HOPF BIFURCATION WITH $\mathbf{S}_N$-SYMMETRY

ANA PAULA S. DIAS AND ANA RODRIGUES

Abstract. We study Hopf bifurcation with $\mathbf{S}_N$-symmetry for the standard absolutely irreducible action of $\mathbf{S}_N$ obtained from the action of $\mathbf{S}_N$ by permutation of N coordinates. Stewart (Symmetry methods in collisionless many-body problems, *J. Nonlinear Sci.* **6** (1996) 543-563) obtains a classification theorem for the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. We use this classification to prove the existence of branches of periodic solutions with $\mathbf{C}$-axial symmetry in systems of ordinary differential equations with $\mathbf{S}_N$-symmetry posed on a direct sum of two such $\mathbf{S}_N$-absolutely irreducible representations, as a result of a Hopf bifurcation occurring as a real parameter is varied. We determine the (generic) conditions on the coefficients of the fifth order $\mathbf{S}_N \times \mathbf{S}^1$-equivariant vector field that describe the stability and criticality of those solution branches. We finish this paper with an application to the cases $N = 4$ and $N = 5$.

1. Introduction

The general theory of Hopf Bifurcation with symmetry was developed by Golubitsky and Stewart [13] and by Golubitsky, Stewart, and Schaeffer [16]. Golubitsky and Stewart [14] applied the theory of Hopf bifurcation with symmetry to systems of ordinary differential equations having the symmetries of a regular polygon (this is, with $\mathbf{D}_n$-symmetry). They studied the existence and stability of symmetry-breaking branches of periodic solutions in such systems. Finally, they applied their results to a general system of n nonlinear oscillators, coupled symmetrically in a ring, and describe the generic oscillation patterns. Since the development of the theory, some examples were studied with detail (see for example, [1],[7]-[9],[10, Chapter 5],[11],[12],[17], [19],[21],[23]).

In this paper we study one of the few classic problems in the theory of Hopf bifurcation with symmetry that has not been completely investigated: Hopf bifurcation with $\mathbf{S}_N$-symmetry. This problem is relevant to, for example, the behaviour of all-to-all coupled nonlinear oscillators. See for example the group theoretic work done by Aronson *et al.* [2] on period doubling with $\mathbf{S}_N$-symmetry and its application to coupled arrays of Josephson junctions; and the work of Ashwin and Swift [4] on the analysis of networks of identical dissipative oscillators weakly coupled.

The basic existence theorem for Hopf bifurcation in the symmetric case is the Equivariant Hopf Theorem, which involves $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ (in this case), this is, isotropy subgroups with two-dimensional fixed-point subspace. Stewart [22] obtains a classification theorem for $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. We use this classification and the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions in systems of ordinary differential equations with $\mathbf{S}_N$-symmetry taking the restriction of the standard action of $\mathbf{S}_N$ on $\mathbf{C}^N$ onto a $\mathbf{S}_N$-simple space. Moreover, we determine (generic) conditions on the coefficients of the fifth order $\mathbf{S}_N \times \mathbf{S}^1$-equivariant

Date: January 16, 2009.

2000 *Mathematics Subject Classification.* 37G40, 37C23, 34C25.

Key words and phrases. Hopf bifurcation, periodic solution, spatio-temporal symmetry.

The second author was supported by FCT Grant SFRH/BD/18631/2004.

Research of APSD and AR supported in part by Centro de Matemática da Universidade do Porto (CMUP) financed by FCT through the programmes POCTI and POSI, with Portuguese and European Community structural funds.

vector field that describe the stability of the different types of bifurcating periodic solutions.

Consider the natural action of $\mathbf{S}_N$ on $\mathbf{C}^N$ where $\sigma \in \mathbf{S}_N$ acts by permutation of coordinates:

$$\sigma(z_1, \ldots, z_N) \;=\; \left(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(N)}\right) \tag{1}$$

where $(z_1, \ldots, z_N) \in \mathbf{C}^N$. Observe the following decomposition of $\mathbf{C}^N$ into invariant subspaces for this action:

$$\mathbf{C}^N \;\cong\; \mathbf{C}^{N,0} \oplus V_1$$

where

$$\mathbf{C}^{N,0} \;=\; \{(z_1, \ldots, z_N) \in \mathbf{C}^N : z_1 + \cdots + z_N = 0\}$$

and

$$V_1 \;=\; \{(z, \ldots, z) : \; z \in \mathbf{C}\} \;\cong\; \mathbf{C}.$$

The action of $\mathbf{S}_N$ on V_1 is trivial and the space $\mathbf{C}^{N,0}$ is $\mathbf{S}_N$-simple:

$$\mathbf{C}^{N,0} \;\cong\; \mathbf{R}^{N,0} \oplus \mathbf{R}^{N,0}$$

where $\mathbf{S}_N$ acts absolutely irreducibly on

$$\mathbf{R}^{N,0} \;=\; \{(x_1, \ldots, x_N) \in \mathbf{R}^N : x_1 + \cdots + x_N = 0\} \;\cong\; \mathbf{R}^{N-1}.$$

We say that a *representation* of a group Γ on a vector space V is *absolutely irreducible*, or the *space* V is said to be *absolutely irreducible*, if the only linear mappings on V that commute with Γ are the scalar multiples of the identity.

If we have a local Γ-equivariant Hopf bifurcation problem, generically the centre subspace at the Hopf bifurcation point is Γ-simple (see [16, Proposition XVI 1.4]). We make that assumption here. Thus we consider a general $\mathbf{S}_N$-equivariant system of ordinary differential equations (ODEs)

$$\frac{dz}{dt} = f(z, \lambda), \tag{2}$$

where $z \in \mathbf{C}^{N,0}, \lambda \in \mathbf{R}$ is the bifurcation parameter and $f : \mathbf{C}^{N,0} \times \mathbf{R} \to \mathbf{C}^{N,0}$ is smooth and commutes with the restriction of the natural action (1) of $\mathbf{S}_N$ on $\mathbf{C}^N$ to the $\mathbf{S}_N$-simple space $\mathbf{C}^{N,0}$. Observe that $f(0, \lambda) \equiv 0$ since $\mathrm{Fix}_{\mathbf{C}^{N,0}}(\mathbf{S}_N) = \{0\}$.

We study Hopf bifurcation of (2) from the trivial equilibrium, say, at $\lambda = 0$, and so we assume that $(df)_{0,0}$ has purely imaginary eigenvalues $\pm i$ (after rescaling time if necessary). Thus if we denote the eigenvalues of $(df)_{0,\lambda}$ by $\sigma(\lambda) \pm i\rho(\lambda)$ then $\sigma(0) = 0, \rho(0) = 1$ (see [16, Lemma XVI 1.5]) and we make the standard hypothesis of the Equivariant Hopf Theorem:

$$\sigma'(0) \neq 0.$$

Under the above hypothesis, we can assume that the action of $\mathbf{S}^1$ on the centre space $\mathbf{C}^{N,0}$ of $(df)_{0,0}$ (that can be identified with the exponential of $(df)_{0,0}$) is given by multiplication by $e^{i\theta}$:

$$\theta(z_1, \ldots, z_N) \;=\; e^{i\theta}\,(z_1, \ldots, z_N) \tag{3}$$

for $\theta \in \mathbf{S}^1, \; (z_1, \ldots, z_N) \in \mathbf{C}^{N,0}$.

Structure of the Paper. In Section 2 we recall the key points for Hopf bifurcation theory of symmetric systems. In Section 3 we describe the classification of the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ given by Stewart [22]. There are two types of $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$: $\Sigma^I_{q,p}$ and Σ^{II}_q (Theorem 3.1). We also obtain the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of each of these groups (Table 4). In Section 4 we use the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions with these symmetries of (2) by Hopf bifurcation from the trivial equilibrium at $\lambda = 0$ for a bifurcation problem with symmetry $\Gamma = \mathbf{S}_N$. The main result of this paper is Theorem 4.1 determining the directions of branching and the stability of periodic solutions guaranteed by the Equivariant Hopf Theorem. For solutions with symmetry Σ^{II}_q the terms of the degree three truncation of the vector field determine the criticality of the branches and also the stability of these solutions (near the origin). However, for solutions with symmetry $\Sigma^I_{p,q}$, although the criticality of the branches is determined by the terms of degree three, the stability of solutions in some directions is not. Moreover, in one particular direction, we show that even the degree five truncation is too degenerate. In Section 5 we present our results for the cases $N = 4$ and $N = 5$. While for $N = 4$ we only need the degree three truncation of the vector field in order to determine the branching equations and the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem, for $N = 5$ the degree five truncation is necessary (and sufficient). We determine explicitly the directions of branching and the stability of these solutions in both cases. Furthermore, we observe the (generic) existence of periodic solutions with submaximal symmetry for the $N = 4$ case. In Section 6 we prove the main result of this paper on the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem, Theorem 4.1. In Appendix A we state the main technical details for the computation of the fifth order truncation of the vector field equivariant under $\mathbf{S}_N \times \mathbf{S}^1$ defined on $\mathbf{C}^{N,0}$. Finally, in Appendix B we present the bifurcation diagrams for the periodic solutions with maximal isotropy for Hopf bifurcation with $\mathbf{S}_4 \times \mathbf{S}^1$-symmetry.

2. Background

In this section we review some key points related to Hopf bifurcation theory of symmetric systems. For the basics of equivariant bifurcation theory see, for example, Golubitsky *et al.*[16, Chapter XVI].

Consider a system of ODEs

$$\frac{dx}{dt} = f(x, \lambda), \qquad f(0,0) = 0, \tag{4}$$

where $x \in V$, $\lambda \in \mathbf{R}$ is the bifurcation parameter, $f : V \times \mathbf{R} \to V$ is a smooth ($\mathcal{C}^\infty$) mapping and $f(0,\lambda) \equiv 0$ for all $\lambda \in \mathbf{R}$. We say that (4) undergoes a *Hopf Bifurcation* at $\lambda = 0$ if $(df)_{0,0}$ has a pair of purely imaginary eigenvalues. Here, $(df)_{0,0}$ denotes the $n \times n$ Jacobian matrix of the derivatives of f with respect to the variables x_j, evaluated at $(x, \lambda) = (0, 0)$.

Suppose that Γ is a compact Lie group with a linear action on $V = \mathbf{R}^n$ and f *commutes* with Γ (or it is Γ-*equivariant*), this is, $f(\gamma x, \lambda) = \gamma f(x, \lambda)$ for all $\gamma \in \Gamma, x \in V, \lambda \in \mathbf{R}$.

We are interested in branches of periodic solutions of (4) occurring by Hopf bifurcation from the trivial solution $(x, \lambda) = (0, 0)$. We say that a representation V of Γ is Γ-*simple* if either $V \cong W \oplus W$, where W is absolutely irreducible for Γ, or V is irreducible, but not absolutely irreducible for Γ. Suppose that $(df)_{0,0}$ has purely imaginary eigenvalues $\pm i\omega$. Then, generically, the corresponding real generalized eigenspace of $(df)_{0,0}$ is Γ-simple (see [16, Proposition XVI 1.4]). Assuming these conditions and supposing that $\mathbf{R}^n$ is Γ-simple, after an equivariant change of coordinates and a rescaling of time if necessary, we can assume that $(df)_{0,0}$ has the form

$$(df)_{0,0} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} = J$$

where I_m is the $m \times m$ identity matrix and $m = n/2$. This is due to the fact that if we assume that $\mathbf{R}^n$ is Γ-simple, the mapping f is Γ-equivariant and $(df)_{0,0}$ has i as an eigenvalue, then the eigenvalues of $(df)_{0,\lambda}$ consist of a complex conjugate pair $\sigma(\lambda) \pm i\rho(\lambda)$, each with multiplicity m. Moreover, σ and ρ are smooth functions of λ and there is an invertible linear map $S : \mathbf{R}^n \to \mathbf{R}^n$, commuting with Γ, such that $(df)_{0,0} = SJS^{-1}$ (see [16, Lemma XVI 1.5]).

We define the *isotropy subgroup* of $x \in V$ in Γ as

$$\Sigma_x = \{\gamma \in \Gamma \ : \ \gamma x = x\} \subseteq \Gamma$$

and the *fixed-point space* of a subgroup $\Sigma \subseteq \Gamma$ is the subspace of V defined by

$$\mathrm{Fix}(\Sigma) = \{x \in V \ : \ \gamma x = x, \ \forall \, \gamma \in \Sigma\}.$$

For any Γ-equivariant mapping f and any subgroup $\Sigma \subseteq \Gamma$ we have

$$f(\mathrm{Fix}(\Sigma) \times \mathbf{R}) \subseteq \mathrm{Fix}(\Sigma).$$

Identify the circle $\mathbf{S}^1$ with $\mathbf{R}/2\pi\mathbf{Z}$ and suppose that $x(t)$ is a periodic solution of (4) in t of period 2π. A symmetry of $x(t)$ is an element $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ such that

$$\gamma x(t) = x(t - \theta).$$

The set of all symmetries of $x(t)$ forms a subgroup

$$\Sigma_{x(t)} = \{(\gamma, \theta) \in \Gamma \times \mathbf{S}^1 : \gamma x(t) = x(t - \theta)\}.$$

Take the natural action of $\Gamma \times \mathbf{S}^1$ on the space $\mathcal{C}_{2\pi}$ of 2π-periodic functions from $\mathbf{R}$ into V, given by

$$(\gamma, \theta) \cdot x(t) = \gamma x(t + \theta).$$

Thus, the action of Γ on $\mathcal{C}_{2\pi}$ is induced through its spatial action on V and $\mathbf{S}^1$ acts by phase shift.

This way, the initial definition of symmetry of the periodic solution $x(t)$ may be rewritten as

$$(\gamma, \theta) \cdot x(t) = x(t)$$

and with respect to this action, $\Sigma_{x(t)}$ is the isotropy subgroup of $x(t)$.

So if we assume (4) where f commutes with Γ and $(df)_{0,0} = L$ has purely imaginary eigenvalues, we can apply a Liapunov-Schmidt reduction, preserving symmetries, that will induce a different action of $\mathbf{S}^1$ on a finite-dimensional space, which can be identified with the exponential of $L|_{E_i}$ acting on the imaginary eigenspace E_i of L. The reduced function of f will commute with $\Gamma \times \mathbf{S}^1$ (see [16, Chapter XVI Section 3]).

Consider the system of ODEs (4), where $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is smooth and commutes with a compact Lie group Γ. Assume the generic hypothesis that $\mathbf{R}^n$ is Γ-simple and $(df)_{0,0}$ has i as eigenvalue. Thus, after a change of coordinates, we can assume that $(df)_{0,0} = J$, where $m = n/2$. The eigenvalues of $(df)_{0,\lambda}$ are $\sigma(\lambda) \pm i\rho(\lambda)$ each with multiplicity m. Therefore $\sigma(0) = 0$ and $\rho(0) = 1$. Furthermore, assume that $\sigma'(0) \neq 0$, that is, the eigenvalues of $(df)_{0,\lambda}$ cross the imaginary axis with nonzero speed. Let $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ be an isotropy subgroup such that $\dim\mathrm{Fix}(\Sigma) = 2$. Then by the Equivariant Hopf Theorem (see [16, Theorem XVI 4.1]) there exists a unique branch of small-amplitude periodic solutions to (4) with period near 2π, having Σ as their group of symmetries.

The basic idea in the Equivariant Hopf Theorem is that small amplitude periodic solutions of (4) of period near 2π correspond to zeros of a reduced equation $\varphi(x, \lambda, \tau) = 0$ where τ is the period-perturbing parameter. To find periodic solutions of (4) with symmetries Σ is equivalent to find zeros of the reduced equation with isotropy Σ and they correspond to the zeros of the reduced equation restricted to $\mathrm{Fix}(\Sigma)$.

The main tool for calculating the stabilities of the periodic solutions (including those guaranteed by the Equivariant Hopf Theorem) is to use a Birkhoff normal form of f: by a suitable coordinate change, up to any given order k, the vector field f can be made to commute with Γ and $\mathbf{S}^1$ (in the Hopf case). This result is the equivariant version of the Poincaré-Birkhoff normal form Theorem.

Suppose that the vector field f in (4) is in Birkhoff normal form. Then it is possible to perform a Liapunov-Schmidt reduction on (4) such that the reduced equation φ has the explicit form

$$\varphi(x, \lambda, \tau) = f(x, \lambda) - (1 + \tau)Jx,$$

where τ is the period-scaling parameter (see [16, Theorem XVI 10.1]). Let (x_0, λ_0, τ_0) be a solution to $\varphi = 0$ with isotropy Σ, and let $x(t)$ be the corresponding solution of (4). Then $x(t)$ is orbitally stable if the $n - d_\Sigma$ (where $d_\Sigma = \mathrm{dim}\Gamma + 1 - \mathrm{dim}\Sigma$) eigenvalues of $(d\varphi)_{x_0,\lambda_0,\tau_0}$ which are not forced to be zero by the group action have negative real parts (see [16, Corollary XVI 10.2]).

Thus, the assumptions of Birkhoff normal form implies that we can apply the standard Hopf Theorem to $\dot{x} = f(x, \lambda)$ restricted to $\mathrm{Fix}(\Sigma) \times \mathbf{R}$. In this case, exchange of stability happens, so that if the trivial steady-state solution is stable subcritically, then a subcritical branch of periodic solutions with isotropy subgroup Σ is unstable. Supercritical branches may be stable or unstable depending on the signs of the real part of the eigenvalues on the complement of $\mathrm{Fix}(\Sigma)$.

Call the system

$$\dot{y} = Ly + g_2(y) + \cdots + g_k(y)$$

the (kth order) *truncated Birkhoff normal form*. The dynamics of the truncated Birkhoff normal form are related to, but not identical with, the local dynamics of the system (4) around the equilibrium $x = 0$. On the other hand, in general, it is not possible to find a single change of coordinates that puts f into normal form for all orders. And if it is, then there is the problem of the first 'tail'.

When discussing the stability of the solutions found using the Equivariant Hopf Theorem we suppose that the kth order truncation of f commutes also with $\mathbf{S}^1$. Thus we are ignoring terms of higher order that do not commute necessarily with $\mathbf{S}^1$ and that can change the dynamics. However, we use a result (that we state below) that guarantees that the stability results for the periodic solutions hold even when f is of the form

$$\tilde{f}(x, \lambda) + o(\|x\|^k),$$

where $\tilde{f}$ commutes with $\Gamma \times \mathbf{S}^1$ but $o(\|x\|^k)$ commutes only with Γ, provided k is large enough (see [16, Theorem XVI 11.2]). Here we use $h(x) = o(\|x\|^k)$ to mean that $h(x)/\|x\|^k \to 0$ as $\|x\| \to 0$.

Before we state the result, we recall the definition of p-determined stability of an isotropy subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$. Suppose that dim $\mathrm{Fix}(\Sigma) = 2$. Following [16, Definition XVI 11.1], we say that Σ has *p-determined stability* if all eigenvalues of $(d\tilde{f})_{(x_0,\lambda_0)} - (1 + \tau_0)J$, other than those forced to zero by Σ, have the form

$$\mu_j = \alpha_j a^{m_j} + o(a^{m_j})$$

on a periodic solution $x(s)$ of

$$\dot{x} = \tilde{f}(x, \lambda) \tag{5}$$

such that $\|x(s)\| = a$, where α_j is a $\mathbf{C}$-valued function of the Taylor coefficients of terms of degree lower or equal p in $\tilde{f}$. We expect that the real parts of the α_j to be generically nonzero: these are the nondegeneracy conditions on the Taylor coefficients of $\tilde{f}$ at the origin that are obtained when computing stabilities along the branches. In this case, we say that $\tilde{f}$ is *nondegenerate* for Σ.

Suppose that the hypotheses of the Equivariant Hopf Theorem hold, and the isotropy subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ has p-determined stability. Let $k \geq p$ and assume that $f(x,\lambda) = \tilde{f}(x,\lambda) + o(\|x\|^k)$ where $\tilde{f}$ commutes with $\Gamma \times \mathbf{S}^1$ and is nondegenerate for Σ. Then for λ sufficiently near 0, the stabilities of a periodic solution of $\dot{x} = f(x,\lambda)$ with isotropy Σ are given by the same expressions in the coefficients of f as those that define the stability of a solution of the truncated Birkhoff normal form $\dot{x} = \tilde{f}(x,\lambda)$ with isotropy subgroup Σ (see [16, Theorem XVI 11.2]). As it has been said there always exists a polynomial change putting f in the form $\tilde{f}(x,\lambda) + o(\|x\|^k)$. Thus, if the p-determined stability condition holds, the stability analysis for f is completed.

3. C-Axial Subgroups of $\mathbf{S}_N \times \mathbf{S}^1$

Recalling (1) and (3), we consider the following action of $\mathbf{S}_N \times \mathbf{S}^1$ on $\mathbf{C}^{N,0}$:

$$(\sigma,\theta)(z_1,\ldots,z_N) \; = \; e^{i\theta}\left(z_{\sigma^{-1}(1)},\ldots,z_{\sigma^{-1}(N)}\right) \tag{6}$$

where $(\sigma,\theta) \in \mathbf{S}_N \times \mathbf{S}^1$ and $(z_1,\ldots,z_N) \in \mathbf{C}^{N,0}$. In order to apply the Equivariant Hopf Theorem we require information on the $\mathbf{C}$-axial isotropy subgroups (this is, on the isotropy subgroups with two-dimensional fixed-point subspace) of $\mathbf{S}_N \times \mathbf{S}^1$ for this action. In this section we recall the classification obtained by Stewart [22] of the $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. We use then the form of such isotropy subgroups to obtain the isotypic decomposition of $\mathbf{C}^{N,0}$ under the action of each of these groups. We recall that if $\Sigma \subseteq \mathbf{S}_N \times \mathbf{S}^1$, we can decompose $\mathbf{C}^{N,0}$ into isotypic components

$$\mathbf{C}^{N,0} = U_1 \oplus \cdots \oplus U_r \tag{7}$$

where each U_j is the isotypic component of type V_j for the action of Σ on $\mathbf{C}^{N,0}$. Here $V_1,\ldots,V_r$ are distinct Σ-irreducible spaces. Thus if W is a Γ-invariant subspace of $\mathbf{C}^{N,0}$ and Σ-isomorphic to V_j then $W \subseteq U_j$. These decompositions will play an important role at the calculation of the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem

3.1. C-Axial Subgroups of $\mathbf{S}_N \times \mathbf{S}^1$.

Isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ (acting on the $\mathbf{S}_N$-simple space $\mathbf{C}^{N,0}$) are of the type $H^\theta = \{(h,\theta(h)) : h \in H\}$ where $H \subseteq \mathbf{S}_N$ and $\theta: \; H \to \mathbf{S}^1$ is a group homomorphism (see [16, Definition XVI 7.1, Proposition XVI 7.2]). Also the $\mathbf{C}$-axial subgroups are maximal with respect to fixing a complex line $\mathbf{C}z = \{\mu z : \; \mu \in \mathbf{C}\}$, where $z \neq 0$. A vector z such that the isotropy subgroup Σ_z in $\mathbf{S}_N \times \mathbf{S}^1$ fixes only $\mathbf{C}z$ is called an *axis*. Stewart [22] computes the $\mathbf{C}$-axial isotropy subgroups, up to conjugacy, by describing the axes, that is, the orbit representatives.

Theorem 3.1 (**Stewart** [22])**.** *Suppose that $N \geq 2$. Then the axes of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ have orbit representatives as follows:*
Type I
Let $N = qk + p$ where $2 \leq k \leq N$, $q \geq 1$, $p \geq 0$. Let $\xi = e^{2\pi i/k}$ and set

$$z \; = \; \left(\underbrace{1,\ldots,1}_{q};\,\underbrace{\xi,\ldots,\xi}_{q};\,\underbrace{\xi^2,\ldots,\xi^2}_{q};\,\ldots;\,\underbrace{\xi^{k-1},\ldots,\xi^{k-1}}_{q};\,\underbrace{0,\ldots,0}_{p}\right). \tag{8}$$

Type II
Let $N = q + p$, $1 \leq q < N/2$ *and set*

$$z = \left(\underbrace{1,\ldots,1}_{q};\ \underbrace{a,\ldots,a}_{p}\right) \tag{9}$$

where $a = -q/p$.

Proof. See Stewart [22, Theorem 7]. □

Next we consider the corresponding isotropy subgroups as in [22]. For type I we have $\mathbf{C}$-axial subgroups $H^\theta = \Sigma_z$ where

$$\Sigma_z = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p \stackrel{\text{def}}{=} \Sigma^I_{q,p}. \tag{10}$$

Here $\wr$ denotes the wreath product (see Hall [18, p. 81]) and the tilde indicates that $\mathbf{Z}_k$ is twisted into $\mathbf{S}^1$. Let

$$K = \ker(\theta) = \mathbf{S}^1_q \times \cdots \times \mathbf{S}^k_q \times \mathbf{S}_p, \tag{11}$$

where $\mathbf{S}^j_q$ is the symmetric group on $B_j = \{(j-1)q+1, \ldots, jq\}$ and $\mathbf{S}_p$ is the symmetric group on $B_0 = \{kq+1, \ldots, N\}$. Now we have the action of the twist in each of the k blocks of q elements. Let $\alpha = (1,\ q+1,\ 2q+1,\ \ldots,\ (k-1)q+1)$. Then $\Sigma^I_{q,p}$ is generated by $(\alpha, 2\pi/k)$ and K.

For the type II, the isotropy subgroup is

$$\Sigma_z = S_q \times S_p \stackrel{\text{def}}{=} \Sigma^{II}_q \tag{12}$$

where the respective factors are the symmetric groups on $\{1, \ldots, q\}$ and $\{q+1, \ldots, N\}$.

Table 1 lists the $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ and the corresponding fixed-point subspaces.

Remark 3.2. In terms of all-to-all coupled nonlinear cells (systems of ordinary differential equations), one interpretation is that solutions with $\Sigma^I_{q,p}$-symmetry have k groups, each group comprising q cells with the same waveform and the same phase, and one group of p cells also with the same waveform and the same phase. Cells from the k groups oscillate identically except for phase shifts of jT/k, for $j = 0, 1, \ldots, k$ and T is the period, between each group. Cells in the group of p cells oscillate with kth the frequency of the cells of the other groups. Solutions with Σ^{II}_q correspond to two groups consisting of p and q cells. At each group, cells have the same waveform and same phase. Each group oscillates with a different wave form. ♢

Example 3.3. We apply Theorem 3.1 to the cases $N = 4$ and $N = 5$.
(i) For $N = 4$, note that the number of partitions of an element of $\mathbf{C}^{4,0}$ into k blocks of q equal elements each, plus a group of p null elements with $4 = kq+p$ (where $2 \leq k \leq 4$, $q \geq 1$ and $p \geq 0$) is four: thus we obtain, up to conjugacy, four isotropy subgroups of type I. Now if $4 = q + p$ where $1 \leq q < 2$ then $q = 1,\ p = 3$ and so we get, up to conjugacy, one isotropy subgroup of type II. See Table 2 for the $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$, the corresponding generators and the fixed-point subspaces. Observe that we have used Table 1 with the triplets (q, k, p) and the pairs (q, p) corresponding to each isotropy subgroup of a given form as follows:

$$\begin{array}{lll} \Sigma_1: \ q = k = 2,\ p = 0; & & \Sigma_2: \ q = 1,\ k = 2,\ p = 2; \\ \Sigma_3: \ q = 1,\ k = 3,\ p = 1; & & \Sigma_4: \ q = 1,\ k = 4,\ p = 0; \\ \Sigma_5: \ q = 1,\ p = 3\,. & & \end{array} \tag{13}$$

Isotropy Subgroup	Fixed-Point Subspace
$\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ $N = kq+p,\ 2 \leq k \leq N,$ $q \geq 1, p \geq 0$	$\left\{ \left(\underbrace{z_1, \ldots}_{q}; \underbrace{\xi z_1, \ldots}_{q}; \ldots; \underbrace{\xi^{k-1} z_1, \ldots}_{q}; \underbrace{0, \ldots, 0}_{p} \right) : z_1 \in \mathbf{C} \right\}$
$\Sigma^{II}_q = \mathbf{S}_q \times \mathbf{S}_p$ $N = q+p,\ 1 \leq q < \frac{N}{2}$	$\left\{ \left(\underbrace{z_1, \ldots, z_1}_{q}; \underbrace{-\frac{q}{p} z_1, \ldots, -\frac{q}{p} z_1}_{p} \right) : z_1 \in \mathbf{C} \right\}$

TABLE 1. $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ acting on $\mathbf{C}^{N,0}$ and fixed-point subspaces. Here $\xi = e^{2\pi i/k}$.

Isotropy Subgroup	Generators	Orbit Representative	Fixed-Point Subspace
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$	$((1423), \pi)$, $((13)(24), \pi)$	$(1, 1, -1, -1)$	$\{(z_1, z_1, -z_1, -z_1) : \ z_1 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$	(34), $((12), \pi)$	$(1, -1, 0, 0)$	$\{(z_1, -z_1, 0, 0) : \ z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3$	$\left((123), \frac{2\pi}{3}\right)$	$(1, \xi, \xi^2, 0)$	$\left\{(z_1, \xi z_1, \xi^2 z_1, 0) : \ z_1 \in \mathbf{C}\right\}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$	$\left((1234), \frac{\pi}{2}\right)$	$(1, i, -1, -i)$	$\{(z_1, iz_1, -z_1, -iz_1) : \ z_1 \in \mathbf{C}\}$
$\Sigma_5 = \mathbf{S}_3$	(23), (24)	$(1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$	$\left\{(z_1, -\frac{1}{3} z_1, -\frac{1}{3} z_1, -\frac{1}{3} z_1) : \ z_1 \in \mathbf{C}\right\}$

TABLE 2. $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ acting on $\mathbf{C}^{4,0}$, generators, orbit representatives and fixed-point subspaces. Here $\xi = e^{2\pi i/3}$.

(ii) For $N = 5$ we have five isotropy subgroups of type I and two of type II. See Table 3. Specifically, we have that $\Sigma_i, i = 1, \ldots, 5$ are of the form $\Sigma^I_{q,p}$ and Σ_6, Σ_7 are of the form Σ^{II}_q with:

$$
\begin{array}{llll}
\Sigma_1: & q = k = 2,\ p = 1; & \Sigma_2: & q = 1,\ k = 2,\ p = 3; \\
\Sigma_3: & q = 1,\ k = 3,\ p = 2; & \Sigma_4: & q = 1,\ k = 4,\ p = 1; \\
\Sigma_5: & q = 1,\ k = 5,\ p = 0; & \Sigma_6: & q = 2,\ p = 3; \\
\Sigma_7: & q = 1,\ p = 4\,. & &
\end{array} \tag{14}
$$

◊

Isotropy Subgroup	Generators	Fixed-Point Subspace
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$	(12), (34), $((13)(24), \pi)$	$\{(z_1, z_1, -z_1, -z_1, 0) : z_1 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_3$	(34), (35), $((12), \pi)$	$\{(z_1, -z_1, 0, 0, 0) : z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3 \times \mathbf{S}_2$	(45), $\left((123), \frac{2\pi}{3}\right)$	$\left\{(z_1, \xi z_1, \xi^2 z_1, 0, 0) : z_1 \in \mathbf{C}\right\}$, $\xi = e^{2\pi i/3}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$	$\left((1234), \frac{\pi}{2}\right)$	$\{(z_1, iz_1, -z_1, -iz_1, 0) : z_1 \in \mathbf{C}\}$
$\Sigma_5 = \widetilde{\mathbf{Z}}_5$	$\left((12345), \frac{2\pi}{5}\right)$	$\left\{(z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1) : z_1 \in \mathbf{C}\right\}$, $\xi = e^{2\pi i/5}$
$\Sigma_6 = \mathbf{S}_2 \times \mathbf{S}_3$	(12), (34), (35)	$\left\{(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1) : z_1 \in \mathbf{C}\right\}$
$\Sigma_7 = \mathbf{S}_4$	(23), (24), (25)	$\left\{(z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1) : z_1 \in \mathbf{C}\right\}$

TABLE 3. $\mathbf{C}$-axial isotropy subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$ acting on $\mathbf{C}^{5,0}$, generators and fixed-point subspaces.

3.2. **Isotypic decomposition of $\mathbf{C}^{N,0}$ under the action of the C-Axial Subgroups of $\mathbf{S}_N \times \mathbf{S}^1$.** For the two types of isotropy subgroups $\Sigma^I_{q,p}$ and Σ^{II}_q, we decompose $\mathbf{C}^{N,0}$ into subspaces, each of which is invariant under a different representation of the corresponding isotropy subgroup. The isotypic components for the action of $\Sigma^I_{q,p}$ and Σ^{II}_q on $\mathbf{C}^{N,0}$ are listed in Table 4.

Specifically, for $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ we form the isotypic decomposition

$$\mathbf{C}^{N,0} = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus \sum_{j=2}^{k-1} P_j \tag{15}$$

where $W_0 = \mathrm{Fix}(\Sigma^I_{q,p}), W_1$ and the $k-2$ subspaces $P_j, j = 2, \cdots, k-1$ are complex one-dimensional subspaces, invariant under $\Sigma^I_{q,p}$. Moreover, W_2 and W_3 are complex invariant subspaces of dimension respectively $p-1$ and $k(q-1)$.

Note that if $p = 0$ we have $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k}$ and then W_1, W_2 do not occur in the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$. Moreover, we only have the occurrence of W_2 in the isotypic decomposition if $p \geq 2$. Furthermore, we only have the isotypic component W_3 if $q \geq 2$ and P_j if $k \geq 3$.

For $\Sigma^{II}_q = \mathbf{S}_q \times \mathbf{S}_p$ we form the isotypic decomposition

$$\mathbf{C}^{N,0} = W_0 \oplus W_1 \oplus W_2 \tag{16}$$

where $W_0 = \mathrm{Fix}(\Sigma^{II}_q)$ and W_1, W_2 are complex invariant subspaces of dimension respectively $q-1$ and $p-1$ that are the sum of two isomorphic real absolutely irreducible representations of dimension respectively $q-1$ and $p-1$ of Σ^{II}_q.

Type of Isotropy Subgroup	Isotypic components
$\Sigma^I_{q,p}$	$W_0 = \{(\underbrace{z_1,\ldots,z_1}_{q};\underbrace{\xi z_1,\ldots,\xi z_1}_{q};\ldots;\underbrace{\xi^{k-1}z_1,\ldots,\xi^{k-1}z_1}_{q};\underbrace{0,\ldots,0}_{p}) : z_1 \in \mathbf{C}\}$
$(N = kq + p$ $2 \leq k \leq N$ $q \geq 1,\ p \geq 0)$	$W_1 = \{(\underbrace{z_1,\ldots,z_1}_{kq};\underbrace{-\frac{kq}{p}z_1,\ldots,-\frac{kq}{p}z_1}_{p}) : z_1 \in \mathbf{C}\}$ (if $p \geq 1$)
	$W_2 = \{(0,\ldots,0;\underbrace{z_1,\ldots,z_{p-1},-z_1-\cdots-z_{p-1}}_{p}) : z_1,\ldots,z_{p-1} \in \mathbf{C}\}$ (if $p \geq 2$)
	$W_3 = \{(\underbrace{z_1,\ldots,z_{q-1},z_q}_{q};\ldots;\underbrace{z_{q(k-1)+1},\ldots,z_{kq-1},z_{kq}}_{q};\underbrace{0,\ldots,0}_{p})\}$ (if $q \geq 2$)
	$P_j = \{(\underbrace{z_1,\ldots}_{q};\underbrace{\xi^j z_1,\ldots}_{q};\ldots;\underbrace{\xi^{j(k-1)}z_1,\ldots}_{q};\underbrace{0,\ldots,0}_{p}) : z_1 \in \mathbf{C}\}$ (if $k \geq 3$) for $j = 2,\ldots,k-1$
Σ^{II}_q $(N = q + p$ $1 \leq q < \frac{N}{2})$	$W_0 = \left\{\left(\underbrace{z_1,\ldots,z_1}_{q};\underbrace{-\frac{q}{p}z_1,\ldots,-\frac{q}{p}z_1}_{p}\right) : z_1 \in \mathbf{C}\right\}$
	$W_1 = \{(z_1,\ldots,z_{q-1},-z_1-\cdots-z_{q-1},0,\ldots,0) : z_1,\ldots,z_{q-1} \in \mathbf{C}\}$ (if $q \geq 2$)
	$W_2 = \{(0,\ldots,0,z_{q+1},\ldots,z_{N-1},-z_{q+1}-\cdots-z_{N-1}) : z_{q+1},\ldots,z_{N-1} \in \mathbf{C}\}$ (if $p \geq 2$)

TABLE 4. Isotypic components of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$ and Σ^{II}_q. Here, in W_3 we have $z_q = -z_1-\cdots-z_{q-1},\ldots,z_{kq} = -z_{q(k-1)+1}-\cdots-z_{kq-1}$ and $z_1,\ldots,z_{q-1},\ldots,z_{q(k-1)+1},\ldots,z_{kq-1} \in \mathbf{C}$.

Example 3.4. We return to the cases $N = 4$ and $N = 5$.
(i) Recalling Table 2, equation (13) and Table 4, for the three isotropy subgroups Σ_i, for $i = 2,3,4$, of $\mathbf{S}_4 \times \mathbf{S}^1$, the isotypic decomposition takes, respectively, the form

$$\mathbf{C}^{4,0} = W_0 \oplus W_1 \oplus W_2, \quad \mathbf{C}^{4,0} = W_0 \oplus W_1 \oplus P_2, \quad \mathbf{C}^{4,0} = W_0 \oplus P_2 \oplus P_3$$

where $W_0 = \mathrm{Fix}(\Sigma_i), W_1, W_2, P_2$ and P_3 are the complex one-dimensional isotypic components for the action of Σ_i on $\mathbf{C}^{4,0}$. For Σ_1 and Σ_5 we obtain that $\mathbf{C}^{4,0} = W_0 \oplus W_3$ and $\mathbf{C}^{4,0} = W_0 \oplus W_2$, where W_3, W_2 are complex two-dimensional invariant subspaces. See Table 5.
(ii) For $N = 5$, we recall Table 3 for the $\mathbf{C}$-axial subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$, equation (14) and Table 4. We have that Σ_i, for $i = 1,\ldots,5$ are of the form $\Sigma^I_{q,p}$ and Σ_6, Σ_7 are of the form Σ^{II}_q. We decompose $\mathbf{C}^{5,0}$ into isotypic components for the action of each isotropy subgroup Σ_i, see Table 6. ◊

Isotropy subgroup and Orbit Representative	Isotypic components of $\mathbf{C}^{4,0}$
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$ $z = (z_1, z_1, -z_1, -z_1)$	$W_0 = \mathrm{Fix}(\Sigma_1) = \{(z_1, z_1, -z_1, -z_1):\ z_1 \in \mathbf{C}\}$ $W_3 = \{(z_1, -z_1, z_2, -z_2):\ z_1, z_2 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$ $z = (z_1, -z_1, 0, 0)$	$W_0 = \mathrm{Fix}(\Sigma_2) = \{(z_1, -z_1, 0, 0) : z_1 \in \mathbf{C}\}$ $W_1 = \{(z_1, z_1, -z_1, -z_1):\ z_1 \in \mathbf{C}\}$ $W_2 = \{(0, 0, z_1, -z_1):\ z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3$ $z = (z_1, \xi z_1, \xi^2 z_1, 0)$	$W_0 = \mathrm{Fix}(\Sigma_3) = \left\{(z_1, \xi z_1, \xi^2 z_1, 0):\ z_1 \in \mathbf{C}\right\}$ $W_1 = \{(z_1, z_1, z_1, -3z_1):\ z_1 \in \mathbf{C}\}$ $P_2 = \left\{(z_1, \xi^2 z_1, \xi z_1, 0):\ z_1 \in \mathbf{C}\right\}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$ $z = (z_1, iz_1, -z_1, -iz_1)$	$W_0 = \mathrm{Fix}(\Sigma_4) = \{(z_1, iz_1, -z_1, -iz_1):\ z_1 \in \mathbf{C}\}$ $P_2 = \{(z_1, -z_1, z_1, -z_1):\ z_1 \in \mathbf{C}\}$ $P_3 = \{(z_1, -iz_1, -z_1, iz_1):\ z_1 \in \mathbf{C}\}$
$\Sigma_5 = \mathbf{S}_3$ $z = \left(z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1\right)$	$W_0 = \mathrm{Fix}(\Sigma_5) = \left\{\left(z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1, -\frac{1}{3}z_1\right):\ z_1 \in \mathbf{C}\right\}$ $W_2 = \{(0, z_2, z_3, -z_2 - z_3):\ z_2, z_3, \in \mathbf{C}\}$

TABLE 5. Isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of each of the isotropy subgroups listed in Table 2. Here $\xi = e^{2\pi i/3}$.

Remark 3.5. Ashwin and Swift [4] present a framework for analysing arbitrary networks of identical dissipative oscillators assuming weak coupling. When every oscillator is connected to every other one with equal coupling, the network has $\mathbf{S}_N$-symmetry. Assume a $\mathbf{S}_N$-symmetric network of N identical oscillators, each having an asymptotically stable limit cycle, with coupling parameter ϵ. For $\epsilon = 0$ the dynamics of the system reduces to a linear flow on an N-torus $\mathbf{T}^N$ which is normally hyperbolic, and hence persists for $\epsilon \neq 0$. For weak coupling the equations can be averaged, leading to a $\mathbf{S}_N \times \mathbf{T}^1$-equivariant flow on the torus $\mathbf{T}^N$. In [4], the authors classify the possible spatio-temporal symmetry groups of any periodic oscillation by computing all isotropy subgroups for the action of $\mathbf{S}_N \times \mathbf{T}^1$ on $\mathbf{T}^N$. These groups (up to conjugacy) are in one-to-one correspondence with the ways of writing $N = m(k_1 + \cdots + k_l)$ with integers $m \geq 1$ and $k_1 \geq k_2 \geq \cdots \geq k_l \geq 1$. They also compute the stability and predict the generic bifurcations of some of the periodic orbits (which can have submaximal symmetry). Now the action of $\mathbf{S}_N \times \mathbf{T}^1$ on $\mathbf{T}^N$ can be seen as a linear action of $\mathbf{S}_N \times \mathbf{T}^1$ on $\mathbf{C}^N$, restricted to $\mathbf{T}^N$. It follows then that some of the groups described by [4], not necessarily maximal, are in correspondence with the maximal groups $\Sigma^I_{q,0}$ (for $k_1 = q,\ l = 1,\ N = mq$) and Σ^{II}_q (for $m = 1,\ k_1 = q,\ k_2 = p = N - q$) of $\mathbf{S}_N \times \mathbf{S}^1$ (action on $\mathbf{C}^{N,0}$) obtained by Stewart [22]. Moreover, for those groups, the description of the stability patterns that is made by [4] using the symmetry properties is equivalent to the one we obtain considering the decomposition of the space into isotypic components. In future work we plan to look for solutions with submaximal symmetry, where we hope that some of work done in [22] can be used. ◊

Isotropy subgroup and Orbit Representative	Isotypic components of $\mathbf{C}^{5,0}$
$\Sigma_1 = \widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}$ $z = (z_1, z_1, -z_1, -z_1, 0)$	$W_0 = \mathrm{Fix}(\Sigma_1) = \{(z_1, z_1, -z_1, -z_1, 0) : z_1 \in \mathbf{C}\}$ $W_1 = \{(z_1, z_1, z_1, z_1, -4z_1) : z_1 \in \mathbf{C}\}$ $W_3 = \{(z_1, -z_1, z_2, -z_2, 0) : z_1, z_2 \in \mathbf{C}\}$
$\Sigma_2 = \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_3$ $z = (z_1, -z_1, 0, 0, 0)$	$W_0 = \mathrm{Fix}(\Sigma_2) = \{(z_1, -z_1, 0, 0, 0) : z_1 \in \mathbf{C}\}$ $W_1 = \left\{(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1) : z_1 \in \mathbf{C}\right\}$ $W_2 = \{(0, 0, z_1, z_2, -z_1 - z_2) : z_1 \in \mathbf{C}\}$
$\Sigma_3 = \widetilde{\mathbf{Z}}_3 \times \mathbf{S}_2$ $z = (z_1, \xi z_1, \xi^2 z_1, 0, 0)$ $\xi = e^{2\pi i/3}$	$W_0 = \mathrm{Fix}(\Sigma_3) = \left\{(z_1, \xi z_1, \xi^2 z_1, 0, 0) : z_1 \in \mathbf{C}\right\}$ $W_1 = \left\{(z_1, z_1, z_1, -\frac{3}{2}z_1, -\frac{3}{2}z_1) : z_1 \in \mathbf{C}\right\}$ $W_2 = \{(0, 0, 0, z_1, -z_1) : z_1 \in \mathbf{C}\}$ $P_2 = \left\{(z_1, \xi^2 z_1, \xi^4 z_1, 0, 0) : z_1 \in \mathbf{C}\right\}$
$\Sigma_4 = \widetilde{\mathbf{Z}}_4$ $z = (z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, 0)$ $\xi = i$	$W_0 = \mathrm{Fix}(\Sigma_4) = \left\{(z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, 0) : z_1 \in \mathbf{C}\right\}$ $W_1 = \{(z_1, z_1, z_1, z_1, -4z_1) : z_1 \in \mathbf{C}\}$ $P_2 = \left\{(z_1, \xi^2 z_1, \xi^4 z_1, \xi^6 z_1, 0) : z_1 \in \mathbf{C}\right\}$ $P_3 = \left\{(z_1, \xi^3 z_1, \xi^6 z_1, \xi^9 z_1, 0) : z_1 \in \mathbf{C}\right\}$
$\Sigma_5 = \widetilde{\mathbf{Z}}_5$ $z = (z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1)$ $\xi = e^{2\pi i/5}$	$W_0 = \mathrm{Fix}(\Sigma_5) = \left\{(z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1) : z_1 \in \mathbf{C}\right\}$ $P_2 = \left\{(z_1, \xi^2 z_1, \xi^4 z_1, \xi^6 z_1, \xi^8 z_1) : z_1 \in \mathbf{C}\right\}$ $P_3 = \left\{(z_1, \xi^3 z_1, \xi^6 z_1, \xi^9 z_1, \xi^{12} z_1) : z_1 \in \mathbf{C}\right\}$ $P_4 = \left\{(z_1, \xi^4 z_1, \xi^8 z_1, \xi^{12} z_1, \xi^{16} z_1) : z_1 \in \mathbf{C}\right\}$
$\Sigma_6 = \mathbf{S}_2 \times \mathbf{S}_3$ $z = \left(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1\right)$	$W_0 = \mathrm{Fix}(\Sigma_6) = \left\{\left(z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1\right) : z_1 \in \mathbf{C}\right\}$ $W_1 = \{(z_1, -z_1, 0, 0, 0) : z_1, z_2, \in \mathbf{C}\}$ $W_2 = \{(0, 0, z_1, z_2, -z_1 - z_2) : z_1, z_2, \in \mathbf{C}\}$
$\Sigma_7 = \mathbf{S}_4$ $z = \left(z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1\right)$	$W_0 = \mathrm{Fix}(\Sigma_7) = \left\{\left(z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1\right) : z_1 \in \mathbf{C}\right\}$ $W_2 = \{(0, z_2, z_3, z_4, -z_2 - z_3 - z_4) : z_2, z_3, z_4, \in \mathbf{C}\}$

TABLE 6. Isotypic decomposition of $\mathbf{C}^{5,0}$ for the action of each of the isotropy subgroups listed in Table 3.

4. Periodic Solutions with Maximal Isotropy

Consider the system of ODEs

$$\frac{dz}{dt} = f(z, \lambda), \tag{17}$$

where $f : \mathbf{C}^{N,0} \times \mathbf{R} \to \mathbf{C}^{N,0}$ is smooth, commutes with $\Gamma = \mathbf{S}_N$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0, \rho(0) = 1$ and $\sigma'(0) \neq 0$.

Our aim is to study periodic solutions of (17) obtained by Hopf bifurcation from the trivial equilibrium. Note that we are assuming that f satisfies the conditions of the Equivariant Hopf Theorem.

By Theorem 3.1 we have (up to conjugacy) the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$. See Table 1. Therefore, we can use the Equivariant Hopf Theorem to prove the existence of periodic solutions with these symmetries for the bifurcation problem (17) with symmetry $\Gamma = \mathbf{S}_N$.

As stated in Section 2, periodic solutions of (17) of period $2\pi/(1+\tau)$ are in one-to-one correspondence with the zeros of $g(z,\lambda,\tau)$, the reduced function obtained by the Lyapunov-Schmidt procedure where τ is the period-perturbing parameter. Assuming that f commutes with $\Gamma \times \mathbf{S}^1$, $g(z,\lambda,\tau)$ has the explicit form

$$g(z,\lambda,\tau) \;=\; f(z,\lambda) - (1+\tau)iz. \tag{18}$$

(see [16, Theorem XVI 10.1]). Throughout denote by $\nu(\lambda) = \mu(\lambda) - (1+\tau)i$. If $z(t)$ is a periodic solution of (17) with $\lambda = \lambda_0$ and $\tau = \tau_0$, and (z_0, λ_0, τ_0) is the corresponding solution of (18), then there is a correspondence between the Floquet multipliers of $z(t)$ and the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ such that a multiplier lies inside (respectively outside) the unit circle if and only if the corresponding eigenvalue has negative (respectively positive) real part (see [16, Corollary XVI 10.2]). So, we determine the orbital stability of each type of bifurcating periodic orbit by calculating the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ (to the lowest order in z) that are not forced by the symmetry to be zero.

As g commutes with $\Gamma \times \mathbf{S}^1$, it maps $\mathrm{Fix}(\Sigma)$ into itself (where Σ is either of type $\Sigma^I_{q,p}$ or Σ^{II}_q described in Table 1). By the Equivariant Hopf Theorem, for each of the conjugacy classes $\Sigma^I_{q,p}$ and Σ^{II}_q, we have a distinct branch of periodic solutions of (17) that are in correspondence with the zeros of g with isotropy $\Sigma^I_{q,p}$ and Σ^{II}_q. These zeros are found by solving $g|_{\mathrm{Fix}(\Sigma^I_{q,p})} = 0$ and $g|_{\mathrm{Fix}(\Sigma^{II}_q)} = 0$ (and $\mathrm{Fix}(\Sigma^I_{q,p}), \mathrm{Fix}(\Sigma^{II}_q)$ are two-dimensional). Note that to find the zeros of g, it suffices to look at representative points on $\Gamma \times \mathbf{S}^1$ orbits.

Let $\Sigma_{z_0} \subset \Gamma$ be the isotropy subgroup of z_0. Then, for $\sigma \in \Sigma_{z_0}$ we have

$$(dg)_{z_0}\sigma = \sigma(dg)_{z_0}.$$

That is, $(dg)_{z_0}$ commutes with the isotropy subgroup Σ of z_0. For the two types of isotropy subgroups $\Sigma^I_{q,p}$ and Σ^{II}_q, it is possible to put the Jacobian matrix $(dg)_{z_0}$ into block diagonal form. We do this using the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of each $\mathbf{C}$-axial $\Sigma^I_{q,p}$ and Σ^{II}_q on $\mathbf{C}^{N,0}$ obtained in Section 3.2 and listed in Table 4. We recall that each isotypic component is invariant under a different representation of the corresponding isotropy subgroup Σ_{z_0} and so it is left invariant by $(dg)_{z_0}$. (See for example [16, Theorem XII 3.5].)

We organize the rest of this section in the following way. Below we present the general form of a $\mathbf{S}_N \times \mathbf{S}^1$-equivariant bifurcation problem, up to degree 5. Using that we obtain the form of g in (18). Then we describe the branching equations and the stability of the periodic solutions of (17) obtained by Hopf bifurcation from the trivial equilibrium guaranteed by the Equivariant Hopf Theorem – Theorem 4.1.

General Form of a $\mathbf{S}_N \times \mathbf{S}^1$-equivariant Hopf Bifurcation Problem. We present the general form of the $\mathbf{S}_N \times \mathbf{S}^1$-equivariant bifurcation problem (17), up to degree 5, leaving the details to Appendix A. Our stability results stated in Theorem 4.1 below show that the degree 5 terms of f in (17) are necessary to describe the stability of some of the periodic solutions guaranteed by the Equivariant Hopf Theorem.

If we suppose that the Taylor series of degree five of f around $z = 0$ commutes also with $\mathbf{S}^1$, then we can write

$$\begin{array}{rcl} f(z) & = & (f_1(z), f_1((12)z), \ldots, f_1((1N)z)) \\ & = & (f_1(z_1,\ldots,z_N,\lambda), f_1(z_2,z_1,\ldots,z_N,\lambda), \ldots, f_1(z_N,z_2,\ldots,z_1,\lambda)) \end{array} \tag{19}$$

where

$$f_1(z_1,\ldots,z_N,\lambda) = \mu(\lambda)z_1 + f_1^{(3)}(z_1,\ldots,z_N,\lambda) + f_1^{(5)}(z_1,\ldots,z_N,\lambda) + \cdots$$

and

$$\begin{array}{rcl} f_1^{(3)}(z_1,\ldots,z_N,\lambda) & = & A_1\left[|z_1|^2z_1 - \frac{1}{N}\sum_{k=1}^N |z_k|^2 z_k\right] + \\ & & A_2\,\overline{z}_1\sum_{k=1}^N z_k^2 \;+\; A_3\,z_1\sum_{k=1}^N |z_k|^2 \end{array}$$

$$\begin{array}{rcl} f_1^{(5)}(z_1,\ldots,z_N,\lambda) & = & A_4\left[|z_1|^4z_1 - \frac{1}{N}\sum_{k=1}^N |z_k|^4 z_k\right] + A_5 z_1\sum_{i=1}^N |z_i|^4 + \\ & & A_6 z_1\sum_{i=1}^N z_i^2\sum_{j=1}^N \overline{z}_j^2 + A_7 z_1\sum_{i=1}^N |z_i|^2\sum_{j=1}^N |z_j|^2 + \\ & & A_8\left[z_1^2\sum_{j=1}^N |z_j|^2\overline{z}_j - \frac{1}{N}\sum_{i=1}^N z_i^2\sum_{j=1}^N |z_j|^2\overline{z}_j\right] + \\ & & A_9\left[z_1^3\sum_{j=1}^N \overline{z}_j^2 - \frac{1}{N}\sum_{k=1}^N z_k^3\sum_{j=1}^N \overline{z}_j^2\right] + \\ & & A_{10}\left[\overline{z}_1\sum_{i=1}^N |z_i^2|\sum_{j=1}^N z_j^2\right] \;+\; A_{11}\overline{z}_1\sum_{i=1}^N |z_i^2| z_i^2 + \\ & & A_{12}\left[\overline{z}_1^2\sum_{j=1}^N z_j^3 - \frac{1}{N}\sum_{i=1}^N \overline{z}_i^2\sum_{j=1}^N z_j^3\right] + \\ & & A_{13}\left[|z_1|^2\sum_{k=1}^N |z_k|^2 z_k - \frac{1}{N}\sum_{i=1}^N |z_i|^2\sum_{j=1}^N |z_j|^2 z_j\right] + \\ & & A_{14}\left[|z_1|^2 z_1\sum_{k=1}^N |z_k|^2 - \frac{1}{N}\sum_{i=1}^N |z_i|^2 z_i\sum_{j=1}^N |z_j|^2\right] + \\ & & A_{15}\left[|z_1|^2\overline{z}_1\sum_{k=1}^N z_k^2 - \frac{1}{N}\sum_{i=1}^N |z_i|^2\overline{z}_i\sum_{j=1}^N z_j^2\right] \end{array}$$

with $z_N = -z_1 - \cdots - z_{N-1}$. The coefficients A_i, for $i = 1,\ldots,15$ are complex smooth functions of λ, $\mu(0) = i$ and $\mathrm{Re}(\mu'(0)) \neq 0$. Suppose that $\mathrm{Re}(\mu'(0)) > 0$. Rescaling λ if necessary we can suppose that

$$\mathrm{Re}(\mu(\lambda)) = \lambda + \cdots$$

where $+\cdots$ stands for higher order terms in λ. Thus the trivial solution of (17) is stable for λ negative and unstable for λ positive (near zero).

Throughout, subscripts r and i on the coefficients $A_1,\ldots,A_{15}$ refer to real and imaginary parts.

Stability Result. The main result of this paper is the following theorem which we prove in Section 6:

Theorem 4.1. *Consider the system (17) where f is as in (19) and $N \geq 4$. Assume that* $\mathrm{Re}(\mu'(0)) > 0$*, such that the trivial equilibrium is stable if $\lambda < 0$ and it is unstable if $\lambda > 0$ (near the origin). For each type of the isotropy subgroups of the form $\Sigma_{q,p}^I$ and Σ_q^{II} listed in Table 1, consider the corresponding isotypic decomposition of $\mathbf{C}^{N,0}$ presented in Table 4. Let $\Delta_0,\ldots,\Delta_r$ be the functions of $A_1,\ldots,A_{15}$ evaluated at $\lambda = 0$ and listed in: Table 8; Table 9 if the isotropy subgroup has type $\Sigma_{p,q}^I$ for $k = 2$, $k = 3$ or Σ_q^{II}; Table 10 if the isotropy subgroup has type $\Sigma_{p,q}^I$, for $3 < k \leq N$. Then:*

1. *(1) For each Σ_i the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$. Table 7 lists the branching equations.*
2. *(2) For each Σ_i, if $\Delta_j > 0$ for some $j = 0,\ldots,r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all j, then the branch of periodic solutions is stable near $\lambda = 0$ and $z = 0$.*

The application of this result to study Hopf bifurcation with $\mathbf{S}_N$-symmetry for a specific value of N consists mainly into the following steps:

- (i) To consider the general form of the Hopf bifurcation problem with $\mathbf{S}_N \times \mathbf{S}^1$-symmetry (17) where f is given by (19).
- (ii) To enumerate the $\mathbf{C}$-axial subgroups of $\mathbf{S}_N \times \mathbf{S}^1$ using Table 1.

Isotropy Subgroup	Branching Equations
$\Sigma^I_{q,p},\ 2 < k \leq N$ $N = kq + p,$ $q \geq 1, p \geq 0$	$\lambda \;=\; -\,(A_{1r} + kqA_{3r})\lvert z\rvert^2 \;+\; \cdots$
$\Sigma^I_{q,p}\ ,\ k = 2$ $N = 2q + p,$ $q \geq 1, p \geq 0$	$\lambda \;=\; -\,[A_{1r} + 2q(A_{2r} + A_{3r})]\lvert z\rvert^2 \;+\; \cdots$
Σ^{II}_q $N = q + p,$ $1 \leq q < \frac{N}{2}$	$\lambda \;=\; -\,A_{1r}\left[1 - \frac{q}{N}\left(1 - \frac{q^2}{p^2}\right)\right]\lvert z\rvert^2 -$ $(A_{2r} + A_{3r})q\left(1 + \frac{q}{p}\right)\lvert z\rvert^2 + \cdots$

TABLE 7. Branching equations for $\mathbf{S}_N$ Hopf bifurcation. Subscript r on the coefficients refer to the real part and $+\cdots$ stands for higher order terms.

Isotropy Subgroup	Δ_0
$\Sigma^I_{q,p},\ 2 < k \leq N$ $N = kq + p,\ q \geq 1, p \geq 0$	$A_{1r} + kqA_{3r}$
$\Sigma^I_{q,N-2q},\ k = 2$ $N = 2q + p,\ q \geq 1, p \geq 0$	$A_{1r} + 2q(A_{2r} + A_{3r})$
Σ^{II}_q $N = q + p,\ 1 \leq q < \frac{N}{2}$	$A_{1r}\left[1 - \frac{q}{N}\left(1 - \frac{q^2}{p^2}\right)\right] + (A_{2r} + A_{3r})q\left(1 + \frac{q}{p}\right)$

TABLE 8. Stability for $\mathbf{S}_N$ Hopf bifurcation in the direction of $W_0 = \mathrm{Fix}(\Sigma)$. For each group, the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$.

(iii) For each $\mathbf{C}$-axial subgroup Σ:
  (c.i) To describe the isotypic decomposition for its action on $\mathbf{C}^{N,0}$ according Table 4.
  (c.ii) Using Table 7, to obtain the equation of the branch of periodic solutions with Σ-symmetry for the Hopf bifurcation problem guaranteed by the Equivariant Hopf Theorem. The criticality of the branch is given by the sign of Δ_0 listed in Table 8: it is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$.
  (c.iii) To enumerate the functions of the coefficients $A_1, \ldots, A_{15}$ evaluated at $\lambda = 0$ of f that determine the stability of the corresponding periodic solutions (near the origin): using Table 8, Table 9 (if the isotropy subgroup has type $\Sigma^I_{p,q}$ for $k = 2,\ k = 3$ or Σ^{II}_q) and Table 10 (if the isotropy subgroup has type

$\Sigma^I_{p,q}$, for $3 < k \leq N$). Observe that in Tables 9 and 10, we consider only the expressions determining the stability in the directions of the isotypic components that occur in the isotypic decomposition of $\mathbf{C}^{N,0}$ under the action of Σ. If $\Delta_j < 0$ for all j, then the branch of periodic solutions is stable near $\lambda = 0$ and $z = 0$. If for some j we have $\Delta_j > 0$ then the branch of periodic solutions is unstable. In particular, if the branch is subcritical ($\Delta_j > 0$), then the solutions are unstable.

5. Two examples: $N = 4$ and $N = 5$

In this section we apply the results of sections 3 and 4 to study Hopf bifurcation with $\mathbf{S}_N$-symmetry for the special cases $N = 4$ and $N = 5$.

When $N = 4$, we show that the directions in which we need the fifth degree truncation of the vector field do not appear in the isotypic decomposition for the action of each $\mathbf{C}$-axial group on $\mathbf{C}^{4,0}$. This means this is the case (in fact the only one for values of $N \geq 4$) where the degree three truncation of the vector field determines generically the stability of the solutions guaranteed by the Equivariant Hopf Theorem.

When $N = 5$, we have that the directions in which we need the degree five truncation of the vector field are present in the isotypic decomposition for some of the $\mathbf{C}$-axial isotropy subgroups. Moreover, the degree five terms determine completely the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem.

5.1. **$\mathbf{S}_4$ Hopf Bifurcation.** We consider the action of $\mathbf{S}_4 \times \mathbf{S}^1$ on $\mathbf{C}^{4,0}$ given by (6) for $N = 4$. We study Hopf bifurcation with $\mathbf{S}_4$-symmetry by considering

$$\frac{dz}{dt} = f(z, \lambda), \tag{20}$$

where $f : \mathbf{C}^{4,0} \times \mathbf{R} \to \mathbf{C}^{4,0}$ is smooth, commutes with $\mathbf{S}_4$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0, \rho(0) = 1$ and $\sigma'(0) \neq 0$. Assuming that f also commutes with $\mathbf{S}^1$ we obtain the general form of f given by (19) with $N = 4$.

Example 3.3 describes the $\mathbf{C}$-axial subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$: we obtain four isotropy subgroups of the type $\Sigma^I_{q,p}$ and one of type Σ^{II}_q – see Table 2. Moreover, recall the correspondence between the notation of Table 1 and the $\mathbf{C}$-axial subgroups Σ_i of $\mathbf{S}_4 \times \mathbf{S}^1$ given by (13). Now in Table 5, we have the isotypic decomposition of $\mathbf{C}^{4,0}$ for the action of each of these isotropy subgroups.

The Equivariant Hopf Theorem guarantees that for each $\mathbf{C}$-axial group we have a branch of periodic solutions with that symmetry of (20) obtained by Hopf bifurcation from the trivial equilibrium (since we are assuming that f satisfies the conditions of the Equivariant Hopf Theorem). For each value of p, q and k (if applicable) associated with Σ_i, using Table 7, we get the branching equations listed on Table 11. We compute now from Tables 8, 9 and 10 the criticality and the stability of the solutions guaranteed by the Equivariant Hopf Theorem.

The expressions for Δ_0 for each isotropy subgroup follow from (13) and Table 8. We obtain now the expressions for $\Delta_1, \ldots, \Delta_r$. For the groups $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_5 in Table 5, we apply Table 9: note for example that Σ_1 is $\Sigma^I_{p,q}$ with $q = k = 2$ and $p = 0$. This corresponds to an isotropy subgroup $\Sigma^I_{p,q}$ with $k = 2$ on Table 9. Since $p = 0$ we get Δ_1, Δ_2 from the last two expressions associated with the stability in the directions of the isoypic component W_3. Now Σ_2 is $\Sigma^I_{p,q}$ with $q = 1,\ k = 2,\ p = 2$, the group Σ_3 is $\Sigma^I_{p,q}$ with $q = 1,\ k = 3,\ p = 1$, and Σ_5 is Σ^{II}_q with $q = 1$ and $p = 3$.

For the group Σ_4 in Table 5, which is $\Sigma^I_{p,q}$ with $q = 1,\ k = 4,\ p = 0$, we use now Table 10. Note that as $q = 1$, we have that W_3 does not occur in the isotypic decomposition of $\mathbf{C}^{4,0}$ for its action. See Table 12 for the complete stability analysis

Isotropy Subgroup	Isotypic Component	$\Delta_1, \ldots, \Delta_r$
$\Sigma^I_{p,q}, k=2$	W_1	$\begin{cases} \left(1-\frac{4q}{N}\right)A_{1r}-2qA_{2r}, \\ -\left\lvert\left(1-\frac{4q}{N}\right)A_1-2qA_2\right\rvert^2+\left\lvert\left(1-\frac{2q}{N}\right)A_1+2qA_2\right\rvert^2 \end{cases}$ (if $p \geq 1$)
$N=2q+p$		
$q \geq 1, p \geq 0$	W_2	$\begin{cases} -A_{1r}-2qA_{2r}, \\ -\left(\lvert A_1+2qA_2\rvert^2-\lvert 2qA_2\rvert^2\right) \end{cases}$ (if $p \geq 2$)
	W_3	$\begin{cases} A_{1r}-2qA_{2r}, \\ -\left(\lvert A_1-2qA_2\rvert^2-\lvert A_1+2qA_2\rvert^2\right) \end{cases}$ (if $q \geq 2$)
$\Sigma^I_{p,q},\ k=3$	W_1	$\begin{cases} \left(1-\frac{6q}{N}\right)A_{1r}, \\ -\lvert\left(1-\frac{6q}{N}\right)A_1\rvert^2+\lvert A_1\rvert^2 \end{cases}$ (if $p \geq 1$)
$N=3q+p$	W_2	$\begin{cases} A_{1r}, \\ -\lvert A_1\rvert^2 \end{cases}$ (if $p \geq 2$)
$q \geq 1, p \geq 0$	W_3	$-\left(-3q+\frac{6q}{N}\right)\mathrm{Re}(A_1\overline{A}_{12})$ (if $q \geq 2$)
	P_2	$\begin{cases} A_{1r}+6A_{2r} \\ -\left(\lvert A_1+6A_2\rvert^2-\lvert\left(1-\frac{3}{N}\right)A_1\rvert^2\right) \end{cases}$
Σ^{II}_q	W_1	$\begin{cases} \left(1+\frac{q}{N}-\frac{q^3}{Np^2}\right)A_{1r}-q\left(1+\frac{q}{p}\right)A_{2r} \\ -\left\lvert\left(1+\frac{q}{N}-\frac{q^3}{Np^2}\right)A_1-q\left(1+\frac{q}{p}\right)A_2\right\rvert^2+\left\lvert A_1+q\left(1+\frac{q}{p}\right)A_2\right\rvert^2 \end{cases}$ (if $q \geq 2$)
$N=q+p$		
$1 \leq q \leq \frac{N}{2}$	W_2	$\begin{cases} \left(-1+\frac{q}{N}-\frac{q^3}{Np^2}+\frac{2q^2}{p^2}\right)A_{1r}-q\left(1+\frac{1}{p}\right)A_{2r} \\ -\left\lvert\left(-1+\frac{q}{N}-\frac{q^3}{Np^2}+\frac{2q^2}{p^2}\right)A_1-q\left(1+\frac{1}{p}\right)A_2\right\rvert^2+\left\lvert\frac{q^2}{p^2}A_1+q\left(1+\frac{q}{p}\right)A_2\right\rvert^2 \end{cases}$ (if $p \geq 2$)

TABLE 9. Stability for $\mathbf{S}_N$ Hopf bifurcation in the directions of each isotypic component.

and Appendix B for the bifurcation diagrams (for the periodic solutions with **C**-axial symmetry).

Remark 5.1. (i) The stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem obtained in Table 12 depends only on the coefficients A_1, A_2, A_3 of the degree three truncation of the vector field f given by (19) for $N=4$.

Isotropy Subgroup	Isotypic Component	$\Delta_1,\ldots,\Delta_r$	
$\Sigma^I_{p,q},\ 3<k\le N$, $N=kq+p$, $q\ge 1,\ p\ge 0$	W_1	$\begin{cases} \left(1-\frac{2kq}{N}\right)A_{1r}, \\ -\lvert\left(1-\frac{2kq}{N}\right)A_1\rvert^2+\lvert A_1\rvert^2 \end{cases}$	(if $p\ge 1$)
	W_2	$\begin{cases} A_{1r}, \\ -\lvert A_1\rvert^2 \end{cases}$	(if $p\ge 2$)
	W_3	the fifth degree truncation is too degenerate to determine the stability in the directions in W_3	(if $k\ge 4, q\ge 2$)
	P_2	$\begin{cases} A_{1r} \\ -\left(\lvert A_1\rvert^2-\lvert\left(1-\frac{kq}{N}\right)A_1\rvert^2\right) \end{cases}$	
	P_{k-1}	$\begin{cases} A_{1r}+2kqA_{2r}, \\ -(\lvert A_1+2kqA_2\rvert^2-\lvert A_1\rvert^2) \end{cases}$	(if $k\ge 4$)
	$P_j\ (j=3,\ldots,k-2)$	$-\mathrm{Re}(A_1\overline{\xi}_1)+\mathrm{Re}(2A_1\overline{A}_4+kqA_1\overline{A}_{14})$	(if $k\ge 5$)
		$-\mathrm{Re}(A_1\overline{\xi}_2)+\mathrm{Re}(2A_1\overline{A}_4+kqA_1\overline{A}_{14})$	(if $k\ge 6$)

TABLE 10. Stability for $\mathbf{S}_N$ Hopf bifurcation in the directions of each isotypic component. Here $\xi_1=2A_4+3kqA_{12}+q(kq-1)\left(2-\frac{2kq}{N}\right)A_{13}+kqA_{14}+q(kq-1)\left(1-\frac{2kq}{N}\right)A_{14}+2q(kq-1)A_{15}$ and $\xi_2=\xi_1-3kqA_{12}-kqA_{14}$.

(ii) Observe that periodic solutions with Σ_3-symmetry are always unstable since generically $\Delta_2=|A_1|^2>0$. If $A_{1r}>0$, then solutions with symmetry Σ_4 are unstable and if $A_{2r}<0$, then solutions with symmetry Σ_2 are also unstable. ◊

Periodic solutions with submaximal isotropy. We look now for possible branches of periodic solutions that can bifurcate for the system (20) (with $N=4$) with submaximal isotropy.

We have that the groups $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_2$ listed in Table 13 are submaximal isotropy subgroups of $\mathbf{S}_4\times\mathbf{S}^1$. In fact, using the results of Ashwin and Podvigina [3] (see remark below), these are the only isotropy subgroups of $\mathbf{S}_4\times\mathbf{S}^1$ with fixed-point subspace of complex dimension 2

As it was stated before, when f is supposed to commute also with $\mathbf{S}^1$, then the problem of finding periodic solutions of $\dot z=f(z,\lambda)$ can be transformed to the problem of finding the zeros of $g(z,\lambda,\tau)=0$ where $g=f-(1+\tau)iz$. However, for the branches of periodic solutions with submaximal isotropy that are found here, we can no longer guarantee that they exist for (20) if f commutes only with $\mathbf{S}_4$ (even with the third order Taylor series commuting with $\mathbf{S}^1$). These solutions branches are guaranteed only for the third order truncation with which we work from now on. Consider the truncation of f as in (19)

Isotropy Subgroup	Branching Equations
Σ_1	$\lambda = -\left(A_{1r} + 4A_{2r} + 4A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_2	$\lambda = -\left(A_{1r} + 2A_{2r} + 2A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_3	$\lambda = -\left(A_{1r} + 3A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_4	$\lambda = -\left(A_{1r} + 4A_{3r}\right)\lvert z\rvert^2 + \cdots$
Σ_5	$\lambda = -\frac{1}{3}\left(\frac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r}\right)\lvert z\rvert^2 + \cdots$

Table 11. Branching equations for $\mathbf{S}_4$ Hopf bifurcation. Subscript r on the coefficients refer to the real part and $+\cdots$ stands for higher order terms.

(with $N = 4$) of degree three and the respective reduced vector field $g = f - (1+\tau)iz$ of the same degree.

Let $\Delta = \widetilde{\mathbf{Z}}_2$. We study $g|_{\mathrm{Fix}(\Delta)}$. Consider the normalizer of Δ in $\mathbf{S}_4 \times \mathbf{S}^1$, $N_{\mathbf{S}_4\times\mathbf{S}^1}(\Delta)$, defined by

$$N_{\mathbf{S}_4\times\mathbf{S}^1}(\Delta) = \{\gamma \in \mathbf{S}_4 \times \mathbf{S}^1 : \gamma\Delta\gamma^{-1} = \Delta\}\,.$$

Now $N_{\mathbf{S}_4\times\mathbf{S}^1}(\Delta)$ is the largest subgroup of $\mathbf{S}_4 \times \mathbf{S}^1$ acting on Fix(Δ). (See for example Chossat and Lauterbach [5, Lemma 2.1.9].) Thus $g|_{\mathrm{Fix}(\Delta)}$ is $N_{\mathbf{S}_4\times\mathbf{S}^1}(\Delta)$-equivariant. Easy computations show that

$$N_{\mathbf{S}_4\times\mathbf{S}^1}(\Delta) \cong \mathbf{D}_4 \times \mathbf{S}^1\,.$$

One way of proving this is the following. We recall that the isotropy subgroups $\Delta \subseteq \Gamma \times \mathbf{S}^1$ are always of the form $G^\theta = \{(g, \theta(g)) \in \Gamma \times \mathbf{S}^1 : g \in G\}$ where $G \subseteq \Gamma$ and $\theta : G \to \mathbf{S}^1$ is a group homomorphism. Denote by $K = \mathrm{K}er(\theta)$. Now by Golubitsky and Stewart [15, Lemma 2.5], we have that $N_{\Gamma\times\mathbf{S}^1}(G^\theta) = C(G,K) \times \mathbf{S}^1$ where $C(G,K) = \{\gamma \in \Gamma : \gamma g\gamma^{-1}g^{-1} \in K, \forall g \in G\}$. Taking $G^\theta = \Delta = \widetilde{\mathbf{Z}}_2 = \{Id, ((13)(24), \pi)\}$, the projection of G^θ into $\mathbf{S}_4$ is the group $G = \{Id, (13)(24)\}$, and θ is the homomorphism $\theta :\ G \to \mathbf{S}^1$ (with trivial kernel) such that $\theta((13)(24)) = \pi$. It follows then that for this case

$$C(G,K) = \{\mathrm{I}d, (24), (12)(34), (1432), (13)(24), (1234), (14)(23), (13)\} \cong \mathbf{D}_4\,.$$

When we restrict g to $\mathrm{Fix}(\widetilde{\mathbf{Z}}_2) = \{(z_1, z_2, -z_1, -z_2) :\ z_1, z_2 \in \mathbf{C}\}$, we obtain the following $\mathbf{D}_4 \times \mathbf{S}^1$-equivariant system:

$$\begin{aligned} \dot{z}_1 &= z_1\left(\lambda + i\omega + A(|z_1|^2 + |z_2|^2) + B|z_1|^2\right) + C\overline{z}_1 z_2^2 \\ \dot{z}_2 &= z_2\left(\lambda + i\omega + A(|z_1|^2 + |z_2|^2) + B|z_2|^2\right) + C\overline{z}_2 z_1^2 \end{aligned} \tag{21}$$

where $(z_1, z_2) \in \mathbf{C}^2$, $A = 2A_3, B = A_1 + 2A_2$ and $C = 2A_2$. This is the normal form for the generic Hopf bifurcation problem with symmetry $\mathbf{D}_4$ studied by Swift [23].

The nontrivial solutions in the space $\mathrm{Fix}(\widetilde{\mathbf{Z}}_2)$ with maximal isotropy are the solutions with symmetry $\widetilde{\mathbf{S}_2 \wr \mathbf{Z}_2}, \widetilde{\mathbf{Z}}_4, \widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$ (recall Table 2), corresponding, respectively, to zeros of type $z_1 = z_2$, $z_1 = iz_2$ and $z_1 = 0$. Note that for solutions corresponding to the

Isotropy Subgroup	Δ_0	$\Delta_1, \ldots, \Delta_r$
Σ_1	$A_{1r} + 4A_{2r} + 4A_{3r}$	$A_{1r} - 4A_{2r}$ $-\left(\lvert A_1 - 4A_2\rvert^2 - \lvert A_1 + 4A_2\rvert^2\right)$
Σ_2	$A_{1r} + 2A_{2r} + 2A_{3r}$	$-A_{1r} - 2A_{2r}$ $-\left(\lvert A_1 + 2A_2\rvert^2 - \lvert 2A_2\rvert^2\right)$ $-A_{2r}$ $-\left(4\lvert A_2\rvert^2 - \lvert\frac{1}{2}A_1 + 2A_2\rvert^2\right)$
Σ_3	$A_{1r} + 3A_{3r}$	$-A_{1r}$ $\lvert A_1\rvert^2$ $A_{1r} + 6A_{2r}$ $-\left(\lvert A_1 + 6A_2\rvert^2 - \lvert\frac{1}{4}A_1\rvert^2\right)$
Σ_4	$A_{1r} + 4A_{3r}$	A_{1r} $-\lvert A_1\rvert^2$ $A_{1r} + 8A_{2r}$ $-\left(\lvert A_1 + 8A_2\rvert^2 - \lvert A_1\rvert^2\right)$
Σ_5	$\frac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r}$	$-5A_{1r} - 12A_{2r}$ $-\left(\lvert 5A_1 + 12A_2\rvert^2 - \lvert A_1 + 12A_2\rvert^2\right)$

TABLE 12. Stability for $\mathbf{S}_4$ Hopf bifurcation. For each Σ_i, the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$. If $\Delta_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all j, then the solutions are stable near $\lambda = 0$ and $z = 0$. Note that solutions with Σ_3-symmetry are always unstable.

Isotropy Subgroup	Generators	Fixed-Point Subspace
$\Delta_1 = \widetilde{\mathbf{Z}}_2$	$((13)(24), \pi)$	$\{(z_1, z_2, -z_1, -z_2) : \ z_1, z_2 \in \mathbf{C}\}$
$\Delta_2 = \mathbf{S}_2$	(23)	$\{(z_1, z_2, z_2, -z_1 - 2z_2) : \ z_1, z_2 \in \mathbf{C}\}$

TABLE 13. Generators and fixed-point subspaces corresponding to the isotropy subgroups of $\mathbf{S}_4 \times \mathbf{S}^1$ with fixed-point subspaces of complex dimension two.

isotropy subgroup $\widetilde{\mathbf{Z}}_2 \times \mathbf{S}_2$ we have that $(z_1, 0, -z_1, 0)$ is conjugated to $(z_1, -z_1, 0, 0)$. Their stability properties are studied in [14], [16] and [23].

By [23], in addition to these periodic solutions, there can be a fourth branch of periodic solutions to (21) with $z_1 \neq z_2$ and $z_1 z_2 \neq 0$. Thus, these correspond to $\widetilde{\mathbf{Z}}_2$-symmetric solutions of (20) where f is as in (19) (with $N = 4$) truncated to the third order. Moreover, this solution branch exists if

$$\left|\mathrm{Re}[2(A_1 + 2A_2)\overline{A}_2]\right| < |2A_2|^2 < |A_1 + 2A_2|^2$$

and the solutions are generically unstable.

Remark 5.2. In [3], Ashwin and Podvigina considered Hopf bifurcation with the group $\mathbf{O}$ of rotational symmetries of the cube. The group $\mathbf{O}$ is isomorphic to $\mathbf{S}_4$ and it has two non-isomorphic real irreducible representations of dimension three. In [3] they consider the irreducible representation of $\mathbf{O}$ corresponding to rotational symmetries of a cube in $\mathbf{R}^3 = W$. When studying Hopf bifurcation, they take two copies of this irreducible representation. Specifically, they consider the action of $\mathbf{O} \times \mathbf{S}^1$ on $W \oplus W$ generated by:

$$\begin{array}{l} \rho_{111}(z_1, z_2, z_3) = (z_2, z_3, z_1) \\ \rho_{001}(z_1, z_2, z_3) = (z_2, -z_1, z_3) \\ \gamma_\theta(z_1, z_2, z_3) = e^{i\theta}(z_1, z_2, z_3) \ (\theta \in \mathbf{S}^1). \end{array} \tag{22}$$

Although the permutation group $\mathbf{S}_4$ is isomorphic to the group of rotations of a cube, the action of $\mathbf{O}$ on W and the natural action of $\mathbf{S}_4$ on $\mathbf{R}^{4,0}$ are not isomorphic. Recall that $\mathbf{R}^{4,0} = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$. However, the action of $\mathbf{O} \times \mathbf{S}^1$ on $W \oplus W$ and the action of $\mathbf{S}_4 \times \mathbf{S}^1$ on $\mathbf{C}^{4,0}$ are isomorphic (see [3]). Thus from the point of view of Hopf bifurcation, the two non-isomorphic actions of $\mathbf{S}_4$ give rise to the same results.

In [3], the isotropy lattice for the action of $\mathbf{O} \times \mathbf{S}^1$ on $\mathbf{C}^3$ is obtained and the isotropy subgroups with fixed-point subspaces of complex dimension two have normalizers given, respectively, by $\mathbf{D}_4 \times \mathbf{S}^1$ and $\mathbf{D}_2 \times \mathbf{S}^1$. These are in correspondence with the normalizers of $\widetilde{\mathbf{Z}}_2$ and $\mathbf{S}_2$ for the action of $\mathbf{S}_4 \times \mathbf{S}^1$ on $\mathbf{C}^{4,0}$ considered here.

$\diamondsuit$

5.2. **$\mathbf{S}_5$ Hopf Bifurcation.** We consider the action of $\mathbf{S}_5 \times \mathbf{S}^1$ on $\mathbf{C}^{5,0}$ given by (6) for $N = 5$. We study Hopf bifurcation with $\mathbf{S}_5$-symmetry by considering

$$\frac{dz}{dt} = f(z, \lambda), \tag{23}$$

where $f : \mathbf{C}^{5,0} \times \mathbf{R} \to \mathbf{C}^{5,0}$ is smooth, commutes with $\mathbf{S}_5$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0, \rho(0) = 1$ and $\sigma'(0) \neq 0$. Assuming that f also commutes with $\mathbf{S}^1$ we obtain the general form of f given by (19) with $N = 5$.

Example 3.3 describes the $\mathbf{C}$-axial subgroups of $\mathbf{S}_5 \times \mathbf{S}^1$ acting on $\mathbf{C}^{5,0}$, together with their generators and fixed-point subspaces, given by Table 3: we have used Table 1 where $\Sigma_i, i = 1, \ldots, 5$ are of the form $\Sigma^I_{q,p}$ and Σ_6, Σ_7 are of the form Σ^{II}_q. The correspondence between the notation of Table 1 and the $\mathbf{C}$-axial subgroups Σ_i of $\mathbf{S}_5 \times \mathbf{S}^1$ is given by (14). Table 6 gives the isotypic decomposition of $\mathbf{C}^{5,0}$ for each of the isotropy subgroups Σ_i listed in Table 3.

Proceeding the same way that we did for $N = 4$ we get Tables 14 and 15, which give respectively the branching equations and the stability for Hopf bifurcation with $\mathbf{S}_5$-symmetry. Note that in particular it follows that solutions with Σ_3 and with Σ_4-symmetry are always unstable.

Isotropy Subgroup	Branching Equations
Σ_1	$\lambda \;=\; -(A_{1r}+4A_{2r}+4A_{3r})\|z\|^2 \;+\; \cdots$
Σ_2	$\lambda \;=\; -(A_{1r}+2A_{2r}+2A_{3r})\|z\|^2 \;+\; \cdots$
Σ_3	$\lambda \;=\; -(A_{1r}+3A_{3r})\|z\|^2 \;+\; \cdots$
Σ_4	$\lambda \;=\; -(A_{1r}+4A_{3r})\|z\|^2 \;+\; \cdots$
Σ_5	$\lambda \;=\; -(A_{1r}+5A_{3r})\|z\|^2 \;+\; \cdots$
Σ_6	$\lambda \;=\; -\frac{1}{3}\left(\frac{7}{3}A_{1r}+10A_{2r}+10A_{3r}\right)\|z\|^2 \;+\; \cdots$
Σ_7	$\lambda \;=\; -\frac{1}{4}\left(\frac{13}{4}A_{1r}+5A_{2r}+5A_{3r}\right)\|z\|^2 \;+\; \cdots$

TABLE 14. Branching equations for $\mathbf{S}_5$ Hopf bifurcation. Subscript r on the coefficients refer to the real part and $+\cdots$ stands for higher order terms.

6. Proof of Theorem 4.1

In this section we prove the main result of this paper, Theorem 4.1. We consider the system (17) where f is as in (19). Assume that $\mathrm{Re}(\mu'(0)) > 0$, such that the trivial equilibrium is stable if $\lambda < 0$ and it is unstable if $\lambda > 0$ (near the origin). Thus f satisfies the conditions of the Equivariant Hopf Theorem. It follows that for each $\mathbf{C}$-axial subgroup Σ of $\mathbf{S}_N \times \mathbf{S}^1$ in Table 1, we have a branch of periodic solutions to (17) with that symmetry obtained by Hopf bifurcation from the trivial equilibrium. Moreover, since we are assuming that f in (19) commutes also with $\mathbf{S}^1$, as stated in Section 2, periodic solutions of (17) of period $2\pi/(1+\tau)$ are in one-to-one correspondence with the zeros of $g(z,\lambda,\tau)$, where

$$g(z,\lambda,\tau) \;=\; f(z,\lambda) - (1+\tau)iz. \tag{24}$$

is the explicit form of the reduced function obtained by the Lyapunov-Schmidt procedure. Here τ is the period-perturbing parameter. Throughout denote by $\nu(\lambda) = \mu(\lambda) - (1+\tau)i$.

In order to determine the stability of such solutions, we recall that there is a correspondence between the Floquet multipliers of $z(t)$ and the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$, if $z(t)$ is a periodic solution of (17) with $\lambda = \lambda_0$ and $\tau = \tau_0$, and (z_0, λ_0, τ_0) is the corresponding solution of (18): a multiplier lies inside (respectively outside) the unit circle if and only if the corresponding eigenvalue has negative (respectively positive) real part. So, we determine the stability of each type of bifurcating periodic orbit by calculating the eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ (to the lowest order in z). Let $\Sigma_{z_0} \subset \Gamma$ be the isotropy subgroup of z_0.

Now for each $\mathbf{C}$-axial subgroup Σ_{z_0} of $\mathbf{S}_N \times \mathbf{S}^1$ in Table 1, we have that g maps $\mathrm{Fix}(\Sigma_{z_0})$ into itself since g commutes with $\mathbf{S}_N \times \mathbf{S}^1$. Periodic solutions of (17) with Σ_{z_0}-symmetry are in correspondence with the zeros of g with isotropy Σ_{z_0} which are obtained by solving $g|_{\mathrm{Fix}(\Sigma_{z_0})} = 0$. Note that to find the zeros of g, it suffices to look at representative points on $\mathbf{S}_N \times \mathbf{S}^1$ orbits. We obtain Table 7.

Isotropy Subgroup	Δ_0	$\Delta_1, \ldots, \Delta_r$
Σ_1	$A_{1r} + 4A_{2r} + 4A_{3r}$	$-\frac{3}{5}A_{1r} - 4A_{2r}$ $-\left(\lvert -\frac{3}{5}A_1 - 4A_2\rvert^2 - \lvert\frac{1}{5}A_1 + 4A_2\rvert^2\right)$ $A_{1r} - 4A_{2r}$ $-\left(\lvert A_1 - 4A_2\rvert^2 - \lvert A_1 + 4A_2\rvert^2\right)$
Σ_2	$A_{1r} + 2A_{2r} + 2A_{3r}$	$\frac{1}{5}A_{1r} - 2A_{2r}$ $-\left(\lvert\frac{1}{5}A_1 - 2A_2\rvert^2 - \lvert\frac{3}{5}A_1 + 2A_2\rvert^2\right)$ $-A_{1r} - 2A_{2r}$ $-\left(\lvert A_1 + 2A_2\rvert^2 - \lvert 2A_2\rvert^2\right)$
Σ_3	$A_{1r} + 3A_{3r}$	$-A_{1r}$ $-\left(\lvert A_1 + 6A_2\rvert^2 - \lvert\frac{2}{5}A_1\rvert^2\right)$ $\lvert A_1\rvert^2$ $A_{1r} + 6A_{2r}$ A_{1r} $-\lvert A_1\rvert^2$
Σ_4	$A_{1r} + 4A_{3r}$	$-\lvert A_1\rvert^2$ $-A_{1r}$ A_{1r} $-\left(\lvert A_1 + 8A_2\rvert^2 - \lvert A_1\rvert^2\right)$ $A_{1r} + 8A_{2r}$
Σ_5	$A_{1r} + 5A_{3r}$	$-\lvert A_1\rvert^2$ A_{1r} $-\mathrm{Re}[A_1(\bar{\xi}_1 - \bar{\xi}_2)]$ $A_{1r} + 10A_{2r}$ $-\left(\lvert A_1 + 10A_2\rvert^2 - \lvert A_1\rvert^2\right)$
Σ_6	$\frac{7}{3}A_{1r} + 10A_{2r} + 10A_{3r}$	$\frac{11}{3}A_{1r} - 10A_{2r}$ $-\left(\lvert\frac{1}{3}\left(\frac{11}{3}A_1 - 10A_2\right)\rvert^2 - \lvert A_1 + \frac{10}{3}A_2\rvert^2\right)$ $\frac{1}{3}A_{1r} - 8A_{2r}$ $-\left(\lvert\frac{1}{3}\left(\frac{1}{3}A_1 - 8A_2\right)\rvert^2 - \lvert\frac{1}{3}\left(\frac{4}{3}A_1 + 10A_2\right)\rvert^2\right)$
Σ_7	$\frac{13}{4}A_{1r} + 5A_{2r} + 5A_{3r}$	$\mathrm{Re}(-\frac{55}{80}A_1 - \frac{5}{4}A_2)$ $-\left(\lvert -\frac{55}{80}A_1 - \frac{5}{4}A_2\rvert^2 - \lvert\frac{1}{16}A_1 + \frac{5}{4}A_2\rvert^2\right)$

TABLE 15. Stability for $\mathbf{S}_5$ Hopf bifurcation. Here $\xi_1 = 2A_4 + 10A_{14}$ and $\xi_2 = 2A_4 + 5A_{11} + 5A_{14}$. For each Σ_i, the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$. If $\Delta_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all j, then the solutions are stable near $\lambda = 0$ and $z = 0$. Note that solutions with Σ_3 and Σ_4 symmetry are always unstable.

The Jacobian $(dg)_{z_0}$ commutes with Σ_{z_0}. Now using the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of each $\mathbf{C}$-axial Σ_{z_0} obtained in Table 4, we have that each isotypic component is left invariant by $(dg)_{z_0}$. Thus it is possible to put the Jacobian matrix $(dg)_{z_0}$ into block diagonal. That is, the stability of the periodic solutions is determined by the restrictions of $(dg)_{z_0}$ to each of the isotypic components of $\mathbf{C}^{N,0}$ for the action of Σ_{z_0}.

Also note that as the group action forces some of the Floquet multipliers to be equal to one, it also forces the corresponding eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to be equal to zero. (Recall [16, Theorem XVI 6.2].) The eigenvectors associated with these eigenvalues are the tangent vectors to the orbit of $\mathbf{S}_N \times \mathbf{S}^1$ through z_0. If the solution z_0 has symmetry Σ_{z_0}, then the group orbit has the dimension of $(\Gamma \times \mathbf{S}^1)/\Sigma_{z_0}$ and so the number of zero eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}$ forced by the group action is

$$d_{\Sigma_{z_0}} = 1 - \dim(\Sigma_{z_0})$$

since $\dim(\mathbf{S}_N \times \mathbf{S}^1) = 1$. The groups $\Sigma_{z_0} = \Sigma^I_{q,p}$ and $\Sigma_{z_0} = \Sigma^{II}_q$ are discrete, then there is one eigenvalue forced by the symmetry to be zero (this is, we get $d_{\Sigma_{z_0}} = 1$).

To compute the eigenvalues it is convenient to use the complex coordinates. We take co-ordinate functions on $\mathbf{C}^N$: $z_1, \overline{z}_1, z_2, \overline{z}_2, \ldots, z_N, \overline{z}_N$. These correspond to a basis B for $\mathbf{C}^N$ with elements denoted by $b_1, \overline{b}_1, b_2, \overline{b}_2, \ldots, b_N, \overline{b}_N$.

Recall that an $\mathbf{R}$-linear mapping on $\mathbf{C} \equiv \mathbf{R}^2$ has the form

$$\omega \mapsto \alpha\omega + \beta\overline{\omega} \tag{25}$$

where $\alpha, \beta \in \mathbf{C}$. The matrix of this mapping in these coordinates,

$$M = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \tag{26}$$

has

$$\mathrm{tr}(M) \;=\; 2\mathrm{Re}(\alpha), \qquad \det(M) \;=\; |\alpha|^2 - |\beta|^2.$$

The eigenvalues of this matrix are

$$\frac{\mathrm{tr}(M)}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(M)}{2}\right)^2 - \det(M)}.$$

If one eigenvalue is zero, then $\det(M) = 0$ and the sign of the other eigenvalue (if it is not zero) is given by the sign of the real part of α. If M has no zero eigenvalues, then the eigenvalues have negative real part if and only if the determinant is positive and the trace is negative.

$$\left(\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p, \text{where } N = qk + p,\ 2 \leq k \leq N,\ q \geq 1,\ p \geq 0\right)$$

The fixed-point subspace of $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ is

$$\mathrm{Fix}\left(\Sigma^I_{q,p}\right) \;=\; \{(\underbrace{z,\ldots,z}_{q};\underbrace{\xi z,\ldots,\xi z}_{q};\ldots;\underbrace{\xi^{k-1}z,\ldots,\xi^{k-1}z}_{q};\underbrace{0,\ldots,0}_{p}) : z \in \mathbf{C}\}$$

where $\xi = e^{2\pi i/k}$. Using the equation (18) where f is as in (19), after dividing by z we have if $k \neq 2$

$$\nu(\lambda) + (A_1 + kqA_3)|z|^2 \;+\; \cdots \;=\; 0$$

where $+\ \cdots$ denotes terms of higher order in z and $\overline{z}$, and taking the real part of this equation, we obtain,

$$\lambda = -(A_{1r} + kqA_{3r})|z|^2 \;+\; \cdots$$

It follows that if $A_1 + kqA_3 < 0$, then the branch bifurcates supercritically.

In the particular case $k = 2$ we have

$$\nu(\lambda) + [A_1 + 2q(A_2 + A_3)]|z|^2 \; + \; \cdots \; = \; 0$$

and taking the real part of this equation,

$$\lambda = -[A_{1r} + 2q(A_{2r} + A_{3r})]|z|^2 + \cdots$$

where the functions A_{ir} for $i = 1, 2, 3$ are evaluated at $\lambda = 0$. It follows in this case that if $A_{1r} + 2q(A_{2r} + A_{3r}) < 0$, then the branch bifurcates supercritically.

Throughout we denote by (z_0, λ_0, τ_0) a zero of $g(z, \lambda, \tau) = 0$ with $z_0 \in \mathrm{Fix}(\Sigma)$. Specifically, we wish to calculate $(dg)_{(z_0,\lambda_0,\tau_0)}$.

Recall the generators for $\Sigma^I_{q,p}$ given in Section 3. With respect to the basis B, any "real" matrix commuting with $\Sigma^I_{q,p} = \widetilde{\mathbf{S}_q \wr \mathbf{Z}_k} \times \mathbf{S}_p$ has the form

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} M_1 & M_3 & M_4 & \ldots & M_{k+1} & M_{k+2} \\ M^{\xi^2}_{k+1} & M^{\xi^2}_1 & M^{\xi^2}_3 & \ldots & M^{\xi^2}_k & M^{\xi^2}_{k+2} \\ \vdots & \ddots & & \vdots & & \\ M^{\xi^{2(k-1)}}_3 & & & \ldots & M^{\xi^{2(k-1)}}_1 & M^{\xi^{2(k-1)}}_{k+2} \\ M_{k+3} & M^{\xi^2}_{k+3} & M^{\xi^4}_{k+3} & \ldots & M^{\xi^{2(k-1)}}_{k+3} & M_{k+4} \end{pmatrix}$$

where M_1 commutes with $\mathbf{S}_q$, M_{k+4} commutes with $\mathbf{S}_p$ and the other matrices are defined below.

Suppose M is a square matrix of order a with rows $l_1, \ldots, l_a$ and commuting with S_a. It follows then that $M = (l_1, (12) \cdot l_1, \cdots, (1a) \cdot l_1)^t$, where if $l_1 = (m_1, \ldots, m_a)$ then $(1i) \cdot l_1 = (m_i, m_2, \ldots, m_{i-1}, m_1, m_{i+1}, \ldots, m_a)$. Moreover, l_1 is invariant under S_{a-1} in the last $a - 1$ entries and so it has the following form: $(m_1, m_2, \ldots, m_2)$. Applying this to M_1 and M_{k+4} we get

$$M_1 \; = \; \begin{pmatrix} C_1 & C_2 & \ldots & C_2 \\ C_2 & C_1 & \ldots & C_2 \\ \vdots & & \ddots & \vdots \\ C_2 & C_2 & \ldots & C_1 \end{pmatrix}, \quad M_{k+4} \; = \; \begin{pmatrix} C_{k+4} & C_{k+5} & \ldots & C_{k+5} \\ C_{k+5} & C_{k+4} & \ldots & C_{k+5} \\ \vdots & & \ddots & \vdots \\ C_{k+5} & C_{k+5} & \ldots & C_{k+4} \end{pmatrix},$$

where M_1 is a $2q \times 2q$ matrix and M_{k+4} is a $2p \times 2p$ matrix.

The other symmetry restrictions on the M_i, for $i = 3, \ldots, k+3$, imply that each have one identical entry,

$$M_i \; = \; \begin{pmatrix} C_i & \ldots & C_i \\ & \ddots & \\ C_i & \ldots & C_i \end{pmatrix}.$$

Note that each M_i for $i = 1, \ldots, k+1$ is a $2q \times 2q$ matrix and M_{k+2}, M_{k+3} are, respectively, $2q \times 2p$ and $2p \times 2q$ matrices. Furthermore, we have

$$M^{\xi^j}_1 \; = \; \begin{pmatrix} C^{\xi^j}_1 & C^{\xi^j}_2 & \ldots & C^{\xi^j}_2 \\ C^{\xi^j}_2 & C^{\xi^j}_1 & \ldots & C^{\xi^j}_2 \\ \vdots & & \ddots & \vdots \\ C^{\xi^j}_2 & C^{\xi^j}_2 & \ldots & C^{\xi^j}_1 \end{pmatrix}$$

for $j = 2, \ldots, 2(k-1)$ and

$$M_l^{\xi^j} = \begin{pmatrix} C_l^{\xi^j} & C_l^{\xi^j} & \cdots & C_l^{\xi^j} \\ C_l^{\xi^j} & C_l^{\xi^j} & \cdots & C_l^{\xi^j} \\ \vdots & & \ddots & \vdots \\ C_l^{\xi^j} & C_l^{\xi^j} & \cdots & C_l^{\xi^j} \end{pmatrix}$$

for $l = 3, \ldots, k+3$ and $j = 2, \ldots, 2(k-1)$.

Now, each C_i is of the type

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c_i'} & \overline{c_i} \end{pmatrix}, \quad C_i^{\xi^j} = \begin{pmatrix} c_i & \xi^j c_i' \\ \overline{\xi}^j \overline{c_i'} & \overline{c_i} \end{pmatrix}$$

for $i = 1, \ldots, k+3, j = 2, \ldots, 2(k-1)$ and

$$C_{k+2} = \begin{pmatrix} c_{k+2} & c_{k+2}' \\ \overline{c_{k+2}'} & \overline{c_{k+2}} \end{pmatrix}, \quad C_{k+4} = \begin{pmatrix} c_{k+4} & c_{k+4}' \\ \overline{c_{k+4}'} & \overline{c_{k+4}} \end{pmatrix}, \quad C_{k+5} = \begin{pmatrix} c_{k+5} & c_{k+5}' \\ \overline{c_{k+5}'} & \overline{c_{k+5}} \end{pmatrix},$$

where

$$\begin{array}{llll} c_1 = \frac{\partial g_1}{\partial z_1}, & c_1' = \frac{\partial g_1}{\partial \overline{z}_1}, & c_2 = \frac{\partial g_1}{\partial z_2}, & c_2' = \frac{\partial g_1}{\partial \overline{z}_2}, \\ c_3 = \frac{\partial g_1}{\partial z_{q+1}}, & c_3' = \frac{\partial g_1}{\partial \overline{z}_{q+1}}, & \ldots & c_{k+1} = \frac{\partial g_1}{\partial z_{q(k-1)+1}}, \quad c_{k+1}' = \frac{\partial g_1}{\partial \overline{z}_{q(k-1)+1}}, \\ c_{k+2} = \frac{\partial g_1}{\partial z_{kq+1}}, & c_{k+2}' = \frac{\partial g_1}{\partial \overline{z}_{kq+1}}, & c_{k+3} = \frac{\partial g_{kq+1}}{\partial z_1}, & c_{k+3}' = \frac{\partial g_{kq+1}}{\partial \overline{z}_1}, \\ c_{k+4} = \frac{\partial g_N}{\partial z_N}, & c_{k+4}' = \frac{\partial g_N}{\partial \overline{z}_N}, & c_{k+5} = \frac{\partial g_N}{\partial z_{N-1}}, & c_{k+5}' = \frac{\partial g_N}{\partial \overline{z}_{N-1}}, \end{array}$$

calculated at (z_0, λ_0, τ_0).

Throughout we denote by $(dg)_{(z_0,\lambda_0,\tau_0)}|W_k$ the restriction of $(dg)_{(z_0,\lambda_0,\tau_0)}$ to the subspace W_k. And we recall the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma_{q,p}^I$ given by (15) with the components listed in Table 4.

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ where

$$W_0 = \{(\underbrace{z_1, \ldots, z_1}_q; \underbrace{\xi z_1, \ldots, \xi z_1}_q; \cdots; \underbrace{\xi^{k-1} z_1, \ldots, \xi^{k-1} z_1}_q; \underbrace{0, \ldots, 0}_p) : z_1 \in \mathbf{C}\}.$$

The tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector

$$(\underbrace{iz, \ldots, iz}_q; \underbrace{i\xi z, \ldots, i\xi z}_q; \cdots; \underbrace{i\xi^{k-1} z, \ldots, i\xi^{k-1} z}_q; \underbrace{0, \ldots, 0}_p).$$

Note that

$$\frac{d}{dt}(e^{it}z, \ldots, e^{it}z, \ldots, e^{it}\xi^{k-1}z, \ldots, e^{it}\xi^{k-1}z)\big|_{t=0} = (iz, \ldots, iz, \ldots, i\xi^{k-1}z, \ldots, i\xi^{k-1}z).$$

Now since $g(\mathrm{Fix}(\Sigma_{q,p}^I)) \subseteq \mathrm{Fix}(\Sigma_{q,p}^I)$ we have that $g(\mathrm{Fix}(\Sigma_{q,p}^I))$ is two-dimensional. Thus, $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ is as in (25) and the matrix of this mapping has the form (26). The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is given by

$$2\mathrm{Re}(\alpha) = \ 2\mathrm{Re}(A_1 + kqA_3)|z|^2 + \cdots$$

if $k \geq 3$, whose sign is determined by $A_{1r} + kqA_{3r}$ if it is assumed nonzero (where $A_{1r} + kqA_{3r}$ is calculated at zero). In the particular case $k = 2$, the nonzero eigenvalue is given by

$$2\mathrm{Re}(\alpha) = \ 2\mathrm{Re}[A_1 + 2q(A_2 + A_3)]|z|^2 + \cdots$$

whose sign is determined by $A_{1r} + 2q(A_{2r} + A_{3r})$ if it is assumed nonzero (where $A_{1r} + 2q(A_{2r} + A_{3r})$ is calculated at zero).

We compute $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ where

$$W_1 \ = \ \{(\underbrace{z_1, \ldots, z_1}_{kq}; \underbrace{-\frac{kq}{p}z_1, \ldots, -\frac{kq}{p}z_1}_{p}) : z_1 \in \mathbf{C}\}.$$

We have $\left((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\right) z \to \alpha z + \beta \overline{z}$ where

$$\begin{aligned} \alpha &= c_1 + (q-1)c_2 + qc_3 - 2qc_4, \\ \beta &= c_1' + (q-1)c_2' + qc_3' - 2qc_4', \end{aligned}$$

for $k = 2$. Recall that this case is special case since the branching equation is different from the one we obtain for $k \geq 3$. Thus, we study this case separately. We get for $k = 2$ (see [20, Chapter 4, Section 4, p.71] for the explicit expressions for $c_1, \ldots, c_4, c_1', \ldots, c_4'$) that

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = 2\mathrm{Re}\left[\left(1 - \tfrac{4q}{N}\right) A_1 - 2qA_2\right] |z|^2 + \cdots ,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = \left(\left|\left(1 - \tfrac{4q}{N}\right) A_1 - 2qA_2\right|^2 - \left|\left(1 - \tfrac{2q}{N}\right) A_1 + 2qA_2\right|^2\right) |z|^4 + \cdots .$$

If $k \geq 3$ we have

$$\begin{aligned} \alpha &= c_1 + (q-1)c_2 + qc_3 + \cdots + qc_{k+1} - kqc_{k+2}, \\ \beta &= c_1' + (q-1)c_2' + qc_3' + \cdots + qc_{k+1}' - kqc_{k+2}'. \end{aligned}$$

and we get

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = 2\mathrm{Re}\left[\left(1 - \tfrac{2kq}{N}\right) A_1\right] |z|^2 + \cdots ,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = \left(\left|\left(1 - \tfrac{2kq}{N}\right) A_1\right|^2 - |A_1|^2\right) |z|^4 + \cdots .$$

(see [20, Chapter 4, Section 4, p.72] for the explicit expressions for $c_1, \ldots, c_{k+2}, c_1', \ldots, c_{k+2}'$).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ where

$$W_2 \ = \ \{(0, \ldots, 0; \underbrace{z_1, \ldots, z_{p-1}, -z_1 - \cdots - z_{p-1}}_{p}) : z_1, \ldots, z_{p-1} \in \mathbf{C}\}.$$

Recall that we only have this isotypic component in the decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma_{q,p}^I$ when $p > 1$. Recall (11). The action of $K \subset \Sigma_{q,p}^I$ on W_2 decomposes in the following way:

$$W_2 = W_2^1 \oplus W_2^2$$

where

$$W_2^1 \ = \ \{(0, \ldots, 0; \underbrace{x_1, \ldots, x_{p-1}, -x_1 - \cdots - x_{p-1}}_{p}) : x_1, \ldots, x_{p-1} \in \mathbf{R}\},$$

$$W_2^2 \ = \ \{(0, \ldots, 0; \underbrace{ix_1, \ldots, ix_{p-1}, -ix_1 - \cdots - ix_{p-1}}_{p}) : x_1, \ldots, x_{p-1} \in \mathbf{R}\}.$$

Moreover, the actions of K on W_2^1 and on W_2^2 are K-isomorphic and K-absolutely irreducible. Thus, it is possible to choose a basis of W_2 such that $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the new coordinates has the form

$$\begin{pmatrix} a\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} & b\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} \\ c\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} & d\ \mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)} \end{pmatrix} \tag{27}$$

where $\mathrm{Id}_{(\mathrm{p}-1)\times(\mathrm{p}-1)}$ is the $(p-1)\times(p-1)$ identity matrix. Furthermore, the eigenvalues of (27) are the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ each with multiplicity $p-1$.

With respect to the basis B' of W_2 given by

$$b_{kq+1}-b_N, \overline{b}_{kq+1}-\overline{b}_N, b_{kq+2}-b_N, \overline{b}_{kq+2}-\overline{b}_N, \ldots, b_{N-1}-b_N, \overline{b}_{N-1}-\overline{b}_N,$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the following block diagonal form

$$(dg)_{(z_0,\lambda_0,\tau_0)}|W_2 = \mathrm{diag}(C_{k+4}-C_{k+5}, \ldots, C_{k+4}-C_{k+5}).$$

The eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ are the eigenvalues of $C_{k+4}-C_{k+5}$, each with multiplicity $p-1$. The eigenvalues of $C_{k+4}-C_{k+5}$ have negative real part if and only if

$$\mathrm{tr}(C_{k+4}-C_{k+5}) \ < \ 0 \text{ and } \det(C_{k+4}-C_{k+5}) \ > \ 0.$$

If $k=2$ then

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = 2\mathrm{Re}\,(-A_1-2qA_2)\,|z|^2+\cdots,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \left(|A_1+2qA_2|^2-|2qA_2|^2\right)|z|^4+\cdots.$$

Moreover, if $k\geq 3$ we have

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = 2\mathrm{Re}\,(A_1)\,|z|^2+\cdots,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = |A_1|^2\,|z|^4+\cdots.$$

(see [20, Chapter 4, Section 4, p.73] for the explicit expressions for $c_{k+4}, c_{k+5}, c'_{k+4}, c'_{k+5}$ in both cases).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_3$ where

$$W_3 \ = \ \{(\underbrace{z_1,\ldots,z_{q-1},z_q}_{q};\ldots;\underbrace{z_{q(k-1)+1},\ldots,z_{kq-1},z_{kq}}_{q};\underbrace{0,\ldots,0}_{p}) : z_1,\ldots,z_{kq}\in\mathbf{C}\}$$

with $z_q=-z_1-\cdots-z_{q-1},\ldots,z_{kq}=-z_{q(k-1)+1}-\cdots-z_{kq-1}$. Recall that we only have this isotypic component in the decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$ when $q\geq 2$.

With respect to the basis B' of W_3 given by

$$b_1-b_q, \overline{b}_1-\overline{b}_q, \ldots, b_{q-1}-b_q, \overline{b}_{q-1}-\overline{b}_q,$$

$$b_{q+1}-b_{2q}, \overline{b}_{q+1}-\overline{b}_{2q}, \ldots, b_{2q-1}-b_{2q}, \overline{b}_{2q-1}-\overline{b}_{2q},$$

$$\cdots,$$

$$b_{q(k-1)+1}-b_{kq}, \overline{b}_{q(k-1)+1}-\overline{b}_{kq}, \ldots, b_{kq-1}-b_{kq}, \overline{b}_{kq-1}-\overline{b}_{kq},$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_3$ in the following block diagonal form:

$$(dg)_{(z_0,\lambda_0,\tau_0)}|W_3 = \mathrm{diag}(\underbrace{C_1-C_2,\ldots}_{q-1};\underbrace{C_1^{\xi^2}-C_2^{\xi^2},\ldots}_{q-1};\ldots;\underbrace{C_1^{\xi^{2(k-1)}}-C_2^{\xi^{2(k-1)}},\ldots}_{q-1}).$$

Note that we have $\mathrm{tr}(C_1^{\xi^j} - C_2^{\xi^j}) = \mathrm{tr}(C_1 - C_2)$ and $\det(C_1^{\xi^j} - C_2^{\xi^j}) = \det(C_1 - C_2)$. We get for $k = 2$

$$\mathrm{tr}(C_1 - C_2) = 2\mathrm{Re}\,(A_1 - 2qA_2)\,|z|^2 + \cdots\,,$$

$$\det(C_1 - C_2) = \left(|A_1 - 2qA_2|^2 - |A_1 + 2qA_2|^2\right)|z|^4 + \cdots\,.$$

Furthermore, for $k \geq 3$ we have

$$\det(C_1 - C_2) = 0.$$

Thus, the degree three truncation is too degenerate (it originates a null eigenvalue which is not forced by the symmetry of the problem). We consider now the degree five truncation and we get that $k = 3$ is a particular case. Note that the fifth degree truncation of the branching equations are different in the cases $k = 3$ and $k > 3$, thus, we get different expressions for the derivatives. We study the case $k = 3$ first. We have

$$\begin{aligned} \mathrm{tr}(C_1 - C_2) = \quad & 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots\,, \\ \det(C_1 - C_2) = \quad & \left|A_1 + (2A_4 - 3qA_{12} + 3qA_{14})|z|^2\right|^2 |z|^4 - \\ & \left|A_1 + (2A_4 - \tfrac{6q}{N}A_{12} + 3qA_{14})|z|^2\right|^2 |z|^4 + \cdots = \\ & \left(-3q + \tfrac{6q}{N}\right) 2\mathrm{Re}\left(A_1\overline{A_{12}}\right)|z|^6 + \cdots\,. \end{aligned}$$

Now, for $k > 3$ we get that $\det(C_1 - C_2) = 0$. In this case, when $k > 3$ and when this component appears in the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of $\Sigma^I_{q,p}$, the five degree truncation is too degenerate in order to determine the stability of the system. See [20, Chapter 4, Section 4, p.75 and p.76] for the explicit computation of c_1, c_2, c_1', c_2'.

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|P_j$ where

$$P_j \;=\; \{(\underbrace{z_1, \ldots, z_1}_{q}; \underbrace{\xi^j z_1, \ldots, \xi^j z_1}_{q}; \ldots; \underbrace{\ldots, \xi^{j(k-1)} z_1}_{q}; \underbrace{0, \ldots, 0}_{p}) : z_1 \in \mathbf{C}\}$$

and $2 \leq j \leq k-1$. We have $\left((dg)_{(z_0,\lambda_0,\tau_0)}|P_j\right) z \to \alpha z + \beta\overline{z}$ where

$$\begin{aligned} \alpha &= c_1 + (q-1)c_2 + q\xi^j c_3 + \cdots + q\xi^{(k-1)j} c_{k+1}, \\ \beta &= c_1' + (q-1)c_2' + q\overline{\xi^j} c_3' + \cdots + q\overline{\xi^{(k-1)j}} c_{k+1}'. \end{aligned}$$

When we substitute the expressions for the derivatives (see [20, Chapter 4, Section 4, p.76 to p.78] for the explicit computations) we get that the case $k = 3$ is a particular case. If $j = 2$ and $k \geq 4$ we have

$$\mathrm{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots\,,$$

$$\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = \left(|A_1|^2 - \left|\left(1 - \tfrac{kq}{N}\right)A_1\right|^2\right)|z|^4 + \cdots\,,$$

but it the particular case $k = 3$ it follows that

$$\mathrm{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = 2\mathrm{Re}\,(A_1 + 6A_2)\,|z|^2 + \cdots\,,$$

$$\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_2\right) = \left(|A_1 + 6A_2|^2 - \left|\left(1 - \tfrac{3}{N}\right)A_1\right|^2\right)|z|^4 + \cdots\,.$$

Consider now $j = k-1$. It follows that

$$\mathrm{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-1}\right) = 2\mathrm{Re}\,(A_1 + 2kqA_2)\,|z|^2 + \cdots\,,$$

$$\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-1}\right) = \left(|A_1 + 2kqA_2|^2 - |A_1|^2\right)|z|^4 + \cdots\,.$$

Moreover, if we consider $2 < j \leq k-2$, then we obtain that $\det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_j\right) = 0$. Thus, we need to consider the five degree truncation of (19). We have for $j = k-2$ (note that we only have this isotypic component when $k \geq 5$) that

$$\begin{array}{rcl} \mathrm{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-2}\right) & = & 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots\,, \\ \det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_{k-2}\right) & = & \left(|A_1+\xi_1|z|^2|^2 - |A_1 + (2A_4+kqA_{14})|z|^2|^2\right)|z|^4 + \cdots \\ & = & [2\mathrm{Re}(A_1\overline{\xi}_1) - 2\mathrm{Re}(2A_1\overline{A}_4 + kqA_1\overline{A}_{14}]|z|^6 + \cdots\,, \end{array}$$

where

$$\begin{array}{rl} \xi_1 = & 2A_4 + 3kqA_{12} + q(kq-1)\left(2 - \frac{2kq}{N}\right)A_{13} + kqA_{14} + \\ & +q(kq-1)\left(1-\frac{2kq}{N}\right)A_{14} + 2q(kq-1)A_{15}. \end{array}$$

Furthermore, for $3 \leq j \leq k-3$ (note that we only have this isotypic component when $k \geq 6$) we get

$$\begin{array}{rcl} \mathrm{tr}\left((df)_{(z_0,\lambda_0,\tau_0)}|P_j\right) & = & 2\mathrm{Re}\,(A_1)\,|z|^2 + \cdots\,, \\ \det\left((df)_{(z_0,\lambda_0,\tau_0)}|P_j\right) & = & \left(|A_1+\xi_2|z|^2|^2 - |A_1 + (2A_4+kqA_{14})|z|^2|^2\right)|z|^4 + \cdots \\ & = & [2\mathrm{Re}(A_1\overline{\xi}_2) - 2\mathrm{Re}(2A_1\overline{A}_4 + kqA_1\overline{A}_{14}]|z|^6 + \cdots\,, \end{array}$$

with

$$\xi_2 = \xi_1 - 3kqA_{12} - kqA_{14}.$$

$\left(\Sigma_q^{II} = \mathbf{S}_q \times \mathbf{S}_p,\ \text{where } N = q+p,\ 1 \leq q < \frac{N}{2}\right)$

The fixed-point subspace of $\Sigma_q^{II} = \mathbf{S}_q \times \mathbf{S}_p$ is

$$\mathrm{Fix}\left(\Sigma_{q,p}^{I}\right) = \left\{ \left(\underbrace{z,\ldots,z}_{q};\ \underbrace{-\frac{q}{p}z,\ldots,-\frac{q}{p}z}_{p} \right) : z \in \mathbf{C} \right\}.$$

Using the equation (18) where f is as in (19), after dividing by z we have

$$\nu(\lambda) + A_1\left[1 - \frac{q}{N}\left(1-\frac{q^2}{p^2}\right)\right]|z|^2 + (A_2+A_3)q\left(1+\frac{q}{p}\right)|z|^2 + \cdots = 0$$

where $+\cdots$ denotes terms of higher order in z and $\overline{z}$, and taking the real part of this equation, we obtain,

$$\lambda = -A_{1r}\left[1 - \frac{q}{N}\left(1-\frac{q^2}{p^2}\right)\right]|z|^2 - (A_{2r}+A_{3r})q\left(1+\frac{q}{p}\right)|z|^2 + \cdots\,.$$

It follows that if $A_{1r}\left[1-\frac{q}{N}\left(1-\frac{q^2}{p^2}\right)\right]|z|^2 + (A_{2r}+A_{3r})q\left(1+\frac{q}{p}\right) < 0$, then the branch bifurcates supercritically.

Let $\Sigma_q^{II} = \mathbf{S}_q \times \mathbf{S}_p$ be the isotropy subgroup of $z_0 = \left(z,\ldots,z;\ -\frac{q}{p}z,\ldots,-\frac{q}{p}z\right)$. Recall the generators for Σ_q^{II} given in Section 3.

Suppose M is a square $(q+p)\times(q+p)$ matrix with rows $l_1,\ldots,l_q,l_{q+1},\ldots,l_{q+p}$ and commuting with $S_q \times S_p$. Then

$$M = (l_1, (12)\cdot l_1, \ldots, (1q)\cdot l_1; l_{q+1}, (q+1\ q+2)\cdot l_{q+1}, \ldots, (q+1\ q+p)\cdot l_{q+1})$$

where if $l_1 = (m_1,\ldots,m_{q+p})$ then

$$(1i)\cdot l_1 = (m_i, m_2, \ldots, m_{i-1}, m_1, m_{i+1}, \ldots, m_{q+p}).$$

Moreover, l_1 is $S_{q-1} \times S_p$-invariant and l_{q+1} is $S_q \times S_{p-1}$-invariant. Applying this to $(dg)_{(z_0,\lambda_0,\tau_0)}$ we have

$$(dg)_{(z_0,\lambda_0,\tau_0)} = \begin{pmatrix} C_1 & & C_6 & C_2 & & C_2 \\ & \ddots & & & \ddots & \\ C_6 & & C_1 & C_2 & & C_2 \\ C_3 & & C_3 & C_4 & & C_5 \\ & \ddots & & & \ddots & \\ C_3 & & C_3 & C_5 & & C_4 \end{pmatrix}$$

where C_i for $i = 1, \ldots, 5$ are the 2×2 matrices

$$C_i = \begin{pmatrix} c_i & c_i' \\ \overline{c}_i' & \overline{c}_i \end{pmatrix}$$

and

$$c_1 = \frac{\partial g_1}{\partial z_1}, \quad c_1' = \frac{\partial g_1}{\partial \overline{z}_1}, \quad c_6 = \frac{\partial g_1}{\partial z_2}, \quad c_6' = \frac{\partial g_1}{\partial \overline{z}_2}, \quad c_2 = \frac{\partial g_1}{\partial z_{q+1}}, \quad c_2' = \frac{\partial g_1}{\partial \overline{z}_{q+1}},$$

$$c_3 = \frac{\partial g_{q+1}}{\partial z_1}, \quad c_3' = \frac{\partial g_{q+1}}{\partial \overline{z}_1}, \quad c_4 = \frac{\partial g_{q+1}}{\partial z_{q+1}}, \quad c_4' = \frac{\partial g_{q+1}}{\partial \overline{z}_{q+1}}, \quad c_5 = \frac{\partial g_{q+1}}{\partial z_{q+2}}, \quad c_5' = \frac{\partial g_{q+1}}{\partial \overline{z}_{q+2}},$$

calculated at (z_0, λ_0, τ_0).

Recall the isotypic decomposition of $\mathbf{C}^{N,0}$ for the action of Σ_q^{II} given by (16) with the components listed in Table 4.

We begin by computing $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$. In coordinates $z, \overline{z}$ we have $((dg)_{(z_0,\lambda_0,\tau_0)}|W_0)z = \alpha z + \beta \overline{z}$ where

$$\begin{aligned} \alpha &= c_1 + (q-1)c_6 - [q(N-q)/p]c_2, \\ \beta &= c_1' + (q-1)c_6' - [q(N-q)/p]c_2'. \end{aligned}$$

The tangent vector to the orbit of $\Gamma \times \mathbf{S}^1$ through z_0 is the eigenvector

$$\left(iz, \ldots, iz, -i\frac{q}{p}z, \ldots, -i\frac{q}{p}z \right).$$

Note that

$$\frac{d}{dt}\left(e^{it}z, \ldots, e^{it}z, -e^{it}\frac{q}{p}z, \ldots, -e^{it}\frac{q}{p}z \right)\Big|_{t=0} = \left(iz, \ldots, iz, -i\frac{q}{p}z, \ldots, -i\frac{q}{p}z \right).$$

The matrix $(dg)_{(z_0,\lambda_0,\tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\mathrm{Re}(\alpha) = 2\mathrm{Re}\left[A_1 \left(1 - \frac{q}{N} + \frac{q^3}{Np^2} \right) + (A_2 + A_3)q \left(1 + \frac{q}{p} \right) \right] |z|^2 + \cdots$$

whose sign is determined by

$$A_{1r} \left[1 - \frac{q}{N} \left(1 - \frac{q^2}{p^2} \right) \right] + (A_{2r} + A_{3r})q \left(1 + \frac{q}{p} \right)$$

if it is assumed nonzero (where A_{1r}, A_{2r}, A_{3r} are calculated at zero).

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ where

$$W_1 = \left\{ \left(z_1, \ldots, z_{q-1}, -z_1 - \cdots - z_{q-1}; \underbrace{0, \ldots, 0}_{p} \right) : z_1, \ldots, z_{q-1} \in \mathbf{C} \right\}.$$

The action of Σ_q^{II} on W_1 decomposes in the following way

$$W_1 = W_1^1 \oplus W_1^2$$

where

$$W_1^1 = \left\{ \left(x_1, \ldots, x_{q-1}, -x_1 - \cdots - x_{q-1}; \underbrace{0, \ldots, 0}_{p} \right) : x_1, \ldots, x_{q-1} \in \mathbf{R} \right\},$$

$$W_1^2 = \left\{ \left(ix_1, \ldots, ix_{q-1}, -ix_1 - \cdots - ix_{q-1}; \underbrace{0, \ldots, 0}_{p} \right) x_1, \ldots, x_{q-1} \in \mathbf{R} \right\}.$$

Moreover, the actions of Σ_q^{II} on W_1^1 and on W_1^2 are Σ_q^{II}-isomorphic and are Σ_q^{II}-absolutely irreducible. Thus, it is possible to choose a basis of W_1 such that $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ in the new coordinates has the form

$$\begin{pmatrix} a\ \mathrm{Id}_{(q-1)\times(q-1)} & b\ \mathrm{Id}_{(q-1)\times(q-1)} \\ c\ \mathrm{Id}_{(q-1)\times(q-1)} & d\ \mathrm{Id}_{(q-1)\times(q-1)} \end{pmatrix} \tag{28}$$

where $\mathrm{Id}_{(q-1)\times(q-1)}$ is the $(q-1)\times(q-1)$ identity matrix. Furthermore, the eigenvalues of (28) are the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ each with multiplicity $q-1$.

With respect to the basis B' of W_1 given by

$$b_1 - b_q, \overline{b}_1 - \overline{b}_q, b_2 - b_q, \overline{b}_2 - \overline{b}_q, \ldots, b_{q-1} - b_q, \overline{b}_{q-1} - \overline{b}_q$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ in the following block diagonal form

$$(dg)_{(z_0,\lambda_0,\tau_0)}|W_1 = \mathrm{diag}(C_1 - C_6, C_1 - C_6, \ldots, C_1 - C_6).$$

The eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}|W_1$ are the eigenvalues of $C_1 - C_6$, each with multiplicity $q-1$. The eigenvalues of $C_1 - C_6$ have negative real part if and only if

$$\mathrm{tr}(C_1 - C_6) < 0 \text{ and } \det(C_1 - C_6) > 0.$$

We get

$$\begin{aligned} \mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = &\quad 2\mathrm{Re}\left[\left(1 + \frac{q}{N} - \frac{q^3}{Np^2}\right) A_1 - q\left(1 + \frac{q}{p}\right) A_2 \right] |z|^2 + \cdots, \\ \det((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = &\quad \left| \left(1 + \frac{q}{N} - \frac{q^3}{Np^2}\right) A_1 - q\left(1 + \frac{q}{p}\right) A_2 \right|^2 |z|^4 - \\ &\quad \left| A_1 + q\left(1 + \frac{q}{p}\right) A_2 \right|^2 |z|^4 + \cdots, \end{aligned}$$

(see [20, Chapter 4, Section 4, p.81] for the explicit expressions for c_1, c_6, c_1', c_6').

We compute now $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ where

$$W_2 = \{(0, \ldots, 0, z_{q+1}, \ldots, z_{N-1}, -z_{q+1} - \cdots - z_{N-1}) : z_{q+1}, \ldots, z_{N-1} \in \mathbf{C}\}.$$

The action of Σ_q^{II} on W_2 decomposes in the following way

$$W_2 = W_2^1 \oplus W_2^2$$

where

$$W_2^1 = \left\{ \left(\underbrace{0, \ldots, 0}_{q}, x_{q+1}, \ldots, x_{N-1}, -x_{q+1} - \cdots - x_{N-1}, \right) : x_{q+1}, \ldots, x_{N-1} \in \mathbf{R} \right\},$$

$$W_2^2 = \left\{ \left(\underbrace{0, \ldots, 0}_{q}, ix_{q+1}, \ldots, ix_{N-1}, -ix_{q+1} - \cdots - ix_{N-1}, \right) : x_{q+1}, \ldots, x_{N-1} \in \mathbf{R} \right\}.$$

Moreover, the actions of Σ_q^{II} on W_2^1 and on W_2^2 are Σ_q^{II}-isomorphic and are Σ_q^{II}-absolutely irreducible. Thus, it is possible to choose a basis of W_2 such that $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the new coordinates has the form

$$\begin{pmatrix} a\ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} & b\ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} \\ c\ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} & d\ \mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})} \end{pmatrix} \tag{29}$$

where $\mathrm{Id}_{(\mathrm{N-q-1})\times(\mathrm{N-q-1})}$ is the $(N-q-1)\times(N-q-1)$ identity matrix. Furthermore, the eigenvalues of (29) are the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ each with multiplicity $N-q-1$.

With respect to the basis B' of W_2 given by

$$b_{q+1} - b_N, \bar{b}_{q+1} - \bar{b}_N, b_{q+2} - b_N, \bar{b}_{q+2} - \bar{b}_N, \ldots, b_{N-1} - b_N, \bar{b}_{N-1} - \bar{b}_N,$$

we can write $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ in the following block diagonal form

$$(df)_{(z_0,\lambda_0,\tau_0)}|W_2 = \mathrm{diag}(C_4 - C_5, C_4 - C_5, \ldots, C_4 - C_5).$$

The eigenvalues of $(dg)_{(z_0,\lambda_0,\tau_0)}|W_2$ are the eigenvalues of $C_4 - C_5$, each with multiplicity $N-q-1$. The eigenvalues of $C_4 - C_5$ have negative real part if and only if

$$\mathrm{tr}(C_4 - C_5) < 0 \text{ and } \det(C_4 - C_5) > 0.$$

We have

$$\mathrm{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \quad 2\mathrm{Re}\left[\left(-1 + \frac{q}{N} - \frac{q^3}{Np^2} + \frac{2q^2}{p^2} \right) A_1 - q\left(1 + \frac{1}{p}\right) A_2 \right] |z|^2 + \cdots,$$

$$\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \left| \left(-1 + \frac{q}{N} - \frac{q^3}{Np^2} + \frac{2q^2}{p^2} \right) A_1 - q\left(1 + \frac{1}{p}\right) A_2 \right|^2 |z|^4 - \left| \frac{q^2}{p^2} A_1 + q\left(1 + \frac{q}{p}\right) A_2 \right|^2 |z|^4 + \cdots,$$

(see [20, Chapter 4, Section 4, p.82] for the explicit expressions for c_4, c_5, c_4', c_5').

Appendix A. Equivariant Vector Field

In this section we compute the general form of a $\mathbf{S}_N \times \mathbf{S}^1$- equivariant Hopf bifurcation problem, up to degree 5, given by (19) of Section 4. We follow Dias, Matthews and Rodrigues [6, Section 7] and Rodrigues [20, Section 4].

Theorem A.1 ([6]). *Consider the action of $\mathbf{S}_N \times \mathbf{S}^1$ on $\mathbf{C}^{N,0}$ defined by (6) and the following functions $H_i : \mathbf{C}^{N,0} \to \mathbf{C}^{N,0}$, for $i = 1, \ldots, 15$:*

$$H_i(z) = (h_i(z), h_i((12)z), \ldots, h_i((1N)z))$$

where $z = (z_1, \ldots, z_N) \in \mathbf{C}^{N,0}$ *and*

$$h_1(z) = |z_1|^2 z_1 - \frac{1}{N}\sum_{j=1}^N |z_j|^2 z_j, \qquad h_2(z) = \overline{z}_1 \sum_{j=1}^N z_j^2, \qquad h_3(z) = z_1 \sum_{j=1}^N |z_j|^2;$$

$$\begin{array}{rclrcl}
h_4(z) & = & |z_1|^4 z_1 - \frac{1}{N}\sum_{i=1}^N |z_i|^4 z_i, & h_5(z) & = & \sum_{i=1}^N |z_i|^4 z_1, \\
h_6(z) & = & \sum_{i=1}^N z_i^2 \sum_{j=1}^N \overline{z}_j^2\, z_1, & h_7(z) & = & \left(\sum_{i=1}^N |z_i|^2\right)^2 z_1, \\
h_8(z) & = & \sum_{j=1}^N |z_j|^2 \overline{z}_j\, z_1^2 - \frac{1}{N}\sum_{j=1}^N |z_j|^2 \overline{z}_j \sum_{i=1}^N z_i^2, & h_9(z) & = & \sum_{j=1}^N \overline{z}_j^2 z_1^3 - \frac{1}{N}\sum_{j=1}^N \overline{z}_j^2 \sum_{i=1}^N z_i^3, \\
h_{10}(z) & = & \sum_{i=1}^N |z_i|^2 \sum_{j=1}^N z_j^2\, \overline{z}_1, & h_{11}(z) & = & \sum_{i=1}^N |z_i|^2 z_i^2\, \overline{z}_1, \\
h_{12}(z) & = & \sum_{j=1}^N z_j^3 \overline{z}_1^2 - \frac{1}{N}\sum_{j=1}^N z_j^3 \sum_{i=1}^N \overline{z}_i^2, & h_{13}(z) & = & \sum_{j=1}^N |z_j|^2 z_j |z_1|^2 - \frac{1}{N}\sum_{j=1}^N |z_j|^2 z_j \sum_{i=1}^N |z_i|^2, \\
h_{14}(z) & = & \sum_{j=1}^N |z_j|^2 |z_1|^2 z_1 - \frac{1}{N}\sum_{j=1}^N |z_j|^2 \sum_{i=1}^N |z_i|^2 z_i, & h_{15}(z) & = & \sum_{j=1}^N z_j^2 |z_1|^2 \overline{z}_1 - \frac{1}{N}\sum_{j=1}^N z_j^2 \sum_{i=1}^N |z_i|^2 \overline{z}_i\,.
\end{array}$$

Then:
(i) If $N \geq 4$*, the functions* H_i *for* $i = 1, 2, 3$ *constitute a basis of the complex vector space of the* $\mathbf{S}_N \times \mathbf{S}^1$*-equivariant functions with homogeneous polynomial components of degree* 3*;*
(ii) If $N \geq 6$*, the functions* H_i *for* $i = 4, \ldots, 15$ *constitute a basis of the complex vector space of the* $\mathbf{S}_N \times \mathbf{S}^1$*-equivariant functions with homogeneous polynomial components of degree* 5*.*

Proof. See [6, Section 7]. We outline the proof for completeness. We start by making two observations. The first is that the $\mathbf{S}_N \times \mathbf{S}^1$-equivariant functions with homogeneous polynomial components of degree k are obtained by restriction to $\mathbf{C}^{N,0}$ and projection onto $\mathbf{C}^{N,0}$ of the $\mathbf{S}_N \times \mathbf{S}^1$-equivariant functions from $\mathbf{C}^N$ to $\mathbf{C}^N$ with homogeneous polynomial components of degree k. Also note that with respect to the direct sum decomposition of $\mathbf{C}^N$ into $\mathbf{S}_N$-invariant spaces,

$$\mathbf{C}^N = \{(z, z, \ldots, z) : \ z \in \mathbf{R}\} \oplus \mathbf{C}^{N,0},$$

the projection vector of $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$ onto $\mathbf{C}^{N,0}$ is:

$$z - \frac{1}{N}\left(z_1 + \cdots + z_N\right)(1, \ldots, 1)\,.$$

Thus given a $\mathbf{S}_N \times \mathbf{S}^1$-equivariant function $f : \ \mathbf{C}^N \to \mathbf{C}^N$ where $f = (f_1, \ldots, f_N)$ for $f_i : \ \mathbf{C}^N \to \mathbf{C}$, the restriction of f to $\mathbf{C}^{N,0}$ and projection onto $\mathbf{C}^{N,0}$ is given by

$$f|_{\mathbf{C}^{N,0}} - \frac{1}{N}\sum_{i=1}^N f_i|_{\mathbf{C}^{N,0}}\,(1, \ldots, 1)\ .$$

The second observation is that if $f : \ \mathbf{C}^N \to \mathbf{C}^N$ is $\mathbf{S}_N \times \mathbf{S}^1$-equivariant, then the equivariance of f under $\mathbf{S}_N$ is equivalent to the invariance say of the first component f_1 under $\mathbf{S}_{N-1}$ in the last $N-1$ coordinates $z_2, \ldots, z_N$, and then

$$f(z) = \left(f_1(z), f_1\left((12)z\right), \ldots, f_1\left((1N)z\right)\right)\ . \tag{30}$$

That is,

$$f(z) = \left(f_1(z_1, z_2, \ldots, z_{N-1}, z_N), f_1(z_2, z_1, \ldots, z_{N-1}, z_N), \ldots, f_1(z_N, z_2, \ldots, z_{N-1}, z_1)\right)\ .$$

This follows from the equivariance conditions

$$f\left((1i)(z_1,z_2,\ldots,z_N)\right) \;=\; (1i)f\left(z_1,z_2,\ldots,z_N\right) \tag{31}$$

for $i=2,3,\ldots,N$. Note that for each i, the equality (31) implies that $f_i(z)=f_1((1i)z)$. Now for example if we take $i=2$ in (31), we obtain $f_1\left((1q)z\right)=f_1\left((1q)(12)z\right)$ for any $q\geq 3$ and so $f_1(y)=f_1\left((1q)(12)(1q)y\right)=f_1\left((2q)y\right)$. Thus f_1 is $\mathbf{S}_{N-1}$-invariant in the last $N-1$ coordinates. Obviously if we take f as in (30) where f_1 satisfies this $\mathbf{S}_{N-1}$-invariance condition then f is $\mathbf{S}_N$-equivariant.

Now using the $\mathbf{S}^1$-equivariance, for $z=(z_1,\ldots,z_N)$, taking $\overline{z}=(\overline{z}_1,\ldots,\overline{z}_N)$ and using multi-indices, we have that f_1 can be written as

$$f_1(z) \;=\; \sum a_{\alpha\beta}z^{\alpha}\overline{z}^{\beta}$$

where each $a_{\alpha\beta}\in\mathbf{C}$ and $\alpha,\beta\in\left(\mathbf{Z}_0^+\right)^N$ and satisfies

$$f_1(e^{i\theta}z)=e^{i\theta}f_1(z) \quad \left(\theta\in\mathbf{S}^1,\; z\in\mathbf{C}^N\right). \tag{32}$$

Thus each $a_{\alpha\beta}=0$ unless $|\alpha|=|\beta|+1$. The rest of the proof consists in characterizing the first component f_1. That is, we describe the homogeneous polynomials of degree k, for $k=3,5$, that are $\mathbf{S}_{N-1}$-invariant in the last $N-1$-coordinates $z_2,\ldots,z_N$ and are $\mathbf{S}^1$-equivariant. Specifically, we consider the $\mathbf{S}_N\times\mathbf{S}^1$-equivariants where the first component is an homogeneous polynomial of degree k which can be written as

$$z_1^a\overline{z_1}^b p(z_2,\ldots,z_N)$$

where $a,b\in\mathbf{Z}_0^+, a+b\geq 0$, p is $\mathbf{S}_{N-1}$-invariant and satisfies (32).

As an example, for the degree three polynomials, we consider f_1 given by monomials of the following types: $z_1^2\overline{z}_1=z_1|z_1|^2$, $z_1^2\sum_{j=2}^N\overline{z}_j$, $z_1\overline{z}_1\sum_{j=2}^N z_j$, $z_1p(z_2,\ldots,z_N)$ where $p(z_2,\ldots,z_N)$ has degree two in $z,\overline{z}$ and it is $\mathbf{S}_{N-1}\times\mathbf{S}^1$-invariant, $\overline{z}_1p(z_2,\ldots,z_N)$ where $p(z_2,\ldots,z_N)$ has degree two in $z_2,\ldots,z_N$, it is $\mathbf{S}_{N-1}$-invariant and does not depend on the $\overline{z}_j$ and $p(z_2,\ldots,z_N)$ where p is $\mathbf{S}_{N-1}$-invariant and satisfies (32). This way we obtain eleven $\mathbf{S}_N\times\mathbf{S}^1$-equivariant functions from $\mathbf{C}^N$ to $\mathbf{C}^N$. A list with the eleven equivariants may be found in [20, Theorem 4.2]. Now we restrict to $\mathbf{C}^{N,0}$ and project onto $\mathbf{C}^{N,0}$ the $\mathbf{S}_N\times\mathbf{S}^1$-equivariant functions from $\mathbf{C}^N$ to $\mathbf{C}^N$. Note that if $z\in\mathbf{C}^{N,0}$ then $z_1+\cdots+z_N=0$ and $\overline{z}_1+\cdots+\overline{z}_N=0$.

For the equivariants under $\mathbf{S}_N\times\mathbf{S}^1$ with homogeneous components of degree 5, there are 52 and a list can be found at [20, Theorems 4.5]. See [20, Theorems 4.2, 4.5-4.6, 4.10] for details. □

Remark A.2. (i) The complex vector space of $\mathbf{S}_N\times\mathbf{S}^1$-equivariants with homogeneous polynomial components of degree one has dimension one and so is generated for example by the identity on $\mathbf{C}^{N,0}$. To see that, note that the first component f_1 of any given f has to be a linear combination of the monomials z_1 and $z_2+\cdots+z_N$ which are invariant under $\mathbf{S}_{N-1}$ in the last $N-1$ coordinates and satisfy (32). Equivalently, it is a linear combination of z_1 and $z_1+z_2+\cdots+z_N$. Now at restriction to $\mathbf{C}^{N,0}$ we obtain only z_1.
(ii) For $N=5$ we have

$$\begin{aligned} H_{12}(z) \;=\;& 30H_4(z)-\tfrac{9}{2}H_5(z)+\tfrac{3}{4}H_6(z)+\tfrac{3}{2}H_7(z)-3H_8(z)-\tfrac{3}{2}H_9(z)+\\ & \tfrac{3}{2}H_{10}(z)-3H_{11}(z)-6H_{13}(z)-9H_{14}(z)-\tfrac{9}{2}H_{15}(z)\end{aligned}$$

where $z=(z_1,z_2,z_3,z_4,z_5)\in\mathbf{C}^{5,0}$ and so we obtain (over the complex field) only eleven linearly independent $\mathbf{S}_5\times\mathbf{S}^1$-equivariants with homogeneous polynomial components of degree five. ◊

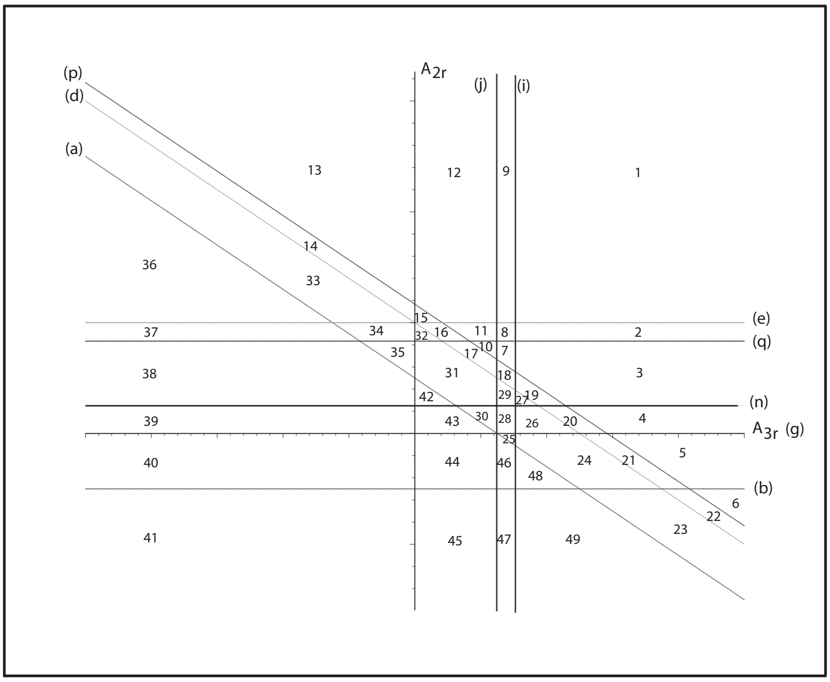

Figure 1. Regions of the (A_{3r}, A_{2r})-parameter space defined by the lines corresponding to the equations (34). Here we assume $A_{1r} < 0$. Lines are labelled according to which of the corresponding expressions on (34) vanishes on them.

Appendix B. Bifurcation diagrams for $N = 4$

In this section we present the bifurcation diagrams concerning the periodic solutions with maximal isotropy for Hopf bifurcation with $\mathbf{S}_4$-symmetry (note that we do not include in the diagrams the possible periodic solutions with submaximal isotropy obtained in Section 5).

For $N = 4$ the solution stabilities depend on the following coefficients

$$A_1, A_2, A_{3r} \tag{33}$$

of the degree three truncation of the vector field f (see Section 5).

Recall the stability results for these solutions summarized in Table 12. From this we obtain the following non-degeneracy conditions:

$$\begin{array}{ll}
\text{(a) } A_{1r} + 4A_{2r} + 4A_{3r} \neq 0, & \text{(b) } A_{1r} - 4A_{2r} \neq 0, \\
\text{(c) } |A_1 - 4A_2|^2 - |A_1 + 4A_2|^2 \neq 0, & \text{(d) } A_{1r} + 2A_{2r} + 2A_{3r} \neq 0, \\
\text{(e) } A_{1r} + 2A_{2r} \neq 0, & \text{(f) } |A_1 + 2A_2|^2 - 4|A_2|^2 \neq 0, \\
\text{(g) } A_{2r} \neq 0, & \text{(h) } \left(4|A_2|^2 - |\tfrac{1}{2}A_1 + 2A_2|^2\right) \neq 0, \\
\text{(i) } A_{1r} + 3A_{3r} \neq 0, & \text{(j) } A_{1r} + 4A_{3r} \neq 0, \\
\text{(k) } A_{1r} + 6A_{2r} \neq 0, & \text{(l) } A_{1r} \neq 0, \\
\text{(n) } A_{1r} + 8A_{2r} \neq 0, & \text{(o) } |A_1 + 8A_2|^2 - |A_1|^2 \neq 0, \\
\text{(p) } \tfrac{7}{3}A_{1r} + 4A_{2r} + 4A_{3r} \neq 0, & \text{(q) } 5A_{1r} + 12A_{2r} \neq 0, \\
\text{(r) } |5A_1 + 12A_2|^2 - |A_1 + 12A_2|^2 \neq 0, & \text{(s) } |A_1 + 6A_2|^2 - |\tfrac{1}{4}A_1|^2 \neq 0\,.
\end{array} \tag{34}$$

The inequalities (34) divide the parameter space (33) into regions characterized by (possibly) distinct bifurcation diagrams. In Figures 1 and 2 we assume, respectively, $A_{1r} < 0$ and $A_{1r} > 0$ and we consider the various regions of the (A_{2r}, A_{3r})-parameter space defined by (34).

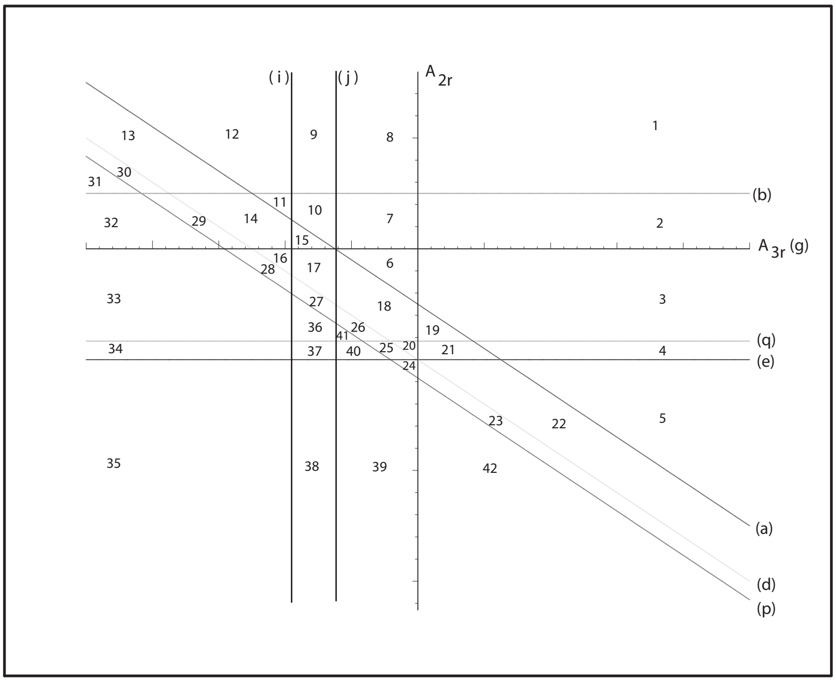

FIGURE 2. Regions of the (A_{3r}, A_{2r})-parameter space defined by the lines corresponding to the equations (34). Here we assume $A_{1r} > 0$. Lines are labelled according to which of the corresponding expressions on (34) vanishes on them.

Figures 3-4 show the bifurcation diagrams corresponding to the regions of parameter space of Figure 1. An asterisk on solution indicates that it is possible for the solution to be unstable, depending on the sign of

$$(35) \qquad \begin{array}{ll} (*) & |A_1 - 4A_2|^2 - |A_1 + 4A_2|^2, \\ (**) & |A_1 + 2A_2|^2 - 4|A_2|^2 \text{ and } 4|A_2|^2 - |\frac{1}{2}A_1 + 2A_2|^2, \\ (***) & |A_1 + 8A_2|^2 - |A_1|^2, \\ (****) & |5A_1 + 12A_2|^2 - |A_1 + 12A_2|^2 \ . \end{array}$$

Furthermore, note that the Σ_3 solution is never stable.

On Figures 5-6 we show the bifurcation diagrams concerning regions of the parameter space of Figure 2.

ACKNOWLEDGEMENTS

We thank the referees of this paper for the comments and suggestions.

We are grateful to Paul Matthews for very helpful discussions and suggestions and we also thank the Isaac Newton Institute, Cambridge, where this work was partially done for the hospitality and additional financial support.

REFERENCES

[1] S.M.C. Abreu and A.P.S. Dias. Hopf bifurcation on Hemispheres, *Nonlinearity* **11** (1998) 247-264.

[2] D.G. Aronson, M. Golubitsky and M. Krupa. Coupled arrays of Josephson junctions and bifurcation of maps with $\mathbf{S}_N$-symmetry, *Nonlinearity* **4** (1991) 861-902.

[3] P. Ashwin and O. Podvigina. Hopf bifurcation with rotational symmetry of the cube and instability of ABC flow, *Proc. Roy. Soc. A* **459** (2003) 1801-1827.

[4] P. Ashwin and J.W. Swift. The dynamics of n weakly coupled identical oscillators, *J. Nonlinear Sci.* **2** (1992) 69-108.

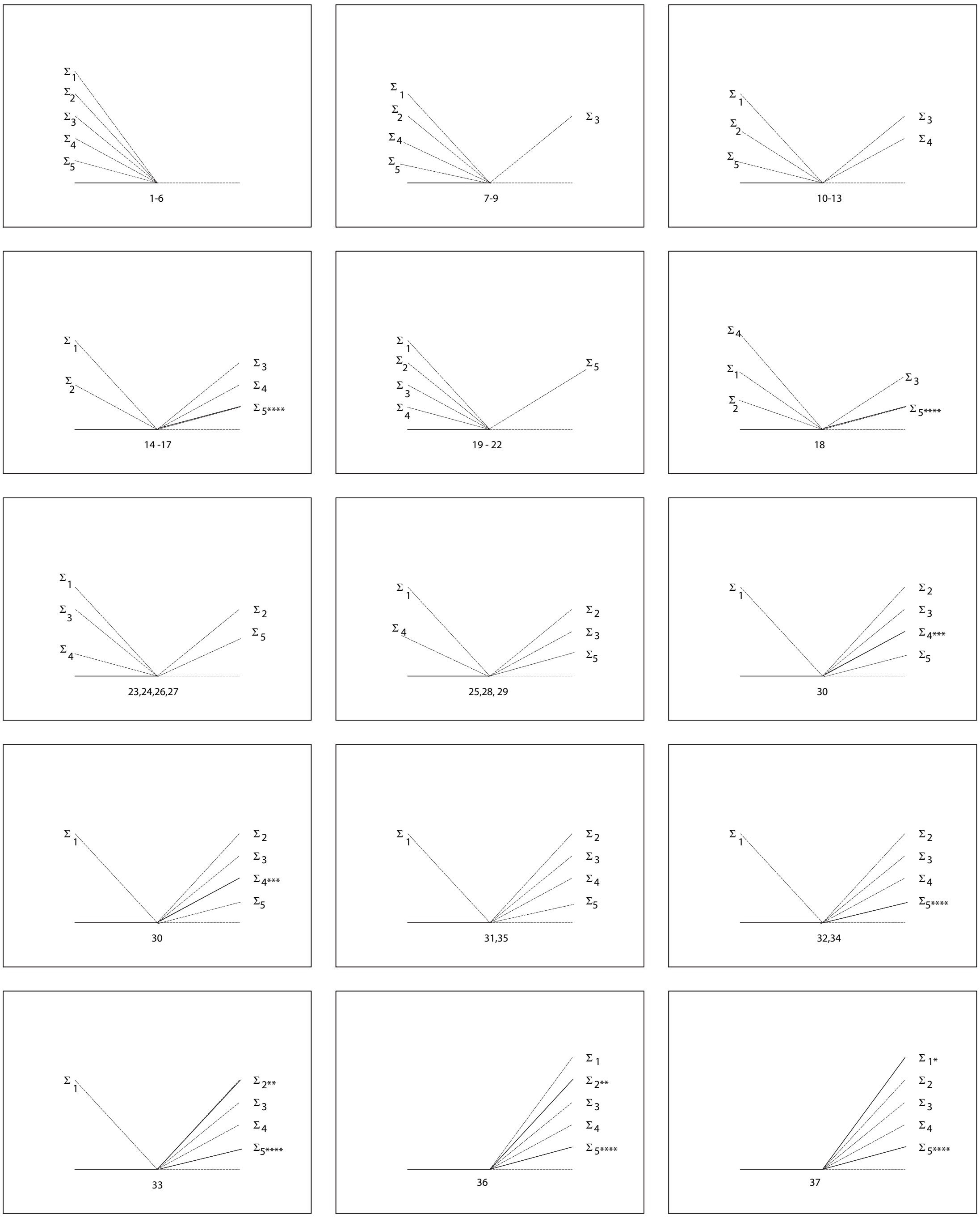

FIGURE 3. Bifurcation diagrams for the nondegenerate Hopf bifurcation with $\mathbf{S}_4$ symmetry. Broken (unbroken) bifurcation curves indicate unstable (stable) solutions. An asterisk on solution indicates that it is possible for the solution to be unstable, depending on the sign of (35). The diagrams are plotted for$A_{1r} < 0$.

[5] P. Chossat and R. Lauterbach. *Methods in Equivariant Bifurcations and Dynamical Systems*, Advanced Series in Nonlinear Dynamics **15**, World Scientific Publishing Co., Inc., River Edge, NJ 2000.

[6] A.P.S. Dias, P.C. Matthews and A. Rodrigues. Generating functions for Hopf bifurcation with $\mathbf{S}_N$-symmetry, CMUP - *preprint* 2008-16.

[7] A.P.S. Dias and R.C. Paiva. Hopf bifurcation with $\mathbf{S}_3$-symmetry, *Portugalia Mathematica* **63** (2) (2005) 127-155.

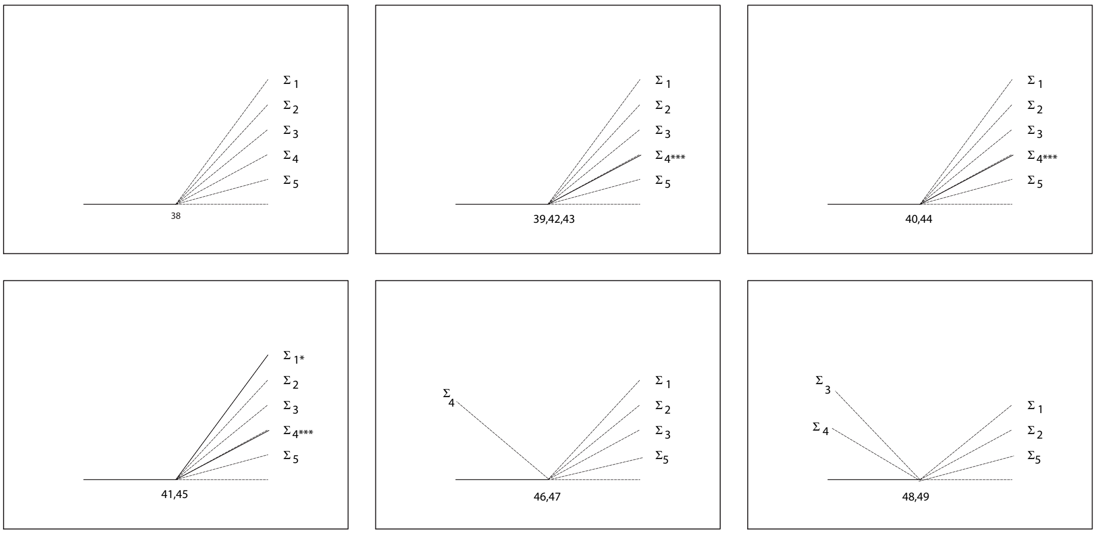

FIGURE 4. Continuation of Figure 3.

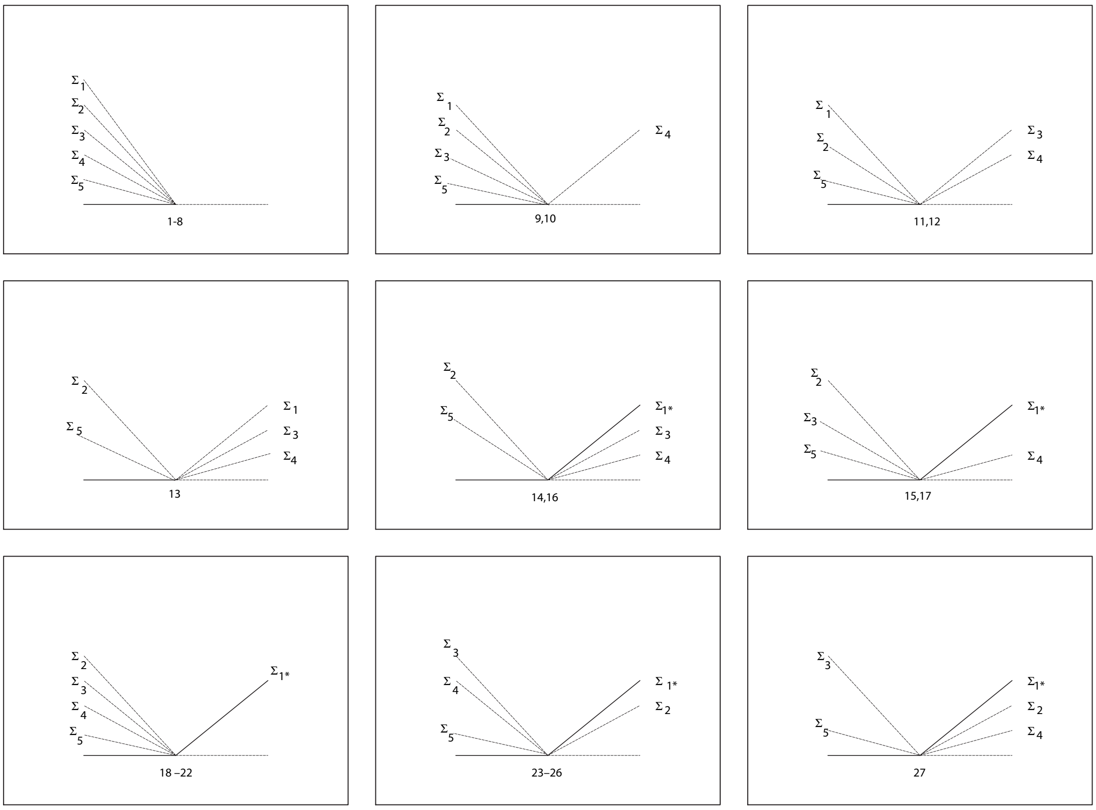

FIGURE 5. Bifurcation diagrams for the nondegenerate Hopf bifurcation with $\mathbf{S}_4$ symmetry. Broken (unbroken) bifurcation curves indicate unstable (stable) solutions. An asterisk on solution indicates that it is possible for the solution to be unstable, depending on the sign of (35). The diagrams are plotted for $A_{1r} > 0$.

[8] A.P.S. Dias and R.C. Paiva. A note on Hopf bifurcation with dihedral group symmetry, *Glasgow Mathematical Journal* **48** (2006) 41-51.
[9] A.P.S. Dias and I. Stewart. Hopf bifurcation on a simple cubic lattice, *Dynamics and Stability of Systems* **14** (1999) 3-55.

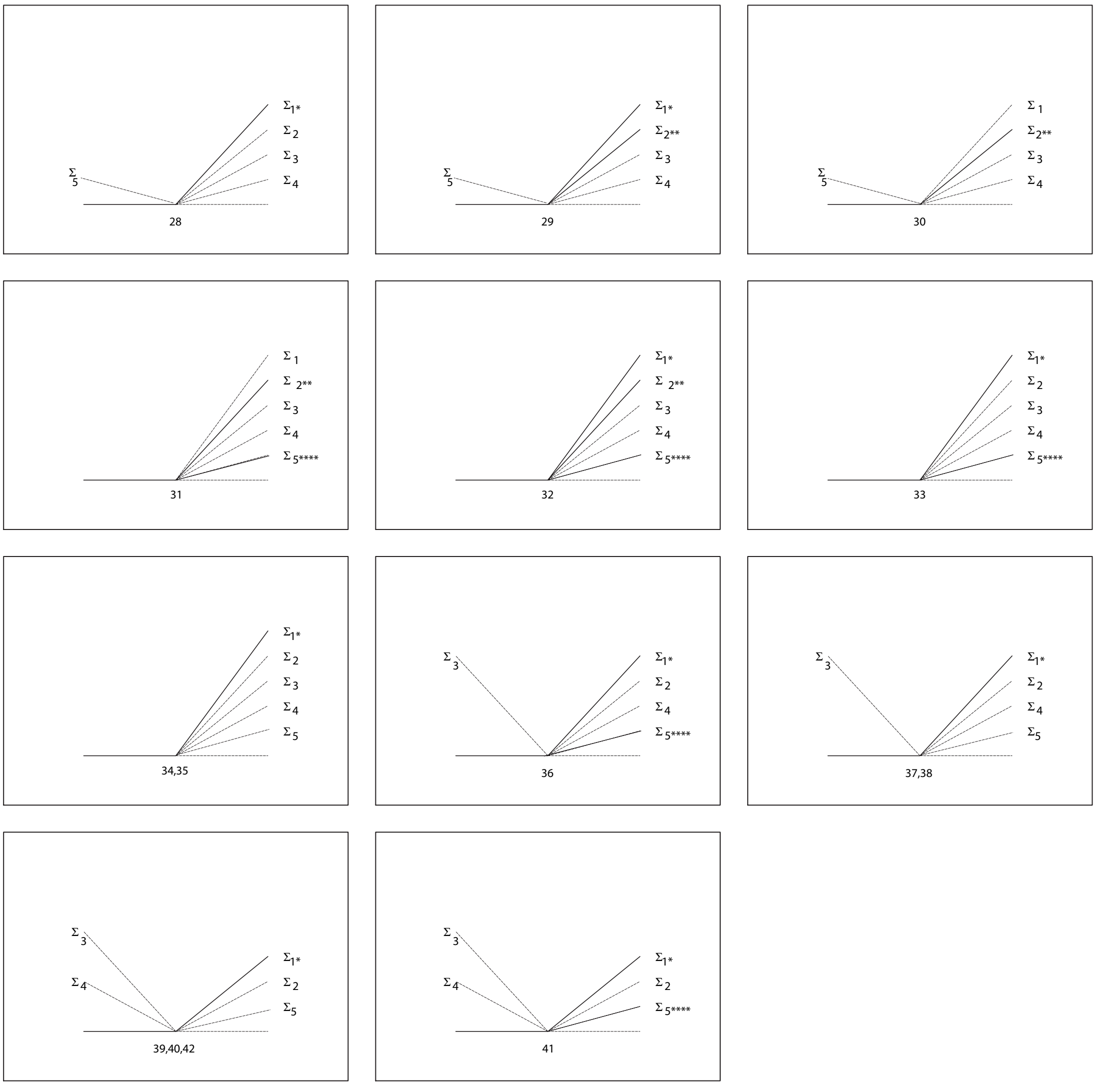

FIGURE 6. Continuation of Figure 5.

[10] M.J. Field. *Dynamics and symmetry*, ICP Advanced Texts in Mathematics, 3, Imperial College Press, London 2007.

[11] M.J. Field and J.W. Swift. Hopf bifurcation and the Hopf fibration, *Nonlinearity* **7** (1994) 385-402.

[12] S.A. van Gils and M. Golubitsky. A Torus Bifurcation Theorem with Symmetry, *J. Dyn. Differ. Equations* **2** (1990) 133-162.

[13] M. Golubitsky and I. Stewart. Hopf bifurcation in the presence of symmetry, *Arch. Rat. Mech. Anal.* **87** (1985) 107-165.

[14] M. Golubitsky and I. Stewart. Hopf bifurcation with dihedral group symmetry: coupled nonlinear oscillators. In: *Multiparameter Bifurcation Theory* (M. Golubitsky and J.M. Guckenheimer eds), Contemp. Math. **56**, Amer. Math. Soc., Providence, RI, 1986, pages 131-173.

[15] M. Golubitsky and I. Stewart. An algebraic criterion for symmetric Hopf bifurcation, *Proc. R. Soc. London A* **440** (1993) 727-732.

[16] M. Golubitsky, I. Stewart, and D. Schaeffer. *Singularities and Groups in Bifurcation Theory*, Vol. II, Applied Mathematical Sciences **69**, Springer-Verlag, New-York 1988.

[17] H. Haaf, M. Roberts and I. Stewart. A Hopf bifurcation with spherical symmetry, *Z. Angew. Math. Phys.* **43** (1993) 793-826.

[18] M. Hall. *The Theory of Groups*, Macmillan, New York 1959.

[19] G. Iooss and M. Rossi. Hopf bifurcation in the presence of spherical symmetry: Analytical results, *SIAM J. Math. Anal. 20* **3** (1989) 511-532.

[20] A. Rodrigues. *Bifurcations of Dynamical Systems with Symmetry*, PhD Thesis, Faculdade de Ciências da Universidade do Porto 2007.

[21] M. Silber and E. Knobloch. Hopf bifurcation on a square lattice, *Nonlinearity* **4** (1991) 1063-1106.
[22] I. Stewart. Symmetry methods in collisionless many-body problems, *J. Nonlinear Sci.* **6** (1996) 543-563.
[23] J. Swift. Hopf bifurcation with the symmetry of the square, *Nonlinearity* **1** (1988) 333-377.

Ana Paula S. Dias, CMUP and Departamento de Matemática Pura, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal.
E-mail address: apdias@fc.up.pt

Ana Rodrigues, CMUP and Departamento de Matemática Pura, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal.
E-mail address: ana.rodrigues@fc.up.pt