathcal{O} \oplus \mathbf{Z}_2^c) \dot{+} \mathbf{T}^3$, which as stated before, is isomorphic to the wreath product group $\mathbf{O}(2) \wr \mathbf{S}_3$.

We start by fixing some notation and by reviewing the classification results obtained by [4, 8] concerning the isotropy lattice of $\Gamma \times \mathbf{S}^1$, where Γ is a general wreath product group. We will use these results for $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_3$.

3.1 Isotropy subgroups of wreath product groups

Let $\mathcal{G}$ be a transitive subgroup of $\mathbf{S}_N$ and $\mathcal{L} \subseteq \mathbf{O}(k)$ be a group acting on $V = \mathbf{R}^k$. Consider the group $\Gamma = \mathcal{L} \wr \mathcal{G}$ acting on the space V^N by

$$(l, \sigma).(X_1, \ldots, X_N) = (l_1 X_{\sigma^{-1}(1)}, \ldots, l_N X_{\sigma^{-1}(N)}),$$

where $l = (l_1, \ldots, l_N) \in \mathcal{L}^N$, $\sigma \in \mathcal{G}$ and $(X_1, \ldots, X_N) \in V^N$. Thus the permutations act on $l \in \mathcal{L}^N$ by

$$\sigma(l) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(N)})$$

and it follows that the group multiplication in $\mathcal{L} \wr \mathcal{G}$ is given by

$$(h, \tau)(l, \sigma) = (h\tau(l), \tau\sigma).$$

See for example [26].

Suppose that $\mathcal{L}^N$ acts nontrivially on $U = V^N$. When considering Hopf bifurcation in ODEs commuting with Γ, we may assume that Γ acts Γ-simply on the centre subspace: this is a generic hypothesis [13]. Recall that a representation U of Γ is Γ-*simple* if either $U \cong W \oplus W$ where W is absolutely irreducicle for Γ, or U is non-absolutely irreducible for Γ. Dionne *et al.* prove that, when Γ is a wreath product group $\mathcal{L} \wr \mathcal{G}$, then the space $U = V^N$ is Γ-simple if and only if V is $\mathcal{L}$-simple.

We denote by Σ_w the *isotropy subgroup* of $w \in U$ in $\Gamma \times \mathbf{S}^1$, that is,

$$\Sigma_w = \{\gamma \in \Gamma \times \mathbf{S}^1 \ : \ \gamma \cdot w = w\},$$

and by $\mathrm{Fix}(\Sigma_w)$ the *fixed-point subspace* of $\Sigma_w \subset \Gamma \times \mathbf{S}^1$ in U defined by

$$\mathrm{Fix}(\Sigma_w) = \{w \in U \ : \ \gamma \cdot w = w, \ \forall \, \gamma \in \Sigma_w\}.$$

We note that points on the same group orbit by the group $\Gamma \times \mathbf{S}^1$ have conjugate isotropy subgroups.

An important result when studying the dynamics of ODEs involving a vector field commuting with $\Gamma \times \mathbf{S}^1$ is that the spaces $\mathrm{Fix}(\Sigma_w)$ are invariant under the dynamics. We use this result to find periodic solutions with specific symmetries. Moreover, when we restrict a vector field to a subspace $\mathrm{Fix}(\Sigma_w)$, this restricted vector field is equivariant by the *normalizer* of Σ_w in $\Gamma \times \mathbf{S}^1$, that we denote by $\mathcal{N}(\Sigma_w)$. Recall that

$$\mathcal{N}(\Sigma_w) = \{\gamma \in \Gamma \times \mathbf{S}^1 \ : \ \gamma^{-1}\Sigma_w\gamma = \Sigma_w\}.$$

In order to apply the equivariant Hopf theorem we are interested in finding the $\mathbf{C}$-axial groups of $\Gamma \times \mathbf{S}^1$. As mentioned before, we say that a subgroup $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-*axial* if it is an isotropy subgroup having a two-dimensional fixed-point subspace (over $\mathbf{R}$).

In the classification of $\mathbf{C}$-axial subgroups of wreath product groups $\mathcal{L} \wr \mathcal{G}$, the structure of these subgroups is determined by the possible blocks that are derived from the permutation group $\mathcal{G}$, where a subset of indices $J \subseteq \{1, \ldots, N\}$ is said to be a *block*, if there exists a subgroup $\mathcal{H}$ of $\mathcal{G}$ that leaves J invariant and acts transitively on J.

To each block J we can associate the permutation subgroup

$$Q_J = \{\sigma \in \mathcal{G} : \sigma(J) = J\}$$

which acts transitively on J. Assuming that V^N is Γ-simple, consider the natural action of $\mathbf{S}^1$ on this space obtained by giving it a complex structure as in [13]. Consider a block J and let B^ψ be a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$ (acting on V) where $\psi : B \to \mathbf{S}^1$ is an homomorphism and

$$B^\psi = \{(b, \psi(b)) : b \in B\}.$$

Following [13] we call the group B^ψ a *twisted* subgroup of $\mathcal{L} \times \mathbf{S}^1$. The image $\psi(B)$ is a closed subgroup of $\mathbf{S}^1$. The closed subgroups of $\mathbf{S}^1$ are $\mathbf{1}$, $\mathbf{Z}_n$ ($n = 2, 3, 4, \ldots$) and $\mathbf{S}^1$. We say that B^ψ is of *finite* twist type if the image $\psi(B)$ is not $\mathbf{S}^1$. Also B^ψ is said to be of type $\mathbf{Z}_n$ or $\mathbf{S}^1$ if $\psi(B) = \mathbf{Z}_n$ or $\mathbf{S}^1$ respectively. Recall that proper isotropy subgroups of $\Gamma \times \mathbf{S}^1$ acting on a Γ-simple space are twisted subgroups [13].

We are interested in the cases where $\mathcal{L}$ does not act trivially on V and we note that for these actions, $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$.

We describe now subgroups that we denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$. In [4], Dias proves that these subgroups are $\mathbf{C}$-axial and that any $\mathbf{C}$-axial subgroup is conjugate to one of these.

The group $\Sigma(B^\psi, J, \sigma, J_1, p)$

Consider a block $J \subseteq \{1, \ldots, N\}$ and let Q_J be the subgroup of $\mathcal{G}$ that leaves J invariant. Suppose

$$J = \{1, \ldots, s\}.$$

Let J_1 be a subset of J such that for some permutation $\sigma \in Q_J$

$$J = J_1 \dot{\cup} \sigma(J_1) \dot{\cup} \cdots \dot{\cup} \sigma^{s'-1}(J_1),$$

where $\dot{\cup}$ is disjoint union and

$$\sigma^{s'}(J_1) = J_1.$$

In particular it follows that $|J| = s'|J_1|$.

Choose notation so that

$$J_{i+1} = \sigma^i(J_1), \quad i = 1, \ldots, s' - 1$$

and let

$$Q_{J,J_1} = \{\tau \in Q_J : \tau(J_i) = J_i,\ i = 1, \ldots, s'\}.$$

Suppose Q_{J,J_1} acts transitively on J_1. This implies that Q_{J,J_1} acts transitively on all J_i.

Note that by definition of block the group Q_J acts transitively on J. Therefore $\sigma = 1$ and $J_1 = J$ are under those conditions.

Define

$$Q_{J,J_k} = \{\tau \in Q_J : \tau(J_j) = \sigma^{k-1}(J_j),\ j = 1, \ldots, s'\},$$

for $k = 2, \ldots, s'$. That is, each permutation in Q_{J,J_k} interchanges the subsets J_i of J in the same way as σ^{k-1}.

Let B^ψ be a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$ of finite twist type $\mathbf{Z}_r$, and let $\hat{B}$ be the subgroup of B^s defined by

$$\hat{B} = \{(b_1, \cdots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Let $\mathbf{Z}_p = < \xi_p >$ be a cyclic subgroup of $\mathbf{S}^1$ such that

$$s' = \min_{i>0} \{\xi_p^i \in \mathbf{Z}_r\}.$$

Call $\xi_{r'} = \xi_p^{s'}$. It follows that $\mathbf{Z}_p = \mathbf{Z}_{s'r'}$ where $\mathbf{Z}_{r'} \subseteq \mathbf{Z}_r$.

Define B_k the subgroup of B^s by

$$B_k = \left\{ \quad (b_1, \ldots, b_s) \in B^s : \psi(b_j) = \quad \begin{cases} \xi_{r'}, & \text{if } j \in J_1 \cup \cdots \cup J_{k-1}, \\ 0, & \text{if } j \in J_k \cup \cdots \cup J_{s'} \end{cases} \quad \right\}$$

Finally denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$ the subgroup of $\Gamma \times \mathbf{S}^1$ generated by the following groups:

$$\Sigma(B^\psi, J, \sigma, J_1, p) = ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) \ + \ ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \ +$$

$$+ \ (\mathbf{1}^N, Q_{J,J_1}, 0) \ + \ \bigcup_{k=2,\ldots,s'} ((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1}),$$

where + indicates 'group generated by' as in [8]. Note that this group depends on the block J, the permutation σ (and so on J_1). Also the group Q_{J,J_1} has to act transitively on J_1. Finally, it depends on B^ψ (a **C**-axial subgroup of $\mathcal{L}\times\mathbf{S}^1$) and on the cyclic subgroup $\mathbf{Z}_p$ of $\mathbf{S}^1$ (where some divisor r' of r divides p).

In [4] it is proved that an isotropy subgroup Σ of $\Gamma\times\mathbf{S}^1$ is **C**-axial if and only if it is conjugate to a (**C**-axial) group of the type $\Sigma(B^\psi, J, \sigma, J_1, p)$, for some **C**-axial group B^ψ of $\mathcal{L}\times\mathbf{S}^1$, a block J, a permutation σ of $\mathcal{G}$, a subset J_1 of J, and a nonnegative integer p.

Consider now the groups of the type

$$\Sigma(B^\psi, J) = (\mathbf{1}^N, Q_J, 0) + ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) + ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi).$$

In the above notation, we have $\Sigma(B^\psi, J) = \Sigma(B^\psi, J, 1, J, r)$ if $\psi(B) = \mathbf{Z}_r$. As it is proved in [8], these are **C**-axial subgroups of $(\mathcal{L}\wr\mathcal{G})\times\mathbf{S}^1$. Moreover, in [4], it is proved that a **C**-axial group $\Sigma = \Sigma_w$ of $(\mathcal{L}\wr\mathcal{G})\times\mathbf{S}^1$ is conjugate to a group of the type $\Sigma(B^\psi, J)$ for some block J (where w_1 is assumed nonzero and $B^\psi = \Sigma_{w_1}$) if and only if Σ and Σ_{w_1} are of the same twist type. This case happens for example when Σ_{w_1} is of twist type $\mathbf{S}^1$. Moreover, it is proved that a **C**-axial group Σ_w of $\Gamma\times\mathbf{S}^1$ is of type $\mathbf{S}^1$ if and only if Σ_{w_1} of $\mathcal{L}\times\mathbf{S}^1$ is of type $\mathbf{S}^1$.

In fact any isotropy subgroup of $(\mathcal{L}\wr\mathcal{G})\times\mathbf{S}^1$ can be described using groups with a structure similar to this one obtained for the **C**-axial groups. Take a partition of $\{1,\ldots,N\}$ in p blocks

$$\{1,\ldots,N\} = J_1\cup\cdots\cup J_p,$$

and let $\Sigma_1,\ldots,\Sigma_p$ be isotropy subgroups in $\Gamma\times\mathbf{S}^1$ of the type

$$\Sigma(B_i^{\psi_i}, J_i, \sigma_i, J_1^i, p_i) \tag{6}$$

where now $B_i^{\psi_i}$ is any isotropy subgroup in $\mathcal{L}\times\mathbf{S}^1$, for a part J_1^i of J_i, a permutation σ_i in $\mathcal{G}$ and a nonnegative integer p_i. Let

$$\Sigma = \bigcap_{i=1}^{p}\Sigma_i.$$

In [4] is proved that Σ is an isotropy subgroup of $\Gamma\times\mathbf{S}^1$ acting on V^N and every isotropy subgroup of $\Gamma\times\mathbf{S}^1$ is conjugate to such a Σ.

3.2 Group action

Throughout let $\Gamma = \mathbf{O}(2)\wr\mathbf{S}_3$ and $V = \mathbf{C}\oplus\mathbf{C}$. We consider the standard action of $\mathbf{O}(2)\times\mathbf{S}^1$ on V [10]:

$$\begin{array}{ll}
\theta(z_1,z_2) = (e^{i\theta}z_1, e^{i\theta}z_2), & (\theta\in\mathbf{S}^1)\\
\kappa(z_1,z_2) = (z_2,z_1), & (\kappa = \text{flip in } \mathbf{O}(2))\\
\psi(z_1,z_2) = (e^{-i\psi}z_1, e^{i\psi}z_2) & (\psi\in\mathbf{SO}(2)).
\end{array}$$

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(a, 0),\ a > 0$	$\widetilde{\mathbf{SO}}(2) = \{(\theta, \theta)\}$	$\{(z_1, 0)\}$
$(a, a),\ a > 0$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c = \{(1, 0), (\kappa, 0), (\pi, \pi), (\kappa\pi, \pi)\}$	$\{(z_1, z_1)\}$

Table 1: Orbit representatives, isotropy subgroups and fixed-point subspaces for the standard action of $\mathbf{O}(2) \times \mathbf{S}^1$ on $\mathbf{C} \oplus \mathbf{C}$.

Here $\mathbf{S}_3$ is the group of the permutations of the set $\{1, 2, 3\}$. Also the group multiplication in $\Gamma \times \mathbf{S}^1$ is given by

$$(h, \tau, \theta_1)(l, \sigma, \theta_2) = (h\tau(l), \tau\sigma, \theta_1\theta_2)$$

and the action of $\Gamma \times \mathbf{S}^1$ on V^3 is given by:

$$((l_1, l_2, l_3), \sigma, \theta)w = ((l_1, \theta)w_{\sigma^{-1}(1)}, (l_2, \theta)w_{\sigma^{-1}(2)}, (l_3, \theta)w_{\sigma^{-1}(3)}),$$

for $(l_1, l_2, l_3) \in \mathbf{O}(2)^3,\ \sigma \in \mathbf{S}_3$ and $\theta \in \mathbf{S}^1$ (with $w = (w_1,\ w_2,\ w_3) \in V^3$).

Note that as V is $\mathbf{O}(2)$-simple, also V^3 is Γ-simple by [8].

The action of $\Gamma \times \mathbf{S}^1$ on the six-dimensional complex vector space spanned by the six travelling waves (2) is related to the above action on V^3 in the following way. The action of $\mathbf{SO}(2)^3$ on $(\mathbf{C} \oplus \mathbf{C})^3$ can be described in terms of translations: identify (ψ_1, ψ_2, ψ_3) with the translation $u(X, t) \to u(X + d, t)$, where $d \cdot \omega_i = \psi_i / 2\pi$, for $i = 1, 2, 3$. Also $\theta \in \mathbf{S}^1$ is identified with the phase-shift in time, i.e., $u(X, t) \to u(x, t + \theta/2\pi)$. The elements of $\mathbf{S}_3$ permute the vectors ω_i and the flips $\kappa_i \in \mathbf{O}(2)^3$ act as $\kappa_i \omega_i = -\omega_i$, for $i = 1, 2, 3$.

3.3 Isotropy lattice

In order to find the $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$ using the results stated in section 3.1, we begin by calculating the $\mathbf{C}$-axial subgroups of $\mathbf{O}(2) \times \mathbf{S}^1$. By [13] for example (proposition XVII 1.1.) we have (up to conjugacy) two types of $\mathbf{C}$-axial subgroups. See table 1 and note that $\widetilde{\mathbf{SO}}(2)$ is of twist type $\mathbf{S}^1$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ of type $\mathbf{Z}_2$. We can now compute the $\mathbf{C}$-axial groups of $\Gamma \times \mathbf{S}^1$.

Theorem 3.1 *There are eight conjugacy classes of $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$ with the above action on V^3. They are listed, together with their orbit representatives and fixed-point subspaces in table 2. In table 3 we present the generators of these groups. Also a denotes a real positive number, $\xi = \frac{2\pi}{3} \in \mathbf{S}^1$, $z_i \in \mathbf{C}$, for $i = 1, \ldots, 6$, and $\kappa_1 = (\kappa, 1, 1)$, $\kappa_2 = (1, \kappa, 1)$, $\kappa_3 = (1, 1, \kappa) \in \mathbf{O}(2)^3$.*

Proof. Up to conjugacy we need to consider only the blocks

$$J = \{1\}, \quad J = \{1, 2\}, \quad J = \{1, 2, 3\},$$

and look for $\mathbf{C}$-axial subgroups Σ_w with

$$\begin{aligned}w &= (w_1, 0, 0),\\ w &= (w_1, \theta_2 w_1, 0),\\ w &= (w_1, \theta_2 w_1, \theta_3 w_1),\end{aligned}$$

where $\theta_2,\ \theta_3 \in \mathbf{S}^1$ and $w_1 \in V$ is such that

$$\dim \mathrm{Fix}_V\ (\Sigma_{w_1}) = 2.$$

Here Σ_{w_1} is the isotropy subgroup of w_1 in $\mathbf{O}(2) \times \mathbf{S}^1$.

For Σ_{w_1}, up to conjugacy, we have two choices: $\widetilde{\mathbf{SO}}(2)$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$. As the first one is of twist type $\mathbf{S}^1$, if $w_1 \in \mathrm{Fix}_V(\widetilde{\mathbf{SO}}(2))$, then we can assume (up to conjugacy) that w has equal nonzero components and we get $\Sigma_1,\ \Sigma_3$ and Σ_6: for example, as $(\theta_2, \theta_2) \in \widetilde{\mathbf{SO}}(2)$, then $w = (w_1, \theta_2 w_1, 0) = (w_1, (\theta_2^{-1}\theta_2, \theta_2)w_1, 0) = ((1, \theta_2^{-1}, 1), 1, 0)(w_1, w_1, 0)$ and so the isotropy subgroup of $(w_1, \theta_2 w_1, 0)$ is conjugate to the isotropy subgroup of $w = (w_1, w_1, 0)$.

Let now $w_1 \in \mathrm{Fix}_V\ (\mathbf{Z}_2 \oplus \mathbf{Z}_2^c)$. Note that $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ is of twist type $\mathbf{Z}_2$. It follows that $\Sigma_{(w_1,0,0)}$ is Σ_2 and if $\theta_2 \in \mathbf{Z}_2$, then $\Sigma_{(w_1,\theta_2 w_1,0)}$ is conjugate to Σ_4 by the same reason as before. Now we need to consider only Σ_5 for the case where $\theta_2 \notin \mathbf{Z}_2$ and $w = (w_1, \theta_2 w_1, 0)$. Note that, once we fixe $s^{'} = 2$, the possibilities for p are 2 and 4. But for $p = 2$ we must have $\theta_2 \in \mathbf{Z}_2$.

Similarly, if θ_2 and θ_3 are in $\mathbf{Z}_2$, then we can conjugate $\Sigma_{(w_1,\theta_2 w_1,\theta_3 w_1)}$ to Σ_7. If some θ_2 or θ_3 is not in $\mathbf{Z}_2$, then the only possibility for $w = (w_1, \theta_2 w_1, \theta_3 w_1)$ to be fixed by a $\mathbf{C}$-axial subgroup (up to conjugacy) is if $\theta_2 = \xi$ and $\theta_3 = \xi^2$ and we have Σ_8 (note that now with $s^{'} = 3$ we only need to consider $p = 3$ or $p = 6$ since the corresponding isotropy subgroups are conjugate). □

Remark 3.2 *If* $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_N$ *is acting on* V^N*, then the* $\mathbf{C}$*-axial groups of* $(\mathbf{O}(2) \wr \mathbf{S}_{N-1}) \times \mathbf{S}^1$ *(acting on* V^{N-1}*) are, with appropriate adjustment, included in those of* $\Gamma \times \mathbf{S}^1$*. As an example, it is the case of the group* $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$ *presented in [27] that can be seen as the group* $(\mathbf{O}(2) \wr \mathbf{S}_2) \times \mathbf{S}^1$*. Note that the* $\mathbf{C}$*-axial subgroups* $\Sigma_1, \ldots, \Sigma_5$ *are in precise correspondence with the* $\mathbf{C}$*-axial subgroups obtained in [27].*

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(a,0,0,0,0,0)$	$\Sigma_1 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1\})$	$\{(z_1,0,0,0,0,0)\}$
$(a,a,0,0,0,0)$	$\Sigma_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1\})$	$\{(z_1,z_1,0,0,0,0)\}$
$(a,0,a,0,0,0)$	$\Sigma_3 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1,2\})$	$\{(z_1,0,z_1,0,0,0)\}$
$(a,a,a,a,0,0)$	$\Sigma_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\})$	$\{(z_1,z_1,z_1,z_1,0,0)\}$
$(a,a,ia,ia,0,0)$	$\Sigma_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\},(12),\{1\},4)$	$\{(z_1,z_1,iz_1,iz_1,0,0)\}$
$(a,0,a,0,a,0)$	$\Sigma_6 = \Sigma(\widetilde{\mathbf{SO}}(2),\{1,2,3\})$	$\{(z_1,0,z_1,0,z_1,0)\}$
(a,a,a,a,a,a)	$\Sigma_7 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2,3\})$	$\{(z_1,z_1,z_1,z_1,z_1,z_1)\}$
$(a,a,\xi a,\xi a,\xi^2 a,\xi^2 a)$	$\Sigma_8 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2,3\},(123),\{1\},3)$	$\{(z_1,z_1,\xi z_1,\xi z_1,\xi^2 z_1,\xi^2 z_1)\}$

Table 2: Orbit representatives, **C**-axial groups and fixed-point subspaces of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

Isotropy subgroup	Generators
$\Sigma_1 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1\})$	$\mathbf{1} \times \mathbf{O}(2)^2,\ \{((\theta,1,1),1,\theta)\},\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Sigma_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\})$	$\mathbf{1} \times \mathbf{O}(2)^2,\ \kappa_1,\ ((\pi,1,1),1,\pi),\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Sigma_3 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1,2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \{((\theta,\theta,1),1,\theta)\},\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Sigma_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ ((\pi,\pi,1),1,\pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Sigma_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ \kappa_2,\ ((\pi,\pi,1),1,\pi),$ $((\pi,1,1),(12),\frac{\pi}{2})$
$\Sigma_6 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1,2,3\})$	$\{((\theta,\theta,\theta),1,\theta)\},\ \mathbf{S}_3$
$\Sigma_7 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2,3\})$	$\kappa_1,\ ((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_3$
$\Sigma_8 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2,3\}, (123), \{1\}, 3)$	$\kappa_1,\ \kappa_2,\ \kappa_3,\ ((\pi,\pi,\pi),1,\pi),\ (1^3,(123),\xi)$

Table 3: Generators of the $\mathbf{C}$-axial groups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

The isotropy lattice of $\mathbf{O}(2)\times\mathbf{S}^1$ is given in figure 1. If we consider the action of $\mathbf{O}(2) \times \mathbf{S}^1$ on the space $\mathcal{C}_{2\pi}$ of 2π-periodic functions $\mathbf{R} \to \mathbf{R}^4$ be defined by $(\gamma,\theta)\cdot v(t) = \gamma\cdot v(t+\theta)$, for $\gamma \in \mathbf{O}(2)$ and $\theta \in \mathbf{S}^1$, then we can refer the symmetry of $v(t)$ as the isotropy subgroup of v with respect to this action. As (π,π) acts trivially on V, every periodic solution satisfies $\mathrm{R}_\pi \cdot v(t) = v(t+\pi)$, that is the effect of a spatial rotation of $v(t)$ through angle π is the same as shifting the phase of $v(t)$ by half a period. Usually periodic solutions with symmetry $\widetilde{\mathbf{SO}}(2)$ are called *rotating waves* and those with symmetry $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ are *standing waves.* The rotating waves also satisfy $\mathrm{R}_\theta \cdot v(t) = v(t+\theta)$, for all θ and the standing waves satisfy $\kappa \cdot v(t) = v(t)$.

$$
\begin{array}{ccc}
 & \mathbf{O}(2) \times \mathbf{S}^1 & \\
 & \nearrow \ \nwarrow & \\
\widetilde{\mathbf{SO}}(2) & & \mathbf{Z}_2 \oplus \mathbf{Z}_2^c \\
 & \nwarrow \ \nearrow & \\
 & \mathbf{Z}_2^c = \{(1,0),\ (\pi,\pi)\} &
\end{array}
$$

Figure 1: Isotropy lattice of $\mathbf{O}(2) \times \mathbf{S}^1$.

Isotropy subgroup	Fixed-point subspace
$\Delta_1 = \Sigma(\mathbf{Z}_2^c, \{1\})$	$\{(z_1, z_2, 0, 0, 0, 0)\}$
$\Delta_2 = \Sigma(\mathbf{Z}_2^c, \{1, 2\})$	$\{(z_1, z_2, z_1, z_2, 0, 0)\}$
$\Delta_3 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}, (12), \{1\}, 4)$	$\{(z_1, z_2, iz_1, iz_2, 0, 0)\}$
$\Delta_4 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1, \}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{2\})$	$\{(z_1, 0, z_2, 0, 0, 0)\}$
$\Delta_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\})$	$\{(z_1, z_1, z_2, z_2, 0, 0)\}$
$\Delta_6 = \Sigma(\mathbf{Z}_2^c, \{1, 2, 3\})$	$\{(z_1, z_2, z_1, z_2, z_1, z_2)\}$
$\Delta_7 = \Sigma(\mathbf{Z}_2^c, \{1, 2, 3\}, (123), \{1\}, 3)$	$\{(z_1, z_2, \xi z_1, \xi z_2, \xi^2 z_1, \xi^2 z_2)\}$
$\Delta_8 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1, 2\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, 0, z_1, 0, z_2, 0)\}$
$\Delta_9 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_1, z_1, z_2, z_2)\}$
$\Delta_{10} = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\}, (12), \{1\}, 4) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, z_1, iz_1, iz_1, z_2, 0)\}$
$\Pi_1 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\{(z_1, z_1, z_2, z_3, 0, 0)\}$
$\Pi_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_2, z_2, z_3, z_3)\}$
$\Pi_3 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{2\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, 0, z_2, 0, z_3, 0)\}$
$\Pi_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2, 3\})$	$\{(z_1, z_1, z_2, z_3, z_2, z_3)\}$
$\Pi_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_1, z_1, z_2, z_3)\}$
$\Pi_6 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}, (12), \{1\}, 4) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(z_1, z_2, iz_1, iz_2, z_3, 0)\}$

Table 4: Fixed-point subspaces corresponding to the submaximal isotropy subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

Isotropy subgroup	Generators
$\Delta_1 = \Sigma(\mathbf{Z}_2^c, \{1\})$	$\mathbf{1} \times \mathbf{O}(2)^2,\ ((\pi,1,1),1,\pi),\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Delta_2 = \Sigma(\mathbf{Z}_2^c, \{1,2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ ((\pi,\pi,1),1,\pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Delta_3 = \Sigma(\mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$	$\mathbf{1}^2 \times \mathbf{O}(2),\ ((\pi,\pi,1),1,\pi),$ $((\pi,1,1),(12),\frac{\pi}{2})$
$\Delta_4 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1,\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \{(\theta,\theta,1),1,\theta)\}$
$\Delta_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ \kappa_2,\ ((\pi,\pi,1),1,\pi)$
$\Delta_6 = \Sigma(\mathbf{Z}_2^c, \{1,2,3\})$	$((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_3$
$\Delta_7 = \Sigma(\mathbf{Z}_2^c, \{1,2,3\}, (123), \{1\}, 3)$	$((\pi,\pi,\pi),1,\pi),\ (1^3,(123),\xi)$
$\Delta_8 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1,2\}) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\{(\theta,\theta,\theta),1,\theta)\},\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Delta_9 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ \kappa_3,\ ((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Delta_{10} = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$ $\cap\ \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$\kappa_1,\ \kappa_2,\ ((\pi,\pi,\pi),1,\pi),$ $((\pi,1,\frac{\pi}{2}),(12),\frac{\pi}{2})$
$\Pi_1 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ \kappa_1,\ ((\pi,\pi,1),1,\pi)$
$\Pi_2 = \bigcap_{i=1,2,3} \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{i\})$	$\kappa_1,\ \kappa_2,\ \kappa_3,\ ((\pi,\pi,\pi),1,\pi)$
$\Pi_3 = \bigcap_{i=1,2,3} \Sigma(\widetilde{\mathbf{SO}}(2), \{i\})$	$\{((\theta,\theta,\theta),1,\theta)\}$
$\Pi_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2,3\})$	$\kappa_1,\ ((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_1 \times \mathbf{S}_2$
$\Pi_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ ((\pi,\pi,\pi),1,\pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Pi_6 = \Sigma(\mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4) \cap \Sigma(\widetilde{\mathbf{SO}}(2), \{3\})$	$((\pi,\pi,\pi),1,\pi),\ ((\pi,1,\frac{\pi}{2}),(12),\frac{\pi}{2})$

Table 5: Generators of the submaximal isotropy subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

Isotropy subgroup	Fixed-point subspace
$\Lambda_1 = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\{(z_1, z_2, z_3, z_4, 0, 0)\}$
$\Lambda_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_2, z_2, z_3, z_4)\}$
$\Lambda_3 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_2, z_1, z_2, z_3, z_4)\}$
$\Phi = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_1, z_2, z_3, z_4, z_5)\}$
$T = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\{(z_1, z_2, z_3, z_4, z_5, z_6)\}$

Table 4: Continuation.

Isotropy subgroup	Generators
$\Lambda_1 = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\})$	$\mathbf{1}^2 \times \mathbf{O}(2),\ ((\pi, \pi, 1), 1, \pi)$
$\Lambda_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ \kappa_2,\ ((\pi, \pi, \pi), 1, \pi)$
$\Lambda_3 = \Sigma(\mathbf{Z}_2^c, \{1, 2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$((\pi, \pi, \pi), 1, \pi),\ \mathbf{S}_2 \times \mathbf{S}_1$
$\Phi = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$\kappa_1,\ ((\pi, \pi, \pi), 1, \pi)$
$T = \Sigma(\mathbf{Z}_2^c, \{1\}) \cap \Sigma(\mathbf{Z}_2^c, \{2\}) \cap \Sigma(\mathbf{Z}_2^c, \{3\})$	$((\pi, \pi, \pi), 1, \pi)$

Table 5: Continuation.

Theorem 3.3 *The complete isotropy lattice of* $\Gamma \times \mathbf{S}^1$ *up to conjugacy is given by the groups listed in table 2 together with the isotropy subgroups in table 4 (these are the groups with fixed-point subspaces with dimension higher than two). The groups* Δ_i *have four-dimensional fixed-point subspaces,* $\mathrm{Fix}(\Pi_i)$ *are six-dimensional,* $\mathrm{Fix}(\Lambda_i)$ *are eight-dimensional,* $\mathrm{Fix}(\Phi)$ *is ten-dimensional. Finally the group* T *acts trivially on the all space* V^3*. In table 5 we present the generators of the subgroups of table 4. Again* ξ *denotes* $\frac{2\pi}{3} \in \mathbf{S}^1$*.*

Proof. We can apply the results stated before. Let $\Sigma = \Sigma_w$ be an isotropy subgroup of $\Gamma \times \mathbf{S}^1$, where $w \in V^3$ is nonzero. The subgroup Σ is not $\Gamma \times \mathbf{S}^1$ since $\mathrm{Fix}_{\mathrm{V}^3}(\Gamma \times \mathbf{S}^1) = \{0\}$. The possible partitions of the set of indices $\{1, 2, 3\}$

in blocks that we need to consider are:

$$\{1,2,3\} = \{1\} \cup \{2\} \cup \{3\},$$

$$\{1,2,3\} = \{1,2\} \cup \{3\},$$

$$\{1,2,3\} = \{1,2,3\}.$$

Suppose that $w = (w_1, w_2, w_3)$ where $w_1,\ w_2,\ w_3 \in V$. We have to look for isotropy subgroups of elements of the kind

$(a.1)$ $(w_1, 0, 0)$,

$(b.1)$ $(w_1, \theta_2 w_1, 0)$, $\qquad$ $(b.2)$ $(w_1, w_2, 0)$,

$(c.1)$ $(w_1, \theta_2 w_1, \theta_3 w_1)$, $\quad$ $(c.2)$ $(w_1, \theta_2 w_1, w_2)$, $\quad$ $(c.3)$ (w_1, w_2, w_3),

where Σ_{w_i} for $i = 1,2,3$ are isotropy subgroups of $\mathbf{O}(2) \times \mathbf{S}^1$ (and distinct from $\mathbf{O}(2) \times \mathbf{S}^1$, i.e., the vectors $w_1,\ w_2$ and w_3 are non-null) and so

$$\Sigma_{w_1},\ \Sigma_{w_2},\ \Sigma_{w_3} \in \{\widetilde{\mathbf{SO}}(2),\ \mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\ \mathbf{Z}_2^c\}.$$

Also $\theta_2,\ \theta_3 \in \mathbf{S}^1$, and if $\Pi_{\mathcal{G}} : \Gamma \times \mathbf{S}^1 \to \mathcal{G}$ denotes projection on $\mathcal{G}$, then for each case we have:

$(a.1)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{1\}) = \{1\}$,

$(b.1)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{1,2\}) = \{1,2\}$, $\quad$ $(b.2)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{i\}) = \{i\},\ i = 1,2,3$,

$(c.2)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{1,2\}) = \{1,2\}$, $\quad$ $(c.3)$ $\Pi_{\mathcal{G}}(\Sigma_w)(\{i\}) = \{i\},\ i = 1,2,3$,

i.e., the elements $w_1,\ w_2,\ w_3 \in V \setminus \{0\}$ are in distinct $\mathbf{O}(2) \times \mathbf{S}^1$ orbits.

Case $(a.1)$
We get Σ_1 and Σ_2 (as we have seen in theorem 3.1) and Δ_1.

Case $(b.1)$
As in theorem 3.1, we get $\Sigma_3,\ \Sigma_4$ and Σ_5 if we consider Σ_{w_1} to be maximal. See now the case $\Sigma_{w_1} = \mathbf{Z}_2^c$. As $\mathbf{Z}_2^c = \{(1,0),\ (\pi,\pi)\} \subset \mathbf{O}(2) \times \mathbf{S}^1$ is of type $\mathbf{Z}_2$, we only need to consider up to conjugacy, the groups $\Sigma(\mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$ and $\Sigma(\mathbf{Z}_2^c, \{1,2\})$, i.e., we obtain Δ_2 and Δ_3.

Case $(b.2)$
We have

$$\Sigma_{(w_1,w_2,0)} = \Sigma_{(w_1,0,0)} \cap \Sigma_{(0,w_2,0)}.$$

The isotropy subgroup $\Sigma_{(w_1,0,0)}$ is of type $\Sigma(A, \{1\})$, where

$$A \in \{\widetilde{\mathbf{SO}}(2),\ \mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\ \mathbf{Z}_2^c\}.$$

Similarly, the isotropy group $\Sigma_{(0,w_2,0)}$ is of type $\Sigma(B,\{2\})$, where B belongs to the same set as A. Considering the possible intersections, we get $\Delta_4,\ \Delta_5,\ \Pi_1$ and Λ_1.

Case $(c.1)$

This case is similar to case $(b.1)$: we obtain the maximal $\Sigma_6,\ \Sigma_7$ and Σ_8 (as before) and again, because $\mathbf{Z}_2^c$ is of twist type $\mathbf{Z}_2$, we consider only $\Sigma(\mathbf{Z}_2^c,\{1,2,3\},(123),\{1\},3)$ and $\Sigma(\mathbf{Z}_2^c,\{1,2,3\})$, i.e., the groups Δ_7 and Δ_6 respectively.

Case $(c.2)$

Now

$$\Sigma_{(w_1,\theta_2 w_1,w_2)} = \Sigma_{(w_1,\theta_2 w_1,0)} \cap \Sigma_{(0,0,w_2)}.$$

Up to conjugacy, the group $\Sigma_{(w_1,\theta_2 w_1,0)}$ is of type $\Sigma(A,\{1,2\}$ with

$$A \in \{\widetilde{\mathbf{SO}}(2),\ \mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\ \mathbf{Z}_2^c\}$$

or

$$\Sigma_{(w_1,\theta_2 w_1,0)} \in \{\Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c,\{1,2\},(12),\{1\},4),\ \Sigma(\mathbf{Z}_2^c,\{1,2\},(12),\{1\},4)\}.$$

We note that if $\Sigma_{w_1} = \widetilde{\mathbf{SO}}(2)$, then as it is of twist type $\mathbf{S}^1$, the group $\Sigma_{(w_1,\theta_2 w_1,0)}$ is always conjugate to $\Sigma_{(w_1,w_1,0)}$ and so it is included in the first possibility.

The group $\Sigma_{(0,0,w_2)}$ can be $\Sigma(B,\{3\})$ where again B is maximal or $\mathbf{Z}_2^c$. We obtain, up to conjugacy, the groups $\Delta_8,\ \Delta_9,\ \Delta_{10}$ and $\Pi_4,\ \Pi_5,\ \Pi_6$ and Λ_3.

Case $(c.3)$

We have the intersection

$$\Sigma_{(w_1,w_2,w_3)} = \Sigma_{(w_1,0,0)} \cap \Sigma_{(0,w_2,0)} \cap \Sigma_{(0,0,w_3)},$$

where the groups $\Sigma_{(w_1,0,0)},\ \Sigma_{(0,w_2,0)}$ and $\Sigma_{(0,0,w_3)}$ can be $\Sigma(A_1,\{1\}),\ \Sigma(A_2,\{2\})$ and $\Sigma(A_3,\{3\})$ respectively, with A_i maximal or $\mathbf{Z}_2^c$ (in $\mathbf{O}(2)\times\mathbf{S}^1$). We obtain $\Pi_2,\ \Pi_3,\ \Lambda_2,\ \Phi$ and T. $\square$

Figure 2 contains the isotropy lattice of $\Gamma\times\mathbf{S}^1$ where each entry represents the entire conjugacy class of that group. We use $H \to K$ to mean that $g^{-1}Hg \subseteq K$ for some $g \in \Gamma\times\mathbf{S}^1$ where $H,\ K$ are subgroups of $\Gamma\times\mathbf{S}^1$. In terms of fixed-point subspaces, this implies that $g(\mathrm{Fix(K)}) \subseteq \mathrm{Fix(H)}$. For example, $((1,\pi/2,1),1,0)(\mathrm{Fix}(\Delta_8)) \subseteq \mathrm{Fix}(\Pi_6)$ and so we have the arrow $\Pi_6 \to \Delta_8$.

3.4 Planforms

We now show some diagrams of planforms that are time-periodic and spatially periodic with respect to the primitive cubic lattice considered in section 2. We consider here functions $u(X,t)$ with $X \in \mathbf{R}^3$ corresponding to the maximal solutions with symmetry $\Sigma_1,\ldots,\Sigma_8$ obtained in theorem 3.1.

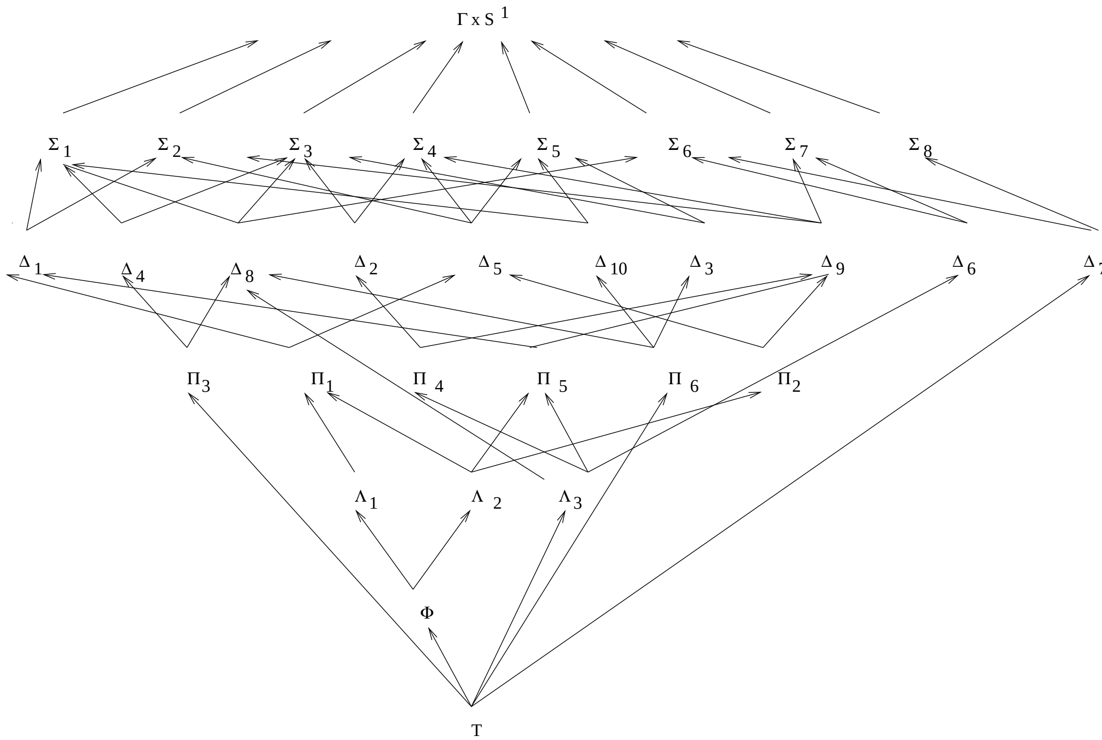


Figure 2: Isotropy lattice of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

It is well known that near a Hopf bifurcation point of a PDE, we can approximate the relevant periodic solutions of the nonlinear PDE by a suitable superposition of linearized eigenfunctions. For similar reasons, planforms of periodic solutions of the nonlinear PDE, near an equivariant Hopf bifurcation point, are well approximated by planforms of suitable linearized eigenfunctions. Here, we consider the Fourier sum (3) plotted at discrete times for representative elements $(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{C}^6$ of the fixed-point subspaces listed in table 2. We recall the identification of $(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{C}^6$ with the function $u : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ defined by

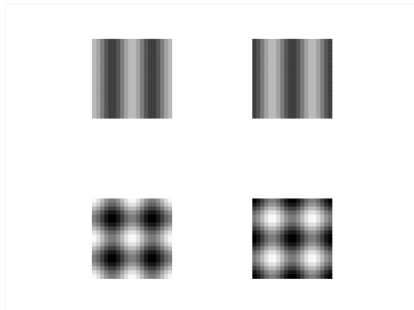
$$\begin{aligned} u(X,t) = \mathrm{Re}\{ & z_1 e^{i2\pi(t-X_1)} + z_3 e^{i2\pi(t-X_2)} + z_5 e^{i2\pi(t-X_3)} + \\ & z_2 e^{i2\pi(t+X_1)} + z_4 e^{i2\pi(t+X_2)} + z_6 e^{i2\pi(t+X_3)} \}. \end{aligned} \tag{7}$$

The groups Σ_1, Σ_3 and Σ_6 are of twist type $\mathbf{S}_1$. The corresponding planforms move at constant velocity but remain the same. The groups Σ_2, Σ_4 and Σ_7 are of twist type $\mathbf{Z}_2$. The corresponding planforms also remain the same for all the time. Their symmetries are only spatial symmetries (except $((\pi, 1, 1), 1, \pi)$ in Σ_2, $((\pi, \pi, 1), 1, \pi)$ in Σ_4 and $((\pi, \pi, \pi), 1, \pi)$ in Σ_7). The groups Σ_5 and Σ_8 have non-trivial spatial and spatial-temporal symmetries. Thus their planforms change with the time.

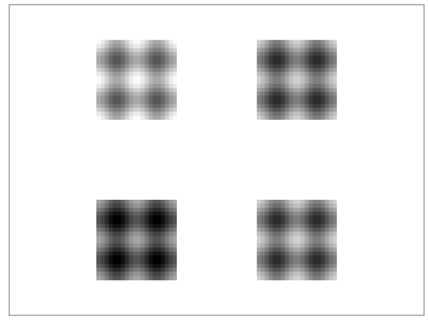
In each case, we consider the spatial domain containing four unit cells (cubes) and the density plot of the planforms $u(X,t)$ is projected in the planar domain (X_1, X_2) (or (X_1, X_3) when indicated), thus for four unit squares. The grey scale indicates the magnitude of $u(X,t)$, with white denoting maximum and black minimum. The planforms corresponding to the symmetries Σ_i for $i = 1, \ldots, 5$ are two-dimensional planforms in the sense that they do not depend on the third spatial variable X_3. These are the planforms obtained by Silber and Knobloch [27] and we adopt the same names as the ones considered there. Planforms $u(X,t)$ corresponding to periodic solutions with symmetry Σ_2 and Σ_4 are called *standing rolls* and *standing squares* respectively, and are shown in figure 3. Those corresponding to periodic solutions with symmetry Σ_7, we call *standing cubes* and are also shown in figure 3. Planforms $u(X,t)$ corresponding to periodic solutions with symmetry Σ_1 and Σ_3 are called *travelling rolls* and *travelling squares* respectively, and appear in figure 3. Those with symmetry Σ_6, we call *travelling cubes* and are shown in figures 3 and 4. Planforms $u(X,t)$ corresponding to periodic solutions with symmetry Σ_5 are called *alternating rolls* and are shown in figure 4. Finally, we name planforms $u(X,t)$ corresponding to periodic solutions with symmetry Σ_8 *alternating cubes.* These appear in figures 4, ..., 6.

4 Equivariant vector fields

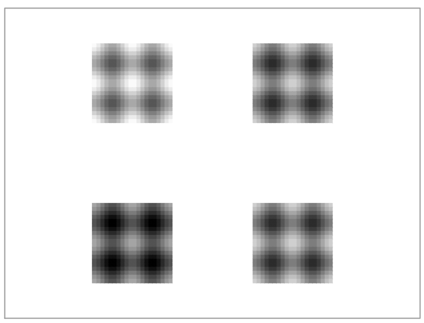
In order to determine the direction of branching and the stability of the bifurcating branches of periodic solutions of (4), we must compute the general form of an $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-equivariant bifurcation problem, i.e., we need to find the invariants and the equivariants by the group $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ for the action



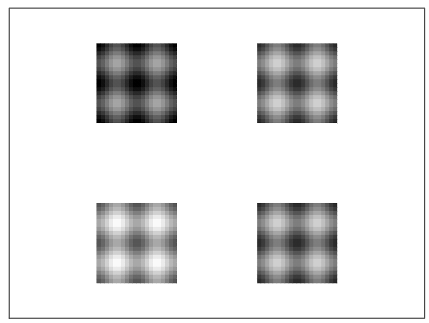
(a) Standing rolls on the top and standing squares above, both for $t = 0$ and $t = 1/2$.



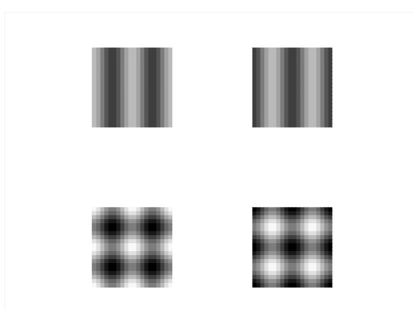
(b) Standing cubes: $t = 0$ and $X_2 = 0,\ 1/4,\ 1/2,\ 3/4$.



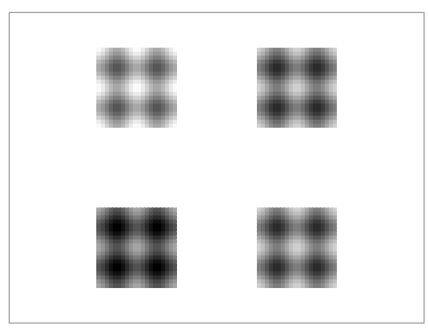
(c) Standing cubes: $t = 0$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.



(d) Standing cubes: $t = 1/2$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.

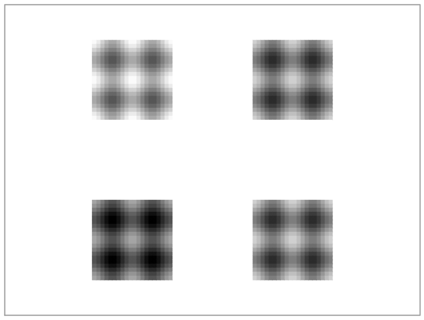


(e) Travelling rolls on the top and travelling squares above, both for $t = 0$ and $t = 1/2$.

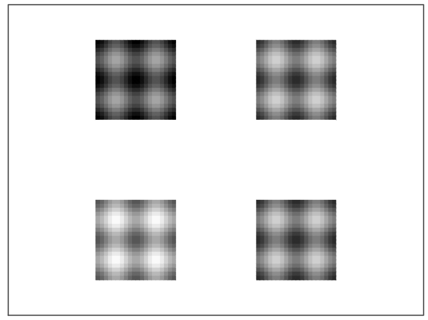


(f) Travelling cubes: $t = 0$ and $X_2 = 0,\ 1/4,\ 1/2,\ 3/4$.

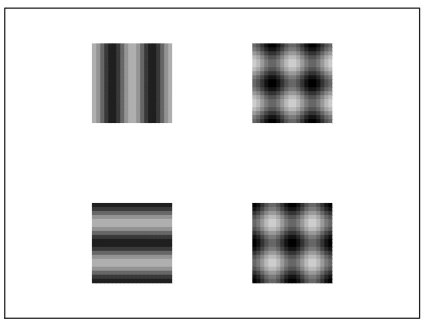
Figure 3: Planforms associated with the solutions guaranteed by the equivariant Hopf theorem.



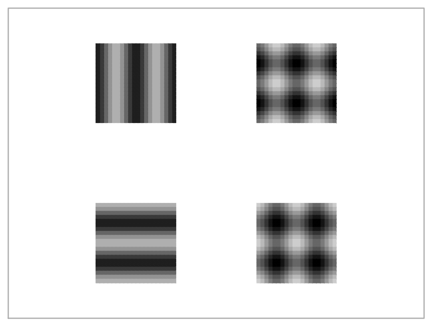
(a) Travelling cubes: $t = 0$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.



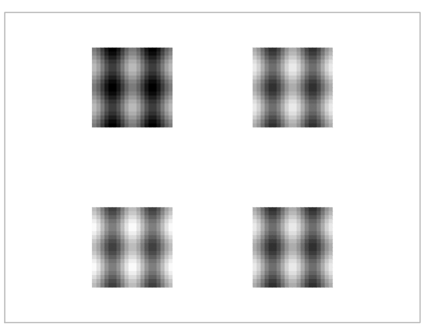
(b) Travelling cubes: $t = 1/2$ and $X_3 = 0,\ 1/4,\ 1/2,\ 3/4$.



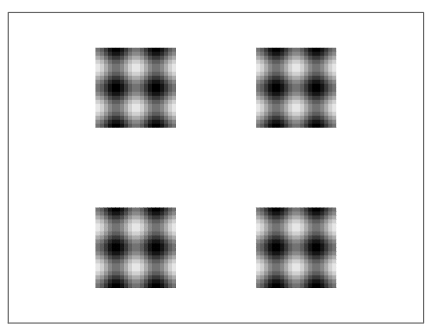
(c) Alternating rolls: $t = 0,\ 1/8,\ 1/4,\ 3/8$.



(d) Alternating rolls: $t = 1/2,\ 5/8,\ 3/4,\ 7/8$.

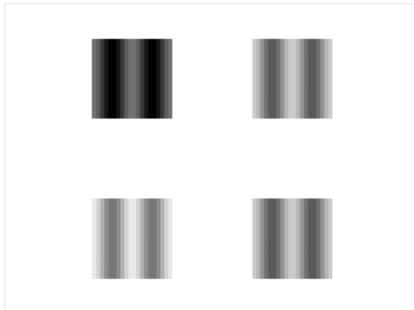


(e) Alternating cubes: $t = 0$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.

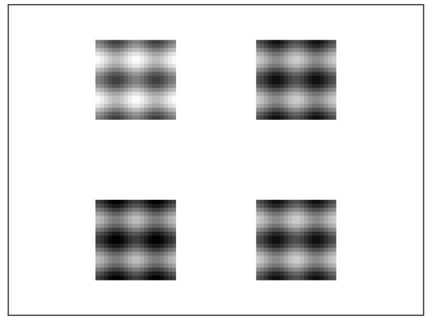


(f) Alternating cubes: $t = 1/12$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.

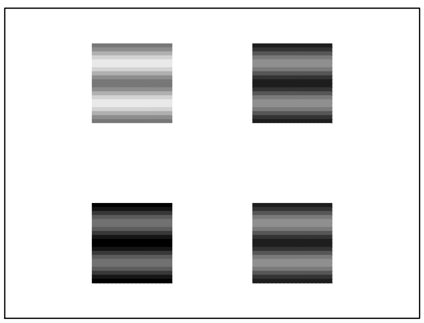
Figure 4: Planforms (continuation).



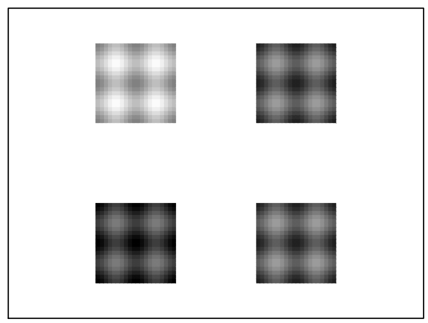
(a) $t = 1/12$ and $X_2 = 0,\ 1/3,\ 1/2,\ 2/3$.



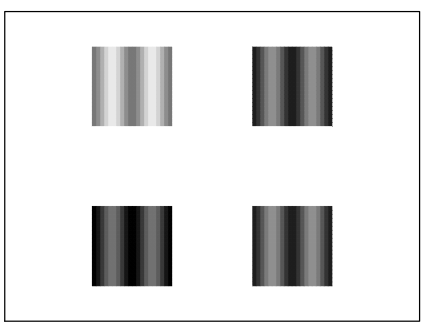
(b) $t = \frac{1}{6}$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.



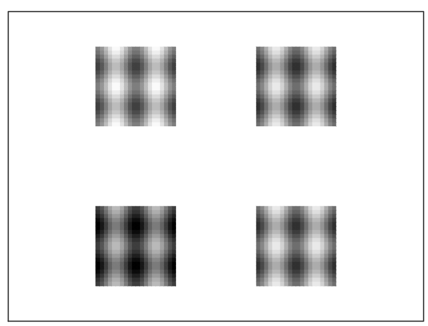
(c) $t = \frac{1}{4}$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$.



(d) $t = \frac{1}{3}$ and $X_3 = 0,\ 1/3,\ 1/2,\ 2/3$. Same pattern as for $t = 0$ and $X_1 = 0,\ 1/3,\ 1/2,\ 2/3$.

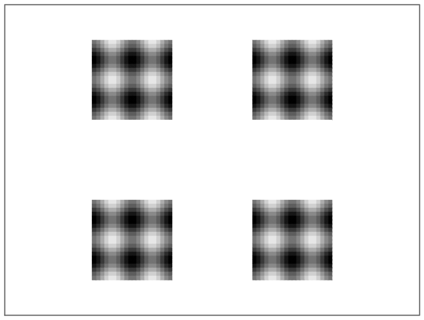


(e) $t = \frac{5}{12}$ and $X_3 = 0\ ,1/3,\ 1/2,\ 2/3$.

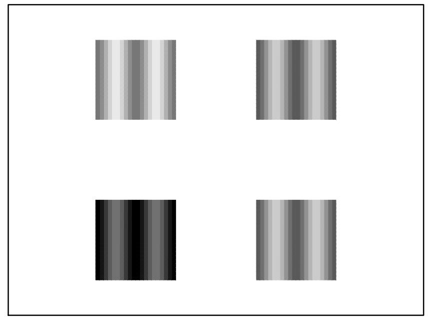


(f) $t = \frac{1}{2}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.

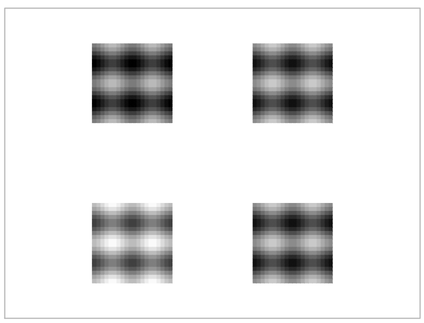
Figure 5: Planforms: alternating cubes.



(a) $t = \frac{7}{12}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.

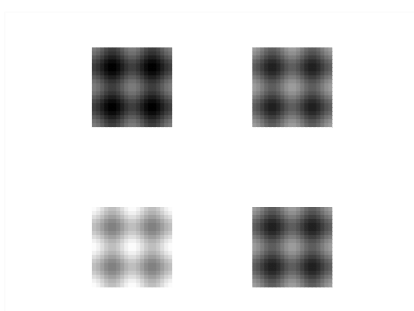


(b) $t = \frac{7}{12}$ and $X_2 = 0,\ 1/6,\ 1/2,\ 5/6$.

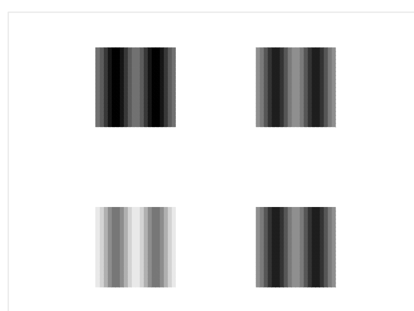


(c) $t = \frac{2}{3}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.

(d) $t = \frac{3}{4}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.



(e) $t = \frac{5}{6}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.



(f) $t = \frac{11}{12}$ and $X_3 = 0,\ 1/6,\ 1/2,\ 5/6$.

Figure 6: Planforms: alternating cubes.

considered in section 3.2. We calculate the general form of an $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-equivariant vector field with polynomial components of degree less or equal to three. Higher order equivariants can be found in [5]. We show in the next section that the third order terms of the Taylor expansion around the bifurcation point of a general vector field commuting with $(\mathbf{O}(2) \wr \mathbf{S}_3)$ (that can also be supposed to commute with $\mathbf{S}^1$ till third order) determine the branching directions and the stabilities of the solutions corresponding to the bifurcating branches found in this work.

Proposition 4.1 *Let $V = \mathbf{C} \oplus \mathbf{C}$ and $f : V^3 \to V^3$ be a polynomial mapping of degree three which commutes with the action of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ described in section 3.2. Then we can write f as $(f_1, f_2, f_3, f_4, f_5, f_6)$, with*

$$\begin{aligned} f_2(z) &= f_1(z_2, z_1, z_3, z_4, z_5, z_6), \\ f_3(z) &= f_1(z_3, z_4, z_1, z_2, z_5, z_6), \\ f_4(z) &= f_1(z_4, z_3, z_1, z_2, z_5, z_6), \\ f_5(z) &= f_1(z_5, z_6, z_3, z_4, z_1, z_2), \\ f_6(z) &= f_1(z_6, z_5, z_3, z_4, z_1, z_2) \end{aligned}$$

and

$$f_1(z) = \mu z_1 + (a|z|^2 + b|z_1|^2 + c|z_2|^2)z_1 + d\overline{z}_2(z_3 z_4 + z_5 z_6),$$

where the complex coefficients μ, a, b, c and d depend on any parameters that f may depend on. Here $z = (z_1, z_2, z_3, z_4, z_5, z_6)$ and $|z|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2$.

In order to prove this proposition, we begin by proving first two lemmas involving only the invariance under $\mathbf{SO}(2)^N \times \mathbf{S}^1$ and $\mathbf{O}(2)^N \times \mathbf{S}^1$.

Lemma 4.2 *Every polynomial germ $f : (\mathbf{C} \oplus \mathbf{C})^N \to \mathbf{C}$ invariant under $\mathbf{O}(2)^N \times \mathbf{S}^1$ has the form*

$$f(z) = P(u_1, \ldots, v_1, \ldots, c_{12}, \ldots, \overline{c_{12}}, \ldots),$$

where

$$\begin{aligned} u_i &= |z_{2i-1}|^2 |z_{2i}|^2, & i &= 1, \ldots, N \\ v_i &= |z_{2i-1}|^2 + |z_{2i}|^2, & i &= 1, \ldots, N \\ c_{ij} &= z_{2i-1} z_{2i} \overline{z_{2j-1} z_{2j}}, & 1 &\leq i < j \leq N. \end{aligned}$$

Proof Let $f : \mathbf{C}^{2N} \to \mathbf{C}$ be written as

$$f(z) = \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta,$$

where $a_{\alpha\beta} \in \mathbf{C}$, α, $\beta \in (\mathbf{Z}_0^+)^{2N}$ and $z \in \mathbf{C}^{2N}$. Here we use multi-indices. In order f to be $\mathbf{SO}(2)^N \times \mathbf{S}^1$ invariant, i.e.,

$$f((\varphi_1, \ldots, \varphi_N, \theta)z) = f(z),$$

for all $(\varphi_1, \ldots, \varphi_N, \theta) \in \mathbf{SO}(2)^N \times \mathbf{S}^1$, we have that for each $\alpha,\ \beta \in (\mathbf{Z}_0^+)^{2N}$ such that $a_{\alpha\beta}$ is non-null, the coefficients α and β must satisfy

$$\theta(|\alpha| - |\beta|) + \sum_{i=1}^{N} \varphi_i(\alpha_{2i} - \beta_{2i} + \beta_{2i-1} - \alpha_{2i-1}) = 0.$$

As this equality must hold for all $(\varphi_1, \ldots, \varphi_N, \theta)$, then

$$|\alpha| = |\beta| \qquad \wedge \qquad \alpha_{2i} - \beta_{2i} = \alpha_{2i-1} - \beta_{2i-1},\ i = 1, \ldots, N,$$

which is equivalent to have

$$\sum_{i=1}^{N} \beta_{2i-1} = \sum_{i=1}^{N} \alpha_{2i-1} \quad \wedge \quad \alpha_{2i} - \beta_{2i} = \alpha_{2i-1} - \beta_{2i-1},\ i = 1, \ldots, N. \qquad (8)$$

In addition, we have

$$f(z) = f(z_2, z_1, z_3, z_4, \ldots) = \cdots = f(z_1, z_2, \ldots, z_{2N}, z_{2N-1}).$$

So f will be the sum of factors like

$$(z_1^{\alpha_1} z_2^{\alpha_2} \overline{z}_1^{\beta_1} \overline{z}_2^{\beta_2} + z_1^{\alpha_2} z_2^{\alpha_1} \overline{z}_1^{\beta_2} \overline{z}_2^{\beta_1}) \cdots (z_{2N-1}^{\alpha_{2N-1}} z_{2N}^{\alpha_{2N}} \overline{z}_{2N-1}^{\beta_{2N-1}} \overline{z}_{2N}^{\beta_{2N}} + z_{2N-1}^{\alpha_{2N}} z_{2N}^{\alpha_{2N-1}} \overline{z}_{2N-1}^{\beta_{2N}} \overline{z}_{2N}^{\beta_{2N-1}})$$

with $\alpha_{2i-1} \leq \alpha_{2i}$ and $\beta_{2i-1} \leq \beta_{2i}$ for $i = 1, \ldots, N$. Note that here we used the fact that if $\alpha_{2i} - \alpha_{2i-1} \geq 0$, then also $\beta_{2i} - \beta_{2i-1} \geq 0$. Moreover, if (α, β) satisfies (8), then for example $(\alpha_2, \alpha_1, \ldots, \beta_2, \beta_1, \ldots)$ also satisfies (8). Now, if we factor out the largest powers of $z_{2i-1}z_{2i}$ and $\overline{z_{2i-1}z_{2i}}$ and use (8), we will have polynomials terms like

$$(z_1 z_2)^{\alpha_1} (\overline{z_1 z_2})^{\beta_1} \ldots (z_{2N-1} z_{2N})^{\alpha_{2N-1}} (\overline{z_{2N-1} z_{2N}})^{\beta_{2N-1}}$$

with

$$\sum_{i=1}^{N} \alpha_{2i-1} = \sum_{i=1}^{N} \beta_{2i-1}$$

and

$$(|z_{2i-1}|^2)^{k_i} + (|z_{2i}|^2)^{k_i}.$$

By pairing the $z_{2i-1}z_{2i}$ with $\overline{z_{2j-1}z_{2j}}$ we can always write the first one as monomial in the u_i's and c_{ij}'s and the second one as a polynomial in the u_i's and v_i's. $\square$

Note that by [13] lemma XVI 9.2., if we have the $\mathbf{C}$-valued invariants in $z,\ \overline{z}$, then we take the real and imaginary parts of these and we have the $\mathbf{R}$-valued invariants.

Lemma 4.3 *Every polynomial germ $f : (\mathbf{C} \oplus \mathbf{C})^N \to \mathbf{C}$ invariant under $\mathbf{SO}(2)^N \times \mathbf{S}^1$ has the following form*

$$f(z) = P(v_1, \ldots, c_{12}, \ldots),$$

where

$$\begin{array}{ll} v_i = |z_i|^2, & i = 1, \ldots, 2N, \\ c_{ij} = z_{2i-1} z_{2i} \overline{z_{2j-1} z_{2j}}, & 1 \leq i < j \leq N. \end{array}$$

Proof From the proof of lemma 4.2 we know that

$$f(z) = \sum a_{\alpha\beta} z^{\alpha} \overline{z}^{\beta},$$

where $a_{\alpha\beta} \in \mathbf{C},\ \alpha,\ \beta \in (\mathbf{Z}_0^+)^{2N}$ and each (α, β) must satisfy

$$\begin{array}{l} \sum \alpha_{2i} = \sum \beta_{2i} \quad \wedge \quad \sum \alpha_{2i-1} = \sum \beta_{2i-1} \\ \alpha_{2i} - \beta_{2i} = \alpha_{2i-1} - \beta_{2i-1}, \quad i = 1, \ldots, N. \end{array}$$

For $\alpha,\ \beta \in (\mathbf{Z}_0^+)^{2N}$ in those conditions, we can factor out

$$z_{2i-1}^{\alpha_{2i-1}} z_{2i}^{\alpha_{2i}} \overline{z}_{2i-1}^{\beta_{2i-1}} \overline{z}_{2i}^{\beta_{2i}}$$

as

$$(z_{2i-1}\overline{z}_{2i-1})^{\beta_{2i-1}} (z_{2i}\overline{z}_{2i})^{\beta_{2i}} (z_{2i-1} z_{2i})^{\alpha_{2i} - \beta_{2i}}$$

if $\alpha_{2i} \geq \beta_{2i}$ (and so $\alpha_{2i-1} \geq \beta_{2i-1}$), or as

$$(z_{2i-1}\overline{z}_{2i-1})^{\alpha_{2i-1}} (z_{2i}\overline{z}_{2i})^{\alpha_{2i}} (\overline{z_{2i-1} z_{2i}})^{\beta_{2i} - \alpha_{2i}}$$

if $\alpha_{2i} \leq \beta_{2i}$ (and so $\alpha_{2i-1} \leq \beta_{2i-1}$). Also we know that

$$\sum_{\{i:\alpha_{2i} \geq \beta_{2i}\}} (\alpha_{2i} - \beta_{2i}) = \sum_{\{i:\alpha_{2i} < \beta_{2i}\}} (\beta_{2i} - \alpha_{2i})$$

Therefore we only need the v_i's and the c_{ij}'s. □

Remark 4.4 *Using a similar result as [14] theorem 4, we have that if* $I_1, \ldots, I_r$ *generate the* $\mathbf{C}$*-valued invariants by* $\mathbf{SO}(2)^N \times \mathbf{S}^1$*, then the equivariants are generated by the mappings*

$$\text{row } j \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial I_g}{\partial \overline{z}_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $1 \leq j \leq 2N$ *and* $1 \leq g \leq r$*. This follows from the fact that if* $g = (g_1, \ldots, g_{2N})$ *is* $\mathbf{SO}(2)^N \times \mathbf{S}^1$*-equivariant, then* $\overline{z}_j g_j$ *is* $\mathbf{SO}(2)^N \times \mathbf{S}^1$*-invariant (for* $j = 1, \ldots, 2N$*).*

Proof of proposition 4.1 We begin by finding first the polynomials from V^3 to $\mathbf{R}$ of degree less or equal three that are invariant under $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$. For that we use the chain of subgroups

$$\mathbf{O}(2)^3 \times \mathbf{S}^1 \subset (\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$$

and lemma 4.2. We have that f is $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-invariant if and only if it is invariant by $\mathbf{O}(2)^3 \times \mathbf{S}^1$ and $\mathbf{S}_3$. By lemma 4.2, we have the invariants for $\mathbf{O}(2)^3 \times \mathbf{S}^1$. We have to find the polynomials

$$p(u_1, \ldots, v_1, \ldots, c_{12}, \ldots, \overline{c}_{12}, \ldots)$$

that are $\mathbf{S}_3$-invariants. Moreover, we wish to find the homogeneous polynomials up to degree three. We look for the polynomials p of degree one in $v_1,\ v_2,\ v_3$ that are $\mathbf{S}_3$-invariants. As $v_1,\ v_2,\ v_3$ are algebrically independents, we only need to consider the linear combination $v_1 + v_2 + v_3$ and we have $|z|^2$.

Now we calculate the polynomial functions from V^3 to V^3 (of degree less or equal three) that are equivariant under $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ (over the ring of the polynomials that are invariant under the same group). We can use

$$\mathbf{SO}(2)^3 \times \mathbf{S}^1 \subset (\mathbf{SO}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1 \subset (\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$$

and lemma 4.3. Let $f : (\mathbf{C} \oplus \mathbf{C})^3 \to (\mathbf{C} \oplus \mathbf{C})^3$ be $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$-equivariant. With lemma 4.3 and remark 4.4 we can describe a general mapping f equivariant by $\mathbf{SO}(2)^3 \times \mathbf{S}^1$. Therefore we have to look for such f that is also equivariant by the flips in $\mathbf{O}(2)^3$ and by the permutations in $\mathbf{S}_3$. The group $\mathbf{S}_3$ is generated by the transpositions (12) and (13). If we have

$$f = (f_1, f_2, f_3, f_4, f_5, f_6),$$

and imposing the equivariance by the flips and the transpositions, then we must have

$$\begin{aligned}
f_2(z) &= f_1(z_2, z_1, z_3, z_4, z_5, z_6), \\
f_3(z) &= f_1(z_3, z_4, z_1, z_2, z_5, z_6), \\
f_4(z) &= f_1(z_4, z_3, z_1, z_2, z_5, z_6), \\
f_5(z) &= f_1(z_5, z_6, z_3, z_4, z_1, z_2), \\
f_6(z) &= f_1(z_6, z_5, z_3, z_4, z_1, z_2),
\end{aligned}$$

where f_1 satisfies

$$f_1(z) = f_1(z_1, z_2, z_5, z_6, z_3, z_4) = f_1(z_1, z_2, z_4, z_3, z_5, z_6) = f_1(z_1, z_2, z_3, z_4, z_6, z_5).$$

From lemma 4.3 and the remark 4.4

$$f_1(z) = z_1 p_1 + \overline{z_2} z_3 z_4 p_2 + \overline{z_2} z_5 z_6 p_3,$$

for polynomials p_i in $v_i,\ c_{ij}$ and $\overline{c_{ij}}$. For degree one we only need to consider $f_1(z) = a z_1$ for $a \in \mathbf{C}$. We choose $f_1(z) = z_1$. For the degree three,

$$f_1(z) = z_1 p_1(v_1, \ldots) + \overline{z_2} z_3 z_4 b_1 + \overline{z_2} z_5 z_6 b_2$$

for constants $b_1,\ b_2\ \in \mathbf{C}$ and we can select $f_1(z) = z_1 |z_1|^2$, also $f_1(z) = z_1 |z_2|^2$ and $f_1(z) = \overline{z}_2 (z_3 z_4 + z_5 z_6)$. $\square$

5 Periodic solutions

From theorem 3.1 we have (up to conjugacy) the $\mathbf{C}$-axial subgroups of the group $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$. Therefore we can use the equivariant Hopf theorem to prove the existence of periodic solutions with these symmetries.

Proposition 5.1 *Consider the system of ODEs*

$$\dot{z} = f(z, \lambda), \tag{9}$$

where
(i) $f : V^3 \times \mathbf{R} \to V^3$ *is smooth, commuting with* Γ *and*
(ii) $(df)_{0,\lambda}$ *has eigenvalues* $\sigma(\lambda) \mp i\rho(\lambda)$ *with* $\sigma(0) = 0,\ \rho(0) = 1$ *and* $\sigma'(0) \neq 0$.
Then for each isotropy subgroup of $\Gamma \times \mathbf{S}^1$ *conjugate to one of the groups in table 2, there is a unique branch of periodic solutions of (9) bifurcating from* $(0, 0)$ *with period near* 2π *and with symmetry that isotropy subgroup.*

Proof As $\mathrm{Fix}_{V^3}(\Gamma) = \{0\}$, from (i) it follows that

$$f(0, \lambda) = 0$$

and so $(0, \lambda)$ is an equilibrium for all λ that changes stability when λ crosses zero since $\sigma'(0) \neq 0$.

Again we note that as V is $\mathbf{O}(2)$-simple, the space V^3 is Γ-simple. After assuming (i), the assumption (ii) imposes the eigenvalue crossing condition of the equivariant Hopf theorem (and scales the eigenvalues of $(df)_{0,0}$ to $\mp i$). From theorem 3.1 we have the $\mathbf{C}$-axial subgroups of $\Gamma \times \mathbf{S}^1$. □

Our aim in this section is to determine the conditions on the coefficients of the lowest order terms of f that are necessary for each of the different types of bifurcating solutions (with maximal symmetry) to be stable. That is, we wish to calculate the stability of these periodic solutions in terms of the coefficients of the lowest degree terms in the Taylor series expansion of f.

We note that the theory of Birkhoff normal form [13] asserts that for any positive integer k the Taylor series of degree k of a vector field commuting with Γ can be made to commute with the $\mathbf{S}^1$-action by a change of coordinates in V^3 that commutes with the action of Γ. Thus we can assume that the degree k Taylor series also commutes with $\mathbf{S}^1$. We point out that this does not imply that the original vector field commutes with $\mathbf{S}^1$.

Here we assume that the Taylor series of degree three of f commutes also with $\mathbf{S}^1$. We show that the $\mathbf{C}$-axial groups are three-determined and that the stability of the bifurcating periodic solutions of (9) with maximal symmetry depends only on the coefficients of this Taylor series of degree three.

Let f be as in (9). If we suppose that the Taylor series of degree three of f around $z = 0$ commutes also with $\mathbf{S}^1$, then by proposition 4.1 we can write $f = (f_1, \ldots, f_6)$, where

$$f_1(z, \lambda) = [\mu(\lambda) + a|z|^2 + b|z_1|^2 + c|z_2|^2] z_1 + d\bar{z}_2(z_3 z_4 + z_5 z_6) + \text{ terms of degree } \geq 5 \tag{10}$$

and

$$\begin{aligned} f_2(z,\lambda) &= f_1(z_2,z_1,z_3,z_4,z_5,z_6),\\ f_3(z,\lambda) &= f_1(z_3,z_4,z_1,z_2,z_5,z_6),\\ f_4(z,\lambda) &= f_1(z_4,z_3,z_1,z_2,z_5,z_6),\\ f_5(z,\lambda) &= f_1(z_5,z_6,z_3,z_4,z_1,z_2),\\ f_6(z,\lambda) &= f_1(z_6,z_5,z_3,z_4,z_1,z_2). \end{aligned}$$

Also, the coefficients a, b, c, d are complex smooth functions of λ and as in proposition 5.1

$$\mu(0) = i \quad \wedge \quad \mathrm{Re}(\mu^{'}(0)) \neq 0.$$

Suppose that

$$\mathrm{Re}(\mu'(0)) > 0.$$

Rescaling λ if necessary we can suppose that

$$\mathrm{Re}(\mu(\lambda)) = \lambda + \text{ higher order terms in } \lambda.$$

Thus the trivial solution of (9) is stable for λ negative and unstable for λ positive (near zero).

We show now that the coefficients a, b, c and d determine (generically) the directions of branching and the stability of the periodic solutions guaranteed by the equivariant Hopf theorem. However we note that the periodic solutions whose existence is guaranteed by this theorem are not necessarily the only periodic solutions bifurcating from $(0,0)$. We seek in the next section periodic solutions with submaximal isotropy.

Throughout, subscripts r and I on the coefficients a, b, c and d refer to real and imaginary parts.

Theorem 5.2 *Consider the system (9) where f is as in (10). For each symmetry group Σ_i listed in table 2, let $s_0, \ldots, s_r$ be the functions of a, b, c and d listed in table 8 evaluated at $\lambda = 0$.*

(1) For each Σ_i the corresponding branch of periodic solutions is supercritical if $s_0 < 0$ and subcritical if $s_0 > 0$. Tables 6 and 7 list the branching equations.

(2) For each Σ_i, if $s_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $s_j < 0$ for all j, then the branch of periodic solutions is (orbitally) stable near $\lambda = 0$ and $z = 0$.

Proof By the equivariant Hopf theorem we know that we can apply a Liapunov-Schmidt reduction to a map $g(z, \lambda, \tau)$ from $V^3 \times \mathbf{R} \times \mathbf{R}$ to V^3 (note that V^3 is the $\pm i$- real eigenspace of $(df)_{0,0}$) commuting with $\Gamma \times \mathbf{S}^1$, whose zeros are in one-to-one correspondence with periodic solutions of (9) of period $2\pi/(1+\tau)$, and where τ corresponds to the period-perturbing parameter.

Moreover, if we assume that f also commutes with $\mathbf{S}^1$ then by [13] theorem XVI 10.1, the Liapunov-Schmidt reduction function g of (9) has the explicit form

$$g(z,\lambda,\tau) = f(z,\lambda) - (1+\tau)iz \tag{11}$$

(and so the branching equations for f are those for g since the τ-dependence enters in a simple way in those equations), and the asymptotic stability of a

Isotropy subgroup	Branching equation
Σ_1	$\nu + (a+b)\|z\|^2 + \cdots = 0$
Σ_2	$\nu + (2a+b+c)\|z\|^2 + \cdots = 0$
Σ_3	$\nu + (2a+b)\|z\|^2 + \cdots = 0$
Σ_4	$\nu + (4a+b+c+d)\|z\|^2 + \cdots = 0$
Σ_5	$\nu + (4a+b+c-d)\|z\|^2 \cdots = 0$
Σ_6	$\nu + (3a+b)\|z\|^2 + \cdots = 0$
Σ_7	$\nu + (6a+b+c+2d)\|z\|^2 + \cdots = 0$
Σ_8	$\nu + (6a+b+c-d)\|z\|^2 + \cdots = 0$

Table 6: Branching equations for $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$ Hopf bifurcation.

Isotropy subgroup	Branching equation
Σ_1	$\lambda = -(a_r + b_r)\|z\|^2 + \cdots$
Σ_2	$\lambda = -(2a_r + b_r + c_r)\|z\|^2 + \cdots$
Σ_3	$\lambda = -(2a_r + b_r)\|z\|^2 + \cdots$
Σ_4	$\lambda = -(4a_r + b_r + c_r + d_r)\|z\|^2 + \cdots$
Σ_5	$\lambda = -(4a_r + b_r + c_r - d_r)\|z\|^2 + \cdots$
Σ_6	$\lambda = -(3a_r + b_r)\|z\|^2 + \cdots$
Σ_7	$\lambda = -(6a_r + b_r + c_r + 2d_r)\|z\|^2 + \cdots$
Σ_8	$\lambda = -(6a_r + b_r + c_r - d_r)\|z\|^2 + \cdots$

Table 7: Branches of periodic solutions for $\mathbf{O}(2) \wr \mathbf{S}_3$ Hopf bifurcation.

Symmetry of the solution	s_0	$s_1, \ldots, s_r$
Σ_1	$a_r + b_r$	$-b_r + c_r$ $-b_r$
Σ_2	$2a_r + b_r + c_r$	$b_r - c_r$ $-b_r - c_r$ $-(\|b+c\|^2 - \|d\|^2)$
Σ_3	$2a_r + b_r$	b_r $-b_r + c_r + d_r$ $-b_r + c_r - d_r$ $-b_r$
Σ_4	$4a_r + b_r + c_r + d_r$	$b_r - c_r - d_r$ $-b_r - c_r - d_r$ $-[\|b+c\|^2 - 3\|d\|^2 + 2\mathrm{Re}((b+c)\overline{d})]$ $b_r + c_r - 3d_r$ $-[\|d\|^2 - \mathrm{Re}((b+c)\overline{d})]$
Σ_5	$4a_r + b_r + c_r - d_r$	$b_r - c_r + d_r$ $b_r + c_r + 3d_r$ $-[\|d\|^2 + \mathrm{Re}((b+c)\overline{d})]$ $-b_r - c_r + d_r$
Σ_6	$3a_r + b_r$	b_r $-b_r + c_r + 2d_r$ $-b_r + c_r - d_r$
Σ_7	$6a_r + b_r + c_r + 2d_r$	$b_r - c_r - 2d_r$ $b_r + c_r - 4d_r$ $-[\|d\|^2 - \mathrm{Re}((b+c)\overline{d})]$
Σ_8	$6a_r + b_r + c_r - d_r$	$b_r - c_r + d_r$ $b_r + c_r + 2d_r \mp \mathrm{Re}(\delta_2 + i\delta_2')^{1/2}$ where $\delta_2 = (b_r + c_r + 2d_r)^2 - 9d_I^2 +$ $6\|d\|^2 - 6\mathrm{Re}((b+c)\overline{d})$ $\delta_2' = -6d_r(2d_I + b_I + c_I)$

Table 8: Stability for $\mathbf{O}(2) \wr \mathbf{S}_3$ Hopf bifurcation.

bifurcating solution of (9) is equivalent to the linearized stability of the corresponding solution of

$$g(z, \lambda, \tau) = 0. \tag{12}$$

By [13] corollary XVI 10.2, if $z(t)$ is a periodic solution of (9) with $\lambda = \lambda_0$ and $\tau = \tau_0$, and (z_0, λ_0, τ_0) is the corresponding solution of (12), then there is a correspondence between the Floquet multipliers of $z(t)$ and the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ such that a multiplier lies inside (respectively outside) the unit circle if and only if the corresponding eigenvalue has negative (respectively positive) real part. So to determine the stability of each type of bifurcating periodic orbit we can calculate the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ (to the lowest order in z). As g commutes with $\Gamma \times \mathbf{S}^1$, it maps $\mathrm{Fix}(\Sigma)$ into itself, and for each of the conjugacy classes Σ_i in table 2, we have a distinct branch of periodic solutions of (9) that are in correspondence with the zeros of g with isotropy Σ_i. These zeros are found by solving $g\,|_{\mathrm{Fix}(\Sigma_i)} = 0$ (and $\mathrm{Fix}(\Sigma_i)$ is two-dimensional). Note that to find the zeros of g, it suffices to look at representative points on $\Gamma \times \mathbf{S}^1$ orbits. We do the following: we assume that the initial vector field f commutes also with $\mathbf{S}^1$ and so we can apply the results stated above to determine the stability. We show that the stability and the branching equations for the periodic solutions with symmetry the groups in table 2 are completely determined by the Taylor series of f of degree three. Moreover, the groups are three-determined. That is, if $\Sigma_{z_0} = \Sigma_i$ for some $i = 1, \ldots, 8$, then all eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$, other than those forced to zero by Σ_i, have the form

$$\mu_j = \alpha_j |z_0|^2 + o(|z_0|^3)$$

where α_j is a $\mathbf{C}$-valued function of the Taylor coefficients of terms of degree lower or equal three of f. Then we use [13] theorem XVI 11.2 to argue that the same conditions determine the stability and the direction of the branches of the periodic solutions with maximal symmetry of (9) even if f does not commute with $\mathbf{S}^1$. Note that by [13] theorem XVI 5.8 we can always choose a coordinates change such that the third order truncated normal form of f commutes with $\Gamma \times \mathbf{S}^1$. Thus there is no loss of generality in assuming that f has the form (10).

If we consider the Taylor series of f around $z = 0$ as in (10) we may write g as $(g_1, g_2, g_3, g_4, g_5, g_6)$, where

$$g_1(z, \lambda, \tau) = [\nu + a|z|^2 + b|z_1|^2 + c|z_2|^2] z_1 + d\bar{z}_2(z_3 z_4 + z_5 z_6) + \cdots,$$

and

$$\begin{aligned} g_2(z, \lambda) &= g_1(z_2, z_1, z_3, z_4, z_5, z_6), \\ g_3(z, \lambda) &= g_1(z_3, z_4, z_1, z_2, z_5, z_6), \\ g_4(z, \lambda) &= g_1(z_4, z_3, z_1, z_2, z_5, z_6), \\ g_5(z, \lambda) &= g_1(z_5, z_6, z_3, z_4, z_1, z_2), \\ g_6(z, \lambda) &= g_1(z_6, z_5, z_3, z_4, z_1, z_2), \end{aligned}$$

with $\nu(\lambda, \tau) = \mu(\lambda) - (1 + \tau)i$.

When restricted to a two-dimensional fixed-point subspace $\mathrm{Fix}(\Sigma)$, equation (12) gives a single complex scalar equation (table 6) and hence two real ones.

Use the imaginary equation (involving $\mathrm{Im}(\nu(\lambda,\tau))$) to solve for τ. Now the other equation we can solve for z as a function of λ by the implicit function theorem.

To determine for each values of the coefficients of the Taylor expansion of g (around zero), and so f, the periodic solutions predicted by the equivariant Hopf theorem bifurcate supercritically or subcritically, we have to solve the real part of (12) restricted to $\mathrm{Fix}(\Sigma)$ and we get table 7.

To determine the stability of each type of the bifurcating periodic orbit, we calculate the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ (to the lowest order in z) in terms of the coefficients of the Taylor expansion of g. First we find conditions on the matrix $(dg)_{(z_0,\lambda_0,\tau_0)}$ which must be satisfied as a consequence of the fact that it commutes with Σ. Recall table 3. Second we use these conditions to find a basis for V^3 with respect to which the matrix is block diagonal with 2×2 blocks. Another way of doing this could be by decomposing V^3 into subspaces, each of each is invariant under a different representation of the isotropy subgroup Σ_{z_0}. That is, we could form the isotypic decomposition $V^3 = W_1 \oplus \cdots \oplus W_k$ where each isotypic component W_i may be further decomposed into subspaces each of which transforms according to the ith irreducible representation of Σ_{z_0} (each W_i would be the sum of all Σ_{z_0}-irreducible subspaces of V^3 that are Σ_{z_0}-isomorphic). We can always take $W_1 = \mathrm{Fix}(\Sigma_{z_0})$ so that W_1 is the sum of all subspaces of V^3 on which Σ_{z_0} acts trivially. Note that the orbit of z_0 by the group $\mathbf{S}^1$ is in W_1, so that there is a zero eigenvalue of $(dg)_{(z_0,\lambda_0,\tau_0)}$ restricted to W_1 for each solution z_0. Thus for the solutions with $\mathbf{C}$-axial symmetry, the nonzero eigenvalue of $(dg)_{(z_0,\lambda_0,\tau_0)}$ restricted to $\mathrm{Fix}(\Sigma_{z_0})$ is given by the trace of $(dg)_{(z_0,\lambda_0,\tau_0)}\,|_{\mathrm{Fix}(\Sigma_{z_0})}$ and it is associated with the eigenvector in the plane of the limit cycle and transverse to it. This eigenvalue depends on the direction of the bifurcating branch. Finally, we calculate the entries in the matrix in terms of the Taylor coefficients and obtain the signs of the real parts of the eigenvalues.

Note that as the group action forces some of the Floquet multipliers to be equal to one, it also forces the corresponding eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to be equal to zero. The eigenvectors of these eigenvalues at the point z_0 are the tangent vectors to the orbit of $\mathbf{SO}(2)^3\times\mathbf{S}^1$ through z_0. In fact, if the solution z_0 has symmetry Σ_{z_0}, then the group orbit has the dimension of $(\Gamma\times\mathbf{S}^1)/\Sigma_{z_0}$ and so the number of zero eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ forced by the group action is

$$d_{\Sigma_{z_0}} = 4 - \dim(\Sigma_{z_0}),$$

since $\dim\Gamma\times\mathbf{S}^1 = 4$. Thus if the group Σ_{z_0} is discrete, then there are four zero eigenvalues: three associated with the $\mathbf{O}(2)^3$ symmetry and one associated with the phase-shift $\mathbf{S}^1$-symmetry. This is the case for Σ_7 and Σ_8. We have $d_{\Sigma_i} = 3$, for $i = 4,5,6$, also $d_{\Sigma_i} = 2$ for $i = 2,3$ and finally $d_{\Sigma_1} = 1$.

We take co-ordinate functions on V^3:

$$z_1, \overline{z_1}, \ldots, z_6, \overline{z_6}.$$

These correspond to a basis B for V^3, the elements of which we will denote by:

$$l_1, \overline{l_1}, \ldots, l_6, \overline{l_6}.$$

An $\mathbf{R}$-linear mapping on $\mathbf{C}$ has the form

$$w \longmapsto \alpha w + \beta \overline{w},$$

where α and β are complex and the matrix M of this mapping in these coordinates has

$$\mathrm{Tr}(M) = 2\mathrm{Re}(\alpha) \ \wedge \ \mathrm{Det}(M) = |\alpha|^2 - |\beta|^2.$$

The eigenvalues of this matrix are

$$\frac{\mathrm{tr}(M)}{2} \mp \sqrt{\left(\frac{\mathrm{Tr}(M)}{2}\right)^2 - \mathrm{Det}(M)}.$$

If one eigenvalue is zero, then $\mathrm{Det}(M) = 0$ and the sign of the other eigenvalue (if it is not zero) is given by the sign of the real part of α. If there are no zero eigenvalues, then the eigenvalues have negative real part if and only if the determinant is positive and the trace is negative.

(Σ_1)

The fixed-point subspace is $z_2 = \cdots = z_6 = 0$ and $z_1 = z$. Using the equation (12) after dividing by z we have

$$\nu(\lambda) + (a+b)|z|^2 + \cdots = 0$$

where $+\cdots$ denotes terms of higher order in z and $\overline{z}$, and taking the real part of this equation, we obtain

$$\lambda = -(a_r + b_r)|z|^2 + \cdots$$

where the functions a_r, b_r are evaluated at $\lambda = 0$ and $+\cdots$ indicates higher order terms in z, $\overline{z}$ and λ. It follows that if $a_r + b_r < 0$ ($a_r + b_r > 0$), then the branch bifurcates supercritically (subcritically).

Throughout we denote by (z_0, λ_0, τ_0) a zero of (12) with $z_0 \in \mathrm{Fix}(\Sigma_i)$ and we wish to calculate $(dg)_{(z_0,\lambda_0,\tau_0)}$.

Let $\Sigma = \Sigma_1$ be the isotropy subgroup of $z_0 = (z, 0, 0, 0, 0, 0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$\mathrm{diag}(A, B, C, C, C, C),$$

where A, B and C are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 \\ 0 & \overline{b}_1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix}$$

and

$$a_1 = \tfrac{\partial g_1}{\partial z_1}, \quad a_1^{'} = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad b_1 = \tfrac{\partial g_2}{\partial z_2}, \quad c_1 = \tfrac{\partial g_3}{\partial z_3}$$

calculated at (z_0, λ_0, τ_0).

A tangent vector to the orbit of $\Gamma\times\mathbf{S}^1$ through z_0 is the eigenvector $(iz,0,0,0,0,0)$. Thus the matrix A has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(a_1) = 2(a_r + b_r)|z|^2 + \cdots,$$

whose sign is determined by

$$a_r + b_r$$

if it is assumed nonzero (where $a_r + b_r$ is calculated at zero).

The eigenvalues of B are the conjugate complex numbers

$$b_1,\ \overline{b}_1$$

and

$$b_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,z,0,0,0,0,\lambda_0,\tau_0)} = (-b+c)|z|^2 + \cdots$$

Thus the sign of $\mathrm{Re}(b_1)$ is determined by the sign of

$$-b_r + c_r.$$

The matrix C has eigenvalues

$$c_1,\ \overline{c}_1$$

four times each one and as

$$c_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,z,0,0,0,\lambda_0,\tau_0)} = -b|z|^2 + \cdots,$$

then $\mathrm{Re}(c_1)$ is determined by

$$-b_r.$$

(Σ_2)

Let $\Sigma = \Sigma_2$ be the isotropy subgroup of $z_0 = (z,z,0,0,0,0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & B & & & & \\ B & A & & & & \\ & & C & D & & \\ & & D & C & & \\ & & & & C & D \\ & & & & D & C \end{bmatrix},$$

where $A,\ B,\ C$ and D are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix},\quad B = \begin{bmatrix} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{bmatrix},\quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix},\quad D = \begin{bmatrix} 0 & d_1^{'} \\ \overline{d}_1^{'} & 0 \end{bmatrix}$$

and

$$a_1 = \frac{\partial g_1}{\partial z_1}, \quad a_1' = \frac{\partial g_1}{\partial \bar{z}_1}, \quad b_1 = \frac{\partial g_1}{\partial z_2}, \quad b_1' = \frac{\partial g_1}{\partial \bar{z}_2}, \quad c_1 = \frac{\partial g_3}{\partial z_3}, \quad d_1' = \frac{\partial g_3}{\partial \bar{z}_4}$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis B':

$$\begin{array}{l} l_1 + l_2,\ c.c.,\ l_1 - l_2,\ c.c.,\ l_3 + l_4,\ c.c., \\ l_3 - l_4,\ c.c.,\ l_5 + l_6,\ c.c.,\ l_5 - l_6,\ c.c., \end{array}$$

the matrix becomes block diagonal with 2×2 blocks:

$$\operatorname{diag}(A+B,\ A-B,\ C+D,\ C-D,\ C+D,\ C-D).$$

That is, the matrices are similar:

$$\operatorname{diag}(A{+}B,\ A{-}B,\ C{+}D,\ C{-}D,\ C{+}D,\ C{-}D) = S \begin{bmatrix} A & B & & & & \\ B & A & & & & \\ & & C & D & & \\ & & D & C & & \\ & & & & C & D \\ & & & & D & C \end{bmatrix} S^{-1},$$

where

$$S = S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} Id & Id & & & & \\ Id & -Id & & & & \\ & & Id & Id & & \\ & & Id & -Id & & \\ & & & & Id & Id \\ & & & & Id & -Id \end{bmatrix}.$$

The null vectors can be chosen to be

$$\begin{array}{l} (iz, iz, 0, 0, 0, 0), \\ (-iz, iz, 0, 0, 0, 0). \end{array}$$

Zero is an eigenvalue of $A+B$ and $A-B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1 + b_1),\ 2\mathrm{Re}(a_1 - b_1),$$

and since

$$\begin{array}{l} a_1 + b_1 = (2a + b + c)|z|^2 + \cdots \\ a_1 - b_1 = (b - c)|z|^2 + \cdots, \end{array}$$

their signs depend on

$$2a_r + b_r + c_r,\ b_r - c_r.$$

For $C+D$ and $C-D$, the eigenvalues are

$$\mathrm{Re}(c_1) \mp \sqrt{|d_1'|^2 - \mathrm{Im}^2(c_1)}$$

each four times, and these eigenvalues have negative real part if and only if

$$\mathrm{Tr}(C+D) = \mathrm{Tr}(C-D) < 0 \ \wedge \ \mathrm{Det}(C+D) = \mathrm{Det}(C-D) > 0.$$

As

$$c_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,z,z,0,0,\lambda_0,\tau_0)} \qquad d_1^{'} = \left(\frac{\partial g_1}{\partial \overline{z}_2}\right)_{(0,0,z,z,0,0,\lambda_0,\tau_0)}$$

we have

$$\begin{array}{l} \mathrm{Tr}(C+D) = 2\mathrm{Re}(c_1) = -2(b_r + c_r)|z|^2 + \cdots \\ \mathrm{Det}(C+D) = |c_1|^2 - |d_1^{'}|^2 = (|b+c|^2 - |d|^2)|z|^4 + \cdots \end{array}$$

(Σ_3)

Let $\Sigma = \Sigma_3$ be the isotropy subgroup of $z_0 = (z,0,z,0,0,0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & & B & & & \\ & C & & D & & \\ B & & A & & & \\ & D & & C & & \\ & & & & E & \\ & & & & & E \end{bmatrix},$$

where $A,\ B,\ C,\ D$ and E are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{bmatrix},$$
$$E = \begin{bmatrix} e_1 & 0 \\ 0 & \overline{e}_1 \end{bmatrix}$$

and

$$a_1 = \tfrac{\partial g_1}{\partial z_1}, \quad a_1^{'} = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad b_1 = \tfrac{\partial g_1}{\partial z_3}, \quad b_1^{'} = \tfrac{\partial g_1}{\partial \overline{z}_3}, \quad c_1 = \tfrac{\partial g_2}{\partial z_2}, \quad d_1 = \tfrac{\partial g_2}{\partial z_4}, \quad e_1 = \tfrac{\partial g_5}{\partial z_5}$$

calculated at (z_0,λ_0,τ_0).

With respect to the new basis $B^{'}$:

$$\begin{array}{l} l_1 + l_3,\ c.c.,\ l_2 + l_4,\ c.c.,\ l_1 - l_3,\ c.c., \\ l_2 - l_4,\ c.c.,\ l_5,\ c.c.,\ l_6,\ c.c., \end{array}$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A+B,\ C+D,\ A-B,\ C-D,\ E,\ E).$$

The null vectors can be

$$\begin{array}{l} (iz,0,iz,0,0,0), \\ (iz,0,-iz,0,0,0), \end{array}$$

and zero is an eigenvalue (two times) of $A+B$ and $A-B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1+b_1),\ 2\mathrm{Re}(a_1-b_1),$$

where

$$\begin{array}{l} a_1=(a+b)|z|^2+\cdots \\ b_1=a|z|^2+\cdots \end{array}$$

and so their signs depend on

$$2a_r+b_r,\ b_r.$$

For $C+D$ and $C-D$, the eigenvalues are

$$c_1\mp d_1,\ c.c.$$

and for E are

$$e_1,\ \overline{e}_1,$$

each occurring two times. Since

$$c_1=\left(\frac{\partial g_1}{\partial z_1}\right)_{(0,z,z,0,0,0,\lambda_0,\tau_0)}\quad d_1=\left(\frac{\partial g_1}{\partial z_4}\right)_{(0,z,z,0,0,0,\lambda_0,\tau_0)}\quad e_1=\left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,z,0,z,0,\lambda_0,\tau_0)}$$

we have

$$\begin{array}{l} \mathrm{Re}(c_1+d_1)=(-b_r+c_r+d_r)|z|^2+\cdots \\ \mathrm{Re}(c_1-d_1)=(-b_r+c_r-d_r)|z|^2+\cdots \\ \mathrm{Re}(e_1)=-b_r|z|^2+\cdots \end{array}$$

(Σ_4)

Let $\Sigma=\Sigma_4$ be the isotropy subgroup of $z_0=(z,z,z,z,0,0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)}=\left[\begin{array}{cccccc} A & B & C & C & & \\ B & A & C & C & & \\ C & C & A & B & & \\ C & C & B & A & & \\ & & & & D & E \\ & & & & E & D \end{array}\right],$$

where $A,\ B,\ C,\ D$ and E are the 2×2 matrices:

$$A=\left[\begin{array}{cc} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{array}\right],\quad B=\left[\begin{array}{cc} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{array}\right],\quad C=\left[\begin{array}{cc} c_1 & c_1^{'} \\ \overline{c}_1^{'} & \overline{c}_1 \end{array}\right],\quad D=\left[\begin{array}{cc} d_1 & 0 \\ 0 & \overline{d}_1 \end{array}\right],$$
$$E=\left[\begin{array}{cc} 0 & e_1^{'} \\ \overline{e}_1^{'} & 0 \end{array}\right]$$

and

$$a_1=\frac{\partial g_1}{\partial z_1},\quad a_1^{'}=\frac{\partial g_1}{\partial \overline{z}_1},\quad b_1=\frac{\partial g_1}{\partial z_2},\quad b_1^{'}=\frac{\partial g_1}{\partial \overline{z}_2},\quad c_1=\frac{\partial g_1}{\partial z_3},\quad c_1^{'}=\frac{\partial g_1}{\partial \overline{z}_3},\quad d_1=\frac{\partial g_5}{\partial z_5},\quad e_1^{'}=\frac{\partial g_5}{\partial \overline{z}_6}$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis $B^{'}$:

$$l_1 - l_2,\ c.c.,\ l_3 - l_4,\ c.c.,\ l_1 + l_2 + l_3 + l_4,\ c.c.,$$
$$l_1 + l_2 - l_3 - l_4,\ c.c.,\ l_5 + l_6,\ c.c.,\ l_5 - l_6,\ c.c.,$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A - B,\ A - B,\ A + B + 2C,\ A + B - 2C, D + E,\ D - E).$$

The null vectors are

$$(iz, iz, iz, iz, 0, 0),$$
$$(-iz, iz, 0, 0, 0, 0),$$
$$(0, 0, -iz, iz, 0, 0)$$

and zero is an eigenvalue $A + B + 2C$ and $A - B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1 - b_1),\ 2\mathrm{Re}(a_1 + b_1 + 2c_1)$$

and

$$\mathrm{Re}(a_1 - b_1) = (b_r - c_r - d_r)|z|^2 + \cdots$$
$$\mathrm{Re}(a_1 + b_1 + 2c_1) = (4a_r + b_r + c_r + d_r)|z|^2 + \cdots$$

The matrix $A + B - 2C$ has eigenvalues complex conjugates whose real part is negative if and only if

$$\mathrm{Tr}(A + B - 2C) < 0\ \wedge\ \mathrm{Det}(A + B - 2C) > 0.$$

We have

$$\mathrm{Tr}(A + B - 2C) = 2\mathrm{Re}(a_1 + b_1 - 2c_1) = 2(b_r + c_r - 3d_r)|z|^2 + \cdots$$
$$\mathrm{Det}(A + B - 2C) = |a_1 + b_1 - 2c_1|^2 - |a_1^{'} + b_1^{'} - 2c_1^{'}|^2 = 8[|d|^2 - \mathrm{Re}((b + c)\overline{d})]|z|^4 + \cdots,$$

Finally, the eigenvalues of $D + E$ and $D - E$ are

$$\mathrm{Re}(d_1) \mp \sqrt{|e_1^{'}|^2 - \mathrm{Im}^2(d_1)},$$

which have negative real part if and only if

$$\mathrm{Tr}(D + E) = 2\mathrm{Re}(d_1) < 0\ \wedge\ \mathrm{Det}(D + E) = |d_1|^2 - |e_1^{'}|^2 > 0.$$

Since

$$d_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,z,z,z,z,\lambda_0,\tau_0)} \quad e_1^{'} = \left(\frac{\partial g_1}{\partial \overline{z}_2}\right)_{(0,0,z,z,z,z,\lambda_0,\tau_0)}$$

we have

$$\mathrm{Re}(d_1) = -(b_r + c_r + d_r)|z|^2 + \cdots$$
$$|d_1|^2 - |e_1^{'}|^2 = [|b + c|^2 - 3|d|^2 + 2\mathrm{Re}((b + c)\overline{d})]|z|^4 + \cdots$$

(Σ_5)

Let $\Sigma = \Sigma_5$ be the isotropy subgroup of $z_0 = (z, z, iz, iz, 0, 0)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & B & C^{'} & C^{'} & & \\ B & A & C^{'} & C^{'} & & \\ C & C & A^{'} & B^{'} & & \\ C & C & B^{'} & A^{'} & & \\ & & & & D & \\ & & & & & D \end{bmatrix},$$

where $A,\ B,\ C,\ D,\ A^{'},\ B^{'},\ C^{'}$ and D are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_1^{'} \\ \overline{b}_1^{'} & \overline{b}_1 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 & c_1^{'} \\ \overline{c}_1^{'} & \overline{c}_1 \end{bmatrix}, \qquad D = \begin{bmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{bmatrix},$$

$$A^{'} = \begin{bmatrix} a_1 & -a_1^{'} \\ -\overline{a}_1^{'} & \overline{a}_1 \end{bmatrix},\ B^{'} = \begin{bmatrix} b_1 & -b_1^{'} \\ -\overline{b}_1^{'} & \overline{b}_1 \end{bmatrix},\ C^{'} = \begin{bmatrix} -c_1 & c_1^{'} \\ \overline{c}_1^{'} & -\overline{c}_1 \end{bmatrix}$$

and

$$a_1 = \tfrac{\partial g_1}{\partial z_1},\quad a_1^{'} = \tfrac{\partial g_1}{\partial \overline{z}_1},\quad b_1 = \tfrac{\partial g_1}{\partial z_2},\quad b_1^{'} = \tfrac{\partial g_1}{\partial \overline{z}_2},\quad -c_1 = \tfrac{\partial g_1}{\partial z_3},\quad c_1^{'} = \tfrac{\partial g_1}{\partial \overline{z}_3},\quad d_1 = \tfrac{\partial g_5}{\partial z_5}$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis $B^{'}$:

$$\begin{array}{l} l_1 - l_2,\ c.c.,\ l_3 - l_4,\ c.c.,\ l_1 + l_2 + i(l_3 + l_4),\ c.c., \\ l_1 + l_2 - i(l_3 + l_4),\ c.c.,\ l_5,\ c.c.,\ l_6,\ c.c., \end{array}$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A - B,\ A^{'} - B^{'},\ M,\ N, D,\ D),$$

where

$$M = \begin{bmatrix} a_1 + b_1 - 2ic_1 & a_1^{'} + b_1^{'} - 2ic_1^{'} \\ \overline{a}_1^{'} + \overline{b}_1^{'} + 2i\overline{c}_1^{'} & \overline{a}_1 + \overline{b}_1 + 2i\overline{c}_1 \end{bmatrix}$$

and

$$N = \begin{bmatrix} a_1 + b_1 + 2ic_1 & a_1^{'} + b_1^{'} + 2ic_1^{'} \\ \overline{a}_1^{'} + \overline{b}_1^{'} - 2i\overline{c}_1^{'} & \overline{a}_1 + \overline{b}_1 - 2i\overline{c}_1 \end{bmatrix}.$$

The null vectors are:

$$\begin{array}{l} (iz, iz, -z, -z, 0, 0), \\ (-iz, iz, 0, 0, 0, 0), \\ (0, 0, z, -z, 0, 0) \end{array}$$

and zero is an eigenvalue of $M,\ A - B$ and $A^{'} - B^{'}$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1 - b_1),\ 2\mathrm{Re}(a_1 + b_1 - 2ic_1),$$

and

$$\begin{array}{l}\mathrm{Re}(a_1 - b_1) = (b_r - c_r + d_r)|z|^2 + \cdots \\ \mathrm{Re}(a_1 + b_1 - 2ic_1) = (4a_r + b_r + c_r - d_r)|z|^2 + \cdots\end{array}$$

The matrix N has eigenvalues complex conjugates whose real part is negative if and only if

$$\mathrm{Tr}(N) < 0 \ \wedge \ \mathrm{Det}(N) > 0,$$

and

$$\begin{array}{l}\mathrm{Tr}(N) = 2\mathrm{Re}(a_1 + b_1 + 2ic_1) = 2(b_r + c_r + 3d_r)|z|^2 + \cdots \\ \mathrm{Det}(N) = |a_1 + b_1 + 2ic_1|^2 - |a_1' + b_1' + 2ic_1'|^2 = 8[|d|^2 + \mathrm{Re}((b+c)\overline{d})]|z|^4 + \cdots\end{array}$$

Finally, the matrix D has the eigenvalues

$$d_1, \ \overline{d}_1$$

and

$$d_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,0,iz,iz,z,z,\lambda_0,\tau_0)}$$

Thus

$$\mathrm{Re}(d_1) = (-b_r - c_r + d_r)|z|^2 + \cdots$$

(Σ_6)

Let $\Sigma = \Sigma_6$ be the isotropy subgroup of $z_0 = (z, 0, z, 0, z, 0)$. With respect to the basis B, any matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & & B & & B & \\ & C & & D & & D \\ B & & A & & B & \\ & D & & C & & D \\ B & & B & & A & \\ & D & & D & & C \end{bmatrix},$$

where $A,\ B,\ C$ and D are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1' \\ \overline{a}_1' & \overline{a}_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_1' \\ \overline{b}_1' & \overline{b}_1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{bmatrix}$$

and

$$a_1 = \tfrac{\partial g_1}{\partial z_1}, \quad a_1' = \tfrac{\partial g_1}{\partial \overline{z}_1}, \quad b_1 = \tfrac{\partial g_1}{\partial z_3}, \quad b_1' = \tfrac{\partial g_1}{\partial \overline{z}_3}, \quad c_1 = \tfrac{\partial g_2}{\partial z_2}, \quad d_1 = \tfrac{\partial g_2}{\partial z_4}$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis B':

$$\begin{array}{l} l_1 + l_3 + l_5,\ c.c.,\ l_2 + l_4 + l_6,\ c.c.,\ l_1 - 2l_3 + l_5,\ c.c., \\ l_2 - 2l_4 + l_6,\ c.c.,\ l_1 - l_3,\ c.c.,\ l_2 - l_4,\ c.c., \end{array}$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A+2B,\ C+2D,\ A-B,\ C-D, A-B,\ C-D).$$

The null vectors can be

$$\begin{array}{l}(iz,0,iz,0,iz,0),\\ (iz,0,-iz,0,0,0),\\ (iz,0,-2iz,0,iz,0)\end{array}$$

and zero is an eigenvalue (three times) of $A+2B$ and $A-B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1+2b_1),\ 2\mathrm{Re}(a_1-b_1)$$

and

$$\begin{array}{l}\mathrm{Re}(a_1+2b_1) = (3a_r+b_r)|z|^2+\cdots\\ \mathrm{Re}(a_1-b_1) = b_r|z|^2+\cdots\end{array}$$

The matrices $C+2D$ and $C-D$ have eigenvalues complex conjugates

$$c_1+2d_1,\ c.c.,\ c_1-d_1,\ c.c.$$

As

$$c_1 = \left(\frac{\partial g_1}{\partial z_1}\right)_{(0,z,z,0,z,0,\lambda_0,\tau_0)} \qquad d_1 = \left(\frac{\partial g_1}{\partial z_4}\right)_{(0,z,z,0,z,0,\lambda_0,\tau_0)}$$

then

$$\begin{array}{l}\mathrm{Re}(c_1+2d_1) = (-b_r+c_r+2d_r)|z|^2+\cdots\\ \mathrm{Re}(c_1-d_1) = (-b_r+c_r-d_r)|z|^2+\cdots\end{array}$$

(Σ_7)

Let $\Sigma = \Sigma_7$ be the isotropy subgroup of $z_0 = (z,z,z,z,z,z)$. With respect to the basis B, any matrix commuting with Σ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix} A & B & C & C & C & C\\ B & A & C & C & C & C\\ C & C & A & B & C & C\\ C & C & B & A & C & C\\ C & C & C & C & A & B\\ C & C & C & C & B & A\end{bmatrix},$$

where A, B and C are the 2×2 matrices:

$$A = \begin{bmatrix} a_1 & a_1^{'}\\ \overline{a}_1^{'} & \overline{a}_1\end{bmatrix},\quad B = \begin{bmatrix} b_1 & b_1^{'}\\ \overline{b}_1^{'} & \overline{b}_1\end{bmatrix},\quad C = \begin{bmatrix} c_1 & c_1^{'}\\ \overline{c}_1^{'} & \overline{c}_1\end{bmatrix}$$

and

$$a_1 = \frac{\partial g_1}{\partial z_1},\quad a_1^{'} = \frac{\partial g_1}{\partial \overline{z}_1},\quad b_1 = \frac{\partial g_1}{\partial z_2},\quad b_1^{'} = \frac{\partial g_1}{\partial \overline{z}_2},\quad c_1 = \frac{\partial g_1}{\partial z_3},\quad c_1^{'} = \frac{\partial g_1}{\partial \overline{z}_3}$$

calculated at (z_0, λ_0, τ_0).

With respect to the new basis B':

$$
\begin{aligned}
&l_1+l_2+l_3+l_4+l_5+l_6,\ c.c.,\ l_1+l_2-(l_5+l_6),\ c.c.,\\
&l_1-l_2+l_3-l_4,\ c.c.,\ l_3-l_4+l_5-l_6,\ c.c.,\ l_1-l_2+l_5-l_6,\ c.c.,\\
&l_1+l_2-(l_3+l_4),\ c.c.,
\end{aligned}
$$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(A+B+4C,\ A+B-2C,\ A-B,\ A-B,\ A-B,\ A+B-2C).$$

The null vectors are:

$$
\begin{aligned}
&(iz, iz, iz, iz, iz, iz),\\
&(-iz, iz, -iz, iz, 0, 0),\\
&(0, 0, -iz, iz, -iz, iz),\\
&(-iz, iz, 0, 0, -iz, iz)
\end{aligned}
$$

and zero is an eigenvalue of $A+B+4C$ and $A-B$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1+b_1+4c_1),\ 2\mathrm{Re}(a_1-b_1)$$

and

$$
\begin{aligned}
&\mathrm{Re}(a_1+b_1+4c_1) = (6a_r+b_r+c_r+2d_r)|z|^2+\cdots\\
&\mathrm{Re}(a_1-b_1) = (b_r-c_r-2d_r)|z|^2+\cdots
\end{aligned}
$$

The matrix $A+B-2C$ has eigenvalues complex conjugates that have negative real part if and only if

$$
\begin{aligned}
&\mathrm{Tr}(A+B-2C) = 2\mathrm{Re}(a_1+b_1-2c_1) < 0,\\
&\mathrm{Det}(A+B-2C) = |a_1+b_1-2c_1|^2 - |a_1'+b_1'-2c_1'|^2 > 0.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\mathrm{Re}(a_1+b_1-2c_1) = (b_r+c_r-4d_r)|z|^2+\cdots\\
&|a_1+b_1-2c_1|^2 - |a_1'+b_1'-2c_1'|^2 = 12[|d|^2 - \mathrm{Re}((b+c)\overline{d})]|z|^4+\cdots
\end{aligned}
$$

(Σ_8)

Let $\Sigma = \Sigma_8$ be the isotropy subgroup of $z_0 = (z, z, \xi z, \xi z, \xi^2 z, \xi^2 z)$. With respect to the basis B, any 'real' matrix commuting with Σ has the form

$$
(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{bmatrix}
A_1 & A_2 & B_2^{\xi} & B_2^{\xi} & B_1^{\xi^2} & B_1^{\xi^2}\\
A_2 & A_1 & B_2^{\xi} & B_2^{\xi} & B_1^{\xi^2} & B_1^{\xi^2}\\
B_1 & B_1 & A_1^{\xi} & A_2^{\xi} & B_2^{\xi^2} & B_2^{\xi^2}\\
B_1 & B_1 & A_2^{\xi} & A_1^{\xi} & B_2^{\xi^2} & B_2^{\xi^2}\\
B_2 & B_2 & B_1^{\xi} & B_1^{\xi} & A_1^{\xi^2} & A_2^{\xi^2}\\
B_2 & B_2 & B_1^{\xi} & B_1^{\xi} & A_2^{\xi^2} & A_1^{\xi^2}
\end{bmatrix},
$$

where $A_1,\ A_2,\ B_1,\ B_2,\ A_1^{\xi},\ A_1^{\xi^2},\ldots$, are the 2×2 matrices:

$$A_1=\begin{bmatrix} a_1 & a_1^{'} \\ \overline{a_1}' & \overline{a_1} \end{bmatrix},\quad A_2=\begin{bmatrix} b_1 & b_1^{'} \\ \overline{b_1}' & \overline{b_1} \end{bmatrix},\qquad B_1=\begin{bmatrix} c_1 & c_1^{'} \\ \overline{c_1}' & \overline{c_1} \end{bmatrix}$$

$$B_2=\begin{bmatrix} d_1 & d_1^{'} \\ \overline{d}_1^{'} & \overline{d}_1 \end{bmatrix},\quad A_1^{\xi}=\begin{bmatrix} a_1 & \xi^2 a_1^{'} \\ \xi\overline{a}_1^{'} & \overline{a}_1 \end{bmatrix},\quad A_1^{\xi^2}=\begin{bmatrix} a_1 & \xi a_1^{'} \\ \xi^2\overline{a}_1^{'} & \overline{a}_1 \end{bmatrix}$$

With respect to the new basis $B^{'}$:

$l_1+l_2+\xi(l_3+l_4)+\xi^2(l_5+l_6),\ c.c.,\ l_1-l_2,\ c.c.,\ l_3-l_4,\ c.c.,$
$l_5-l_6,\ \ c.c.,\ l_1+l_2+l_3+l_4+l_5+l_6,\ \overline{l}_1+\overline{l}_2+\xi(\overline{l}_3+\overline{l}_4)+\xi^2(\overline{l}_5+\overline{l}_6),$
$l_1+l_2+\xi^2(l_3+l_4)+\xi(l_5+l_6),\ \overline{l}_1+\overline{l}_2+\overline{l}_3+\overline{l}_4+\overline{l}_5+\overline{l}_6,$

the matrix becomes block diagonal with 2×2 blocks:

$$\mathrm{diag}(M_1,\ A_1-A_2,\ A_1^{\xi}-A_2^{\xi},\ A_1^{\xi^2}-A_2^{\xi^2},\ M_2,\ M_3),$$

where

$$M_1=\begin{bmatrix} a_1+b_1+2\xi d_1+2\xi^2 c_1 & a_1^{'}+b_1^{'}+2\xi d_1^{'}+2\xi^2 c_1^{'} \\ \overline{a}_1^{'}+\overline{b}_1^{'}+2\xi^2\overline{d}_1^{'}+2\xi\overline{c}_1^{'} & \overline{a}_1+\overline{b}_1+2\xi^2\overline{d}_1+2\xi\overline{c_1} \end{bmatrix},$$

$$M_2=\begin{bmatrix} a_1+b_1+2c_1+2d_1 & a_1^{'}+b_1^{'}+2c_1^{'}+2d_1^{'} \\ \overline{a}_1^{'}+\overline{b}_1^{'}+2\xi\overline{d}_1^{'}+2\xi^2\overline{c}_1^{'} & \overline{a}_1+\overline{b}_1+2\xi\overline{d}_1+2\xi^2\overline{c}_1 \end{bmatrix},$$

$$M_3=\begin{bmatrix} a_1+b_1+2\xi c_1+2\xi^2 d_1 & a_1^{'}+b_1^{'}+2\xi c_1^{'}+2\xi^2 d_1^{'} \\ \overline{a}_1^{'}+\overline{b}_1^{'}+2\overline{c}_1^{'}+2\overline{d}_1^{'} & \overline{a}_1+\overline{b}_1+2\overline{d}_1+2\overline{c}_1 \end{bmatrix}$$

and

$$a_1=\tfrac{\partial g_1}{\partial z_1},\quad a_1^{'}=\tfrac{\partial g_1}{\partial \overline{z}_1},\quad b_1=\tfrac{\partial g_1}{\partial z_2},\quad b_1^{'}=\tfrac{\partial g_1}{\partial \overline{z}_2},$$
$$c_1=\tfrac{\partial g_1}{\partial z_5},\quad \xi c_1^{'}=\tfrac{\partial g_1}{\partial \overline{z}_5},\quad d_1=\tfrac{\partial g_1}{\partial z_3},\quad \xi^2 d_1^{'}=\tfrac{\partial g_1}{\partial \overline{z}_3}$$

calculated at (z_0,λ_0,τ_0).

The null eigenvectors can be

$$(iz,iz,i\xi z,i\xi z,i\xi^2 z,i\xi^2 z),$$
$$(-iz,iz,0,0,0,0),$$
$$(0,0,-i\xi z,i\xi z,0,0),$$
$$(0,0,0,0,-i\xi^2 z,i\xi^2 z),$$

and so zero is an eigenvalue of the matrices $M_1,\ A_1-A_2,\ A_1^{\xi}-A_2^{\xi}$ and $A_1^{\xi^2}-A_2^{\xi^2}$. The others eigenvalues of these matrices are

$$2\mathrm{Re}(a_1+b_1+2\xi d_1+2\xi^2 c_1),\ 2\mathrm{Re}(a_1-b_1),$$

and

$$\mathrm{Re}(a_1+b_1+2\xi d_1+2\xi^2 c_1)=(6a_r+b_r+c_r-d_r)|z|^2+\cdots$$
$$\mathrm{Re}(a_1-b_1)=(b_r-c_r+d_r)|z|^2+\cdots$$

The eigenvalues of M_i for $i = 2,\ 3$ are

$$T_i \mp \sqrt{T_i^2 - D_i}$$

where $T_i = \frac{\mathrm{Tr}(M_i)}{2}$ and $D_i = \mathrm{Det} M_i$, and they have negative real part if and only if

$$\mathrm{Re}(T_i \mp \sqrt{T_i^2 - D_i}) < 0.$$

Note that

$$\begin{array}{l} T_2 = \mathrm{Re}(a_1) + \mathrm{Re}(b_1) + (c_1 + \xi^2 \overline{c}_1) + (d_1 + \xi \overline{d}_1), \\ T_3 = \mathrm{Re}(a) + \mathrm{Re}(b) + (\overline{c} + \xi c) + (\overline{d} + \xi^2 d), \\ D_2 = (a_1 + b_1 + 2c_1 + 2d_1)(\overline{a}_1 + \overline{b}_1 + 2\xi^2 \overline{c}_1 + 2\xi \overline{d}_1) \\ \qquad -(a_1' + b_1' + 2c_1' + 2d_1')(\overline{a}_1' + \overline{b}_1' + 2\xi^2 \overline{c}_1' + 2\xi \overline{d}_1'), \\ D_3 = (a_1 + b_1 + 2\xi c_1 + 2\xi^2 d_1)(\overline{a}_1 + \overline{b}_1 + 2\overline{c}_1 + 2\overline{d}_1) \\ \qquad -(a_1' + b_1' + 2\xi c_1' + 2\xi^2 d_1')(\overline{a}_1' + \overline{b}_1' + 2\overline{c}_1' + 2\overline{d_1'}), \end{array}$$

and so $T_3 = \overline{T}_2$ and $D_3 = \overline{D}_2$. The eigenvalues of M_3 are the conjugates of the eigenvalues of the matrix M_2. As

$$\begin{array}{l} T_2 = (b_r + c_r + 2d_r - i3d_I)|z|^2 + \cdots \\ D_2 = 6[(b + c)\overline{d} - |d|^2]|z|^4 + \cdots, \end{array}$$

it follows the expressions in the table 8. $\square$

6 More periodic solutions

In the previous section we considered the possible branches of periodic solutions with maximal isotropy that could generically bifurcate for the system (9). We wish now to look for possible bifurcation to branches of periodic solutions with submaximal isotropy.

As was stated, when f is supposed to commute also with $\mathbf{S}^1$, then the periodic solutions of $\dot{z} = f(z, \lambda)$ are in one-to-one correspondence with the zeros of $\dot{z} = g(z, \lambda, \tau)$ where $g = f - (1 + \tau)iz$. Also the stability of these zeros gives the stability of the corresponding solutions. As before we consider the third order truncation of f commuting with $\Gamma \times \mathbf{S}^1$. However, we point out that for the branches of periodic solutions with submaximal isotropy that are found here, we can no longer guarantee that they exist for (9) if f commutes only with Γ (even with the third order Taylor series commuting with $\mathbf{S}^1$). These solution branches are guaranteed only for the third order truncation, with which we work from now on. Throughout, consider the truncation of f as in (10) of degree three and the respective reduced vector field $g = f - (1 + \tau)iz$ of the same degree. Thus let $g = (g_1, g_2, g_3, g_4, g_5, g_6)$ with

$$g_1(z, \lambda, \tau) = [\nu + a|z|^2 + b|z_1|^2 + c|z_2|^2]z_1 + d\overline{z}_2(z_3 z_4 + z_5 z_6),$$

where $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in V^3$, the coefficients a, b, c, d are complex and depend smoothly on λ. Also, $\nu = \mu(\lambda) - (1+\tau)i$, where $\mu(0) = i$ and λ is scaled such that $\mathrm{Re}(\mu) = \lambda + o(\lambda^2)$). Thus, as before the trivial solution is stable for λ negative and unstable for λ positive (near zero), and the nondegeneracy condition of the equivariant Hopf theorem is satisfied.

Since g is $\Gamma \times \mathbf{S}^1$ equivariant, each fixed-point subspace is invariant under the dynamics, i.e., $g(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma)$, if Σ is an isotropy subgroup of $\Gamma \times \mathbf{S}^1$. Recall tables 2 and 4, and figure 2 for the isotropy lattice of $\Gamma \times \mathbf{S}^1$.

We analyse symmetric solutions by analysing the zeros of the vector field g restricted to each fixed-point subspace, that is, we study

$$\dot{v} = g\,|_{\mathrm{Fix}(\Sigma)}\,(v), \qquad v \in \mathrm{Fix}(\Sigma).$$

$\mathrm{Fix}(\Sigma_i)$, $i = 1, \ldots, 8$.

As we saw before, for each Σ_i for $i = 1, \ldots, 8$, the equivariant Hopf theorem guarantees the existence of a branch of small-amplitude periodic solutions with symmetry Σ_i bifurcating from the trivial solution at $\lambda = 0$. For each two-dimensional $\mathrm{Fix}(\Sigma_i)$, the dynamics of $\dot{z}_1 = g\,|_{\mathrm{Fix}(\Sigma_i)}\,(z_1)$ are governed by

$$\dot{z}_1 = \nu z_1 + A|z_1|^2 z_1, \quad z_1 \in \mathbf{C}. \tag{13}$$

Note that the assumption of Birkhoff normal form implies that we can apply the standard Hopf theorem to the problem restricted to the two-dimensional fixed-point subspace (the vector field restricetd to the fixed-point subspace commutes with $\mathbf{S}^1$). Thus as we are assuming that $\mathrm{Re}(\mu^{'}(0)) > 0$, the trivial solution is stable for λ negative and the stability of the periodic orbit in the fixed-point subspace is determined by the real part A_r of A. If $A_r < 0$, then the Hopf bifurcation is supercritical and the solution is stable to small perturbations within $\mathrm{Fix}(\Sigma_i)$. If $A_r > 0$, then the Hopf bifurcation is subcritical and the solution is unstable. See table 9.

Consider now the dynamics of $\dot{z} = g(z, \lambda)$ restricted to the four-dimensional fixed-point subspaces Δ_i (recall table 4).

$\mathrm{Fix}(\Delta_5)$

When we restrict g to $\mathrm{Fix}(\Delta_5)$ we obtain the system:

$$\begin{aligned} \dot{z}_1 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_1|^2]z_1 + C\overline{z}_1 z_2^2, \\ \dot{z}_2 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_2|^2]z_2 + C\overline{z}_2 z_1^2, \end{aligned} \tag{14}$$

where $A = 2a$, $B = b+c$ and $C = d$. This is the normal form for the generic Hopf bifurcation problem with symmetry $\mathbf{D}_4$ studied in [13, 11, 29]. The nontrivial solutions in the space $\mathrm{Fix}(\Delta_5)$ with maximal isotropy are: the (SR) solutions with symmetry Σ_2 corresponding to the zeros of (14) of type $z_2 = 0$, the (SS) solutions with symmetry Σ_4 corresponding to zeros satisfying $z_1 = z_2$ and the (AR) solutions with symmetry Σ_5, where the zeros satisfy $z_2 = iz_1$. Their

Symmetry	Name (abbreviation)	A_r (stability)
Σ_1	Travelling rolls (TR)	$a_r + b_r$
Σ_2	Standing rolls (SR)	$2a_r + b_r + c_r$
Σ_3	Travelling squares (TS)	$2a_r + b_r$
Σ_4	Standing squares (SS)	$4a_r + b_r + c_r + d_r$
Σ_5	Alternating rolls (AR)	$4a_r + b_r + c_r - d_r$
Σ_6	Travelling cubes (TC)	$3a_r + b_r$
Σ_7	Standing cubes (SC)	$6a_r + b_r + c_r + 2d_r$
Σ_8	Alternating cubes (AC)	$6a_r + b_r + c_r - d_r$

Table 9: Stability of the periodic solution of the Hopf equation for each of the **C**-axial groups of $(\mathbf{O}(2) \wr \mathbf{S}_3) \times \mathbf{S}^1$.

Solution	Eigenvalues	Expressions for stability from the full analysis
Σ_2 (SR) $(z_1, z_2) = (z_1, 0)$	$2(A_r + B_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -2B_r\lvert z_1\rvert^2$ $\mu_1\mu_2 = (\lvert B\rvert^2 - \lvert C\rvert^2)\lvert z_1\rvert^4$	$2a_r + b_r + c_r$ $-b_r - c_r$ $-(\lvert b+c\rvert^2 - \lvert d\rvert^2)$
Σ_4 (SS) $(z_1, z_2) = (z_1, z_1)$	$2(2A_r + B_r + C_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = 2(B_r - 3C_r)\lvert z_1\rvert^2$ $\mu_1\mu_2 = 8[\lvert C\rvert^2 - \mathrm{Re}(B\overline{C})]\lvert z_1\rvert^4$	$4a_r + b_r + c_r + d_r$ $b_r + c_r - 3d_r$ $-[\lvert d\rvert^2 - \mathrm{Re}((b+c)\overline{d})]$
Σ_5 (AR) $(z_1, z_2) = (z_1, iz_1)$	$2(2A_r + B_r - C_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = 2(B_r + 3C_r)\lvert z_1\rvert^2$ $\mu_1\mu_2 = 8[\lvert C\rvert^2 + \mathrm{Re}(B\overline{C})]\lvert z_1\rvert^4$	$4a_r + b_r + c_r - d_r$ $b_r + c_r + 3d_r$ $-[\lvert d\rvert^2 + \mathrm{Re}((b+c)\overline{d})]$

Table 10: Stability of the periodic solutions with maximal isotropy in $\mathbf{D}_4$-Hopf bifurcation.

stability properties in this subspace are summarized in table 10. Note that $B_r = 0$ is a degenerate condition only when $|B|^2 > |C|^2$ (in this case the eigenvalues $\mu_1,\ \mu_2$ are conjugate purely imaginary numbers). Similar, the conditions $B_r = \pm 3C_r$ are degenerate when $|C|^2 \mp \mathrm{Re}(B\overline{C}) > 0$. In table 10 we also show the corresponding conditions on the coefficients of f that were obtained in the full stability analysis in the previous section and that determine the stability of these three solution types in the space $\mathrm{Fix}(\Delta_5)$. As (14) is equivariant under the transformation (parameter symmetry [29])

$$(z_1, z_2; C) \to (z_1, iz_2; -C), \tag{15}$$

the stability conditions for the (AR) solution are obtained from those for the (SS) solution by letting $C \to -C$ (see table 10). In each case, the first two eigenvalues are in the radial and phase directions, respectively, and $\mu_1,\ \mu_2$ are in directions transverse to the plane containing the periodic orbit.

By [29], in addition to these periodic solutions, there can be a fourth branch of periodic solutions to (14) with $|z_1| \neq |z_2|$ and $z_1 z_2 \neq 0$. This solution branch exists if

$$|\mathrm{Re}((b+c)\overline{d})| < |d|^2 < |b+c|^2,$$

and the solutions are generically unstable.

In figure 7 we reproduce the possible bifurcation diagrams obtained by [13] for this problem for the branches of solutions with maximal isotropy, assuming that the expressions of table 10 are nonzero. The bifurcation diagrams are plotted for $C_r > 0$. The parameter symmetry (15) allows to infer the bifurcation diagrams for $C_r < 0$ (the same diagrams with (SS) and (AR) labels interchanged). There are regions of the parameter space (A, B, C) (and so of (a, b, c, d)) where a stable supercritical branch and one subcritical branch bifurcate from the origin. If two solutions branches bifurcate subcritically, then all the three solutions are unstable. Also, there are regions where all of the three solutions branches are supercritical and two of them are stable. However, it is not possible to have the three branches stable. Another possible bifurcation diagram arises when all branches are supercritical and unstable. Note that the stability of the Σ_4 and Σ_5 branches implies the instabiltiy of the Σ_2 branch: if $b_r + c_r + 3d_r$ and $b_r + c_r - 3d_r$ are both negative then $b_r + c_r$ is also negative.

$\mathrm{Fix}(\Delta_i)$, $i = 1, 2, 3, 4, 6, 7$.

If we restrict the initial vector field f to each $\mathrm{Fix}(\Delta_i)$ for $i = 1, 2, 3, 4, 6, 7$, then we obtain the normal form for the generic Hopf bifurcation problem with $\mathbf{O}(2)$ symmetry (studied by [13]):

$$\begin{aligned} \dot{z}_1 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_1|^2]z_1, \\ \dot{z}_2 &= [\nu + A(|z_1|^2 + |z_2|^2) + B|z_2|^2]z_2. \end{aligned} \tag{16}$$

See table 11 with the coefficients A and B in terms of the coefficients of f. The two nontrivial periodic solutions of (16) satisfy the conditions $z_2 = 0$ and $z_1 = z_2$, the first ones we call by TW (travelling waves) and the second ones SW (standing waves).

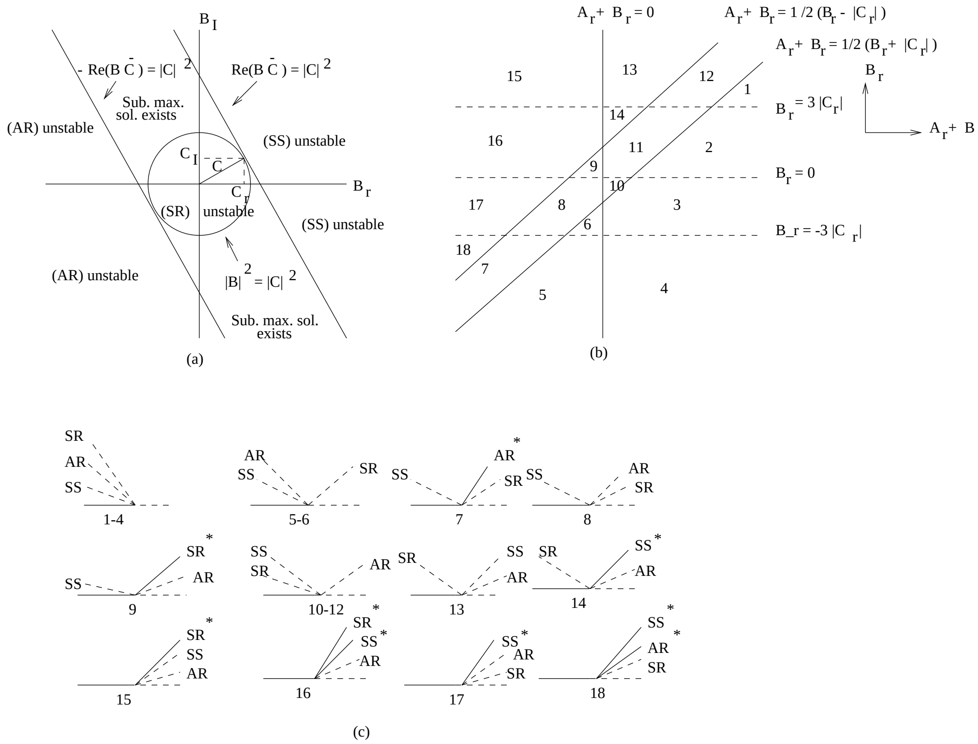


Figure 7: Bifurcation diagram for the nondegenerate Hopf bifurcation with $\mathbf{D}_4$ symmetry. (a) The complex coefficient C is plotted in the complex B plane. (c) Broken bifurcation curves indicate unstable solutions. An asterisk on solution indicates that it is possible the solution to be stable and it depends on where B lies in (a). The diagrams are plotted for $C_r > 0$.

Δ_i	$\mathrm{Fix}(\Delta_i)$	TW	SW	A	B
Δ_1	$\{(z_1, z_2, 0, 0, 0, 0)\}$	Σ_1 (TR)	Σ_2 (SR)	$a+c$	$b-c$
Δ_2	$\{(z_1, z_2, z_1, z_2, 0, 0)\}$	Σ_3 (TS)	Σ_4 (SS)	$2a+c+d$	$b-c-d$
Δ_3	$\{(z_1, z_2, iz_1, iz_2, 0, 0)\}$	Σ_3 (TS)	Σ_5 (AR)	$2a+c-d$	$b-c+d$
Δ_4	$\{(z_1, 0, z_2, 0, 0, 0)\}$	Σ_1 (TR)	Σ_3 (TS)	a	b
Δ_6	$\{(z_1, z_2, z_1, z_2, z_1, z_2)\}$	Σ_6 (TC)	Σ_7 (SC)	$3a+c+2d$	$b-c-2d$
Δ_7	$\{(z_1, z_2, \xi z_1, \xi z_2, \xi^2 z_1, \xi^2 z_2)\}$	Σ_6 (TC)	Σ_8 (AC)	$3a+c-d$	$b-c+d$

Table 11: Correspondence between solutions in $\mathbf{O}(2)$ Hopf bifurcation and solutions of $\dot{z} = f(z, \lambda)$ restricted to the appropriate fixed-point subspace.

Note that for the space $\mathrm{Fix}(\Delta_3)$, the TW solution has symmetry a conjugate subgroup of Σ_3 since $(z_1, 0, iz_1, 0, 0, 0)$ and $(z_1, 0, z_1, 0, 0, 0)$ are in the same orbit by the group $\Gamma \times \mathbf{S}^1$. The same happens for the TW solution contained in $\mathrm{Fix}(\Delta_7)$ with isotropy conjugate to Σ_6. We recall that solutions in the same orbit by the group $\Gamma \times \mathbf{S}^1$ have the same stability. The stability properties of the solutions in these subspaces are given in table 12. As (16) is a special case of (14) (take $C = 0$), this table is just obtained from table 10 with $C = 0$. We also show the conditions on the coefficients of g obtained in the full stability analysis for the stability in each of the subspaces $\mathrm{Fix}(\Delta_i)$ for $i = 1, 2, 3, 4, 6, 7$.

Note that the SW and TW solutions of table 12 cannot be both stable. In fact, one of the two branches of solutions is stable only if both branches of the bifurcating solutions are supercritical, in which case, generically, precisely one branch is stable. Thus, for example, the Σ_3 solution cannot be stable if any of the Σ_1, Σ_4 or Σ_5 solutions is stable. In figure 8 we reproduce the bifurcation diagram for the nondegenerate Hopf bifurcation with $\mathbf{O}(2)$ symmetry [13].

$\mathrm{Fix}(\Delta_8)$

Proposition 6.1 *Consider the equations for f restricted to* $\mathrm{Fix}(\Delta_8)$*. Generically, there are only branches of periodic solutions with symmetry Σ_1, Σ_3 and Σ_6. Table 13 contains the list of the eigenvalues depending on the coefficients of f that generically determine the stability of these solutions in the space* $\mathrm{Fix}(\Delta_8)$.

Proof Consider g restricted to $\mathrm{Fix}(\Delta_8)$. The equations are governed by

$$\begin{aligned} \dot{z}_1 &= [\nu + a(2|z_1|^2 + |z_2|^2) + b|z_1|^2]z_1, \\ \dot{z}_2 &= [\nu + a(2|z_1|^2 + |z_2|^2) + b|z_2|^2]z_2, \end{aligned} \tag{17}$$

Space solution	Eigenvalues	Solution	Eigenvalues
(TW) $(z_1, z_2) = (z_1, 0)$	$2(A_r + B_r)\|z_1\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -2B_r\|z_1\|^2,$ $\mu_1\mu_2 = \|B\|^2\|z_1\|^4;$	(SW) $(z_1, z_2) = (z_1, z_1)$	$2(2A_r + B_r)\|z_1\|^2,\ 0;$ $2B_r\|z_1\|^2,\ 0;$
Fix(Δ_1) $\Sigma_1\ (TR)$	$a_r + b_r$ $-b_r + c_r$	$\Sigma_2\ (SR)$	$2a_r + b_r + c_r$ $b_r - c_r$
Fix(Δ_2) $\Sigma_3\ (TS)$	$2a_r + b_r$ $-b_r + c_r + d_r$	$\Sigma_4\ (SS)$	$4a_r + b_r + c_r + d_r$ $b_r - c_r - d_r$
Fix(Δ_3) $\Sigma_3\ (TS)$	$2a_r + b_r$ $-b_r + c_r - d_r$	$\Sigma_5\ (AR)$	$4a_r + b_r + c_r - d_r$ $b_r - c_r + d_r$
Fix(Δ_4) $\Sigma_1\ (TR)$	$a_r + b_r$ $-b_r$	$\Sigma_3\ (TS)$	$2a_r + b_r$ b_r
Fix(Δ_6) $\Sigma_6\ (TC)$	$3a_r + b_r$ $-b_r + c_r + 2d_r$	$\Sigma_7\ (SC)$	$6a_r + b_r + c_r + 2d_r$ $b_r - c_r - 2d_r$
Fix(Δ_7) $\Sigma_6\ (TC)$	$3a_r + b_r$ $-b_r + c_r - d_r$	$\Sigma_8\ (AC)$	$6a_r + b_r + c_r - d_r$ $b_r - c_r + d_r$

Table 12: Stability of the periodic solutions in $\mathbf{O}(2)$ Hopf bifurcation.

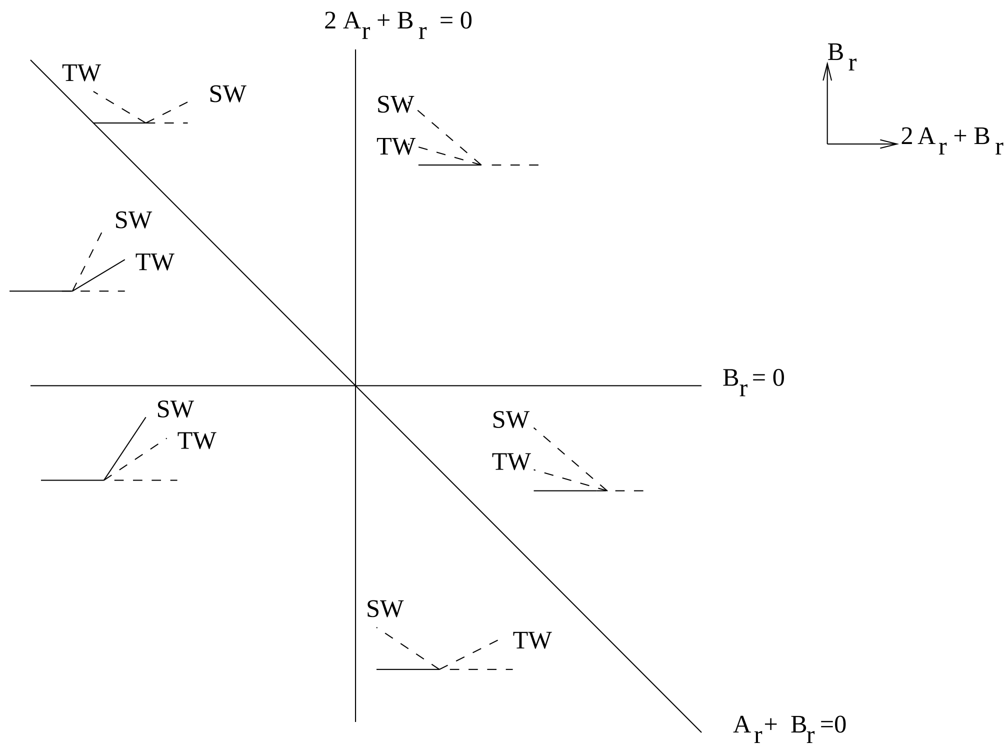


Figure 8: Bifurcation diagram for the nondegenerate Hopf bifurcation with $\mathbf{O}(2)$ symmetry. Broken bifurcation curves indicate unstable solutions and unbroken curves indicate stable solutions.

for $(z_1, z_2) \in \mathbf{C}^2$. We have by the equivariant Hopf theorem periodic solutions with symmetry Σ_1, Σ_3 and Σ_6. We show now that these are the only periodic solutions. Let $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ in (17). Since these equations commute with $\mathbf{SO}(2) \times \mathbf{S}^1$, the (r_1, r_2)-equations decouple from the phases (ϕ_1, ϕ_2). We obtain the following equations:

$$\begin{aligned} \dot{r}_1 &= [\lambda + (2a_r + b_r)r_1^2 + a_r r_2^2] r_1, \\ \dot{r}_2 &= [\lambda + (a_r + b_r)r_2^2 + 2a_r r_1^2] r_2. \end{aligned} \tag{18}$$

Here we are taking $\mathrm{Re}(\nu) = \lambda$. Note that we are studying bifurcations near $\lambda = 0$, and so there is no loss of generality in this assumption.

The equilibria of this new system correspond to the zeros of (17) (and so to periodic solutions of the original problem). The periodic solutions with symmetry Σ_1 correspond to the fixed-points of (18) of the type $(r_1, r_2) = (0, r_2)$, the (TS) solutions to the fixed-points of the type $(r_1, r_2) = (r_1, 0)$ and the (TC) solutions to the zeros of the type $(r_1, r_2) = (r_1, r_1)$. If $b_r \neq 0$, then there are no fixed-points besides the corresponding to these maximal solutions. We show in

Solution	Eigenvalues	Solution	Eigenvalues
Σ_1 (TR) $(z_1, z_2) = (0, z_2)$	$2(a_r + b_r)\lvert z_2\rvert^2,\ 0;$ $-2b_r\lvert z_2\rvert^2,\ 0;$	Σ_3 (TS) $(z_1, z_2) = (z_1, 0)$	$2(2a_r + b_r)\lvert z_1\rvert^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1 + \mu_2 = -2b_r\lvert z_1\rvert^2,$ $\mu_1\mu_2 = \lvert b\rvert^2\lvert z_1\rvert^4;$
Σ_6 (TC) $(z_1, z_2) = (z_1, z_1)$	$2(3a_r + b_r)\lvert z_1\rvert^2,\ 0;$ $2b_r\lvert z_1\rvert^2, 0.$		

Table 13: Stability of the periodic solutions of $\dot{z} = f(z, \lambda)$ restricted to the space $\mathrm{Fix}(\Delta_8)$.

table 13 the stability of these solutions in the space $\mathrm{Fix}(\Delta_8)$. □

$\mathrm{Fix}(\Delta_{10})$

Proposition 6.2 *Consider the equations for f restricted to* $\mathrm{Fix}(\Delta_{10})$. *Generically, there are only branches of periodic solutions with symmetry Σ_1 and Σ_5. In table 14 we list the eigenvalues depending on the coefficients of f that determine (in the nondegenerate case) the stability of these solutions in the space* $\mathrm{Fix}(\Delta_{10})$.

Proof Consider g restricted to $\mathrm{Fix}(\Delta_{10})$. The equations are governed by

$$\begin{aligned} \dot{z}_1 &= [\nu + a(4|z_1|^2 + |z_2|^2) + (b + c - d)|z_1|^2]z_1, \\ \dot{z}_2 &= [\nu + a(4|z_1|^2 + |z_2|^2) + b|z_2|^2]z_2, \end{aligned} \tag{19}$$

for $(z_1, z_2) \in \mathbf{C}^2$. We note that by the equivariant Hopf theorem there are periodic solutions with symmetry Σ_5 in the space $z_2 = 0$ and with conjugate symmetry to Σ_1 in the space $z_1 = 0$.

We wish to know now if there are any zeros of (19) with $z_1 z_2 \neq 0$. Let $(z_1, z_2) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2})$ in (19) and note that these equations commute with $\mathbf{SO}(2) \times \mathbf{S}^1$. Zeros of (19) with $z_1 z_2 \neq 0$ satisfy

$$\begin{aligned} &\nu + (4a + b + c - d)r_1^2 + ar_2^2 = 0, \\ &\nu + (a + b)r_2^2 + 4ar_1^2 = 0, \end{aligned} \tag{20}$$

with $r_1 r_2 \neq 0$, and so

$$(4a + b + c - d)r_1^2 + ar_2^2 = (a + b)r_2^2 + 4ar_1^2,$$

i.e.,

$$(b + c - d)r_1^2 = br_2^2. \tag{21}$$

Solution	Eigenvalues
Σ_5 (AR) $(z_1,z_2)=(z_1,0)$	$2(4a_r+b_r+c_r-d_r)\|z_1\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1+\mu_2=2(-b_r-c_r+d_r)\|z_1\|^2,$ $\mu_1\mu_2=\|-b-c+d\|^2\|z_1\|^4;$
Σ_1 (TR) $(z_1,z_2)=(0,z_2)$	$2(a_r+b_r)\|z_2\|^2,\ 0;$ $-2b_r\|z_2\|^2,\ 0;$

Table 14: Stability of the periodic solutions of $\dot{z}=f(z,\lambda)$ restricted to $\mathrm{Fix}(\Delta_{10})$.

If $r_1=r_2$, then $(c-d)r_1^2=0$ and so generically there are no solutions for this case. If $r_1\neq r_2$ (and $r_1r_2\neq 0$), then the real and imaginary parts of (21) give two equations for $(r_1/r_2)^2$, and again (generically) there are no solutions for this case.

We show in table 14 the stability of the maximal solutions in the space $\mathrm{Fix}(\Delta_{10})$ depending on the coefficients of f. □

$\mathrm{Fix}(\Delta_9)$

Proposition 6.3 *Consider the equations for f restricted to* $\mathrm{Fix}(\Delta_9)$. *Then there are branches of periodic solutions with symmetry Σ_4, Σ_2 and Σ_7. We list in table 15 the eigenvalues depending on the coefficients of f that generically determine the stability of these solutions in the space* $\mathrm{Fix}(\Delta_9)$.

Generically, there can be branches of periodic solutions with submaximal isotropy. These correspond to zeros of g restricted to $\mathrm{Fix}(\Delta_9)$ *of the type (z_1,z_2) with $|z_1|\neq|z_2|$ and $z_1z_2\neq 0$. The number of the branches depends on the real zeros in $]0,\pi[$ of the following equation in $\phi=arg(z_2\overline{z}_1)$:*

$$Im(d\overline{f})(1-\cos(2\phi))+3Re(d\overline{f})\sin(2\phi)+|d|^2\sin(2\phi)(1-4\cos(2\phi))=0 \quad (22)$$

with $f=b+c$, satisfying the conditions

$$f_r+d_r-2Re(\overline{d}e^{i2\phi})\neq 0,$$
$$f_I+d_I+2Im(\overline{d}e^{i2\phi})\neq 0$$

and

$$\frac{f_r-Re(de^{i2\phi})}{f_r+d_r-2Re(\overline{d}e^{i2\phi})}>0,\neq 1.$$

Proof The equations $g\mid_{\mathrm{Fix}(\Delta_9)}=0$ are:

$$\begin{array}{l}\left[\nu+2a(2|z_1|^2+|z_2|^2)+(b+c+d)|z_1|^2\right]z_1+d\overline{z}_1z_2^2=0,\\ \left[\nu+2a(2|z_1|^2+|z_2|^2)+(b+c)|z_2|^2\right]z_2+2d\overline{z}_2z_1^2=0,\end{array} \quad (23)$$

Solution	Eigenvalues
Σ_2 (SR) $(z_1,z_2)=(0,z_2)$	$2(2a_r+b_r+c_r)\|z_2\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1+\mu_2=-2(b_r+c_r)\|z_2\|^2,$ $\mu_1\mu_2=(\|b+c\|^2-\|d\|^2)\|z_2\|^4;$
Σ_4 (SS) $(z_1,z_2)=(z_1,0)$	$2(4a_r+b_r+c_r+d_r)\|z_1\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1+\mu_2=-(b_r+c_r+d_r)\|z_1\|^2,$ $\mu_1\mu_2=[\|b+c\|^2-3\|d\|^2+2\mathrm{Re}((b+c)\overline{d})]\|z_1\|^4;$
Σ_7 (SC) $(z_1,z_2)=(z_1,z_1)$	$2(6a_r+b_r+c_r+2d_r)\|z_1\|^2,\ 0;$ $\mu_1,\ \mu_2:$ $\mu_1+\mu_2=2(b_r+c_r-4d_r)\|z_1\|^2,$ $\mu_1\mu_2=12[\|d\|^2-\mathrm{Re}((b+c)\overline{d})]\|z_1\|^4;$

Table 15: Stability of the periodic solutions of $\dot{z}=f(z,\lambda)$ restricted to $\mathrm{Fix}(\Delta_9)$.

with $(z_1,z_2)\in\mathbf{C}^2$. The existence of periodic solutions with symmetry Σ_2, Σ_4 and Σ_7 is guaranteed by the equivariant Hopf theorem. These solutions correspond to the zeros of this system of the type $z_1=0$ (and $z_2\neq 0$), $z_2=0$ (and $z_1\neq 0$) and $z_1=z_2\neq 0$ respectively. The stability of these solutions in the space $\mathrm{Fix}(\Delta_9)$ is given in table 15.

In order to find submaximal solutions for the equations for f restricted to $\mathrm{Fix}(\Delta_9)$, we have to look for zeros (z_1,z_2) of the equations (23) such that $z_1\neq z_2$ and $z_1z_2\neq 0$. Let $(z_1,z_2)=(r_1e^{i\phi_1},r_2e^{i\phi_2})$ in (23) and denote by $\phi=\phi_2-\phi_1=\arg(z_2\overline{z_1})$. Zeros of (23) (with $z_1z_2\neq 0$) satisfy

$$\begin{array}{l}\nu+(4a+b+c+d)r_1^2+2ar_2^2+de^{i2\phi}r_2^2=0,\\ \nu+(2a+b+c)r_2^2+4ar_1^2+2de^{-i2\phi}r_1^2=0\end{array}$$

and so, if we denote by $f=b+c$, we get

$$f(r_1^2-r_2^2)+dr_1^2+d(e^{i2\phi}r_2^2-2e^{-i2\phi}r_1^2)=0. \tag{24}$$

The real and imaginary parts of (24) correspond to the equations

$$\begin{array}{l}r_1^2(f_r+d_r-2d_r\cos(2\phi)-2d_I\sin(2\phi))+r_2^2(-f_r+d_r\cos(2\phi)-d_I\sin(2\phi))=0,\\ r_1^2(f_I+d_I-2d_I\cos(2\phi)+2d_r\sin(2\phi))+r_2^2(-f_I+d_I\cos(2\phi)+d_r\sin(2\phi))=0.\end{array}$$

These equations may be solved for $(r_1/r_2)^2$ obtaining

$$\left(\frac{r_1}{r_2}\right)^2=\frac{f_r-d_r\cos(2\phi)+d_I\sin(2\phi)}{f_r+d_r-2d_r\cos(2\phi)-2d_I\sin(2\phi)}, \tag{25}$$

provided

$$f_r + d_r - 2\mathrm{Re}(\overline{d}e^{i2\phi}) \neq 0$$

and one equation for $\cos(2\phi)$:

$$\mathrm{Im}(d\overline{f})(1-\cos(2\phi)) + 3\mathrm{Re}(d\overline{f})\sin(2\phi) + |d|^2\sin(2\phi)(1-4\cos(2\phi)) = 0, \quad (26)$$

provided

$$f_I + d_I + 2\mathrm{Im}(\overline{d}e^{i2\phi}) \neq 0.$$

Note that the expression for $(r_1/r_2)^2$ in (25) has to be positive and different from 1. This last because if $r_1^2 = r_2^2$ in (24), then ϕ satisfies $\cos(2\phi) = 1$ and $\sin(2\phi) = 0$ and we obtain a maximal solution. □

We can find regions of the parameter space (b, c, d) where there can co-exist two distinct branches of submaximal solutions, one branch and no branch of submaximal solutions for the equations for f restricted to $\mathrm{Fix}(\Delta_9)$. For example, fixing $b = c = 0.5 + i$, $d_r = 1$ and varying d_I we can find these three cases. We did not find cases where there were three branches of periodic solutions with submaximal isotropy (in the space $\mathrm{Fix}(\Delta_9)$). Note that solving (26) for $\sin(2\phi)$ and substituting in $\sin^2(2\phi) + \cos^2(2\phi) = 1$, we get an equation in ϕ of the type $\cos^3(2\phi) + A\cos^2(2\phi) + B\cos(2\phi) + C = 0$ with real coefficients A, B and C.

Remark 6.4 *With the conditions of proposition 6.3, for parameter values for which there are branches of periodic solutions for f restricted to $\mathrm{Fix}(\Delta_9)$ with submaximal isotropy, the branching equation is*

$$\nu = -(4a + b + c + d)r_1^2 - 2ar_2^2 - de^{i2\phi}r_2^2,$$

where $r_1 = |z_1|$ and $r_2 = |z_2|$ (and $z = (z_1, z_1, z_1, z_1, z_2, z_2)$ in $\mathrm{Fix}(\Delta_9)$) satisfy

$$\left(\frac{r_1}{r_2}\right)^2 = \frac{f_r - d_r\cos(2\phi) + d_I\sin(2\phi)}{f_r + d_r - 2d_r\cos(2\phi) - 2d_I\sin(2\phi)},$$

for $\phi = arg(z_2\overline{z}_1)$ solution of the equation

$$Im(d\overline{f})(1-\cos(2\phi)) + 3Re(d\overline{f})\sin(2\phi) + |d|^2\sin(2\phi)(1-4\cos(2\phi)) = 0.$$

$\mathrm{Fix}(\Pi_3)$

Proposition 6.5 *Consider the equations for f restricted to the space $\mathrm{Fix}(\Pi_3)$. Then generically, only branches of periodic solutions with symmetry (conjugate to) Σ_1, Σ_3 and Σ_6 can bifurcate at the origin.*

Proof If we take coordinates $(z_1, 0, z_2, 0, z_3, 0) = (r_1e^{i\phi_1}, 0, r_2e^{i\phi_2}, 0, r_3e^{i\phi_3}, 0)$ in $\mathrm{Fix}(\Pi_3)$, then from $g\,|_{\mathrm{Fix}(\Pi_3)} = 0$, we get

$$\begin{array}{l}(\nu + ar^2 + br_1^2)r_1 = 0,\\ (\nu + ar^2 + br_2^2)r_2 = 0,\\ (\nu + ar^2 + br_3^2)r_3 = 0,\end{array} \quad (27)$$

where $r^2 = r_1^2 + r_2^2 + r_3^2$. Generically, the only nontrivial zeros of (27) are the corresponding to the zeros with symmetry $\Sigma_1,\ \Sigma_3$ and Σ_6. $\square$

Fix(Π_6)

Proposition 6.6 *Consider the equations for f restricted to the space* Fix(Π_6)*. Then generically, only branches of periodic solutions with symmetry (conjugate to) $\Sigma_1,\ \Sigma_3,\ \Sigma_5$ and Σ_6 can bifurcate at the origin.*

Proof Take $(z_1, z_2, iz_1, iz_2, z_3, 0) = (r_1e^{i\phi_1}, r_2e^{i\phi_2}, ir_1e^{i\phi_1}, ir_2e^{i\phi_2}, r_3e^{i\phi_3}, 0)$ in Fix(Π_6). From $g\ |_{\mathrm{Fix}(\Pi_6)} = 0$, we obtain

$$\begin{array}{l} \left[\nu + ar^2 + br_1^2 + (c-d)r_2^2\right] r_1 = 0, \\ \left[\nu + ar^2 + br_2^2 + (c-d)r_1^2\right] r_2 = 0, \\ \left[\nu + ar^2 + br_3^2\right] r_3 = 0, \end{array} \tag{28}$$

where $r^2 = 2r_1^2 + 2r_2^2 + r_3^2$. If some r_i is zero, then we obtain zeros of $g\ |_{\mathrm{Fix}(\Pi_6)} = 0$ corresponding to periodic solutions with symmetry conjugate to $\Sigma_1,\ \Sigma_3,\ \Sigma_5$ and Σ_6. If $r_i \neq 0$, for $i = 1, 2, 3$, then from (28) we get

$$br_1^2 + (c-d)r_2^2 = br_2^2 + (c-d)r_1^2 = br_3^2.$$

From the first equality, we obtain

$$(b - c + d)(r_1^2 - r_2^2) = 0$$

and so, generically (i.e., assuming $b - c + d \neq 0$) we have that $r_1^2 = r_2^2$. From the second equality, we get

$$(b + c - d)r_1^2 = br_3^2$$

and so two equations (real and imaginary) for $(r_1/r_3)^2$. $\square$

Fix(Λ_1)

When we restrict f to the space Fix(Λ_1), the restricted ODEs are equivariant under the group $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$. The dynamics of a vector field commuting with $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ on V^2 was studied by Silber and Knobloch in [27]. This group is the wreath product group $\mathbf{O}(2) \wr \mathbf{S}_2$. It was proved that generically, the only branches of periodic solutions that can bifurcate at the origin are those corresponding to the solutions with symmetry (conjugate to) $\Sigma_1, \ldots, \Sigma_5$ or with submaximal isotropy obtained by Swift [29] (in Fix(Δ_5)). The stability of these solutions in the space Fix(Λ_1) is determined by the derivative (dg) calculated at the corresponding zero of g, and restricted to the fixed-point subspace of Λ_1. Table 16 contains the conditions on the coefficients of f that determine the stability

Symmetry of the solution	Eigenvalues
Σ_1 (TR)	$a_r + b_r$ $-b_r + c_r$ $-b_r$
Σ_2 (SR)	$2a_r + b_r + c_r$ $b_r - c_r$ $-b_r - c_r$ $-(\lvert b+c\rvert^2 - \lvert d\rvert^2)$
Σ_3 (TS)	$2a_r + b_r$ b_r $-b_r + c_r + d_r$ $-b_r + c_r - d_r$
Σ_4 (SS)	$4a_r + b_r + c_r + d_r$ $b_r + c_r - 3d_r$ $-[\lvert d\rvert^2 - \mathrm{Re}((b+c)\overline{d})]$ $b_r - c_r - d_r$
Σ_5 (AR)	$4a_r + b_r + c_r - d_r$ $b_r + c_r + 3d_r$ $-[\lvert d\rvert^2 + \mathrm{Re}((b+c)\overline{d})]$ $b_r - c_r + d_r$

Table 16: Stability of the periodic solutions of the square lattice problem.

(this table is contained in the table 8 that lists the stability conditions in the all space V^3). These conditions agree with the results of [27].

We note that the stability of the solutions with symmetry Σ_1 and Σ_2 is the same in the space $\mathrm{Fix}(\Lambda_1)$ as in the all space V^3. Note that $\mathbf{S}_1 \times \mathbf{S}_2$ is contained in the groups Σ_1 and Σ_2: this justifies the repetition in the case of the derivative for the (TR) solution of the matrix

$$\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$$

and in the case of the (SR) solution of the matrix

$$\begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

obtained in the proof of proposition 5.2. However, the solutions with symmetry Σ_3 in $\mathrm{Fix}(\Lambda_1)$ can be stable, but in the full space the branch is generically unstable: there are eigenvalues in the transverse direction to $\mathrm{Fix}(\Lambda_1)$ with real parts whose sign is determined by $-b_r$.

In $\mathrm{Fix}(\Lambda_1)$ the stability for (AR) can be inferred from the stability for (SS) letting $d \mapsto -d$ because of the parameter symmetry [29], that is, the map $g\,|_{\mathrm{Fix}(\Lambda_1)}$ is equivariant under the transformation

$$(z_1, z_2, z_3, z_4; d) \mapsto (z_1, z_2, iz_3, iz_4; -d).$$

However, in the all space, the stability for the (SS) solution is also determined by the eigenvalues such that the sum has sign determined by $-b_r - c_r - d_r$ and product is determined by $|b+c|^2 - 3|d|^2 + 2\mathrm{Re}((b+c)\overline{d})$. For the (AR) solution, similar case happens but now the extra eigenvalues are complex conjugate (or equal and real) with real part determined $-b_r - c_r + d_r$.

In [27], Silber and Knobloch also show that an unstable branch of periodic solutions with submaximal symmetry can generically bifurcate from the origin. This solution branch corresponds to the one obtained in the restricted problem to the space $\mathrm{Fix}(\Delta_5)$ which has $\mathbf{D}_4$ symmetry and that it was studied in [29].

All the possible bifurcation diagrams (when considering the expressions in table 16 nonzero) are described in [27]. They show that it is possible to have the five branches to bifurcate supercritically with two of them being stable. The Σ_4 solution can be stable even with two supercritical branches. For some parameter region all five branches bifurcate supercritically with none being stable. They also explore the possibility of a primary bifurcation to a structurally stable heteroclinic cycle. We return to this matter in the next section.

$\mathrm{Fix}(\Lambda_3)$

Theorem 6.7 *Consider the equations for f restricted to the space* $\mathrm{Fix}(\Lambda_3)$*. Then there are branches of periodic solutions with symmetry Σ_1, Σ_2, Σ_3, Σ_4, Σ_6 and Σ_7.*

Generically there can be branches of periodic solutions with submaximal symmetry and these correspond to the ones obtained in proposition 6.3.

Proof The equations $g\mid_{\mathrm{Fix}(\Lambda_3)}=0$ are:

$$\begin{array}{l}\left[\nu+a|z|^2+(a+b)|z_1|^2+(a+c+d)|z_2|^2\right]z_1+d\overline{z}_2z_3z_4=0,\\ \left[\nu+a|z|^2+(a+b)|z_2|^2+(a+c+d)|z_1|^2\right]z_2+d\overline{z}_1z_3z_4=0,\\ \left[\nu+2a|z|^2+(-a+b)|z_3|^2+(-a+c)|z_4|^2\right]z_3+2d\overline{z}_4z_1z_2=0,\\ \left[\nu+2a|z|^2+(-a+b)|z_4|^2+(-a+c)|z_3|^2\right]z_4+2d\overline{z}_3z_1z_2=0,\end{array}\qquad(29)$$

where $|z|^2=|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2$. The only cases, up to conjugacy, that we need to consider are:

$$\begin{array}{l}(1)\ z_1\neq 0\ \wedge\ z_2=z_3=z_4=0\\ (2)\ z_3\neq 0\ \wedge\ z_1=z_2=z_4=0\\ (3)\ z_1\neq 0\ \wedge z_3\neq 0\ \wedge z_2=z_4=0\\ (4)\ z_1\neq 0\ \wedge z_2\neq 0\ \wedge\ z_3=z_4=0\\ (5)\ z_3\neq 0\ \wedge z_4\neq 0\ \wedge\ z_1=z_2=0\\ (6)\ z_1z_2\overline{z}_3\overline{z}_4\neq 0.\end{array}$$

For the cases (1)-(5) we obtain the solutions with symmetry Σ_i for $i=1,2,3,4,6$ (apply proposition 6.1 and recall that f restricted to $\mathrm{Fix}(\Delta_2)$ and $\mathrm{Fix}(\Delta_1)$ is $\mathbf{O}(2)$-symmetric). Consider now the case (6). Zeros of (29) satisfy

$$G=\left(\begin{array}{l}\left[\nu+ar^2+(a+b)r_1^2+(a+c+d)r_2^2\right]r_1+dr_2r_3r_4e^{i2\psi}\\ \left[\nu+ar^2+(a+b)r_2^2+(a+c+d)r_1^2\right]r_2+dr_1r_3r_4e^{i2\psi}\\ \left[\nu+2ar^2+(-a+b)r_3^2+(-a+c)r_4^2\right]r_3+2dr_1r_2r_4e^{-i2\psi}\\ \left[\nu+2ar^2+(-a+b)r_4^2+(-a+c)r_3^2\right]r_4+2dr_1r_2r_3e^{-i2\psi}\end{array}\right)=0\quad(30)$$

where $r_i=|z_i|$ for $i=1,\ldots,4$ and $r^2=r_1^2+r_2^2+r_3^2+r_4^2$. Also, $2\psi=\arg(\overline{z}_1\overline{z}_2z_3z_4)$. Note that in this case $r_i\neq 0$ for $i=1,\ldots,4$. As

$$\begin{array}{l}r_1G_1-r_2G_2=[\nu+ar^2+(a+b)(r_1^2+r_2^2)](r_1^2-r_2^2)=0,\\ r_3G_3-r_4G_4=[\nu+2ar^2+(-a+b)(r_3^2+r_4^2)](r_3^2-r_4^2)=0,\end{array}$$

then case (6) is subdivided into four cases:

$$\begin{array}{l}(6.a)\ r_1^2=r_2^2\ \wedge\ r_3^2=r_4^2\\ (6.b)\ r_1^2=r_2^2\ \wedge\ \nu+2ar^2+(-a+b)(r_3^2+r_4^2)=0\\ (6.c)\ r_3^2=r_4^2\ \wedge\ \nu+ar^2+(a+b)(r_1^2+r_2^2)=0\\ (6.d)\ \nu+ar^2+(a+b)(r_1^2+r_2^2)=0\ \wedge\ \nu+2ar^2+(-a+b)(r_3^2+r_4^2)=0.\end{array}$$

Consider first the case $(6.a)$. If $r_1=r_2=r_3=r_4$, then from (30) we get that $\cos(2\psi)=1$ and $\sin(2\psi)=0$ and we have the solution with symmetry Σ_7. Now, if $r_1=r_2\neq r_3=r_4$ in (30), we get

$$\begin{array}{l}\nu+(4a+b+c+d)r_1^2+2ar_3^2+dr_3^2e^{i2\psi}=0,\\ \nu+(2a+b+c)r_3^2+4ar_1^2+2dr_1^2e^{-i2\psi}=0\end{array}$$

and so we have the submaximal solutions obtained in proposition 6.3. Note that, an element in $\mathrm{Fix}(\Lambda_3)$ such that $r_1=r_2\neq r_3=r_4$ and $r_1r_3\neq 0$ is conjugate to an element in $\mathrm{Fix}(\Delta_9)$. That is, the solutions obtained from the above equations have symmetry (conjugate to) Δ_9.

For the case (6.b): we can use $\nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0$ in $G_3 = 0$ of (30) and we get

$$r_3 r_4 e^{i2\psi} = \frac{2d}{b-c} r_1 r_2 \tag{31}$$

(assuming $b - c \neq 0$). We use the conditions $\nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0$ and $r_1 = r_2$ in $G_1 = 0$ of (30), and substituting $r_3 r_4 e^{i2\psi}$ as in (31), we obtain

$$\left(b + c + d + \frac{2d^2}{b-c}\right) r_1^2 = b(r_3^2 + r_4^2). \tag{32}$$

The real and imaginary parts of this equation give two equations for the ratio $r_1^2/(r_3^2 + r_4^2)$. Thus generically, there are no solutions in case (6.b).

The case (6.c) is similar. We can use $\nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0$ in $G_1 = 0$ of (30) and we get

$$r_1 r_2 e^{-i2\psi} = \frac{d}{b-c-d} r_3 r_4 \tag{33}$$

(assuming $b - c - d \neq 0$). Again, in $G_3 = 0$ of (30) we use $r_3 = r_4$ and $\nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0$. Also substituting $r_1 r_2 e^{-i2\psi}$ as in (33), we obtain

$$\left(b + c + \frac{2d^2}{b-c-d}\right) r_3^2 = b(r_1^2 + r_2^2). \tag{34}$$

The real and imaginary parts of this equation give two equations for the ratio $r_3^2/(r_1^2 + r_2^2)$ and so, generically, there are no solutions in this case.

Finally, case (6.d). Using the condition $\nu + ar^2 + (a+b)(r_1^2 + r_2^2) = 0$ in $G_1 = 0$ of (30), we get

$$(-b + c + d) r_1 r_2 + d r_3 r_4 e^{i2\psi} = 0, \tag{35}$$

and in $G_3 = 0$ the condition $\nu + 2ar^2 + (-a+b)(r_3^2 + r_4^2) = 0$, we obtain

$$2d r_1 r_2 + (c-b) r_3 r_4 e^{i2\psi} = 0. \tag{36}$$

Since, we are supposing $r_1 r_2 r_3 r_4 \neq 0$, the conditions (35) and (36) imply that

$$(b-c-d)(b-c) = 2d^2. \tag{37}$$

Thus, generically there are no solutions in case (6.d). □

In figure 9 we list some interesting bifurcation digrams. We include only the branches of solutions with maximal symmetry. Observe that it is possible to have two branches of stable solutions with two subcritical branches. Also, for some parameter values, all the branches can bifurcate supercritically, either being all unstable or having just one or two stable branches. We note that the solutions with symmetry Σ_4, Σ_5 cannot be both stable: from $b_r + c_r + 3d_r$, $-b_r -$

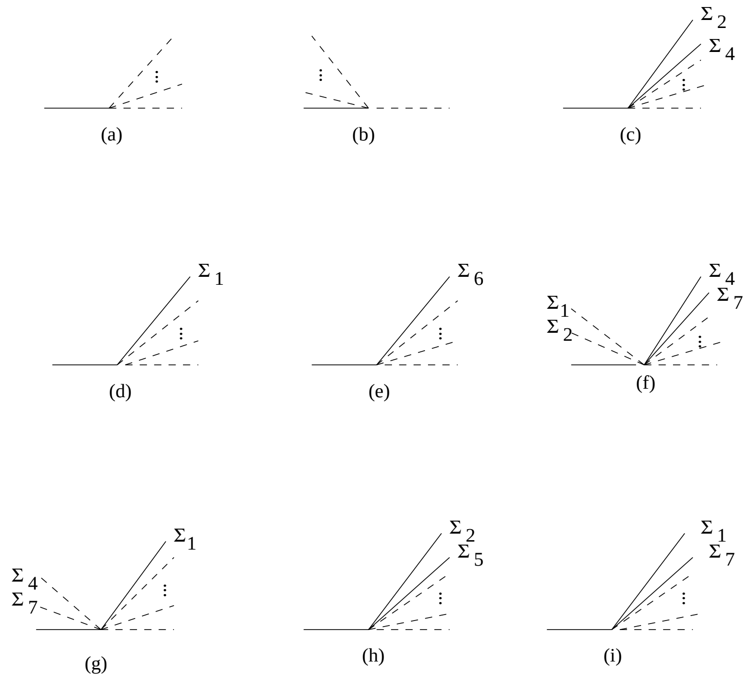


Figure 9: Possible bifurcation diagrams for the nondegenerate Hopf bifurcation with $\mathbf{O}(2)\wr\mathbf{S}_3$ symmetry. Broken (unbroken) bifurcation curves indicate unstable (stable) solutions.

$c_r + d_r < 0$ we get $d_r < 0$ and from $b_r + c_r - 3d_r,\ b_r + c_r + 3d_r < 0$ we have $b_r + c_r < 0$ and so $-b_r - c_r - d_r > 0$. This contrasts with the generic $\mathbf{D}_4$-Hopf bifurcation problem. See figure 7 where (SS) and (AR) stand for the solutions with symmetry Σ_4 and Σ_5 respectively. Similarly, it cannot happen that the first branch bifurcates subcritically with the second being stable, again as it happens for some parameter values in the Hopf problem with $\mathbf{D}_4$ symmetry.

The solutions with symmetry Σ_3 are always unstable. Also the solutions with symmetry Σ_6 and Σ_8 cannot both be stable. The same happens for the solutions with symmetry Σ_1 and Σ_6.

7 Heteroclinic cycles

In addition to the periodic solutions described in the previous section, there can exist quasiperiodic and heteroclinic cycle solutions to

$$\dot{z} = f(z, \lambda). \tag{38}$$

These solutions can exist in open regions of the coefficient space (a, b, c, d) and bifurcate directly from the trivial solution.

The existence of quasiperiodic solutions follows from the fact that the vector field f restricted to the subspace $\mathrm{Fix}(\Delta_5)$ has $\mathbf{D}_4 \times \mathbf{S}^1$ symmetry. Swift [29] proves that there exists a branch of quasiperiodic solutions to this bifurcation problem.

We have explored several possibilities for heteroclinic cycles involving periodic solutions. For that we used the symmetry of the solutions, that is, we considered some cases attending some relevant parts of the isotropy lattice of $\Gamma \times \mathbf{S}^1$. In the most of the cases, we found that generically those cycles cannot happen. As an example, consider the possibility of the cycle involving the solutions (TR), (SR) and (AR). The heteroclinic cycle would consist in three periodic solutions together with the heteroclinic orbits connecting them. The relevant part of the isotropy lattice of $\Gamma \times \mathbf{S}^1$ is:

$$\begin{array}{ccccccc} \Sigma_1 & & \Sigma_2 & & \Sigma_5 & & \Sigma_1 \\ & \nwarrow\nearrow & & \nwarrow\nearrow & & \nwarrow\nearrow & \\ & \Delta_1 & & \Delta_5 & & \Delta_{10} & \end{array}$$

Thus we are interested in the cycle $(TR) \to (SR) \to (AR) \to (TR)$. We need to show the existence of heteroclinic orbits $(TR) \to (SR)$ (in $\mathrm{Fix}(\Delta_1)$), $(SR) \to (AR)$ (in $\mathrm{Fix}(\Delta_5)$) and $(AR) \to (TR)$ (in $\mathrm{Fix}(\Delta_{10})$). We begin with the heteroclinic connections $(TR) \to (SR)$ and $(AR) \to (TR)$. Restricting the dynamics of g to the space $\mathrm{Fix}(\Delta_1)$, we obtain (16), the normal form for a generic Hopf bifurcation problem with symmetry $\mathbf{O}(2) \times \mathbf{S}^1$, where $A = a + c$ and $B = b - c$. If we use the coordinates $(z_1, z_2, z_3, z_4, z_5, z_6) = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, 0, 0, 0, 0)$ in $\mathrm{Fix}(\Delta_1)$, then the (r_1, r_2)-equations decouple from the phases (ϕ_1, ϕ_2) and are given by

$$\begin{aligned} \dot{r}_1 &= [\lambda + (a_r + b_r)r_1^2 + (a_r + c_r)r_2^2]r_1 \\ \dot{r}_2 &= [\lambda + (a_r + b_r)r_2^2 + (a_r + c_r)r_1^2]r_2. \end{aligned} \qquad (39)$$

The (TR) solutions correspond to zeros of (39) of the type $\xi_1 = (r_1, 0)$ (where $r_1 \neq 0$) and the (SR) solutions to zeros of the type $\xi_2 = (r_1, r_1)$ with $r_1 \neq 0$. See table 12 for stability. Suppose that both of the branches of these solutions bifurcate supercritically, i.e., $a_r + b_r < 0$ and $2a_r + b_r + c_r < 0$. Using [13] lemma XVII 4.1, a zero of the amplitude equations is asymptotically stable if and only if the corresponding solution is (orbitally) asymptotically stable. In this case the sign of the nonradial eigenvalue for the zeros corresponding to (TR) and (SR) is determined by $c_r - b_r$: if we assume $b_r - c_r < 0$, then a zero corresponding to a (TR) solution is a saddle and the one corresponding to (SR) is a sink. That there is a saddle-sink connection $(TR) \to (SR)$ follows easily: the equations (39) admit no equilibria other than those corresponding to (TR) and (SR) solutions in the region $r_1 > 0,\ r_2 \geq 0$; the unstable manifold $W^u(\xi_1)$ remains within $O(\sqrt{\lambda})$ from the origin, by [16] or [20] proposition 2.6; the Poincaré-Bendixson theorem (see for example [15]) implies the existence of the heteroclinic connection $\xi_1 \to \xi_2$.

The saddle-sink connection $(AR) \to (TR)$ is proved in a similar way. Provided that $b_r + c_r - d_r$ and $-b_r$ have the same sign, these are the only solutions for g restricted to $\mathrm{Fix}(\Delta_{10})$ (recall proposition 6.2). See also table 14 for stability and branching directions. Considering that both of the branches of solutions corresponding to (TR) and (AR) bifurcate supercritically, that is, $a_r + b_r < 0$

and $4a_r + b_r + c_r - d_r < 0$, if $-b_r - c_r + d_r > 0$ and $-b_r < 0$, then there is a saddle-sink connection between those solutions. We recall the conditions on the coefficients of f that we need for the saddle-sink connections $(TR) \to (SR)$ and $(AR) \to (TR)$:

$$\begin{array}{lll} a_r + b_r < 0, & 2a_r + b_r + c_r < 0, & 4a_r + b_r + c_r - d_r < 0, \\ b_r - c_r < 0, & b_r + c_r - d_r < 0, & -b_r < 0 \end{array}$$

and we note that these conditions imply that $a_r < 0$ and $b_r,\ c_r,\ d_r > 0$.

Finally, we have to prove that there is an heteroclinic connection between the (SR) solution and the (AR) solution (for parameter values for which there are the connections $(TR) \to (SR)$ and $(AR) \to (TR)$). As we saw previously, the dynamics of f restricted to the subspace $\mathrm{Fix}(\Delta_5)$ are given by (14), the normal form of a Hopf bifurcation problem with symmetry $\mathbf{D}_4 \times \mathbf{S}^1$. By [29] there are periodic solutions besides (SR), (SS) and (AR) (in our problem) provided that

$$|\mathrm{Re}((b+c)\overline{d})| < |d|^2 < |b+c|^2.$$

Swift [29] projects the dynamics on the space $\mathrm{Fix}(\Delta_5)$ onto the surface of the unit sphere, obtaining a system in polar coordinates (r, θ, ϕ) not depending on the total phase $\psi = \arg(z_1 \overline{z}_2)$. Since the corresponding $(\dot{\theta}, \dot{\phi})$-equations depend on r only through an overall multiplicative factor which can be removed by introducing a new time τ such that $\dot{\tau} = r,\ \tau(0) = 0$, Swift proves that when solving the associated spherical system on the sphere S^2, there is a straight correspondence between the fixed-points and limit cycles of this associated spherical system and the original system (14). Recall the stability for the solutions (SR), (SS), (AR) in the space $\mathrm{Fix}(\Delta_5)$ given in the previous section in table 10. Again we are considering supercritical bifurcations and the eigenvalues $\mu_1^*,\ \mu_2^*$ for the corresponding fixed-points of this reduced spherical system have products and sums with signs that depend on the same expressions as the eigenvalues $\mu_1,\ \mu_2$ obtained in table 10. For the fixed-point corresponding to (SR), say ξ_1^* the sum depends on $-b_r - c_r$ and the product on $|b+c|^2 - |d|^2$, for the one corresponding to (SS), that we call ξ_2^*, the sum depends on $b_r + c_r - 3d_r$ and the product on $|d|^2 - \mathrm{Re}((b+c)\overline{d})$. Finally for (AR), call ξ_3^*, the sum depends on $b_r + c_r + 3d_r$ and the product on $|d|^2 + \mathrm{Re}((b+c)\overline{d})$. In order ξ_1^* to be a saddle, since the sum of the eigenvalues depends on $-b_r - c_r$ that is always negative in the parameter region in which we wish to work, the best we can have is when the product of the eigenvalues is negative, that is, when $|b+c|^2 - |d|^2 < 0$. Note that with this condition, in this parameter region there are no submaximal solutions for (14). Now ξ_3^* has eigenvalues with sum determined by $b_r + c_r + 3d_r$ that is always positive in the parameter set we are working (since $b_r,\ c_r,\ d_r > 0$). Thus only if $|d|^2 + \mathrm{Re}((b+c)\overline{d}) < 0$, then this equilibrium has one negative eigenvalue and one positive. Thus it is a saddle. Generically a heteroclinic connection between these two fixed-points cannot happen. Heteroclinic connections saddle-saddle can happen just as a codimension one phenomenon. So the heteroclinic cycle does not occur generically.

Other possibilities were explored that lead to a similar conclusion. For example the possible cycle $(TR) \to (SR) \to (SS) \to (TS) \to (TR)$ also cannot happen generically. Similar conclusion we arrived for the possibilities of the cycles $(TR) \to (SR) \to (AR) \to (TS) \to (TR)$ or $(TR) \to (AR) \to (SR) \to (TR)$ and $(TR) \to (TS) \to (SS) \to (SR) \to (TR)$. Also the cycle $(TR) \to (TS) \to (AR) \to (TR)$ cannot happen.

In [27], Silber and Knobloch explore the possibilty for the heteroclinic cycle involving the solutions (TS), (SS), (AR). The sugesting part of the isotropy lattice of $\Gamma \times \mathbf{S}^1$ is:

$$\begin{array}{ccccccc} \Sigma_3 & & \Sigma_4 & & \Sigma_5 & & \Sigma_3 \\ & \nwarrow\nearrow & & \nwarrow\nearrow & & \nwarrow\nearrow & \\ & \Delta_2 & & \Delta_5 & & \Delta_3 & \end{array}$$

Thus the cycle is formed by the orbits denoted by $(TS) \to (SS) \to (AR) \to (TS)$. They need to show the heteroclinic orbits $(TS) \to (SS)$ (in $\mathrm{Fix}(\Delta_2)$), $(SS) \to (AR)$ (in $\mathrm{Fix}(\Delta_5)$) and $(AR) \to (TS)$ (in $\mathrm{Fix}(\Delta_3)$). The heteroclinic connections $(TS) \to (SS)$ and $(AR) \to (TS)$ can occur by [20]. Note that the dynamics of f restricted to the spaces $\mathrm{Fix}(\Delta_2)$ and $\mathrm{Fix}(\Delta_3)$ are $\mathbf{O}(2)$-symmetric. Recall table 12. The conditions needed for the saddle-sink connections to happen are:

$$2a_r + b_r < 0, \quad 4a_r + b_r + c_r + d_r < 0, \quad 4a_r + b_r + c_r - d_r < 0,$$
$$-b_r + c_r + d_r > 0, \quad b_r - c_r + d_r > 0.$$

The heteroclinic connection between the (SS) solution and the (AR) solution (for parameter values for which there are the connections $(TS) \to (SS)$ and $(AR) \to (TS)$) is conjectured based on the results of [29]. As we saw before, the dynamics of f restricted to the subspace $\mathrm{Fix}(\Delta_5)$ are given by the normal form of a Hopf bifurcation problem with symmetry $\mathbf{D}_4 \times \mathbf{S}^1$: equations (14) with $A = 2a$, $B = b + c$ and $C = d$. The possibility for the connection $(SS) \to (AR)$ is translated to the possibilty of a saddle-sink orbit of the spherical reduced system in the sphere. In this reduced system the stability for the fixed-point corresponding to the (SS) solution is determined by eigenvalues with trace with sign determined by $b_r + c_r - 3d_r$ and determinant $|d|^2 - \mathrm{Re}((b+c)\overline{d})$ (see table 10). For the fixed-point corresponding to (AR) the eigenvalues have trace with sign determined by $b_r + c_r + 3d_r$ and determinant $|d|^2 + \mathrm{Re}((b+c)\overline{d})$. It is possible to choose values of a, b, c, d such that the first is a saddle and the second is a sink. And using the work of [29] they conjecture that there is a saddle-sink connection in the sphere.

8 Review

We proved that, generically, up to conjugacy, there are eight branches of periodic solutions with maximal isotropy that bifurcate from the trivial solution at a generic Hopf bifurcation with symmetry $\mathbf{O}(2) \wr \mathbf{S}_3$. The stability for these solutions depends only on the terms of degree three of the vector field (and the truncated vector field can be supposed to commute with $\mathbf{S}^1$). Solutions with submaximal isotropy can also exist for an open region of the parameter space.

We are able to prove the existence of these branches only for the Birkhoff normal form of the original vector field of degree three. The existence of one branch of periodic solutions with submaximal isotropy is justified by [29], since it is a solution branch corresponding to the truncated vector field restricted to a four-dimensional fixed-point subspace, where this restricted vector field commutes with $\mathbf{D}_4 \times \mathbf{S}^1$. Swift [29] proves that these solutions are generically unstable. Regions of the parameter space were found for which one or two new branches of solutions with submaximal isotropy can bifurcate from the trivial solution. We did not discuss the stability of these solutions: the equations defining these solutions are complicated.

Althought we did not discuss the full $\mathbf{O}(2) \wr \mathbf{S}_3$-equivariant Hopf problem, we can use the classification obtained by [27], propositions 6.5, 6.6 and theorem 6.7 to have a partial classification of the dynamics of the $\mathbf{O}(2) \wr \mathbf{S}_3$-equivariant Hopf problem.

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