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Hopf Bifurcation in Coupled Cell Networks

Departamento de Matemática Pura
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To my parents
and to Teresa.

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Declaration

I declare, that, to the best of my knowledge and unless where otherwise stated, all the work presented on this thesis is original and was done under the supervision of Professor Ana Paula Dias.

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Abstract

The aim of this thesis is to study Hopf bifurcation in symmetric systems of ordinary differential equations and in coupled cell systems symmetric and with interior symmetry. We continue the works of Golubitsky and Stewart (Hopf bifurcation with dihedral group symmetry: Coupled nonlinear oscillators. In: *Multiparameter Bifurcation Series* (M. Golubitsky and J. Guckenheimer, eds.) Contemporary Mathematics **46**, Am. Math. Soc., Providence, R.I.1986, 131-173) and Gils and Valkering (Hopf bifurcation and symmetry: standing and travelling waves in a circular chain, *Japan J. Appl. Math.* **3** (1986) 207-222). We provide the complete study of generic existence of branches of periodic solutions that bifurcate from the trivial solution of ordinary differential equations with $\mathbf{D}_n$-symmetry, depending on one real parameter, that present Hopf bifurcation. In coupled cell systems with interior symmetry we extend the work of Golubitsky, Pivato and Stewart (Interior symmetry and local bifurcation in coupled cell networks, *Dynamical Systems* **19** (4) (2004) 389-407) obtaining a full analogue of Equivariant Hopf Theorem for networks with symmetries in the context of networks with interior symmetries. This completes the program of generalising the two main results from equivariant bifurcation theory to the class of networks with interior symmetries. We expand the work of Dias and Lamb (Local bifurcation in symmetric coupled cell networks: linear theory, *Physica D* **223** (2006) 93-108). We consider coupled cell networks of differential equations with finite symmetry group, where the group is abelian and permutes cells transitively, and describe how the structure of the coupled cell network can be taken into account in the study of types of codimension one local bifurcations when the phase space of the cells is one-dimensional.

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Chapter 1

Introduction

The aim of this thesis is to study Hopf bifurcation in symmetric systems of ordinary differential equations and in coupled cell systems with symmetry and interior symmetry.

Consider a system of ordinary differential equations (ODEs)

$$\dot{x} = f(x, \lambda), \quad f(0, \lambda) \equiv 0, \tag{1.1}$$

where $x \in V = \mathbf{R}^n$, $\lambda \in \mathbf{R}$ and $f : V \times \mathbf{R} \to V$ is C^∞. We say that (1.1) presents Hopf bifurcation at $\lambda = 0$ if $(df)_{(0,0)}$ has a pair of purely imaginary eigenvalues. Here $(df)_{(x,\lambda)}$ represents the Jacobian matrix of f relatively to x, evaluated at (x, λ). Under additional nondegeneracy hypotheses, the system (1.1) presents a branch of periodic solutions – Hopf Theorem (Golubitsky and Schaeffer [16] Theorem VIII 3.1. See Theorem 2.2.1). The fundamental nondegeneracy hypotheses of Hopf Theorem are that the pair of imaginary eigenvalues of $(df)_{(0,0)}$ is simple and the eigenvalues of $(df)_{(0,\lambda)}$ cross the imaginary axis with non null speed. These hypotheses allow the use of a Liapunov-Schmidt reduction (Golubitsky *et al.* [16] Chapter VII) on an operator ϕ constructed using f and obtain a single scalar reduced equation whose zeros near the origin are in one-to-one correspondence with the periodic solutions of (1.1) with period near 2π. Chapter 2 of this work is dedicated to this subject. We follow Golubitsky *et al.* [16] Chapter VIII.

Suppose now that in (1.1), f *commutes* with a compact Lie group Γ (or it is Γ-*equivariant*), say

$$\Gamma \subseteq \mathbf{O}(n) = \left\{ A \in M_{n \times n}(\mathbf{R}) : A^t A = \mathrm{Id}_{n \times n} \right\}.$$

Thus

$$f(\gamma x, \lambda) = \gamma f(x, \lambda), \forall \gamma \in \Gamma, x \in V, \lambda \in \mathbf{R}. \tag{1.2}$$

We are interested in branches of periodic solutions of (1.1) where f commutes with the group Γ occurring by Hopf bifurcation from the trivial solution

$(x, \lambda) = (0, 0)$. However, the symmetry of f imposes restrictions on the bifurcations that can occur and the main aim of the theory is to understand the effect that these restrictions have. Suppose then that $(df)_{(0,0)}$ has a pair of imaginary eigenvalues $\mp \omega i$. By rescaling time and choosing appropriate coordinates we may assume that $\omega = 1$. It follows that the corresponding imaginary eigenspace E_i contains a Γ-*simple* subspace W of V (Golubitsky, Stewart and Schaeffer [21] Lemma XVI 1.2). Thus $W \cong W_1 \oplus W_1$ where W_1 is *absolutely irreducible* for Γ or W is *irreducible* but non-absolutely irreducible for Γ. Here W is irreducible if it is Γ-invariant and only admits trivial invariant subspaces; W is absolutely irreducible if it is irreducible and the only linear maps from W to W commuting with Γ are the real multiples of the identity on W. Moreover, generically the imaginary eigenspace of $(df)_{(0,0)}$ is itself Γ-simple and coincides with the corresponding real generalized eigenspace; in this case $L = (df)_{(0,0)}$ has only one pair of complex conjugate eigenvalues on the imaginary axis that can be forced by the action of the group to be multiple (Golubitsky *et al.* [21] Proposition XVI 1.4. See Proposition 3.2.4.). Consequently, the Standard Hopf Theorem cannot be applied directly. The main result in the study of bifurcations to periodic solutions in systems commuting with a compact Lie group Γ is then the Equivariant Hopf Theorem (Golubitsky *et al.* [21] Theorem XVI 4.1. See Theorem 3.2.6.) which guarantees, with certain nondegeneracy conditions, that for each isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ with a two dimensional fixed-point subspace (called $\mathbf{C}$-*axial*) there exists a branch of periodic solutions with that symmetry. We recall that if $x(t)$ is a Σ-symmetric periodic solution then $\gamma x(t) = x(t + \theta)$ for all elements (γ, θ) of the *group of symmetries* Σ. We call (γ, θ) a *spatio-temporal symmetry* of the solution $x(t)$. This theorem reduces part of the existence problem for Hopf bifurcations to an algebraic problem: the classification of $\mathbf{C}$-axial subgroups.

The basic idea in the Equivariant Hopf Theorem is that small amplitude periodic solutions of (1.1) of period near 2π correspond to zeros of a reduced equation $\phi(x, \lambda, \tau) = 0$ where τ is the period-perturbing parameter. The reduced equation is obtained by a Liapunov-Schmidt reduction preserving symmetries that will induce a different action of $\mathbf{S}^1$ on a finite-dimensional space, which can be identified with the exponential of $L|_{E_i}$ acting on the imaginary eigenspace E_i of L. Moreover the reduced equation of f commutes with $\Gamma \times \mathbf{S}^1$. See Golubitsky *et al.* [21] Lemma XXVI 3.2. To find periodic solutions of (1.1) with symmetries Σ is equivalent to find zeros of the reduced equation restricted to $\mathrm{Fix}(\Sigma)$. See Golubitsky *et al.* [21] Chapter XVI Section 4.

However, the periodic solutions whose existence is guaranteed by Equivariant Hopf Theorem are not necessarily the only ones. When looking for branches of periodic solutions obtained by Hopf bifurcation in symmetric systems that are not guaranteed by the Equivariant Hopf Theorem, a useful tool is to assume the vector field in Birkhoff normal form. The idea is that by an adequate coordinate change, up to an order k, the vector field f commutes not only with Γ but also with a group $\mathbf{S}$ determined by $(df)_{(0,0)}$. On Hopf bifurcation $\mathbf{S}$ is the circular group $\mathbf{S}^1$. This introduces an extra symmetry that can be explored on the study of the generic existence and stability of branches of periodic solutions of (1.1). In chapter 3 we outline the concepts and results involved in the study of (1.1) in presence of symmetry. We follow Golubitsky *et al.* [21] Chapter XVI.

The first question we address in this thesis is the generic existence of branches of periodic solutions in symmetric systems of ODEs occurring by Hopf bifurcation from the trivial solution $(x, \lambda) = (0, 0)$ that are not guaranteed by the Equivariant Hopf Theorem. We raise the question: which methods and techniques to use? In chapter 4 we address this subject by considering systems with symmetry $\mathbf{D}_n$, $n \geq 3$, the dihedral group of order $2n$. Specifically, we consider the standard action of $\mathbf{D}_n$ on $V = \mathbf{C} \oplus \mathbf{C}$ (see section 4.1). That is, V is the sum of two (isomorphic) absolutely irreducible representations where $\mathbf{D}_n$ acts on $\mathbf{C} \equiv \mathbf{R}^2$ in the standard way as symmetries of the regular n-gon. Although $\mathbf{D}_n$ has many distinct two-dimensional irreducible representations there is no loss of generality in making this assumption. Essentially it is possible to arrange for a standard action by relabeling the group elements and dividing by the kernel of the action. Golubitsky and Stewart [18] and van Gils and Valkering [35] (see also Golubitsky *et al.* [21] Chapter XVIII) prove the generic existence of three branches of periodic solutions, up to conjugacy, of (1.1) bifurcating from the trivial solution. These solutions are found by using the Equivariant Hopf Theorem. They thus correspond to three (conjugacy classes of) *maximal* isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ (acting on V), each having a two-dimensional fixed-point subspace. We prove in Theorem 4.3.2 that if f in (1.1) satisfies the conditions of the Equivariant Hopf Theorem and f is in Birkhoff normal form then, when $n \neq 4$ and $n \geq 3$, the *only* branches of small-amplitude periodic solutions of period near 2π of (1.1) that bifurcate from the trivial equilibrium are the branches of solutions guaranteed by the Equivariant Hopf Theorem. The case when $n = 4$ differs markedly from those other n. Swift [34] studies the dynamics of all possible square-symmetric *codimension one* Hopf bifurcations (with one parameter). In particular, it is shown that periodic solutions with *submaximal* symmetry bifurcate from the origin for open regions of the parameter space of the cubic coefficients in the Birkhoff normal form. Essentially we use two techniques: assume that f is in Birkhoff normal form to all orders and so f commutes also with $\mathbf{S}^1$ and use the Morse Lemma (see for example Poston and Stewart [29] Theorem 4.2). Observe that these techniques could be used in the study of Hopf bifurcation in systems of ODEs with other symmetry groups.

Standard examples of dynamical systems include networks of coupled *cells*, that is, coupled ODEs. Here, a network is represented by a directed graph whose nodes and edges are classified according to associated labels or *types*. The nodes (or cells) of a network $\mathcal{G}$ represent dynamical systems, and the edges (*arrows*) represent couplings. Cells with the same label have 'identical' internal dynamics; arrows with the same label correspond to 'identical' couplings. We follow the theory developed by Stewart, Golubitsky and Pivato [33] and Golubitsky, Stewart and Török [22]. The *input set* of a cell is the set of edges directed to that cell. Coupled cell systems are dynamical systems compatible with the architecture or topology of a directed graph representing the network. Formally, they are defined in the following way. Each cell c is equipped with a phase space P_c, and the *total phase space* of the network is the cartesian product $P = \prod_c P_c$. A vector field f is called *admissible* (or $\mathcal{G}$-*admissible*) if its component f_c for cell c depends only on variables associated with the input set of c, and if its components for cells c, d that have isomorphic input sets are identical up to a suitable permutation of the relevant variables.

In the study of network dynamics there is an important class of networks, namely, networks that possess a group of symmetries. In this context there is a group of permutations of the cells (and arrows) that preserves the network structure (including cell-types and arrow-types) and its action on P is by permutation of cell coordinates. In between the class of general networks and the class of symmetric networks lies an interesting class of non-symmetric networks, where group theoretic methods still apply, namely, networks admitting *interior symmetries*. In this case there is a group of permutations of a subset $\mathcal{S}$ of the cells (and edges directed to $\mathcal{S}$) that partially preserves the network structure (including cell-types and edges-types) and its action on P is by permutation of cell coordinates. In other words, the cells in $\mathcal{S}$ together with all the edges directed to them form a subnetwork which possesses a nontrivial group of symmetry $\Sigma_{\mathcal{S}}$. For example, network $\mathcal{G}_1$ of Figure 1.1 (left) has exact $\mathbf{S}_3$-symmetry, whereas network $\mathcal{G}_2$ of Figure 1.1 (right) has $\mathbf{S}_3$-interior symmetry on the set of cells $S = \{1, 2, 3\}$. This notion was introduced and investigated by Golubitsky, Pivato and Stewart [15]. The presence of interior symmetries places some restrictions on the structure of the network. In this thesis we restrict the study to Hopf bifurcation in coupled cell systems associated with symmetric and interior-symmetric networks. We give the background of this subject in chapter 5.

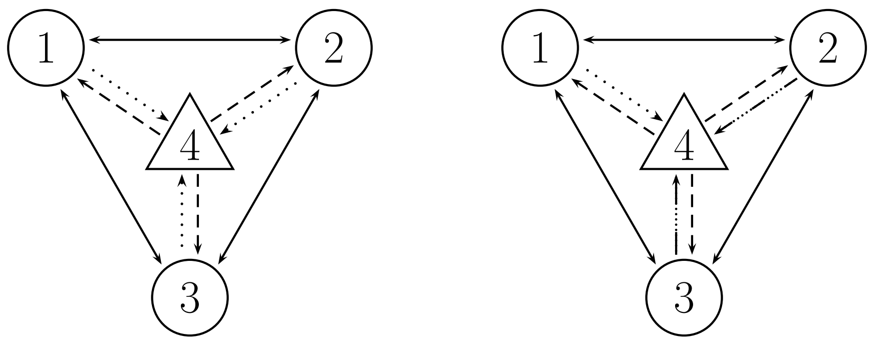

Figure 1.1: (Left) Network $\mathcal{G}_1$ with exact $\mathbf{S}_3$-symmetry. (Right) Network $\mathcal{G}_2$ with $\mathbf{S}_3$-interior symmetry on $\mathcal{S} = \{1, 2, 3\}$.

Coupled cell systems (ODEs) associated with Γ-symmetric networks are of the form (1.1) where $V = P$ and the $\mathcal{G}$-admissible vector field f is equivariant under the action of the group Γ on phase space P. The theory of equivariant dynamical systems (see Golubitsky *et al.* [19, 21]) can be applied to such dynamical systems. The Equivariant Hopf Theorem guarantees, with certain nondegeneracy conditions, that for each $\mathbf{C}$-axial subgroup Σ of $\Gamma \times \mathbf{S}^1$ there exists a branch of periodic solutions of (1.1) with that symmetry. Consider, for example, the network $\mathcal{G}_1$ (Figure 1.1 (left)) that has exact $\mathbf{S}_3$-symmetry (where $\mathbf{S}_3 \cong \mathbf{D}_3$). Applying the Equivariant Hopf Theorem to the coupled cell systems encoded by this network we obtain, under certain nondegeneracy conditions, branches of periodic solutions having a group of symmetries associated to one of the three *conjugacy classes* of $\mathbf{C}$-axial subgroups of $\mathbf{S}_3 \times \mathbf{S}^1$: $\mathbf{Z}_2 = \big\langle((1\,2), \mathbf{1})\big\rangle$, $\tilde{\mathbf{Z}}_2 = \big\langle((1\,2), \pi)\big\rangle$ and $\tilde{\mathbf{Z}}_3 = \big\langle((1\,2\,3), 2\pi/3)\big\rangle$. Periodic solutions with $\mathbf{Z}_2$-symmetry have cells $1, 2$ oscillating synchronously, solutions with $\tilde{\mathbf{Z}}_2$-symmetry have cells $1, 2$ oscillating a half-period out of phase while cell 3 oscillates at twice the frequency, $\tilde{\mathbf{Z}}_3$-symmetric periodic solutions have cells $1, 2, 3$ oscillating with the same wave form but with a one-third period phase shift between cells.

Golubitsky *et al.* [15] provided an analogue of the Equivariant Hopf Theorem for coupled cell systems with interior symmetries. They prove the existence of states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatial symmetries*. However, their result has novel and rather restrictive features. For example, in the coupled cell systems associated to the network $\mathcal{G}_2$ (Figure 1.1 (right)) it only predicts the periodic states associated to the *spatially* $\mathbf{C}$-axial subgroup $\mathbf{Z}_2 = \big\langle((1\,2), \mathbf{1})\big\rangle$.

The second main question addressed in this thesis is Hopf bifurcation in coupled cell systems associated to networks with interior symmetries. In chapter 6 we obtain the full analogue of the Equivariant Hopf Theorem for networks with symmetries (Theorem 6.1.3). We extend the result of Golubitsky *et al.* [15] obtaining states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having spatio-temporal symmetries, that is, corresponding to *interiorly* $\mathbf{C}$-axial subgroups of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$. This new version of the Hopf Theorem with interior symmetries includes the previous version as a special case and is in complete analogy with the Equivariant Hopf Theorem (see Theorem 6.1.3). We do the proof through two approaches. The first uses a modification of the Liapunov-Schmidt reduction to arrive at a situation where the proof of the Standard Hopf Bifurcation Theorem can be applied (recall Theorem 2.2.1). The second uses the center manifold reduction to reach a phase in where the Standard Hopf Theorem gives the result. This completes the program of generalising the main result from equivariant Hopf bifurcation theory to the class of networks with interior symmetries. As an example, consider the coupled cell systems associated to the network $\mathcal{G}_2$ (Figure 1.1 (right)) that has $\mathbf{S}_3$-interior symmetry on $\mathcal{S} = \{1, 2, 3\}$. Our result predicts, under certain nondegeneracy conditions, that there are three branches of *synchronously modulated* Δ-symmetric waves associated to the three conjugacy classes of interiorly $\mathbf{C}$-axial subgroups of $\mathbf{S}_3 \times \mathbf{S}^1$ (see Table 1.1). The first periodic state of Table 1.1 is associated to a spatially $\mathbf{C}$-axial subgroup and so is predicted by Theorem 3 of Golubitsky *et al.* [15]. The third periodic state of Table 1.1 is an approximate rotating wave.

Subgroup	Form of solution to lowest order in λ
$\mathbf{Z}_2 = \langle ((1\,2), \mathbf{1}) \rangle$	$(w_1(t) + u(t), w_1(t) + u(t), w_2(t) + u(t), v(t))$
$\tilde{\mathbf{Z}}_2 = \langle ((1\,2), \pi) \rangle$	$(w_1(t) + u(t), w_1(t + \frac{1}{2}) + u(t), \hat{w}(t) + u(t), v(t))$
$\tilde{\mathbf{Z}}_3 = \langle ((1\,2\,3), \frac{2\pi}{3}) \rangle$	$(w_1(t) + u(t), w_1(t + \frac{1}{3}) + u(t), w_1(t + \frac{2}{3}) + u(t), v(t))$

Table 1.1: Branches of synchronously modulated Δ-symmetric waves supported by the network $\mathcal{G}_2$ of Figure 1.1 (right) and the associated subgroup. The hat over a variable indicates that $\hat{w}$ has twice the frequency.

One of the main questions in the theory of coupled cell networks is the following: in what way the network architecture may affect the kinds of bifurcations that are expected to occur in a coupled cell network? In the last part of this thesis we address this question by focussing on networks with a symmetry group that permutes cells transitively. Observe that distinct networks (with the same number of cells) can have the same symmetry group. In addressing this question, the first concern is with the spectrum of the linearized vector field (Jacobian matrix) at the equilibrium solution when parameters are varied, in particular with the analysis of how eigenvalues typically cross the imaginary axis. Dias and Lamb [7] address this question in the case of abelian symmetric networks. They obtain a result that states that in an abelian symmetric connected coupled cell network $\mathcal{G}$ where the symmetry group acts transitively by permutation of the cells of the network, if the phase space of cells has dimension greater than one, the codimension one eigenvalue movements across the imaginary axis of the linearization of the $\mathcal{G}$-admissible vector fields at a fully symmetric equilibrium x_0 are independent of the network structure and are identical to the corresponding eigenvalue movements in general equivariant vector fields. We show that this result is incomplete when we study codimension one Hopf bifurcation assuming that the phase space of the cells is one-dimensional. For example, in one-parameter coupled cell systems associated to the network of nine cells with $\mathbf{Z}_3 \times \mathbf{Z}_3$ symmetry represented in Figure 1.2, can generically occur Hopf bifurcation associated to the crossings with the imaginary axis of two distinct pairs of complex eigenvalues of linearization of the vector field at a fully symmetric equilibrium, that is, a Hopf/Hopf *mode interaction*. Generically,

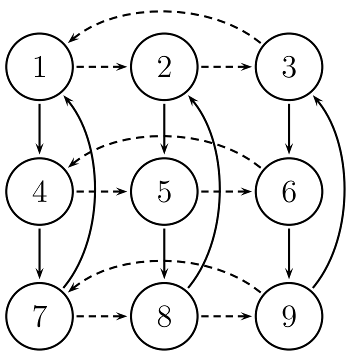

Figure 1.2: Network with $\mathbf{Z}_3 \times \mathbf{Z}_3$-symmetry.

in general equivariant linear systems, in case of codimension one eigenvalue crossings with the imaginary axis, this phenomenon is not expected. In chapter 7 we describe the abelian groups that can present this phenomenon in terms of complex characters of abelian groups. Essentially, each cell of the network $\mathcal{G}$ corresponds to a unique element of the abelian symmetry group Γ. With this identification, we associate with $\mathcal{G}$ a set $S \subseteq \Gamma$ that depends on the group elements corresponding to the present couplings. In case $\mathcal{G}$ is connected, S generates Γ. Moreover, the eigenvalues of the linearization of the $\mathcal{G}$-admissible vector fields at a fully symmetric equilibrium depend on the characters of Γ evaluated on the elements of S. We determine, using this fact, generic conditions involving the complex characters of Γ that permit in one parameter the occurrence of Hopf bifurcation associated to the crossings with the imaginary axis of two or more distinct pairs of complex eigenvalues of linearization (see Theorem 7.5.7).

This thesis is organized as follows. In chapter 2 we begin by introducing the phenomenon of Hopf bifurcation in systems of ODEs without symmetry. We present the Standard Hopf Theorem.

In chapter 3 we formulate the same problem of chapter 2 – search of branches of periodic solutions of (1.1) by bifurcation from the trivial equilibrium, but assuming that the vector field commutes with a compact Lie group. We describe the consequences of such hypothesis and present the main existence result: the Equivariant Hopf Theorem (see Theorem 3.2.6). We finish the chapter with some considerations about Birkhoff Normal Form needed for looking for branches of periodic solutions obtained by Hopf bifurcation in symmetric systems that are not guaranteed by the Equivariant Hopf Theorem.

Chapter 4 studies Hopf bifurcation with $\mathbf{D}_n$ symmetry. We consider the standard action of $\mathbf{D}_n$ on $V = \mathbf{C} \oplus \mathbf{C}$ (see section 4.1). In section 4.1 we describe the conjugacy classes of $\mathbf{D}_n \times \mathbf{S}^1$ (with standard action on V) obtained by Golubitsky *et al.* [21]. For each n, there are five conjugacy classes and three of them correspond to isotropy subgroups with two-dimensional fixed point subspaces. In section 4.2 we give the general form of a vector field that commutes with $\mathbf{D}_n \times \mathbf{S}^1$ obtained by Golubitsky *et al.* [21]. Finally in section 4.3 we obtain the main result of this chapter – Theorem 4.3.2. We prove that when $n \neq 4$ and $n \geq 3$ generically the only branches of small-amplitude periodic solutions of (1.2) that bifurcate from the trivial equilibrium are those guaranteed by the Equivariant Hopf Theorem. The proof of this theorem relies mostly in the general form of f and the use of Morse Lemma.

In chapter 5 we study Hopf bifurcation with interior symmetry. Section 5.1 gives the formal definition of a coupled cell network and the associated dynamical systems, and states some basic facts, including the concept of a balanced equivalence relation (colouring). We also discuss the symmetry group of a network. Section 5.2 gives the definition of interior symmetry given by Golubitsky *et al.* [15]. We also describe an equivalent condition, in terms of symmetries of a subnetwork, which in some cases (no multiple edges and no self-connections) amounts to finding the symmetries of a subnetwork (see Proposition 5.2.3). We also analyse the structure of networks with interior symmetry and discuss some features of the admissible vector fields associated to such class of networks. Section 5.3 gives the notion of synchrony-breaking bifurcation in coupled cell networks. Then we specialise to networks with interior symmetries where group theoretic concepts play a significant role, focusing on the important case of codimension-one synchrony-breaking bifurcations.

In chapter 6 we present the main result of this thesis: the full analogue of the Equivariant Hopf Theorem for networks with symmetries (Theorem 6.1.3). We prove this theorem in two different ways: the Liapunov-Schmidt approach and the center manifold approach. We illustrate all the concepts and results by a running example of the simplest network with $\mathbf{S}_3$ interior symmetry and the closely related network with exact $\mathbf{S}_3$ symmetry (recall Figure 1.1). Finally, we present a numerical simulation of the states provided by the Theorem 6.1.3 in the case of our running example.

In chapter 7 we study linear theory for local bifurcation in abelian symmetric coupled cell networks. We begin by illustrating the problem with an example. In section 7.2 we give some background following James and Liebeck [24] and Dias and Lamb [7] which we use throughout this chapter. In section 7.3, in order to describe which class of abelian groups Γ and Γ-symmetric coupled cell networks $\mathcal{G}$ can exhibit Hopf bifurcation associated to the crossings with the imaginary axis of two or more distinct pairs of complex eigenvalues of linearization of the $\mathcal{G}$-admissible vector fields at a fully symmetric equilibrium x_0, we present some results and an algorithm involving characters of abelian groups. In that, we relate a general abelian group Γ with a set $S \subseteq \Gamma$ that describes the structure of network. In section 7.5 we establish the connection between the theory developed in section 7.3 and codimension one eigenvalue movements through the imaginary axis for coupled cell networks. Theorem 7.5.7 gives sufficient conditions for the generic existence of the nondegenerate phenomenon described above. We illustrate all the concepts and results by two running examples of the simplest networks with $\mathbf{Z}_3 \times \mathbf{Z}_3$ and $\mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ interior symmetries (see figures 1.2 and 7.3). We finish the chapter studying the class of coupled cell networks with abelian symmetry whose set of admissible vector fields coincide with the set of equivariant vector fields.

Chapter 2

Hopf Bifurcation

Consider the system of ordinary differential equations (ODEs)

$$\dot{u} = F\,(u, \lambda) \tag{2.1}$$

where $u \in V = \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is the bifurcation parameter, $F : V \times \mathbf{R} \to V$ is C^∞ and assume that $F(0, \lambda) \equiv 0$ so that $u = 0$ is a steady-state solution to (2.1) for all λ. In this chapter we introduce the phenomenon of Hopf bifurcation in which the steady-state solution $u = 0$, say at $\lambda = 0$, evolves into a periodic orbit as the bifurcation parameter λ is varied. We follow Golubitsky and Schaeffer [16] Chapter VIII.

The chapter is divided into two sections. In section 2.1 we introduce the phenomenon of Hopf Bifurcation by describing some examples. In section 2.2 we present a result that provides sufficient conditions for the existence of a family of periodic solutions to (2.1) parameterized by the bifurcation parameter λ – the Hopf Theorem (Theorem 2.2.1) – and give a sketch of the proof.

2.1 Examples

Consider the system (2.1). Hopf showed that a one-parameter family of periodic solutions to (2.1) emanating from $(u, \lambda) = (0, 0)$ could be found if two hypotheses on F were satisfied. Let $A\,(\lambda) = (dF)_{(0,\lambda)}$ be the $n \times n$ Jacobian matrix of F along the steady state solutions $(0, \lambda)$. The first Hopf assumption is:

(i) $A(0)$ has simple eigenvalues $\pm i$ and

(ii) $A(0)$ has no other eigenvalues lying on the imaginary axis. (2.2)

Remarks 2.1.1 (i) Note that if we rescale the time t in (2.1) by setting $t = ks$ for k fixed and positive, (2.1) changes to

$$\frac{du}{ds} = kF(u, \lambda)$$

where the linearization of the vector field on $(0, \lambda)$ is the matrix $A(\lambda)$ multiplied by k. As a result, we may interpret the hypothesis (i) of (2.2) as stating that $A(0)$ has a pair of nonzero, purely imaginary eigenvalues which have been rescaled to equal $\pm i$.
(ii) There is no difficulty in proving that periodic orbits for (2.1) exist if $A(0)$ has other eigenvalues on the imaginary axis, provided none of these is an integer multiple of $\pm i$. However, the hypothesis (ii) of (2.2) is vital for the analysis of stability. For simplicity, we make the assumption (ii) of (2.2) throughout.

The matrix $A(\lambda)$ has simple eigenvalues of the form $\sigma(\lambda) \pm i\omega(\lambda)$, where $\sigma(0) = 0$, $\omega(0) = 1$ and σ and ω are smooth. This follows from the fact that $A(\lambda)$ has real entries which depend smoothly on λ and that the eigenvalues $\pm i$ of $A(0)$ are simple. The second Hopf assumption is:

$$\sigma'(0) \neq 0, \tag{2.3}$$

that is, the imaginary eigenvalues of $A(\lambda)$ cross the imaginary axis with nonzero speed as λ crosses zero. The Hopf Theorem states that there is a one-parameter family of periodic solutions to (2.1) if assumptions (2.2) and (2.3) hold.

Example 2.1.2 An elementary and instructive example that illustrates well the Hopf Theorem is the simplest linear example in the plane defined by

$$F(u, \lambda) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} u \tag{2.4}$$

where $u \in \mathbf{R}^2$. We can compute the phase portraits for the system (2.1) as λ varies solving the equations explicitly. With initial conditions $u(0) = (a, 0)$, the solution to (2.1) is given by $u(t) = ae^{\lambda t}(\cos t, \mathrm{sen} t)$. The phase portraits for this system are given in Figure 2.1. For $\lambda < 0$ the steady state $u = 0$ is stable (i.e., orbits spiral into the origin), while for $\lambda > 0$ the steady state $u = 0$ is unstable (i.e., orbits spiral away from the origin). However, for $\lambda = 0$ the steady state $u = 0$ is neutrally stable, and there is the one parameter family of periodic orbits guaranteed by Hopf Theorem where each orbit is 2π-periodic. We may parametrize these orbits by their amplitudes.

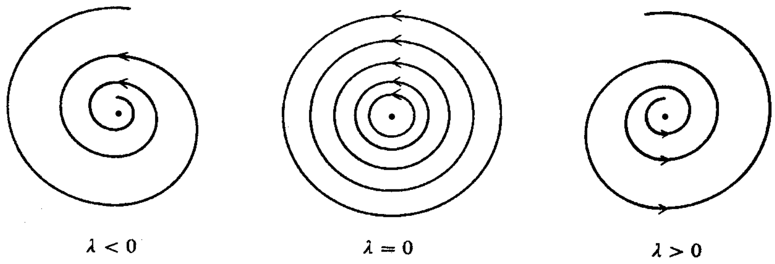

Figure 2.1: Phase portraits for the linear system (2.1) with F as in (2.4).

On the above example the family of periodic solutions guaranteed by Hopf Theorem occurs when $\lambda = 0$. In a sense, Hopf Theorem states that this family of periodic solutions persists even when higher-order terms in u and λ are added to F. However, this one-parameter family of periodic solutions need not remain in the plane $\lambda = 0$. In fact, the generic situation is that when higher-order terms are added to F, for each fixed λ there is at most one periodic orbit remaining near the origin.

For example, consider the system defined by

$$F(u,\lambda) = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} u - |u|^2 u \tag{2.5}$$

where $|u|^2 = x^2 + y^2$, if $u = (x, y)$. The system written in polar coordinates is

$$\begin{cases} \dot{r} = r\left(\lambda - r^2\right) \\ \dot{\theta} = 1 \end{cases}$$

and (2.1) with F as in (2.5) admits for each $\lambda > 0$ the periodic solution $u(t) = \sqrt{\lambda}\,(\cos t, \mathrm{sen}t)$ with initial condition $u(0) = \left(\sqrt{\lambda}, 0\right)$. See the phase portraits for the system in Figure 2.2.

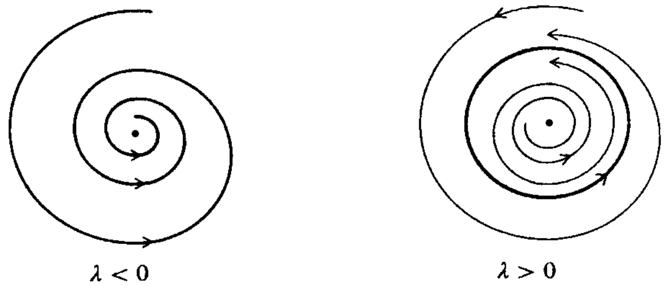

Figure 2.2: Phase portraits for the nonlinear system (2.1) with F as in (2.5).

In this example, for each $\lambda > 0$ there is exactly one periodic solution to (2.1). (Moreover, this periodic solution is stable: all nearby orbits approach this periodic solution.) $\diamond$

2.2 Hopf Theorem

In this section we state the Hopf Theorem and sketch its proof.

In analysing (2.1) it will be convenient to allow the equation to depend on auxiliary parameters from the start. Let $F : \mathbf{R}^n \times \mathbf{R}^{k+1} \to \mathbf{R}^n$ and consider the equation

$$\frac{du}{dt} = F(u, \alpha) \tag{2.6}$$

where $\alpha = (\alpha_0, \ldots, \alpha_k)$ and $\alpha_0 = \lambda$ is the bifurcation parameter. Suppose that

$$F(0, \alpha) \equiv 0$$

and that $A(\alpha)$ satisfies (2.2), where $A(\alpha) = (dF)_{0,\alpha}$.

The Hopf Theorem involves conditions that guarantee the existence of a family of periodic solutions of (2.6) which can be parameterized by the bifurcation parameter λ. We present here only the existence result:

Theorem 2.2.1 (Standard Hopf Theorem [16]) *Let the system of ODEs* (2.6) *satisfy:*
(H1) *the simple eigenvalue condition* (2.2); *and*
(H2) *the eigenvalues crossing condition* (2.3) *(i.e., $\sigma_\lambda(0) \neq 0$).*
Then there is a $k+1$-parameter family of periodic orbits of (2.6) *bifurcating from the steady-state solution $u = 0$ at $\alpha = 0$.*

We dedicate the rest of this section to the sketch of the proof of this result. We start by stating a result that guarantees that orbits of small amplitude of (2.6) with period near 2π are in one-to-one correspondence with zeros of a scalar equation

$$g(x, \alpha) = 0 \tag{2.7}$$

provided the simple eigenvalue hypothesis (2.2) holds:

Theorem 2.2.2 ([16]) *Assume that the system* (2.6) *satisfies the simple eigenvalue hypothesis* (2.2). *Then there exists a smooth germ $g(x, \alpha)$ of the form*

$$g(x, \alpha) = r\left(x^2, \alpha\right) x, \quad r(0, 0) = 0$$

such that locally solutions to $g(x, \alpha) = 0$ *with* $x \geq 0$ *and near to* $(0,0)$ *are in one-to-one correspondence with orbits of small amplitude periodic solutions to the system* (2.6) *with period near* 2π.

Proof: See Golubitsky *et al.* [16] Theorem VIII 2.1. The proof of this theorem uses the Liapunov-Schmidt reduction: from (2.6) is constructed a mapping $\phi : \mathbf{C} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{C}$ where the solutions of $\phi = 0$ are in one-to-one correspondence with the periodic solutions of (2.6) of period near 2π. Moreover, this reduction can be performed to be symmetric by the standard action on $\mathbf{C}$ of the group $\mathbf{S}^1$, that is, we can assume that

$$\phi(e^{i\theta} z, \alpha, \tau) = e^{i\theta}\phi(z, \alpha, \tau) \tag{2.8}$$

where $\theta \in \mathbf{S}^1$. Here τ is the period-scaling parameter. It follows that

$$\phi(z, \alpha, \tau) = p(|z|^2, \alpha, \tau)z + q(|z|^2, \alpha, \tau)iz \tag{2.9}$$

where p, q are real-valued smooth functions satisfying $p(0) = q(0) = 0$. Using (2.8) we need only look for solutions where $z = x \in \mathbf{R}$. Hence solutions to $\phi = 0$ are of two types: $x = 0$ (the trivial equilibrium) and solutions to the system $p = q = 0$ (the desired small-amplitude periodic solutions).

In Hopf bifurcation, a calculation shows that $q_\tau(0) = -1$ ([16] Proposition VIII 2.3). Hence, the equation $q = 0$ can be solved by the Implicit Function Theorem for $\tau = \tau(x^2, \alpha)$, where $\tau(0) = 0$, and small amplitude periodic solutions to (2.6) are found by solving

$$r(x^2, \alpha) \equiv p\left(x^2, \alpha, \tau\left(x^2, \alpha\right)\right) = 0.$$

Thus

$$g(x, \alpha) = r\left(x^2, \alpha\right)x, \quad r(0,0) = 0.$$

□

The function $g(x, \alpha)$ has the form $r(x^2, \alpha)\,x$ for some function r; the nontrivial solutions of (2.7) may be obtained by solving

$$r\left(x^2, \alpha\right) = 0. \tag{2.10}$$

The information on Hopf Theorem is readily obtained if the function r in (2.10) is available; the difficult point is to derive the information directly from the ordinary differential equation (2.6). Suppose that r is known and that

$$r_\lambda(0,0) \neq 0, \tag{2.11}$$

where $\lambda = \alpha_0$ is the bifurcation parameter. Here r_λ denotes the partial derivative of r with respect to λ. Then, by the Implicit Function Theorem, we may solve (2.10) for λ as function of x^2 and $\alpha' = (\alpha_1, \ldots, \alpha_k)$; in symbols

$$\lambda = \mu \left(x^2, \alpha'\right).$$

In other words, if (2.11) holds, then (2.6) has a $(k+1)$-parameter family of periodic solutions which bifurcate from the trivial solution. Assuming (2.2) holds, the matrices $A(\alpha) = (dF)_{0,\alpha}$ for α close to the origin have simple eigenvalues close to $\pm i$ that vary smoothly with α; we let

$$\Sigma(\alpha) = \sigma(\alpha) - i\omega(\alpha)$$

be the eigenvalue of $A(\alpha)$ satisfying $\sigma(0) = 0$, $\omega(0) = 1$. It can be proved that

$$r_\lambda(0,0) = \sigma_\lambda(0)$$

(see Golubitsky *et al.* [16] Proposition VIII 3.3). Thus, (2.11) holds if and only if (2.3) holds and the Hopf Theorem is stated.

Chapter 3

Hopf Bifurcation with Symmetry

Consider a system of ordinary differential equations (ODEs)

$$\dot{v} = f(v, \lambda), \quad f(0, \lambda) \equiv 0, \tag{3.1}$$

where $v \in V = \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is the bifurcation parameter and $f : V \times \mathbf{R} \to V$ is C^∞. In this chapter we analyze the problem (3.1) where f is symmetric and $(df)_{(0,0)}$ has a pair of purely imaginary eigenvalues. We describe hypotheses of nondegeneracy that imply the occurrence of branches of periodic solutions. The symmetry of the problem imposes restrictions on the bifurcations that can occur. In particular, symmetry can force multiple eigenvalues and so the fundamental nondegeneracy hypothesis in the standard Hopf Theorem concerning the imaginary eigenvalues to be simple is not satisfied. Also, in another cases it is not possible to arrange for the eigenvalues of $(df)_{(0,0)}$ to be purely imaginary. There are nevertheless techniques that exploit the symmetry of the problem. We follow Golubitsky *et al.* [21] Chapter XVI.

This chapter divides into two parts. In section 3.1 we introduce some background involving symmetry that we use throughout this chapter. In the section 3.2 we address the same problem of the chapter 2 – search of branches of periodic solutions near the bifurcation point – but assuming that the vector field commutes with a compact Lie group. We describe some of the consequences of such hypothesis. We then present one of the main results in the study of Hopf bifurcation in symmetric systems: the Equivariant Hopf Theorem (Theorem 3.2.6).

3.1 Symmetry

Given a system of ODEs as (3.1) where f is symmetric, the symmetry of f imposes restrictions on the form how the solutions bifurcate from the trivial solution and often forces the eigenvalues of $(df)_{(0,0)}$ to be multiple. Although the symmetries complicate the analysis by forcing multiple eigenvalues, they also potentially simplify it by placing restrictions on the form of the mapping f. There are techniques as invariant theory and restriction to fixed-point subspaces that simplify the analysis exploiting the symmetry. In order to describe some of these techniques we begin by presenting a few results concerning representation of compact Lie groups.

Linear Actions of Compact Groups

Let Γ be a Lie group and V a finite-dimensional real vector space. We say that Γ *acts* (*linearly*) *on* V if there is a continuous mapping (*the action*)

$$\begin{aligned} \Gamma \times V &\to V \\ (\gamma, v) &\mapsto \gamma \cdot v \end{aligned}$$

such that:
(a) For each $\gamma \in \Gamma$ the mapping $\rho_\gamma : V \to V$ defined by $\rho_\gamma(v) = \gamma \cdot v$ is linear;
(b) If $\gamma_1, \gamma_2 \in \Gamma$ then $\gamma_1 \cdot (\gamma_2 \cdot v) = (\gamma_1 \gamma_2) \cdot v$ for all $v \in V$.

The function $\rho : \Gamma \to \mathbf{GL}(V)$ such that $\rho(\gamma) = \rho_\gamma$ is called a *representation of* Γ *on* V. (We denote the group of invertible linear transformations from V to V by $\mathbf{GL}(V)$.) When Γ is compact, there exists a Γ-invariant inner product on V where by choosing an orthonormal basis of V with respect to that inner product, for all $\gamma \in \Gamma$, the matrix of the linear function ρ_γ relatively to that basis of V, say A_γ, is orthogonal. That is, $A_\gamma^t \cdot A_\gamma = \mathrm{Id}_n$, where n is the dimension of V. (See for example Golubitsky *et al.*[21] Proposition XII 1.3.) Here we denote by A_γ^t the transpose matrix of A_γ. In this work we will assume that Γ is compact. Without loss of generality, we can then suppose that Γ acts linearly and orthogonally on V, and then $A_\gamma \in \mathbf{O}(n)$ where

$$\mathbf{O}(n) = \{A \in M_{n\times n}(\mathbf{R}) : \ A \cdot A^t = Id_n\}$$

(called the n-dimensional *orthogonal group*).

Orbits and Isotropy Subgroups

Given $v \in V$, the *orbit* of the action of Γ on v is the set

$$\Gamma v = \{\gamma \cdot v : \ \gamma \in \Gamma\}$$

and the *isotropy subgroup* of v, Σ_v, is the subgroup of Γ defined by

$$\Sigma_v = \{\gamma \in \Gamma : \ \gamma \cdot v = v\}.$$

Given a subgroup $\Sigma \subseteq \Gamma$ and $\gamma \in \Gamma$, the subgroup of Γ defined by

$$\gamma\Sigma\gamma^{-1} = \left\{\gamma\sigma\gamma^{-1} : \ \sigma \in \Sigma\right\}$$

is said to be *conjugate to* Σ. The *conjugacy class of* Σ consists of all subgroups of Γ that are conjugate to Σ. An isotropy subgroup Σ of Γ is called *maximal* if there does not exist an isotropy subgroup Δ of Γ such that $\Sigma \subset \Delta \subset \Gamma$.

Vectors on the same orbit of Γ have *conjugate isotropy subgroups*. More precisely,

$$\Sigma_{\gamma \cdot v} = \gamma\Sigma_v\gamma^{-1}.$$

To see this, note that if $v \in V$, $\sigma \in \Sigma_v$ and $\gamma \in \Gamma$, then $\gamma\sigma\gamma^{-1} \cdot (\gamma \cdot v) = \gamma \cdot v$ and so $\gamma\sigma\gamma^{-1} \in \Sigma_{\gamma \cdot v}$. Thus $\gamma\Sigma_v\gamma^{-1} \subseteq \Sigma_{\gamma \cdot v}$. Replacing v by γv and γ by γ^{-1} we obtain $\gamma^{-1}\Sigma_{\gamma \cdot v}\gamma \subseteq \Sigma_v$.

Equivariance

We introduce now the notion of symmetry of a system. Given a function $f : V \to V$, we say that f *commutes* with the action of Γ on V (or f is Γ-*equivariant*) if

$$f(\gamma \cdot v) = \gamma \cdot f(v), \quad \forall \gamma \in \Gamma, v \in V. \tag{3.2}$$

Remarks 3.1.1
(i) If $x(t)$ is a solution of the system (3.1) then $\gamma \cdot x(t)$ is also a solution for all $\gamma \in \Gamma$: if $x(t)$ is a solution of (3.1) where f commutes with Γ then

$$\frac{dx(t)}{dt} = f(x(t), \lambda) \Leftrightarrow \gamma\frac{dx(t)}{dt} = \gamma f(x(t), \lambda) \Leftrightarrow \frac{d\gamma x(t)}{dt} = f(\gamma x(t), \lambda).$$

(ii) Applying the chain rule to the equality $f(\gamma \cdot x, \lambda) = \gamma \cdot f(x, \lambda)$ we conclude that $(df)_{(0,\lambda)}\gamma = \gamma(df)_{(0,\lambda)}$, that is, the matrix $A(\lambda) = (df)_{(0,\lambda)}$ commutes with Γ.

Invariant Theory

We say that a real valued function $g : V \to \mathbf{R}$ is Γ-*invariant* if

$$g(\gamma \cdot v) = g(v),$$

for all $\gamma \in \Gamma$ and $v \in V$.

We denote by $\mathcal{P}(\Gamma)$ $(\mathcal{E}(\Gamma))$ the ring of Γ-invariant polynomials (C^∞ functions) from V to $\mathbf{R}$.

If there is a finite set of Γ-invariant polynomials such that every Γ-invariant polynomial may be written as a polynomial function of them, this set is said to generate or to form a *Hilbert basis* of $\mathcal{P}(\Gamma)$.

Theorem 3.1.2 (Theorem of Hilbert-Weyl) *Let Γ be a compact Lie group acting on V. Then there exists a finite Hilbert basis of $\mathcal{P}(\Gamma)$.*

Proof: See Golubitsky *et al.* [21] Theorem XII 4.2. □

Theorem 3.1.3 (Theorem of Schwarz) *Let Γ be a compact Lie group acting on V. Let $\{\mu_1, \ldots, \mu_s\}$ be an Hilbert basis for $\mathcal{P}(\Gamma)$. Let $f \in \mathcal{E}(\Gamma)$. Then there exists a smooth germ $h : \mathbf{R}^s \to \mathbf{R}$ such that*

$$f(v) = h\left(\mu_1(v), \ldots, \mu_s(v)\right).$$

Proof: See Schwarz [30]. □

Definition 3.1.4 (i) A set of Γ-invariant polynomials $\{\mu_1(v), \ldots, \mu_s(v)\}$ has a *relation* if there exists a nonzero polynomial $r\left(y_1, \ldots, y_s\right)$ such that $r\left(\mu_1(v), \ldots, \mu_s(v)\right) \equiv 0$.
(ii) The ring $\mathcal{P}(\Gamma)$ is a *polynomial ring* if it has a Hilbert basis of $\mathcal{P}(\Gamma)$ without relations.

Lemma 3.1.5 *Let $f \in \mathcal{E}(\Gamma)$ and let g be a Γ-equivariant function from V to V. Then $f \cdot g : V \to V$ is Γ-equivariant.*

Proof: Let $v \in V$ and $\gamma \in \Gamma$. Then

$$(fg)(\gamma \cdot v) = f(\gamma \cdot v) g(\gamma \cdot v) = f(v)\left(\gamma \cdot g(v)\right) = \gamma \cdot \left(f\left(v\right) g\left(v\right)\right) = \gamma \cdot \left(fg\right)\left(v\right).$$

□

Denote by $\overrightarrow{\mathcal{P}_V}(\Gamma)$ the real vector space of Γ-equivariant polynomial mappings of V into V, and let $\overrightarrow{\mathcal{E}_V}(\Gamma)$ be the space of Γ-equivariant smooth mappings of V into V.

We say that the Γ-equivariant polynomial mappings $g_1, \ldots, g_r$ of V into V *generate* the module $\overrightarrow{\mathcal{P}_V}(\Gamma)$ (or $\overrightarrow{\mathcal{E}_V}(\Gamma)$) over the ring $\mathcal{P}(\Gamma)$ (or $\mathcal{E}(\Gamma)$) if every $f \in \overrightarrow{\mathcal{P}_V}(\Gamma)$ $(\overrightarrow{\mathcal{E}_V}(\Gamma))$ may be written as $f = f_1 g_1 + \cdots + f_r g_r$ where $f_j \in \mathcal{P}(\Gamma)$ $(\mathcal{E}(\Gamma))$.

Theorem 3.1.6 ([21]) *Let Γ be a compact Lie group acting on V. Then there exists a finite set of Γ-equivariant mappings from V to V with polynomials components that generate $\overrightarrow{\mathcal{P}_V}(\Gamma)$ over the ring $\mathcal{P}(\Gamma)$.*

Proof: See Golubitsky *et al.* [21] Theorem XII 5.2. $\square$

Theorem 3.1.7 (Poénaru) *Let Γ be a compact Lie group with a linear action defined on V and suppose that $g_1, \ldots, g_r$ generate the module $\overrightarrow{\mathcal{P}_V}(\Gamma)$ over the ring $\mathcal{P}(\Gamma)$. Then $g_1, \ldots, g_r$ generate $\overrightarrow{\mathcal{E}_V}(\Gamma)$ over the ring $\mathcal{E}(\Gamma)$.*

Proof: See Poénaru [28]. $\square$

Fixed-point Subspaces

The *fixed-point subspace* of a subgroup Σ of Γ is the vector subspace of V defined by

$$\mathrm{Fix}\,(\Sigma) = \{v \in V : \ \gamma \cdot v = v, \ \forall \gamma \in \Sigma\}\,.$$

Remark 3.1.8 If $f : V \to V$ is Γ-equivariant and $\Sigma \subseteq \Gamma$ then

$$f\,(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma).$$

To prove this, note that, given $v \in \mathrm{Fix}\,(\Sigma)$ and $\gamma \in \Sigma$ then $\gamma \cdot f(v) = f(\gamma \cdot v) = f(v)$. Thus $f\,(v) \in \mathrm{Fix}\,(\Sigma)$.

Irreducibility

A subspace $W \subseteq V$ is called Γ-*irreducible* if it is Γ-invariant (that is, $\gamma \cdot w \in W$ for all $\gamma \in \Gamma$, $w \in W$) and the only Γ-invariant subspaces of W are $\{0_V\}$ and W. A Γ-invariant subspace W of V such that the only linear mappings from W to W that commute with Γ are the scalar multiples of identity is called Γ-*absolutely irreducible*. In fact, if W is Γ-absolutely irreducible then it is Γ-irreducible (see Golubitsky *et al.* [21] Lemma XII 3.3). Two Γ-invariant vector spaces W_1, W_2 are Γ-isomorphic if the corresponding representations, say ρ_1 and ρ_2, are equivalent. That is, there exists an invertible linear transformation S from W_2 to W_1 such that $\rho_1(\gamma) = S\rho_2(\gamma)S^{-1}$ for all $\gamma \in \Gamma$.

Theorem 3.1.9 (Theorem of Complete Reducibility [21]) *Let Γ be a compact Lie Group acting on V. There exist Γ-irreducible subspaces $V_1, \ldots, V_s$ of V such that*

$$V = V_1 \oplus \cdots \oplus V_s.$$

Proof: See Golubitsky *et al.* [21] Corollary XII 2.2. $\square$

Theorem 3.1.10 ([21]) *Let Γ be a compact Lie group acting on the vector space V.*
(a) *Up to Γ-isomorphism there are a finite number of distinct Γ-irreducible subspaces of V. Call these $U_1, \ldots, U_k$.*
(b) *Define W_j to be the sum of all Γ-irreducible subspaces W of V such that W is Γ-isomorphic to U_j, for $j = 1, \ldots, k$. Then $V = W_1 \oplus \cdots \oplus W_k$.*

Proof: See Golubitsky *et al.* [21] Theorem XII 2.5. $\square$

The subspaces W_j, $j = 1, \ldots, k$ are called the *isotypic components* of V of the type U_j for the action of Γ on V. The isotypic decomposition is unique.

Theorem 3.1.11 ([21]) *Let Γ be a compact Lie group acting on the vector space V. Decompose V into isotypic components*

$$V = W_1 \oplus \cdots \oplus W_k.$$

Let $A : V \to V$ be a linear mapping commuting with Γ. Then

$$A\left(W_i\right) \subseteq W_i$$

for $i = 1, \ldots, k$.

Proof: See Golubitsky *et al.* [21] Theorem XII 3.5. $\square$

3.2 Equivariant Hopf Theorem

We say that the system (3.1) presents a Hopf bifurcation at $\lambda = 0$ if $(df)_{(0,0)}$ has a pair of purely imaginary eigenvalues. We saw in section 2.2 that under additional hypotheses of nondegeneracy the system (2.1) has a branch of periodic solutions bifurcating at $\lambda = 0$ from the trivial equilibrium.

In this section we describe necessary conditions to the occurrence of Hopf bifurcation in systems of ODEs where f commutes with a symmetry group Γ. Although the symmetry in many aspects complicate the study, there are techniques that simplify the analysis of symmetric bifurcation problems by exploiting the symmetries of the problem.

Conditions for Pure Imaginary Eigenvalues

We begin by showing that when f commutes with a symmetry group Γ and $(df)_{(0,0)}$ has a pair of purely imaginary eigenvalues, then the symmetry imposes restrictions to the imaginary eigenspace associated to that pair of purely imaginary eigenvalues.

Definition 3.2.1 A representation V of Γ is Γ-*simple* if either:
(a) $V \cong W \oplus W$ where W is absolutely irreducible for Γ, or
(b) V is non-absolutely irreducible for Γ.

We will see that if $(df)_{(0,0)}$ has purely imaginary eigenvalues then there must be a Γ-simple subspace of V.

Examples 3.2.2 (a) Let $\Gamma = \mathbf{O}(2)$ and consider the standard action of Γ on $V = \mathbf{R}^2 \cong \mathbf{C}$ given by

$$\begin{aligned} &\theta \cdot z = e^{i\theta} z \quad (\theta \in \mathbf{SO}(2)), \\ &k \cdot z = \bar{z}. \end{aligned}$$

This action is absolutely irreducible since the only commuting matrices with this action of Γ are the real scalars of the identity I_2. Here I_2 is the 2×2 identity matrix. Hence, Hopf bifurcation cannot occur in a system of ODEs $\dot{v} = f(v, \lambda)$, where $f : V \times \mathbf{R} \to V$ commutes with the standard action of $\mathbf{O}(2)$ on V.
(b) Suppose, now, that $\mathbf{O}(2)$ acts on $\mathbf{R}^4 \cong \mathbf{C}^2$ by the diagonal action

$$\gamma \cdot (z_1, z_2) = (\gamma \cdot z_1, \gamma \cdot z_2) \tag{3.3}$$

where γ acts on z_1 and z_2 as defined in (a). Then the commuting linear mappings with this action of Γ are of the form

$$\begin{bmatrix} aI_2 & bI_2 \\ cI_2 & dI_2 \end{bmatrix}$$

where $a, b, c, d \in \mathbf{R}$. The eigenvalues of this matrix are those of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

repeated twice. If we take $a = d = 0$ and $b = -1$, $c = 1$, we obtain eigenvalues $\pm i$ (twice). Hence, Hopf bifurcation with $\mathbf{O}(2)$ symmetry, where $\mathbf{O}(2)$ acts on $\mathbf{C}^2$ as in (3.3), is possible. Note that we are forced to have multiple eigenvalues $\pm i$ in this case. $\diamond$

We now analyse the general case by considering an arbitrary linear mapping L commuting with the group Γ. Note that any such L can be realized in the form $(df)_{(0,0)}$ for a Γ-equivariant f – for example, $f(v,\lambda) = Lv$. Thus, Hopf bifurcation can occur only when some commuting mapping L has purely imaginary eigenvalues. Given an action of Γ on $\mathbf{R}^n$, we may decompose $\mathbf{R}^n$ into a direct sum of irreducible Γ-invariant subspaces (recall Theorem 3.1.9):

$$\mathbf{R}^n = V_1 \oplus \cdots \oplus V_k. \tag{3.4}$$

Lemma 3.2.3 ([21]) *Let $L : \mathbf{R}^n \to \mathbf{R}^n$ be a linear map having a nonreal eigenvalue and commuting with Γ. Then either:*
(a) *Some absolutely irreducible representation of Γ occurs at least twice (up to Γ-isomorphism) in the decomposition* (3.4)*, or*
(b) *The action of Γ on some V_j is not absolutely irreducible.*

Proof: Suppose that do not occur neither (a) or (b). Then, all the V_i are absolutely irreducible and nonisomorphic by Γ. Theorem 3.1.11 implies that $L(V_j) \subseteq V_j$ for all j, and by absolute irreducibility $L_{|V_j} = \mu_j I$, where $\mu_j \in \mathbf{R}$. Hence, the eigenvalues of L are just the μ_j, and these are real, contrary to assumption. □

If $f : V \times \mathbf{R} \to V$ is a Γ-equivariant map, then $(df)_{(0,0)}$ is a Γ-equivariant linear map (recall Remark 3.1.1). To occur Hopf bifurcation, the matrix $(df)_{(0,0)}$ must have purely imaginary eigenvalues. By Lemma 3.2.3, there must exist a Γ-simple subspace of V.

We introduce now some notation for eigenspaces. Suppose that $L : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map and $\mu \in \mathbf{C}$ is an eigenvalue of L. We define (real) *eigenspace* E_μ and *generalized eigenspace* G_μ of L as follows:

$$E_\mu = \begin{cases} \{x \in \mathbf{R}^n : (L - \mu I)\, x = 0\}, & \text{if } \mu \in \mathbf{R} \\ \{x \in \mathbf{R}^n : (L - \mu I)\,(L - \bar{\mu} I)\, x = 0\}, & \text{if } \mu \notin \mathbf{R}. \end{cases}$$

$$G_\mu = \begin{cases} \{x \in \mathbf{R}^n : (L - \mu I)^n\, x = 0\}, & \text{if } \mu \in \mathbf{R} \\ \{x \in \mathbf{R}^n : (L - \mu I)^n\,(L - \bar{\mu} I)^n\, x = 0\}, & \text{if } \mu \notin \mathbf{R}. \end{cases}$$

We also define *imaginary eigenspace* of L to be the sum of all E_μ for which μ is purely imaginary.

In studying the bifurcation problem (3.1), where f commutes with a compact group Γ, it is important to know how the eigenvalues of $(df)_{0,\lambda}$ cross the imaginary axis at $\lambda = 0$ and to describe the structure of the associated eigenspace. By Lemma 3.2.3 it follows that if L has a purely imaginary eigenvalue, then $\mathbf{R}^n$ must contain a Γ-simple invariant subspace. Further, this subspace must lie in the imaginary eigenspace of L. The next proposition states that, generically, the imaginary eigenspace is itself Γ-simple:

Proposition 3.2.4 ([21]) *Let $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ be a Γ-equivariant bifurcation problem with $f(0,0) = 0$. Suppose that $(df)_{(0,0)}$ has purely imaginary eigenvalues $\pm i\omega$. Let $G_{i\omega}$ be the corresponding real generalized eigenspace of $(df)_{(0,0)}$. Then generically $G_{i\omega}$ is Γ-simple. Moreover, $G_{i\omega} = E_{i\omega}$.*

Proof: See Golubitsky *et al.* [21] Proposition XVI 1.4. □

In the abstract discussion of symmetric Hopf bifurcation we may, without loss of generality, assume that *all* the eigenvalues of $(df)_{(0,0)}$ are on the imaginary axis and 0 is not an eigenvalue of $(df)_{(0,0)}$. Generically, we may consider that $(df)_{(0,0)}$ has only one pair of complex conjugate eigenvalues on the imaginary axis perhaps of high multiplicity. See [21] page 265.

The next lemma shows that we may assume $(df)_{(0,0)} = J$, where

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix} \tag{3.5}$$

and $m = n/2$ (if we assume that $\mathbf{R}^n$ is Γ-simple):

Lemma 3.2.5 ([21]) *Assume that $\mathbf{R}^n$ is Γ-simple, f is Γ-equivariant and C^∞. Suppose that $(df)_{(0,0)}$ has i as an eigenvalue. Then:*
(a) *The eigenvalues of $(df)_{0,\lambda}$ consist of a complex conjugate pair $\sigma(\lambda) \mp i\rho(\lambda)$, each of multiplicity m. Moreover, σ and ρ are smooth functions of λ;*
(b) *There is an invertible linear map $S : \mathbf{R}^n \to \mathbf{R}^n$, commuting with Γ, such that*

$$(df)_{(0,0)} = SJS^{-1}.$$

Proof: We only prove this lemma under the assumption that $\mathbf{R}^n = W \oplus W$ with W absolutely irreducible by Γ, as in Definition 3.2.1 (a) (see the case 3.2.1 (b) in Golubitsky *et al.* [21] Lemma XVI 1.5).

Let L be a linear map $W \oplus W \to W \oplus W$, commuting with Γ. Write L in block form as

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

for $m \times m$ matrices A, B, C, D. Since L commutes with the diagonal action of Γ on W, each A, B, C, D commutes with the action of Γ on V. By absolute irreducibility we have

$$L = \begin{bmatrix} aI_m & bI_m \\ cI_m & dI_m \end{bmatrix}.$$

If A, B, C, D commute (as here) then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det (AD - BC) .$$

Therefore, the characteristic polynomial of L is

$$\det (L - \mu I_n) = [(a - \mu)(d - \mu) - bc]^m . \tag{3.6}$$

Thus, each eigenvalue of L occurs with multiplicity at least m. Now $(df)_{(0,0)}$ commutes with Γ and has a pair of (nonzero) complex conjugate purely imaginary eigenvalues. Therefore, each occurs with multiplicity m. The smoothness of σ and ρ also follows from (3.6). Let

$$(df)_{(0,0)} = \begin{bmatrix} aI_m & bI_m \\ cI_m & dI_m \end{bmatrix} .$$

For i to be an eigenvalue, (3.6) implies that $a + d = 0$ and $ad - bc = 1$. Assuming $a \neq 0$, define

$$R_\theta = \begin{bmatrix} \cos\theta I_m & -\mathrm{sen}\theta I_m \\ \mathrm{sen}\theta I_m & \cos\theta I_m \end{bmatrix}$$

which commutes with Γ. Choose θ so that $\mathrm{cotg}\,(2\theta) = (b + c)/2a$. Then

$$R_\theta \,(df)_{(0,0)}\, R_\theta^{-1} = \begin{bmatrix} 0 & hI_m \\ -h^{-1}I_m & 0 \end{bmatrix}$$

for $h \in \mathbf{R}$. Finally note that

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -hI_m \end{bmatrix} \begin{bmatrix} 0 & hI_m \\ -h^{-1}I_m & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & -h^{-1}I_m \end{bmatrix} .$$

Thus $S = R_\theta^{-1} \begin{bmatrix} I_m & 0 \\ 0 & -h^{-1}I_m \end{bmatrix}$ provides the required similarity. $\square$

It follows then that one of the difficulties in applying the Standard Hopf Theorem to the system (3.1) when f commutes with a compact Lie group is that the purely imaginary eigenvalues of $(df)_{(0,0)}$ may have high multiplicity.

Symmetry of a Periodic Solution

One method for finding periodic solutions to a system (3.1) rests on prescribing in advance the symmetry of solutions we seek. In many cases, this corresponds to reduce the original problem to a lower-dimensional problem – a subspace where the eigenvalues of the linearization of the restricted problem are simple. The main result we present in this section in this direction is the Equivariant Hopf Theorem. Before we state it we describe precisely the concept of symmetry of a periodic solution and the context in which to select the subspace.

Suppose that we have a system of ODEs

$$\dot{v} = f(v, \lambda) \tag{3.7}$$

with $v \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is the bifurcation parameter, $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is smooth and commutes with a compact Lie group Γ and $f(0, \lambda) \equiv 0$ for all $\lambda \in \mathbf{R}$.

Identify the circle $\mathbf{S}^1$ with $\mathbf{R}/2\pi\mathbf{Z}$ and suppose that $v(t)$ is a 2π-periodic solution in t of (3.7). The *symmetry* of $v(t)$ is an element $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ such that

$$\gamma \cdot v(t) = v(t - \theta);$$

that is, the spacial action of γ on V may be exactly compensated by a phase shift. Note that $\mathbf{S}^1$ acts on the space of 2π-mappings $v(t)$, not in $\mathbf{R}^n$. We call this action of $\mathbf{S}^1$ the *phase-shift* action. The collection of all symmetries for $v(t)$ forms a subgroup:

$$\Sigma_{v(t)} = \{(\gamma, \theta) \in \Gamma \times \mathbf{S}^1 : \ \gamma \cdot v(t) = v(t - \theta)\}.$$

There is a natural action of $\Gamma \times \mathbf{S}^1$ on $C_{2\pi}$ of 2π-periodic mappings from $\mathbf{R}$ to $\mathbf{R}^n$, defined by

$$(\gamma, \theta) \cdot v(t) = \gamma \cdot v(t + \theta).$$

That is, the action of Γ on $C_{2\pi}$ is induced from its spacial action on $\mathbf{R}^n$ and $\mathbf{S}^1$ acts by phase shift. Consequently we can rewrite the definition of a symmetry of the periodic solution $v(t)$ as: $(\gamma, \theta) \cdot v(t) = v(t)$. This shows that $\Sigma_{v(t)}$ is just the isotropy subgroup of $v(t)$ with respect to this action of $\Gamma \times \mathbf{S}^1$.

If we assume in (3.7) that $(df)_{(0,0)} = L$ has purely imaginary eigenvalues, we can apply a Liapunov-Schmidt reduction preserving symmetries that will induce a different action of $\mathbf{S}^1$ on a finite-dimensional space, which can be identified with the exponential of $L|_{E_i(L)}$ acting on the imaginary eigenspace $E_i(L)$ of L. The reduced function of f will commute with $\Gamma \times \mathbf{S}^1$ (See Golubitsky *et al.* [21] p. 270-275).

Hopf Theorem with Symmetry

An isotropy subgroup $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-*axial* if $\dim \mathrm{Fix}_{E_i(L)}(\Sigma) = 2$. Basically, the Equivariant Hopf Theorem states that for each $\mathbf{C}$-axial isotropy subgroup of $\Gamma \times \mathbf{S}^1$ there exists a unique branch of periodic solutions of (3.7) with that symmetry (assuming the nondegeneracy crossing condition of the eigenvalues). To state this result we need both actions of $\mathbf{S}^1$ described above.

Theorem 3.2.6 (Equivariant Hopf Theorem) *Consider the system of ODEs*

$$\frac{dv}{dt} = f(v, \lambda) \tag{3.8}$$

where $f : \mathbf{R}^{2m} \times \mathbf{R} \to \mathbf{R}^{2m}$ *is smooth and commutes with a compact Lie group* Γ.

Assume the generic hypothesis that $\mathbf{R}^{2m}$ *is* Γ*-simple and that* $(df)_{(0,0)} = J$ *where* J *is defined in* (3.5). *Using Lemma 3.2.5, the eigenvalues of* $(df)_{0,\lambda}$ *are of the form* $\sigma(\lambda) \mp i\rho(\lambda)$, *each with multiplicity* m. *Therefore* $\sigma(0) = 0$ *and* $\rho(0) = 1$. *Assume now that*

$$\sigma'(0) \neq 0,$$

that is, the eigenvalues of $(df)_{0,\lambda}$ *cross the imaginary axis with nonzero speed.*

Let $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ *be an isotropy subgroup such that*

$$\dim \mathrm{Fix}(\Sigma) = 2.$$

Then there exists a unique branch of small-amplitude periodic solutions to (3.8) *with period near* 2π *having* Σ *as their group of symmetries.*

Proof: See Golubitsky *et al.* [21] Theorem XVI 4.1 or Golubitsky and Stewart [19] p.91. □

The main idea in the Equivariant Hopf Theorem is that small-amplitude periodic solutions of (3.8) of period near 2π correspond to zeros of a reduced equation $\phi(v, \lambda, \tau) = 0$ where τ is the period-perturbing parameter. Finding periodic solutions of (3.8) with symmetries Σ is equivalent to find zeros of the reduced equation with isotropy Σ and they correspond to the zeros of the reduced equation restricted to $\mathrm{Fix}(\Sigma)$.

Remark 3.2.7 The solutions guaranteed by the Equivariant Hopf Theorem are not necessarily the only ones.

Birkhoff Normal Form

A tool for seeking periodic solutions that are not guaranteed by the Equivariant Hopf Theorem and also for calculating the stabilities of the periodic solutions is to use a *Birkhoff normal form* of f: by a suitable coordinate change, up to any given order k, the vector field f can be made to commute not only with Γ but also with $\mathbf{S}^1$ (in the Hopf case) in the case of $(df)_{(0,0)} = J$. See Golubitsky *et al.* [21] Theorems XVI 5.8 and XVI 5.9. The dynamics of the truncated Birkhoff normal form of f, say of order k, are related but are not equal to the dynamics of f. On the other hand, in general, it is not possible to find a single change of coordinates that puts f into normal form for all orders. If we assume that the original vector field is in Birkhoff normal form (it commutes also with $\mathbf{S}^1$) then it is valid the following result:

Theorem 3.2.8 ([21]) *Suppose that vector the field f in* (3.8) *is in Birkhoff normal form. Then it is possible to perform a Liapunov-Schmidt reduction on* (3.8) *such that the reduced equation ϕ has the form*

$$\phi\,(v, \lambda, \tau) = f\,(v, \lambda) - (1 + \tau)Jv$$

where τ is the period-scaling parameter and $J = (df)_{(0,0)}$.

Proof: See Golubitsky *et al.* [21] Theorem XVI 10.1. $\square$

Chapter 4

Hopf Bifurcation with $\mathbf{D}_n$-symmetry

A paper with the contents of this chapter has been published [8].

The aim of this chapter is to study Hopf bifurcation with $\mathbf{D}_n$-symmetry assuming Birkhoff normal form. Here, $\mathbf{D}_n$ is the dihedral group of order $2n$ and we consider $\mathbf{D}_n$ acting on $\mathbf{C} \equiv \mathbf{R}^2$ as symmetries of the regular n-gon. This representation is absolutely irreducible and so the corresponding Hopf bifurcation occurs on $\mathbf{C} \oplus \mathbf{C} \cong \mathbf{C}^2$. Golubitsky *et al.* [18] and van Gils *et al.* [35] (see also Golubitsky *et al.* [21]) prove the generic existence of three branches of periodic solutions, up to conjugacy, in systems of ordinary differential equations with $\mathbf{D}_n$-symmetry, depending on one real parameter, that present Hopf bifurcation. These solutions are found by using the Equivariant Hopf Theorem. We prove that generically, when $n \neq 4$, these are the *only* branches of periodic solutions that bifurcate from the trivial solution.

This chapter has three sections organized in the following way. Section 4.1 defines the standard action of $\mathbf{D}_n \times \mathbf{S}^1$ on $V = \mathbf{C}^2$ and the corresponding isotropy lattice. We describe the conjugacy classes of $\mathbf{D}_n \times \mathbf{S}^1$ (with action on V) obtained by Golubitsky *et al.* [21]. For each n, there are five conjugacy classes and three of them correspond to $\mathbf{C}$-axial isotropy subgroups, that is, isotropy subgroups with two-dimensional subspaces. In section 4.2 we find the general form of the vector field f that commute with the standard action of $\mathbf{D}_n$ on $\mathbf{C}^2$. We assume that f is in Birkhoff normal form to all orders and so f commutes also with $\mathbf{S}^1$. Specifically, we choose coordinates such that

$$\theta \cdot z = e^{i\theta} z \quad \left(\theta \in \mathbf{S}^1,\ z \in V\right).$$

Finally in Section 4.3 we obtain the main result of the chapter – Theorem 4.3.2. We prove that when $n \neq 4$ and $n \geq 3$ generically the only branches of small-amplitude periodic solutions in systems of ODEs with $\mathbf{D}_n$ symmetry, depending on one parameter, that bifurcate from the trivial equilibrium are those guaranteed by the Equivariant Hopf Theorem (Theorem 3.2.6). The proof of this theorem relies mostly in the general form of f and the use of Morse Lemma.

4.1 The Action of $\mathbf{D}_n \times \mathbf{S}^1$

In this section we define the standard action of $\mathbf{D}_n \times \mathbf{S}^1$ on $\mathbf{C}^2$, and give the corresponding isotropy lattice. We follow Golubitsky *et al.* [21], Chapter XVIII.

Let us assume that $\Gamma = \mathbf{D}_n$ where $n \geq 3$ acts on $\mathbf{C} \equiv \mathbf{R}^2$ in the standard way as symmetries of the regular n-gon. This action is generated by

$$\begin{array}{rcl} \zeta \cdot z & = & e^{i\zeta} z \quad \text{ where } \zeta = 2\pi/n, \\ \kappa \cdot z & = & \overline{z}. \end{array}$$

Thus the cyclic subgroup $\mathbf{Z}_n$ of $\mathbf{D}_n$ consists of rotations of the plane through the angles $0, \zeta, 2\zeta, \ldots, (n-1)\zeta$, the *flip* κ is reflection in the x-axis and $\mathbf{D}_n = \langle \zeta, \kappa \rangle$. Although $\mathbf{D}_n$ has many distinct two-dimensional irreducible representations there is no loss of generality in making this assumption. Essentially it is possible to arrange for a standard action by relabeling the group elements and dividing by the kernel of the action.

Let $V = \mathbf{C}^2$. Suppose now that Γ acts on V by the diagonal action

$$\gamma \cdot (z_1, z_2) = (\gamma \cdot z_1, \gamma \cdot z_2) \quad (\gamma \in \mathbf{D}_n).$$

Note that $\mathbf{C}$ is absolutely irreducible for $\mathbf{D}_n$ and so V is $\mathbf{D}_n$-simple. It is possible to choose coordinates on V such that the action of $\mathbf{D}_n$ on V is generated by

$$\begin{array}{rcl} \zeta \cdot (z_1, z_2) & = & \left(e^{i\zeta} z_1, e^{-i\zeta} z_2\right), \\ \kappa \cdot (z_1, z_2) & = & (z_2, z_1) \end{array} \tag{4.1}$$

(see [21] page 368).

Suppose we have a system of ODEs

$$\dot{x} = f(x, \lambda) \tag{4.2}$$

where $x \in V$, $\lambda \in \mathbf{R}$ is the bifurcation parameter and $f : V \times \mathbf{R} \to V$ is smooth and commutes with $\mathbf{D}_n$. Note that since $\mathrm{Fix}_V(\mathbf{D}_n) = \{0\}$ then as $f(\mathrm{Fix}_V(\mathbf{D}_n)) \subseteq \mathrm{Fix}_V(\mathbf{D}_n)$ we have $f(0, \lambda) \equiv 0$. We assume that $(df)_{(0,0)}$ has eigenvalues $\mp i$. Our aim is to study the generic existence of branches of periodic solutions of (4.2) near the bifurcation point $(x, \lambda) = (0, 0)$. We assume that f is in Birkhoff normal form, that is, f also commutes with $\mathbf{S}^1$, where we may assume that $\mathbf{S}^1$ acts on V by

$$\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2) \quad \left(\theta \in \mathbf{S}^1\right). \tag{4.3}$$

The Isotropy Lattice

Consider the subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ defined by

$$\begin{array}{ll} \widetilde{\mathbf{Z}}_n = \{(\gamma, -\gamma) : \ \gamma \in \mathbf{Z}_n\}, & \mathbf{Z}_2(\kappa) = \{\mathbf{1}, \kappa\}, \\ \mathbf{Z}_2(\kappa, \pi) = \{\mathbf{1}, (\kappa, \pi)\}, & \mathbf{Z}_2(\kappa\zeta) = \{\mathbf{1}, \kappa\zeta\} \end{array} \tag{4.4}$$

where $\zeta = 2\pi/n$ and so $\mathbf{Z}_n = \langle\zeta\rangle$. In the next proposition we describe the isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ and the corresponding fixed-point subspaces.

Proposition 4.1.1 ([21]) *Let $V = \mathbf{C}^2$ and consider the action of $\mathbf{D}_n \times \mathbf{S}^1$ ($n \geq 3$) on V given by* (4.1) *and* (4.3). *For each n there are five conjugacy classes of isotropy subgroups for this action. They are listed, together with their orbit representatives and fixed-point subspaces in Tables 4.1, 4.2 and 4.3.*

Proof: See Golubitsky *et al.* [21], pp. 368-371. □

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(0,0)$	$\mathbf{D}_n \times \mathbf{S}^1$	$\{(0,0)\}$
$(a,0)$	$\widetilde{\mathbf{Z}}_n$	$\{(w,0) : w \in \mathbf{C}\}$
(a,a)	$\mathbf{Z}_2(\kappa)$	$\{(w,w) : w \in \mathbf{C}\}$
$(a,-a)$	$\mathbf{Z}_2(\kappa,\pi)$	$\{(w,-w) : w \in \mathbf{C}\}$
(a,w), $w \neq \pm a, 0$	$\mathbf{1}$	$\mathbf{C}^2$

Table 4.1: Isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ acting on $\mathbf{C}^2$ when n is odd.

Up to conjugacy, for each n, we have three isotropy subgroups with two-dimensional fixed-point subspaces. It follows from the Equivariant Hopf Theorem (Theorem 3.2.6), that there are (*at least*) three branches of periodic solutions occurring generically in Hopf bifurcation with $\mathbf{D}_n$-symmetry. That is, to each isotropy subgroup Σ of $\mathbf{D}_n \times \mathbf{S}^1$ with two-dimensional fixed-point subspace corresponds a unique branch of periodic solutions of (4.2) with period near 2π and with symmetry Σ, obtained by bifurcation from the trivial equilibrium (assuming that f satisfies the conditions of the cited theorem). Let us notice, however, that the periodic solutions whose existence

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(0,0)$	$\mathbf{D}_n \times \mathbf{S}^1$	$\{(0,0)\}$
$(a,0)$	$\widetilde{\mathbf{Z}}_n$	$\{(w,0) : w \in \mathbf{C}\}$
(a,a)	$\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c$	$\{(w,w) : w \in \mathbf{C}\}$
$(a,-a)$	$\mathbf{Z}_2(\kappa,\pi) \oplus \mathbf{Z}_2^c$	$\{(w,-w) : w \in \mathbf{C}\}$
(a,w), $w \neq \pm a, 0$	$\mathbf{Z}_2^c$	$\mathbf{C}^2$

Table 4.2: Isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ acting on $\mathbf{C}^2$ when $n \equiv 2 \pmod 4$. Here $\mathbf{Z}_2^c = \{(0,0), (\pi,\pi)\}$.

Orbit representative	Isotropy subgroup	Fixed-point subspace
$(0,0)$	$\mathbf{D}_n \times \mathbf{S}^1$	$\{(0,0)\}$
$(a,0)$	$\widetilde{\mathbf{Z}}_n$	$\{(w,0) : w \in \mathbf{C}\}$
(a,a)	$\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c$	$\{(w,w) : w \in \mathbf{C}\}$
$\left(a, e^{2\pi i/n}a\right)$	$\mathbf{Z}_2(\kappa\zeta) \oplus \mathbf{Z}_2^c$	$\left\{\left(w, e^{2\pi i/n}w\right) : w \in \mathbf{C}\right\}$
(a,w), $w \neq \pm a, 0$	$\mathbf{Z}_2^c$	$\mathbf{C}^2$

Table 4.3: Isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ acting on $\mathbf{C}^2$ when $n \equiv 0 \pmod 4$. Here $\mathbf{Z}_2^c = \{(0,0), (\pi,\pi)\}$.

is guaranteed by the Equivariant Hopf Theorem are not necessarily the only periodic solutions that bifurcate from $(0,0)$. In Theorem 4.3.2 of section 4.3 we prove that when $n \neq 4$ generically these are the only branches of periodic solutions of (4.2) assuming that f is in Birkhoff normal form.

4.2 Invariant Theory for $\mathbf{D}_n \times \mathbf{S}^1$

In order to look for periodic solutions of (4.2) we calculate now the general form of a $\mathbf{D}_n \times \mathbf{S}^1$-equivariant bifurcation problem. Recall the action of $\mathbf{D}_n \times \mathbf{S}^1$ on $\mathbf{C}^2$ generated by (4.1) and (4.3), and define

$$m = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases} \tag{4.5}$$

Proposition 4.2.1 ([21]) *Let $n \geq 3$ and let m be as in* (4.5)*. Then*
(a) *Every smooth $\mathbf{D}_n \times \mathbf{S}^1$-invariant function $f: \mathbf{C}^2 \to \mathbf{R}$ has the form*

$$f(z_1, z_2) = h(N, P, S, T)$$

where $N = |z_1|^2 + |z_2|^2$, $P = |z_1|^2|z_2|^2$, $S = (z_1\overline{z}_2)^m + (\overline{z}_1 z_2)^m$,

$$T = i\left(|z_1|^2 - |z_2|^2\right)\left((z_1\overline{z}_2)^m - (\overline{z}_1 z_2)^m\right)$$

and $h:\ \mathbf{R}^4 \to \mathbf{R}$ *is smooth.*
(b) *Every smooth* $\mathbf{D}_n \times \mathbf{S}^1$*-equivariant function* $f:\ \mathbf{C}^2 \to \mathbf{C}^2$ *has the form*

$$f(z_1,z_2) = A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B\begin{bmatrix} z_1^2\overline{z}_1 \\ z_2^2\overline{z}_2 \end{bmatrix} + C\begin{bmatrix} \overline{z}_1^{m-1} z_2^m \\ z_1^m \overline{z}_2^{m-1} \end{bmatrix} + D\begin{bmatrix} z_1^{m+1}\overline{z}_2^m \\ \overline{z}_1^m z_2^{m+1} \end{bmatrix}$$

where A, B, C, D *are complex-valued* $\mathbf{D}_n \times \mathbf{S}^1$*-invariant smooth functions.*

Remark 4.2.2 The $\mathbf{D}_n \times \mathbf{S}^1$-invariant polynomials do not form a polynomial ring. There is a relation:

$$T^2 = (4P - N^2)(S^2 - 4P^m). \tag{4.6}$$

Proof: See appendix A. □

4.3 Generic Hopf Bifurcation with $\mathbf{D}_n$-symmetry

In section 4.1 we determined the conjugacy classes of isotropy subgroups for the action of $\mathbf{D}_n \times \mathbf{S}^1$ on $V = \mathbf{C}^2$ (Proposition 4.1.1). Up to conjugacy, for each $n \geq 3$, we have three isotropy subgroups with two-dimensional fixed-point subspaces. It follows from the Equivariant Hopf Theorem, that there are (*at least*) three branches of periodic solutions corresponding to each one of these isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ occurring in generic Hopf bifurcation with $\mathbf{D}_n$-symmetry. We prove in Theorem 4.3.2 bellow that when $n \neq 4$ generically these are the only branches of periodic solutions obtained through bifurcation from the trivial equilibrium in one-parameter bifurcation problems with $\mathbf{D}_n$-symmetry (assuming Birkhoff normal form).

Suppose that the function $f:\ V \times \mathbf{R} \to V$ is $\mathbf{D}_n \times \mathbf{S}^1$-equivariant and smooth, and satisfies the conditions of the Equivariant Hopf Theorem. Thus we assume that

$$(df)_{0,\lambda}(z) = \mu(\lambda) z \tag{4.7}$$

where μ is a smooth function from $\mathbf{R}$ to $\mathbf{C}$ such that

$$\mu(0) = i, \quad \mathrm{Re}(\mu'(0)) \neq 0. \tag{4.8}$$

From Theorem 3.2.8 the small-amplitude periodic solutions of the equation

$$\dot{z} = f(z,\lambda) \tag{4.9}$$

of period near 2π are in one-to-one correspondence with the zeros of the equation

$$g(z, \lambda, \tau) = 0 \tag{4.10}$$

where $g = f - (1 + \tau)iz$ and τ is the period-scaling parameter. From Proposition 4.2.1 the general form of $f = (f_1, f_2)$ is

$$\begin{array}{rcl} f_1(z_1, z_2, \lambda) & = & \mu(\lambda)z_1 + Az_1 + Bz_1^2\overline{z}_1 + C\overline{z}_1^{m-1}z_2^m + Dz_1^{m+1}\overline{z}_2^m \\ f_2(z_1, z_2, \lambda) & = & \mu(\lambda)z_2 + Az_2 + Bz_2^2\overline{z}_2 + Cz_1^m\overline{z}_2^{m-1} + D\overline{z}_1^m z_2^{m+1} \end{array} \tag{4.11}$$

where A, B, C, D are smooth $\mathbf{D}_n \times \mathbf{S}^1$-invariant functions from $V \times \mathbf{R}$ to $\mathbf{C}$ (thus they may depend on λ) and m is defined by (4.5). Since we are assuming (4.7) it follows that $A\,(0, \lambda) \equiv 0$. Let us consider g as in (4.10). From (4.11) g has form

$$\begin{array}{rcl} g_1(z, \lambda, \tau) & = & (\nu + A)\, z_1 + Bz_1^2\overline{z}_1 + C\overline{z}_1^{m-1}z_2^m + Dz_1^{m+1}\overline{z}_2^m \\ g_2(z, \lambda, \tau) & = & (\nu + A)\, z_2 + Bz_2^2\overline{z}_2 + Cz_1^m\overline{z}_2^{m-1} + D\overline{z}_1^m z_2^{m+1} \end{array} \tag{4.12}$$

where $\nu = \mu(\lambda) - (1 + \tau)i$.

Lemma 4.3.1 *Consider f as in* (4.11)*. Let $(z_1, z_2) = \left(r_1 e^{i\phi_1}, r_2 e^{i\phi_2}\right)$ with $r_1, r_2 \in \mathbf{R}$ and let $\phi = \phi_2 - \phi_1$. Then we can write $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ as*

$$\begin{bmatrix} r_1 e^{i\phi_1} h(r_1, r_2, \phi, \lambda) \\ r_2 e^{i\phi_2} h(r_2, r_1, -\phi, \lambda) \end{bmatrix}$$

where h is a smooth function from $\mathbf{R}^4$ to $\mathbf{C}$.

Proof: Let N, P, S and T be as in the Proposition 4.2.1. Taking $(z_1, z_2) = \left(r_1 e^{i\phi_1}, r_2 e^{i\phi_2}\right)$ and $\phi = \phi_2 - \phi_1$ we can write each of the invariant polynomials in the form

$$\begin{array}{rclcrcl} N & = & r_1^2 + r_2^2 & \qquad & P & = & r_1^2 r_2^2 \\ & & & & & & \\ S & = & 2r_1^m r_2^m \cos(m\phi) & & T & = & 2r_1^m r_2^m \sin(m\phi)(r_1^2 - r_2^2). \end{array}$$

Recall now Proposition 4.2.1(b) and denote by

$$X_2 = \begin{bmatrix} z_1^2\overline{z}_1 \\ z_2^2\overline{z}_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} \overline{z}_1^{m-1} z_2^m \\ z_1^m \overline{z}_2^{m-1} \end{bmatrix}, \quad X_4 = \begin{bmatrix} z_1^{m+1}\overline{z}_2^m \\ \overline{z}_1^m z_2^{m+1} \end{bmatrix}.$$

Then

$$X_j = \begin{bmatrix} r_1 e^{i\phi_1} h_j(r_1, r_2, \phi) \\ r_2 e^{i\phi_2} h_j(r_2, r_1, -\phi) \end{bmatrix} \tag{4.13}$$

where

$$
\begin{aligned}
h_2(r_1, r_2, \phi) &= r_1^2 \\
h_3(r_1, r_2, \phi) &= r_1^{m-2} r_2^m \left(\cos(m\phi) + i\sin(m\phi)\right) \\
h_4(r_1, r_2, \lambda) &= r_1^m r_2^m (\cos(m\phi) - i\sin(m\phi)).
\end{aligned}
\tag{4.14}
$$

It follows the result if we consider (4.11). □

Theorem 4.3.2 *Consider* (4.9) *with* f *as in* (4.11) *where* $A(0,\lambda) \equiv 0$ *and* $\mu:\ \mathbf{R} \to \mathbf{C}$ *is smooth and satisfies* (4.8). *Suppose that* $n \neq 4$ *and* $n \geq 3$. *Then, generically, the system* (4.9) *admits only branches of periodic solutions that bifurcate from* $(0,0)$ *corresponding to the isotropy subgroups of* $\mathbf{D}_n \times \mathbf{S}^1$ *with two-dimensional fixed-point subspaces.*

Proof: We have that $\mathrm{Fix}_V(\mathbf{D}_n) = \{0\}$, consequently $f(0,\lambda) \equiv 0$. Therefore $(0,\lambda)$ is an equilibrium point of (4.9) for all values of λ. Since we are assuming that $(df)_{0,\lambda}(z) = \mu(\lambda)z$, where $\mu(0) = i$ and $\mathrm{Re}(\mu'(0)) \neq 0$, the stability of this equilibrium varies when λ crosses zero.

The space V is $\mathbf{D}_n$-simple and we are assuming (4.7) and (4.8) and so the conditions of the Equivariant Hopf Theorem are satisfied. Therefore, for each isotropy subgroup Σ of $\mathbf{D}_n \times \mathbf{S}^1$ with a two-dimensional fixed-point subspace, the system (4.9) admits a unique branch of periodic solutions with symmetry Σ by bifurcation from $(z,\lambda) = (0,0)$. Moreover, this corresponds to a branch of zeros of (4.10) with the corresponding symmetry. We study now the existence of branches of periodic solutions of (4.9) with submaximal symmetry that bifurcate from $(0,0)$. We begin by looking for branches of zeros (z_1, z_2) of (4.12) with $z_1 z_2 \neq 0$. These satisfy

$$
\left\{
\begin{array}{l}
\dfrac{g_1(z,\lambda,\tau)}{z_1} = 0 \\[2ex]
\dfrac{g_2(z,\lambda,\tau)}{z_2} = 0.
\end{array}
\right.
\tag{4.15}
$$

Taking $(z_1, z_2) = \left(r_1 e^{i\phi_1}, r_2 e^{i\phi_2}\right)$ and $\phi = \phi_2 - \phi_1$, by Lemma 4.3.1 we can write f in the form

$$
\left[
\begin{array}{c}
r_1 e^{i\phi_1} h(r_1, r_2, \phi, \lambda) \\
r_2 e^{i\phi_2} h(r_2, r_1, -\phi, \lambda
\end{array}
\right]
$$

and so (4.15) can be written as

$$\begin{cases} \nu + A + Br_1^2 + Cr_1^{m-2} r_2^m \left(\cos(m\phi) + i\sin(m\phi)\right) + D\left(r_1 r_2\right)^m \left(\cos(m\phi) - i\sin(m\phi)\right) = 0 \\ \nu + A + Br_2^2 + Cr_1^m r_2^{m-2} \left(\cos(m\phi) - i\sin(m\phi)\right) + D\left(r_1 r_2\right)^m \left(\cos(m\phi) + i\sin(m\phi)\right) = 0. \end{cases} \tag{4.16}$$

Taking the difference of the equations of (4.16) we obtain

$$B\left(r_1^2 - r_2^2\right) + C\left(r_1 r_2\right)^{m-2} \left(\cos(m\phi)\left(r_2^2 - r_1^2\right) + i\sin(m\phi)\left(r_1^2 + r_2^2\right)\right) - 2iD\left(r_1 r_2\right)^m \sin\left(m\phi\right) = 0 \tag{4.17}$$

and so the real and imaginary parts of (4.17) should verify

$$\begin{cases} \left(r_2^2 - r_1^2\right)\left(C_R\left(r_1 r_2\right)^{m-2}\cos(m\phi) - B_R\right) + \sin(m\phi)\left(r_1 r_2\right)^{m-2}\left(2D_I r_1^2 r_2^2 - C_I\left(r_1^2 + r_2^2\right)\right) = 0 \\ \\ \left(r_2^2 - r_1^2\right)\left(C_I\left(r_1 r_2\right)^{m-2}\cos(m\phi) - B_I\right) + \sin(m\phi)\left(r_1 r_2\right)^{m-2}\left(C_R\left(r_1^2 + r_2^2\right) - 2D_R r_1^2 r_2^2\right) = 0 \end{cases} \tag{4.18}$$

Here we use the notation $B_R = \mathrm{Re}(B),\ B_I = \mathrm{Im}(B), \ldots$

Assume the generic hypothesis

$$B_R(0) \neq 0.$$

Recall that $n \geq 3$ and $n \neq 4$. By (4.5) it follows that $m - 2 \geq 1$ and so

$$\left(B_R - C_R(r_1 r_2)^{m-2}\cos(m\phi)\right)(0) \neq 0.$$

Therefore in a sufficiently small neighborhood of the origin the system (4.18) can be written as

$$\begin{cases} r_2^2 - r_1^2 = \dfrac{\sin(m\phi)(r_1 r_2)^{m-2}(C_I(r_1^2 + r_2^2) - 2D_I r_1^2 r_2^2)}{C_R(r_1 r_2)^{m-2}\cos(m\phi) - B_R} \\ \\ \sin(m\phi)\big(\left(B_I C_I + B_R C_R\right)\left(r_1^2 + r_2^2\right) + P\left(r_1, r_2, \lambda, m\right)\big) = 0 \end{cases} \tag{4.19}$$

where

$$\begin{aligned} P\left(r_1, r_2, \lambda, m\right) \quad = \quad & (r_1 r_2)^{m-2}\cos(m\phi)\big(2(C_I D_I + C_R D_R) r_1^2 r_2^2 - (C_R^2 + C_I^2)(r_1^2 + r_2^2)\big) \\ & -2 r_1^2 r_2^2 (B_R D_R + B_I D_I). \end{aligned}$$

Assume the generic hypothesis

$$\left(B_I C_I + B_R C_R\right)(0) \neq 0.$$

By Morse Lemma (see for example Poston and Stewart [29] Theorem 4.2), the equation

$$(B_I C_I + B_R C_R)\left(r_1^2 + r_2^2\right) + P(r_1, r_2, \lambda, m) = 0$$

in a sufficiently small neighborhood of the origin admits only the trivial solution $(r_1, r_2) = (0, 0)$. Recall (4.5) and note that we are assuming $n \geq 3$ and $n \neq 4$. Thus $m - 2 \geq 1$. It follows that the system (4.19) in a sufficiently small neighborhood of the origin admits only branches of solutions (containing $(r_1, r_2) = (0, 0)$ and) satisfying

$$\begin{cases} \sin(m\phi) = 0 \\ \\ r_1^2 = r_2^2. \end{cases} \tag{4.20}$$

Thus $\phi = \frac{k\pi}{m}$ for some integer k. We show below that these solutions correspond to the branches of periodic solutions of (4.9) guaranteed by the Equivariant Hopf Theorem. Note that the case $n = 4$ and so $m - 2 = 0$ is special. The existence of branches of periodic solutions of (4.9) with submaximal symmetry that bifurcate from $(0, 0)$ in generic Hopf bifurcation with $\mathbf{D}_4$-symmetry is proved by Swift [34].

We show now the correspondence between the solutions of (4.20) and the periodic solutions of (4.9).

(i) We begin with the case when n is odd. We recall that

$$\mathrm{Fix}_{\mathbf{C}^2}\left(\mathbf{Z}_2(k)\right) = \{(w, w) : \ w \in \mathbf{C}\}$$

(see Table 4.1). It follows that $n + 1 \geq 4$ is even and so $k(1 + n)\pi/n \in \mathbf{Z}_n$. Moreover,

$$e^{\frac{ik(1+n)\pi}{n}} \cdot \left(w, e^{\frac{2ik\pi}{n}} w\right) = \left(w e^{\frac{ik(1+n)\pi}{n}}, w e^{\frac{2ik\pi}{n} - \frac{ik(1+n)\pi}{n}}\right)$$

and

$$\frac{2k\pi}{n} - \frac{k(1+n)\pi}{n} - \frac{k(1+n)\pi}{n} = -2k\pi.$$

So, periodic solutions of (4.9) with symmetry (conjugate to) $\mathbf{Z}_2(k)$ correspond to zeros of (4.10) where

$$r_1 = r_2 \ \text{ and } \ \phi = \frac{2k\pi}{n}, \ k \in \mathbf{Z}$$

(or $r_1 = -r_2$ and $\phi = 2k\pi/n - \pi, \ k \in \mathbf{Z}$). In the case of $\mathbf{Z}_2(k, \pi)$, we have

$$\mathrm{Fix}_{\mathbf{C}^2}(\mathbf{Z}_2(k, \pi)) = \{(w, -w) : \ w \in \mathbf{C}\}.$$

Observe that if $n \equiv 3 \pmod 4$

$$e^{\frac{(2k+1)(1+n)\pi i}{2n}} \cdot \left(w, e^{\frac{(2k+1)i\pi}{n}} w\right) = \left(we^{\frac{(2k+1)(1+n)\pi i}{2n}}, we^{\frac{(2k+1)\pi i}{n} - \frac{(2k+1)(1+n)\pi i}{2n}}\right)$$

and

$$\frac{(2k+1)\pi}{n} - \frac{(2k+1)(1+n)\pi}{2n} - \frac{(2k+1)(1+n)\pi}{2n} = -(2k+1)\pi$$

and if $n \equiv 1 \pmod 4$

$$e^{\frac{(2k+1)(1-n)\pi i}{2n}} \cdot \left(w, e^{\frac{(2k+1)i\pi}{n}} w\right) = \left(we^{\frac{(2k+1)(1-n)\pi i}{2n}}, we^{\frac{(2k+1)\pi i}{n} - \frac{(2k+1)(1-n)\pi i}{2n}}\right)$$

and

$$\frac{(2k+1)\pi}{n} - \frac{(2k+1)(1-n)\pi}{2n} - \frac{(2k+1)(1-n)\pi}{2n} = (2k+1)\pi.$$

So, periodic solutions of (4.9) with symmetry (conjugate to) $\mathbf{Z}_2(k, \pi)$ correspond to zeros of (4.10) where

$$r_1 = r_2 \text{ and } \phi = \frac{(2k+1)\pi}{n}, \; k \in \mathbf{Z}$$

(or $r_1 = -r_2$ and $\phi = (2k+1)\pi/n - \pi, \; k \in \mathbf{Z}$).
(ii) We consider now the case where $n \equiv 2 \pmod 4$. We recall that

$$\mathrm{Fix}_{\mathbf{C}^2}\left(\mathbf{Z}_2(k) \oplus \mathbf{Z}_2^c\right) = \{(w, w) : \; w \in \mathbf{C}\}$$

and

$$\mathrm{Fix}_{\mathbf{C}^2}\left(\mathbf{Z}_2(k, \pi) \oplus \mathbf{Z}_2^c\right) = \{(w, -w) : \; w \in \mathbf{C}\}.$$

(see Table 4.2). We can prove, by the same method used in (i), that periodic solutions of (4.9) with symmetry (conjugate to) $\mathbf{Z}_2(k) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.10) where

$$r_1 = r_2 \text{ and } \phi = \frac{2k\pi}{m}, \; k \in \mathbf{Z}$$

and periodic solutions of (4.9) with symmetry (conjugate to) $\mathbf{Z}_2(k, \pi) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.10) where

$$r_1 = r_2 \text{ and } \phi = \frac{(2k+1)\pi}{m}, \; k \in \mathbf{Z}.$$

(iii) Finally we study the case where $n \equiv 0 \pmod 4$ and $n \neq 4$. We recall that

$$\mathrm{Fix}_{\mathbf{C}^2}\left(\mathbf{Z}_2(k) \oplus \mathbf{Z}_2^c\right) = \{(w, w) : \; w \in \mathbf{C}\}$$

and

$$\mathrm{Fix}_{\mathbf{C}^2}\left(\mathbf{Z}_2(k\zeta) \oplus \mathbf{Z}_2^c\right) = \{(w, e^{2\pi i/n}w) : \ w \in \mathbf{C}\}$$

(see Table 4.3). We can prove, by the same method used in (i), that periodic solutions of (4.9) with symmetry (conjugate to) $\mathbf{Z}_2(k) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.10) where

$$r_1 = r_2 \ \text{ and } \ \phi = \frac{2k\pi}{m}, \ k \in \mathbf{Z}$$

and that periodic solutions of (4.9) with symmetry (conjugate to) $\mathbf{Z}_2(k\zeta) \oplus \mathbf{Z}_2^c$ correspond to zeros of (4.10) where

$$r_1 = r_2 \ \text{ and } \ \phi = \frac{(2k+1)\pi}{m}, \ k \in \mathbf{Z}.$$

We finish the proof considering the cases where $z_1 = 0$ and $z_2 \neq 0$. Let N, P, S and T be as in the Proposition 4.2.1. In that case $N = |z_2|^2$, $P = S = T = 0$ and (4.12) takes the form

$$\left\{ \begin{array}{lcl} g_1(z, \lambda, \tau) & = & 0 \\ g_2(z, \lambda, \tau) & = & (\nu + A)\, z_2 + B z_2^2 \overline{z}_2. \end{array} \right.$$

In this case we obtain zeros corresponding to a branch of periodic solutions with symmetry conjugate to $\widetilde{\mathbf{Z}}_n$. If $z_2 = 0$ and $z_1 \neq 0$ the situation is similar to this one. $\square$

Remark 4.3.3 From the above proof, the nondegeneracy conditions (referred in the word "generically") in Theorem 4.3.2 that guarantee that the only branches of periodic solutions with symmetry corresponding to isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$ with two-dimensional fixed-point subspaces can bifurcate at $\lambda = 0$ for the equations (4.9) with f as in (4.11) are

$$B_R(0) \neq 0, \quad (B_I C_I + B_R C_R)(0) \neq 0.$$

Remark 4.3.4 The existence of branches of periodic solutions of (4.9) with submaximal symmetry that bifurcate from $(0,0)$ in generic Hopf bifurcation with $\mathbf{D}_4$-symmetry differs markedly from those other $\mathbf{D}_n$. Swift [34] studies the dynamics of all possible square-symmetric codimension one Hopf bifurcations (with one parameter). In particular, it is shown that periodic solutions with submaximal symmetry bifurcate from the origin for open regions of the parameter space of the cubic coefficients in the Birkhoff normal form. Briefly, in [34] it is used the following method. The $\mathbf{S}^1$-symmetry of the normal form on $\mathbf{C}^2$ ensures that the equation for the average phase of z_1 and z_2 decouples from the rest of the equations. Thus the four-dimensional normal form in $\mathbf{C}^2$ ($\mathbf{D}_n \times \mathbf{S}^1$-equivariant) is then reduced to a three-dimensional system in $\mathbf{R}^3$. Writing this reduced system in $\mathbf{R}^3$ in spherical coordinates, it follows that for the *cubic truncation* the two angular coordinates essentially decouple from the radial coordinate. It is possible then to classify most aspects of the dynamics by studying an ODE on the two-dimensional sphere.

Chapter 5

Hopf Bifurcation in Coupled Cell Networks with Interior Symmetry

A coupled cell system is a network of dynamical systems coupled together. Here, a network is represented by a directed graph $\mathcal{G}$ whose nodes correspond to cells and whose edges represent couplings. Cells with the same label have 'identical' internal dynamics; arrows with the same label correspond to 'identical' couplings. Equip each cell c with a phase space P_c so that the total phase space of the network is the cartesian product $P = \prod_c P_c$. A vector field respecting the topology of the network is called $\mathcal{G}$-*admissible*. We follow the theory developed by Stewart *et al.* [33] and Golubitsky *et al.* [22].

When the network possess a group of symmetries then the associated coupled cell systems (ODE's) are also symmetric. In this context there is a group of permutations of the cells (and arrows) that preserves the network structure (including cell-types and arrow-types) and its action on P is by permutation of cell coordinates. Moreover, the coupled cell systems (ODE's) are of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x),$$

where the $\mathcal{G}$-admissible vector field f is smooth (C^∞) and satisfies

$$f(\gamma x) = \gamma f(x)\,, \qquad \forall\, x \in P, \gamma \in \Gamma.$$

That is, f is equivariant under the action of the group Γ on phase space P. The theory of equivariant dynamical systems (Golubitsky *et al.* [19, 21]. See chapter 3) can be applied to such dynamical systems.

There are non-symmetric networks where group theoretic methods still apply, namely, networks admitting *interior symmetries*. In this case there is a group of permutations of a subset $\mathcal{S}$ of the cells (and edges directed to $\mathcal{S}$) that partially preserves the network structure (including cell-types and edges-types) and its action on P is by permutation of cell coordinates. In other words, the cells in $\mathcal{S}$ together with all the edges directed to them form a subnetwork which possesses a nontrivial group of symmetry $\Sigma_{\mathcal{S}}$. This notion was introduced and investigated by Golubitsky *et al.* [15]. In this chapter we study Hopf bifurcation in coupled cell systems associated with interior symmetric networks.

This chapter is organized as follows. Section 5.1 gives the formal definition of a coupled cell network and the associated dynamical systems, and states some basic features, including the concept of a balanced equivalence relation (colouring). We also discuss the symmetry group of a network. Section 5.2 gives the definition of interior symmetry given by Golubitsky *et al.* [15]. Moreover, we provide an equivalent definition, in terms of symmetries of a subnetwork, which in some cases (no multiple edges and no self-connections) amounts to finding the symmetries of the subnetwork (see Proposition 5.2.3). We also analyse the structure of these networks and discuss some features of the admissible vector fields associated to such class of networks. Section 5.3 gives the notion of synchrony-breaking bifurcation in coupled cell networks. Then we specialise to networks with interior symmetries where group theoretic concepts play a significant role, focusing on the important case of codimension-one synchrony-breaking bifurcations.

5.1 Network Formalism

First, we give the formal definition of a coupled cell network and the associated dynamical systems. For a survey, overview and examples, see Golubitsky and Stewart [20]. The initial definition of coupled cell network given by Stewart *et al.* [33] was modified by Golubitsky *et al.* [22] to permit multiple arrows and self-connections, which turns out to have major advantages. More recently, Stewart [32] extended the formalism introduced by Golubitsky *et al.* [22] to include a large class of infinite networks – the so called *networks of finite type.*

5.1.1 Coupled Cell Networks

In this work we consider *finite networks* and so employ the 'finite multi-arrow' formalism for consistency with the existing literature.

Definition 5.1.1 ([22]) A *coupled cell network* $\mathcal{G}$ comprises the following:

(a) A finite set $\mathcal{C}$ of *nodes* or *cells*.

(b) An equivalence relation $\sim_C$ on cells in $\mathcal{C}$, called *cell-equivalence*. The *type* or *cell label* of cell c is its $\sim_C$-equivalence class.

(c) A finite set $\mathcal{E}$ of *edges* or *arrows*.

(d) An equivalence relation $\sim_E$ on edges in $\mathcal{E}$, called *edge-equivalence* or *arrow-equivalence*. The *type* or *coupling label* of edge e is its $\sim_E$-equivalence class.

(e) Two maps $\mathcal{H} : \mathcal{E} \to \mathcal{C}$ and $\mathcal{T} : \mathcal{E} \to \mathcal{C}$. For $e \in \mathcal{E}$ we call $\mathcal{H}(e)$ the *head* of e and $\mathcal{T}(e)$ the *tail* of e.

We also require a *consistency condition*:

(f) Equivalent arrows have equivalent tails and heads:

$$\mathcal{H}(e_1) \sim_C \mathcal{H}(e_2) \qquad \mathcal{T}(e_1) \sim_C \mathcal{T}(e_2)$$

for all $e_1, e_2 \in \mathcal{E}$ with $e_1 \sim_E e_2$.

Example 5.1.2 We can represent abstract networks by labeled directed graphs. Figure 5.1 shows two examples. Here the node labels, drawn as the three circles and the triangle, indicate the cells; the symbols show that cells $1, 2, 3$ have the same type, whereas cell 4 is different, in both cases. In the network $\mathcal{G}_1$ there are three types of edge label, whereas in the network $\mathcal{G}_2$ there are five types of edge label, drawn as different styles of arrows. The tail and head of each edge is, respectively, indicated by the absence or presence of a tip on one end of the arrow. When an arrow between cells c and d is drawn with tips in both ends then it represents two arrows of the same type with opposite orientation between cells c and d.

$\diamond$

5.1.2 Input Sets and the Symmetry Groupoid

Associated with each cell $c \in \mathcal{C}$ is a canonical set of edges, namely, those that represent couplings into cell c, as described next.

Definition 5.1.3 ([22]) If $c \in \mathcal{C}$, then the *input set* of c is the finite set of edges directed to c,

$$I(c) = \{e \in \mathcal{E} : \mathcal{H}(e) = c\}. \tag{5.1}$$

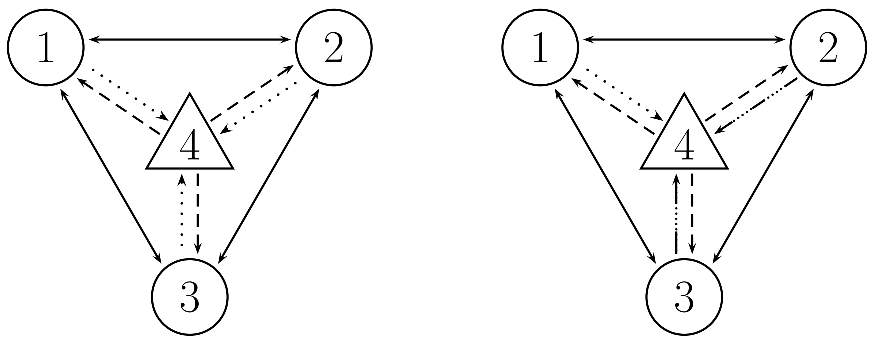

Figure 5.1: (Left) Network $\mathcal{G}_1$ with exact $\mathbf{S}_3$-symmetry. (Right) Network $\mathcal{G}_2$ with $\mathbf{S}_3$-interior symmetry.

Definition 5.1.4 ([22]) The relation $\sim_I$ of *input equivalence* on $\mathcal{C}$ is defined by $c \sim_I d$ if and only if there exists a bijection

$$\beta : I(c) \to I(d) \tag{5.2}$$

such that for every $i \in I(c)$,

$$i \sim_E \beta(i). \tag{5.3}$$

Any such bijection β is called an *input isomorphism* from cell c to cell d. The set $B(c, d)$ denotes the collection of all input isomorphisms from cell c to cell d. The union

$$\mathcal{B}_{\mathcal{G}} = \bigcup_{c,d \in \mathcal{C}} B(c, d) \tag{5.4}$$

is the *symmetry groupoid* of the network $\mathcal{G}$. A coupled cell network is *homogeneous* if all input sets are isomorphic.

The groupoid operation on $\mathcal{B}_{\mathcal{G}}$ is composition of maps, and in general the composition $\beta\alpha$ is defined only when $\alpha \in B(a, b)$ and $\beta \in B(b, c)$ for cells a, b, c. This is why $\mathcal{B}_{\mathcal{G}}$ need not be a group. Observe that for any $c \in \mathcal{C}$, the subset $B(c, c)$ is always non-empty and it is a group.

Example 5.1.5 In our running examples, shown in Figure 5.1, it is easy to see that both networks have only two input isomorphism classes of cells: $\{1, 2, 3\}$ and $\{4\}$. The input sets of cells $1, 2, 3$ are isomorphic, since each one of them contains three edges, two of them drawn as a solid arrow with a circle in the tail and one of them drawn as a dashed arrow with a triangle in the tail. $\diamond$

5.1.3 Admissible Vector Fields

We now explain how to interpret such diagrams as Figure 5.1 as being representative of a class of vector fields.

For each cell in $\mathcal{C}$ choose a *cell phase space* P_c, which we assume to be a nonzero finite-dimensional real vector space. We require

$$c \sim_C d \quad \Rightarrow \quad P_c = P_d,$$

and in this case we employ the same coordinate systems on P_c and P_d. The *total phase space* is then

$$P = \prod_{c \in \mathcal{C}} P_c$$

with a cell-based coordinate system

$$x = (x_c)_{c \in \mathcal{C}}.$$

If $\mathcal{D} = (c_1, \ldots, c_s) \subseteq \mathcal{C}$ is any finite ordered set of cells, then we write

$$P_{\mathcal{D}} = \prod_{d \in \mathcal{D}} P_d$$

and

$$x_{\mathcal{D}} = (x_{c_1}, \ldots, x_{c_s}),$$

where $x_c \in P_c$.

Given $c \in C$, denote by $\mathcal{T}(I(c))$ the ordered set of cells

$$(\mathcal{T}(i_1), \ldots, \mathcal{T}(i_s))$$

where the arrows i_k run through $I(c)$. Suppose that $c \sim_I d$ and consider the ordered sets $\mathcal{D}_1 = \mathcal{T}(I(c))$, $\mathcal{D}_2 = \mathcal{T}(I(d))$ of $\mathcal{C}$. Let $\beta \in B(c,d)$. Then β is a bijection between $I(c)$ and $I(d)$. Moreover, for all $i \in I(c)$ we have $i \sim_E \beta(i)$, and so $\mathcal{T}(i) \sim_C \mathcal{T}(\beta(i))$. We can define the *pullback map*

$$\beta^* : P_{\mathcal{D}_2} \to P_{\mathcal{D}_1}$$

by

$$(\beta^* z)_{\mathcal{T}(i)} = z_{\mathcal{T}(\beta(i))} \tag{5.5}$$

for all $\mathcal{T}(i) \in \mathcal{D}_1$ and $z \in P_{\mathcal{D}_2}$. If $\mathcal{T}(I(c)) = (\mathcal{T}(i_1), \ldots, \mathcal{T}(i_s))$ then $x_{\mathcal{T}(I(c))} = \left(x_{\mathcal{T}(i_1)}, \ldots, x_{\mathcal{T}(i_s)}\right)$ and $\beta^*\left(x_{\mathcal{T}(I(d))}\right) = \left(x_{\mathcal{T}(\beta(i_1))}, \ldots, x_{\mathcal{T}(\beta(i_s))}\right)$.

We use pullback maps to relate different components of a vector field associated with a given coupled cell network. Specifically, the class of vector fields that are encoded by a coupled cell network is given by the following definition.

For a given cell c the *internal phase space* is P_c and the *coupling phase space* is

$$P_{\mathcal{T}(I(c))} = P_{\mathcal{T}(i_1)} \times \cdots \times P_{\mathcal{T}(i_s)}.$$

Definition 5.1.6 ([22]) Let $\mathcal{G}$ be a coupled cell network. For a given choice of P_c, a (smooth) vector field $f : P \to P$ is $\mathcal{G}$-*admissible* if the following hold:

(a) *Domain condition*: For all $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variables x_c and the coupling phase space variables $x_{\mathcal{T}(I(c))}$; that is, there exists a (smooth) function $\hat{f}_c : P_c \times P_{\mathcal{T}(I(c))} \to P_c$ such that

$$f_c(x) \;=\; \hat{f}_c(x_c, x_{\mathcal{T}(I(c))}). \tag{5.6}$$

(b) *Pull-back condition*: For all $c, d \in \mathcal{C}$ and $\beta \in B(c,d)$

$$\hat{f}_d(x_d, x_{\mathcal{T}(I(d))}) = \hat{f}_c(x_d, \beta^* x_{\mathcal{T}(I(d))}) \tag{5.7}$$

for all $x \in P$.

Example 5.1.7 For the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1 the cell phase spaces P_1, P_2 and P_3 are identical and equal to $\mathbf{R}^k$, whereas $P_4 = \mathbf{R}^l$. The general form of the admissible vector fields (ODEs) encoded by the network $\mathcal{G}_1$ is

$$\begin{array}{rcl} \dot{x}_1 &=& f(x_1, \overline{x_2, x_3}, x_4) \\ \dot{x}_2 &=& f(x_2, \overline{x_3, x_1}, x_4) \\ \dot{x}_3 &=& f(x_3, \overline{x_1, x_2}, x_4) \\ \dot{x}_4 &=& g(x_4, \overline{x_1, x_2, x_3}) \end{array} \tag{5.8}$$

where $x_i \in \mathbf{R}^k$ $(i = 1, 2, 3)$, $x_4 \in \mathbf{R}^l$, $f : \mathbf{R}^{3k} \times \mathbf{R}^l \to \mathbf{R}^k$ is a smooth map invariant under permutation of the second and third arguments and $g : \mathbf{R}^{3k} \times \mathbf{R}^l \to \mathbf{R}^l$ is a smooth map invariant under any permutation of the last three arguments. The general form of the admissible vector fields (ODEs) associated with the network $\mathcal{G}_2$ is

$$\begin{array}{rcl} \dot{x}_1 &=& f(x_1, \overline{x_2, x_3}, x_4) \\ \dot{x}_2 &=& f(x_2, \overline{x_3, x_1}, x_4) \\ \dot{x}_3 &=& f(x_3, \overline{x_1, x_2}, x_4) \\ \dot{x}_4 &=& g(x_4, x_1, x_2, x_3) \end{array} \tag{5.9}$$

where $x_i \in \mathbf{R}^k$ $(i = 1, 2, 3)$, $x_4 \in \mathbf{R}^l$, $f : \mathbf{R}^{3k} \times \mathbf{R}^l \to \mathbf{R}^k$ is a smooth map, invariant under permutation of the second and third argument and $g : \mathbf{R}^{3k} \times \mathbf{R}^l \to \mathbf{R}^l$ is a general smooth map. $\diamond$

Remarks 5.1.8 (a) The condition of local finiteness together with the domain condition implies that the components f_c of the admissible maps are functions defined on a finite dimensional vector space.
(b) For certain purposes it is useful to work with more restricted classes of admissible vector fields: linear, polynomial, etc. When such properties are relevant we say that an admissible map f is a *linear* or *polynomial admissible vector field* if and only if each $\hat{f}_c$ is linear or polynomial, respectively.

We give the following characterization of admissible vector fields which is proved for networks of finite type in Stewart [32].

Proposition 5.1.9 ([32]) *Let $\mathcal{G}$ be a coupled cell network. A vector field $f : P \to P$ for a given choice of P_c satisfying the domain condition is $\mathcal{G}$-admissible if and only if for each $\sim_I$-equivalence class $\mathcal{Q}$*

(i) *f_c is invariant under $B(c, c)$ for some $c \in \mathcal{Q}$.*

(ii) *For $d \in \mathcal{Q}$ such that $d \neq c$, given (any) $\beta \in B(c, d)$, we have $f_d\left(x_d, x_{\mathcal{T}(I(d))}\right) = \widehat{f}_c\left(x_d, \beta^*\left(x_{\mathcal{T}(I(d))}\right)\right)$.*

Proof: See Stewart [32]. $\square$

Remark 5.1.10 Let $\mathcal{G}$ be a coupled cell network and P a given choice of the total phase space consistent with $\mathcal{G}$. By the above proposition, every smooth $\mathcal{G}$-admissible vector field on P is determined uniquely by its component f_c where c runs through a set of representatives for the $\sim_I$-equivalence classes. Now f_c depends only on x_c, $x_{\mathcal{T}(I(c))}$ and is invariant under all the bijections in $B(c, c)$. If $d \sim_I c$ then f_d is related to f_c by a pullback map β^* for $\beta \in B(c, d)$.

5.1.4 Balanced Equivalence Relations

Given a coupled cell network, an equivalence relation $\bowtie$ on $\mathcal{C}$ determines a unique partition of $\mathcal{C}$ into $\bowtie$-equivalence classes, which can be interpreted as a colouring of $\mathcal{C}$ in which $\bowtie$-equivalent cells receive the same colour. Conversely, any partition (colouring) determines a unique equivalence relation. The corresponding *polydiagonal* is

$$\triangle_{\bowtie} = \{x \in P : c \bowtie d \Rightarrow x_c = x_d\}. \tag{5.10}$$

A subspace V of P is called *admissibly flow-invariant* if $f(V) \subseteq V$ for all admissible vector field f on P.

Definition 5.1.11 ([22]) Let $\mathcal{G}$ be a coupled cell network. An equivalence relation $\bowtie$ on $\mathcal{C}$ is *balanced* if for every $c, d \in \mathcal{C}$ with $c \bowtie d$ there exists $\beta \in B(c, d)$ such that $\mathcal{T}(i) \bowtie \mathcal{T}(\beta(i))$ for all $i \in I(c)$. The associated colouring is called a *balanced colouring*. In particular, $B(c, d) \neq \emptyset$ implies $c \sim_I d$. Hence, balanced equivalence relations refine input equivalence.

A crucial property of balanced equivalence relations is that they define admissibly flow-invariant subspaces, and conversely the following holds.

Theorem 5.1.12 ([33]) *Let $\bowtie$ be an equivalence relation on a coupled cell network. Then $\triangle_{\bowtie}$ is admissibly flow-invariant if and only if $\bowtie$ is balanced.*

Proof: The proof of the above result for finite networks is given in Golubitsky *et al.*[22] and in Stewart *et al.* [33] and for networks of finite type in Stewart [32]. $\square$

The dynamical implication of such flow-invariance is that $\bowtie$ determines a *robust pattern of synchrony*: there exist trajectories $x(t)$ of the ODE such that

$$c \bowtie d \;\Rightarrow\; x_c(t) = x_d(t)\; \forall t \in \mathbf{R}.$$

Such trajectories arise when initial conditions $x(0)$ lie in $\triangle_{\bowtie}$. Then the entire trajectory, for all positive and negative time, lies in $\triangle_{\bowtie}$ and is a trajectory of the restriction $f|_{\triangle_{\bowtie}}$. The associated dynamics can be steady-state, periodic, even chaotic, depending on f and its restriction to $\triangle_{\bowtie}$. An example of synchronised chaos generated by this mechanism can be found in Golubitsky *et al.* [20].

Since there is always a canonical balanced relation $\sim_I$ on every network, let $\triangle_I$ denote polydiagonal subspace of P associated to the input equivalence relation $\sim_I$, that is,

$$\triangle_I = \{x \in P : c \sim_I d \Rightarrow x_c = x_d\}.$$

Then $\triangle_I$ is a flow invariant subspace. Solution of admissible vector fields contained in $\triangle_I$ represent the states of highest degree of synchrony allowed by the network.

Remark 5.1.13 Whenever self-connections or multiple arrows do not occur it will be convenient to revert to the formalism of Stewart [33], but now considered as a specialisation of the multi-arrow formalism. Since no two distinct arrows have the same head and tail, we can identify an arrow e with the pair of cells $(\mathcal{T}(e), \mathcal{H}(e))$. Now the set $\mathcal{E}$ of arrows identifies with a subset of $\mathcal{C} \times \mathcal{C} \setminus \{(c, c) : c \in \mathcal{C}\}$. Similarly the input set $I(c)$ can be identified with the set of all tail cells of arrows e that have c as head cell.

Example 5.1.14 We continue with our running examples, the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1. There is an equivalence relation $\bowtie$ for which $1 \bowtie 2$; its equivalence classes are $\{1, 2\}$, $\{3\}$ and $\{4\}$. The corresponding polydiagonal is

$$\triangle_{\bowtie} = \{x \in P : x_1 = x_2\} = \{(x, x, y, z)\}.$$

On this subspace the differential equations become

$$\begin{aligned} \dot{x} &= f(x, \overline{x, y}, z) \\ \dot{x} &= f(x, \overline{y, x}, z) \\ \dot{y} &= f(y, \overline{x, x}, z) \\ \dot{z} &= g(z, x, x, y). \end{aligned} \tag{5.11}$$

Since the first two equations are identical (recall that the bar over x, y means that they can be interchanged), $\triangle_{\bowtie}$ is invariant under all admissible vector fields. The relation $\bowtie$ is balanced. The only condition to verify is that cells 1 and 2, which are $\bowtie$-equivalent but distinct, have input sets that are isomorphic by an isomorphism that preserves $\bowtie$-equivalence classes for both networks. In both networks the input sets are

$$I(1) = \{(2, 1), (3, 1), (4, 1)\} \qquad \text{and} \qquad I(2) = \{(1, 2), (3, 2), (4, 2)\},$$

where (c, d) denotes an arrow with tail c and head d (see Remark 5.1.13). The bijection $\beta : I(1) \to I(2)$ with $\beta((2, 1)) = (1, 2)$, $\beta((3, 1)) = (3, 2)$ and $\beta((4, 1)) = (4, 2)$ is an input isomorphism that preserves $\bowtie$-equivalence classes since $1 \bowtie 2$, $3 \bowtie 3$ and $4 \bowtie 4$. That is, $\bowtie$ is a balanced relation as claimed. There are two other balanced equivalence relations (different from $\sim_I$) on the networks $\mathcal{G}_1$ and $\mathcal{G}_2$. In one of them the equivalence classes are $\{2, 3\}$, $\{1\}$ and $\{4\}$. In the other the equivalence classes are $\{1, 3\}$, $\{2\}$ and $\{4\}$. $\diamond$

5.1.5 Symmetry Groups of Networks

We now consider symmetries of networks in the group theoretic ('global') sense.

Definition 5.1.15 ([2]) Let $\mathcal{G}$ be a network. A *symmetry* of $\mathcal{G}$ consists of a pair of bijections $\gamma_C : \mathcal{C} \to \mathcal{C}$ and $\gamma_E : \mathcal{E} \to \mathcal{E}$, where γ_C preserves input equivalence and γ_E preserves edge equivalence; that is, for all $c \in \mathcal{C}$ and $e \in \mathcal{E}$,

$$\gamma_C(c) \sim_I c \qquad \text{and} \qquad \gamma_E(e) \sim_E e. \tag{5.12}$$

In addition, the two bijections must satisfy the consistency conditions

$$\gamma_C(\mathcal{H}(e)) = \mathcal{H}(\gamma_E(e)) \qquad \text{and} \qquad \gamma_C(\mathcal{T}(e)) = \mathcal{T}(\gamma_E(e)) \tag{5.13}$$

for all $e \in \mathcal{E}$. The set of all $\gamma = (\gamma_C, \gamma_E)$ forms a finite group $\mathrm{Aut}(\mathcal{G})$ called the *symmetry group* of the network of $\mathcal{G}$.

Observe that a symmetry γ preserves input sets in a natural sense. Because of the way input sets are defined in the multi-arrow formalism, the precise relation is

$$\gamma_E(I(c)) = I(\gamma_C(c)),$$

where $\gamma = (\gamma_C, \gamma_E) \in \mathrm{Aut}(\mathcal{G})$.

Remark 5.1.16 When the network $\mathcal{G}$ has no self-connections and multiarrows there is a simplification of the notion of symmetry due to the following observation. Given a vertex permutation γ_C, there is a unique edge permutation γ_E satisfying the consistency condition (5.13); that is, γ_E is implicitly defined by γ_C since, by Remark 5.1.13, each arrow e can be identified with a pair of cells $(\mathcal{T}(e), \mathcal{H}(e))$. Thus a *symmetry* of $\mathcal{G}$ is given by a permutation γ of $\mathcal{C}$ such that

(a) $\gamma(c) \sim_I c$ for all $c \in \mathcal{C}$.

(b) $(\gamma(a), \gamma(b)) \in \mathcal{E} \Leftrightarrow (a, b) \in \mathcal{E}$.

(c) $(\gamma(a), \gamma(b)) \sim_E (a, b)\ \forall (a, b) \in \mathcal{E}$.

In this case, the group $\mathrm{Aut}(\mathcal{G})$ of symmetries of the network $\mathcal{G}$ is a subgroup of the group $\mathrm{Sym}(\mathcal{C})$ of permutations on the set of cells of the network. We shall adopt this convention throughout the remainder of the work whenever the network under consideration has no self-connections and multiarrows.

Example 5.1.17 Since the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of our running example of Figure 5.1 do not have multiple arrows and self-connections, Remark 5.1.16 applies. The group $\mathbf{S}_3 \subset \mathbf{S}_4$ consisting of the transpositions $(1\,2)$, $(1\,3)$, $(2\,3)$, the 3-cycle permutations $(1\,2\,3)$, $(1\,3\,2)$ and the identity is the symmetry group of the network $\mathcal{G}_1$. Observe that cell 4 is fixed by the symmetry group.

On the other hand, the network $\mathcal{G}_2$ has only the identity permutation as a symmetry because the arrows $(1,4)$, $(2,4)$ and $(3,4)$ are all different amongst each other. $\diamond$

This last example shows that the definition of symmetry of a network is very rigid. In the next section we will generalise the definition of symmetry of a network by introducing the notion of *interior symmetry*. In this new context the network $\mathcal{G}_2$ of our example admits an action of the permutation group $\mathbf{S}_3$ as a group of interior symmetries. This corresponds to the symmetry group of the subnetwork of $\mathcal{G}_2$ obtained by ignoring the arrows $(1,4)$, $(2,4)$ and $(3,4)$ of $\mathcal{G}_2$.

Next we relate network symmetries to (ODE) symmetries. Recall that a symmetry γ of a differential equation

$$\dot{x} = f(x) \tag{5.14}$$

with phase space P is a linear map $x \mapsto \gamma x$ on P that commutes with f; that is, γ satisfies

$$\gamma f(x) = f(\gamma x) \tag{5.15}$$

for all $x \in P$. Recall from section 3.1 that given a group Γ acting linearly on P, a mapping $f : P \to P$ that satisfies (5.15) for all $\gamma \in \Gamma$ is called Γ-equivariant. The set of all Γ-equivariant mappings is denoted by $\vec{\mathcal{E}}_P(\Gamma)$.

Let $\mathcal{G}$ be a network and fix a total phase space P for $\mathcal{G}$. If $\gamma = (\gamma_C, \gamma_E)$ is a symmetry of $\mathcal{G}$ then γ_C permutes the cells of $\mathcal{G}$ and hence it induces an action of γ on P by permuting the cell coordinates

$$\gamma(x_c)_{c\in\mathcal{C}} = (x_{\gamma_C^{-1}(c)})_{c\in\mathcal{C}}. \tag{5.16}$$

Since the input equivalence preserving cell permutation γ is a network symmetry, there is an edge equivalence preserving arrow permutation γ_E that satisfies (5.13). Then γ commutes with every $\mathcal{G}$-admissible vector field $f : P \to P$; that is, γ and f satisfy (5.15). Finally, note that equivariance holds independently of the choice of γ_E because of the invariance of the component f_c under the group $B(c,c)$ called *vertex group* (recall Definition 5.1.4). Therefore, every $\mathcal{G}$-admissible vector field is $\mathrm{Aut}(\mathcal{G})$-equivariant.

We can construct (some) balanced equivalence relations on a network $\mathcal{G}$ from subgroups of the symmetry group $\mathrm{Aut}(\mathcal{G})$ of $\mathcal{G}$. Namely, suppose that $H \subseteq \mathrm{Aut}(\mathcal{G})$. Define the relation $\bowtie_H$ on the set of cells $\mathcal{C}$ of $\mathcal{G}$ by:

$$c \bowtie_H d \quad \Leftrightarrow \quad \exists\, \gamma = (\gamma_C, \gamma_E) \in H : \gamma_C(c) = d.$$

Then the $\bowtie_H$-classes are the H-orbits of cells where the H-orbit of $c \in \mathcal{C}$ is defined by $\{\gamma_C(c) : \exists\, \gamma = (\gamma_C, \gamma_E) \in H : \gamma_C(c) = d\}$; the corresponding polydiagonal is

$$\triangle_H = \triangle_{\bowtie_H} = \mathrm{Fix}_P(H)$$

where $\mathrm{Fix}_P(H)$ is the fixed-point space of H acting on the total phase space P. This polydiagonal is balanced:

Proposition 5.1.18 *Let $\mathcal{G}$ be a network and let H be any subgroup of* $\mathrm{Aut}(\mathcal{G})$*. Then* $\mathrm{Fix}(H)$ *is a balanced polydiagonal.*

Proof: Let f be a $\mathcal{G}$-admissible vector field. Then f is equivariant under $\mathrm{Aut}(\mathcal{G})$. By Remark 3.1.8, $\mathrm{Fix}(H)$ is flow-invariant for f. Since this holds for any $\mathcal{G}$-admissible f, then by Theorem 5.1.12, $\mathrm{Fix}(H)$ is a balanced polydiagonal. (A more direct proof is possible by routine computations. See Proposition 5.2.5 below.) $\square$

Remark 5.1.19 By Dias and Stewart [10] Proposition 8.16, we have in general that

$$\vec{\mathcal{F}}_P(\mathcal{G}) \;\subsetneq\; \vec{\mathcal{E}}_P(\mathrm{Aut}(\mathcal{G})). \tag{5.17}$$

One of the consequences of this 'gap' (5.17) for finite networks is the existence of *exotic balanced colourings*, that is, a balanced colouring that is not a fixed-point space of a subgroup of the automorphism group of the network. Consider for example the 6-cell network $\mathcal{G}$ of Figure 5.2. The automorphism

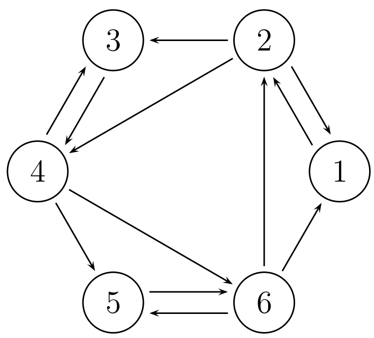

Figure 5.2: Patterns of synchrony in a six-cell $\mathbf{Z}_3$-symmetric network.

group of $\mathcal{G}$ is isomorphic to the cyclic group $\mathbf{Z}_3$. Its action on cells is generated by the permutation (135)(246), and this induces a unique action on edges since there are no multiple arrows. There are precisely two balanced polydiagonals coming from fixed-point spaces of subgroups of $\mathbf{Z}_3$: one is P

itself and the other is $\{(x, y, x, y, x, y)\}$. However, this network has several other balanced polydiagonals: any of the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ can be *independently* identified and the resulting polydiagonal is balanced. Consider for example the $\bowtie$-equivalence relation with classes $\{1, 2\}$, $\{3\}$, $\{4\}$, $\{5\}$ and $\{6\}$. The input sets of cells 1 and 2 are:

$$I(1) = \{(2, 1), (6, 1)\} \qquad \text{and} \qquad I(2) = \{(1, 2), (6, 2)\}.$$

The bijection $\beta : I(1) \to I(2)$ such that $\beta((2, 1)) = (1, 2)$ and $\beta((6, 1)) = (6, 2)$ is an input isomorphism that preserves $\bowtie$-equivalence classes since $1 \bowtie 2$ and $6 \bowtie 6$. Consequently, the equivalence relation $\bowtie$ is a balanced relation. See Antoneli and Stewart [2, 3] for details.

5.2 Interior Symmetry

We present the notion of interior symmetry following Golubitsky *et al.* [15] and give an alternative characterisation in terms of the symmetries of a subnetwork (Proposition 5.2.3).

5.2.1 Interior Symmetry Groups of Networks

Definition 5.2.1 ([15]) Let $\mathcal{G}$ be a coupled cell network. Let $\mathcal{S} \subseteq \mathcal{C}$ be a subset of cells and let $I(\mathcal{S}) = \{e \in \mathcal{E} : \mathcal{H}(e) \in \mathcal{S}\}$. A pair of bijections $\sigma_C : \mathcal{C} \to \mathcal{C}$ and $\sigma_E : \mathcal{E} \to \mathcal{E}$ is an *interior symmetry* of $\mathcal{G}$ (on the subset $\mathcal{S}$) if the following holds:

(a) $\sigma_C : \mathcal{C} \to \mathcal{C}$ is an input equivalence-preserving permutation which is the identity map on the complement $\mathcal{C} \setminus \mathcal{S}$ of $\mathcal{S}$ in $\mathcal{C}$,

(b) $\sigma_E : \mathcal{E} \to \mathcal{E}$ is an edge equivalence-preserving permutation which is the identity map on the complement $\mathcal{E} \setminus I(\mathcal{S})$ of $I(\mathcal{S})$ in $\mathcal{E}$,

(c) the consistency condition

$$\sigma_C(\mathcal{H}(e)) = \mathcal{H}(\sigma_E(e)) \qquad \text{and} \qquad \sigma_C(\mathcal{T}(e)) = \mathcal{T}(\sigma_E(e)) \quad (5.18)$$

is satisfied for every $e \in I(\mathcal{S})$.

The set of all interior symmetries of $\mathcal{G}$ (on the subset $\mathcal{S}$) forms a finite group $\Sigma_{\mathcal{S}}$ called the *group of interior symmetries* of $\mathcal{G}$ (on the subset $\mathcal{S}$).

Note that in Definition 5.2.1 if $\mathcal{S} = \mathcal{C}$, then $\Sigma_{\mathcal{S}} = \mathrm{Aut}(\mathcal{G})$. Hence, the definition of interior symmetry of a network is a generalization of a symmetry of a network. That is why we refer to the elements of $\mathrm{Aut}(\mathcal{G})$ as *global symmetries* of $\mathcal{G}$. The most interesting case is when $\mathrm{Aut}(\mathcal{G})$ is trivial but $\Sigma_{\mathcal{S}}$ is nontrivial for some $\mathcal{S}$.

Example 5.2.2 We continue with our running example, the two networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1. We have seen that the network $\mathcal{G}_1$ is $\mathbf{S}_3$-symmetric and the network $\mathcal{G}_2$ has only the trivial symmetry. However, the group of permutations

$$\mathbf{S}_3 = \{\mathrm{id}, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$$

is the group of interior symmetries of the network $\mathcal{G}_2$ on the subset $\mathcal{S} = \{1, 2, 3\}$. Observe that all elements of $\mathbf{S}_3$ fix cell 4 and

$$I(\mathcal{S}) = \{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (4,1), (4,2), (4,3)\}.$$

If we assume that the permutations in $\mathbf{S}_3$ act as identity on the set of arrows

$$\mathcal{E} \setminus I(\mathcal{S}) = \{(1,4), (2,4), (3,4)\},$$

then $\mathbf{S}_3$ is the group of interior symmetries of the network $\mathcal{G}_2$ on the subset $\mathcal{S} = \{1, 2, 3\}$. $\diamond$

There is an alternative characterization of interior symmetries using the notion of symmetry of a network. The main idea is the following: by "ignoring" some arrows we find a subnetwork whose symmetry group is the group of interior symmetries of the original network.

Let us be more precise. Given a coupled cell network $\mathcal{G}$ and a subset $\mathcal{S} \subset \mathcal{C}$ of cells define $\mathcal{G}_{\mathcal{S}} = (\mathcal{C}, I(\mathcal{S}), \sim_C, \sim_E)$ to be the subnetwork of $\mathcal{G}$ whose set of cells is $\mathcal{C}$ (together with its cell-equivalence $\sim_C$) and whose set of arrows is $I(\mathcal{S})$. The edge-equivalence on $I(\mathcal{S})$ is obtained by the restriction of the edge-equivalence $\sim_E$ on $\mathcal{E}$.

Proposition 5.2.3 *Let $\mathcal{G}$ be a coupled cell network and $\mathcal{S} \subset \mathcal{C}$ be a subset of cells of the set of cells of $\mathcal{G}$. Consider the network $\mathcal{G}_{\mathcal{S}}$ as defined above. Then the group of interior symmetries of the network $\mathcal{G}$ (on the subset $\mathcal{S}$) can be canonically identified with the group of symmetries of the network $\mathcal{G}_{\mathcal{S}}$:*

$$\Sigma_{\mathcal{S}} \cong \mathrm{Aut}(\mathcal{G}_{\mathcal{S}}).$$

Proof: We start by proving that $\Sigma_{\mathcal{S}}$ can be canonically identified with a subset of $\mathrm{Aut}(\mathcal{G}_{\mathcal{S}})$. Let $\sigma = (\sigma_C, \sigma_E) \in \Sigma_{\mathcal{S}}$ be an interior symmetry of $\mathcal{G}$ (on the subset $\mathcal{S}$), as in Definition 5.2.1. Then, because σ_C and σ_E are the identity maps on $\mathcal{C} \setminus \mathcal{S}$ and $\mathcal{E} \setminus I(\mathcal{S})$, respectively, it follows that σ is a symmetry of $\mathcal{G}_{\mathcal{S}}$, according to Definition 5.1.15. Now we show that the above identification is surjective. Let $\gamma = (\gamma_C, \gamma_E) \in \mathrm{Aut}(\mathcal{G}_{\mathcal{S}})$ be a symmetry of $\mathcal{G}_{\mathcal{S}}$ (in the sense of Definition 5.1.15); that is, γ_E is a permutation on the set $I(\mathcal{S})$. Now we can extend γ_E to a permutation σ_E on $\mathcal{E}$ which acts as identity on $\mathcal{E} \setminus I(\mathcal{S})$. The pair $\sigma = (\sigma_C, \sigma_E)$, where $\sigma_C = \gamma_C$ is an interior symmetry of $\mathcal{G}$ (on the subset $\mathcal{S}$) according to Definition 5.2.1. $\square$

The characterisation of interior symmetry provided by Proposition 5.2.3 is particularly useful when the network does not have multiple arrows and/or self-connections, since by Remark 5.1.16, a symmetry is simply a permutation on the set vertices of the underlying graph.

Example 5.2.4 Consider the two networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1. Let $\mathcal{S} = \{1, 2, 3\}$. Note that the network $\mathcal{G}_{\mathcal{S}}$ obtained from $\mathcal{G}_1$ is the same as the one obtained from $\mathcal{G}_2$. In Figure 5.3 we show these three networks. Observe that for the three networks the sets of arrows coming from the set $\mathcal{S} = \{1, 2, 3\}$ and directed to the complement $\mathcal{C} \setminus \mathcal{S} = \{4\}$ are different. $\diamond$

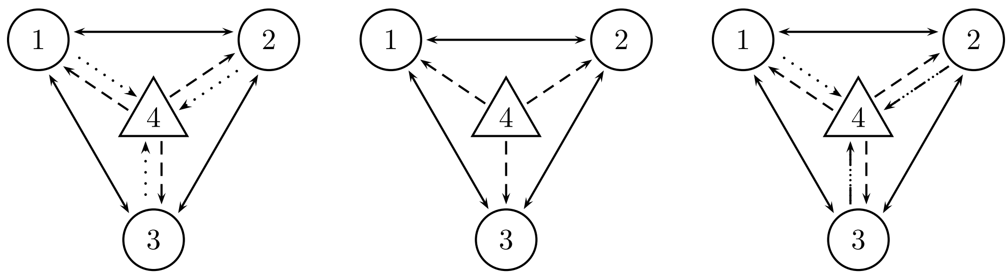

Figure 5.3: (Left) Network $\mathcal{G}_1$. (Center) Network $\mathcal{G}_{\mathcal{S}}$, where $\mathcal{S} = \{1, 2, 3\}$. (Right) Network $\mathcal{G}_2$.

Let $\mathcal{G}$ be a network, and fix a phase space P. Suppose that $\mathcal{G}$ admits nontrivial interior symmetries $\Sigma_{\mathcal{S}}$ on a subset of cells $\mathcal{S}$. Then we can decompose the phase space P as a Cartesian product $P = P_{\mathcal{S}} \times P_{\mathcal{C} \setminus \mathcal{S}}$ where

$$P_{\mathcal{S}} = \prod_{s \in \mathcal{S}} P_s \qquad \text{and} \qquad P_{\mathcal{C} \setminus \mathcal{S}} = \prod_{c \in \mathcal{C} \setminus \mathcal{S}} P_c.$$

For any $x \in P$ we write $x = (x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}})$, where $x_{\mathcal{S}} \in P_{\mathcal{S}}$ and $x_{\mathcal{C}\setminus\mathcal{S}} \in P_{\mathcal{C}\setminus\mathcal{S}}$. If $\sigma = (\sigma_C, \sigma_E) \in \Sigma_{\mathcal{S}}$, then σ_C permutes the cells of $\mathcal{S}$ and induces an action of $\Sigma_{\mathcal{S}}$ on P by permuting the cell coordinates

$$\sigma(x_c)_{c\in\mathcal{C}} = (x_{\sigma_C^{-1}(c)})_{c\in\mathcal{C}}.$$

Since $\Sigma_{\mathcal{S}}$ fixes all cells in $\mathcal{C} \setminus \mathcal{S}$ we can write

$$\sigma(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) = (\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}). \tag{5.19}$$

As in the case of symmetric networks, we can construct (some) balanced equivalence relations on a network $\mathcal{G}$ from subgroups of the interior symmetry group. Suppose that $K \subseteq \Sigma_{\mathcal{S}}$ is a subgroup. Then

$$\mathrm{Fix}_P(K) = \{(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) : \sigma x_{\mathcal{S}} = x_{\mathcal{S}}, \ \ \forall\, \sigma \in K\}.$$

Define the relation $\bowtie_K$ on the cells in $\mathcal{C}$ by

$$c \bowtie_K d \quad \Leftrightarrow \quad \exists\, \sigma = (\sigma_C, \sigma_E) \in K : \sigma_C(c) = d.$$

Then the $\bowtie_K$-classes are the K-orbits on the cells in $\mathcal{S}$, and the corresponding polydiagonal is

$$\triangle_K = \triangle_{\bowtie_K} = \mathrm{Fix}_P(K).$$

The following proposition from Golubitsky *et al.* [15] Proposition 1 (p. 397) is fundamental in the study of coupled cell networks with interior symmetries.

Proposition 5.2.5 ([15]) *Let $\mathcal{G}$ be a network admitting a nontrivial interior symmetry group $\Sigma_{\mathcal{S}}$ and fix a phase space P. Let K be any subgroup of $\Sigma_{\mathcal{S}}$. Then $\bowtie_K$ is a balanced relation on $\mathcal{C}$. In particular,* $\mathrm{Fix}_P(K)$ *is a flow invariant subspace for all $\mathcal{G}$-admissible vector fields.*

Proof: Let s_1 and s_2 be two cells on the same K-orbit. Then there exists an element $\sigma = (\sigma_C, \sigma_E)$ of K such that $\sigma_C(s_1) = s_2$ and by the consistency condition (5.18) it follows that the restriction

$$\sigma_E|_{I(s_1)} : I(s_1) \to I(s_2)$$

is an input isomorphism. Since the $\bowtie_K$-equivalence classes are exactly the K-orbits on $\mathcal{C}$ it follows that the input isomorphism $\sigma_E|_{I(s_1)}$ preserves the $\bowtie_K$ equivalence relation. Hence, by Theorem 5.1.12, it follows that $\triangle_H = \mathrm{Fix}_P(K)$ is a flow invariant subspace for all $\mathcal{G}$-admissible vector fields. $\square$

Example 5.2.6 Consider the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1 and fix a phase space P for both networks. There are two nontrivial conjugacy classes of subgroups of $\mathbf{S}_3$. The first conjugacy class is formed by the subgroup generated by a 3-cycle,

$$\mathbf{Z}_3 = \big\langle (1\,2\,3) \big\rangle.$$

The associated balanced relation has two equivalence classes $\{1, 2, 3\}$ and $\{4\}$ given by the two orbits of $\mathbf{Z}_3$ on the set of cells $\mathcal{C}$. The fixed-point subspace of $\mathbf{Z}_3$ is

$$\mathrm{Fix}_P(\mathbf{Z}_3) = \{(x, x, x, y) : x \in P_{\mathcal{S}}, y \in P_{\mathcal{C}\setminus\mathcal{S}}\} = \mathrm{Fix}_P(\mathbf{S}_3).$$

The second conjugacy class of subgroups is represented, for example, by the subgroup generated by a transposition

$$\mathbf{Z}_2 = \big\langle (1\,2) \big\rangle.$$

The associated balanced relation has three equivalence classes $\{1, 2\}$, $\{3\}$ and $\{4\}$ given by the three orbits of $\mathbf{Z}_2$ on the set of cells $\mathcal{C}$. The fixed-point subspace of $\mathbf{Z}_2$ is

$$\mathrm{Fix}_P(\mathbf{Z}_2) = \{(x, x, y, z) : x, y \in P_{\mathcal{S}}, z \in P_{\mathcal{C}\setminus\mathcal{S}}\}.$$

The other two subgroups in the conjugacy class of $\big\langle (1\,2) \big\rangle$ are the ones generated by $(1\,3)$ and $(2\,3)$. Observe that these three balanced equivalence relations given by orbits of subgroups are exactly the same balanced equivalence relations previously found by direct methods (Example 5.1.14). Therefore, in our running example all flow-invariant subspaces can be given as fixed-point subspaces of subgroups. $\diamond$

Remark 5.2.7 It is not true, even for symmetric networks, that all balanced equivalence relations are given by orbits of subgroups of the symmetry group of the network. Balanced equivalence relations that are not of this type are called *exotic*. For examples of exotic balanced relations, see Antoneli and Stewart [2, 3].

5.2.2 Admissible Vector Fields with Interior Symmetry

Let $\mathcal{G}$ be a network with a nontrivial interior symmetry group $\Sigma_{\mathcal{S}}$ on a subset of cells $\mathcal{S}$, and fix a phase space P. Recall the natural decomposition

$$P = P_{\mathcal{S}} \oplus P_{\mathcal{C}\setminus\mathcal{S}} \tag{5.20}$$

with coordinates $(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}})$. If $f : P \to P$ is a $\mathcal{G}$-admissible vector field, then we can write $f = (f_{\mathcal{S}}, f_{\mathcal{C}\setminus\mathcal{S}})$, where $f_{\mathcal{S}} : P \to P_{\mathcal{S}}$ and $f_{\mathcal{C}\setminus\mathcal{S}} : P \to P_{\mathcal{C}\setminus\mathcal{S}}$. Groupoid-equivariance of the coupled cell system implies that

$$\sigma f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) = f_{\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \tag{5.21}$$

for all $\sigma \in \Sigma_{\mathcal{S}}$.

A $\mathcal{G}$-admissible vector field f can be written as

$$f(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) = \begin{bmatrix} f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix} + \begin{bmatrix} 0 \\ h(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix}, \tag{5.22}$$

where $\tilde{f}_{\mathcal{C}\setminus\mathcal{S}}$, $h : P \to P_{\mathcal{C}\setminus\mathcal{S}}$ and $f_{\mathcal{C}\setminus\mathcal{S}} = \tilde{f}_{\mathcal{C}\setminus\mathcal{S}} + h$. The vector field $\tilde{f} = (f_{\mathcal{S}}, \tilde{f}_{\mathcal{C}\setminus\mathcal{S}})$ is the $\Sigma_{\mathcal{S}}$*-equivariant part* of f; that is, for all $\sigma \in \Sigma_{\mathcal{S}}$,

$$\sigma \tilde{f}(x) = \tilde{f}(\sigma x),$$

or more explicitly,

$$\begin{bmatrix} \sigma f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix} = \begin{bmatrix} f_{\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix}, \tag{5.23}$$

since $\Sigma_{\mathcal{S}}$ acts trivially on $P_{\mathcal{C}\setminus\mathcal{S}}$. Equation (5.22) can be seen as a decomposition of the vector field f as the sum of a $\Sigma_{\mathcal{S}}$-equivariant vector field and a non-equivariant "perturbation" with null components in $\mathcal{S}$.

Example 5.2.8 Consider the network $\mathcal{G}_2$ of Figure 5.1. Recall from Example 5.1.7 the general form of the ODEs associated with the network $\mathcal{G}_2$. Using the decomposition (5.20), we have $x_{\mathcal{S}} = (x_1, x_2, x_3)$ and $x_{\mathcal{C}\setminus\mathcal{S}} = (x_4)$ where $x_i \in \mathbf{R}^k$ $(i = 1, 2, 3)$, $x_4 \in \mathbf{R}^l$. Then by (5.22) we can write a general ODE for the network $\mathcal{G}_2$ as

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_3}, x_4) \\ \dot{x}_2 &= f(x_2, \overline{x_3, x_1}, x_4) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2}, x_4) \\ \dot{x}_4 &= g(x_4, \overline{x_1, x_2, x_3}) + h(x_4, x_1, x_2, x_3), \end{aligned}$$

where $f : \mathbf{R}^{3k} \times \mathbf{R}^l \to \mathbf{R}^k$ is a smooth map invariant under permutation of the second and third argument, $g : \mathbf{R}^l \times \mathbf{R}^{3k} \to \mathbf{R}^l$ is $\mathbf{S}_3$-invariant with respect to (x_1, x_2, x_3) and $h : \mathbf{R}^l \times \mathbf{R}^{3k} \to \mathbf{R}^l$ is a general smooth map. $\diamond$

Now we introduce another set of coordinates on P, adapted to the action of the interior symmetry group. By Proposition 5.2.5 the subspace $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is flow-invariant. Since $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is $\Sigma_{\mathcal{S}}$-invariant and $\Sigma_{\mathcal{S}}$ acts trivially on the cells in $\mathcal{C} \setminus \mathcal{S}$ we have that $P_{\mathcal{C}\setminus\mathcal{S}} \subset \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. Let

$$U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}}). \tag{5.24}$$

The action of the group $\Sigma_{\mathcal{S}}$ decomposes the set $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k,$$

where the sets $\mathcal{S}_i$ $(i = 1, \ldots, k)$ are the orbits of the $\Sigma_{\mathcal{S}}$-action. Let

$$W = \left\{ x \in P : x_c = 0,\ \forall c \in \mathcal{C} \setminus \mathcal{S} \ \text{ and } \sum_{s \in \mathcal{S}_i} x_s = 0 \ \text{ for } 1 \leqslant i \leqslant k \right\}. \tag{5.25}$$

Since W is a $\Sigma_{\mathcal{S}}$-invariant subspace of $P_{\mathcal{S}}$ and $W \cap U = \{0\}$ we can decompose the phase space P as a direct sum of $\Sigma_{\mathcal{S}}$-invariant subspaces

$$P = W \oplus U. \tag{5.26}$$

In particular, (5.25) implies that vectors in W, when written in coupled cell coordinates, have zero components on all cells in $\mathcal{C} \setminus \mathcal{S}$.

We can choose coordinates (w, u) with $w \in W$ and $u \in U$ adapted to the decomposition (5.26) and write any admissible vector field f as

$$f(w, u) = \begin{bmatrix} f_W(w, u) \\ f_U(w, u) \end{bmatrix} + \begin{bmatrix} 0 \\ h(w, u) \end{bmatrix}, \tag{5.27}$$

where f_U, $h : P \to U$ and $f_W : P \to W$ satisfies

$$\sigma f_W(w, u) = f_W(\sigma w, u), \qquad \forall\, \sigma \in \Sigma_{\mathcal{S}}.$$

With respect to the decomposition (5.26), the equivariant part of f is written as $\tilde{f}(w, u) = (f_W(w, u), f_U(w, u))$ and for all $\sigma \in \Sigma_{\mathcal{S}}$ we have

$$\begin{bmatrix} \sigma f_W(w, u) \\ f_U(w, u) \end{bmatrix} = \begin{bmatrix} f_W(\sigma w, u) \\ f_U(\sigma w, u) \end{bmatrix} \tag{5.28}$$

since $\Sigma_{\mathcal{S}}$ acts trivially on $U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$.

Example 5.2.9 Consider the network $\mathcal{G}_2$ of Figure 5.1. With respect to the decomposition (5.20) adapted to the network structure, the total phase space P has coordinates $x_{\mathcal{S}} = (x_1, x_2, x_3)$ and $x_{\mathcal{C}\setminus\mathcal{S}} = (x_4)$ where $x_i \in \mathbf{R}^k$

$(i = 1, 2, 3)$, $x_4 \in \mathbf{R}^l$. Now with respect to the decomposition (5.26) adapted to the $\mathbf{S}_3$-action on P we have that

$$W = \{(w_1, w_2, -w_1 - w_2, 0) : w_1, w_2 \in \mathbf{R}^k\}$$

and

$$U = \mathrm{Fix}_P(\mathbf{S}_3) = \{(u_1, u_1, u_1, u_2) : u_1 \in \mathbf{R}^k,\ u_2 \in \mathbf{R}^l\}.$$

$\diamond$

In the linear case, we may choose a basis of P adapted to the decomposition (5.26) and then a $\mathcal{G}$-admissible linear vector field L can be written as

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{5.29}$$

where $B = L|_U : U \to U$, $C : W \to U$ and $A : W \to W$ satisfies (by (5.28))

$$A\,\sigma = \sigma A, \qquad \forall\, \sigma \in \Sigma_{\mathcal{S}}.$$

Note that Proposition 5.2.5 implies that $U = \mathrm{Fix}_P\,(\Sigma_{\mathcal{S}})$ is invariant under L.

The spectral properties of L in (5.29) are given by Golubitsky *et al.* [15] Lemma 1 (p. 399). Since we will use these results several times we reproduce it here.

Lemma 5.2.10 ([15]) *Let $\mathcal{G}$ be a network admitting a nontrivial group of interior symmetries $\Sigma_{\mathcal{S}}$ and fix a total phase space P. Let $L : P \to P$ be a $\mathcal{G}$-admissible linear vector field and consider the decomposition of L with respect to the decomposition* (5.26) *given by* (5.29)*. Then the following hold:*

- (i) *The eigenvalues of L are the eigenvalues of A together with the eigenvalues of B.*
- (ii) *A vector $u \in U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is an eigenvector of B with eigenvalue ν if and only if u is an eigenvector of L with eigenvalue ν.*
- (iii) *If $w \in W$ is an eigenvector of A with eigenvalue μ, then there exists an eigenvector v of L with eigenvalue μ of the form*

  $$v = w + u$$

  where $u \in U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$.
- (iv) *All eigenspaces of A are $\Sigma_{\mathcal{S}}$-invariant.*

Proof: Parts (i) (ii) and (iii) are consequences of the block form (5.29) of L. Part (iv) follows from the $\Sigma_{\mathcal{S}}$-equivariance of A. $\square$

Example 5.2.11 We continue our running example, the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1. The general form of the admissible linear mappings associated with the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1 are (in cell coordinates)

$$L_1 = \begin{pmatrix} a & b & b & d \\ b & a & b & d \\ b & b & a & d \\ e & e & e & c \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} a & b & b & d \\ b & a & b & d \\ b & b & a & d \\ e_1 & e_2 & e_3 & c \end{pmatrix}$$

where a, b are $k \times k$ matrices, c is a $l \times l$ matrix, d is a $k \times l$ matrix and e, e_1, e_2, e_3 are $l \times k$ matrices. Choosing adequate bases for W and U the linear mappings L_1 and L_2 can be written as

$$L_1 = \begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a+2b & d \\ 0 & 0 & 3e & c \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a+2b & d \\ e_1 - e_3 & e_2 - e_3 & e_1 + e_2 + e_3 & c \end{pmatrix}.$$

$\diamond$

5.3 Synchrony-Breaking Bifurcations

Now we study local bifurcations in coupled cell networks with nontrivial interior symmetries. We are interested in codimension-one synchrony-breaking bifurcations. Steady-state and Hopf bifurcations in coupled cell networks with interior symmetries were studied by Golubitsky *et al.* [15].

5.3.1 Local Bifurcations in Coupled Cell Systems

Let $\mathcal{G}$ be a coupled cell network and fix a phase space P. Let $f : P \times \mathbf{R}^k \to P$ be a smooth k-parameter family of $\mathcal{G}$-admissible vector fields in P and assume that the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \tag{5.30}$$

has a synchronous equilibrium x_0 in $\triangle_I$ (the polydiagonal subspace of P associated with the input equivalence relation $\sim_I$). In the present context we may assume that

$$f(x_0, \lambda) \equiv 0$$

and that a bifurcation occurs at $\lambda = 0$. Let $L = (\mathrm{d}f)_{(x_0,0)}$ be the linearisation of f at $(x_0, 0)$ and denote by E^c the center subspace of L.

Local bifurcations in coupled cell networks can be divided into two types according to E^c is contained or not into the flow-invariant subspace $\triangle_I$.

Definition 5.3.1 We say that a coupled cell system (5.30) undergoes a *synchrony-preserving* bifurcation at a synchronous equilibrium in $\triangle_I$ if $E^c \subset \triangle_I$ and that (5.30) undergoes a *synchrony-breaking* bifurcation if $E^c \not\subset \triangle_I$.

Now we specialize to codimension-one bifurcations; that is, $f : P \times \mathbf{R} \to P$ is a smooth 1-parameter family of $\mathcal{G}$-admissible vector fields in P. These bifurcations fall into two classes: *steady-state* bifurcations ($L|_{E^c}$ has a zero eigenvalue) and *Hopf* bifurcations ($L|_{E^c}$ has a pair of purely imaginary eigenvalues). The new steady-states and periodic solutions that emanate from the synchrony-preserving bifurcations are themselves synchronous solutions. For the remainder of this chapter we will focus on codimension-one synchrony-breaking bifurcations from a synchronous equilibrium.

5.3.2 Local Bifurcations with Interior Symmetry

Interior symmetries introduce genuine restrictions on the form of the linearisation and this structure can be used to study certain kind of synchrony-breaking bifurcations, namely, the bifurcations that break the interior symmetry.

Let $\mathcal{G}$ be a network admitting a nontrivial group of interior symmetries $\Sigma_{\mathcal{S}}$ on $\mathcal{S}$ and fix a phase space P. First, note that the polydiagonal subspace $\triangle_I$ associated to the input equivalence relation $\sim_I$ satisfies

$$\triangle_I \subseteq \mathrm{Fix}_P(\Sigma_{\mathcal{S}}).$$

Since we are interested in synchrony-breaking bifurcations that also break the interior symmetry we may assume that $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ and that the center subspace $E^c(L)$ associated to the critical eigenvalues satisfies

$$E^c(L) \not\subset \mathrm{Fix}_P(\Sigma_{\mathcal{S}}). \tag{5.31}$$

However, this is not enough to exclude the possibility of having critical eigenvectors in $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ in a synchrony-breaking bifurcation. That is, we could have a situation where some critical eigenvectors belong to $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ and the others are outside $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. Indeed, it is well known that (non-symmetric) coupled cell systems generically can exhibit mode interaction in codimension-one bifurcations. See for example Golubitsky and Lauterbach [14] and chapter 7. In this work we make a stronger assumption. We assume

$$E^c(L) \cap \mathrm{Fix}_P(\Sigma_{\mathcal{S}}) = \{0\} \tag{5.32}$$

and so we exclude the possibility of having eigenvectors in $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. This situation corresponds to a synchrony-breaking bifurcation that "breaks *only* the interior symmetry".

Definition 5.3.2 Let $f : P \to P$ be a $\mathcal{G}$-admissible vector field and let $L = (\mathrm{d}f)_{(x_0)}$ be the linearisation of f at x_0. Consider the decomposition (5.26) of P adapted to the $\Sigma_{\mathcal{S}}$-action and write L in block form as

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}.$$

Then the matrix A is called the $\Sigma_{\mathcal{S}}$*-equivariant subblock* of L.

If we write f using coordinates (w, u) adapted to the decomposition $P = W \oplus U$ as

$$f(w,u) = \begin{bmatrix} f_W(w,u) \\ f_U(w,u) \end{bmatrix} + \begin{bmatrix} 0 \\ h(w,u) \end{bmatrix}$$

then

$$A = \left(\mathrm{d}_{(1)} f_W\right)_{(x_0)}$$

where $x_0 = (w_0, u_0)$ and

$$\left(\mathrm{d}_{(1)} f_W\right)_{(x_0)} \cdot w = \left(\mathrm{d} f_W\right)_{(w_0,u_0)} \cdot (w, 0)$$

for all $w \in W$.

Remark 5.3.3 It can be shown that the following three conditions are equivalent:

(a) $E^c(L) \cap \mathrm{Fix}_P(\Sigma_{\mathcal{S}}) = \{0\}$.

(b) $\dim E^c(L) = \dim E^c(A)$.

(c) All the critical eigenvalues of L come from the $\Sigma_{\mathcal{S}}$-equivariant sub--block A of L.

It is obvious that (a) implies both (b) and (c). On the other hand, to prove that (b) implies (a), we observe that by Lemma 5.2.10 (iii), we always have $\dim E^{\mathrm{c}}(A) \leqslant \dim E^{\mathrm{c}}(L)$. Finally, to prove that (c) implies (a), we observe that the block form of L guarantees that no generalized eigenvector associated to an eigenvalue coming from subblock A belong to $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$.

In general f is not $\Sigma_{\mathcal{S}}$-equivariant and L does not commute with $\Sigma_{\mathcal{S}}$. In particular, $E^{\mathrm{c}}(L) \not\subset W$. However, the block matrix A does commute with $\Sigma_{\mathcal{S}}$ and thus $E^{\mathrm{c}}(A) \subset W$ is $\Sigma_{\mathcal{S}}$-invariant. Moreover, if A has purely imaginary eigenvalues there is a natural action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $E^{\mathrm{c}}(A)$, where $\mathbf{S}^1$ acts by $\exp(sA^{\mathrm{t}})$.

Definition 5.3.4 Consider a 1-parameter family of coupled cell systems (5.30) with interior symmetry group $\Sigma_{\mathcal{S}}$ on $\mathcal{S}$ undergoing a codimension--one synchrony-breaking bifurcation at a synchronous equilibrium x_0 when $\lambda = 0$. We say that f undergoes a *codimension-one interior symmetry--breaking bifurcation* if the following conditions hold:

(a) All the critical eigenvalues μ of L come from the $\Sigma_{\mathcal{S}}$-equivariant sub--block A of L.

(b) The critical eigenvalues μ extend uniquely and smoothly to eigenvalues $\mu(\lambda)$ of $(\mathrm{d}f)_{(x_0,\lambda)}$ for λ near 0.

(c) The *eigenvalues crossing condition*:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{Re}(\mu(\lambda))\bigg|_{\lambda=0} \neq 0. \tag{5.33}$$

More specifically, the bifurcation problem (5.30) is called

- A *codimension-one interior symmetry-breaking steady-state bifurcation* if, in addition to the conditions (a), (b), (c) above, the matrix A has a zero eigenvalue and the associated center subspace is given by

$$E_0(A) = \ker(A). \tag{5.34}$$

- A *codimension-one interior symmetry-breaking Hopf bifurcation* if, in addition to the conditions (a), (b), (c) above, the matrix A is non-singular and all the critical eigenvalues (after rescaling time if necessary) have the form $\pm i$ and the associated center subspace is given by

$$E_i(A) = \{x \in W : (A^2 + 1)x = 0\}. \tag{5.35}$$

Example 5.3.5 Consider the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1. Suppose that for all cells c we choose the internal phase space to be $P_c = \mathbf{C}$ and so the total phase space is $P = \mathbf{C}^4$. Consider the decomposition of $P = W \oplus U$ adapted to the $\mathbf{S}_3$-action. Then

$$W = \{(w_1, w_2, -w_1 - w_2, 0) : w_1, w_2 \in \mathbf{C}\}$$

and

$$U = \mathrm{Fix}_P(\mathbf{S}_3) = \{(u_1, u_1, u_1, u_2) : u_1, u_2 \in \mathbf{C}\}.$$

and W is a $\mathbf{S}_3$-simple representation ($W = W_1 \oplus W_2$ where W_1, W_2 are two isomorphic $\mathbf{S}_3$-absolutely irreducible spaces). Now consider a 1-parameter family $f : P \times \mathbf{R} \to P$ of $\mathcal{G}$-admissible vector fields on P undergoing a codimension-one interior symmetry-breaking Hopf bifurcation at an equilibrium point x_0 when $\lambda = 0$. Since W is a $\mathbf{S}_3$-simple representation, one necessarily has that $E^c(A) = W$. Moreover, the action of the circle group $\mathbf{S}^1$ defined by $\exp(sA^{\mathrm{t}})$ is equivalent to the standard action of $\mathbf{S}^1$ on $\mathbf{C}^2$, that is,

$$\theta \cdot (z_1, z_2) = (\mathrm{e}^{i\theta} z_1, \mathrm{e}^{i\theta} z_2),$$

for all $\theta \in \mathbf{S}^1$ and $z_1, z_2 \in \mathbf{C}$. $\diamond$

Chapter 6

Interior Symmetry-Breaking Hopf Theorem

A paper with the contents of this chapter has been published [1].

The Equivariant Hopf Theorem (Theorem 3.2.6) concerns periodic solutions to symmetric differential equations near a point where the linearization has purely imaginary eigenvalues. In particular, we can use this theorem in symmetric coupled cell systems. Golubitsky *et al.* [15] prove an analogue of the Equivariant Hopf Theorem for coupled cell systems with interior symmetries. They prove the existence of states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatial symmetries*. However, their result has novel and rather restrictive features. In this chapter we address the second main question in this thesis that is Hopf bifurcation in coupled cell systems associated to networks with interior symmetries. We generalize the result of Golubitsky *et al.* [15] described above. Specifically, in this chapter we obtain the full analogue of the Equivariant Hopf Theorem for networks with symmetries (Theorem 6.1.3). We extend the result of Golubitsky *et al.* [15] obtaining states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatio-temporal symmetries*, that is, corresponding to *interiorly* $\mathbf{C}$-axial subgroups of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$. This new version of the Hopf Theorem with interior symmetries includes the previous version as a special case and is in complete analogy with the Equivariant Hopf Theorem (see Theorem 6.1.3). In section 6.1 we present this result. The rest of the chapter is dedicated to the proof of this theorem through two approaches. In section 6.2 we use a modification of the Liapunov-Schmidt reduction to arrive at a situation where the proof of the Standard Hopf Bifurcation Theorem (Theorem 2.2.1) can be applied. In section 6.3 we use a center manifold reduction to reach a phase where the Standard Hopf Theorem gives the result. This completes the program of generalising the main result from equivariant Hopf bifurcation theory to the class of networks with interior symmetries.

6.1 Interior Symmetry-Breaking Hopf Theorem

Let $\mathcal{G}$ be a coupled cell network admitting a nontrivial group of interior symmetries $\Sigma_{\mathcal{S}}$ on a subset of cells $\mathcal{S}$ (recall Definition 5.2.1) and choose a total phase space P. Recall the decomposition of P as the cartesian product $P = P_{\mathcal{S}} \times P_{\mathcal{C}\setminus\mathcal{S}}$ where $P_{\mathcal{S}} = \prod_{s\in\mathcal{S}} P_s$ and $P_{\mathcal{C}\setminus\mathcal{S}} = \prod_{c\in\mathcal{C}\setminus\mathcal{S}} P_c$. For any $x \in P$, we write $x = (x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}})$ where $x_{\mathcal{S}} \in P_{\mathcal{S}}$ and $x_{\mathcal{C}\setminus\mathcal{S}} \in P_{\mathcal{C}\setminus\mathcal{S}}$ and we can take the action of $\Sigma_{\mathcal{S}}$ on P given by:

$$\sigma\left(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}\right) = \left(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}\right) \quad (\sigma \in \Sigma_{\mathcal{S}}). \tag{6.1}$$

Here $\Sigma_{\mathcal{S}}$ acts on $x_{\mathcal{S}}$ by permuting the coordinates corresponding to the cells in $\mathcal{S}$. For a subgroup $K \subseteq \Sigma_{\mathcal{S}}$ define

$$\mathrm{Fix}_P(K) = \{(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) : \sigma x_{\mathcal{S}} = x_{\mathcal{S}},\ \ \forall\, \sigma \in K\}.$$

By Proposition 5.2.5 the subspace $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is flow-invariant under any $\mathcal{G}$-admissible vector field on P. Since $\mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is $\Sigma_{\mathcal{S}}$-invariant and $\Sigma_{\mathcal{S}}$ acts trivially on the cells in $\mathcal{C} \setminus \mathcal{S}$ we have that $P_{\mathcal{C}\setminus\mathcal{S}} \subset \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. Let $U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. Recall that the action of the group $\Sigma_{\mathcal{S}}$ decomposes the set $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k$, where the sets $\mathcal{S}_i$ $(i = 1, \ldots, k)$ are the orbits of the $\Sigma_{\mathcal{S}}$-action. As before let

$$W = \left\{ x \in P : x_c = 0,\ \forall\, c \in \mathcal{C} \setminus \mathcal{S} \text{ and } \sum_{s\in\mathcal{S}_i} x_s = 0 \ \text{ for } 1 \leqslant i \leqslant k \right\}. \tag{6.2}$$

Since W is a $\Sigma_{\mathcal{S}}$-invariant subspace of $P_{\mathcal{S}}$ and $W \cap U = \{0\}$ we can decompose the phase space P as a direct sum of $\Sigma_{\mathcal{S}}$-invariant subspaces:

$$P = W \oplus U. \tag{6.3}$$

Consider a smooth 1-parameter family $f : P \times \mathbf{R} \to P$ of $\mathcal{G}$-admissible vector fields on P and assume that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \tag{6.4}$$

has an equilibrium point $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. We can choose coordinates (w, u) with $w \in W$ and $u \in U$ adapted to the decomposition (6.3) and write any admissible vector field f as

$$f(w,u,\lambda) = \begin{bmatrix} f_W(w,u,\lambda) \\ f_U(w,u,\lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ h(w,u,\lambda) \end{bmatrix}, \tag{6.5}$$

where $f_U, h : P \times \mathbf{R} \to U$, $f_W : P \times \mathbf{R} \to W$ and $\tilde{f}(w,u,\lambda) = (f_W(w,u,\lambda), f_U(w,u,\lambda))$ is the $\Sigma_{\mathcal{S}}$-*equivariant part* of f. That is,

$$\begin{bmatrix} \sigma f_W(w,u,\lambda) \\ f_U(w,u,\lambda) \end{bmatrix} = \begin{bmatrix} f_W(\sigma w,u,\lambda) \\ f_U(\sigma w,u,\lambda) \end{bmatrix} \qquad (\forall\, \sigma \in \Sigma_{\mathcal{S}})\,, \tag{6.6}$$

since $\Sigma_{\mathcal{S}}$ acts trivially on U.

In the linear case, we may choose a basis of P adapted to the decomposition (6.3) and then a $\mathcal{G}$-admissible linear vector field L can be written as

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{6.7}$$

where $B = L|_U : U \to U$, $C : W \to U$ and $A : W \to W$ satisfies, by (6.6),

$$A\,\sigma = \sigma A \qquad (\forall\, \sigma \in \Sigma_{\mathcal{S}})\,. \tag{6.8}$$

Recall the spectral properties of L in Lemma 5.2.10.

Before stating the next theorem let us introduce an important concept which generalises the notion of $\mathbf{C}$-axial subgroup from equivariant bifurcation theory.

Definition 6.1.1 Let $\mathcal{G}$ be a coupled cell network admitting a nontrivial group of interior symmetries $\Sigma_{\mathcal{S}}$ on a subset $\mathcal{S}$. Let P denote the total phase space and consider the decomposition (6.3) of P adapted to the $\Sigma_{\mathcal{S}}$-action. Suppose that there is an action of circle group $\mathbf{S}^1$ on W which commutes with the action of $\Sigma_{\mathcal{S}}$. Let $E \subset W$ be a $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-invariant subspace. An isotropy subgroup $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ is called *interiorly* $\mathbf{C}$*-axial (on E)* if

$$\dim_{\mathbf{R}} \mathrm{Fix}_E(\Delta) = 2.$$

Assume that L as in (6.7) has $\pm i$ as eigenvalues that come only from the subblock A of L and that they are the only critical eigenvalues of L. Consider $A^c \equiv A|_{E_i(A)}$. As A has $\pm i$ as eigenvalues there is a natural action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on P, where $\mathbf{S}^1$ acts on $E_i(A)$ by $\exp(s(A^c)^t)$ and trivially on $P \setminus E_i(A)$. The action of $\Sigma_{\mathcal{S}}$ on P is given by (6.1).

Now suppose the family (6.4) undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at the equilibrium x_0 when $\lambda = \lambda_0$ (recall Definition 5.3.4). Then the center subspace $E^c(A) \equiv E_i(A)$ of the $\Sigma_{\mathcal{S}}$-equivariant subblock A of the linearization $L = (\mathrm{d}f)_{(x_0,\lambda_0)}$ of f at (x_0, λ_0) is a $\Sigma_{\mathcal{S}}$-invariant subspace of W. Therefore, the action of the circle group $\mathbf{S}^1$ defined by $\exp(s(A^c)^{\mathrm{t}})$ commutes with the action of $\Sigma_{\mathcal{S}}$. Thus $E^c(A)$ is a $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-invariant subspace and so there is a well-defined action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $E^c(A)$ (and W).

Example 6.1.2 Consider the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of Figure 5.1. Suppose that for all cells c we choose the internal phase space to be $P_c = \mathbf{C}$ and so the total phase space is $P = \mathbf{C}^4$. Suppose that a smooth 1-parameter family $f : P \times \mathbf{R} \to P$ of $\mathcal{G}$-admissible vector fields on P undergoes a codimension--one interior symmetry-breaking Hopf bifurcation at the equilibrium $x_0 = 0$ when $\lambda = 0$. Then $E_i(A) = W$, where A is the $\Sigma_{\mathcal{S}}$-equivariant subblock of the linearization $L = (\mathrm{d}f)_{(0,0)}$ of f at $(0, 0)$. In Example 5.3.5 we have observed that the action of $\mathbf{S}^1$ on W, given by $\exp(sA^{\mathrm{t}})$, can be identified with the standard action of $\mathbf{S}^1$ on $\mathbf{C}^2$. There are three non-trivial conjugacy classes of isotropy subgroups of $\mathbf{S}_3 \times \mathbf{S}^1$ acting on W. The first conjugacy class of subgroups is represented for example by the subgroup

$$\mathbf{Z}_2 = \big\langle ((1\,2), \mathbf{1}) \big\rangle.$$

The fixed-point subspace of $\mathbf{Z}_2$ is

$$\mathrm{Fix}_W(\mathbf{Z}_2) = \{(-w, -w, 2w, 0) : w \in \mathbf{C}\}.$$

The second conjugacy class of subgroups is represented for example by the subgroup

$$\tilde{\mathbf{Z}}_2 = \big\langle ((1\,2), \pi) \big\rangle.$$

The fixed-point subspace of $\tilde{\mathbf{Z}}_2$ is

$$\mathrm{Fix}_W(\tilde{\mathbf{Z}}_2) = \{(w, -w, 0, 0) : w \in \mathbf{C}\}.$$

The third conjugacy class of subgroups is represented for example by the subgroup

$$\tilde{\mathbf{Z}}_3 = \big\langle ((1\,2\,3), \tfrac{2\pi}{3}) \big\rangle.$$

The fixed-point subspace of $\tilde{\mathbf{Z}}_3$ is

$$\mathrm{Fix}_W(\tilde{\mathbf{Z}}_3) = \{(w, \mathrm{e}^{i\frac{2\pi}{3}} w, \mathrm{e}^{i\frac{4\pi}{3}} w, 0) : w \in \mathbf{C}\}.$$

$\diamond$

The main result of this chapter is the interior symmetry-breaking Hopf bifurcation Theorem:

Theorem 6.1.3 *Let $\mathcal{G}$ be a coupled cell network admitting a nontrivial group of interior symmetries $\Sigma_{\mathcal{S}}$ relative to a subset $\mathcal{S}$ of cells and fix a phase space P. Consider* (6.4) *where $f : P \times \mathbf{R} \to P$ is a smooth* 1*-parameter family of $\mathcal{G}$-admissible vector fields on P. Suppose that a codimension-one interior symmetry-breaking Hopf bifurcation (see Definition 5.3.4) occurs at an equilibrium point $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ when $\lambda = 0$. Take $L = (df)_{(x_0,0)}$. Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be an interiorly $\mathbf{C}$-axial subgroup (on $E^c(A)$). Then generically there exists a family of small amplitude periodic solutions of* (6.4) *bifurcating from $(x_0, 0)$ and having period near 2π. Moreover, to lowest order in the bifurcation parameter λ, the solution $x(t)$ is of the form*

$$x(t) \approx w(t) + u(t) \tag{6.9}$$

where $w(t) = \exp(tL)w_0$ ($w_0 \in \mathrm{Fix}_W(\Delta)$) has exact spatio-temporal symmetry Δ on the cells in $\mathcal{S}$ and $u(t) = \exp(tL)u_0$ ($u_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$) is synchronous on the $\Sigma_{\mathcal{S}}$-orbits of cells in $\mathcal{S}$.

We call such a state a *synchronously modulated Δ-symmetric wave* on $\mathcal{S}$.

Remarks 6.1.4

(a) The above theorem asserts no restriction on $u_j(t)$ when $j \in \mathcal{C} \setminus \mathcal{S}$.

(b) Theorem 6.1.3 generalises the interior symmetry Hopf Theorem of Golubitsky *et al.* [15] Theorem 6.3. Given a subgroup $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ we define the *spatial subgroup* of Δ to be $K = \Delta \cap \Sigma_{\mathcal{S}}$. A subgroup Δ is called *spatially* $\mathbf{C}$*-axial* if

$$\dim_{\mathbf{R}} \mathrm{Fix}_{E_i(A)}(\Delta) = \dim_{\mathbf{R}} \mathrm{Fix}_{E_i(A)}(K) = 2$$

where K is the spatial subgroup of Δ. Obviously every spatially $\mathbf{C}$-axial subgroup is interiorly $\mathbf{C}$-axial. Since the interior symmetry Hopf Theorem of Golubitsky *et al.* [15] is proved for all spatially $\mathbf{C}$-axial subgroups, it is a special case of Theorem 6.1.3.

(c) Theorem 6.1.3 holds if the assumption (5.35) of Definition 5.3.4 is generalised to: the matrix A is non-singular, semi-simple and (after rescaling time if necessary) all the critical eigenvalues have the form $k_l i$ ($k_l \in \mathbf{Z}$).

In sections 6.2 and 6.3 we prove Theorem 6.1.3 by two different approaches: the Liapunov-Schmidt approach and the center manifold approach. There are different advantages of each one of them.

6.2 Liapunov-Schmidt Approach

In the Liapunov-Schmidt approach, the proof of Theorem 6.1.3 follows from a couple of lemmas that we state and prove below. We start by setting up the framework.

Consider the system (6.4) and assume that the linearization $L = (\mathrm{d}f)_{(x_0,0)}$ of f at $(x_0, 0)$ is non-singular but has a pair of purely imaginary eigenvalues. Let $C^0_{2\pi}(P)$ be the space consisting of all continuous 2π-periodic mappings from $\mathbf{R}$ to P endowed with the C^0 norm and $C^1_{2\pi}(P)$ be the space consisting of all continuous differentiable 2π-periodic mappings from $\mathbf{R}$ to P endowed with the C^1 norm.

By introducing a perturbed period parameter τ we can rescale time, from t to $s(1+\tau)t$, and consider the operator $\mathcal{F} : C^1_{2\pi}(P) \times \mathbf{R} \times \mathbf{R} \to C^0_{2\pi}(P)$ given by

$$\mathcal{F}(x, \lambda, \tau) = (1+\tau)\frac{\mathrm{d}x}{\mathrm{d}s}(s) - f(x(s), \lambda). \tag{6.10}$$

The 2π-periodic solutions of the equation $\mathcal{F}(x, \lambda, \tau) = 0$ near $(0, 0, 0)$ correspond bijectively to the small amplitude periodic solutions of (6.4) near x_0 and with period near 2π. As it is well known, the operator $\mathcal{F}$ is $\mathbf{S}^1$-equivariant with respect to the *phase shift action* of $\mathbf{S}^1$ on the spaces $C^1_{2\pi}(P)$ and $C^0_{2\pi}(P)$; that is, if $x \in C^0_{2\pi}(P)$ and $\theta \in \mathbf{S}^1$ then

$$\big(\theta \cdot x\big)(s) = x(s+\theta)$$

and thus

$$\theta \cdot \mathcal{F}(x, \tau, \lambda) = \mathcal{F}(\theta \cdot x, \tau, \lambda).$$

The linearization of $\mathcal{F}$ about the origin is

$$\mathcal{L}(x) = \frac{\mathrm{d}x}{\mathrm{d}s}(s) - L\,x(s) \tag{6.11}$$

and $\ker(\mathcal{L})$ consists of all functions $\mathrm{Re}(\mathrm{e}^{is}v)$ where v is an eigenvector of L associated with the eigenvalue i.

In the Standard Hopf Theorem $\ker(\mathcal{L})$ is two-dimensional and Liapunov--Schmidt reduction in the presence of symmetry leads to a reduced equation that can be solved for a unique branch of 2π-periodic solutions as long as the eigenvalues crossing condition is valid (recall section 2.2). In the equivariant context, $\ker(\mathcal{L})$ may be higher-dimensional – generically $\ker(\mathcal{L})$ is a Γ-simple representation (Proposition 3.2.4). The proof of the Equivariant Hopf Bifurcation Theorem proceeds by restricting the Liapunov-Schmidt reduced equation to the fixed-point subspace $\mathrm{Fix}_{E_i(L)}(\Delta)$ of a $\mathbf{C}$-axial subgroup $\Delta \subset$

$\Gamma \times \mathbf{S}^1$, which is two-dimensional. Then the proof is completed as in the standard Hopf Bifurcation Theorem.

That approach does not work in the context of interior symmetries since in general there is no action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on $E_i(L)$, because the original vector field f (and its linearization L) is not $\Sigma_{\mathcal{S}}$-equivariant. Nevertheless, we shall introduce a "modified Liapunov-Schmidt procedure" that does work in the context of interior symmetries.

The decomposition (6.3) described in section 6.1 induces the decompositions

$$C^0_{2\pi}(P) = C^0_{2\pi}(W) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$$

and

$$C^1_{2\pi}(P) = C^1_{2\pi}(W) \oplus C^1_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})).$$

In our modification of the standard Liapunov-Schmidt procedure we consider the following action of the group $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on the spaces $C^0_{2\pi}(P)$ and $C^1_{2\pi}(P)$. Let us write $x \in C^0_{2\pi}(P)$ as $x(s) = (w(s), u(s))$ where $w \in C^0_{2\pi}(W)$ and $u \in C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$. Then for $(\delta, \theta) \in \Sigma_{\mathcal{S}} \times \mathbf{S}^1$,

$$(\delta, \theta) \cdot x(s) = (\delta, \theta) \cdot (w(s), u(s)) = \big(\delta w(s+\theta), u(s)\big). \tag{6.12}$$

The difference from the usual action on the loop space (see, for example Golubitsky *et al.* [21] Chapter XVI Section 3) is that in (6.12) the group $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ acts trivially on $C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$ and $C^1_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$, respectively. A straightforward consequence of the above definition is stated in the next lemma for convenience.

Lemma 6.2.1 *For any subgroup Δ of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ we have the decompositions*

$$\mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta) = C^0_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$$

and

$$\mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) = C^1_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^1_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})).$$

Proof: Let $x \in C^0_{2\pi}(P)$ be written as $x(s) = (w(s), u(s))$ where $w \in C^0_{2\pi}(W)$ and $u \in C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$. Then (6.12) implies that

$$C^0_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})) \subseteq \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta).$$

Now let $(\delta, \theta) \in \Delta$ and suppose that $x \in \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta)$. Then

$$(\delta, \theta) \cdot x(s) = x(s).$$

The decomposition $x(s) = (w(s), u(s))$ yields

$$\big((\delta w)(s+\theta), u(s)\big) = (w(s), u(s));$$

that is, $w(s) \in \mathrm{Fix}_W(\Delta)$ and $u(s) \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ for all $s \in \mathbf{R}$. Hence

$$\mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta) \subseteq C^0_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})).$$

Therefore,

$$\mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta) = C^0_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})).$$

The same argument with $C^1_{2\pi}$ instead of $C^0_{2\pi}$ gives the other equality. $\square$

Lemma 6.2.2 *Let $L : P \to P$ be a $\mathcal{G}$-admissible linear mapping. Let $\mathcal{L} : C^1_{2\pi}(P) \times \mathbf{R} \times \mathbf{R} \to C^0_{2\pi}(P)$ be the linear operator given by equation* (6.11) *and $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be a subgroup. Then we have that*

$$\mathcal{L}\big(C^1_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))\big) \subseteq C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})),$$
$$\mathcal{L}\big(C^1_{2\pi}(\mathrm{Fix}_W(\Delta))\big) \subseteq \big(C^0_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))\big)$$

and

$$\mathcal{L}\big(\mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta)\big) \subseteq \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta). \tag{6.13}$$

In particular, we can define a linear operator

$$\mathcal{L}_\Delta : \mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) \longrightarrow \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta) \tag{6.14}$$

by restriction.

Proof: Note that since the circle group $\mathbf{S}^1$ acts on the domain of the mappings, all the decompositions above are $\mathbf{S}^1$-invariant.

First suppose $x(s) = (0, u(s))$ with $u(s) \in C^1_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$. Consider the block form of L given in (6.7) and recall (6.12). Then

$$\mathcal{L}(x) = \frac{\mathrm{d}u}{\mathrm{d}s}(s) - L(0, u(s)).$$

If $\sigma \in \Sigma_{\mathcal{S}}$, then

$$\begin{aligned}\sigma\mathcal{L}(x) &= \sigma\frac{\mathrm{d}u}{\mathrm{d}s}(s) - \sigma L\,(0, u(s))\\ &= \frac{\mathrm{d}\sigma u}{\mathrm{d}s}(s) - L\,(0, u(s))\\ &= \frac{\mathrm{d}u}{\mathrm{d}s}(s) - L\,(0, u(s))\\ &= \mathcal{L}(x).\end{aligned}$$

The second equality above follows from the fact that

$$\sigma\big(L\,(0,u)\big) = L\,(0,u)$$

for all $\sigma \in \Sigma_{\mathcal{S}}$. Therefore, we have $\mathcal{L}(x(s)) \in C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}}))$.

Next suppose that $x(s) = (w(s), 0)$ with $w(s) \in C^1_{2\pi}(\mathrm{Fix}_W(\Delta))$. Since $w(s) \in \mathrm{Fix}_W(\Delta)$ for all $s \in \mathbf{R}$, we have that

$$(\delta, \theta) \cdot w(s) = \delta w(s + \theta) = w(s) \tag{6.15}$$

for all $(\delta, \theta) \in \Delta$, $s \in \mathbf{R}$. Write

$$\mathcal{L}(x) = \big([\mathcal{L}(x)]_1(s), [\mathcal{L}(x)]_2(s)\big),$$

with

$$[\mathcal{L}(x)]_1(s) \in W \quad \text{for all } s \in \mathbf{R}$$

and

$$[\mathcal{L}(x)]_2(s) \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}}) \quad \text{for all } s \in \mathbf{R}.$$

Then

$$[\mathcal{L}(x)]_1(s) = \frac{\mathrm{d}w}{\mathrm{d}s}(s) - A\,w(s)$$

and

$$[\mathcal{L}(x)]_2(s) = -[C\,w(s) + B\,0] = -Cw(s).$$

Clearly, $[\mathcal{L}(x)]_2(s) \in U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. Let $(\delta, \theta) \in \Delta$; then

$$\begin{aligned}
(\delta, \theta) \cdot [\mathcal{L}(x)]_1(s) &= (\delta, \theta) \cdot \frac{\mathrm{d}w}{\mathrm{d}s}(s) - (\delta, \theta) \cdot A\,w(s) \\
&= \delta\,\frac{\mathrm{d}w}{\mathrm{d}s}(s + \theta) - \delta A\,w(s + \theta) \\
&= \frac{\mathrm{d}\delta w}{\mathrm{d}s}(s + \theta) - A\delta\,w(s + \theta) \qquad \text{(by (6.8))} \\
&= \frac{\mathrm{d}w}{\mathrm{d}s}(s) - A\,w(s) \qquad \text{(by (6.15))} \\
&= [\mathcal{L}(x)]_1(s)
\end{aligned}$$

and thus $[\mathcal{L}(x)]_1(s) \in \mathrm{Fix}_W(\Delta)$. Therefore

$$\mathcal{L}(x) \in C^0_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})).$$

Thus by linearity of $\mathcal{L}$ and Lemma 6.2.1 we have

$$\mathcal{L}\big(\mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta)\big) \subseteq \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta).$$

$\square$

Consider now a 1-parameter family of $\mathcal{G}$-admissible vector fields $f(x,\lambda)$ such that $L = (\mathrm{d}f)_{(x_0,0)}$ satisfies the conditions of the definition of interior symmetry-breaking Hopf bifurcation (see Definition 5.3.4), where A is the $\Sigma_{\mathcal{S}}$-equivariant subblock of L.

Lemma 6.2.3 *Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be a subgroup. Let $\mathcal{L}_\Delta : \mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) \to \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta)$ be the operator given by equation (6.14) with $L = (\mathrm{d}f)_{(x_0,0)}$. Then*

$$\dim_{\mathbf{R}} \ker(\mathcal{L}_\Delta) = \dim_{\mathbf{R}} \mathrm{Fix}_{E_i(A)}(\Delta).$$

Proof: By Lemma 5.2.10 and assumption (5.35) of Definition 5.3.4, $\ker(\mathcal{L}_\Delta)$ consists of all functions $\mathrm{Re}(\mathrm{e}^{is} v_0)$ where v_0 is an eigenvector of L associated to the eigenvalue i which can be decomposed as

$$v_0 = w_0 + u_0,$$

where $u_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ is uniquely determined by an eigenvector $w_0 \in \mathrm{Fix}_W(\Delta)$ of A with purely imaginary eigenvalue and

$$(\delta,\theta) \cdot \mathrm{Re}(\mathrm{e}^{is} w_0) = \mathrm{Re}(\mathrm{e}^{i(s+\theta)}\, \delta w_0) = \mathrm{Re}(\mathrm{e}^{is} w_0),$$

for all $(\delta,\theta) \in \Delta$. Hence

$$w_0 \in \mathrm{Fix}_W(\Delta) \cap E_i(A) = \mathrm{Fix}_{E_i(A)}(\Delta).$$

By uniqueness of the decomposition $v_0 = w_0 + u_0$ and the dimension condition (b) of Remark 5.3.3 we have

$$\dim_{\mathbf{R}} \ker(\mathcal{L}_\Delta) = \dim_{\mathbf{R}} \mathrm{Fix}_{E_i(A)}(\Delta).$$

$\square$

Lemma 6.2.4 *Let us write the 1-parameter family of $\mathcal{G}$-admissible vector fields $f(x,\lambda)$ in the form*

$$f(x,\lambda) = \begin{bmatrix} f_{\mathcal{S}}(x,\lambda) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(x,\lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ h(x,\lambda) \end{bmatrix} \tag{6.16}$$

where

$$\tilde{f}(x,\lambda) = \begin{bmatrix} f_{\mathcal{S}}(x,\lambda) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(x,\lambda) \end{bmatrix}$$

is the $\Sigma_{\mathcal{S}}$-equivariant part of f. Let $\mathcal{F}$, $\tilde{\mathcal{F}}$ be operators on $C^1_{2\pi}(P) \times \mathbf{R} \times \mathbf{R} \to C^0_{2\pi}(P)$ defined by formula (6.10) *using f and $\tilde{f}$, respectively. Define*

$$\mathcal{H}(x, \tau, \lambda) = h(x(s), \lambda)$$

so that

$$\mathcal{F}(x, \tau, \lambda) = \tilde{\mathcal{F}}(x, \tau, \lambda) - \mathcal{H}(x, \tau, \lambda).$$

Then

$$\mathcal{F}\big(\mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) \times \mathbf{R} \times \mathbf{R}\big) \subseteq \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta). \tag{6.17}$$

In particular, we may define the operator

$$\mathcal{F}_\Delta : \mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta) \tag{6.18}$$

by restriction and the linearization of $\mathcal{F}_\Delta$ about the origin is the linear operator $\mathcal{L}_\Delta$ given by the formula (6.14)*, where $L = (\mathrm{d}f)_{(x_0,0)}$.*

Proof: The $\Sigma_{\mathcal{S}}$-equivariance of $\tilde{f}$ implies that $\tilde{\mathcal{F}}$ is $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-equivariant (see Golubitsky *et al.* [21] Lemma XVI 3.2). It follows then that

$$\tilde{\mathcal{F}}\big(\mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) \times \mathbf{R} \times \mathbf{R}\big) \subseteq \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta).$$

Then it is enough to show that

$$\mathcal{H}\big(\mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) \times \mathbf{R} \times \mathbf{R}\big) \subseteq \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta).$$

Now let $x(s) \in \mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta)$. Recall that $h : P \to P_{\mathcal{C}\setminus\mathcal{S}}$ and $P_{\mathcal{C}\setminus\mathcal{S}} \subset \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. Therefore,

$$\mathcal{H}(x, \tau, \lambda)(s) = h(x(s), \lambda) \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}}) \quad (s \in \mathbf{R})$$

for all $\lambda, \tau \in \mathbf{R}$. By Lemma 6.2.1 we have that

$$C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})) \subset C^0_{2\pi}(\mathrm{Fix}_W(\Delta)) \oplus C^0_{2\pi}(\mathrm{Fix}_P(\Sigma_{\mathcal{S}})) = \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta),$$

and the result follows. □

Remark 6.2.5 Equation (6.13) of Lemma 6.2.2 can be derived directly from the above lemma.

Proof of Theorem 6.1.3 (Liapunov-Schmidt approach) Consider the operator

$$\mathcal{F}_\Delta : \mathrm{Fix}_{C^1_{2\pi}(P)}(\Delta) \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathrm{Fix}_{C^0_{2\pi}(P)}(\Delta).$$

The linearization of $\mathcal{F}_\Delta$ about the origin is the linear operator $\mathcal{L}_\Delta$. Now we invoke the assumption that Δ is $\mathbf{C}$-axial for the natural $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-action on $E_i(A)$, which together with Lemma 6.2.3 implies that

$$\dim_{\mathbf{R}} \ker(\mathcal{L}_\Delta) = 2.$$

Now we may proceed as in the proof of the standard Hopf Bifurcation Theorem (Theorem 2.2.1). If we identify $\ker(\mathcal{L}_\Delta) \cong \mathbf{C}$ then the action of $\mathbf{S}^1$ on $\ker(\mathcal{L}_\Delta)$ is equivalent to the standard action of $\mathbf{S}^1$ on $\mathbf{C}$. The Liapunov-Schmidt reduction applied to $\mathcal{F}_\Delta$ yields a $\mathbf{S}^1$-equivariant bifurcation equation

$$\phi : \mathbf{C} \times \mathbf{R} \times \mathbf{R} \to \mathbf{C}.$$

Moreover, the assumptions of the definition of codimension-one interior symmetry-breaking bifurcation are exactly the conditions necessary to carry out the proof. □

Example 6.2.6 Consider the network $\mathcal{G}_2$ of Figure 5.1. Suppose that for all cells c we choose the internal phase space to be $P_c = \mathbf{C}$. Then the total phase space is $P = \mathbf{C}^4$. Suppose that a smooth 1-parameter family $f : P \times \mathbf{R} \to P$ of $\mathcal{G}$-admissible vector fields on P undergoes a codimension--one interior symmetry-breaking Hopf bifurcation at the equilibrium $x_0 = 0$ when $\lambda = 0$. Then $E_i(A) = W$, where A is the $\Sigma_{\mathcal{S}}$-equivariant subblock of the linearization $L = (\mathrm{d}f)_{(0,0)}$ of f at $(0,0)$. By Theorem 6.1.3 there are three branches of synchronously modulated Δ-symmetric waves associated to the three conjugacy classes of interiorly $\mathbf{C}$-axial subgroups of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ (see Table 6.1). Observe that the first periodic state of Table 6.1 is associated

Subgroup	Form of solution to lowest order in λ
$\mathbf{Z}_2$	$(w_1(t)+u(t), w_1(t)+u(t), w_2(t)+u(t), v(t))$
$\tilde{\mathbf{Z}}_2$	$(w_1(t)+u(t), w_1(t+\frac{1}{2})+u(t), \hat{w}(t)+u(t), v(t))$
$\tilde{\mathbf{Z}}_3$	$(w_1(t)+u(t), w_1(t+\frac{1}{3})+u(t), w_1(t+\frac{2}{3})+u(t), v(t))$

Table 6.1: Branches of synchronously modulated Δ-symmetric waves supported by the network $\mathcal{G}_2$ of Figure 5.1 and the associated subgroup. The hat over a variable indicates that $\hat{w}$ has twice the frequency.

to a spatially **C**-axial subgroup, as it is predicted by Golubitsky *et al.* [15] Theorem 6.3. The third periodic state of Table 6.1 is an approximate rotating wave. $\diamond$

6.3 Center Manifold Reduction Approach

In this section, we present an alternative proof of Theorem 6.1.3 using a center manifold reduction. This approach can be useful in the development of normal form theory aiming at the study of the stability of the periodic solutions guaranteed by Theorem 6.1.3. The proof of Theorem 6.1.3 using center manifold reduction approach follows from a couple of lemmas that we state and prove.

Consider the system (6.4) as defined in section 6.1. Recall that under the hypotheses of Theorem 6.1.3, $L = (df)_{(x_0,0)}$ written as in (6.7) has $\pm i$ as eigenvalues that come only from the subblock A of L and that they are the only critical eigenvalues of L. Considering $A^c \equiv A|_{E_i(A)}$, as A has $\pm i$ as eigenvalues, there is a natural action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on P, where $\mathbf{S}^1$ acts on $E_i(A)$ by $\exp(s(A^c)^t)$ and trivially on $P \setminus E_i(A)$. The action of $\Sigma_{\mathcal{S}}$ on P is given by (6.1) which implies that with respect to the decomposition (6.3):

$$\sigma(w, u) = (\sigma w, u)$$

for $\sigma \in \Sigma_{\mathcal{S}}$, $w \in W$, $u \in U = \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$.

Lemma 6.3.1 *Consider L as in* (6.7). *Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be an isotropy subgroup for the action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on P defined above. Then*

$$\dim_{\mathbf{R}}(E_i(A)) = \dim_{\mathbf{R}}(E_i(L))$$

and

$$\dim_{\mathbf{R}}(\mathrm{Fix}_{E_i(A)}(\Delta)) = \dim_{\mathbf{R}}(\mathrm{Fix}_P(\Delta) \cap E_i(L)).$$

Proof: Consider $x = (w, u) \in P$ where $w \in W, u \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$. Assume $\dim_{\mathbf{R}} W = k$ and $\dim_{\mathbf{R}} \mathrm{Fix}_P(\Sigma_{\mathcal{S}}) = l$. As

$$(L^2 + I_{k+l})x = 0 \quad \Longleftrightarrow \quad \begin{bmatrix} A^2 + I_k & 0 \\ CA + BC & B^2 + I_l \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and B does not have $\pm i$ as eigenvalues, we get

$$E_i(L) = \{(w, -(B^2 + I_l)^{-1}(CA + BC)w), w \in E_i(A)\}.$$

In particular, it follows that $\dim_{\mathbf{R}}(E_i(A)) = \dim_{\mathbf{R}}(E_i(L))$. As $\mathrm{Fix}_P(\Delta) = \mathrm{Fix}_W(\Delta) \oplus \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$, we have

$$\mathrm{Fix}_P(\Delta) \cap E_i(L) = \{(v, -(B^2 + I_l)^{-1}(CA + BC)v), v \in \mathrm{Fix}_{E_i(A)}(\Delta)\}$$

and so $\dim_{\mathbf{R}}(\mathrm{Fix}_P(\Delta) \cap E_i(L)) = \dim_{\mathbf{R}}(\mathrm{Fix}_{E_i(A)}(\Delta))$. □

Lemma 6.3.2 ([26]) *Let the vector field $f : \mathbf{R}^n \to \mathbf{R}^n$ have a flow invariant subspace V with an equilibrium at $x_0 \in V$. Let E^c be the center subspace at x_0. Then a center manifold reduction $f^c : E^c \to E^c$ can be chosen so that the subspace $E^c \cap V$ is flow invariant for f^c. Moreover, if $\sigma : V \to V$ is a symmetry of $f|V$ that leaves $E^c \cap V$ invariant, then the center manifold reduction f^c may be chosen so that $\sigma|E^c \cap V$ is a symmetry for $f^c|E^c \cap V$.*

Proof: See Leite and Golubitsky [26] Lemma 4.12. □

Definition 6.3.3 Consider f as in (6.5). We say that f is in *interior normal form (to all orders)* near λ_0, if $\tilde{f}(\cdot, \lambda)$ is in normal form (to all orders) near λ_0, that is, $\tilde{f}$ commutes with the action of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ on P defined above, for λ near λ_0.

Now we prove Theorem 6.1.3 using center manifold reduction under the additional hypothesis that f is in interior normal form near $\lambda = 0$.

Proof of Theorem 6.1.3 (Center manifold reduction approach) Consider f written in the coordinates (w, u) as in (6.5). Thus

$$\tilde{f}(w, u, \lambda) = (f_W(w, u, \lambda), f_U(w, u, \lambda))$$

is $\Sigma_{\mathcal{S}}$-equivariant. By hypothesis, a codimension-one interior symmetry-breaking Hopf bifurcation occurs at an equilibrium point $x_0 \in \mathrm{Fix}_P(\Sigma_{\mathcal{S}})$ when $\lambda = 0$. Since f is in interior normal form near $\lambda = 0$, $\tilde{f}$ is $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-equivariant and so $\tilde{f}(\mathrm{Fix}_P(\Delta) \times \mathbf{R}) \subseteq \mathrm{Fix}_P(\Delta)$ for every $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$. As $h : P \times \mathbf{R} \to P_{C \setminus S}$ and $P_{C \setminus S} \subseteq \mathrm{Fix}_P(\Delta)$ we have

$$f(\mathrm{Fix}_P(\Delta) \times \mathbf{R}) \subseteq \mathrm{Fix}_P(\Delta). \tag{6.19}$$

In our case, $E^c(L) = E_i(L)$ since the only critical eigenvalues of L are $\pm i$ and these come only from the subblock A of L. Then, under the condition (6.19), Lemma 6.3.2 grants that a center manifold reduction $f^c : E_i(L) \to E_i(L)$ can be chosen so that

$$f^c(E_i(L) \cap \mathrm{Fix}_P(\Delta)) \subseteq E_i(L) \cap \mathrm{Fix}_P(\Delta).$$

By hypothesis $\dim_{\mathbf{R}}(\mathrm{Fix}_{E_i(A)}(\Delta)) = 2$. Then, by Lemma 6.3.1, it follows that

$$\dim_{\mathbf{R}}(E_i(L) \cap \mathrm{Fix}_P(\Delta)) = 2.$$

Finally, Theorem 2.2.1 gives the result. □

Remark 6.3.4 We can also end the previous proof using an alternative argument. Consider that $W_i^c(x_0)$ represents the center manifold associated to $E_i(L)$ on x_0. From (6.19) and from the flow invariance of $W_i^c(x_0)$ (see Vanderbauwhede [36] Theorem 4.1 for details) it follows that

$$f(W_i^c(x_0) \cap \mathrm{Fix}_P(\Delta)) \subseteq W_i^c(x_0) \cap \mathrm{Fix}_P(\Delta). \tag{6.20}$$

By hypothesis $\dim_{\mathbf{R}}(\mathrm{Fix}_{E_i(A)}(\Delta)) = 2$. By Lemma 6.3.1 we have $\dim_{\mathbf{R}}(E_i(L) \cap \mathrm{Fix}_P(\Delta)) = 2$ and consequently

$$\dim_{\mathbf{R}}(W_i^c(L) \cap \mathrm{Fix}_P(\Delta)) = 2.$$

Theorem 2.2.1 guarantees now the pretended result.

Remark 6.3.5 Observe that there is a correspondence between the center manifold reduction approach and the Liapunov-Schmidt approach in the proof of the Theorem 6.1.3. Lemma 6.2.3 is in correspondence with Lemma 6.3.1 because

$$\ker(\mathcal{L}_\Delta) \cong \mathrm{Fix}_P(\Delta) \cap E_i(L).$$

Equation (6.17) of Lemma 6.2.4 corresponds to (6.19).

6.4 Numerical Simulation

In this last section we illustrate the conclusions of Example 6.2.6 with a numerical simulation. In order to write down an explicit coupled cell system associated to network $\mathcal{G}_2$ of Figure 5.1 we choose the internal phase space of all four cells to be $P_c = \mathbf{C} \cong \mathbf{R}^2$. Thus the total phase space is $P = (\mathbf{R}^2)^4$.

Consider the coupled cell system

$$\begin{array}{rcl} \dot{x_1} & = & g(x_1, x_2, x_3) + 2x_4, \\ \dot{x_2} & = & g(x_2, x_3, x_1) + 2x_4, \\ \dot{x_3} & = & g(x_3, x_1, x_2) + 2x_4, \\ \dot{x_4} & = & -x_4 + e_1x_1 + e_2x_2 + e_3x_3, \end{array} \tag{6.21}$$

where $g : (\mathbf{R}^2)^3 \to \mathbf{R}^2$ is given by

$$\begin{aligned} g(x,y,z) \; &= \; -x + (a - 2b_2)\, x\|x\|^2 + b_1\,(y+z) + b_2\,(y\|y\|^2 + z\|z\|^2) \\ &\quad + a\,(x\|y\|^2 + x\|z\|^2) + b_3\,(y\|y\|^4 + z\|z\|^4) \end{aligned}$$

and $a, b_1(\lambda), b_2, b_3, e_1, e_2, e_3$ are 2×2 matrices with b_1 depending smoothly on a real parameter λ. Note that $g(x,y,z) = g(x,z,y)$ for all $x, y, z \in \mathbf{R}^2$. Let f be the vector field defined by (6.21). Thus f is $\mathcal{G}_2$-admissible. Observe that the origin is an equilibrium point for all λ

$$f(0, \lambda) \equiv 0.$$

The linearization of f at $(0, \lambda)$ is given by (as 2×2 block matrix)

$$L(\lambda) = \begin{pmatrix} -1 & b_1 & b_1 & 2 \\ b_1 & -1 & b_1 & 2 \\ b_1 & b_1 & -1 & 2 \\ e_1 & e_2 & e_3 & -1 \end{pmatrix},$$

where $\pm c$ represents $\pm\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$.

We need to choose the coefficients b_1 and e_1, e_2, e_3 in order to have purely imaginary eigenvalues for some λ coming from the subblock A when L is written in the form (6.7). The following values will do the work:

$$b_1(\lambda) = \begin{pmatrix} -1-\lambda & -1.5 \\ 1.5 & -1 \end{pmatrix}$$

and any values between -1 and 1 for the entries of the matrices e_1, e_2, e_3.

The spectrum of the matrix $L(\lambda)$ has the following properties:

(1) For $\lambda < 0$ all eigenvalues of $L(\lambda)$ have negative real parts.

(2) For $\lambda = 0$ the matrix $L = L(0)$ has two pairs of eigenvalues $\pm i$ and the remaining eigenvalues have negative real parts. Moreover, the eigenvectors associated to the purely imaginary eigenvalues are not in $\mathrm{Fix}_P(\mathbf{S}_3)$.

(3) For $\lambda > 0$ all eigenvalues of $L(\lambda)$ whose associated eigenvectors are in $\mathrm{Fix}_P(\mathbf{S}_3)$ have negative real parts and the remaining eigenvalues have positive real parts.

Thus (6.21) undergoes a interior symmetry-breaking Hopf bifurcation when $\lambda = 0$ giving rise to one branch of periodic solutions for each one of the three interiorly $\mathbf{C}$-axial subgroups of $\mathbf{S}_3 \times \mathbf{S}^1$ as in Table 6.1, when $\lambda > 0$. However, depending on the choice of the coefficients a, b_2 and b_3 of g, one can make at least one of these periodic solutions to be stable. In our simulations we have chosen the following coefficients:

$$a = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix},$$

(1) for a solution with (interior) symmetry $\tilde{\mathbf{Z}}_3$:

$$b_2 = \begin{pmatrix} 0.6 & 2 \\ 2 & 0.6 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(2) for a solution with (interior) symmetry $\tilde{\mathbf{Z}}_2$:

$$b_2 = \begin{pmatrix} -0.6 & 1 \\ 1 & -0.6 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0.2 & -0.7 \\ -0.7 & 0.2 \end{pmatrix}.$$

(3) for a solution with (interior) symmetry $\mathbf{Z}_2$:

$$b_2 = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0.7 \\ 0.7 & 0 \end{pmatrix}.$$

The coefficients e_1, e_2 and e_3 represent the coupling that breaks the $\mathbf{S}_3$-symmetry. If $e_1 = e_2 = e_3$ then the coupled cell system (6.21) is admissible for the network $\mathcal{G}_1$ of Figure 5.1 and so it is $\mathbf{S}_3$-symmetric. On the other hand, if $e_1 \neq e_2 \neq e_3$ then the coupled cell system (6.21) is admissible for the network $\mathcal{G}_2$ of Figure 5.1 and have genuine $\mathbf{S}_3$-interior symmetry.

In the following we present the results of numerical simulations obtaining the three types of periodic solutions mentioned above, for both of the networks $\mathcal{G}_1$ and $\mathcal{G}_2$ of our running example. In Figures 6.1, 6.2 and 6.3 we superimpose the time series of all four cells, which are identified by colours:

$$1 = \text{blue}, \qquad 2 = \text{red}, \qquad 3 = \text{green}, \qquad 4 = \text{black}.$$

The upper panels show the first components and the lower panels show the second components. The left panels refer to network $\mathcal{G}_1$ with exact $\mathbf{S}_3$-symmetry and the panels on the right refer to network $\mathcal{G}_1$ with $\mathbf{S}_3$-interior symmetry. Finally, in Figure 6.4 we present the solution with interior symmetry $\tilde{\mathbf{Z}}_3$ of network $\mathcal{G}_2$, i.e., the approximate rotating wave from Figure 6.1 (right), viewed in difference coordinates:

$$x_1 - x_2 = \text{blue}, \qquad x_2 - x_3 = \text{green}, \qquad x_3 - x_1 = \text{red}.$$

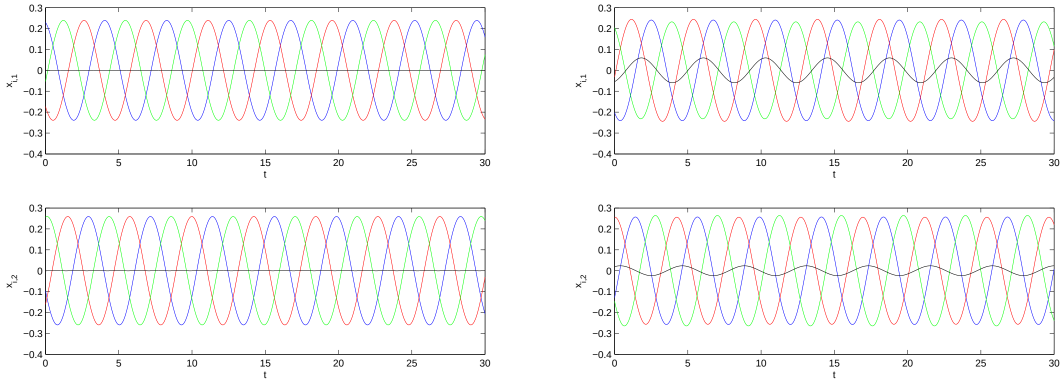

Figure 6.1: Solutions with $\tilde{\mathbf{Z}}_3$ (interior) symmetry. (Left) Network $\mathcal{G}_1$ with exact $\mathbf{S}_3$-symmetry. (Right) Network $\mathcal{G}_2$ with $\mathbf{S}_3$-interior symmetry on $\mathcal{S} = \{1, 2, 3\}$.

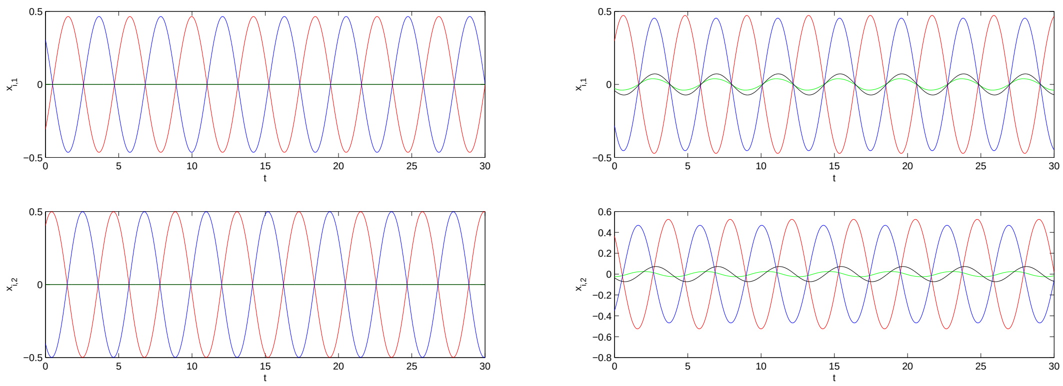

Figure 6.2: Solutions with $\tilde{\mathbf{Z}}_2$ (interior) symmetry. (Left) Network $\mathcal{G}_1$ with exact $\mathbf{S}_3$-symmetry. (Right) Network $\mathcal{G}_2$ with $\mathbf{S}_3$-interior symmetry on $\mathcal{S} = \{1, 2, 3\}$.

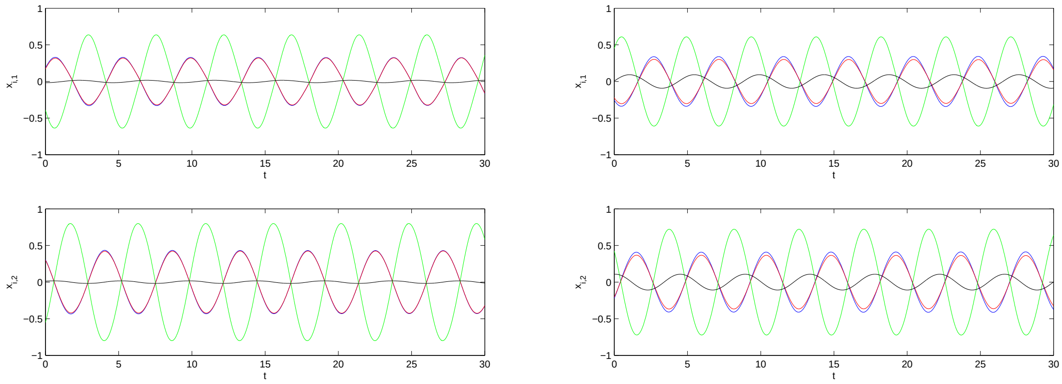

Figure 6.3: Solutions with $\mathbf{Z}_2$ (interior) symmetry. (Left) Network $\mathcal{G}_1$ with exact $\mathbf{S}_3$-symmetry. (Right) Network $\mathcal{G}_2$ with $\mathbf{S}_3$-interior symmetry on $\mathcal{S} = \{1, 2, 3\}$.

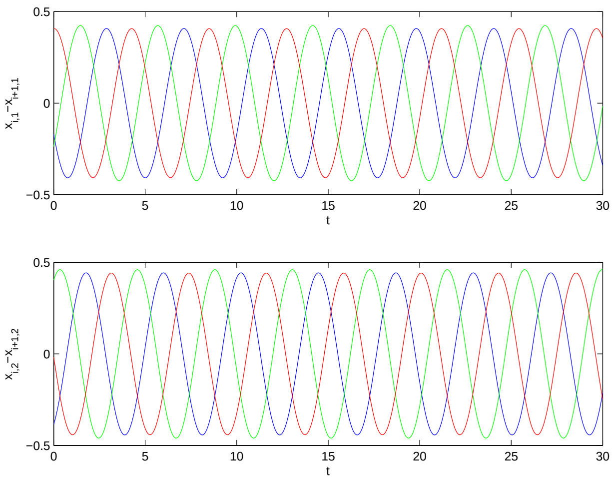

Figure 6.4: Approximate rotating wave in network $\mathcal{G}_2$, viewed in difference coordinates: $x_1 - x_2$, $x_2 - x_3$ and $x_3 - x_1$.

Chapter 7

Local Bifurcation in Symmetric Coupled Cell Networks: Linear Theory

7.1 Introduction

Perhaps the single most important problem in classical bifurcation theory is to identify the ways in which an equilibrium of a system of ODEs can lose stability as a parameter is varied. Roughly speaking, this loss of stability occurs by one of two methods: static bifurcation or Hopf bifurcation. More precisely, let

$$\dot{x} = f(x, \lambda) \tag{7.1}$$

be a system of ODEs where $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$. Assume that $x = 0$ is a "trivial" steady state for all λ, so $f(0, \lambda) \equiv 0$. Assume further that the equilibrium $x = 0$ is asymptotically stable for $\lambda < 0$; that is, all eigenvalues of $(df)_{(0,\lambda)}$ have negative real part when $\lambda < 0$; and that $x = 0$ loses stability at $\lambda = 0$; that is, some eigenvalue of $(df)_{(0,0)}$ lies on imaginary axis. When f has no special structure (such as symmetry or network constraints), generically, there are two possibilities:

(a) $(df)_{(0,0)}$ has a simple zero eigenvalue and no other imaginary eigenvalues;
(b) $(df)_{(0,0)}$ has a pair of simple eigenvalues $\pm\omega i$ ($\omega \neq 0$) and no other imaginary eigenvalues. (7.2)

In many fields of application, the eigenvectors of $(df)_{(0,0)}$ corresponding to simple eigenvalues are called *modes*. A mode whose eigenvalues lie on the imaginary axis is said to be *critical*. As λ varies through the bifurcation point, generically the critical eigenvalues cross the imaginary axis transversely (with nonzero speed), and this property guarantees the local existence of a bifurcating branch of nonzero states. We refer to (7.2(a)) as *steady-state* bifurcation leading to a branch of steady-states and to (7.2(b)) as *Hopf* bifurcation leading to a branch of periodic solutions; Thus we use these terms to describe degeneracies in the linear terms of the vector field f.

If f has additional structure, what is 'generic' may change. The critical eigenvalues still govern local bifurcation and the real/imaginary distinction still applies that we still refer to steady-state and Hopf bifurcation.

In general Γ-equivariant systems, that is, in the cases where f commutes with a linear action of a compact group Γ on $\mathbf{R}^n$, multiple eigenvalues of $(df)_{(0,0)}$ often occur. At this situation, a *steady-state mode* occurs when $(df)_{(0,0)}$ has a zero eigenvalue and the corresponding eigenspace is generically Γ-absolutely irreducible (Golubitsky *et al.* [21] Proposition XIII 3.2). A *Hopf mode* occurs when $(df)_{(0,0)}$ has an imaginary eigenvalue and the corresponding eigenspace is generically Γ-simple (Proposition 3.2.4). We obtain the existence of branches of equilibria and periodic solutions with symmetry groups satisfying the conditions of the Equivariant Branching Lemma (Golubitsky *et al.* [21] Theorem XIII 3.3) or the Equivariant Hopf Theorem (Theorem 3.2.6). Again, generically we expect, in a one parameter system, to have only one critical mode.

The occurrence of multiple critical modes is called *mode interaction.* For example, in systems with more than one parameter, are expected multiple critical modes, see for instance Golubitsky *et al.* [21] Chapter XIX. Since there are two types of critical mode (steady-state and Hopf), there are three types of mode interaction: steady-state/steady-state, Hopf/steady-state and Hopf/Hopf.

In the Γ-equivariant case, for the first type, the 0-eigenspace of the linearization at the bifurcation point decomposes as the direct sum of two Γ-absolutely irreducible subspaces; for the second type, a zero eigenvalue and a purely imaginary one occur simultaneously, and the critical eigenspace is the direct sum of a Γ-absolutely irreducible subspace and a Γ-simple space; for the third type, the critical eigenspace is the sum of two (distinct) Γ-simple spaces.

Suppose $\mathcal{G}$ is a coupled cell network with n cells that is symmetric with respect to a transitive and faithful permutation action of a group $\Gamma \subseteq \mathbf{S}_n$ on the set of cells $\{1, \ldots, n\}$ (recall section 5.1). Throughout this chapter we assume that the networks have not self-connections neither multiarrows. We recall that *transitivity* implies that for each pair $\{i, j\} \subseteq \{1, \ldots, n\}$ there exists $\gamma \in \Gamma$ such that $\gamma(i) = j$, and Γ acts *faithfully* if $\gamma(i) = i$ for all $i = 1, \ldots, n$ implies that γ is the identity element of Γ.

Let the phase space of each cell be $\mathbf{R}$ and so the total phase space is $V = \mathbf{R}^n$. Assume then that, with respect to the cell coordinates $v_1, \ldots, v_n$, the action of Γ on V is given by permutation of indexes. Consider (7.1) where f represents a smooth 1-parameter family

$$f : V \times \mathbf{R} \to V \tag{7.3}$$

of $\mathcal{G}$-admissible vector fields. In particular, it follows that f commutes with the permutation action of Γ on V and $x = 0$ is a fully symmetric equilibrium.

When $\mathcal{G}$ is a Γ-network (see Definition 7.7.1), Antoneli and Stewart [3] proved that f often is a general Γ-equivariant vector field. Using the results of [21] stated above we have that, in this case, generically, mode interactions are not expected to occur in codimension one local bifurcations. Here, we say that a type of local bifurcation has codimension m if the corresponding set of (smooth) vector fields satisfying the bifurcation condition has codimension m in the ambient space. Coupled cell networks which are not Γ-networks possess a structure that is independent of the symmetry. This should naturally be taken into account when analyzing the (typical) dynamics of coupled cell networks. We would like to address the question how the network structure may affect the kinds of bifurcations that can be expected to occur in a coupled cell network. It turns out that the problem is quite a complicated one. Dias and Lamb [7] focus on networks with an abelian symmetry group that permutes cells transitively, and local bifurcation from a fully symmetric equilibrium solution. They prove that if the dimension of the phase space of cells is greater than one, the codimension one eigenvalue movements across the imaginary axis of $(df)_{(x_0,\lambda)}$ at a fully symmetric equilibrium x_0 are independent of the network structure and are identical to the corresponding eigenvalue movements in general equivariant vector fields. However, this result is incomplete when we study codimension one Hopf bifurcation assuming that the phase space of the cells is one-dimensional. In this case Hopf/Hopf mode interaction can occur which is not generically expected in general equivariant vector fields. Before discussing this in more detail, we first illustrate the problem with an example.

Example 7.1.1 An example of the difficulties that can arise when we study codimension one Hopf bifurcation on coupled cell networks is given by the ring of nine cells on the right of the Figure 7.1 with symmetry

$$\Gamma = \langle (123)(456)(789), (147)(258)(369) \rangle \cong \mathbf{Z}_3 \times \mathbf{Z}_3.$$

We assume that the phase space of the cells is one-dimensional and that the network dynamics have a group invariant equilibrium. Then, a general

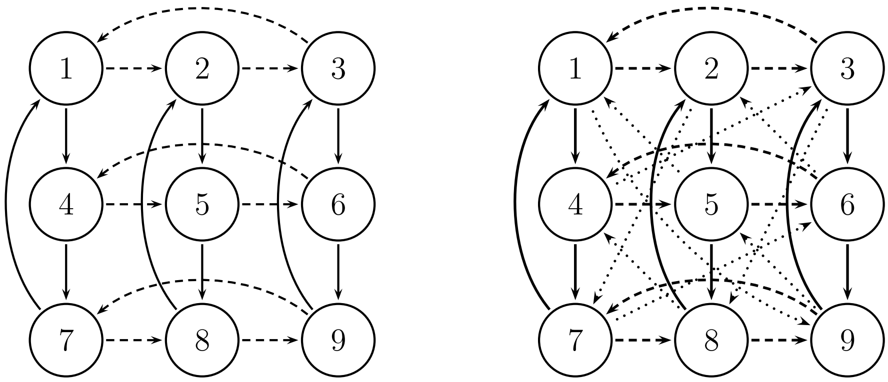

Figure 7.1: Networks $\mathcal{G}_1$ (left) and $\mathcal{G}_2$ (right) with $\mathbf{Z}_3 \times \mathbf{Z}_3$-symmetry.

$\mathcal{G}_2$-admissible linear map at such an equilibrium has the form

$$L = \begin{bmatrix} A & C & B \\ B & A & C \\ C & B & A \end{bmatrix} \tag{7.4}$$

where

$$A = \begin{bmatrix} a & 0 & c \\ c & a & 0 \\ 0 & c & a \end{bmatrix}, \; B = \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}, \; C = \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & d \\ d & 0 & 0 \end{bmatrix}$$

and a, b, c, d are real-valued smooth functions of λ. The eigenvalues of L are

$$\begin{aligned}
\lambda_1 &= a + b + c + d \\
\lambda_{2,\pm} &= a + b - \tfrac{1}{2}(c + d) \pm i(c - d)\tfrac{\sqrt{3}}{2} \\
\lambda_{3,\pm} &= a - \tfrac{1}{2}(b + c + d) \pm i(b + c + d)\tfrac{\sqrt{3}}{2} \\
\lambda_{4,\pm} &= a + c - \tfrac{1}{2}(b + d) \pm i(b - d)\tfrac{\sqrt{3}}{2} \\
\lambda_{5,\pm} &= a - \tfrac{1}{2}(b + c) + d \pm i(b - c)\tfrac{\sqrt{3}}{2}.
\end{aligned}$$

Consider the network $\mathcal{G}_1$ ($d = 0$) shown in Figure 7.1 (left). The eigenvalues of L are then

$$\begin{aligned}
\lambda_1 &= a + b + c \\
\lambda_{2,\pm} &= a + b - \tfrac{1}{2}c \pm ic\tfrac{\sqrt{3}}{2} \\
\lambda_{3,\pm} &= a - \tfrac{1}{2}(b + c) \pm i(b + c)\tfrac{\sqrt{3}}{2} \\
\lambda_{4,\pm} &= a + c - \tfrac{1}{2}b \pm ib\tfrac{\sqrt{3}}{2} \\
\lambda_{5,\pm} &= a - \tfrac{1}{2}(b + c) \pm i(b - c)\tfrac{\sqrt{3}}{2}.
\end{aligned}$$

Observe that for the network $\mathcal{G}_1$ if $b = 2a - c$ then we have a codimension one Hopf/Hopf mode interaction given by $\lambda_{3,\pm} = \pm ia\sqrt{3}$ and $\lambda_{5,\pm} = \pm i(a-c)\sqrt{3}$. This situation cannot generically occur for the network $\mathcal{G}_2$. $\diamond$

In this chapter we describe which abelian groups Γ and Γ-symmetric coupled cell networks can origin, under generic conditions, the phenomenon of Hopf/Hopf mode interaction and other degenerate phenomenon in codimension one local bifurcations of type (7.3) where f is admissible for the networks. We do that using character theory of finite abelian groups. Essentially, when the symmetry group of the network is abelian and acts transitively (and faithfully) by permutation on the cells of the network, the number n of cells equals the order of Γ. In this case we may identify cells uniquely with group elements once we have identified for example cell 1 with the identity element e in Γ. Moreover, the structure of the network is fully determined by the set S of cells that are connected to cell e by an edge. If the network is connected then $< S >= \Gamma$. The eigenvalues of $(df)_{(x_0,\lambda)}$ depend on the characters of Γ evaluated at the elements of S. We determine, using this fact, generic conditions that permit in one parameter families of admissible vector fields, the occurrence of Hopf bifurcation associated to the crossings with the imaginary axis (away from zero) of two or more distinct pairs of complex eigenvalues of linearization.

7.2 Background

The crucial step in this chapter is the description of the restrictions on the eigenvalue structure of $\mathcal{G}$-admissible linear vector fields on $\mathbf{R}^n$. Here we consider transitive and faithful actions of Γ on $\mathbf{R}^n$ and the restrictions are imposed by the network structure of $\mathcal{G}$. For that, we begin by complexifying the state space to $V = \mathbf{C}^n$ in order to use the theory of complex representations of finite groups. See for example James and Liebeck [24] for the basic definitions and results on this subject, which we use throughout this section. In section 7.5 we then interpret the results in terms of real representations.

Let Γ be a subgroup of the symmetric group $\mathbf{S}_n$ permuting transitively and faithfully the set $\{1, \ldots, n\}$. Consider a *n-dimensional complex vector space* V, a basis $b = \{e_1, \ldots, e_n\}$ of V and the action of Γ on V given by permutation of the corresponding coordinates. Thus we can assume that $V = \mathbf{C}^n$ and this action corresponds to a representation T of Γ on V through a linear homomorphism from Γ to the group $\mathrm{GL}(V)$ of invertible linear transformations on V defined by

$$T(\gamma)(v_1, \ldots, v_n) = \left(v_{\gamma^{-1}(1)}, \ldots, v_{\gamma^{-1}(n)}\right), \gamma \in \Gamma, (v_1, \ldots, v_n) \in V. \qquad (7.5)$$

Since Γ is finite there appear in V at most s distinct complex irreducible representations, where s is the number of conjugacy classes of Γ. Denote those that appear by $V_1, \dots, V_r$ and so $r \leq s$. We can decompose V into isotypic components

$$V = U_1 \oplus \cdots \oplus U_r$$

where each U_j is the isotypic component of type V_j for the action of Γ on V. Thus if W is a Γ-invariant subspace of V and Γ-isomorphic to V_j then $W \subseteq U_j$. Suppose now that $M \in \mathrm{gl}(V)$ commutes with Γ:

$$MT(\gamma) = T(\gamma)M, \forall \gamma \in \Gamma.$$

Since M commutes with Γ, it preserves the isotypic components for the action of Γ on V. Thus $M(U_j) \subseteq U_j$ for $j = 1, \dots, r$. Denote by M^j the restriction of M to U_j:

$$M^j \equiv M|_{U_j} : U_j \to U_j.$$

It follows that M^j commutes with Γ.

Given an irreducible Γ-invariant vector space V_j, then the *character* of V_j is the function $\chi_j : \Gamma \to \mathbf{C}$ defined by

$$\chi_j(\gamma) = \mathrm{tr}\left(T(\gamma)|_{V_j}\right), \gamma \in \Gamma.$$

Since V_j is irreducible, we also say that χ_j is irreducible. The dimension of V_j is called the *dimension* (or *degree*) of χ_j. Characters of dimension 1 are called *linear characters*. We review the following properties of the characters:
(a) If e denotes the identity element of the group Γ then $\chi_j(e) = \dim_{\mathbf{C}} V_j$;
(b) If $\gamma \in \Gamma$ has order m, then $\chi_j(\gamma)$ is a sum of mth roots of unity;
(c) If $\chi_j(\gamma)$ is linear then it is a homomorphism from Γ to the multiplicative group of non-zero complex numbers $\{z \in \mathbf{C} : |z| = 1\}$.

Definition 7.2.1 Define the projection operator of V onto the Γ-isotypic component U_j by

$$P^j = \frac{\dim V_j}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_j(\gamma)} T(\gamma),$$

where χ_j is the character corresponding to the irreducible V_j.

Remark 7.2.2 ([7]) Observe that if $\Gamma \subseteq \mathbf{S}_n$ is abelian and acts transitively on $\{1, ..., n\}$ then if $\gamma(i) = i$ for some i, then $\gamma(j) = j$ for all j. That is, γ is the identity. To verify this point use transitivity to choose $\delta \in \Gamma$ such that $\delta(i) = j$. Since Γ is abelian, it follows that

$$\gamma(j) = \gamma\delta(i) = \delta\gamma(i) = \delta(i) = j.$$

We have then that $|\Gamma| = n$. Moreover, all the irreducible Γ-invariant vector spaces are one-dimensional and $r = n$, see for example James and Liebeck [24]

Proposition 9.5. We obtain that $U_j = V_j$ for $j = 1, \dots, n$, where the V_j form a complete set of non-isomorphic irreducible and one-dimensional Γ-invariant vector spaces. The representation V is called the *regular representation* of Γ.

Example 7.2.3 Let $\mathbf{Z}_n$ be a cyclic group of order n generated by an element a satisfying $a^n = e$. Denote by $\omega = e^{i\frac{2\pi}{n}}$. The group $\mathbf{Z}_n$ has n distinct linear characters χ_j, $j = 1, \dots, n$, given by

$$\chi_j(a^r) = \omega^{(j-1)r}, j = 1, \dots, n.$$

Here $r \in \{0, \dots, n-1\}$ where $a^0 = e$. Consider the subgroup Γ of $\mathbf{S}_n$ isomorphic to $\mathbf{Z}_n$ that permuts transitively (and faithfully) the set $\{1, \dots, n\}$ which is generated by

$$\alpha = (12 \dots n).$$

Let $V = \mathbf{C}^n$ and $b = \{e_1, \dots, e_n\}$ a basis of V and consider the action of Γ on V given by permutation of the corresponding coordinates (recall (7.5)). Thus if $\gamma_k = \alpha^{k-1}$ for $k = 1, \dots, n$ then $e_k = T(\gamma_k)e_1$. The action of Γ on V corresponds to the regular representation of $\Gamma \cong \mathbf{Z}_n$: each distinct Γ-irreducible appears in the Γ-isotypic decomposition of V, with multiplicity one. Thus

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

where each irreducible V_j has character type χ_j. $\diamond$

7.3 Representation Theory of Finite Abelian Groups

In this section we study representation theory of finite abelian groups. We focus on abelian subgroups of $\mathbf{S}_n$ with order n that act transitively (and faithfully) on the set $\{1, \dots, n\}$.

The next result is a major structure theorem for finite abelian groups.

Theorem 7.3.1 ([12]) *Every finite abelian group Γ is isomorphic to a direct product of cyclic groups*

$$\mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_r} \tag{7.6}$$

where n_i, $1 \leq i \leq r$ are powers of primes not necessarily distinct. The values of $n_1, \dots, n_r$ are (up to rearranging the indexes) uniquely determined by Γ.

Proof: See Fraleigh [12] Chapter I. $\square$

Using Theorem 7.3.1, we obtain the irreducible representations of all finite abelian groups by determining the irreducible representations of all direct products (7.6).

Recall that $\mathbf{Z}_{nm}$ is isomorphic to the direct product of $\mathbf{Z}_n$ and $\mathbf{Z}_m$ if and only if m and n are coprime.

Theorem 7.3.2 ([24]) *Let $\chi_1, \ldots, \chi_n$ be the distinct irreducible characters of a group G and let $\psi_1, \psi_2, \ldots, \psi_m$ be the distinct irreducible characters of a group H. Then $G \times H$ has precisely $n \times m$ irreducible characters, and these are*

$$\chi_i \times \psi_j \ \ (1 \leq i \leq n,\ 1 \leq j \leq m).$$

Proof: See James and Liebeck [24] Theorem 19.18 for details. $\square$

Consider an abelian group $\Gamma \subseteq \mathbf{S}_n$ of order n that acts transitively (and faithfully) on the set $\{1, \ldots, n\}$. By Theorem 7.3.1 we can consider

$$\Gamma \cong \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_r} \tag{7.7}$$

where $\mathbf{Z}_{n_k}$ is a cyclic group of order $n_k \geq 2$; also $n_1 \cdots n_r = n$ with $n_1 \geq n_2 \geq \cdots \geq n_r$ and the n_k are powers of primes, not necessarily distinct. Moreover, $\mathbf{Z}_{n_k} \cong < a_k >$ for $a_k \in \mathbf{S}_n$. Indeed, it is well known that a_k is not unique. Choose a set $G = \{a_1, \ldots, a_r\}$ of permutation generators for the given abelian group $\Gamma \subseteq \mathbf{S}_n$. Define $\vec{s} = (s_1, \ldots, s_r)$ where $s_k \in \{0, \ldots, n_k - 1\}$. We can write (enumerate) the elements of Γ as:

$$\Gamma = \{\gamma_{\vec{s}} = a_1^{s_1} a_2^{s_2} \cdots a_r^{s_r},\ s_k \in \{0, \ldots, n_k - 1\}\} \tag{7.8}$$

where $< a_k > \cong \mathbf{Z}_{n_k}$, $a_k a_l = a_l a_k$, $\forall k, l$ and $a_k^0 = e$. Let $V = \mathbf{C}^n$, $B = \{e_1, \ldots, e_n\}$ be a basis of V over $\mathbf{C}$ and consider the action of Γ on V given by (7.5). By Remark 7.2.2 we may identify group elements uniquely with numbers $l \in \{1, \ldots, n\}$ once we have identified for example cell 1 with the identity element in Γ. In particular, for $i = 1, \ldots, n$, we identify cell i by the element of Γ, say $\gamma_{\vec{s}_i}$, such that

$$\gamma_{\vec{s}_i}(1) = i. \tag{7.9}$$

Thus $\gamma_{\vec{0}} \equiv \gamma_{\vec{s}_1} \equiv e$ and

$$e_i = T\left(\gamma_{\vec{s}_i}\right) e_1,\ i = 1, \ldots, n.$$

This action of Γ on V corresponds to the regular representation of Γ: each distinct Γ-irreducible appears in the Γ-isotypic decomposition with multiplicity one. Thus

$$V = V_1 \oplus \cdots \oplus V_n \tag{7.10}$$

where each irreducible V_i has character type χ_i. We call i the *character index of* χ_i. By Theorem 7.3.2 the group Γ has $n = n_1 \cdots n_r$ linear distinct irreducible characters that can be constructed from the n_j distinct linear characters of $\mathbf{Z}_{n_j}$, for $j = 1, \ldots, r$.

Recall that if $\mathbf{Z}_{n_j} =< a_j >$ and $\omega_j = e^{2\pi i/n_j}$, then the n_j linear characters of $\mathbf{Z}_{n_j}$, say $\psi_1, \ldots, \psi_{n_j}$ are determined by their values at the generators, $\psi_i(a_j) = \omega_j^{(i-1)}$, $i = 1, \ldots, n_j$.

We can also enumerate each character of Γ using (7.9). Let $\vec{s_i} = (s_{i_1}, \ldots, s_{i_r})$ be such that

$$\gamma_{\vec{s}_i}(1) = i. \tag{7.11}$$

We denote by χ_i the linear irreducible character of Γ determined by:

$$\chi_i(a_1) = \omega_1^{s_{i_1}}, \ \ldots, \ \chi_i(a_r) = \omega_r^{s_{i_r}} \tag{7.12}$$

where as before $\omega_j = e^{i2\pi/n_j}$, for $j = 1, \ldots, r$. Thus

$$\chi_i\left(a_1 \cdots a_r\right) = \omega_1^{s_{i_1}} \cdots \omega_r^{s_{i_r}}$$

and

$$\chi_i\left(a_1^{p_1} \cdots a_r^{p_r}\right) = \omega_1^{p_1 s_{i_1}} \cdots \omega_r^{p_r s_{i_r}} \tag{7.13}$$

where $p_j \in \{0, \ldots, n_j - 1\}$, for $j = 1, \ldots, r$. We also write

$$\chi_i \equiv \chi_{\vec{s}_i} \equiv \chi_{(s_{i_1}, \ldots, s_{i_r})}.$$

Because Γ is abelian, we know that if $\gamma_{\vec{s}_i} = a_1^{s_{i_1}} \cdots a_r^{s_{i_r}}$, then $\left(\gamma_{\vec{s}_i}\right)^{-1} = a_1^{n_1 - s_{i_1}} \cdots a_r^{n_r - s_{i_r}}$ and so

$$\overline{\chi_i} = \chi_{\left(\gamma_{\vec{s}_i}\right)^{-1}(1)} \equiv \chi_{\left(\gamma_{\vec{s}_i}\right)^{-1}}.$$

We say that i and $\left(\gamma_{\vec{s}_i}\right)^{-1}(1)$ are *conjugated indexes.*

Definition 7.3.3 Consider an abelian group $\Gamma \subseteq \mathbf{S}_n$ of order n that acts transitively (and faithfully) on the set $\{1, \ldots, n\}$. Let Γ be as in (7.7) with the group elements enumerated as in (7.8) for the given choice of generators set $\{a_1, \ldots, a_r\}$ of Γ. Consider $S = \{b_1, b_2, \ldots, b_q\} \subseteq \Gamma$ with $q \geq r \geq 2$, $< S >= \Gamma$ and the irreducible characters χ_i of Γ enumerated as in (7.11)–(7.13). A character index $i \in \{1, \ldots, n\}$ is *special with multiplicity* $\widehat{m} \geq 4$

over S if there are $\widehat{m}$ character indexes $i = i_1 < \cdots < i_{\widehat{m}}$ such that the vectors

$$\widetilde{v}_{i_l} = (\mathrm{Re}\,(\chi_{i_l}\,(b_1))\,,\ldots,\mathrm{Re}\,(\chi_{i_l}\,(b_q))) \tag{7.14}$$

are equal for $l = 1, \ldots, \widehat{m}$ and $i = i_1$ and $i_{\widehat{m}}$ are respectively the lowest and largest indexes that verify this condition. We call the vector $v_i = (\chi_i\,(b_1)\,,\ldots,\chi_i\,(b_q))$ a *special vector over* S. The indexes $i_1,\ldots,i_{\widehat{m}}$ and the vectors $v_{i_1},\ldots,v_{i_{\widehat{m}}}$ are respectively the *indexes* and the *vectors over* S *associated with the special index* i. If there are special indexes over S we say that Γ is *special* over S or simply that S is *special*.

Observe that if i is special with multiplicity $\widehat{m}$ over S then half of the indexes $i = i_1 < \cdots < i_{\widehat{m}}$ are conjugated from the others. If we choose from each pair of conjugated indexes the smallest we obtain the *representative indexes of the indexes associated with* i. The corresponding vectors are the *representative vectors of the vectors over* S *associated with the special index* i.

Example 7.3.4 Consider the abelian group $\Gamma \subseteq \mathbf{S}_9$ of order 9 such that $\Gamma =< a_1, a_2 >\cong \mathbf{Z}_3\times\mathbf{Z}_3$ where $a_1 = (123)(456)(789)$ and $a_2 = (147)(258)(369)$. Thus Γ acts transitively and faithfully on the set $\{1,\ldots,9\}$. Using the notation of (7.7), we have $n_1 = n_2 = 3$ and $r = 2$. The character table of Γ is represented in Table 7.1 where in the first column we follow the notation determined by (7.11) and (7.12): $\gamma_{(0,0)}\,(1) = (a_1^0 a_2^0)\,(1) = 1$, $\gamma_{(1,0)}\,(1) = (a_1^1 a_2^0)\,(1) = 2$, etc.

	$a_1^{s_1}a_2^{s_2}$	a_1	a_2
$\chi_1 = \chi_{(0,0)}$	1	1	1
$\chi_2 = \chi_{(1,0)}$	ζ^{s_1}	ζ	1
$\chi_3 = \chi_{(2,0)}$	ζ^{2s_1}	ζ^2	1
$\chi_4 = \chi_{(0,1)}$	ζ^{s_2}	1	ζ
$\chi_5 = \chi_{(1,1)}$	$\zeta^{s_1+s_2}$	ζ	ζ
$\chi_6 = \chi_{(2,1)}$	$\zeta^{2s_1+s_2}$	ζ^2	ζ
$\chi_7 = \chi_{(0,2)}$	ζ^{2s_2}	1	ζ^2
$\chi_8 = \chi_{(1,2)}$	$\zeta^{s_1+2s_2}$	ζ	ζ^2
$\chi_9 = \chi_{(2,2)}$	$\zeta^{2s_1+2s_2}$	ζ^2	ζ^2

Table 7.1: Character table of $\mathbf{Z}_3 \times \mathbf{Z}_3$. Here $\zeta = e^{i\frac{2\pi}{3}}$ and $s_1, s_2 \in \{0, 1, 2\}$.

Consider $S = \{a_1, a_2\}$. Then

$$\widetilde{v}_i = (\mathrm{Re}\,(\chi_i\,(a_1))\,,\mathrm{Re}\,(\chi_i\,(a_2))) = \left(\cos\frac{2\pi}{3}, \cos\frac{2\pi}{3}\right)$$

for $i=5,6,7,8$. Because $\widetilde{v}_5=\widetilde{v}_6=\widetilde{v}_7=\widetilde{v}_8$, the vectors

$$\begin{array}{l} v_5=\left(\chi_5\left(a_1\right),\chi_5\left(a_2\right)\right)=\left(\zeta,\zeta\right)\\ v_6=\left(\chi_6\left(a_1\right),\chi_6\left(a_2\right)\right)=\left(\zeta^2,\zeta\right)\\ v_8=\left(\chi_8\left(a_1\right),\chi_8\left(a_2\right)\right)=\left(\zeta,\zeta^2\right)\\ v_9=\left(\chi_9\left(a_1\right),\chi_9\left(a_2\right)\right)=\left(\zeta^2,\zeta^2\right)\end{array}$$

are the vectors associated with the special index 5 over S. Moreover, as $\overline{\chi_5}=\chi_9$ and $\overline{\chi_6}=\chi_8$ then v_5 and v_6 are the representative vectors of the vectors associated with the special index 5. $\diamond$

Theorem 7.3.5 *Consider Γ described by* (7.7) *and* (7.8) *for a given choice $a_1,\ldots,a_r$ of generators, with $r\geq 2$, the irreducible characters of Γ enumerated by χ_i as in* (7.11)–(7.13) *and that the decomposition* (7.7) *has at least two cyclic groups with order greater than two. Let $I=\{n_1,\ldots,n_r\}$. Consider that $m_1,\ldots,m_q$ are the even numbers different from two in I, $m_{q+1},\ldots,m_{\tilde r}$ are the odd numbers in I and that $n_k=2$ for all $k\geq\tilde r+1$. Then there are*

$$\begin{cases} \frac{1}{2^{2\tilde r-r}}\prod\limits_{k=1}^{\tilde r}\left(m_k-1\right), & \text{if } q=0;\\ \frac{1}{2^{2\tilde r-r}}\prod\limits_{k=1}^{q}\left(m_k-2\right)\times\prod\limits_{k=q+1}^{\tilde r}\left(m_k-1\right), & \text{if } q\geq 1.\end{cases}$$

special vectors over $S=\{a_1,a_2,\ldots,a_r\}$ with $\tilde r$ non real components and with multiplicity $2^{\tilde r}$.

In general if, for $i\in I$, the number of real components of the vector $(\chi_i(a_1),\ldots,\chi_i(a_r))$ is $N\leq r-2$ then i corresponds to an element $l\in\{1,\ldots,n\}$ that is special over S with multiplicity $\widehat{m}=2^{r-N}$.

Proof: Suppose $i=\left(a_1^{j_1}\cdots a_r^{j_r}\right)(1)$ where $0\leq j_k\leq n_k-1$ for $k=1,\ldots,r$. Write

$$\begin{array}{l}\left(\chi_i\left(a_1\right),\ldots,\chi_i\left(a_{\tilde r}\right),\chi_i\left(a_{\tilde r+1}\right),\ldots,\chi_i\left(a_r\right)\right)\\ =\left(\omega_1^{j_1},\ldots,\omega_{\tilde r}^{j_{\tilde r}},\left(-1\right)^{j_{\tilde r+1}},\ldots,\left(-1\right)^{j_r}\right)\end{array}$$

where $\omega_k=e^{2\pi/n_k}$. If $q=0$, then for each k such that $1\leq k\leq\widetilde{r}$, the $\omega_k^{j_k}$, for $0\leq j_k\leq m_k-1$, are the m_kth roots of unity, where only one is real and $(m_k-1)/2$ have positive imaginary part. The other $(m_k-1)/2$ are conjugated of these. For each vector $(j_1,\ldots,j_r)$, where $1\leq j_k\leq(m_k-1)/2$ for $k\in\{1,\ldots,\tilde r\}$, and $j_k\in\{0,1\}$ for $k\in\{\tilde r+1,\ldots,r\}$, the vectors

$$\left(\omega_1^{\pm j_1},\ldots,\omega_{\tilde r}^{\pm j_{\tilde r}},\left(-1\right)^{j_{\tilde r+1}},\ldots,\left(-1\right)^{j_r}\right)$$

correspond to different special vectors with multiplicity $2^{\tilde{r}}$ and with $\tilde{r}$ non real components. Then, there are

$$\left(\prod_{k=1}^{\tilde{r}} \frac{m_k - 1}{2}\right) 2^{r-\tilde{r}} = \frac{1}{2^{2\tilde{r}-r}} \prod_{k=1}^{\tilde{r}} (m_k - 1)$$

special vectors, each one with multiplicity $2^{\tilde{r}}$.

When $q > 0$, for each k such that $1 \leq k \leq q$, we have that $(e^{i2\pi/m_k})^{j_k}$, for $0 \leq j_k \leq m_k - 1$, are the m_kth roots of unity. Moreover, two are real and $(m_k - 2)/2$ have positive imaginary part. The other $(m_k - 2)/2$ are conjugated of these. The vectors

$$\left(\left(e^{i\frac{2\pi}{m_1}}\right)^{\pm j_1}, \ldots, \left(e^{i\frac{2\pi}{m_q}}\right)^{\pm j_q}, \left(e^{i\frac{2\pi}{m_{q+1}}}\right)^{\pm j_{q+1}}, \ldots, \left(e^{i\frac{2\pi}{m_{\tilde{r}}}}\right)^{\pm j_{\tilde{r}}}, (-1)^{j_{\tilde{r}+1}}, \ldots, (-1)^{j_r}\right),$$

for each vector $(j_1, \ldots, j_r)$, where $1 \leq j_k \leq (m_k-2)/2$ for $k \in \{1, \ldots, q\}$, $1 \leq j_k \leq (m_k - 1)/2$ for $k \in \{q+1, \ldots, \tilde{r}\}$ and $j_k \in \{0, 1\}$ for $k \in \{\tilde{r}+1, \ldots, r\}$, are the vectors associated with a special vector with multiplicity $2^{\tilde{r}}$ and with $\tilde{r}$ non real components. Then, there are

$$\left(\prod_{k=1}^{q} \frac{m_k - 2}{2}\right)\left(\prod_{k=q+1}^{\tilde{r}} \frac{m_k - 1}{2}\right) 2^{r-\tilde{r}}$$

special vectors, each one with multiplicity $2^{\tilde{r}}$.

We prove now the second part of the theorem. If the number of real components of the vector $(\chi_i(a_1), \ldots, \chi_i(a_r))$ is $N \leq r - 2$, then for each of the $r - N$ non real components we can consider its conjugated. The set of all of these vectors correspond to 2^{r-N} character indexes. The smallest index of this set is special with multiplicity $\widehat{m} = 2^{r-N}$. □

We specialize now our results to cyclic groups. Observe that if Γ is cyclic then $\Gamma \cong \mathbf{Z}_{P_1^{m_1}} \times \cdots \times \mathbf{Z}_{P_r^{m_r}}$ where $P_1, \ldots, P_r$ are distinct prime numbers since $P_1^{m_1}, \ldots, P_r^{m_r}$ must be coprime. See Fraleigh [12] Theorem I 1.18.

Corollary 7.3.6 *Consider a cyclic group $\Gamma \cong \mathbf{Z}_{P_1^{m_1}} \times \cdots \times \mathbf{Z}_{P_r^{m_r}}$ where $P_1, \ldots, P_r$ are distinct prime numbers and generated by $a_1, \ldots, a_r$. Then Γ admits special indexes over $S = \{a_1, a_2, \ldots, a_r\}$ if and only if $r \geq 2$ and at least two of the powers $P_1^{m_1}, \ldots, P_r^{m_r}$ are greater than two.*

Proof: If $r \geq 2$ and at least two of the powers $P_1^{m_1}, \ldots, P_r^{m_r}$ are greater than two then Theorem 7.3.5 guarantees the existence of special indexes over S.

Consider now χ_i as in (7.13) and assume that there are not two powers in $P_1^{m_1}, \ldots, P_r^{m_r}$ that are greater than two. Let $n_1 = P_1^{m_1}$ with $P_1 \neq 2$. If $r \neq 1$, because $P_1, \ldots, P_r$ are different prime numbers then $r = 2$ and $\Gamma = \langle a_1, a_2 \rangle \cong \mathbf{Z}_{n_1} \times \mathbf{Z}_2$. For $i = 1, \ldots, n$ the vectors $(\chi_i(a_1), \chi_i(a_2))$ are always of the form $\left(e^{i\frac{2\pi}{n_1}j_1}, (-1)^{j_2}\right)$ with $0 \leq j_1 \leq n_1 - 1$, $j_2 \in \{0, 1\}$, and consequently Γ does not have special indexes over S. If $r = 1$ then $\Gamma = \mathbf{Z}_{P_1^{m_1}}$ and obviously these are no special indexes over S. $\square$

Example 7.3.7 Consider an abelian group $\Gamma \subseteq \mathbf{S}_{18}$ of order 18 that acts transitively (and faithfully) on the set $\{1, \ldots, 18\}$ such that $\Gamma =< a_1, a_2 >\cong \mathbf{Z}_{18} \cong \mathbf{Z}_9 \times \mathbf{Z}_2$ where $< a_1 >\cong \mathbf{Z}_9$ and $< a_2 >\cong \mathbf{Z}_2$. Then, by Corollary 7.3.6, $\mathbf{Z}_{18}$ is not special over $S = \{a_1, a_2\}$.
By the same corollary, a group $\Gamma \subseteq \mathbf{S}_{12}$ of order 12 that acts transitively (and faithfully) on the set $\{1, \ldots, 12\}$ such that $\Gamma =< a_1, a_2 >\cong \mathbf{Z}_{12} \cong \mathbf{Z}_4 \times \mathbf{Z}_3$ is special over $S = \{a_1, a_2\}$ where $< a_1 >\cong \mathbf{Z}_4$ and $< a_2 >\cong \mathbf{Z}_3$. $\diamond$

Corollary 7.3.8 *A group Γ as in (7.7) and (7.8) does not have special indexes over $S = \{a_1, a_2, \ldots, a_r\}$ if and only if $\Gamma \cong \mathbf{Z}_{n_1} \times (\mathbf{Z}_2)^{r-1}$ or $\Gamma \cong (\mathbf{Z}_2)^r$ with $r \in \mathbb{N}$ and $n_1 \geq 3$ a power of a prime.*

Proof: Consider χ_i as in (7.13). If

$$\Gamma = \langle a_1, a_2, \ldots, a_r \rangle \cong \mathbf{Z}_{n_1} \times (\mathbf{Z}_2)^{r-1},$$

the vectors

$$(\chi_i(a_1), \chi_i(a_2), \ldots, \chi_i(a_r)) \tag{7.15}$$

are always of the form $\left(e^{i\frac{2\pi}{n_1}j_1}, (-1)^{j_2}, \ldots, (-1)^{j_r}\right)$ with $0 \leq j_1 \leq n_1 - 1$, $j_2, \ldots, j_r \in \{0, 1\}$ and hence cannot correspond to a special vector. The case $\Gamma \cong (\mathbf{Z}_2)^r$ is similar.

If an abelian group Γ does not have special indexes over S then, by Theorem 7.3.5, the cyclic decomposition (7.7) does not have at least two groups different from $\mathbf{Z}_2$. Then it must be of the form $\Gamma \cong (\mathbf{Z}_{n_1}) \times (\mathbf{Z}_2)^{r-1}$ or $\Gamma \cong (\mathbf{Z}_2)^r$. $\square$

Corollary 7.3.9 *Consider Γ as in (7.7) and (7.8). If $\Gamma \cong \mathbf{Z}_{n_1} \times (\mathbf{Z}_2)^{r-1}$ with $n \geq 1$ and $n_1 \geq 3$ a power of a prime then Γ does not have special indexes over any $S \subseteq \Gamma$ such that $< S >= \Gamma$.*

Proof: Let $S = \{b_1, \ldots, b_q\}$ and $i = \left(a_1^{j_1} \cdots a_r^{j_r}\right)(1)$ where $0 \leq j_1 \leq n_1 - 1$ and $j_2, \ldots, j_r \in \{0, 1\}$. If $b_l \in < a_2, \ldots, a_r >$ then $\chi_i(b_l) = \pm 1$ for $i = 1, \ldots, n$. Consider now $b_l = a_1^{p_1} \cdots a_r^{p_r}$ and $b_m = a_1^{q_1} \cdots a_r^{q_r}$ with $0 < p_1, q_1 \leq n_1 - 1$ and $p_2, q_2, \ldots p_r, q_r \in \{0, 1\}$. It follows that $\chi_i(b_l) = \pm e^{i\frac{2\pi}{n_1}p_1 j_1}$ and $\chi_i(b_m) = \pm e^{i\frac{2\pi}{n_1}q_1 j_1}$. If $\chi_j(b_l) = \overline{\chi_i(b_l)}$ then we have $\chi_j(b_l) = \pm e^{i\frac{2\pi}{n_1}p_1(n_1 - j_1)}$ and consequently $\chi_j(b_m) = \overline{\chi_i(b_m)} = \pm e^{i\frac{2\pi}{n_1}q_1(n_1 - j_1)}$. It turns out that $\chi_j(\gamma) = \overline{\chi_i(\gamma)}$ for all $\gamma \in S$ and that Γ does not have special indexes over S. □

Example 7.3.10 If $\Gamma \subset \mathbf{S}_{10}$ is an abelian group of order 10 that acts transitively (and faithfully) on the set $\{1, \ldots, 10\}$ such that $\Gamma \cong \mathbf{Z}_{10}$ then, because $\mathbf{Z}_{10} \cong \mathbf{Z}_5 \times \mathbf{Z}_2$, Corollary 7.3.9 guarantees that Γ has not special indexes over any set S such that $< S >= \Gamma$.

For another example, every abelian group $\Gamma \subset \mathbf{S}_8$ of order 8, is isomorphic to either $\mathbf{Z}_8$, $\mathbf{Z}_4 \times \mathbf{Z}_2$, or $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 = (\mathbf{Z}_2)^3$. By Corollary 7.3.9 there is no abelian group of order 8 that acts transitively (and faithfully) on the set $\{1, \ldots, 8\}$, special over any set S such that $< S >= \Gamma$. ◇

Note that for the results stated above, the presence of $\mathbf{Z}_2$ in the decomposition (7.7) always implied particular analysis. The next example illustrates the effect that $\mathbf{Z}_2$ has at the existence of special indexes.

Example 7.3.11 Consider the abelian group $\Gamma \subseteq \mathbf{S}_{18}$ of order 18 generated by

$$\begin{aligned}
a_1 &= (123)(456)(789)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18), \\
a_2 &= (147)(258)(369)(10\ 13\ 16)(11\ 14\ 17)(12\ 15\ 18), \\
a_3 &= (1\ 10)(2\ 11)(3\ 12)\cdots(9\ 18).
\end{aligned}$$

It acts transitively (and faithfully) on the set $\{1, \ldots, 18\}$ and $\Gamma \cong \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_2$. Using the notation of (7.7) we have then that $r = 3$, $n_1 = n_2 = 3$ and $n_3 = 2$.

Consider $S = \{a_1, a_2, a_3\}$ and the character table of Γ represented in Table 7.2. Then

$$\widetilde{v}_i = \left(\mathrm{Re}\left(\chi_i\left(a_1\right)\right), \mathrm{Re}\left(\chi_i\left(a_2\right)\right), \mathrm{Re}\left(\chi_i\left(a_2\right)\right)\right) = \left(\cos\frac{2\pi}{3}, \cos\frac{2\pi}{3}, 1\right)$$

	$a_1^{s_1}a_2^{s_2}a_3^{s_3}$	a_1	a_2	a_3
$\chi_1 = \chi_{(0,0,0)}$	1	1	1	1
$\chi_2 = \chi_{(1,0,0)}$	ζ^{s_1}	ζ	1	1
$\chi_3 = \chi_{(2,0,0)}$	ζ^{2s_1}	ζ^2	1	1
$\chi_4 = \chi_{(0,1,0)}$	ζ^{s_2}	1	ζ	1
$\chi_5 = \chi_{(1,1,0)}$	$\zeta^{s_1+s_2}$	ζ	ζ	1
$\chi_6 = \chi_{(2,1,0)}$	$\zeta^{2s_1+s_2}$	ζ^2	ζ	1
$\chi_7 = \chi_{(0,2,0)}$	ζ^{2s_2}	1	ζ^2	1
$\chi_8 = \chi_{(1,2,0)}$	$\zeta^{s_1+2s_2}$	ζ	ζ^2	1
$\chi_9 = \chi_{(2,2,0)}$	$\zeta^{2s_1+2s_2}$	ζ^2	ζ^2	1
$\chi_{10} = \chi_{(0,0,1)}$	1	1	1	-1
$\chi_{11} = \chi_{(1,0,1)}$	$\zeta^{s_1}(-1)^{s_3}$	ζ	1	-1
$\chi_{12} = \chi_{(2,0,1)}$	$\zeta^{2s_1}(-1)^{s_3}$	ζ^2	1	-1
$\chi_{13} = \chi_{(0,1,1)}$	$\zeta^{s_2}(-1)^{s_3}$	1	ζ	-1
$\chi_{14} = \chi_{(1,1,1)}$	$\zeta^{s_1+s_2}(-1)^{s_3}$	ζ	ζ	-1
$\chi_{15} = \chi_{(2,1,1)}$	$\zeta^{2s_1+s_2}(-1)^{s_3}$	ζ^2	ζ	-1
$\chi_{16} = \chi_{(0,2,1)}$	$\zeta^{2s_2}(-1)^{s_3}$	1	ζ^2	-1
$\chi_{17} = \chi_{(1,2,1)}$	$\zeta^{s_1+2s_2}(-1)^{s_3}$	ζ	ζ^2	-1
$\chi_{18} = \chi_{(2,2,1)}$	$\zeta^{2s_1+2s_2}(-1)^{s_3}$	ζ^2	ζ^2	-1

Table 7.2: Character table of $\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_2$. Here $\zeta = e^{i\frac{2\pi}{3}}$, $s_1, s_2 \in \{0,1,2\}$ and $s_3 \in \{0,1\}$.

for $i = 5,6,8,9$. Because $\widetilde{v}_5 = \widetilde{v}_6 = \widetilde{v}_7 = \widetilde{v}_8$, the vectors

$$\begin{aligned}
v_5 &= (\chi_5(a_1), \chi_5(a_2), \chi_5(a_3)) = (\zeta, \zeta, 1)\\
v_6 &= (\chi_6(a_1), \chi_6(a_2), \chi_6(a_3)) = (\zeta^2, \zeta, 1)\\
v_8 &= (\chi_8(a_1), \chi_8(a_2), \chi_8(a_3)) = (\zeta, \zeta^2, 1)\\
v_9 &= (\chi_9(a_1), \chi_9(a_2), \chi_9(a_3)) = (\zeta^2, \zeta^2, 1)
\end{aligned}$$

are the vectors associated with the special index 5 over S. Moreover, because $v_8 = \overline{v_6}$ and $v_9 = \overline{v_5}$ we have that v_5 and v_6 are the representative vectors of the vectors associated with the special index 5. The presence of $\mathbf{Z}_2$ adds the special index 14 which has

$$\begin{aligned}
v_{14} &= (\chi_{14}(a_1), \chi_{14}(a_2), \chi_{14}(a_3)) = (\zeta, \zeta, -1),\\
v_{15} &= (\chi_{15}(a_1), \chi_{15}(a_2), \chi_{15}(a_3)) = (\zeta^2, \zeta, -1),\\
v_{17} &= (\chi_{17}(a_1), \chi_{17}(a_2), \chi_{17}(a_3)) = (\zeta, \zeta^2, -1),\\
v_{18} &= (\chi_{18}(a_1), \chi_{18}(a_2), \chi_{18}(a_3)) = (\zeta^2, \zeta^2, -1)
\end{aligned}$$

as vectors associated with them. Now, because $v_{17} = \overline{v_{15}}$ and $v_{18} = \overline{v_{14}}$ we have that v_{14} and v_{15} are the representative vectors of the vectors associated with the special index 14. $\diamond$

Recall the notation introduced above that we use in the next lemmas. Let

$$\Gamma \cong \mathbf{Z}_{n_1} \times \cdots \times \mathbf{Z}_{n_r}$$

be a subgroup of the symmetric group $\mathbf{S}_n$ acting transitively on $\{1, \ldots, n\}$ where $n = n_1 \cdots n_r$. Let $\{a_1, \ldots, a_r\}$ be a set of generators of Γ as in (7.8). Given $j \in \{1, \ldots, n\}$ then

$$(s_{j_1}, \ldots, s_{j_r})$$

denotes the unique vector such that

$$a_1^{s_{j_1}} \cdots a_r^{s_{j_r}}(1) = j$$

($s_{j_k} \in \{0, \ldots, n_k - 1\}$ for $k = 1, \ldots, r$) and χ_j is the irreducible character of Γ defined by:

$$\chi_j\left(a_1^{p_1} \cdots a_r^{p_r}\right) = \omega_1^{p_1 s_{j_1}} \cdots \omega_r^{p_r s_{j_r}}$$

where $p_k \in \{0, \ldots, n_k - 1\}$, $\omega_k = e^{i2\pi/n_k}$, for $k = 1, \ldots, r$.

Lemma 7.3.12 *Consider Γ and the irreducible characters χ_j of Γ described above. Set $S_1 = \{a_1, \ldots, a_r\}$. Assume that the index $i \in \{1, \ldots, n\}$ is special with multiplicity $\widehat{m} \geq 4$ over S_1 and that $v_j = (\chi_j(a_1), \ldots, \chi_j(a_r))$, for $j \in I \subseteq \{1, \ldots, n\}$ are the vectors over S associated with the special index i. Thus I has cardinality $\widehat{m}$.*

Then there is a special index $i' \in I$ over

$$S_2 = \{a_1, \ldots, a_r, a_1^{p_1} \cdots a_r^{p_r}\}$$

with multiplicity ≥ 4 and lower or equal $\widehat{m}$ if and only if there exists $I' \subseteq I$ such that $i' \in I'$, I' has cardinality ≥ 4 (and lower or equal $\widehat{m}$) and $(p_1, \ldots, p_r)$ verifies: for all $j,\ t \in I'$,

$$\frac{p_1}{n_1}(s_{j_1} - s_{t_1}) + \cdots + \frac{p_r}{n_r}(s_{j_r} - s_{t_r}) = k \tag{7.16}$$

or

$$\frac{p_1}{n_1}(s_{j_1} + s_{t_1}) + \cdots + \frac{p_r}{n_r}(s_{j_r} + s_{t_r}) = k \tag{7.17}$$

for some $k \in \mathbf{Z}$.

Proof: Suppose that $i' \in I$ is special over S_2. Say with multiplicity $\widehat{m}'$. Then there must exist a set I' such that $i' \in I'$, with cardinality $\widehat{m}'$ and such that the vectors

$$(\mathrm{Re}\left(\chi_l\left(a_1\right)\right), \ldots, \mathrm{Re}\left(\chi_l\left(a_r\right)\right), \mathrm{Re}\left(\chi_l\left(a_1^{p_1} \cdots a_r^{p_r}\right)\right))$$

for $l \in I'$ are equal. Moreover, i' is the lowest index of I'. Because $i' \in I$,

$$(\mathrm{Re}\,(\chi_{i'}\,(a_1))\,,\ldots,\mathrm{Re}\,(\chi_{i'}\,(a_r))) = (\mathrm{Re}\,(\chi_l\,(a_1))\,,\ldots,\mathrm{Re}\,(\chi_l\,(a_r)))$$

for $l \in I$. Thus $I' \subseteq I$, as I is the largest set of indexes containing i' that make $(\mathrm{Re}\,(\chi_l\,(a_1))\,,\ldots,\mathrm{Re}\,(\chi_l\,(a_r)))$ be equal.

As

$$\chi_j\left(a_1^{p_1}\cdots a_r^{p_r}\right) = e^{i\frac{2\pi}{n_1}p_1 s_{j_1}}\cdots e^{i\frac{2\pi}{n_r}p_r s_{j_r}}$$

and

$$\chi_t\left(a_1^{p_1}\cdots a_r^{p_r}\right) = e^{i\frac{2\pi}{n_1}p_1 s_{t_1}}\cdots e^{i\frac{2\pi}{n_r}p_r s_{t_r}}$$

then

$$\mathrm{Re}\left(\chi_j\left(a_1^{p_1}\cdots a_r^{p_r}\right)\right) = \mathrm{Re}\left(\chi_t\left(a_1^{p_1}\cdots a_r^{p_r}\right)\right)$$

for all $j, t \in I'$ if and only if

$$\cos\left[2\pi\left(\frac{p_1}{n_1}s_{j_1}+\cdots+\frac{p_r}{n_r}s_{j_r}\right)\right] = \cos\left[2\pi\left(\frac{p_1}{n_1}s_{t_1}+\cdots+\frac{p_r}{n_r}s_{t_r}\right)\right]$$

for all $j, t \in I'$. Solving these equations we obtain the conditions (7.16) and (7.17).

Suppose now that exists $I' \subseteq I$ such that for each $j, t \in I'$, $(p_1, \ldots, p_r)$ verifies at least one of the conditions (7.16) or (7.17). Then $\mathrm{Re}\left(\chi_j\left(a_1^{p_1}\cdots a_r^{p_r}\right)\right) = \mathrm{Re}\left(\chi_t\left(a_1^{p_1}\cdots a_r^{p_r}\right)\right)$ for all $j, t \in I'$. As $I' \subseteq I$ we have the result. That is, there is a special index $i' \in I' \subseteq I$ over S_2. $\square$

Lemma 7.3.13 *Consider Γ and the irreducible characters of Γ as described above. Let $S_1 = \{a_1, \ldots, a_r\}$. Assume that*

$$v_i = (\chi_i\,(a_1)\,,\ldots,\chi_i\,(a_r)) \tag{7.18}$$

is a special vector over S_1 with multiplicity $\widehat{m}$. Then the vector

$$\left(\chi_i\,(a_1)\,,\ldots,\chi_i\,(a_r)\,,\chi_i\left(a_1^{p_1}\right),\ldots,\chi_i\left(a_r^{p_r}\right)\right) \tag{7.19}$$

corresponds to a special vector over $S_2 = \{a_1, \ldots, a_r, a_1^{p_1}, \ldots, a_r^{p_r}\}$ associated with the special index i with multiplicity $\widehat{m}$. Here $p_k \in \{0, \ldots, n_k - 1\}$.

Proof: Denote by I the set of indexes associated with the special index i over S_1. Then $\mathrm{Re}\,(\chi_j\,(a_k)) = \mathrm{Re}\,(\chi_t\,(a_k)) = \cos\left(\frac{2\pi}{n_k}s_{j_k}\right)$ for all $j, t \in I$ and $k = 1, \ldots, r$. It turns out that because $\chi_i\left(a_k^{p_k}\right) = (\chi_i\,(a_k))^{p_k}$, $\mathrm{Re}\left(\chi_j\left(a_k^{p_k}\right)\right) = \mathrm{Re}\left(\chi_t\left(a_k^{p_k}\right)\right) = \cos\left(\frac{2\pi}{n_k}s_{j_k}p_k\right)$ for all $t \in I$ and the result follows. $\square$

As we saw above, the study of the existence of special characters involves complicated conditions even when S is a set with few elements. Observe that in some of the previous results we consider $S = \{a_1, \ldots, a_r\}$. It is the simplest set that generates Γ. The following algorithm determines the sets $I_1, \ldots, I_j$ of indexes associated with the special indexes $\min\{I_1\}, \ldots, \min\{I_j\}$ over a general set $S \subseteq \Gamma$ such that $< S >= \Gamma$ and the respective sets $\widetilde{I}_1, \ldots, \widetilde{I}_j$ of representative indexes. For $k = 1, \ldots, j$, each set I_k corresponds to the special index $\min\{I_k\}$ and to $\# I_k \geq 4$ equal rows on the table of the real parts of the character table of Γ restricted to S.

Algorithm 7.3.14 Given a group $\Gamma \subseteq \mathbf{S}_n$ of order n described by (7.7) and (7.8) for a given choice $a_1, \ldots, a_r$ of generators of Γ, $S = \{b_1, \ldots, b_q\} \subseteq \Gamma$ such that $< S >= \Gamma$, this algorithm finds the special indexes over S, the set of special indexes associated with each one of them and the respective sets of representative indexes.

1. [Character table restricted to S] Make the character table of Γ restricted to S enumerating the irreducible characters of Γ as in (7.13).

2. [Character table of real parts restricted to S] Make the table of real parts of the table of step 1.

3. [Define a matrix corresponding to the table of step 2] For $i = 1, \ldots, n$ set
$$v_i = (\mathrm{Re}\,(\chi_i\,(b_1))\,, \ldots, \mathrm{Re}\,(\chi_i\,(b_q)))\,.$$
Let A be the $n \times q$ matrix with rows $v_1, \ldots, v_n$.

4. [Initialize] Set $p = 0$, $j = 1$, $I_1 = \emptyset$, $\mathrm{CA} = \{2, \ldots, n\}$.

5. [Compute the sets of special indexes] Set $k = \min \mathrm{CA}$. Set $v = v_k$.

   For $l \in \mathrm{CA}$ do

   If $v_l = v$ then set $I_j = I_j \cup \{l\}$.

   Set $\mathrm{CA} = \mathrm{CA} \backslash I_j$

   If $\# I_j \geq 4$ then set $j = j + 1$, $p = p + 1$. Else set $I_j = \emptyset$.

   If $\#\mathrm{CA} < 4$ then go to step 6.

Go to the beginning of step 5.

6. [Compute sets of representative indexes of the indexes associated with the special indexes] If $p = 0$ go to step 7. If $p \geq 1$, for $k = 1, \ldots, p$ set $I = I_k$, $\widetilde{I_k} = \emptyset$ and do the following: repeat the steps $l = \min\{I\}$, $\widetilde{I_k} = \widetilde{I_k} \cup \{l\}$, $I = I - \{l, \gamma_l^{-1}(1)\}$, where γ_l is the element of Γ such that $\gamma_l(1) = l$, until $I = \emptyset$. Then go to step 7.

7. [Output] If $p = 0$ then does not exist special indexes. If $p \geq 1$, then output the sets $I_1, \ldots, I_p$ and $\widetilde{I_1}, \ldots, \widetilde{I_p}$. Each I_k, $1 \leq k \leq p$ is the set of indexes associated with the special index $\min\{I_k\}$ which has $\widetilde{I_k}$ as the set of representative indexes. Terminate the algorithm.

$\diamond$

Example 7.3.15 Consider the abelian group $\Gamma \subset \mathbf{S}_{60}$ of order 60 acting transitively (and faithfully) on the set $\{1, \ldots, 60\}$ such that $\Gamma \cong \mathbf{Z}_{60} \cong \mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ and generated by a_1, a_2, a_3 where

$$\begin{aligned} a_1 &= (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)\cdots(56\ 57\ 58\ 59\ 60),\\ a_2 &= (1\ 6\ 11\ 16)(2\ 7\ 12\ 17)\cdots(45\ 50\ 55\ 60),\\ a_3 &= (1\ 21\ 41)(2\ 22\ 42)\cdots(20\ 40\ 60). \end{aligned}$$

Let $S = \{a_1, a_2, a_3\}$. Now we follow the steps of Algorithm 7.3.14 to obtain the sets $I_1, \ldots, I_j$ of indexes associated with the special indexes $\min\{I_1\}, \ldots, \min\{I_j\}$ over S and the respective sets of representative indexes. Step 1 consists in obtaining the character table for Γ restricted to S enumerating the characters types as in (7.11)–(7.13). In step 2 we obtain the table of real parts of the table of step 1. See tables 7.3 and 7.4 respectively.

Following the remainder steps of the Algorithm 7.3.14 we obtain as output the sets of indexes associated with the special indexes over S and the respective sets of representative indexes:

$$\begin{aligned}
&I_1 = \{7, 10, 17, 20\} && \widetilde{I_1} = \{7, 10\}\\
&I_2 = \{8, 9, 18, 19\} && \widetilde{I_2} = \{8, 9\}\\
&I_3 = \{22, 25, 42, 45\} && \widetilde{I_3} = \{22, 25\}\\
&I_4 = \{23, 24, 43, 44\} && \widetilde{I_4} = \{23, 24\}\\
&I_5 = \{26, 36, 46, 56\} && \widetilde{I_5} = \{26, 36\}\\
&I_6 = \{27, 30, 37, 40, 47, 50, 57, 60\} && \widetilde{I_6} = \{27, 30, 37, 40\}\\
&I_7 = \{28, 29, 38, 39, 48, 49, 58, 59\} && \widetilde{I_7} = \{28, 29, 38, 39\}\\
&I_8 = \{32, 35, 52, 55\} && \widetilde{I_8} = \{32, 35\}\\
&I_9 = \{33, 34, 53, 54\} && \widetilde{I_9} = \{33, 34\}.
\end{aligned}$$

	a_1	a_2	a_3		a_1	a_2	a_3
$\chi_1 = \chi_{(0,0,0)}$	1	1	1	$\chi_{31} = \chi_{(0,2,1)}$	1	-1	$e^{i\frac{2\pi}{3}}$
$\chi_2 = \chi_{(1,0,0)}$	$e^{i\frac{2\pi}{5}}$	1	1	$\chi_{32} = \chi_{(1,2,1)}$	$e^{i\frac{2\pi}{5}}$	-1	$e^{i\frac{2\pi}{3}}$
$\chi_3 = \chi_{(2,0,0)}$	$e^{i\frac{4\pi}{5}}$	1	1	$\chi_{33} = \chi_{(2,2,1)}$	$e^{i\frac{4\pi}{5}}$	-1	$e^{i\frac{2\pi}{3}}$
$\chi_4 = \chi_{(3,0,0)}$	$e^{-i\frac{4\pi}{5}}$	1	1	$\chi_{34} = \chi_{(3,2,1)}$	$e^{-i\frac{4\pi}{5}}$	-1	$e^{i\frac{2\pi}{3}}$
$\chi_5 = \chi_{(4,0,0)}$	$e^{-i\frac{2\pi}{5}}$	1	1	$\chi_{35} = \chi_{(4,2,1)}$	$e^{-i\frac{2\pi}{5}}$	-1	$e^{i\frac{2\pi}{3}}$
$\chi_6 = \chi_{(0,1,0)}$	1	i	1	$\chi_{36} = \chi_{(0,3,1)}$	1	$-i$	$e^{i\frac{2\pi}{3}}$
$\chi_7 = \chi_{(1,1,0)}$	$e^{i\frac{2\pi}{5}}$	i	1	$\chi_{37} = \chi_{(1,3,1)}$	$e^{i\frac{2\pi}{5}}$	$-i$	$e^{i\frac{2\pi}{3}}$
$\chi_8 = \chi_{(2,1,0)}$	$e^{i\frac{4\pi}{5}}$	i	1	$\chi_{38} = \chi_{(2,3,1)}$	$e^{i\frac{4\pi}{5}}$	$-i$	$e^{i\frac{2\pi}{3}}$
$\chi_9 = \chi_{(3,1,0)}$	$e^{-i\frac{4\pi}{5}}$	i	1	$\chi_{39} = \chi_{(3,3,1)}$	$e^{-i\frac{4\pi}{5}}$	$-i$	$e^{i\frac{2\pi}{3}}$
$\chi_{10} = \chi_{(4,1,0)}$	$e^{-i\frac{2\pi}{5}}$	i	1	$\chi_{40} = \chi_{(4,3,1)}$	$e^{-i\frac{2\pi}{5}}$	$-i$	$e^{i\frac{2\pi}{3}}$
$\chi_{11} = \chi_{(0,2,0)}$	1	-1	1	$\chi_{41} = \chi_{(0,0,2)}$	1	1	$e^{-i\frac{2\pi}{3}}$
$\chi_{12} = \chi_{(1,2,0)}$	$e^{i\frac{2\pi}{5}}$	-1	1	$\chi_{42} = \chi_{(1,0,2)}$	$e^{i\frac{2\pi}{5}}$	1	$e^{-i\frac{2\pi}{3}}$
$\chi_{13} = \chi_{(2,2,0)}$	$e^{i\frac{4\pi}{5}}$	-1	1	$\chi_{43} = \chi_{(2,0,2)}$	$e^{i\frac{4\pi}{5}}$	1	$e^{-i\frac{2\pi}{3}}$
$\chi_{14} = \chi_{(3,2,0)}$	$e^{-i\frac{4\pi}{5}}$	-1	1	$\chi_{44} = \chi_{(3,0,2)}$	$e^{-i\frac{4\pi}{5}}$	1	$e^{-i\frac{2\pi}{3}}$
$\chi_{15} = \chi_{(4,2,0)}$	$e^{-i\frac{2\pi}{5}}$	-1	1	$\chi_{45} = \chi_{(4,0,2)}$	$e^{-i\frac{2\pi}{5}}$	1	$e^{-i\frac{2\pi}{3}}$
$\chi_{16} = \chi_{(0,3,0)}$	1	$-i$	1	$\chi_{46} = \chi_{(0,1,2)}$	1	i	$e^{-i\frac{2\pi}{3}}$
$\chi_{17} = \chi_{(1,3,0)}$	$e^{i\frac{2\pi}{5}}$	$-i$	1	$\chi_{47} = \chi_{(1,1,2)}$	$e^{i\frac{2\pi}{5}}$	i	$e^{-i\frac{2\pi}{3}}$
$\chi_{18} = \chi_{(2,3,0)}$	$e^{i\frac{4\pi}{5}}$	$-i$	1	$\chi_{48} = \chi_{(2,1,2)}$	$e^{i\frac{4\pi}{5}}$	i	$e^{-i\frac{2\pi}{3}}$
$\chi_{19} = \chi_{(3,3,0)}$	$e^{-i\frac{4\pi}{5}}$	$-i$	1	$\chi_{49} = \chi_{(3,1,2)}$	$e^{-i\frac{4\pi}{5}}$	i	$e^{-i\frac{2\pi}{3}}$
$\chi_{20} = \chi_{(4,3,0)}$	$e^{-i\frac{2\pi}{5}}$	$-i$	1	$\chi_{50} = \chi_{(4,1,2)}$	$e^{-i\frac{2\pi}{5}}$	i	$e^{-i\frac{2\pi}{3}}$
$\chi_{21} = \chi_{(0,0,1)}$	1	1	$e^{i\frac{2\pi}{3}}$	$\chi_{51} = \chi_{(0,2,2)}$	1	-1	$e^{-i\frac{2\pi}{3}}$
$\chi_{22} = \chi_{(1,0,1)}$	$e^{i\frac{2\pi}{5}}$	1	$e^{i\frac{2\pi}{3}}$	$\chi_{52} = \chi_{(1,2,2)}$	$e^{i\frac{2\pi}{5}}$	-1	$e^{-i\frac{2\pi}{3}}$
$\chi_{23} = \chi_{(2,0,1)}$	$e^{i\frac{4\pi}{5}}$	1	$e^{i\frac{2\pi}{3}}$	$\chi_{53} = \chi_{(2,2,2)}$	$e^{i\frac{4\pi}{5}}$	-1	$e^{-i\frac{2\pi}{3}}$
$\chi_{24} = \chi_{(3,0,1)}$	$e^{-i\frac{4\pi}{5}}$	1	$e^{i\frac{2\pi}{3}}$	$\chi_{54} = \chi_{(3,2,2)}$	$e^{-i\frac{4\pi}{5}}$	-1	$e^{-i\frac{2\pi}{3}}$
$\chi_{25} = \chi_{(4,0,1)}$	$e^{-i\frac{2\pi}{5}}$	1	$e^{i\frac{2\pi}{3}}$	$\chi_{55} = \chi_{(4,2,2)}$	$e^{-i\frac{2\pi}{5}}$	-1	$e^{-i\frac{2\pi}{3}}$
$\chi_{26} = \chi_{(0,1,1)}$	1	i	$e^{i\frac{2\pi}{3}}$	$\chi_{56} = \chi_{(0,3,2)}$	1	$-i$	$e^{-i\frac{2\pi}{3}}$
$\chi_{27} = \chi_{(1,1,1)}$	$e^{i\frac{2\pi}{5}}$	i	$e^{i\frac{2\pi}{3}}$	$\chi_{57} = \chi_{(1,3,2)}$	$e^{i\frac{2\pi}{5}}$	$-i$	$e^{-i\frac{2\pi}{3}}$
$\chi_{28} = \chi_{(2,1,1)}$	$e^{i\frac{4\pi}{5}}$	i	$e^{i\frac{2\pi}{3}}$	$\chi_{58} = \chi_{(2,3,2)}$	$e^{i\frac{4\pi}{5}}$	$-i$	$e^{-i\frac{2\pi}{3}}$
$\chi_{29} = \chi_{(3,1,1)}$	$e^{-i\frac{4\pi}{5}}$	i	$e^{i\frac{2\pi}{3}}$	$\chi_{59} = \chi_{(3,3,2)}$	$e^{-i\frac{4\pi}{5}}$	$-i$	$e^{-i\frac{2\pi}{3}}$
$\chi_{30} = \chi_{(4,1,1)}$	$e^{-i\frac{2\pi}{5}}$	i	$e^{i\frac{2\pi}{3}}$	$\chi_{60} = \chi_{(4,3,2)}$	$e^{-i\frac{2\pi}{5}}$	$-i$	$e^{-i\frac{2\pi}{3}}$

Table 7.3: Character table of $\mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ restricted to $S = \{a_1, a_2, a_3\}$.

The indexes $7, 8, 22, 23, 26, 32$ and 33 are special over S with multiplicity 4. The indexes 27 and 28 are special over S with multiplicity 8. Observe that these conclusions are consistent with Theorem 7.3.5 and Corollary 7.3.6. $\diamond$

Example 7.3.16 Consider $\Gamma = \langle a_1, a_2, a_3 \rangle \cong \mathbf{Z}_{60} \cong \mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ as in Example 7.3.15. In this example we address the question of existence of special indexes by a different way. Let us notice, however, that this only works in some particular cases.

Table 7.5 includes the four vectors associated to a general special index i over $S = \{a_3^{s_3}, a_2^{s_2} a_1^{s_1}, a_3^{s_3} a_2^2\}$ of multiplicity four. Here $s_1 \in \{0, 1, 2, 3, 4\}$, $s_2 \in \{0, 1, 2, 3\}$ and $s_3 \in \{0, 1, 2\}$. Each $\vec{j} = (j_1, j_2, j_3)$ with $j_1 \in \{1, 2, 3, 4\}$, $j_2 \in \{1, 2, 3\}$ and $j_3 \in \{1, 2\}$ corresponds to a special index i with multiplicity

	a_1	a_2	a_3		a_1	a_2	a_3
$\mathrm{Re}(\chi_1)$	1	1	1	$\mathrm{Re}(\chi_{31})$	1	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_2)$	$\cos\left(\frac{2\pi}{5}\right)$	1	1	$\mathrm{Re}(\chi_{32})$	$\cos\left(\frac{2\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_3)$	$\cos\left(\frac{4\pi}{5}\right)$	1	1	$\mathrm{Re}(\chi_{33})$	$\cos\left(\frac{4\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_4)$	$\cos\left(\frac{4\pi}{5}\right)$	1	1	$\mathrm{Re}(\chi_{34})$	$\cos\left(\frac{4\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_5)$	$\cos\left(\frac{2\pi}{5}\right)$	1	1	$\mathrm{Re}(\chi_{35})$	$\cos\left(\frac{2\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_6)$	1	0	1	$\mathrm{Re}(\chi_{36})$	1	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_7)$	$\cos\left(\frac{2\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{37})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_8)$	$\cos\left(\frac{4\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{38})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_9)$	$\cos\left(\frac{4\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{39})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{10})$	$\cos\left(\frac{2\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{40})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{11})$	1	-1	1	$\mathrm{Re}(\chi_{41})$	1	1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{12})$	$\cos\left(\frac{2\pi}{5}\right)$	-1	1	$\mathrm{Re}(\chi_{42})$	$\cos\left(\frac{2\pi}{5}\right)$	1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{13})$	$\cos\left(\frac{4\pi}{5}\right)$	-1	1	$\mathrm{Re}(\chi_{43})$	$\cos\left(\frac{4\pi}{5}\right)$	1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{14})$	$\cos\left(\frac{4\pi}{5}\right)$	-1	1	$\mathrm{Re}(\chi_{44})$	$\cos\left(\frac{4\pi}{5}\right)$	1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{15})$	$\cos\left(\frac{2\pi}{5}\right)$	-1	1	$\mathrm{Re}(\chi_{45})$	$\cos\left(\frac{2\pi}{5}\right)$	1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{16})$	1	0	1	$\mathrm{Re}(\chi_{46})$	1	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{17})$	$\cos\left(\frac{2\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{47})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{18})$	$\cos\left(\frac{4\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{48})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{19})$	$\cos\left(\frac{4\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{49})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{20})$	$\cos\left(\frac{2\pi}{5}\right)$	0	1	$\mathrm{Re}(\chi_{50})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{21})$	1	1	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{51})$	1	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{22})$	$\cos\left(\frac{2\pi}{5}\right)$	1	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{52})$	$\cos\left(\frac{2\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{23})$	$\cos\left(\frac{4\pi}{5}\right)$	1	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{53})$	$\cos\left(\frac{4\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{24})$	$\cos\left(\frac{4\pi}{5}\right)$	1	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{54})$	$\cos\left(\frac{4\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{25})$	$\cos\left(\frac{2\pi}{5}\right)$	1	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{55})$	$\cos\left(\frac{2\pi}{5}\right)$	-1	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{26})$	1	0	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{56})$	1	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{27})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{57})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{28})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{58})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{29})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{59})$	$\cos\left(\frac{4\pi}{5}\right)$	0	$-\frac{1}{2}$
$\mathrm{Re}(\chi_{30})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$	$\mathrm{Re}(\chi_{60})$	$\cos\left(\frac{2\pi}{5}\right)$	0	$-\frac{1}{2}$

Table 7.4: Table of the real parts of the entries of Table 7.3.

	$a_3^{s_3}$	$a_2^{s_2}a_1^{s_1}$	$a_3^{s_3}a_2^2$
$\chi(j_1,j_2,j_3)$	$e^{i\frac{2\pi s_3 j_3}{3}}$	$e^{i2\pi\left(\frac{s_2 j_2}{4}+\frac{s_1 j_1}{5}\right)}$	$-e^{i\frac{2\pi}{3}s_3 j_3}$
$\chi(5-j_1,4-j_2,j_3)$	$e^{i\frac{2\pi s_3 j_3}{3}}$	$e^{-i2\pi\left(\frac{s_2 j_2}{4}+\frac{s_1 j_1}{5}\right)}$	$-e^{i\frac{2\pi}{3}s_3 j_3}$
$\chi(j_1,j_2,3-j_3)$	$e^{-i\frac{2\pi s_3 j_3}{3}}$	$e^{i2\pi\left(\frac{s_2 j_2}{4}+\frac{s_1 j_1}{5}\right)}$	$-e^{-i\frac{2\pi}{3}s_3 j_3}$
$\chi(5-j_1,4-j_2,3-j_3)$	$e^{-i\frac{2\pi s_3 j_3}{3}}$	$e^{-i2\pi\left(\frac{s_2 j_2}{4}+\frac{s_1 j_1}{5}\right)}$	$-e^{-i\frac{2\pi}{3}s_3 j_3}$

Table 7.5: Four irreducible characters of $\mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ restricted to $S = \{a_3^{s_3}, a_2^{s_2}a_1^{s_1}, a_3^{s_3}a_2^2\}$ determining the four vectors associated to a special index over S with multiplicity four. Here $s_1 \in \{0,1,2,3,4\}$, $s_2 \in \{0,1,2,3\}$, $s_3 \in \{0,1,2\}$ and $j_1 \in \{1,2,3,4\}$, $j_2 \in \{1,2,3\}$, $j_3 \in \{1,2\}$.

$\widehat{m} = 4$.

Table 7.6 contains the eight vectors associated to a general special index i over $S = \{a_1^{s_1}, a_2^{s_2}, a_3^{s_3}, a_3^{s_3}a_2^2, a_2^2a_1^{s_1}\}$ of multiplicity eight. Here $s_1 \in \{0,1,2,3,4\}$, $s_2 \in \{0,1,2,3\}$ and $s_3 \in \{0,1,2\}$. In this case, each $\vec{j} = (j_1, j_2, j_3)$ with $j_1 \in \{1,2,3,4\}$, $j_2 \in \{1,2,3\}$ and $j_3 \in \{1,2\}$ corresponds to a special index i with multiplicity $\widehat{m} = 8$.

	$a_1^{s_1}$	$a_2^{s_2}$	$a_3^{s_3}$	$a_3^{s_3}a_2^2$	$a_2^2a_1^{s_1}$
$\chi_{(j_1,j_2,j_3)}$	$e^{i\frac{2\pi s_1 j_1}{5}}$	$e^{i\frac{2\pi s_2 j_2}{4}}$	$e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_1 j_1}{5}}$
$\chi_{(5-j_1,4-j_2,j_3)}$	$e^{-i\frac{2\pi s_1 j_1}{5}}$	$e^{-i\frac{2\pi s_2 j_2}{4}}$	$e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_1 j_1}{5}}$
$\chi_{(j_1,4-j_2,j_3)}$	$e^{i\frac{2\pi s_1 j_1}{5}}$	$e^{-i\frac{2\pi s_2 j_2}{4}}$	$e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_1 j_1}{5}}$
$\chi_{(5-j_1,j_2,j_3)}$	$e^{-i\frac{2\pi s_1 j_1}{5}}$	$e^{i\frac{2\pi s_2 j_2}{4}}$	$e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_1 j_1}{5}}$
$\chi_{(5-j_1,4-j_2,3-j_3)}$	$e^{-i\frac{2\pi s_1 j_1}{5}}$	$e^{-i\frac{2\pi s_2 j_2}{4}}$	$e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_1 j_1}{5}}$
$\chi_{(j_1,j_2,3-j_3)}$	$e^{i\frac{2\pi s_1 j_1}{5}}$	$e^{i\frac{2\pi s_2 j_2}{4}}$	$e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_1 j_1}{5}}$
$\chi_{(j_1,4-j_2,3-j_3)}$	$e^{i\frac{2\pi s_1 j_1}{5}}$	$e^{-i\frac{2\pi s_2 j_2}{4}}$	$e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{i\frac{2\pi s_1 j_1}{5}}$
$\chi_{(5-j_1,j_2,3-j_3)}$	$e^{-i\frac{2\pi s_1 j_1}{5}}$	$e^{i\frac{2\pi s_2 j_2}{4}}$	$e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_3 j_3}{3}}$	$-e^{-i\frac{2\pi s_1 j_1}{5}}$

Table 7.6: Eight irreducible characters of $\mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ restricted to $S = \{a_1^{s_1}, a_2^{s_2}, a_3^{s_3}, a_3^{s_3}a_2^2, a_2^2a_1^{s_1}\}$ determining the eight vectors associated to a special index over S with multiplicity eight. Here $s_1 \in \{0,1,2,3,4\}$, $s_2 \in \{0,1,2,3\}$, $s_3 \in \{0,1,2\}$ and $j_1 \in \{1,2,3,4\}$, $j_2 \in \{1,2,3\}$, $j_3 \in \{1,2\}$.

In Table 7.7 are represented the vectors associated to a general special index i over $S = \{a_3^{s_3}a_2^{s_2}, a_1^{s_1}, a_2^2a_1^{s_1}\}$ where $s_1 \in \{0,1,2,3,4\}$, $s_2 \in \{0,1,2,3\}$ and $s_3 \in \{0,1,2\}$. Each $\vec{j} = (j_1, j_2, j_3)$ with $j_1 \in \{1,2,3,4\}$, $j_2 \in \{1,2,3\}$ and $j_3 \in \{1,2\}$ corresponds to a special index i with multiplicity $\widehat{m} = 4$.

	$a_3^{s_3}a_2^{s_2}$	$a_1^{s_1}$	$a_2^2a_1^{s_1}$
$\chi_{(j_1,j_2,j_3)}$	$e^{i2\pi\left(\frac{s_3j_3}{3}+\frac{s_2j_2}{4}\right)}$	$e^{i\frac{2\pi s_1 j_1}{5}}$	$-e^{i\frac{2\pi}{5}s_1j_1}$
$\chi_{(5-j_1,4-j_2,3-j_3)}$	$e^{-i2\pi\left(\frac{s_3j_3}{3}+\frac{s_2j_2}{4}\right)}$	$e^{-i\frac{2\pi s_1 j_1}{5}}$	$-e^{-i\frac{2\pi}{5}s_1j_1}$
$\chi_{(j_1,4-j_2,3-j_3)}$	$e^{-i2\pi\left(\frac{s_3j_3}{3}+\frac{s_2j_2}{4}\right)}$	$e^{i\frac{2\pi s_1 j_1}{5}}$	$-e^{i\frac{2\pi}{5}s_1j_1}$
$\chi_{(5-j_1,j_2,j_3)}$	$e^{i2\pi\left(\frac{s_3j_3}{3}+\frac{s_2j_2}{4}\right)}$	$e^{i\frac{-2\pi s_1 j_1}{5}}$	$-e^{-i\frac{2\pi}{5}s_1j_1}$

Table 7.7: Four irreducible characters of $\mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ restricted to $S = \{a_3^{s_3}a_2^{s_2}, a_1^{s_1}, a_2^2a_1^{s_1}\}$ determining the four vectors associated to a special index over S with multiplicity four. Here $s_1 \in \{0,1,2,3,4\}$, $s_2 \in \{0,1,2,3\}$, $s_3 \in \{0,1,2\}$ and $j_1 \in \{1,2,3,4\}$, $j_2 \in \{1,2,3\}$, $j_3 \in \{1,2\}$.

◇

7.4 Remarks about Generators of Finite Abelian Groups

Consider an abelian group $\Gamma \subseteq \mathbf{S}_n$ of order n that acts transitively (and faithfully) on the set $\{1,\ldots,n\}$. By Theorem 7.3.1 we can consider

$$\Gamma \cong \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_r} \tag{7.20}$$

where $\mathbf{Z}_{n_k}$ is a cyclic group of order $n_k \geq 2$, and $n_1 \cdots n_r = n$ with $n_1 \geq n_2 \geq \cdots \geq n_r$ where the n_k are powers of primes, not necessarily distinct. Moreover $\mathbf{Z}_{n_k} \cong < a_k >$ for $a_k \in \mathbf{S}_n$. It is well known that a_k is not unique and in principle there are many subgroups of $\mathbf{S}_n$ isomorphic to Γ that act transitively on the set $\{1,\ldots,n\}$. All of them determine the regular representation of Γ on $\mathbf{C}^n$ (or $\mathbf{R}^n$). Thus there is no loss of generality in fixing one of these groups and that is what we do in the next algorithm.

Definition 7.4.1 Consider $\rho = (a_{1,1}\ \cdots\ a_{1,m_1})\cdots(a_{k,1}\ \cdots\ a_{k,m_k}) \in \mathbf{S}_n$ with $m_i > 1$. For $y \in \{1,\ldots,n-1\}$ such that $a_{j,l} + y \leq n$ for all $a_{j,l}$ we define

$$(\rho \diamond y) = (a_{1,1}+y\ \cdots\ a_{1,m_1}+y)\cdots(a_{k,1}+y\ \cdots\ a_{k,m_k}+y)\,.$$

Algorithm 7.4.2 Given an abelian group Γ of order n as in (7.20) this algorithm finds a set of generators $G = \{a_1,\ldots,a_r\}$ of a subgroup $\widetilde{\Gamma} \subseteq \mathbf{S}_n$ acting transitively (and faithfully) on the set $\{1,\ldots,n\}$ and such that $\widetilde{\Gamma} \cong \Gamma$.

1. [Initialize] Consider Γ has is (7.20). Set

   $$a_1 = (1\ 2\ \cdots\ n_1)\,(n_1+1\ \ n_1+2\ \cdots\ 2n_1)\cdots(n-n_1+1\ \ n-n_1+2\ \cdots\ n)$$

   and for $\widetilde{r} = 2$ until r set
   $c_{\widetilde{r}} = n_1 \cdots n_{\widetilde{r}-1}$,
   $a_{\widetilde{r}} = (1\ \ 1+c_{\widetilde{r}}\ \ 1+2c_{\widetilde{r}}\ \cdots\ 1+(n_{\widetilde{r}}-1)\,c_{\widetilde{r}})\,.$
   Set $k = 2$ and $l = 1$.

2. [Compute set G] If $r = 1$ go to step 3. If $r \geq 2$,

   If $l = 1$ then set

   $$a_k = a_k\,(a_k \diamond 1)\,(a_k \diamond 2)\cdots(a_k \diamond (n_1 \cdots n_{k-1} - 1))\,,$$

   $l = l + 1$ and go to the beginning of step 2.

If $2 \le l+k-2 \le r-1$ then set

$$\begin{array}{rl} a_k = & a_k\left(a_k \diamond (n_1 \cdots n_{l+k-2})\right)\left(a_k \diamond (2n_1 \cdots n_{l+k-2})\right) \\ & \cdots \left(a_k \diamond ((n_{l+k-1}-1)\, n_1 \cdots n_{l+k-2})\right), \end{array}$$

$l = l+1$ and go to the beginning of step 2.

If $l+k-2 = r$ and $k < r$ then set $k = k+1$, $l = 1$ and go to the beginning of step 2.

If $l+k-2 = r$ and $k = r$ then set $G = \{a_1, \ldots, a_r\}$ and go to step 3.

3. [Output] The set G contains for each $k = 1, \ldots, r$ an element $a_k \in \mathbf{S}_n$ that is unique and such that $\mathbf{Z}_{n_k} \cong < a_k >$. Output $G = \{a_1, \ldots, a_r\}$ and terminate the algorithm.

$\diamond$

We call G the *special set of generators of* $\widetilde{\Gamma}$. The elements of G are the *special generators of* $\widetilde{\Gamma}$.

Example 7.4.3 Consider the abelian group $\mathbf{Z}_3 \times \mathbf{Z}_3$ of order 9. We have then $r = 2$ and $n_1 = n_2 = 3$. Applying Algorithm 7.4.2 we obtain the group $\Gamma \subseteq \mathbf{S}_9$ with the special set of generators $G = \{a_1, a_2\}$ where $a_1 = (123)(456)(789)$ and $a_2 = (147)(258)(369)$. This group acts transitively on the set $\{1, \ldots, 9\}$. $\diamond$

7.5 Codimension one Eigenvalue Movements

Suppose $\mathcal{G}$ is a coupled cell network with n cells. Assume that the network is symmetric with respect to a transitive and faithful permutation action of an abelian group Γ on the set of cells $\{1, \ldots, n\}$. Recall from section 7.3 that if

$$\Gamma \cong \mathbf{Z}_{n_1} \times \cdots \times \mathbf{Z}_{n_r}$$

where $n_1 \cdots n_r = n$, $n_1 \ge \cdots \ge n_r \ge 2$, we consider a set $\{a_1, \ldots, a_r\}$ of generators of Γ such that $< a_k > \cong \mathbf{Z}_{n_k}$ and enumerate the elements of Γ as:

$$\Gamma = \{\gamma_{\vec{s}} = a_1^{s_1} a_2^{s_2} \cdots a_r^{s_r},\ s_k \in \{0, \ldots, n_k - 1\}\} \tag{7.21}$$

where $\vec{s} = (s_1, \ldots, s_r)$ and $a_k^0 = e$.

Let $V = \mathbf{R}^n$ and consider a smooth 1-parameter family $f : V \times \mathbf{R} \to V$ of $\mathcal{G}$-admissible vector fields on V. Here we are assuming that the cell phase-space of all cells is $\mathbf{R}$. The symmetry Γ of the network $\mathcal{G}$ implies that f commutes with Γ. Moreover, we can assume that, with respect to the cell coordinates $x_1, \ldots, x_n$, the action of Γ corresponds to the linear homomorphism from Γ to the group $\mathrm{GL}(V)$ defined by:

$$T(\gamma)(x_1, \ldots, x_n) = \left(x_{\gamma^{-1}(1)}, \ldots, x_{\gamma^{-1}(n)}\right), \gamma \in \Gamma, (x_1, \ldots, x_n) \in V. \quad (7.22)$$

Assume that the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \lambda) \quad (7.23)$$

has a fully symmetric equilibrium x_0. We study now the codimension one eigenvalue movements across the imaginary axis of $(df)_{(x_0,\lambda)}$.

As mentioned before, to facilitate the analysis we first consider the complexification $\mathbf{C}^n$ of $\mathbf{R}^n$ and later deduce the consequences implied by the fact that the phase space is real. Assume then that the phase space of $\mathcal{G}$ is $V = \mathbf{C}^n$. Observe that we can make Γ act on $\mathbf{C}^n$ by extending the action on $\mathbf{R}^n$ via linearity over $\mathbf{C}$: let $z \in \mathbf{C}^n$ and write it as $z = x + iy$ where $x, y \in \mathbf{R}^n$. Define a Γ-action on $\mathbf{C}^n$ by

$$\gamma \cdot z = \gamma \cdot x + i\gamma \cdot y. \quad (7.24)$$

Each transformation

$$\begin{array}{rl} T(\gamma): & \mathbf{C}^n \to \mathbf{C}^n \\ & z \ \mapsto \gamma \cdot z \end{array}$$

is $\mathbf{C}$-linear.

The isotypic decomposition of V under the action of Γ is

$$V_1 \oplus \cdots \oplus V_n \quad (7.25)$$

where $V_1, \ldots, V_n$ form a set of Γ-nonisomorphic irreducible spaces. By Remark 7.2.2, the number n of cells equals the order of Γ. Moreover, the representation of Γ on V corresponds to the regular representation. We consequently may identify cells uniquely with group elements once we have identified for example cell 1 with the identity element in Γ. In particular, we may label each cell i by a unique element $\gamma_i \in \Gamma$ such that $\gamma_i(1) = i$, so that $\gamma_1 = e$, etc. For $i = 1, \ldots, n$, let $\vec{s_i} = (s_{i_1}, \ldots, s_{i_r})$ be such that

$$a_1^{s_{i_1}} \cdots a_r^{s_{i_r}}(1) = i$$

where $s_{j_k} \in \{0, \ldots, n_k-1\}$. Thus $\gamma_i \equiv \gamma_{\vec{s}_i}$. Denote by χ_i the linear irreducible character of Γ given by:

$$\chi_i \left(a_1^{p_1} \cdots a_r^{p_r}\right) = \omega_1^{p_1 s_{i_1}} \cdots \omega_r^{p_r s_{i_r}} \tag{7.26}$$

where $p_k \in \{0, \ldots, n_k - 1\}$ and $\omega_k = e^{i2\pi/n_k}$ for $k = 1 \ldots, r$. Thus, $\chi_i(a_k) = \omega_k^{s_{i_k}}$ for $k = 1, \ldots, r$. We assume then that the irreducible spaces V_i of the isotypic decomposition (7.25) are ordered so that V_i has character type χ_i.

The linearization

$$M = (\mathrm{d}f)_{(x_0,\lambda)} \tag{7.27}$$

of f at (x_0, λ) is Γ-equivariant and hence M leaves each isotypic component V_i invariant. We denote by M^i the restriction of M to V_i. Then,

$$M = [M_{ij}],\ 1 \leq i, j \leq n$$

represents de matrix $(df)_{(x_0,\lambda)}$ with respect to the canonical basis of $\mathbf{C}^n$ whereas

$$\mathrm{diag}(M^1, \ldots, M^n)$$

represents $(df)_{(x_0,\lambda)}$ written according the isotypic decomposition of $\mathbf{C}^n$ described above.

Since Γ is abelian, each irreducible V_i has complex dimension one. In view of the complexification, there are two types of irreducible representations.
(i) Either χ_i is real and V_i corresponds to the complexification of an irreducible real space W_i with character χ_i and the real commuting matrices on W_i are the real scalar multiples of the identity on W_i. In this case, W_i is called Γ-*absolutely irreducible* and V_i is said to be of *real type*.
(ii) The other case is when χ_i is complex. Then the conjugate $\overline{\chi_i}$ is also an irreducible character distinct from χ_i, associated say with $\overline{V_i}$. More precisely, suppose $V_i = \mathbf{C}\{u + iv\}$ where $u, v \in \mathbf{R}^n$. We have that

$$T\left(\gamma\right)\left(u + iv\right) = \gamma \cdot u + i\gamma \cdot v = \chi_i(\gamma)(u + iv),\ \forall \gamma \in \Gamma. \tag{7.28}$$

Then

$$\begin{aligned} \overline{T\left(\gamma\right)\left(u + iv\right)} &= \overline{\gamma \cdot u + i\gamma \cdot v} \\ &= \gamma \cdot u - i\gamma \cdot v \quad (\text{since } \gamma \cdot u \in \mathbf{R}^n \text{ and } \gamma \cdot v \in \mathbf{R}^n) \\ &= T\left(\gamma\right)\left(u - iv\right) \quad (\text{by } (7.24)). \end{aligned}$$

Also $\overline{T\left(\gamma\right)\left(u + iv\right)} = \overline{\chi_i(\gamma)}(u - iv)$ by (7.28). We conclude then that

$$T\left(\gamma\right)\left(u - iv\right) = \overline{\chi_i(\gamma)}(u - iv)$$

for all $\gamma \in \Gamma$ and $\overline{V_i} \equiv \mathbf{C}\{u - iv\}$ has character $\overline{\chi_i}$. Moreover, $V_i \oplus \overline{V_i}$ is a real Γ-irreducible with character $\chi_i + \overline{\chi_i}$ and the vector space of the real commuting matrices defined on $V_i \oplus \overline{V_i}$ is isomorphic to $\mathbf{C}$. In this case V_i is said to be of *complex type* and $V_i \oplus \overline{V_i}$ is Γ-*simple*. For details, see the next lemma. Recall that a space W is called Γ-*simple* if W is the direct sum of two isomorphic absolutely irreducible spaces, or if it is irreducible of complex type.

Lemma 7.5.1 *Let V_i be a Γ-irreducible subspace of $\mathbf{C}^n$ of complex type with character χ_j and suppose that $V_j = \mathbf{C}\{u+iv\}$ where $u, v \in \mathbf{R}^n$. Then $\mathbf{R}\{u, v\}$ is a Γ-irreducible subspace of $\mathbf{R}^n$ with character $\chi_j + \overline{\chi_j}$. Moreover,*

$$T(\gamma)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (\chi_j)_R(\gamma) & -(\chi_j)_I(\gamma) \\ (\chi_j)_I(\gamma) & (\chi_j)_R(\gamma) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Proof: Let us write $\chi_j \in \mathbf{C}$ as $\chi_j = (\chi_j)_R + i(\chi_j)_I$ with $(\chi_j)_R, (\chi_j)_I \in \mathbf{R}$. We know that

$$\begin{aligned} T(\gamma)(u+iv) &= \chi_j(\gamma)(u+iv) \\ &= \big((\chi_j)_R(\gamma) + i(\chi_j)_I(\gamma)\big)(u+iv) \\ &= \big[(\chi_j)_R(\gamma)u - (\chi_j)_I(\gamma)v\big] + \big[(\chi_j)_R(\gamma)v + (\chi_j)_I(\gamma)u\big]i. \end{aligned}$$

On the other hand, by (7.24) we have $T(\gamma)(u+iv) = \gamma \cdot u + i\gamma \cdot v$. Then

$$\begin{aligned} \gamma \cdot u &= (\chi_j)_R(\gamma)u - (\chi_j)_I(\gamma)v, \\ \gamma \cdot v &= (\chi_j)_R(\gamma)v + (\chi_j)_I(\gamma)u. \end{aligned}$$

That is, $W = \mathbf{R}\{u, v\} \subseteq \mathbf{R}^n$ is Γ-invariant: we have

$$T(\gamma)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (\chi_j)_R(\gamma) & -(\chi_j)_I(\gamma) \\ (\chi_j)_I(\gamma) & (\chi_j)_R(\gamma) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

and W has character $2(\chi_j)_R = \chi_j + \overline{\chi_j}$. Moreover, $W = \mathbf{R}\{u, v\}$ is irreducible. To see that suppose that W is invariant but not irreducible. Then $W = W_1 \oplus W_2$ where W_1 and W_2 are one-dimensional irreducible subspaces. In this case, Proposition 23.6 of James and Liebeck [24] implies that W_1 or W_2 has character χ_j. This is an absurd because in this case χ_j would have to be real and we are assuming that χ_j is of complex type. $\square$

Example 7.5.2 Consider the abelian group $\Gamma = \langle(123)\rangle \cong \mathbf{Z}_3$ and the action of Γ on $\mathbf{R}^3$ given by:

$$(123) \cdot (x_1, x_2, x_3) = (x_3, x_1, x_2). \tag{7.29}$$

Consider the complexification $\mathbf{C}^3$ of $\mathbf{R}^3$ and the action of Γ on $\mathbf{C}^3$ as described above. The isotypic decomposition of $\mathbf{C}^3$ under the action of Γ is

$$V_1 \oplus V_2 \oplus V_3$$

where each irreducible V_j has character type determined by

$$\chi_j\left((123)\right) = e^{i\frac{2\pi}{3}(j-1)}, \quad j = 1, 2, 3.$$

Then V_1 is an irreducible space of real type (where the group acts trivially) and V_2 and V_3 are irreducible spaces of complex type. Moreover, because $\overline{\chi_2} = \chi_3$, we have $V_3 = \overline{V_2}$ and $V_2 \oplus \overline{V_2}$ is a real Γ-irreducible with character $\chi_2 + \overline{\chi_2}$.

The isotypic decomposition of $\mathbf{R}^3$ under the action (7.29) is then

$$W_1 \oplus W_2$$

where W_1 is the trivial representation of dimension one and $W_1 = V_2 \oplus \overline{V_2}$ is a real Γ-irreducible. $\diamond$

In the case of general equivariant linear vector fields, it is well known that codimension one eigenvalue movements through the imaginary axis can be characterized by the following conditions (Proposition 3.2.4 and Golubitsky *et al.* [21] Proposition XIII 3.2): a one-parameter family $M(\lambda)$, where $M(0)$ satisfies:

(a) The critical eigenspace E^c of $M(0)$ is Γ-simple (in case eigenvalues intersect $i\mathbf{R}$ away from 0) or Γ-absolutely irreducible (in the case eigenvalues intersect at 0).

(b) The eigenvalues $\mu(\lambda)$ of $M(\lambda)$ such that $\mathrm{Re}(\mu(0)) = 0$ satisfy:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{Re}(\mu(\lambda))|_{\lambda=0} \neq 0.$$

We now consider how the codimension one movement of eigenvalues through the imaginary axis is affected by the network structure.

Recall that we say that a coupled cell network is *connected* if there exists a path (formed by concatenation of edges, not necessarily uni-directional) connecting i and j (for all $i \neq j$). We say that the network is *disconnected* otherwise. Note that if Γ is transitive on $\{1, \ldots, n\}$ then the network is connected if and only if there are directed paths from any cell to any other cell. We may summarize the connectivity information of a n-cell network in a $n \times n$ connectivity matrix C, where

$$C_{i,j} = \begin{cases} 1, & \text{if there is a connection from } j \text{ to } i, \\ 0, & \text{otherwise.} \end{cases}$$

As the network is Γ-equivariant, we have $C_{\gamma(i),\gamma(j)} = C_{i,j}$ for all $\gamma \in \Gamma$. We note that because of the transitivity of the action of Γ, the connectivity matrix C is fully determined by its first row (or first column). With the identification of cell i by the unique element $\gamma_i \in \Gamma$ such that $\gamma_i(1) = i$, we have

$$C_{1,\gamma(1)} \equiv C_{e,\gamma}.$$

There exists a group theoretic description of connectedness for a network with transitive symmetry group Γ.

Lemma 7.5.3 ([7]) *A network with transitive and abelian symmetry group Γ is connected if and only if $\Gamma =< S >$, where*

$$S = \{\gamma \in \Gamma : C_{e,\gamma} = 1\}. \tag{7.30}$$

Proof: See Dias and Lamb [7] Lemma 4.1. □

Lemma 7.5.4 ([7]) *Let S denote the set of group elements corresponding to the present couplings, so that if $\gamma \in S$ then there is a coupling from cell γ to cell e. Then*

$$\left(M^1, \ldots, M^n\right) = \sum_{\gamma \in S} c(\gamma) \left(\overline{\chi_1(\gamma)}, \ldots, \overline{\chi_n(\gamma)}\right) \tag{7.31}$$

where $c : S \to \mathbf{C}$ is arbitrary.

Proof: See Dias and Lamb [7] Lemma 4.2 for details. □

We make the following observation. Since all the irreducible characters χ_i are linear, we have that

$$T(\gamma)v_i = \chi_i(\gamma^{-1})v_i = \overline{\chi_i(\gamma)}v_i$$

for $\gamma \in S$ and $v_i \in V_i$, for $i = 1, \ldots, n$. It follows from equation (7.31) that

$$M = \sum_{\gamma \in S} c(\gamma) M_\gamma$$

where each M_γ denotes the $n \times n$ matrix associated with $T(\gamma)$ with respect to the canonical basis of $\mathbf{C}^n$.

We incorporate now the fact that in the context of coupled cell networks we work with real linear maps, rather than with their complexification. Equation (7.31) gives the eigenvalues $M^1, \ldots, M^n$ of M as function of the irreducible characters of Γ evaluated on S. We decomplexify now the answer obtained in (7.31). Thus we return to the system (7.23) with $V = \mathbf{R}^n$ and take $M = (df)_{(x_0,\lambda)} \in M_{n\times n}(\mathbf{R})$.

Let V_j be an isotypic component for the action of Γ on $\mathbf{C}^n$. Then:

- If V_j is of real type, then M^j should be interpreted as a real number.
- If V_j is of complex type, then there is another isotypic component V_k so that $\chi_k = \overline{\chi_j}$ and $M^k = \overline{M^j}$. Also we write $V_k = \overline{V_j}$. The decomplexification acts on $\widehat{V_j}$ where $\widehat{V_j} = V_j \oplus V_k \cong \mathbf{R}^2$, as

$$\widehat{M^j} = \begin{pmatrix} M_R^j & -M_I^j \\ M_I^j & M_R^j \end{pmatrix} \tag{7.32}$$

  where

$$M^j = M_R^j + iM_I^j.$$

(Recall Lemma 7.5.1.) For each pair V_j and $V_k = \overline{V_j}$ we consider $\widehat{M}^{\min\{j,k\}}$.

It is then possible to put M into block diagonal form with 2×2 and 1×1 blocks. Applying (7.31) it follows that

$$M = \sum_{\gamma\in S} c(\gamma) M_\gamma \tag{7.33}$$

where now $c : S \to \mathbf{R}$. Moreover, if $V_j \cong \mathbf{R}$ is of real type with character χ_j then

$$M^j \equiv M|_{V_j} \approx \sum_{\gamma\in S} c(\gamma)\chi_j(\gamma);$$

if $\widehat{V_j} = V_j \oplus \overline{V_j} \cong \mathbf{R}^2$ where V_j is of complex type then by Lemma 7.5.1

$$\widehat{M^j} \equiv M|_{\widehat{V_j}} \approx \sum_{\gamma\in S} c(\gamma) \begin{pmatrix} (\chi_j)_R(\gamma) & -(\chi_j)_I(\gamma) \\ (\chi_j)_I(\gamma) & (\chi_j)_R(\gamma) \end{pmatrix}.$$

Thus in the first case the (real) eigenvalue of $M|_{V_j}$ is

$$\sum_{\gamma\in S} c(\gamma)\chi_j(\gamma);$$

In the second case, the eigenvalues of $M|_{\widehat{V}_j}$ are

$$\sum_{\gamma \in S} c(\gamma)\,(\chi_j)_R(\gamma) \pm i\left(\sum_{\gamma \in S} c(\gamma)\,(\chi_j)_I(\gamma)\right).$$

It follows that if for j, l we have $\widehat{V}_j \neq \widehat{V}_l$ and

$$(\chi_j)_R(\gamma) = (\chi_l)_R(\gamma)$$

for all $\gamma \in S$, then

$$\sum_{\gamma \in S} c(\gamma)\,(\chi_j)_R(\gamma) = \sum_{\gamma \in S} c(\gamma)\,(\chi_l)_R(\gamma)$$

and so $M|_{\widehat{V}_j}$ and $M|_{\widehat{V}_l}$ have (complex) eigenvalues with equal real part.

Example 7.5.5 Recalling Example 7.1.1, consider the graph $\mathcal{G}_1$ of 9 cells represented in Figure 7.1 (left). In this case $\mathcal{G}_1$ has symmetry

$$\Gamma = \langle a_1, a_2 \rangle \cong \mathbf{Z}_3 \times \mathbf{Z}_3$$

where $a_1 = (123)(456)(789)$ and $a_2 = (147)(258)(369)$. Consider the action of Γ on $\mathbf{R}^9$ as in (7.22). Complexify $\mathbf{R}^9$ to get $\mathbf{C}^9$ and extend the action (7.22) of Γ to $\mathbf{C}^9$ as in (7.24). We have then the regular representation of Γ and the isotypic decomposition of $\mathbf{C}^9$ is

$$V_1 \oplus V_2 \oplus \cdots \oplus V_9 \tag{7.34}$$

where each V_j is Γ-irreducible with character χ_j. Recall Table 7.1. Because $\chi_3 = \overline{\chi_2}$, $\chi_7 = \overline{\chi_4}$, $\chi_8 = \overline{\chi_6}$ and $\chi_9 = \overline{\chi_5}$ then $V_3 = \overline{V_2}$, $V_7 = \overline{V_4}$, $V_8 = \overline{V_6}$ and $V_9 = \overline{V_5}$. Then, for $i = 2, 4, 5, 6$, $\widehat{V_i} = V_i \oplus \overline{V_i}$ is a real Γ-irreducible (with real dimension 2) with character $\chi_i + \overline{\chi_i}$. The isotypic decomposition of $\mathbf{R}^9$ is then

$$V_1 \oplus \widehat{V}_2 \oplus \widehat{V}_4 \oplus \widehat{V}_5 \oplus \widehat{V}_6. \tag{7.35}$$

Observe that there are directed edges from cells 3 and 7 to cell 1. Let S be as in (7.30). In the present case, because $e(1) = 1$, $a_1^2(1) = 3$ and $a_2^2(1) = 7$, we have $S = \{e, a_1^2, a_2^2\}$. Recall from Example 7.1.1 the 1-parameter $\mathcal{G}$-admissible linear map L and the corresponding eigenvalues. By equation (7.31) we obtain the same eigenvalues $M^1, \ldots, M^9$ of L where

$$\begin{aligned}
M^1 &= a + c\overline{\chi_1(a_1^2)} + b\overline{\chi_1(a_2^2)} = a + c + b,\\
M^2 &= a + c\overline{\chi_2(a_1^2)} + b\overline{\chi_2(a_2^2)} = a - \tfrac{1}{2}c + b + i\tfrac{\sqrt{3}}{2}c,\\
M^4 &= a + c\overline{\chi_4(a_1^2)} + b\overline{\chi_4(a_2^2)} = a + c - \tfrac{1}{2}b + i\tfrac{\sqrt{3}}{2}b,\\
M^5 &= a + c\overline{\chi_5(a_1^2)} + b\overline{\chi_5(a_2^2)} = a - \tfrac{1}{2}c - \tfrac{1}{2}b + \tfrac{\sqrt{3}}{2}(b+c)i,\\
M^6 &= a + c\overline{\chi_6(a_1^2)} + b\overline{\chi_6(a_2^2)} = a - \tfrac{1}{2}c - \tfrac{1}{2}b + \tfrac{\sqrt{3}}{2}(b-c)i
\end{aligned}$$

and $M^3 = \overline{M^2}$, $M^7 = \overline{M^4}$, $M^8 = \overline{M^6}$, $M^9 = \overline{M^5}$. Here $a, b, c \in \mathbf{R}$. Then

$$\begin{aligned}
\widehat{M^2} &= aI_2 + cA + bI_2 \\
\widehat{M^4} &= aI_2 + cI_2 + bA \\
\widehat{M^5} &= aI_2 + cA + bA \\
\widehat{M^6} &= aI_2 + cA^T + bA
\end{aligned}$$

and

$$L \approx \operatorname{diag}\left(a + c + b, \widehat{M^2}, \widehat{M^4}, \widehat{M^5}, \widehat{M^6}\right)$$

where I_2 represents the 2×2 identity matrix and

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Moreover, because $(\chi_5)_R(\gamma) = (\chi_6)_R(\gamma)$ for all $\gamma \in S$, the eigenvalues M^5, $\overline{M^5}$ and M^6, $\overline{M^6}$ of the blocks $\widehat{M_5}$ and $\widehat{M_6}$ respectively have equal real part. $\diamond$

In general equivariant systems with abelian symmetry, we have the following result of Golubitsky *et al.* [21] Proposition XIII 3.2 and Proposition 3.2.4. If in a one-parameter family of real linear equivariant vector fields, an eigenvalue crosses the imaginary axis then, typically, one of the following scenarios applies:

(a) The eigenvalues are restricted to the real axis, crossing the imaginary axis at zero, and the associated eigenvectors lie in one absolutely irreducible representation of Γ. The number of eigenvalues simultaneously crossing the imaginary axis is equal to the dimension of the irreducible representation, and they all have the same value.

(b) The eigenvalues are not restricted to lie on the real axis, crossing the imaginary axis at $\pm i\omega$ ($\omega \neq 0$). The associated eigenvectors lie in the direct sum of two isomorphic absolutely irreducible representations of Γ. The number of eigenvalues simultaneously crossing the imaginary axis is equal to twice the (complex) dimension of the irreducible representation, half taking one same value and the remaining half its complex conjugate.

(c) The eigenvalues are not restricted to lie on the real axis, crossing the imaginary axis at $\pm i\omega$ ($\omega \neq 0$). The associated eigenvectors lie in one irreducible representation of Γ of complex type. The number of eigenvalues simultaneously crossing the imaginary axis is equal to twice the (complex) dimension of the irreducible representation, half assuming one value and the remaining half its complex conjugate.

The eigenvalue movement in (a) is associated with steady-state bifurcation, and the remaining cases (b) and (c) with Hopf bifurcation. We note however that when the abelian group representation is regular it cannot have two isomorphic real representations and so case (b) does not occur in this case.

Consider the system (7.23). Dias and Lamb [7] prove that the codimension one eigenvalue movements across the imaginary axis of $(df)_{(x_0,\lambda)}$ are independent of the network structure and are identical to the corresponding eigenvalue movements in general equivariant vector fields in the case (a). In the case (c) the same is valid when the dimension of the phase space of each cell of the network is greater than one.

We prove in the rest of this section that, when the phase space of the cells of the network is one-dimensional, the codimension one eigenvalue movements across the imaginary axis of $(df)_{(x_0,\lambda)}$ depends on the network structure. In fact, for general abelian symmetry groups Γ, $(df)_{(x_0,\lambda)}$ can have two or more distinct and non conjugated purely imaginary eigenvalues crossing the imaginary axis when $\lambda = 0$. We give sufficient conditions for that to occur using our results of section 7.3.

Definition 7.5.6 Consider the system (7.23) where f is a one-parameter family of $\mathcal{G}$-admissible vector fields and $L_{(x_0,\lambda)} = (df)_{(x_0,\lambda)}$ where x_0 is a fully symmetric equilibrium of (7.23) for all $\lambda \in \mathbf{R}$. We say that $L_{(x_0,\lambda)}$ *is special* if $L_{(x_0,\lambda)}$ has generically two or more pairs of eigenvalues crossing the imaginary axis when $\lambda = 0$.

In the next theorem we consider Γ an abelian group decomposed as in (7.7) and the (complex) irreducible character of Γ enumerated as in (7.11)–(7.13).

Theorem 7.5.7 *Let $\mathcal{G}$ be a symmetric coupled cell network with respect to a transitive and faithful permutation action of an abelian group Γ on the set of cells $\{1, \ldots, n\}$ and fix a phase space $V = \mathbf{R}^n$. Consider the system (7.23) where $f : V \times \mathbf{R} \to V$ is a smooth 1-parameter family of $\mathcal{G}$-admissible vector fields on V and $L_{(x_0,\lambda)} = (\mathrm{d}f)_{(x_0,\lambda)}$ where x_0 is Γ-symmetric. Let $M^1, \ldots, M^n$ be the eigenvalues of $L_{(x_0,\lambda)}$ given by (7.31). Let S be as in (7.30) and assume $< S >= \Gamma$. Suppose S is special (recall Definition 7.3.3) and that $i \in \{1, \ldots, n\}$ is a special character index with multiplicity $\widehat{m} \geq 4$ over S. If $M^i(0)$ is purely imaginary then $L_{(x_0,\lambda)}$ is special.*

Proof: If i is special over $S \subseteq \Gamma$ and $i_1, \ldots, i_{\widehat{m}}$ are the correspondent associated indexes (recall Definition 7.3.3) then

$$(\chi_{i_1})_R(\gamma) = \cdots = (\chi_{i_{\widehat{m}}})_R(\gamma) = (\chi_i)_R(\gamma) \tag{7.36}$$

for all $\gamma \in S$. Observe that the characters $\chi_{i_1}, \ldots, \chi_{i_{\widehat{m}}}$ (and χ_i) must be of complex type. If one of these linear characters was of real type, then the others would have also to be of real type since $< S >= \Gamma$ and (7.36) holds for all $\gamma \in S$. Again, from (7.36) we would have $\chi_{i_1}(\gamma) = (\chi_{i_1})_R(\gamma) = \cdots = \chi_{i_{\widehat{m}}}(\gamma) = (\chi_{i_{\widehat{m}}})_R(\gamma)$ for all $\gamma \in S$. But again as $< S >= \Gamma$, it would follow that $\chi_{i_1} = \chi_{i_2} = \cdots = \chi_{i_{\widehat{m}}}$, a contradiction since $\chi_{i_1}, \ldots, \chi_{i_{\widehat{m}}}$ are distinct irreducible characters. Now, by (7.31) and (7.33) we have

$$M^j = \sum_{\gamma \in S} c(\gamma)\overline{\chi_j(\gamma)}$$

where $c : S \to \mathbf{R}$. Thus

$$M_R^{i_1} = M_R^{i_2} = \cdots = M_R^{i_{\widehat{m}}} = M_R^i = \sum_{\gamma \in S} c(\gamma)(\chi_i)_R(\gamma).$$

Half of the eigenvalues $M^{i_1}, \ldots, M^{i_{\widehat{m}}}$ of $L_{(x_0,\lambda)}$ assume one value and the remaining half are its complex conjugate. Subsequently, if $M_R^i(0) = 0$, the spectrum of $L_{(x_0,0)}$ includes these $\widehat{m}/2$ pairs of critical eigenvalues. Then generically $L_{(x_0,\lambda)}$ is special over S.

$\square$

7.6 Examples

We apply now our results to two network examples. In the first example we consider a $\mathbf{Z}_3 \times \mathbf{Z}_3$-symmetric network of nine cells. In the second one we take a $\mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$-symmetric network of 60 cells.

To motivate our first example, we recall the definition given by Golubitsky and Lauterbach [13] of product network of regular networks. A homogeneous network is *regular* if all couplings are of the same type.

Definition 7.6.1 ([13]) Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are two regular networks of sizes r_1 and r_2 with cells $c_1, \ldots, c_{r_1}$ and $d_1, \ldots, d_{r_2}$ respectively. Form the *product network* $\mathcal{G} = \mathcal{G}_1 \boxtimes \mathcal{G}_2$ of size $r = r_1 r_2$, where each node c_i in network $\mathcal{G}_1$ is replaced by a copy of network $\mathcal{G}_2$. Let p_{ij} be the jth cell in the copy of network $\mathcal{G}_2$ that replaces cell c_i. So p_{ij} is a copy of cell d_j. We assume that there is an arrow from cell p_{ij} to cell p_{lj} if and only if there is an arrow from cell c_i to cell c_l in network $\mathcal{G}_1$. We also assume that there is an arrow from cell p_{ij} to cell p_{il} if and only if there is an arrow from cell d_j to cell d_l in network $\mathcal{G}_2$. Finally, we assume that these two types of arrows are different. So there are two arrow types in the homogeneous network $\mathcal{G}$ – those arrows that connect cells within a given copy of $\mathcal{G}_2$ and those arrows that connect cells between copies of $\mathcal{G}_2$.

Example 7.6.2 We continue the Example 7.5.5 considering the graph $\mathcal{G}$ of nine cells represented in Figure 7.2 (right) and the 1-parameter $\mathcal{G}$-admissible linear map L. This network is the product network of the two $\mathbf{Z}_3$-symmetric

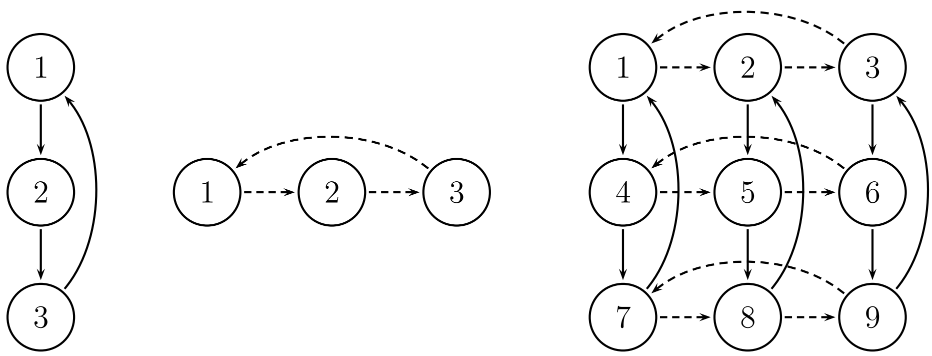

Figure 7.2: (Left and center) 3-cell regular networks with $\mathbf{Z}_3$-symmetry; (right) network with $\mathbf{Z}_3 \times \mathbf{Z}_3$-symmetry that is the product of the two networks pictured on its left.

networks represented in Figure 7.2 (left and center) and has symmetry

$$\Gamma = \langle a_1, a_2 \rangle \cong \mathbf{Z}_3 \times \mathbf{Z}_3$$

where $a_1 = (123)(456)(789)$ and $a_2 = (147)(258)(369)$. Recall from Example 7.3.4 that 5 is a special index over $\{a_1^2, a_2^2\}$ and obviously is also special over $S = \{e, a_1^2, a_2^2\}$ and 5 and 6 are the representative indexes of the indexes associated with 5. Then, by Theorem 7.5.7 and Table 7.1, if $M_R^5(0) = 0$, that is

$$a(0) - \frac{1}{2}c(0) - \frac{1}{2}b(0) = 0,$$

the spectrum of L at $\lambda = 0$ includes the critical eigenvalues

$$\begin{aligned} \pm i M_I^5(0) &= \pm i\left(a(0) + c(0)(\chi_5)_I(a_1^2) + b(0)(\chi_5)_I(a_2^2)\right) \\ &= \pm i a(0)\sqrt{3} \end{aligned}$$

and

$$\begin{aligned} \pm i M_I^6\,(0) &= \pm i\,(a\,(0) + c\,(0)\,(\chi_6)_I\,(a_1^2) + b\,(0)\,(\chi_6)_I\,(a_2^2)) \\ &= \pm i\,(a-c)\,(0)\,\sqrt{3}. \end{aligned}$$

$\diamond$

Remark 7.6.3 Example 7.6.2 does not contradict the result described in section 4 of Golubitsky *et al.* [13] that states, in particular, that the center subspace of the Jacobian J of an admissible system of differential equations associated to a product network generically cannot correspond to two different and non conjugated purely imaginary eigenvalues when the dimension of the network nodes is greater than one. In the Example 7.6.2, $\widehat{V}_5 \oplus \widehat{V}_6$ is the center subspace corresponding to two different and non conjugated purely imaginary eigenvalues but the dimension of the network nodes is one-dimensional.

Example 7.6.4 Consider the symmetric connected coupled cell network $\mathcal{G}$ with 60 cells represented in Figure 7.3. This network has symmetry $\Gamma = \langle a_1, a_2, a_3 \rangle$ where

$$\begin{aligned} a_1 &= (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)\cdots(56\ 57\ 58\ 59\ 60), \\ a_2 &= (1\ 6\ 11\ 16)(2\ 7\ 12\ 17)\cdots(45\ 50\ 55\ 60), \\ a_3 &= (1\ 21\ 41)(2\ 22\ 42)\cdots(20\ 40\ 60) \end{aligned}$$

and $\Gamma \cong \mathbf{Z}_{60} \cong \mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$ acts transitively and faithfully on the set of cells $\{1, \ldots, 60\}$. In this case $S = \{e, a_1, a_2, a_3\}$. That is, cells 2, 6 and 21 have couplings directed to cell 1; identifying each cell i by the unique element $\gamma_i \in \Gamma$ such that $\gamma_i(1) = i$ we have $e(1) = 1$, $a_1(1) = 2$, $a_2(1) = 6$, $a_3(1) = 21$ and then $S = \{e, a_1, a_2, a_3\}$.

Assume now that $V = \mathbf{R}^{60}$ and let $L : \mathbf{R}^{60} \times \mathbf{R} \to \mathbf{R}^{60}$ be a 1-parameter family of $\mathcal{G}$-admissible linear vector fields. Then, with respect to the canonical basis of $\mathbf{R}^n$ we have

$$L = \begin{bmatrix} A & B & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & B & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A & B & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 \\ B & 0 & 0 & A & 0 & 0 & 0 & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & B & 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & B & 0 & 0 & A & 0 & 0 & 0 & C \\ C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 \\ 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B \\ 0 & 0 & 0 & C & 0 & 0 & 0 & 0 & B & 0 & 0 & A \end{bmatrix}$$

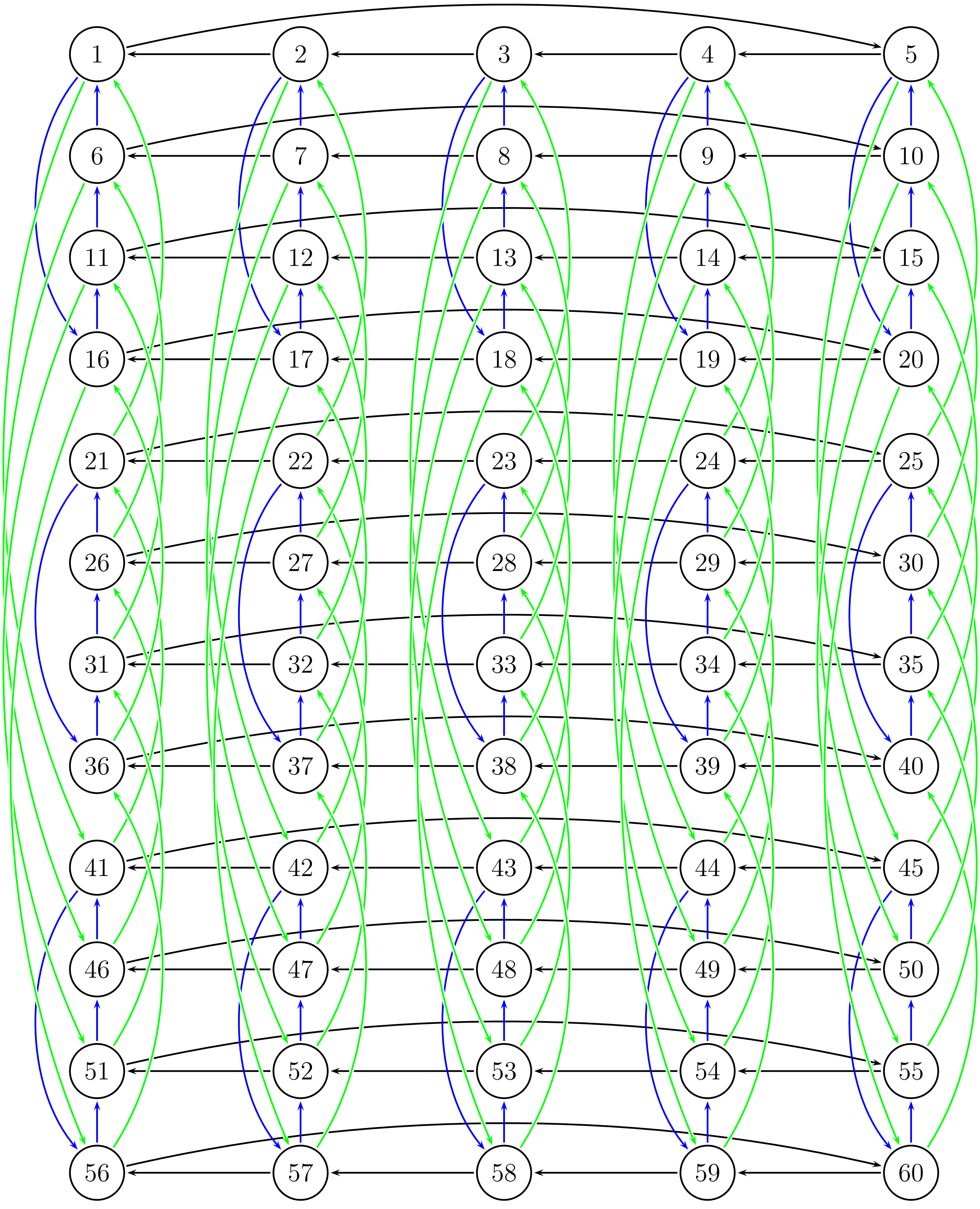

Figure 7.3: 60-cell network with symmetry $\mathbf{Z}_5 \times \mathbf{Z}_4 \times \mathbf{Z}_3$.

where

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & a & b \\ b & 0 & 0 & 0 & a \end{bmatrix}, \quad B = cI_5, \quad C = dI_5$$

and a, b, c, d are real-valued smooth functions of λ.

Recall from Example 7.3.15 the special indexes over $\{a_1, a_2, a_3\}$ and consequently over S and consider, for example, the special index $i = 28$ over S. The corresponding set of associated indexes and representative indexes are $I_7 = \{28, 29, 38, 39, 48, 49, 58, 59\}$ and $\widetilde{I}_7 = \{28, 29, 38, 39\}$ respectively. By equation (7.31) and Table 7.3 we obtain the eigenvalues $M^{28}, M^{29}, M^{38}, M^{39}$ of L:

$$\begin{aligned}
M^{28} &= a + b\overline{\chi_{28}(a_1)} + c\overline{\chi_{28}(a_2)} + d\overline{\chi_{28}(a_3)} = a + be^{-i\frac{4\pi}{5}} - ci + de^{-i\frac{2\pi}{3}}, \\
M^{29} &= a + b\overline{\chi_{29}(a_1)} + c\overline{\chi_{29}(a_2)} + d\overline{\chi_{29}(a_3)} = a + be^{i\frac{4\pi}{5}} - ci + de^{-i\frac{2\pi}{3}}, \\
M^{38} &= a + b\overline{\chi_{38}(a_1)} + c\overline{\chi_{38}(a_2)} + d\overline{\chi_{38}(a_3)} = a + be^{-i\frac{4\pi}{5}} + ci + de^{-i\frac{2\pi}{3}}, \\
M^{39} &= a + b\overline{\chi_{39}(a_1)} + c\overline{\chi_{39}(a_2)} + d\overline{\chi_{39}(a_3)} = a + be^{i\frac{4\pi}{5}} + ci + de^{-i\frac{2\pi}{3}}.
\end{aligned}$$

Then, by Theorem 7.5.7, if $M_R^{28}(0) = 0$, that is

$$a(0) + \cos\left(\frac{4\pi}{5}\right) b(0) - \frac{1}{2}d(0) = 0,$$

then the spectrum of $L(0)$ includes the critical eigenvalues

$$\begin{aligned}
\pm iM_I^{28} &= \pm \left(-\sin\left(\tfrac{4\pi}{5}\right) b(0) - c(0) - \tfrac{\sqrt{3}}{2}d(0)\right) i \\
\pm iM_I^{29} &= \pm \left(\sin\left(\tfrac{4\pi}{5}\right) b(0) + c(0) - \tfrac{\sqrt{3}}{2}d(0)\right) i \\
\pm iM_I^{38} &= \pm \left(-\sin\left(\tfrac{4\pi}{5}\right) b(0) + c(0) - \tfrac{\sqrt{3}}{2}d(0)\right) i \\
\pm iM_I^{39} &= \pm \left(\sin\left(\tfrac{4\pi}{5}\right) b(0) - c(0) - \tfrac{\sqrt{3}}{2}d(0)\right) i.
\end{aligned}$$

Consider, for another example, the special index $i = 26$. The correspondent set of associated indexes and representative indexes are $I_5 = \{26, 36, 46, 56\}$ and $\widetilde{I}_5 = \{26, 36\}$ respectively. By equation (7.31) and Table 7.3 we obtain the eigenvalues M^{26}, M^{36} of L:

$$\begin{aligned}
M^{26} &= a + b\overline{\chi_{26}(a_1)} + c\overline{\chi_{26}(a_2)} + d\overline{\chi_{26}(a_3)} = a + b - ci + de^{-i\frac{2\pi}{3}}, \\
M^{36} &= a + b\overline{\chi_{36}(a_1)} + c\overline{\chi_{36}(a_2)} + d\overline{\chi_{36}(a_3)} = a + b + ci + de^{-i\frac{2\pi}{3}}.
\end{aligned}$$

Then, by Theorem 7.5.7, if $M_R^{26}(0) = 0$, that is

$$a(0) + b(0) - \frac{1}{2}d(0) = 0,$$

then the spectrum of $L(0)$ includes the critical eigenvalues

$$\pm i M_I^{26} = \pm \left(-c\,(0) - \tfrac{1}{2} d\,(0)\,\sqrt{3}\right) i$$
$$\pm i M_I^{36} = \pm \left(c\,(0) - \tfrac{1}{2} d\,(0)\,\sqrt{3}\right) i.$$

$\diamond$

7.7 Correspondence between $\mathcal{G}$-admissible and General Equivariant Maps

Corollary 7.3.8 asserts that, in general, an abelian group Γ has special indexes over $S = \{a_1, \ldots, a_r\}$ where $\{a_1, \ldots, a_r\}$ is a set of generators of Γ. In this case S is the simplest set such that $< S >= \Gamma$. Next we present the opposite situation.

Definition 7.7.1 ([3]) Let Γ be a permutation group acting on $\mathcal{C} = \{1, 2, \ldots, n\}$, so that $\Gamma \subseteq \mathbf{S}_n$. A network $\mathcal{G}$ is a Γ-*network* if

(a) The cells of $\mathcal{G}$ are the elements of $\mathcal{C}$.

(b) $c \sim_C d \Leftrightarrow d = \gamma c$ for some $\gamma \in \Gamma$. That is, the cell types are the Γ-orbits on C.

(c) $\mathcal{E}$ contains exactly one edge (c, d) for each $c \neq d \in \mathcal{C}$, where $\mathcal{H}(c, d) = c$ and $\mathcal{T}(c, d) = d$, and no others. In particular, there are no self-connections and no multiple arrows.

(d) $(c, d) \sim_E (c', d') \Leftrightarrow c' = \gamma c$ and $d' = \gamma d$ for some $\gamma \in \Gamma$. That is, the arrow types are the Γ-orbits on the set A of pairs (c, d), with $c \neq d$.

These conditions determine the network uniquely up to isomorphism. We denote it by $\mathcal{G}_\Gamma$. A network $\mathcal{G}$ is a *group network* if it is a Γ-network for some group Γ.

Proposition 7.7.2 ([3]) *Consider a* Γ-*network* $\mathcal{G}$*, where the symmetry group* Γ *of the network is abelian and acts transitively and faithfully by permutation on the cells of the network. Then*

$$\mathcal{F}_P\,(\mathcal{G}) = \overrightarrow{\mathcal{E}}_P\,(\Gamma)$$

where $\mathcal{F}_P\,(\mathcal{G})$ *and* $\overrightarrow{\mathcal{E}}_P\,(\Gamma)$ *denotes respectively the spaces of* $\mathcal{G}$*-admissible and* Γ*-equivariant maps* $f : P \to P$ *where* P *is the total phase space.*

Proof: The result is a particular case of Proposition 4.8 of Antoneli and Stewart [3] when the symmetry group Γ of the network is abelian and acts transitively and faithfully by permutation on the cells of the network. Then, we only need to prove that

$$\Gamma_c = B(c, c) \tag{7.37}$$

for all $c \in \mathcal{C}$. Here $\Gamma_c = \{\gamma \in \Gamma : \gamma c = c\}$ and $B(c, c)$ is as in Definition 5.1.4.

We begin by proving that $\Gamma_c = \{\mathbf{1}\}$. Consider $c \in \mathcal{C}$ and $\gamma \in \Gamma_c$. By Remark 7.2.2 if $\gamma c = c$ for some $c \in \mathcal{C}$ then $\gamma i = i$ for all $i \in \mathcal{C}$. That is, γ is the identity.

Now we prove that $B(c, c) = \{\mathbf{1}\}$. Consider $c \in \mathcal{C}$ and the edges (a, c) and (b, c) in $I(c)$. By definition of Γ-network $(a, c) \sim_E (b, c)$ if and only if $b = \gamma a$ and $c = \gamma c$ for some $\gamma \in \Gamma$. By Remark 7.2.2, from $c = \gamma c$ we have that γ is the identity. Consequently $a = b$. We conclude that the edges in $I(c)$ are all distinct. Thus $B(c, c) = \{\mathbf{1}\}$ and $\Gamma_c = B(c, c)$. Also observe that by (c) of Definition 7.7.1, for each $d \in \mathcal{C} \setminus c$ there is one edge $(d, c) \in I(c)$.

We have that $\mathcal{F}_P(\mathcal{G}) \subseteq \overrightarrow{\mathcal{E}}_P(\Gamma)$. Since $\mathcal{G}$ is a Γ-network, the orbits of Γ on $\mathcal{C}$ are exactly the equivalence classes for the relation of input equivalence. In the present case there is only one equivalence class. By Stewart *et al.* [33] (Proposition 4.6), the $\mathcal{B}_{\mathcal{G}}$-admissible maps are determined (via pullback) by the component f_c which is $B(c, c)$-invariant. Similarly, the Γ-equivariant maps are determined (via pullback) by the component f_c and the function f_c is Γ_c-invariant. Finally, (7.37) and the observation at the end of the previous paragraph imply that the set of Γ_c-invariant functions and the set of $B(c, c)$-invariant functions are equal. Therefore $\mathcal{F}_P(\mathcal{G}) = \overrightarrow{\mathcal{E}}_P(\Gamma)$.

$\square$

Remark 7.7.3 Let $\mathcal{G}$ be a Γ-network where the symmetry group Γ is abelian and acts transitively (and faithfully) by permutation on the cells $\{1, \ldots, n\}$ of the network. Consider the system (7.23) where $V = \mathbf{R}^n$ and $f : V \times \mathbf{R} \to V$ is a smooth 1-parameter family of $\mathcal{G}$-admissible vector fields. In this case $S = \Gamma$. By Proposition 7.7.2, the codimension one eigenvalue movements across the imaginary axis of $(\mathrm{d}f)_{(x_0,\lambda)} = L_{(x_0,\lambda)}$ are identical to the corresponding eigenvalue movements in general equivariant vector fields. Golubitsky *et al.* [21] proved that, generically, in general 1-parameter family of Γ-equivariant systems, mode interactions do not occur.

Example 7.7.4 Consider the $\mathbf{Z}_3 \times \mathbf{Z}_3$-network represented in Figure 7.4 where

$$\mathbf{Z}_3 \times \mathbf{Z}_3 = \langle (123)(456)(789), (147)(258)(369).$$

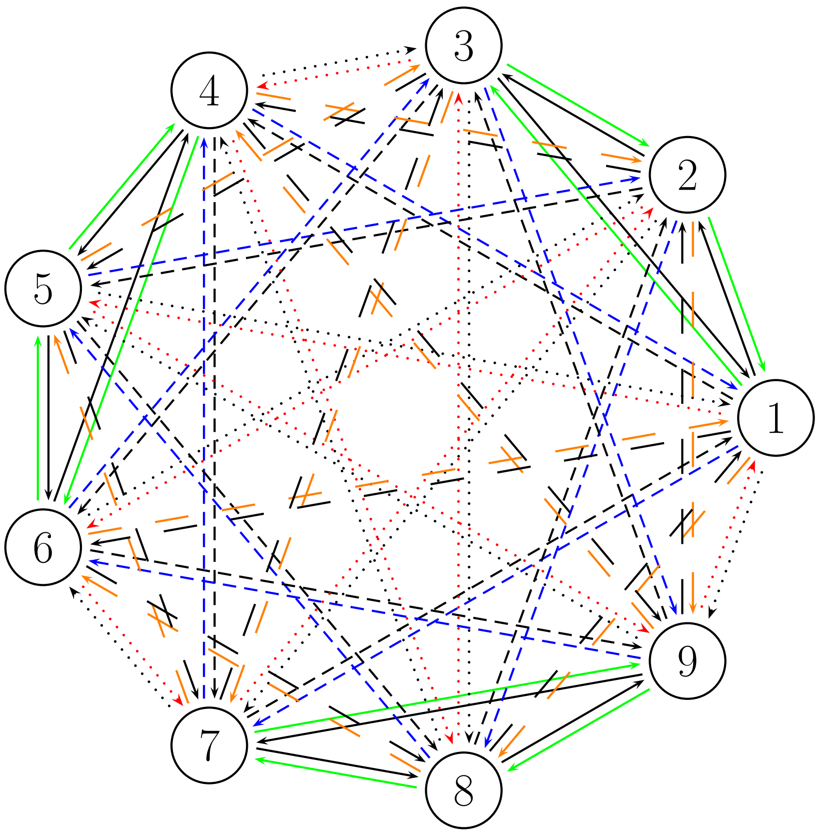

Figure 7.4: $\mathbf{Z}_3 \times \mathbf{Z}_3$-network. Note the eight different types of arrow.

Assume that $P = \mathbf{R}^9$ and that the network dynamics have a group invariant equilibrium. Then, the general $\mathcal{G}$-admissible linear map at such an equilibrium has the form

$$L = \begin{bmatrix} A & C & B \\ B & A & C \\ C & B & A \end{bmatrix} \tag{7.38}$$

where

$$A = \begin{bmatrix} a & f & c \\ c & a & f \\ f & c & a \end{bmatrix}, \; B = \begin{bmatrix} b & h & j \\ j & b & h \\ h & j & b \end{bmatrix}, \; C = \begin{bmatrix} e & d & g \\ g & e & d \\ d & g & e \end{bmatrix}$$

and $a, b, c, d, e, f, g, h, j \in \mathbf{R}$. Note that (7.38) with $f = 0 = h = j = e = g$ corresponds to the matrix (7.4) of Example 7.1.1. Observe that each absence of connections in network $\mathcal{G}$ implies that the correspondent constant in (7.38) is null. In this case $S = \mathbf{Z}_3 \times \mathbf{Z}_3$. By Remark 7.7.3 we can conclude that generically mode interactions do not occur in $\mathcal{G}$-admissible systems. $\diamond$

Chapter 8

Conclusions and Future Work

Consider a system of ODEs

$$\dot{x} = f(x, \lambda) \tag{8.1}$$

where $x \in V = \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is the bifurcation parameter, $f : V \times \mathbf{R} \to V$ is C^∞ and assume that $f(x_0, \lambda) \equiv 0$ where x_0 is a fully symmetric equilibrium.

In chapter 4 we study the generic existence of branches of periodic solutions in symmetric systems of ODEs occurring by Hopf bifurcation from the trivial solution $(x, \lambda) = (x_0, 0)$ that are not guaranteed by the Equivariant Hopf Theorem. Essentially we use two techniques: assume that f is in Birkhoff normal form to all orders and so f commutes also with $\mathbf{S}^1$ and use the Morse Lemma. We address this question by considering systems with symmetry $\mathbf{D}_n$, $n \geq 3$, the dihedral group of order $2n$. These techniques can be used when studying Hopf bifurcation in systems of ODEs with other symmetry groups. However, the application of Morse Lemma depends on the structure of the Birkhoff normal form of f. Consequently it is not always possible to apply these two techniques. The natural next task can be the description of the groups where the combination of these two techniques can be applied.

In chapter 6 we obtain the full analogue of the Equivariant Hopf Theorem for networks with symmetries (Theorem 6.1.3). We extend the result of Golubitsky *et al.* [15] obtaining states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having spatio-temporal symmetries, that is, corresponding to interiorly $\mathbf{C}$-axial subgroups of $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$. In section 6.3 we prove Theorem 6.1.3 using a center manifold reduction. This approach can be useful in the development of normal form theory aiming at the study of the stability of the periodic solutions guaranteed by Theorem 6.1.3. In context of interior symmetry, since f is not symmetric, we cannot use the same group theoretic methods of the symmetric case to do this study. The study of the stability of these periodic solutions represents the natural next task.

In chapter 7 we address one of the main questions in the theory of coupled cell networks: in what way the network architecture may affect the kinds of bifurcations that are expected to occur in a coupled cell network? In general equivariant vector fields, when we have a codimension one bifurcation problem, generically the critical space of $(df)_{(x_0,0)}$ is absolutely irreducible or Γ-simple (see Golubitsky *et al.* [21] Proposition XIII 3.2 and Proposition 3.2.4). In the context of symmetric coupled cell systems this is not necessarily true. We address this question by focussing on networks with an abelian symmetry group permuting cells transitively. We established a relation between the symmetry group of the network, the network architecture and the kinds of bifurcations that are expected to occur when the phase space of the cells of the network is one-dimensional using characters of abelian groups. In particular, we show that for most of the abelian symmetric coupled cell networks Hopf bifurcation can occur associated to the crossings with the imaginary axis of two or more distinct pairs of complex eigenvalues of linearization of the admissible vector fields at a fully symmetric equilibrium x_0. The next step is to study this question for non abelian symmetry groups.

Appendix A

Proof of Proposition 4.2.1

In this chapter we prove Proposition 4.2.1 following Golubitsky *et al.*[21]. To simplify the calculation of $\mathbf{D}_n \times \mathbf{S}^1$-invariants and equivariants we begin by presenting a few results.

Let $V = \mathbf{C}^m$ and consider the action of $\mathbf{S}^1$ on $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$ given by

$$\theta \cdot z = \left(e^{i\theta} z_1, \ldots, e^{i\theta} z_m\right) \quad (\theta \in \mathbf{S}^1). \tag{A.1}$$

For $z = (z_1, \ldots, z_m)$ define $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_m)$. The next lemma reduces the problem of finding invariant polynomials from $\mathbf{C}^m$ to $\mathbf{R}$ to the calculation of invariant polynomials from $\mathbf{C}^m$ to $\mathbf{C}$.

Lemma A.1 ([21]) *Let Γ be a Lie group acting on $\mathbf{C}^m$. Suppose that $N_1, \ldots, N_s$ generate the Γ-invariant polynomials from $\mathbf{C}^m$ to $\mathbf{C}$ in z and $\overline{z}$. Then*

$$\mathrm{Re}\,(N_1)\,, \ldots, \mathrm{Re}\,(N_s)\,, \mathrm{Im}\,(N_1)\,, \ldots, \mathrm{Im}\,(N_s)$$

generate the invariant polynomials from $\mathbf{C}^m$ to $\mathbf{R}$.

Proof: We follow Golubitsky *et al.* [21] Lemma XVI 9.2. The invariant polynomials from $\mathbf{C}^m$ to $\mathbf{R}$ are the invariant polynomials from $\mathbf{C}^m$ to $\mathbf{C}$ whose values happen to lie in $\mathbf{R}$. Hence they are generated by the real and the imaginary parts of monomials $N_1^{\alpha_1}, \ldots, N_s^{\alpha_s}$. But if p and q are polynomials in $z, \overline{z}$ over $\mathbf{C}$ then

$$\begin{aligned}\mathrm{Re}\,(pq) &= \mathrm{Re}(p)\mathrm{Re}(q) - \mathrm{Im}(p)\mathrm{Im}(q)\\ \mathrm{Im}\,(pq) &= \mathrm{Re}(p)\mathrm{Im}(q) + \mathrm{Im}(p)\mathrm{Re}(q)\end{aligned}$$

and an induction completes the proof. $\square$

We now compute the $\mathbf{S}^1$-invariant polynomials and the $\mathbf{S}^1$-equivariant polynomial mappings for the action (A.1) of $\mathbf{S}^1$ on $\mathbf{C}^m$.

Proposition A.2 ([21]) *Consider the action* (A.1) *of* $\mathbf{S}^1$ *on* $\mathbf{C}^m$.
(a) *An Hilbert basis for the smooth* $\mathbf{S}^1$*-invariant functions is given by the* m^2 *quadratics:*

$$\begin{array}{lll} u_j = z_j\overline{z}_j & (1 \leq j \leq m); & \\ Re\ v_{ij}, \ \ Im\ v_{ij} & (1 \leq i < j \leq m) & \textit{where } v_{ij} = z_i\overline{z}_j. \end{array}$$

Relations between these polynomials are given by

$$v_{ij}\overline{v}_{ij} = u_i u_j.$$

(b) *Let* $g = (g_1, \ldots, g_m) : \mathbf{C}^m \to \mathbf{C}^m$ *be* $\mathbf{S}^1$*-equivariant and smooth. Then each* g_j *satisfies*

$$g_j(\theta \cdot z) = e^{i\theta} g_j(z) \tag{A.2}$$

for all $\theta \in \mathbf{S}^1$, $z \in \mathbf{C}^m$. *The module of such* $g_j : \mathbf{C}^m \to \mathbf{C}$ *is generated over the invariant functions by the* $2m$ *mappings*

$$\begin{array}{l} X_j(z) = z_j \\ Y_j(z) = iz_j \end{array}$$

for $1 \leq j \leq m$. *Thus the module of* $\mathbf{S}^1$*-equivariant mappings from* $\mathbf{C}^m$ *to* $\mathbf{C}^m$ *have* $2m^2$ *generators of the form*

$$(0, \ldots, 0, X_j, 0, \ldots, 0)$$

and

$$(0, \ldots, 0, Y_j, 0, \ldots, 0)\,.$$

Proof: We follow Golubitsky *et al.* [21] Lemma XVI 9.3. By the theorems of Schwarz and Poénaru (Theorem 3.1.3 and Theorem 3.1.7) we may restrict the study to polynomials and polynomial mappings.

We begin with the proof of (a). Let $f : \mathbf{C}^m \to \mathbf{C}$ be a polynomial. Using multi-indexes we can write

$$f(z) = \sum_{\alpha,\beta \in \left(\mathbf{Z}_0^+\right)^m} a_{\alpha\beta} z_1^{\alpha_1} \ldots z_m^{\alpha_m} \overline{z}_1^{\beta_1} \ldots \overline{z}_m^{\beta_m}$$

as

$$f(z) = \sum a_{\alpha\beta} z^\alpha \overline{z}^\beta$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$, $z = (z_1, \ldots, z_m)$, $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_m)$ and $a_{\alpha\beta} \in \mathbf{C}$. For f to be $\mathbf{S}^1$-invariant we must have $f(\theta \cdot z) = f(z)$, for all $\theta \in \mathbf{S}^1$ and $z \in \mathbf{C}^m$. This equality is verified if and only if for each (α, β) we have

$$a_{\alpha\beta} = 0 \ \text{ or } \ |\alpha| = |\beta|.$$

By pairing the $z_i s$ with $z_j s$ we can always write $z^\alpha \overline{z}^\beta$ as a monomial in the u_j and v_{ij}. Now use Lemma A.1 and observing that u_j is real, the result is proved.

We prove now (b). A mapping $g = (g_1, \ldots, g_m)$ is $\mathbf{S}^1$-equivariant if and only if, for $j = 1, \ldots, m$, we have $g_j(\theta \cdot z) = e^{i\theta} g_j(z)$ for all $\theta \in \mathbf{S}^1$ and $z \in \mathbf{C}^m$. Writing g_j as

$$g_j(z) = \sum b_{\alpha\beta} z_\alpha \overline{z}_\beta$$

where $b_{\alpha\beta} \in \mathbf{C}$, then $g_j(\theta \cdot z) = e^{i\theta} g_j(z)$ for all $\theta \in \mathbf{S}^1$ and $z \in \mathbf{C}^m$ if and only if for each (α, β) such that $b_{\alpha\beta} \neq 0$ we have

$$|\alpha| - |\beta| = 1 \Leftrightarrow |\alpha| = |\beta| + 1.$$

In particular we have $|\alpha| > 0$. We can thus divide out a z_j for each monomial $z^\alpha \overline{z}^\beta$ and the quotient is a $\mathbf{C}$-valued $\mathbf{S}^1$-invariant. $\square$

Consider the action of $\mathbf{D}_n \times \mathbf{S}^1$ on $(z_1, z_2) \in \mathbf{C}^2$ defined by

$$\begin{array}{ll} \text{(a)} & \gamma \cdot (z_1, z_2) = (e^{i\gamma} z_1, e^{-i\gamma} z_2) \quad (\gamma \in \mathbf{Z}_n) \\ \text{(b)} & k \cdot (z_1, z_2) = (z_2, z_1) \\ \text{(c)} & \theta \cdot (z_1, z_2) = \left(e^{i\theta} z_1, e^{i\theta} z_2\right) \quad\quad (\theta \in \mathbf{S}^1)\,. \end{array} \tag{A.3}$$

Here $\mathbf{Z}_n = \langle \frac{2\pi}{n} \rangle$ and $\mathbf{D}_n = \langle \frac{2\pi}{n}, k \rangle$.

The following results depend on parity of n. Define

$$m = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases} \tag{A.4}$$

Proposition A.3 ([21]) *Let $n \geq 3$ and let m be as in* (A.4). *Then*
(a) *Every smooth $\mathbf{D}_n \times \mathbf{S}^1$-invariant function $f:\ \mathbf{C}^2 \to \mathbf{R}$ has the form*

$$f(z_1, z_2) = h(N, P, S, T)$$

where $N = |z_1|^2 + |z_2|^2$, $P = |z_1|^2 |z_2|^2$, $S = (z_1 \overline{z}_2)^m + (\overline{z}_1 z_2)^m$,

$$T = i\left(|z_1|^2 - |z_2|^2\right)\left((z_1 \overline{z}_2)^m - (\overline{z}_1 z_2)^m\right)$$

and $h:\ \mathbf{R}^4 \to \mathbf{R}$ is smooth.
(b) *Every smooth $\mathbf{D}_n \times \mathbf{S}^1$-equivariant function $f:\ \mathbf{C}^2 \to \mathbf{C}^2$ has the form*

$$f(z_1, z_2) = A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B \begin{bmatrix} z_1^2 \overline{z}_1 \\ z_2^2 \overline{z}_2 \end{bmatrix} + C \begin{bmatrix} \overline{z}_1^{m-1} z_2^m \\ z_1^m \overline{z}_2^{m-1} \end{bmatrix} + D \begin{bmatrix} z_1^{m+1} \overline{z}_2^m \\ \overline{z}_1^m z_2^{m+1} \end{bmatrix}$$

where A, B, C, D are complex-valued $\mathbf{D}_n \times \mathbf{S}^1$-invariant smooth functions.

Remark A.4 The $\mathbf{D}_n \times \mathbf{S}^1$-invariant polynomials do not form a polynomial ring. There is a relation:

$$T^2 = (4P - N^2)(S^2 - 4P^m). \tag{A.5}$$

Proof: We follow Golubitsky *et al.* [21] Proposition XVIII 2.1. We include it for completeness. We find the invariants using Lemma A.1 which let us consider the simpler situation of finding the invariants from $\mathbf{C}^2$ to $\mathbf{C}$.

Consider the chain of subgroups

$$\mathbf{S}^1 \subset \mathbf{Z}_n \times \mathbf{S}^1 \subset \mathbf{D}_n \times \mathbf{S}^1.$$

We can find the $\mathbf{D}_n \times \mathbf{S}^1$-invariant polynomials by climbing the chain: first finding the $\mathbf{S}^1$-invariants, then the $\mathbf{Z}_n \times \mathbf{S}^1$-invariants and finally the $\mathbf{D}_n \times \mathbf{S}^1$-invariants. Since $\mathbf{S}^1$ acts on $\mathbf{C}^2$ by $\theta \cdot (z_1, z_2) = \left(e^{i\theta} z_1, e^{i\theta} z_2\right)$, by Proposition A.2 the $\mathbf{S}^1$-invariants are generated by

$$u_1 = z_1 \overline{z}_1, \quad u_2 = z_2 \overline{z}_2, \quad v = z_1 \overline{z}_2 \ \text{ e } \ \overline{v}, \tag{A.6}$$

with the relation

$$u_1 u_2 = v \overline{v}. \tag{A.7}$$

Next we compute the action of $\mathbf{D}_n$ on the three-dimensional space (u_1, u_2, v). As $\gamma = \frac{2\pi}{n} \in \mathbf{Z}_n$ acts on (z_1, z_2) by $(e^{i\gamma} z_1, e^{-i\gamma} z_2)$ then

$$\gamma \cdot (u_1, u_2, v) = \left(u_1, u_2, e^{2i\gamma} v\right).$$

When n is odd, $e^{2i\gamma}$ generates $\mathbf{Z}_n$, whereas when n is even $e^{2i\gamma}$ generates $\mathbf{Z}_{n/2}$. Thus, with m defined as in (A.4), $\mathbf{Z}_n \subset \mathbf{D}_n$ acts on $v \in \mathbf{C}$ as $\mathbf{Z}_m$. Therefore, the $\mathbf{Z}_n \times \mathbf{S}^1$-invariants are generated by u_1, u_2 and

$$w = v^m, \quad \overline{w}, \quad x = v \overline{v}. \tag{A.8}$$

Recall, however, that because $x = v\overline{v} = u_1 u_2$ then x is redundant. The next step is to compute the action of κ on the $\mathbf{Z}_n \times \mathbf{S}^1$-invariants, obtaining

$$\kappa \cdot (u_1, u_2, w) = (u_2, u_1, \overline{w}). \tag{A.9}$$

It is now straightforward to show that the κ-invariants on (u_1, u_2, w)-space are generated by

$$\begin{array}{ll} \text{(a)} & u_1 + u_2, \quad u_1 u_2 \\ \text{(b)} & w + \overline{w}, \quad w\overline{w} \\ \text{(c)} & (u_1 - u_2)(w - \overline{w}). \end{array} \tag{A.10}$$

We now translate these generators for the $\mathbf{D}_n \times \mathbf{S}^1$-invariant polynomials in (u_1, u_2, w)-space into (z_1, z_2)-coordinates. Note that $w\overline{w} = (v\overline{v})^m = (u_1 u_2)^m$ is redundant. We are left with the four generators

$$\begin{array}{ll} \text{(a)} & u_1 + u_2 = |z_1|^2 + |z_2|^2 = N \\ \text{(b)} & u_1 u_2 = |z_1|^2 |z_2|^2 = P \\ \text{(c)} & w + \overline{w} = (z_1 \overline{z}_2)^m + (\overline{z}_1 z_2)^m = S \\ \text{(d)} & (u_1 - u_2)(w - \overline{w}) = [z_1^{m+1} \overline{z}_1 \overline{z}_2^m + \overline{z}_1^m z_2^{m+1} \overline{z}_2 \\ & \qquad - z_1 \overline{z}_1^{m+1} z_2^m - z_1^m z_2 \overline{z}_2^{m+1}] \\ & \qquad = -iT. \end{array} \tag{A.11}$$

Part (a) of the proposition follows from Lemma A.1 noting that N, P and S are real-valued, whereas $-iT$ is purely imaginary. Also (A.11) implies that

$$T^2 = -(u_1 - u_2)^2 (w - \overline{w})^2 = \left(4P - N^2\right)\left(S^2 - 4P^m\right),$$

yielding the relation (A.5).

We prove now (b). Suppose that $g(z_1, z_2) = (\phi_1(z_1, z_2), \phi_2(z_1, z_2))$ commutes with $\mathbf{D}_n \times \mathbf{S}^1$. Commutativity with κ implies that $\phi_2(z_1, z_2) = \phi_1(z_2, z_1)$. Thus we must determine the mappings $\phi : \mathbf{C}^2 \to \mathbf{C}$ that commute with $\mathbf{Z}_n \times \mathbf{S}^1$. The $\mathbf{S}^1$-equivariant polynomial mappings have the form

$$\phi(z_1, z_2) = p(u, v) z_1 + q(u, v) z_2 \tag{A.12}$$

where $u = (u_1, u_2) \in \mathbf{R}^2$ and $v \in \mathbf{C}$ are defined as (A.6). Also $p, q : \mathbf{R}^2 \times \mathbf{C} \to \mathbf{C}$.

The action of $\mathbf{Z}_n$ on (A.12) produces

$$\phi\left(e^{i\gamma} z_1, e^{-i\gamma} z_2\right) = p\left(u, e^{2i\gamma} v\right) e^{i\gamma} z_1 + q\left(u, e^{2i\gamma} v\right) e^{-i\gamma} z_2. \tag{A.13}$$

Commutativity of g with $\mathbf{Z}_n$ implies that

$$\phi\left(e^{i\gamma} z_1, e^{-i\gamma} z_2\right) = e^{i\gamma} \phi(z_1, z_2). \tag{A.14}$$

From (A.13) and (A.14) we obtain

$$\begin{array}{ll} \text{(a)} & p\left(u, e^{2i\gamma} v\right) = p(u, v) \\ \text{(b)} & q\left(u, e^{2i\gamma} v\right) = e^{2i\gamma} q(u, v). \end{array} \tag{A.15}$$

Identity (A.15(a)) states that p is $\mathbf{Z}_m \times \mathbf{S}^1$-invariant in v, with u as a parameter, hence has the form

$$p(u, v) = A(u, w) \tag{A.16}$$

with $w = v^m = (z_1\overline{z}_2)^m$. Similarly q commutes with $\mathbf{Z}_m \times \mathbf{S}^1$ in v, with u as a parameter, so has the form

$$q(u,v) = \alpha(u, v\overline{v}, v^m)\, v + \beta(u, v\overline{v}, v^m)\, \overline{v}^{m-1}. \tag{A.17}$$

Since $v\overline{v} = u_1 u_2$ is redundant and $v^m = w$ we may rewrite (A.17) as

$$q(u,v) = B(u_1, u_2, w)\, v + C(u_1, u_2, w)\, \overline{v}^{m-1}. \tag{A.18}$$

Using (A.12), (A.16) and (A.18) we obtain

$$\phi(z_1, z_2) = A(u, w)\, z_1 + B(u_1, u_2, w)\, v z_2 + C(u_1, u_2, w)\, \overline{v}^{m-1} z_2.$$

Thus ϕ has the form

$$\phi(z_1, z_2) = A z_1 + B|z_2|^2 z_1 + C\overline{z}_1^{m-1} z_2^m \tag{A.19}$$

where A, B and C are functions of $u_1 = |z_1|^2, u_2 = |z_2|^2$ and $w = (z_1\overline{z}_2)^m$. We now rewrite A, B and C in terms of the $\mathbf{D}_n \times \mathbf{S}^1$-invariants. Every polynomial in u_1, u_2 and w has the form

$$\alpha(u_1, u_2, u_1u_2, w) + \beta(u_1, u_2, u_1u_2, w)\, u_1.$$

Since $u_1|z_2|^2 z_1 = |z_1|^2|z_2|^2 z_1 = P z_1$ and $u_1 \overline{z}_1^{m-1} z_2^m = S z_1 - z_1^{m+1}\overline{z}_2^m$, we may use (A.11(b,c)) to rewrite (A.19) as

$$\phi = \left(A_1 + A_2|z_1|^2\right) z_1 + B_1|z_2|^2 z_1 + C_1\overline{z}_1^{m-1} z_2^m + C_2 z_1^{m+1}\overline{z}_2^m. \tag{A.20}$$

Next, rewrite $A_2|z_1|^2 + B_1|z_2|^2 = \alpha(|z_1|^2 + |z_2|^2) + \beta|z_1|^2$. Then (A.20) has the form

$$\phi = A z_1 + B|z_1|^2 z_1 + C\overline{z}_1^{m-1} z_2^m + D z_1^{m+1}\overline{z}_2^m \tag{A.21}$$

where A, B, C and D are functions of N, P, S and w.

Finally, every function of N, P, S and w may be written as

$$\widetilde{\alpha}\left(N, P, S, w + \overline{w}, (w - \overline{w})^2\right) + \widetilde{\beta}\left(N, P, S, w + \overline{w}, (w - \overline{w})^2\right)(w - \overline{w}).$$

Because $w + \overline{w} = S$ and $(w - \overline{w})^2 = S^2 - 4P^m$, we have that α and β are $\mathbf{D}_n \times \mathbf{S}^1$-invariants. Thus, the module of the $\mathbf{Z}_n \times \mathbf{S}^1$-equivariant polynomial mappings is generated by

$$\begin{array}{ll} \text{(a)} & z_1, |z_1|^2 z_1, \overline{z}_1^{m-1} z_2^m, z_1^{m+1}\overline{z}_2^m, \\ \text{(b)} & (w - \overline{w})\, z_1, (w - \overline{w})\, z_1^2\overline{z}_1, (w - \overline{w})\, \overline{z}_1^{m-1} z_2^m, (w - \overline{w})\, z_1^{m+1}\overline{z}_2^m. \end{array} \tag{A.22}$$

Since $w - \overline{w} = v^m - \overline{v}^m = (z_1\overline{z}_2)^m - (\overline{z}_1 z_2)^m$ the generators in (A.22(b)) are redundant. In particular, the identities

$$\begin{array}{ll} \text{(a)} & (w-\overline{w})\, z_1 = 2z_1^{m+1}\overline{z}_2^m - Sz_1 \\ \text{(b)} & (w-\overline{w})\, |z_1|^2 z_1 = Nz_1^{m+1}\overline{z}_2^m + \frac{1}{2}(-iT)\, z_1 - \frac{1}{2}NSz_1 \\ \text{(c)} & (w-\overline{w})\, \overline{z}_1^{m-1} z_2^m = -NP^{m-1}z_1 + P^{m-1}|z_1|^2 z_1 - S\overline{z}_1^{m-1} z_2^m \\ \text{(d)} & (w-\overline{w})\, z_1^{m+1}\overline{z}_2^m = Sz_1^{m+1}\overline{z}_2^m - 2P^m z_1 \end{array} \tag{A.23}$$

hold.

We have proved that every $\mathbf{D}_n \times \mathbf{S}^1$-equivariant has the form

$$A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B \begin{bmatrix} z_1^2\overline{z}_1 \\ z_2^2\overline{z}_2 \end{bmatrix} + C \begin{bmatrix} \overline{z}_1^{m-1} z_2^m \\ z_1^m \overline{z}_2^{m-1} \end{bmatrix} + D \begin{bmatrix} z_1^{m+1}\overline{z}_2^m \\ \overline{z}_1^m z_2^{m+1} \end{bmatrix}$$

where A, B, C and D are C^∞ functions that depend of N, P, S and, in addition, A depends on T. This statement is slightly stronger than part (b) of Proposition 4.2.1. □

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