Local bifurcation in symmetric coupled cell networks: linear theory

Ana Paula S. Dias[1] and Jeroen S.W. Lamb[2]

[1] Dep. de Matemática Pura, Centro de Matemática, Universidade do Porto,
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
[2] Department of Mathematics, Imperial College London, London SW7 2AZ, UK

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Abstract

We consider a coupled cell network of differential equations with finite symmetry group Γ, where Γ permutes cells transitively. We show how the structure of the coupled cell network, represented by a directed graph whose vertices represent individual cells and edges represent couplings, can be taken into account in the bifurcation analysis of a fully symmetric steady-state solution.

We focus on the analysis of the linearized vector field at a fully symmetric equilibrium and show that in the case of *active* cells, if Γ is Abelian the network structure does not influence the types of codimension one local bifurcations. We also show that beyond this context, when Γ is not Abelian, cells are *passive*, or when considering local bifurcations of higher codimensions, anomalies due to the network structure may arise.

1 Introduction

Coupled cell networks are dynamical systems comprising of components, called *cells*, which are coupled together by *connections*. The corresponding networks structure is specified by a labelled directed graph, see for instance Golubitsky *et al.* [11, 10, 21, 14] and Field [8].

Coupled cell networks naturally arise in the context of many applications in engineering, physics and biology, and the corresponding relevant literature is enormous. For example networks of coupled dynamical systems have been used to model biological oscillators [25, 18, 22, 4], Josephson junction arrays [3, 24], excitable media [9], neural networks [1, 5, 15], spatial games [19], genetic control networks [17] and many other self-organizing systems.

Recently coupled cell networks have received increased attention in view of the question in what way - if any - the network structure influences the dynamics of a coupled cell network, see for instance Wang [23], Stewart [20].

1.1 Coupled cell networks

For the purpose of this paper, coupled cell networks are described by differential equations whose structure is represented by a directed graph, with vertices representing *cells* and directed edges representing the *connections* between cells. We denote the phase space for a coupled cell network as $\Omega_1 \times \cdots \times \Omega_n$, where n denotes the number of cells and Ω_i is a manifold representing the phase space for the internal dynamics of cell i. We represent the state of the network by $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ where $\mathbf{x}_i \in \Omega_i$.

We consider the equations of the jth cell to be given by the differential equation

$$\frac{d}{dt}\mathbf{x}_j = f_j(\mathbf{x}_1, \ldots, \mathbf{x}_n).$$

Graphically, we may represent the coupled cell network by a directed graph with n vertices and a directed edge from vertex i to vertex j if f_j depends on $\mathbf{x}_i$. Note that the absence of a directed edge from vertex i to vertex j thus means that $f_j(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n) = f_j(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, 0, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)$ for all $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n$. We naturally assume that the network is *connected*, in the sense that the network is not a disjoint union of smaller networks.

Instead of drawing edges from a vertex to itself to indicate the fact that the dynamics of cells depend on their own state, we refer to this property as cells being *active*. If cells are not active, we call them *passive*. In the latter case, coupled cell network dynamics preserves volume (as it is readily verified that the vector field has vanishing divergence). As a consequence, in this case it is not surprising that codimension one local bifurcations may not follow those of general Γ-equivariant vector fields (fully connected coupled cell networks). Some illustrations are given below in Examples 1.1 and 1.2. In the context of certain applications, for instance in electronic networks where – in contrast to biological networks – one may have the possibility to control physical properties of components tightly, coupled cell networks with passive components may well be of interest.

Our definition of a coupled cell network appears the most general (and least restrictive) one relating directly to the structure of a directed graph. We note that other concepts of coupled cell systems with additional structure have been proposed in the literature, for instance the networks defined by the "symmetry groupoid" formalism of Golubitsky and Stewart *et al.* [21, 14]. It depends on the properties of the specific network under consideration which modeling assumption should be used. It turns out that for the purpose of the linear analysis discussed in this paper the difference between these types of coupled cell networks and the ones discussed here is irrelevant, as differences in the equations of motion appear only at the level of nonlinear terms.

We are interested in the consequences of a coupled cell network structure on local bifurcations. This appears a hard problem in general, and as a starting point we make an important assumption concerning the homogeneity of the coupled cell network: we assume that the network is such that from the point of view of each individual cell, the network *appears* identical. More precisely, we assume that for each pair of cells $\{i, j\}$ there exists a permutation of the set of cells T, such that the image of cell i under T is cell j, and T leaves the equations of motion invariant. The latter requirement is equivalent to stating that the coupled cell network is T-*equivariant*. This is equivalent to

$$DT(\mathbf{x}) \cdot f(\mathbf{x}) = f(T\mathbf{x})$$

for all $\mathbf{x} \in \Omega_1 \times \cdots \times \Omega_n$, where T acts on $\Omega_1 \times \cdots \times \Omega_n$ by permutation of the Ω_i's.

We thus assume the existence of a permutation group $\Gamma \subseteq \mathbb{S}_n$ of network symmetries acting faithfully and transitively as permutations on the n cells. We recall that *transitivity* implies that for each pair $\{i, j\}$ there exists $\gamma \in \Gamma$ such that $\gamma(i) = j$, and Γ acts *faithfully* if $\gamma(i) = i$ for all $i = 1, \ldots, n$ implies that γ is the identity element of Γ.

During the last decade, there has been a number of studies [6, 7, 10, 11] where assumptions of symmetry properties of a coupled cell network were used to explain the occurrence of spatially and

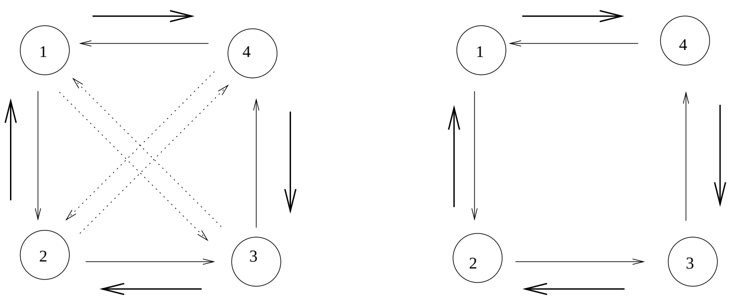

Figure 1: Two examples of graphs representing $\mathbb{Z}_4$-equivariant coupled cell networks. The left one is fully connected, while the right one has nearest neighbour coupling.

spatiotemporally symmetric patterns in coupled cell networks. The formal setting for this theory centred upon the symmetry group of the network, ignoring to a large extend the network structure. However, in case coupled cell networks which are not fully connected, they possess a structure that is independent of the symmetry which should naturally be taken into account when analyzing the (typical) dynamics of coupled cell networks.

We thus would like to address the question how the network architecture may affect the kinds of bifurcations that can be expected to occur in a coupled cell network. It turns out that the problem is quite a complicated one, and in this paper we focus on networks with a symmetry group that permutes cells transitively, and local bifurcation from a fully symmetric equilibrium solution. In this context, we assume without loss of generality that $\Omega_i = \mathbb{R}^l$ for all i and that the equilibrium is represented by $(0, \ldots, 0)$.

The first concern is with the spectrum of the linearized vector field (Jacobian matrix) at the equilibrium solution when parameters are varied, in particular with the analysis of how eigenvalues typically cross the imaginary axis $i\mathbb{R}$. The main aim of this paper is to address this problem. Before discussing this in more detail, we first illustrate the problem with some examples.

Example 1.1 (Networks of four cells with $\mathbb{Z}_4$ symmetry.) An example of the difficulties that can arise is given in a ring of four cells with $\mathbb{Z}_4$ symmetry. We assume that each cell is one-dimensional. The network architecture is shown in Figure 1 (left) with all possible couplings. We assume that the network dynamics have a group invariant equilibrium. The Jacobian matrix at such an equilibrium has the form

$$M = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$$

where a is the linearized internal cell dynamics. The eigenvalues of M are

$$\lambda_1 = a + b + c + d \qquad \lambda_2 = a + c - (b + d) \qquad \lambda_{3,\pm} = a - c \pm i(b - d).$$

Consider the case of nearest-neighbour coupling ($c = 0$) as shown in Figure 1 (right). The eigenvalues of M are then

$$\lambda_1 = a + (b + d) \qquad \lambda_2 = a - (b + d) \qquad \lambda_{3,\pm} = a \pm i(b - d).$$

It is straightforward to show that any one of these three eigenvalues can lie on the imaginary axis while the other two have nonzero real part. Thus, the same three types of bifurcation that can occur as codimension one bifurcations in the fully connected system also occur as codimension one bifurcations in the nearest neighbour coupled system and no others.

Note, however, that anomalous behaviour can occur in codimension two. Consider the steady-state/steady-state mode interaction given by $\lambda_1 = \lambda_2 = 0$ (a mode interaction between the trivial and nontrivial one-dimensional representations of $\mathbb{Z}_4$). In this case $a = 0 = b + d$; these equalities force $\lambda_{3,\pm} = \pm i(b - d)$ to also lie on the imaginary axis, producing, in effect, a steady-state/steady-state/Hopf mode interaction. In the fully connected case (c not constrained by network architecture to be zero), $\lambda_1 = \lambda_2 = 0$ implies that $a + c = 0$ and $b + d = 0$. Then, typically, the eigenvalues $\lambda_{3,\pm} = 2(a \pm ib)$ do not lie on the imaginary axis. We would like to note that it turns out (by a straightforward calculation) that if the internal dynamics of each cell is two-dimensional, the triple mode interaction is no longer forced by the steady-state/steady-state mode interaction.

Let us now assume that the network is passive, i.e. $a = 0$. Then the eigenvalues of M are

$$\lambda_1 = b + c + d, \lambda_2 = c - (b + d), \lambda_{3,\pm} = -c \pm i(b - d).$$

We thus observe that when one of the eigenvalues is passing through the imaginary axis, this does not imply that any of the other eigenvalues passes at the same time, like in the case of general $\mathbb{Z}_4$-equivariant linear systems.

If we consider the passive network ($a = 0$), but now with only nearest neighbour coupling ($c = 0$), the eigenvalues of L are

$$\lambda_1 = b + d, \lambda_2 = -(b + d), \lambda_{3,\pm} = \pm i(b - d),$$

Now, whenever $\lambda_1 = 0$ we have at the same time $\lambda_2 = 0$.

In the case of passive coupled cell networks, when f_j does not depend on $\mathbf{x}_j$, the vector field is divergence free, so that the flow preserves volume. As generic equivariant systems are rarely volume preserving, it is not surprising that correspondence with generic equivariant systems may not be always observed in such networks. This observation raises the question whether eigenvalue bifurcations of symmetric passive coupled cell networks correspond to eigenvalue bifurcations of volume preserving equivariant systems. In this example we observe that in the case of nearest neighbour coupling the pair of eigenvalues $\lambda_{3,\pm}$ is always positioned on the imaginary axis. In generic volume preserving $\mathbb{Z}_4$-equivariant systems one would not expect to observe such behaviour. The consequence of volume preservation is that the sum of the eigenvalues equals zero, which in combination with the equivariance with respect to the $\mathbb{Z}_4$ action in this example never implies that eigenvalues should stick to the imaginary axis. $\diamondsuit$

Example 1.2 (Networks of six cells with $\mathbb{D}_3$ symmetry.) In Figure 2 we present two examples of $\mathbb{D}_3$-equivariant coupled cell networks with six cells, represented by the corresponding directed graphs.

We assume that $\mathbb{D}_3$ permutes the six cells transitively, generated by the two group elements acting on the cells as the following permutations:

$$\alpha = (123)(456), \;\; \beta = (14)(26)(35).$$

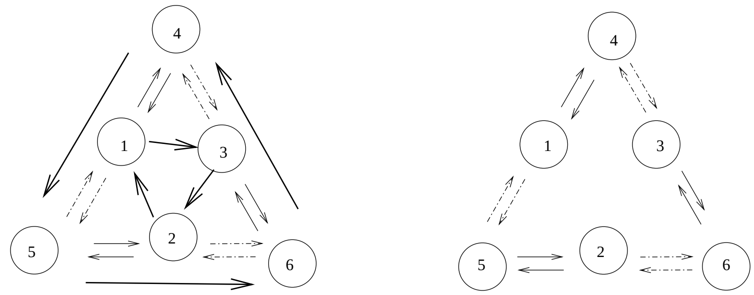

Figure 2: Schematic representations of two $\mathbb{D}_3$-equivariant coupled cell network structures where $\mathbb{D}_3$ acts transitively on six cells (divided in two three-cycles $F_1 = \{1, 2, 3\}$ and $F_2 = \{4, 5, 6\}$). The left network has couplings between cells within each three-cycle (and admits Hopf bifurcation), whereas the right network has not (and does not admit Hopf bifurcation from a $\mathbb{D}_3$-invariant equilibrium in the case of one-dimensional cells).

We suppose that the cells are one-dimensional. The Jacobian matrix at a fully symmetric equilibrium solution for the network of Figure 2 (left) has the following structure:

$$M = \begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{pmatrix}, \qquad B = \begin{pmatrix} c & d & 0 \\ 0 & c & d \\ d & 0 & c \end{pmatrix}$$

and A^T denotes the transpose of the matrix A. The eigenvalues of M are

$$\lambda_1 = a + b + c + d, \qquad \lambda_2 = a + b - c - d, \qquad \lambda_{3,4,\pm} = a - \frac{b}{2} \pm \sqrt{-3b^2 + 4c^2 - 4cd + 4d^2}.$$

The analysis for the network of Figure 2 (right) is obtained by setting $b = 0$. In this case, all eigenvalues of M are real, since $4c^2 - 4cd + 4d^2 = 3c^2 + (2d - c)^2 \geq 0$. This can also be deduced directly from the fact that, if $b = 0$, M is symmetric, that is, $M = M^T$, and eigenvalues of symmetric matrices are always real.

Thus, in the network of Figure 2 (right) a fully symmetric equilibrium cannot undergo a Hopf bifurcation. It is thus possible for the network architecture to suppress the occurrence of Hopf bifurcation. It is readily verified that in the network of Figure 2 (left), when in general $b \neq 0$, no suppression of Hopf bifurcation occurs. $\diamondsuit$

1.2 Main results and discussion

We consider coupled cell networks with symmetry group Γ permuting cells transitively. Identical couplings are thus induced by symmetry only, and no other conditions on the couplings are assumed to hold. We recall that we say that a type of local bifurcation has codimension m if the corresponding set of (smooth) vector fields satisfying the bifurcation condition has codimension m in the ambient space. By usual considerations of transversality, this implies that within an m-parameter family of vector fields the bifurcation condition is typically satisfied at isolated points and that the occurrence of such isolated bifurcation points is persistent (as a consequence of the fact that the m-parameter family is typically transverse to the bifurcation set in a bifurcation point). In the context of this paper, the bifurcation conditions will always involve a statement about the eigenvalues of the Jacobian matrix at a fully symmetric equilibrium.

The main question we address is the following:

> Are the codimension one bifurcations associated to a symmetric coupled cell network dependent on the network structure?

Examples 1.1 and 1.2 illustrate the fact that a general affirmative answer to this question is not possible. Our main result in this paper is that in case the symmetry group is Abelian (meaning that all elements of the symmetry group commute with each other, which holds in the case of $\mathbb{Z}_4$ in Example 1.1 but not in the case of $\mathbb{D}_3$ in Example 1.2), under some mild assumption on the network, this question has an affirmative answer:

Theorem 1.3 *Consider a symmetric connected coupled cell network, where the symmetry group of the network is Abelian and acts transitively by permutation on the cells of the network. Then, codimension one eigenvalue movements across the imaginary axis of the Jacobian matrix at a fully symmetric equilibrium are independent of the network structure if the cells are assumed to be active.*

As a direct consequence of this Theorem, in analogy with general equivariant linear systems [13], typically (codimension zero) eigenvalues do not lie on the imaginary axis, and in the case of codimension one eigenvalue crossings with the imaginary axis, the center subspace is either absolutely irreducible (in the case of steady-state bifurcation) or Γ-simple (in the case of Hopf bifurcation). See Section 4 for more details.

Theorem 1.3 concerns the eigenvalue movements of the linearization of a nonlinear vector field at a fully symmetric equilibrium point. From this we can immediately draw certain consequences about bifurcations at the level of the full (nonlinear) dynamical system. Namely, once the codimension one eigenvalue movements described in Theorem 1.3 occur, one directly obtains the existence of branches of equilibria and periodic solutions whose symmetry groups (leaving solutions setwise invariant) satisfy the conditions of the Equivariant Branching Lemma or the Equivariant Hopf Theorem. For details, we refer the reader to [13, 10].

It should be noted that although the Equivariant Branching Lemma and the Equivariant Hopf Theorem guarantee the existence of branches of solutions with specific symmetry properties, due to possible anomalies in higher order terms, detailed properties of the branches (like direction, growth, stability properties) may depend on the network structure and thus may differ in those aspects from the bifurcations in generic equivariant systems without a coupled cell network structure.

Example 1.2 illustrates that the validity of Theorem 1.3 does not extend in general to coupled cell networks with non-Abelian symmetry. It would be of interest to explore the non-Abelian case further in more detail, but this is beyond the scope of this paper. We note, though in this context, that coupled cell networks with Abelian symmetry have received considerable interest as models of central pattern generators for animal gaits: Golubitsky *et al* [12] have argued the importance of the Abelian nature of the symmetry group to avoid ambiguity due to the simultaneous existence of solution branches of solutions with conjugate (non-identical) symmetry groups.

It is natural to address local bifurcations with codimension higher than one. From Example 1.1 it follows that Theorem 1.3 does not extend in general to local bifurcations of coupled cell networks with Abelian symmetry with codimension higher than one.

Finally, we would like to point out that the occurrence of anomalous local bifurcations can be related to the cell dimension, see for instance the discussion in Example 1.1. It would be worthwhile to further investigate the role of the dimension of the phase space, but this is beyond the scope of this paper.

The remainder of this paper is organized as follows. In Section 2 we discuss in general terms our approach relating absence of connections to conditions on linear maps between isotypic components, followed in Section 3 by technical details about the linear analysis used to derive such conditions. We illustrate in detail the implications of these conditions for networks with cyclic and dihedral symmetry groups. Finally, in Section 4 we prove Theorem 1.3 on codimension one eigenvalue movements across the imaginary axis for coupled cell networks with Abelian symmetry.

2 From cells to isotypic components

In this section we briefly sketch the approach we take to understand the implications of the network structure on the eigenvalues of the linearized vector field $\mathbf{M}$ at a fully symmetric equilibrium.

The vector space $\mathbb{R}^{nl} \simeq \mathbb{R}^l \otimes \mathbb{R}^n$, as the phase space of a coupled cell network whose cells are permuted transitively by a symmetry group Γ, has a natural basis $\{c_j \otimes e_i\}_{j=1,\dots l,\ i=1,\dots n}$ where the index i labels the cells and $\Omega_i = \mathbb{R}\ \{c_j \otimes e_i\}_{j=1,\dots l}$. It is then natural to choose the basis such that a permutation $\gamma \in \Gamma$ acts as $\gamma(c_j \otimes e_i) = (c_j \otimes e_{\gamma^{-1}(i)})$.

The absence of a connection from cell k to cell m implies the following condition on the linear vector field $\mathbf{M} \in \mathrm{gl}(\mathbb{R}^{nl})$ commuting with Γ:

$$\langle c_i \otimes e_m, \mathbf{M}(c_t \otimes e_k)\rangle = 0, \quad i, t = 1, \dots l, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard (Euclidean) inner product on $\mathbb{R}^{nl}$.

The consequences of the Γ-equivariance on $\mathbf{M}$ are best viewed in relation to the Γ-isotypic decomposition of its domain. Recall that an *irreducible* subspace is an indecomposable Γ-invariant subspace of $\mathbb{R}^{nl}$. When Γ is a finite group the number of different (that is, non-isomorphic) irreducible subspaces is finite, and the span of the union of one irreducible subspace V_α together with all others that are isomorphic, is called a Γ-*isotypic component* of $\mathbb{R}^{nl}$. We denote the corresponding *isotypic decomposition* by

$$\mathbb{R}^{nl} = \oplus_\alpha W_\alpha.$$

Because Γ acts essentially on $\mathbb{R}^n$, we have in fact $W_\alpha = \mathbb{R}^l \otimes U_\alpha$ where $\mathbb{R}^n = \oplus_\alpha U_\alpha$ is the isotypic decomposition for the transitive permutation action of Γ on $\mathbb{R}^n$. It follows from a real version of Schur's Lemma [2] that $\mathbf{M}$ preserves the isotypic decomposition, and that the set of Γ-equivariant real matrices in $\mathrm{gl}(\mathbb{R}^l \otimes U_\alpha)$ is isomorphic (as an algebra) to $\mathrm{gl}(ml, \mathbf{k})$ with $\mathbf{k} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ if $\dim U_\alpha = m \dim V_\alpha$ and V_α is of type $\mathbf{K}$. We write the blockdiagonalization $\mathbf{M} = \oplus_\alpha \mathbf{M}^\alpha$.

The equivariant linear vector fields thus decouple into a set of independent linear equivariant vector fields $\mathbf{M}^\alpha$ on isotypic components W_α.

For a symmetric coupled cell network, the absence of certain connections between cells leads to relations between the otherwise independent linear vector fields $\mathbf{M}^\alpha$. These relations can be derived from (2.1), and using projections to isotypic components. Such projections are well known from group representation theory, see for instance James and Liebeck [16]. In the following sections, these relations will be derived, and the consequences – in particular Theorem 1.3 – discussed.

3 Linear analysis

In this section we describe the restrictions on the eigenvalue structure of linear transformations $\mathbf{M} \in \mathrm{gl}(\mathbb{R}^l \otimes \mathbb{R}^n)$ commuting $\Gamma \subseteq \mathbb{S}_n$, where Γ acts trivially on $\mathbb{R}^l$, and transitively and faithfully on $\mathbb{R}^n$, satisfying restrictions of the type

$$\langle c_i \otimes e_1, \mathbf{M}(c_t \otimes e_k)\rangle = 0, \quad i, t = 1, \ldots l.$$

Here $c_i,\ i = 1, \ldots, l$ is a basis of $\mathbb{R}^l$, $e_j,\ j = 1, \ldots, n$ is a basis of $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^l \otimes \mathbb{R}^n$.

We start by addressing the question for $M \in \mathrm{gl}(\mathbb{R}^n)$ commuting with Γ, where Γ acts transitively and faithfully on $\mathbb{R}^n$. We begin by complexifying the state space to $V = \mathbb{C}^n$ in order to use the theory of complex representations of finite groups. See for example [16] for the basic definitions and results on this subject, which we use throughout this section. We then interpret the results in terms of real representations. Finally, we generalize our results to $\mathbb{C}^l \otimes \mathbb{C}^n$ (and $\mathbb{R}^l \otimes \mathbb{R}^n$) and we illustrate them with the cyclic and dihedral groups.

3.1 Linear analysis on $\mathbb{C}^n$

Let Γ be a subgroup of the symmetric group $\mathbb{S}_n$ permuting transitively and faithfully the set $\{1, \ldots, n\}$. Consider a n-dimensional complex vector space V, a basis $b = \{e_1, \ldots, e_n\}$ of V, and the action of Γ on V given by permutation of the corresponding coordinates. Thus we can assume that $V = \mathbb{C}^n$ and this action corresponds to a representation T of Γ on V through a linear homomorphism from Γ to the group $\mathrm{GL}(V)$ of invertible linear transformations on V defined by

$$T(\gamma)(v_1, \ldots, v_n) = \left(v_{\gamma^{-1}(1)}, \ldots, v_{\gamma^{-1}(n)}\right), \quad \gamma \in \Gamma,\ (v_1, \ldots, v_n) \in V. \tag{3.2}$$

A subspace W of V is said to be Γ-*invariant* if $T(\gamma)W \subseteq W$ for all $\gamma \in \Gamma$. If V possesses a proper invariant subspace we say that V is *reducible*, otherwise V is called *irreducible*. Two Γ-invariant vector spaces W_1, W_2 are Γ-*isomorphic* if the corresponding representations, say T_1 and T_2, are *equivalent*. That is, there exists an invertible linear transformation S from W_2 to W_1 such that $T_1(\gamma) = ST_2(\gamma)S^{-1}$ for all $\gamma \in \Gamma$.

Since Γ is finite there appear in V at most s distinct complex irreducible representations, where s is the number of conjugacy classes of Γ. Denote those that appear by $V_1, \ldots, V_r$ and so $r \leq s$. We can decompose V into isotypic components

$$V = U_1 \oplus \cdots \oplus U_r$$

where each U_j is the isotypic component of type V_j for the action of Γ on V. Thus if W is a Γ-invariant subspace of V and Γ-isomorphic to V_j then $W \subseteq U_j$.

Suppose now that $M \in \mathrm{gl}(V)$ commutes with Γ:

$$MT(\gamma) = T(\gamma)M, \quad \forall \gamma \in \Gamma.$$

Since M commutes with Γ, it preserves the isotypic components for the action of Γ on V. Thus $M(U_j) \subseteq U_j$ for $j = 1, \ldots, r$. Denote by M^j the restriction of M to U_j:

$$M^j \equiv M|_{U_j} : \ U_j \to U_j.$$

It follows that M^j commutes with Γ.

Consider the vector space V equipped with the following inner product:

$$\langle (\lambda_1, \ldots, \lambda_n), (\alpha_1, \ldots, \alpha_n) \rangle = \sum_{j=1}^{n} \lambda_j \overline{\alpha}_j$$

where $(\lambda_1, \ldots, \lambda_n)$, $(\alpha_1, \ldots, \alpha_n) \in V$. Thus $\langle e_j, e_k \rangle = 1$ if $j = k$, and 0 otherwise. Observe that the inner product is Γ-*invariant*. That is,

$$\langle T(\gamma)w_1, T(\gamma)w_2 \rangle = \langle w_1, w_2 \rangle$$

for all $\gamma \in \Gamma$ and $w_1, w_2 \in V$.

Given an irreducible Γ-invariant vector space V_j, then the *character* of V_j is the function $\chi_j : \Gamma \to \mathbb{C}$ defined by

$$\chi_j(\gamma) = \mathrm{tr}\left(T(\gamma)|_{V_j} \right), \quad \gamma \in \Gamma.$$

The dimension of V_j is called the *dimension* (or *degree*) of χ_j. Characters of dimension 1 are called *linear characters*. We review the following properties of the characters: if e denotes the identity element of the group Γ then $\chi_j(e) = \dim V_j$; if $\gamma \in \Gamma$ has order m, then $\chi_j(\gamma)$ is a sum of mth roots of unity; if χ_j is linear then it is a homomorphism from Γ to the multiplicative group of non-zero complex numbers $\{z \in \mathbb{C} : \ |z| = 1\}$.

Definition 3.1 Define the projection operator of V onto the Γ-isotypic component U_j by

$$P^j = \frac{\dim V_j}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_j(\gamma)}\, T(\gamma),$$

where χ_j is the character corresponding to the irreducible V_j. $\Diamond$

For $i, j = 1, \ldots, n$ denote

$$M_{ij} \equiv \langle e_i, M e_j \rangle.$$

We use now the projection operators P^j to describe the restrictions on the M^j imposed by a condition of the type

$$M_{1k} = 0. \tag{3.3}$$

Observe that since M commutes with Γ, if $M_{1k} = 0$ then $M_{\gamma(1)\gamma(k)} = 0$ for all $\gamma \in \Gamma$.

Denote by H the subgroup of Γ defined by

$$H = \{\gamma \in \Gamma : \ \gamma(1) = 1\}.$$

Choose permutations $\gamma_2, \ldots, \gamma_n \in \Gamma$ such that

$$j = \gamma_j(1)$$

(recall that Γ acts transitively on $\{1, \ldots, n\}$) and set $\gamma_1 = e$. It follows then that

$$\Gamma = H \,\dot{\cup}\, \gamma_2 H \,\dot{\cup}\, \cdots \,\dot{\cup}\, \gamma_n H$$

where $\dot{\cup}$ denotes disjoint union and

$$e_j = T(\gamma_j) e_1, \quad j = 1, \ldots, n.$$

In the following lemma, given $z \in \mathbb{C}$ we denote $|z|^2 = z\overline{z}$.

Lemma 3.2 *For $j = 1, \ldots, r$, let P^j denote the projection of V onto the isotypic component U_j of type V_j, χ_j the character of the irreducible Γ-invariant vector space V_j and d_j the dimension of V_j. Then:*

$$\begin{aligned} \langle P^j e_1, P^j e_1 \rangle &= \left(\frac{d_j}{|\Gamma|} \right)^2 \sum_{k=1}^{n} \left| \sum_{\gamma \in \gamma_k H} \chi_j(\gamma) \right|^2, \\ P^j e_k &= T(\gamma_k) P^j e_1, \quad k = 1, \ldots, n. \end{aligned}$$

In particular, if χ_j is a linear character of Γ, then

$$P^j e_k = \chi_j(\gamma_k) P^j e_1, \quad k = 1, \ldots, n.$$

Proof: By definition of P^j we have

$$P^j e_1 = \frac{d_j}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_j(\gamma)} T(\gamma) e_1 = \frac{d_j}{|\Gamma|} \left(\sum_{k=1}^{n} \sum_{\gamma \in \gamma_k H} \overline{\chi_j(\gamma)} e_k \right).$$

As $\langle e_i, e_j \rangle = 0$ if $i \neq j$, and $\langle e_i, e_i \rangle = 1$, the formula for $\langle P^j e_1, P^j e_1 \rangle$ follows. Now for the second equality, recall that $e_k = T(\gamma_k) e_1$ for $k = 1, \ldots, n$. Therefore

$$P^j e_k = \frac{d_j}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_j(\gamma)} T(\gamma) T(\gamma_k) e_1 = T(\gamma_k) \left(\frac{d_j}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_j(\gamma_k^{-1} \gamma \gamma_k)} T\left(\gamma_k^{-1} \gamma \gamma_k\right) \right) e_1 = T(\gamma_k) P^j e_1.$$

Moreover, if χ_j is linear, as $P^j e_1 \in U_j$, it follows that $T(\gamma_k) P^j e_1 = \chi_j(\gamma_k) P^j e_1$. $\square$

Proposition 3.3 *Suppose the conditions of Lemma 3.2 and let $I = \{1, \ldots, r\}$. Given $k \in \{1, \ldots, n\}$ then*

$$M_{1k} = 0 \;\Leftrightarrow\; \sum_{j \in I} \langle P^j e_1, M^j P^j e_k \rangle = 0 \;\Leftrightarrow\; \sum_{j \in I} \langle P^j e_1, T(\gamma_k) M^j P^j e_1 \rangle = 0. \tag{3.4}$$

Proof: Recall that $M_{1k} = \langle e_1, M e_k \rangle$, and $U_1, \ldots, U_r$ are the isotypic components for the action of Γ on V, of type $V_1, \ldots, V_r$, respectively. Thus $\sum_{j \in I} P^j = \mathrm{Id}_V$ and so $\langle e_1, M e_k \rangle = \langle \sum_{j \in I} P^j e_1, M \sum_{l \in I} P^l e_k \rangle$. As $P^l e_k \in U_l$, it follows that

$$M \sum_{l \in I} P^l e_k = \sum_{l \in I} M^l P^l e_k.$$

Also $\langle u_j, u_l \rangle = 0$ if $u_j \in U_j,\ u_l \in U_l$ and $j \neq l$. (This property is valid for any Γ-invariant inner product defined on V.) Moreover by Lemma 3.2 and because M^l commutes with Γ we obtain (3.4). $\square$

Remark 3.4 Observe that if $\Gamma \subseteq \mathbb{S}_n$ is Abelian and acts transitively on $\{1, \ldots, n\}$ then if $\gamma(i) = i$ for some i, then $\gamma(j) = j$ for all j. That is, γ is the identity. To verify this point use transitivity to choose $\delta \in \Gamma$ such that $\delta(i) = j$. Since Γ is Abelian, it follows that

$$\gamma(j) = \gamma\delta(i) = \delta\gamma(i) = \delta(i) = j.$$

We have then that $|\Gamma| = n$. Moreover, all the irreducible Γ-invariant vector spaces are one-dimensional and $r = n$, see for example [16, Proposition 9.5]. We obtain that $U_j = V_j$ for $j = 1, \ldots, n$, where the V_j form a complete set of non-isomorphic irreducible and one-dimensional Γ-invariant vector spaces. The representation V is called the *regular representation* of Γ. $\Diamond$

Corollary 3.5 *Suppose the conditions of Lemma 3.2 and assume that Γ is an Abelian group (of order n). Given $k \in \{1, \ldots, n\}$ then*

$$M_{1k} = 0 \ \Leftrightarrow \sum_{j \in \{1, \ldots, n\}} \chi_j(\gamma_k) M^j = 0. \tag{3.5}$$

Proof: By Proposition 3.3 and Remark 3.4 we get

$$M_{1k} = 0 \ \Leftrightarrow \sum_{j \in \{1, \ldots, n\}} \langle P^j e_1, \chi_j(\gamma_k) M^j P^j e_1 \rangle = 0 \ \Leftrightarrow \sum_{j \in \{1, \ldots, n\}} \overline{\chi_j(\gamma_k) M^j} \langle P^j e_1, P^j e_1 \rangle = 0.$$

As $\langle P^j e_1, P^j e_1 \rangle = 1/|\Gamma|$ by Lemma 3.2, formula (3.5) follows. $\square$

3.2 Linear analysis on $\mathbb{C}^l \otimes \mathbb{C}^n$

We extend now the action of Γ on $V = \mathbb{C}^n$ to the space $\mathbb{C}^{ln} \cong \mathbb{C}^l \otimes V$ given by

$$\mathbf{T}(\gamma)(y \otimes v) = y \otimes (T(\gamma)v), \quad \gamma \in \Gamma,\, y \in \mathbb{C}^l,\, v \in V.$$

Thus Γ acts trivially on $\mathbb{C}^l$ and on V as in (3.2). It follows then that the isotypic decomposition of $\mathbb{C}^l \otimes V$ for the action of Γ on $\mathbb{C}^l \otimes V$ is

$$\mathbb{C}^l \otimes V = \left(\mathbb{C}^l \otimes U_1\right) \oplus \cdots \oplus \left(\mathbb{C}^l \otimes U_r\right)$$

where $U_1, \ldots, U_r$ are the isotypic components of the action of Γ on V. Observe that if $\mathbf{P}^j$ denotes the projection operator onto the isotypic component $\mathbb{C}^l \otimes U_j$, we have that

$$\mathbf{P}^j(y \otimes v) = y \otimes P^j(v)$$

where P^j is the projection operator defined on V and onto the isotypic component U_j for the action of Γ on V.

Suppose now that $\mathbf{M} \in \mathrm{gl}(\mathbb{C}^l \otimes V)$ commutes with Γ. Thus we have that

$$\mathbf{M}(\mathbb{C}^l \otimes U_j) \subseteq \mathbb{C}^l \otimes U_j.$$

Denote by $\mathbf{M}^j$ the restriction of $\mathbf{M}$ to $\mathbb{C}^l \otimes U_j$.

We can define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^l \otimes V$ by extending the inner product $\langle \cdot, \cdot \rangle$ on V in the following way. Denote by $c_1 = (1, 0, \ldots, 0, 1)$, $\ldots$, $c_l = (0, 0, \ldots, 0, 1)$. Thus $\{c_1, \ldots, c_l\}$ is the canonical basis of $\mathbb{C}^l$. Define then:

$$\langle c_i \otimes v_1, c_t \otimes v_2 \rangle = \delta_{it} \langle v_1, v_2 \rangle, \quad i, t = 1, \ldots, l; \; v_1, v_2 \in V,$$

where δ_{it} is equal to 1 if $i = t$ and 0 otherwise. This corresponds to the standard (Euclidean) inner product on $\mathbb{C}^{ln}$ and so we use the same symbol.

Given $k \in \{1, \ldots, n\}$, we are now interested in using the projection operators $\mathbf{P}^j$ to describe the restrictions on the $\mathbf{M}^j$ imposed by the set of l^2 conditions of the type

$$\langle c_i \otimes e_1, \mathbf{M}(c_t \otimes e_k) \rangle = 0, \quad i, t = 1, \ldots, l.$$

We have the following generalization of Lemma 3.2:

Lemma 3.6 *For $j = 1, \ldots, r$, let P^j denote the projection of V onto the isotypic component U_j of type V_j, χ_j the character of the irreducible Γ-invariant vector space V_j and d_j the dimension of V_j. Given $i, t \in \{1, \ldots, l\}$ we have:*

$$\begin{aligned}
\langle \mathbf{P}^j(c_i \otimes e_1), \mathbf{P}^j(c_t \otimes e_1) \rangle &= \delta_{it} \langle P^j e_1, P^j e_1 \rangle = \delta_{it} \left(\frac{d_j}{|\Gamma|} \right)^2 \sum_{k=1}^{n} \left| \sum_{\gamma \in \gamma_k H} \chi_j(\gamma) \right|^2, \\
\mathbf{P}^j(c_t \otimes e_k) &= c_t \otimes P^j e_k = c_t \otimes T(\gamma_k) P^j, \quad k = 1, \ldots, n.
\end{aligned}$$

In particular, if χ_j is a linear character of Γ, then

$$\mathbf{P}^j(c_t \otimes e_k) = c_t \otimes \chi_j(\gamma_k) P^j e_1, \quad k = 1, \ldots, n.$$

The generalization of Proposition 3.3 is:

Proposition 3.7 *Suppose the conditions of Lemma 3.6 and let $I = \{1, \ldots, r\}$. Given $k \in \{1, \ldots, n\}$ and $i, t \in \{1, \ldots, l\}$ then*

$$\begin{aligned}
\langle c_i \otimes e_1, \mathbf{M}(c_t \otimes e_k) \rangle = 0 \quad &\Leftrightarrow \quad \sum_{j \in I} \langle \mathbf{P}^j(c_i \otimes e_1), \mathbf{M}^j \mathbf{P}^j(c_t \otimes e_k) \rangle = 0, \\
&\Leftrightarrow \quad \sum_{j \in I} \langle c_i \otimes P^j(e_1), \mathbf{M}^j(c_t \otimes P^j(e_k)) \rangle = 0, \\
&\Leftrightarrow \quad \sum_{j \in I} \langle c_i \otimes P^j(e_1), \mathbf{M}^j(c_t \otimes T(\gamma_k) P^j(e_1)) \rangle = 0.
\end{aligned} \tag{3.6}$$

Corollary 3.8 *Suppose the conditions of Lemma 3.6 and assume that Γ is an Abelian group (of order n). Given $k \in \{1, \ldots, n\}$ and $i, t \in \{1, \ldots, l\}$ then*

$$\langle c_i \otimes e_1, \mathbf{M}(c_t \otimes e_k) \rangle = 0 \;\Leftrightarrow\; \sum_{j \in \{1, \ldots, n\}} \chi_j(\gamma_k) \mathbf{M}^j_{it} = 0 \tag{3.7}$$

where $\mathbf{M}^j_{it} = \langle c_i \otimes P^j(e_1), \mathbf{M}^j \left(c_t \otimes P^j(e_1) \right) \rangle$.

Proof: By Proposition 3.7 and Remark 3.4 we get

$$\begin{aligned}\langle c_i \otimes e_1, \mathbf{M}(c_t \otimes e_k)\rangle = 0 \quad &\Leftrightarrow \sum_{j\in\{1,\ldots,n\}} \langle c_i \otimes P^j(e_1), \mathbf{M}^j(c_t \otimes \chi_j(\gamma_k)P^j(e_1))\rangle = 0, \\ &\Leftrightarrow \sum_{j\in\{1,\ldots,n\}} \overline{\chi_j(\gamma_k)\mathbf{M}^j_{it}} = 0.\end{aligned}$$

□

3.3 Examples

We apply the above results to the cyclic and dihedral groups.

3.3.1 The cyclic group $\mathbb{Z}_n$

Let $\mathbb{Z}_n$ be a cyclic group of order n generated by an element a satisfying $a^n = e$. Put $\omega = e^{i2\pi/n}$. Then $\mathbb{Z}_n$ has n distinct linear characters χ_j, $j = 1, \ldots, n$, given by

$$\chi_j\left(a^r\right) = \omega^{(j-1)r}, \quad j = 1, \ldots, n.$$

Here $r \in \{1, \ldots, n-1\}$ where $a^0 = e$. Consider now Γ the subgroup of $\mathbb{S}_n$ isomorphic to $\mathbb{Z}_n$ permuting transitively (and faithfully) the set $\{1, \ldots, n\}$ and generated by

$$\alpha = (1\ 2\ \ldots\ n).$$

Let $V = \mathbb{C}^n$ and $b = \{e_1, \ldots, e_n\}$ a basis of V and consider the action of Γ on V given by permutation of the corresponding coordinates (recall (3.2)). Thus if $\gamma_k = \alpha^{k-1}$ for $k = 1, \ldots, n$ then $e_k = T(\gamma_k)e_1$. The action of Γ on V corresponds to the regular representation of $\Gamma \cong \mathbb{Z}_n$: each distinct Γ-irreducible appears in the Γ-isotypic decomposition of V, with multiplicity one. Thus

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

where each irreducible V_j has character type χ_j. Direct application of Corollary 3.5 leads to:

Proposition 3.9 *Suppose $M \in \mathrm{gl}(V)$ and assume that M commutes with the above action of Γ. For $k, j = 1, \ldots, n$ denote by $M_{1k} = \langle e_1, Me_k\rangle$ and M^j the restriction of M to V_j. Then*

$$M_{1k} = 0 \;\Leftrightarrow \sum_{j\in\{1,\ldots,n\}} \omega^{(j-1)(k-1)} M^j = 0. \tag{3.8}$$

Corollary 3.10 *Suppose the conditions of Proposition 3.9 and write $M^j = M^j_R + iM^j_I$ where $M^j_R, M^j_I \in \mathbb{R}$. Assume $M \in \mathrm{gl}(\mathbb{R}^n)$.*
(i) If n is odd then

$$M_{1k} = 0 \;\Leftrightarrow\; M^1 + 2\sum_{j=2}^{(n-1)/2+1} \left(\cos\left(\frac{2\pi(k-1)(j-1)}{n}\right) M^j_R - \sin\left(\frac{2\pi(k-1)(j-1)}{n}\right) M^j_I\right) = 0.$$

(ii) If n is even then

$$M_{1k} = 0 \;\Leftrightarrow\; M^1 + \frac{M^{n/2+1}}{(-1)^{k-1}} + 2\sum_{j=2}^{n/2} \left(\cos\left(\frac{2\pi(k-1)(j-1)}{n}\right) M_R^j - \sin\left(\frac{2\pi(k-1)(j-1)}{n}\right) M_I^j \right) = 0.$$

Proof: Let V_j be an isotypic component. Thus V_j is Γ-irreducible and has character χ_j. If χ_j is real, then $M^j \in \mathrm{gl}(\mathbb{R})$. If χ_j is not real , then there is another isotypic component $V_i = \overline{V_j}$, so that $M^j = \overline{M^i}$. The decomplexification acts on $\widehat{V}_j = V_j \oplus V_i$, as

$$\widehat{M^j} = \begin{pmatrix} M_R^j & M_I^j \\ -M_I^j & M_R^j \end{pmatrix}$$

where $M^j = M_R^j + iM_I^j$. Accordingly, let us write $\chi_j \in \mathbb{C}$ as $\chi_j = (\chi_j)_R + i(\chi_j)_I$ with $(\chi_j)_R, (\chi_j)_I \in \mathbb{R}$. In (3.8) the sum $\chi_j M^j + \overline{\chi_j} M^i$ yields $2(\chi_j)_R M_R - 2(\chi_j)_I M_I$, so that

$$\sum_{j\in\{1,\ldots,n\}} \omega^{(j-1)(k-1)} M^j = 0 \Leftrightarrow$$

$$\sum_{\substack{j \text{ s.t.} \\ \chi_j \text{ real}}} \chi_j(\gamma_k) M^j + 2 \sum_{\substack{j \text{ s.t.} \\ \chi_j \text{ complex}}} \Big((\chi_j)_R(\gamma_k) M_R^j - (\chi_j)_I(\gamma_k) M_I^j \Big) = 0, \tag{3.9}$$

where we note that in the latter sum, for each real irreducible $\widehat{V}_j$, only one irreducible representation is taken. □

Example 3.11 We return to Example 1.1 and recall we are assuming the cells are one-dimensional and so the total phase space is $V = \mathbb{R}^4$. This space is decomposed into three isotypic components that in this case are irreducible: two one-dimensional isotypic components V_1, V_3, where $\mathbb{Z}_4$ acts trivially and non-trivially, respectively, and one two-dimensional isotypic component V_2 of complex type. More precisely, V_2 is the two-dimensional irreducible subspace of V where the action of $\mathbb{Z}_4$ is generated by an element that corresponds to the rotation by $\pi/2$ on the plane. We have $V = V_1 \oplus V_2 \oplus V_3$ and $M = M^1 \oplus M^2 \oplus M^3$. Here M^1, M^3 are linear maps corresponding to the restrictions of M to V_1, V_3, respectively and M^2 corresponds to the restriction of M to V_2. Since V_2 is of complex type and M^2 commutes with $\mathbb{Z}_4$, a basis of V_2 can be chosen such that M^2 with respect with this basis is:

$$M^2 = \begin{pmatrix} M_R^2 & M_I^2 \\ -M_I^2 & M_R^2 \end{pmatrix}$$

which has eigenvalues $M_R^2 \pm iM_I^2$. Recall the expressions for the eigenvalues of M obtained at the beginning of Example 1.1: $M^1 \equiv \lambda_1 = a+b+c+d$, $M^3 \equiv \lambda_2 = a+c-(b+d)$ and $M_R^2 + iM_I^2 \equiv \lambda_{3,-} = a-c-i(b-d)$.

Corollary 3.10 gives the following equivalences:

$$\begin{aligned} M^1 + M^3 + 2M_R^2 = 0 &\;\Leftrightarrow\; M_{11} = a = 0, \\ M^1 - M^3 - 2M_I^2 = 0 &\;\Leftrightarrow\; M_{12} = b = 0, \\ M^1 + M^3 - 2M_R^2 = 0 &\;\Leftrightarrow\; M_{13} = c = 0, \\ M^1 - M^3 + 2M_I^2 = 0 &\;\Leftrightarrow\; M_{14} = d = 0. \end{aligned}$$

which are easily verified. If we assume the cells are l-dimensional, the same relations hold, where now the matrices M^1, M^3, M_R^2, M_I^2 are $l \times l$-matrices. $\Diamond$

3.3.2 The dihedral group $\mathbb{D}_m$

Consider $\mathbb{D}_m$ the dihedral group of order $n = 2m$. Recall that $\mathbb{D}_m$ is the symmetry group of a regular m-sided polygon generated by elements a, b satisfying the relations $a^m = b^2 = (ab)^2 = e$. If m is odd then $\mathbb{D}_m$ has $(m+3)/2$ conjugacy classes with representatives $e, a, a^2, \cdots, a^{(m-1)/2}, b$, and so $(m+3)/2$ distinct non-isomorphic irreducible Γ-invariant vector spaces: two of dimension one and $(m-1)/2$ of dimension two. If m is even then $\mathbb{D}_m$ has $m/2+3$ conjugacy classes with representatives $e, a, a^2, \cdots, a^{m/2-1}, a^{m/2}, b, ab$, and so $m/2+3$ distinct non-isomorphic irreducible Γ-invariant vector spaces: four of dimension one and $m/2-1$ of dimension two. See Table 1 where the linear characters are denoted by χ_j and the two-dimensional by ψ_j.

m is odd					
γ	e	$a^t(1 \le t \le (m-1)/2$	b		
$\lvert C_{\mathbb{D}_m}(\gamma)\rvert$	$2m$	m	2		
χ_1	1	1	1		
χ_2	1	1	-1		
ψ_j	2	$\epsilon^{jt} + \overline{\epsilon}^{jt}$	0		
$(j = 1, \ldots, (m-1)/2)$					
m is even					
γ	e	$a^t(1 \le t \le m/2-1)$	$a^{\frac{m}{2}}$	b	ab
$\lvert C_{\mathbb{D}_m}(\gamma)\rvert$	$2m$	m	$2m$	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^t$	$(-1)^{\frac{m}{2}}$	1	-1
χ_4	1	$(-1)^t$	$(-1)^{\frac{m}{2}}$	-1	1
ψ_j	2	$\epsilon^{jt} + \overline{\epsilon}^{jt}$	$2(-1)^j$	0	0
$(j = 1, \ldots, m/2-1)$					

Table 1: Character table for the dihedral group $\mathbb{D}_m$ according to the parity of m [16]. Here $\epsilon = e^{2\pi i/m}$ and $C_{\mathbb{D}_m}(\gamma)$ is the centralizer of γ in $\mathbb{D}_m$.

Throughout let

$$p = \begin{cases} 2 & \text{if } m \text{ is odd}, \\ 4 & \text{if } m \text{ is even}, \end{cases} \qquad q = \begin{cases} \frac{m-1}{2} & \text{if } m \text{ is odd}, \\ \frac{m}{2} - 1 & \text{if } m \text{ is even}. \end{cases} \tag{3.10}$$

There are essentially two ways that $\Gamma \cong \mathbb{D}_m$ acts faithfully and transitively on a finite set. As before, let $H = \{\gamma \in \Gamma : \ \gamma(1) = 1\}$. Either H is the trivial group. This corresponds to the regular representation of $\mathbb{D}_m$ and $\Gamma \subseteq \mathbb{S}_n$ where $n = 2m$. Or $H \cong \{e, a^i b\}$ for some integer i between 0 and $m-1$ and $\Gamma \subseteq \mathbb{S}_m$. Observe that only the trivial group and the subgroups of $\mathbb{D}_m$ of type $\{e, a^i b\}$ do not contain nontrivial normal subgroups of $\mathbb{D}_m$.

Regular representation

Consider Γ the subgroup of $\mathbb{S}_n$ permuting transitively and faithfully the set $\{1,\dots,n\}$ and generated by

$$\begin{array}{rcl}\alpha & = & (1\ 2\ \dots\ m)(m+1\ m+2\ \dots\ 2m),\\ \beta & = & (1\ m+1)(2\ 2m)(3\ 2m-1)\dots(m\ m+2)\end{array}$$

where $n=2m$. Note that Γ is isomorphic to $\mathbb{D}_m$ and the group H is trivial.

Let $V=\mathbb{C}^n$ and $b=\{e_1,\dots,e_n\}$ a basis of V and consider the action of Γ on V given by permutation of the corresponding coordinates (recall (3.2)). Observe that if we denote

$$\gamma_k=\begin{cases}\alpha^{k-1} & \text{if } k=1,\dots,m,\\ \alpha^{k-m-1}\beta & \text{if } k=m+1,\dots,2m,\end{cases}$$

where $\gamma_1=\alpha^0\equiv e$ then

$$e_k=T(\gamma_k)e_1,\quad k=1,\dots,2m.$$

Moreover, the action of Γ on V corresponds to the regular representation of $\Gamma\cong\mathbb{D}_m$. That is, each distinct Γ-irreducible appears in the Γ-isotypic decomposition of V, with multiplicity equals to its dimension. The decomposition of V as a sum of its isotypic components is then given by:

$$V=U_0^1\oplus\cdots\oplus U_0^p\oplus U_1\oplus U_2\oplus\cdots\oplus U_q$$

where $U_0^1,\dots,U_0^p$ are one-dimensional (distinct) irreducible Γ-invariant subspaces of V of type $\chi_1,\dots,\chi_p$, and $U_1,\dots,U_q$ are each the sum of two irreducible Γ-invariant subspaces of V of dimension 2 and of type $\psi_1,\dots,\psi_q$. Recall that p,q are defined by (3.10).

Given j between 1 and q, define

$$w_1^j=\tfrac{1}{m}\left(e_1+\bar{\epsilon}^j e_2+\cdots+\bar{\epsilon}^{(m-1)j}e_m\right),\qquad w_4^j=\tfrac{1}{m}\left(e_1+\epsilon^j e_2+\cdots+\epsilon^{(m-1)j}e_m\right),$$

$$w_2^j=\tfrac{1}{m}\left(e_{m+1}+\epsilon^j e_{m+2}+\cdots+\epsilon^{(m-1)j}e_{2m}\right),\quad w_3^j=\tfrac{1}{m}\left(e_{m+1}+\bar{\epsilon}^j e_{m+2}+\cdots+\bar{\epsilon}^{(m-1)j}e_{2m}\right),$$

where $\epsilon=e^{i2\pi/m}$ and

$$b_j=\{w_1^j,w_2^j,w_3^j,w_4^j\}.$$

The vectors $w_1^j,w_2^j,w_3^j,w_4^j\in V$ are linearly independent, $\langle w_k^j,w_l^j\rangle=0$ if $k\neq l$, and $\langle w_l^j,w_l^j\rangle=1/m$ for $l=1,\dots,4$. Moreover, $\mathbf{C}\left(\{w_1^j,w_2^j\}\right)$ and $\mathbf{C}\left(\{w_3^j,w_4^j\}\right)$ are Γ-isomorphic irreducible subspaces of V of character type ψ_j. It follows then that

$$U_j=\mathbf{C}\left(\{w_1^j,w_2^j,w_3^j,w_4^j\}\right)=\mathbf{C}\left(\{w_1^j,w_2^j\}\right)\oplus\mathbf{C}\left(\{w_3^j,w_4^j\}\right)\cong V_j\oplus V_j.$$

Moreover, if we consider the projection operator $P^j:\ V\to V$ onto the isotypic component U_j:

$$P^j=\frac{2}{|\mathbb{D}_m|}\sum_{\gamma\in\mathbb{D}_m}\overline{\psi_j(\gamma)}T(\gamma)$$

we obtain

$$P^je_k=\begin{cases}\epsilon^{j(k-1)}w_1^j+\bar{\epsilon}^{j(k-1)}w_4^j & \text{if } 1\le k\le m,\\ \epsilon^{j(k-1-m)}w_3^j+\bar{\epsilon}^{j(k-1-m)}w_2^j & \text{if } m+1\le k\le 2m.\end{cases}\tag{3.11}$$

Suppose $M \in \mathrm{gl}(V)$ commutes with Γ. For $j = 1, \ldots, q$ denote by M^j the restriction of M to the isotypic component U_j with respect to the basis b_j. Thus each M^j is a 4×4 matrix with complex entries commuting with Γ of the following form:

$$M^j = (M|_{U_j})_{b_j} = \begin{pmatrix} M^j_{1,1} & 0 & M^j_{1,3} & 0 \\ 0 & M^j_{1,1} & 0 & M^j_{1,3} \\ M^j_{3,1} & 0 & M^j_{3,3} & 0 \\ 0 & M^j_{3,1} & 0 & M^j_{3,3} \end{pmatrix}. \tag{3.12}$$

For $k = 1, \ldots, p$ denote by $M^k_0 = M|_{U^k_0}$. We have the following result:

Proposition 3.12 *(i) If $1 \leq k \leq m$ then*

$$M_{1k} = 0 \;\Leftrightarrow\; \frac{1}{2}\sum_{j=1}^{p} \chi_j(\gamma_k) M^j_0 + \sum_{j=1}^{q}\left(\epsilon^{j(k-1)} M^j_{1,1} + \overline{\epsilon}^{j(k-1)} M^j_{3,3}\right) = 0.$$

(ii) If $m+1 \leq k \leq 2m$ then

$$M_{1k} = 0 \;\Leftrightarrow\; \frac{1}{2}\sum_{j=1}^{p} \chi_j(\gamma_k) M^j_0 + \sum_{j=1}^{q}\left(\epsilon^{j(k-1-m)} M^j_{1,3} + \overline{\epsilon}^{j(k-1-m)} M^j_{3,1}\right) = 0.$$

Proof: We apply Proposition 3.3. Denote by P_j the projections onto the one-dimensional Γ-irreducibles U^j_0, for $j = 1, \ldots, p$. By Lemma 3.2 we have $P_j e_k = \chi_j(\gamma_k) P_j e_1$ and $\langle P_j e_1, P^j e_1 \rangle = 1/|\Gamma| = 1/2m$. Thus

$$\langle P_j e_1, M^j_0 P_j e_k \rangle = \langle P_j e_1, M^j_0 \chi_j(\gamma_k) P_j e_1 \rangle = \overline{\chi_j(\gamma_k) M^j_0} \langle P_j e_1, P_j e_1 \rangle = \frac{1}{2m}\overline{\chi_j(g_k) M^j_0}.$$

Recall that P^j denotes the projection onto the isotypic component U_j, for $j = 1, \ldots, q$, and $M^j = (M|_{U_j})_{b_j}$ is given by (3.12). Using (3.11), if $k \in \{1, \ldots, m\}$ then

$$\begin{aligned} \langle P^j e_1, M^j P^j e_k \rangle &= \overline{\epsilon}^{j(k-1)} \langle w^j_1, M^j w^j_1 \rangle + \epsilon^{j(k-1)} \langle w^j_1, M^j w^j_4 \rangle + \\ &\quad \overline{\epsilon}^{j(k-1)} \langle w^j_4, M^j w^j_1 \rangle + \epsilon^{j(k-1)} \langle w^j_4, M^j w^j_4 \rangle. \end{aligned}$$

As $\langle w^j_i, w^j_i \rangle = 1/m$ for $i = 1, 2, 3, 4$ and $\langle w^j_r, w^j_s \rangle = 0$ if $r \neq s$, it follows that

$$\langle P^j e_1, M^j P^j e_k \rangle = \frac{1}{m}\left(\overline{\epsilon}^{j(k-1)} \overline{M^j_{1,1}} + \epsilon^{j(k-1)} \overline{M^j_{3,3}}\right).$$

The proof of (ii) is similar. $\square$

Corollary 3.13 *Suppose the conditions of Proposition 3.12 where now $M \in gl(\mathbb{R}^n)$.*
(i) If $1 \leq k \leq m$ then

$$M_{1k} = 0 \;\Leftrightarrow\; \frac{1}{2}\sum_{j=1}^{p} \chi_j(\gamma_k) M^j_0 + 2\sum_{j=1}^{q} \mathrm{Re}\left(\epsilon^{j(k-1)} M^j_{1,1}\right) = 0.$$

(ii) If $m+1 \le k \le 2m$ then

$$M_{1k} = 0 \;\Leftrightarrow\; \frac{1}{2}\sum_{j=1}^{p} \chi_j\left(\gamma_k\right) M_0^j + 2\sum_{j=1}^{q} \mathrm{Re}\left(\epsilon^{j(k-1-m)} M_{1,3}^j\right) = 0.$$

Proof: For $j = 1, \ldots, q$ define

$$B_j = \{w_1^j + w_4^j,\, w_2^j + w_3^j,\, i(w_1^j - w_4^j),\, i(w_2^j - w_3^j)\}.$$

It follows that B_j is a basis of U_j. Moreover, since $M \in \mathrm{gl}(\mathbb{R}^n)$ we have that $M|_{U_j}$ with respect to this basis has real entries. Comparing the matrices of $M|_{U_j}$ with respect to b_j (recall (3.12)) and B_j we obtain that

$$M_{3,3}^j = \overline{M}_{1,1}^j, \quad M_{3,1}^j = \overline{M}_{1,3}^j.$$

The result now follows from Proposition 3.12. $\square$

Example 3.14 Revisiting the $\mathbb{D}_3$-symmetric networks of Example 1.2 in the Introduction, observe that by Corollary 3.13 the absence of connections from cells $2, 3$ to cell 1 leads to the following two conditions:

$$\begin{array}{lcl} M_{12} = 0 & \Leftrightarrow & \frac{1}{2}M_0^1 + \frac{1}{2}M_0^2 + 2\mathrm{Re}\left(\epsilon M_{1,1}^1\right) = 0, \\ M_{13} = 0 & \Leftrightarrow & \frac{1}{2}M_0^1 + \frac{1}{2}M_0^2 + 2\mathrm{Re}\left(\epsilon^2 M_{1,1}^1\right) = 0, \end{array}$$

where $\epsilon = e^{i2\pi/3}$. Subtracting these equations, we obtain

$$\mathrm{Re}\left((\epsilon^2 - \epsilon)M_{1,1}^1\right) = 0 \Leftrightarrow \mathrm{Im}(M_{1,1}^1) = 0 \Leftrightarrow M_{1,1}^1 \in \mathbb{R}.$$

Thus the eigenvalues of M^1 as in (3.12) for $j = 1$, and where $M_{1,1}^1 = M_{3,3}^1 \in \mathbb{R}$, $M_{1,3}^1 = \overline{M_{3,1}^1} \in \mathbb{C}$, are the eigenvalues of the matrix

$$\left(\begin{array}{cc} M_{1,1}^1 & M_{1,3}^1 \\ \overline{M_{1,3}^1} & M_{1,1}^1 \end{array} \right)$$

each with multiplicity two. Moreover, this matrix has real eigenvalues. Thus no Hopf bifurcation can occur for coupled cell networks with the structure given by Figure 2 (right) as we had shown before in the Introduction by other method. $\Diamond$

Remark 3.15 More generally, considering the $\mathbb{D}_m$-symmetric networks where $\mathbb{D}_m$ permutes transitively $2m$ cells (recall the beginning of Section 3.3.2) and using Corollary 3.13, we can prove that no Hopf bifurcation can occur for coupled cell systems with one-dimensional cells and associated with those networks that have no connections from cells $m, m-1, \ldots, 2$ to cell 1. Observe that again the Jacobian matrix at a fully symmetric equilibrium solution will be in this case a symmetric matrix.

$\Diamond$

Suppose now that $\mathbf{M} \in \mathrm{gl}(\mathbb{C}^l \otimes V)$ commutes with Γ where the action of Γ on $\mathbb{C}^l$ is trivial.

For $j = 1, \ldots, p$, if $\{w_j\}$ is a basis of U_0^j then $\{c_1 \otimes w_j, \ldots, c_l \otimes w_j\}$ is a basis of $\mathbb{C}^l \otimes U_0^j$. Denote by $\mathbf{M}_0^j$ the restriction of $\mathbf{M}$ to the isotypic component $\mathbb{C}^l \otimes U_0^j$ with respect to this basis which is a $l \times l$ matrix of complex entries.

For $j = 1, \ldots, q$, since $b_j = \{w_1^j, w_2^j, w_3^j, w_4^j\}$ is a basis of U_j, it follows then that

$$\mathbf{b}_j = \{c_1 \otimes w_1^j, \ldots, c_l \otimes w_1^j, c_1 \otimes w_2^j, \ldots, c_l \otimes w_2^j, c_1 \otimes w_3^j, \ldots, c_l \otimes w_3^j, c_1 \otimes w_4^j, \ldots, c_l \otimes w_4^j\}$$

is a basis of $\mathbb{C}^l \otimes U_j$. Denote by $\mathbf{M}^j$ the restriction of $\mathbf{M}$ to the isotypic component $\mathbb{C}^l \otimes U_j$ with respect to this basis which is a $4l \times 4l$ matrix with complex entries of the following form:

$$\mathbf{M}^j \equiv (\mathbf{M}|_{\mathbb{C}^l \otimes U_j})_{\mathbf{b}_j} = \begin{pmatrix} \mathbf{M}_{1,1}^j & 0 & \mathbf{M}_{1,3}^j & 0 \\ 0 & \mathbf{M}_{1,1}^j & 0 & \mathbf{M}_{1,3}^j \\ \mathbf{M}_{3,1}^j & 0 & \mathbf{M}_{3,3}^j & 0 \\ 0 & \mathbf{M}_{3,1}^j & 0 & \mathbf{M}_{3,3}^j \end{pmatrix} \tag{3.13}$$

where $\mathbf{M}_{1,1}^j, \mathbf{M}_{1,3}^j, \mathbf{M}_{3,1}^j, \mathbf{M}_{3,3}^j$ are $l \times l$ matrices with complex entries.

Proposition 3.16 *(i) If $1 \leq k \leq m$ then*

$$\left.\begin{array}{l} (c_i \otimes e_1, \mathbf{M}(c_t \otimes e_k)) = 0 \\ \forall i, t \in \{1, \ldots, l\} \end{array}\right\} \Leftrightarrow \frac{1}{2}\sum_{j=1}^{p} \chi_j\left(\gamma_k\right) \mathbf{M}_0^j + \sum_{j=1}^{q} \left(\epsilon^{j(k-1)} \mathbf{M}_{1,1}^j + \bar{\epsilon}^{j(k-1)} \mathbf{M}_{3,3}^j\right) = 0.$$

(ii) If $m+1 \leq k \leq 2m$ then

$$\left.\begin{array}{l} (c_i \otimes e_1, \mathbf{M}(c_t \otimes e_k)) = 0 \\ \forall i, t \in \{1, \ldots, l\} \end{array}\right\} \Leftrightarrow \frac{1}{2}\sum_{j=1}^{p} \chi_j\left(\gamma_k\right) \mathbf{M}_0^j + \sum_{j=1}^{q} \left(\epsilon^{j(k-1-m)} \mathbf{M}_{1,3}^j + \bar{\epsilon}^{j(k-1-m)} \mathbf{M}_{3,1}^j\right) = 0.$$

Proof: Direct application of Proposition 3.7. □

Nonregular representation

We consider now $\Gamma \cong \mathbb{D}_m$ permuting transitively and faithfully the set $\{1, \ldots, m\}$. Thus $\Gamma \subseteq \mathbb{S}_m$ and $H \cong \{e, a^i b\}$ for some integer i between 0 and $m-1$.

(i) Let m be odd and consider Γ the subgroup of $\mathbb{S}_m$ generated by

$$\begin{array}{rcl} \alpha & = & (1\ 2\ \ldots\ m), \\ \beta & = & (2\ m)(3\ m-1)\ldots((m+1)/2\ \ (m+1)/2+1)\,. \end{array} \tag{3.14}$$

Note that $H = \{e, \beta\}$. Let $V = \mathbb{C}^m$ and $\{e_1, \ldots, e_m\}$ a basis of V and consider the action of Γ on V given by permutation of the corresponding coordinates (3.2).

Observe that if we denote by $\gamma_k = \alpha^{k-1}$, $k = 1, \ldots, m$ where $\gamma_1 = \alpha^0 \equiv e$ then $e_k = T(\gamma_k)e_1$.

We have the following general result:

Lemma 3.17 *Let χ_i be an irreducible character of Γ and H a subgroup of Γ acting trivially on e_1. Then the multiplicity of the irreducible χ_i that appears in the Γ-isotypic decomposition of V is the number $1/|H| \sum_{h \in H} \chi_i(h)$.*

Proof: Denote by ψ_Γ the character of the action of Γ on V as defined above, and let ψ_H be the trivial character of H.

The action of H on $U = \mathbb{C}(\{e_1\})$ is trivial. Thus $U \subseteq V$ is H-invariant and has character ψ_H. Now observe that $V = \mathbb{C}\left(\{\gamma u : \ \gamma \in \Gamma, u \in U\}\right)$. That is, V is the Γ-invariant subspace of V induced from U and it is denoted by $U \uparrow \Gamma$. Also, ψ_Γ is denoted by $\psi_H \uparrow \Gamma$.

Now as V is Γ-invariant, then V is H-invariant and $V \downarrow H$ is called the restriction of V to H. The character of $V \downarrow H$ is obtained from the character χ on the elements of H only and is denoted by $\chi \downarrow H$.

Recall that if χ_j and χ_k are two functions from Γ to $\mathbb{C}$, then

$$\langle \chi_j, \chi_k \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_j(g) \overline{\chi_k(g)}$$

is an inner product on the complex vector space of functions from Γ to $\mathbb{C}$.

The multiplicity of an irreducible with character χ_i appearing in the Γ-isotypic decomposition of V is given by $\langle \psi_\Gamma, \chi_i \rangle_\Gamma = \langle \psi_H \uparrow \Gamma, \chi_i \rangle_\Gamma$. Now by the Frobenius Reciprocity Theorem [16, page 232] we have that $\langle \psi_H \uparrow \Gamma, \chi_i \rangle_\Gamma = \langle \psi_H, \chi_i \downarrow H \rangle_H$. □

Recall Table 1, where χ_1, χ_2 are the one-dimensional characters of $\Gamma \cong \mathbb{D}_m$ (m is odd), and ψ_j for $j = 1, \ldots, q$ where $q = (m-1)/2$ are the two-dimensional irreducible characters of Γ. Taking $H = \{e, \beta\} \cong \{e, b\}$ we obtain

$$\begin{array}{ll} \chi_1(e) = 1, & \chi_1(\alpha^i \beta) = 1, \\ \chi_2(e) = 1, & \chi_2(\alpha^i \beta) = -1, \\ \psi_j(e) = 2, & \psi_j(\alpha^i \beta) = 0, \quad j = 1, \ldots, q. \end{array}$$

Here $i = 0, \ldots, m-1$. Using Lemma 3.17 the irreducibles that appear in V are of type $\chi_1, \psi_1, \ldots, \psi_q$, and all appear once:

$$\psi_\Gamma = \chi_1 + \psi_1 + \cdots + \psi_q$$

and

$$V = U_0^1 \oplus U_1 \oplus U_2 \oplus \cdots \oplus U_q$$

where U_0^1, U_j are irreducible with character type χ_1, ψ_j, respectively. Other transitive and faithful actions of $\Gamma \cong \mathbb{D}_m$ on $\{1, \ldots, m\}$ where m is odd and such that $H = \{e, \alpha^i \beta\} \cong \{e, a^i b\}$ give the same representation.

(ii) Let m be even and consider Γ the subgroup of $\mathbb{S}_m$ isomorphic to $\mathbb{D}_m$ permuting transitively and faithfully the set $\{1, \ldots, m\}$ and generated by

$$\begin{array}{rcl} \alpha & = & (1\ 2\ \ldots\ m), \\ \beta & = & (2\ m)(3\ m-1)\ldots(m/2\ \ m/2+2)\,. \end{array} \tag{3.15}$$

Again, we take $\gamma_k = \alpha^{k-1}$ and so $e_k = T(\gamma_k) e_1$ for $k = 1, \ldots, m$.

Recall Table 1, where $\chi_1, \chi_2, \chi_3, \chi_4$ are the one-dimensional characters of $\Gamma \cong \mathbb{D}_m$ (m is even), and ψ_j for $j = 1, \ldots, q$ where $q = m/2 - 1$ are the two-dimensional irreducible characters of Γ.

Taking $H=\{e,\beta\}\cong\{e,b\}$, we have that

$$\begin{array}{lll}
\chi_1(e)=1, & \chi_1(\alpha^{2i}\beta)=1, & \chi_1(\alpha^{2i+1}\beta)=1,\\
\chi_2(e)=1, & \chi_2(\alpha^{2i}\beta)=-1, & \chi_2(\alpha^{2i+1}\beta)=-1,\\
\chi_3(e)=1, & \chi_3(\alpha^{2i}\beta)=1, & \chi_3(\alpha^{2i+1}\beta)=-1,\\
\chi_4(e)=1, & \chi_4(\alpha^{2i}\beta)=-1, & \chi_4(\alpha^{2i+1}\beta)=1,\\
\psi_j(e)=2, & \psi_j(\alpha^{i}\beta)=0, \quad j=1,\ldots,q. &
\end{array}$$

By Lemma 3.17 the Γ-irreducibles that appear (once) in V are $\chi_1,\chi_3,\psi_1,\ldots,\psi_q$:

$$\psi_\Gamma=\chi_1+\chi_3+\psi_1+\cdots+\psi_q$$

and

$$V=U_0^1\oplus U_0^3\oplus U_1\oplus U_2\oplus\cdots\oplus U_q$$

where U_0^1,U_0^3,U_j are Γ-irreducible with character types χ_1,χ_3,ψ_j respectively.

Taking another transitive and faithful action of $\Gamma\cong\mathbb{D}_m$ on $\{1,\ldots,m\}$ such that $H=\{e,\alpha^{2i}\beta\}\cong\{e,a^{2i}b\}$ corresponds to the same representation of Γ. If $H=\{e,\alpha^{2i+1}\beta\}\cong\{e,a^{2i+1}b\}$ then we obtain a *quasi-equivalent* representation, that is, equivalence composed with an outer automorphism of Γ:

$$\psi_\Gamma=\chi_1+\chi_4+\psi_1+\cdots+\psi_q$$

and

$$V=U_0^1\oplus U_0^4\oplus U_1\oplus U_2\oplus\cdots\oplus U_q$$

where U_0^1,U_0^4,U_j are Γ-irreducible with character type χ_1,χ_4,ψ_j.

Proposition 3.18 *Let $\Gamma\cong\mathbb{D}_m$ and $V=\mathbb{C}^m$ with basis $b=\{e_1,\ldots,e_m\}$, and consider the action of Γ on V by permutation of coordinates given by (3.14) if m is odd, and (3.15) if m is even. Let $M\in\mathrm{gl}(V)$ commuting with Γ. For $1\le k\le m$ let $\gamma_k=\alpha^{k-1}$ and recall that $e_k=T(\gamma_k)e_1$.*
(i) Suppose that m is odd, and let $V=U_0^1\oplus U_1\oplus U_2\oplus\cdots\oplus U_q$, where U_0^1 is irreducible of trivial type and U_j is Γ-irreducible with character type ψ_j. Consider $M_0^1=M|_{U_0^1}$ and $M^j=M|_{U_j}\cong M_{1,1}^j\mathrm{Id}_{2\times2}$. Then

$$M_{1k}=0\;\Leftrightarrow\;M_0^1+\sum_{j=1}^{q}\psi_j(\gamma_k)M_{1,1}^j=0.$$

(ii) Suppose that m is even and let $V=U_0^1\oplus U_0^3\oplus U_1\oplus U_2\oplus\cdots\oplus U_q$, where U_0^1,U_0^3,U_j are Γ-irreducible with character types χ_1,χ_3,ψ_j and $M_0^j=M|_{U_0^j}$, $M^j=M|_{U_j}\cong M_{1,1}^j\mathrm{Id}_{2\times2}$. Then

$$M_{1k}=0\;\Leftrightarrow\;M_0^1+\chi_3(\gamma_k)M_0^3+\sum_{j=1}^{q}\psi_j(\gamma_k)M_{1,1}^j=0.$$

Proof: We apply Proposition 3.3. Denote by P_j the projection of V onto the isotypic component U_0^j with linear character type χ_j. By Lemma 3.2, if χ_j is linear then $P_je_k=\chi_j(\gamma_k)P_je_1$. If $j=1$, or if $j=3$ and m is even, then $\langle P_je_1,P_je_1\rangle=1/m$.

For $j=1,\ldots,q$, consider

$$w_1^j=\frac{1}{m}\left(e_1+\overline{\epsilon}^je_2+\cdots+\overline{\epsilon}^{(m-1)j}e_m\right),\qquad w_4^j=\frac{1}{m}\left(e_1+\epsilon^je_2+\cdots+\epsilon^{(m-1)j}e_m\right)\tag{3.16}$$

where $\epsilon = e^{i2\pi/m}$. Then $\langle w_1^j, w_1^j\rangle = \langle w_4^j, w_4^j\rangle = 1/m$ and $\langle w_1^j, w_4^j\rangle = 0$. Moreover, $\mathbb{C}(\{w_1^j, w_4^j\})$ is an irreducible Γ-invariant subspace of V with character type ψ_j. Thus $U_j = \mathbb{C}(\{w_1^j, w_4^j\})$ and $P^j(e_k) = \epsilon^{j(k-1)} w_1^j + \overline{\epsilon}^{j(k-1)} w_4^j$. The proof now follows as in the proof of Proposition 3.12.

$\square$

Corollary 3.19 *Suppose the conditions of Proposition 3.18 where now $M \in gl(\mathbb{R}^m)$.*
(i) If m is odd then

$$M_{1k} = 0 \;\Leftrightarrow\; M_0^1 + \sum_{j=1}^{q} 2\cos\left(\frac{2\pi j(k-1)}{m}\right) M_{1,1}^j = 0.$$

(ii) If m is even then

$$M_{1k} = 0 \;\Leftrightarrow\; M_0^1 + (-1)^{k-1} M_0^3 + \sum_{j=1}^{q} 2\cos\left(\frac{2\pi j(k-1)}{m}\right) M_{1,1}^j = 0.$$

Proof: Recall Table 1. Note also that from (3.16) we get that $\{w_1^j + w_4^j, i(w_1^j - w_4^j)\}$ is a basis of $U_j = \mathbb{C}(\{w_1^j, w_4^j\})$ where $M^j = M|_{U_j}$ has real entries. Comparing with $M_{1,1}^j \mathrm{Id}_{2\times 2}$ we obtain that $M_{1,1}^j \in \mathbb{R}$. $\square$

4 Codimension one eigenvalue movements

In this section we show that codimension one eigenvalue movement through the imaginary axis for coupled cell networks with Abelian symmetry are independent of the network structure, if the network is connected and the cells are active, thus proving Theorem 1.3.

Suppose G is a coupled cell network with n cells, and assume that the phase space of the cells is $\mathbb{R}^l$ and so the total phase space is $\mathbb{R}^{nl}$. To facilitate the analysis we first consider the complexification $\mathbb{C}^{nl}$ of $\mathbb{R}^{nl}$ and later deduce the consequences implied by the fact that the phase space is real.

Assume that the network is equivariant with respect to a transitive and faithful permutation action of an Abelian group Γ on the set of cells $\{1, \ldots, n\}$. Thus the total phase space is $\mathbb{C}^l \otimes V$ where $V = \mathbb{C}^n$ and we assume that with respect to the cell coordinates $v_1, \ldots, v_n$ the action of Γ is given by the homomorphism $\mathbf{T}$ from Γ to $\mathrm{GL}(\mathbb{C}^l \otimes V)$ defined by

$$\mathbf{T}(\gamma)(y \otimes v) = y \otimes (T(\gamma)v), \quad \gamma \in \Gamma,\, y \in \mathbb{C}^l,\, v \in V,$$

where $T:\ \Gamma \to \mathrm{GL}(V)$ is

$$T(\gamma)(v_1, \ldots, v_n) = \left(v_{\gamma^{-1}(1)}, \ldots, v_{\gamma^{-1}(n)}\right), \quad \gamma \in \Gamma,\ (v_1, \ldots, v_n) \in V. \tag{4.17}$$

Following the notation of Section 3, the isotypic decomposition of $\mathbb{C}^l \otimes V$ under the action of Γ is

$$\mathbb{C}^l \otimes V = (\mathbb{C}^l \otimes V_1) \oplus \cdots \oplus (\mathbb{C}^l \otimes V_n)$$

where $V_1, \ldots, V_n$ form a complete set of Γ-isomorphic irreducible spaces under the action of Γ on V. Recall Remark 3.4.

The linearization $\mathbf{M} \in \mathrm{gl}(\mathbb{C}^l \otimes V)$ at a fully symmetric equilibrium of a system of ordinary differential equations defined on $\mathbb{C}^l \otimes V$ corresponding to the network G is assumed to be Γ-equivariant and hence $\mathbf{M}$ leaves each isotypic component $\mathbb{C}^l \otimes V_j$ invariant. As before we denote by $\mathbf{M}^j$ the restriction of $\mathbf{M}$ to $\mathbb{C}^l \otimes V_j$.

Since Γ is Abelian, each irreducible V_j has complex dimension one. In view of the complexification, we need to be aware that there are two types of irreducible representations. Either χ_j is real and V_j corresponds to the complexification of an irreducible real space W_j with character χ_j and the real commuting matrices on W_j are the real scalar multiples of the identity on W_j. In this case, W_j is called Γ-*absolutely irreducible* and V_j is said to be of *real type*. The other case is when χ_j is complex. Then the conjugate $\overline{\chi_j}$ is also an irreducible character distinct from χ_j, associated say with $\overline{V_j}$. Moreover, $V_j \oplus \overline{V_j}$ is a real Γ-irreducible with character $\chi_j + \overline{\chi_j}$ and the vector space of the real commuting matrices defined on $V_j \oplus \overline{V_j}$ is isomorphic to $\mathbb{C}$. In this case V_j is said to be of *complex type*. For details, see for instance [16]. A space W is called Γ-*simple* if W is the direct sum of two isomorphic absolutely irreducible spaces, or if it is irreducible of complex type.

In the case of general equivariant linear vector fields, it is well known that codimension one eigenvalue movements through the imaginary axis can be characterized by the following conditions [13]: a one-parameter family $\mathbf{M}(\mu)$, where $\mathbf{M}(0)$ satisfies:

(a) The critical eigenspace E_c of $\mathbf{M}(0)$ is Γ-simple (in case eigenvalues intersect $i\mathbb{R}$ away from 0) or Γ-absolutely irreducible (in the case eigenvalues intersect at 0).

(b) The eigenvalues $\lambda(\mu)$ of $\mathbf{M}(\mu)$ such that $\mathrm{Re}(\lambda(0)) = 0$ satisfy:

$$\frac{d}{d\mu}\mathrm{Re}(\lambda(\mu))|_{\mu=0} \neq 0.$$

We now consider how the codimension one movement of eigenvalues through the imaginary axis is affected by the absence of network connections.

Recall that we say that a coupled cell network is *connected* if there exists a path (formed by concatenation of edges, not necessarily uni-directional) connecting i and j (for all $i \neq j$). We say that the network is *disconnected* otherwise. Note that if Γ is transitive then the network is connected if and only if there are directed paths from any cell to any other cell.

We may summarize the connectivity information of an n-cell network in an $n \times n$ connectivity matrix C, where

$$C_{i,j} = \begin{cases} 1 & \text{if there is a connection from } j \text{ to } i, \\ 0 & \text{otherwise.} \end{cases}$$

As the network is Γ-equivariant, we have $C_{\gamma(i),\gamma(j)} = C_{i,j}$ for all $\gamma \in \Gamma$. We note that because of the transitivity of the action of Γ, the connectivity matrix C is fully determined by its first row (or first column).

By Remark 3.4, the number n of cells equals the order of Γ. Moreover, the representation of Γ on V corresponds to the regular representation. We consequently may identify cells uniquely with group elements once we have identified for example cell 1 with the identity element in Γ. In particular, we may label each cell i by a unique element $\gamma_i \in \Gamma$ such that $\gamma_i(1) = i$, so that $\gamma_1 = e$, etc. With this identification we have $C_{1,\gamma(1)} \equiv C_{e,\gamma}$.

There exists a group theoretic description of connectedness for a network with transitive symmetry group Γ.

Lemma 4.1 *The network with transitive and Abelian symmetry group Γ is connected if and only if $\Gamma =< S >$, where*

$$S = \{\gamma \in \Gamma : \; C_{e,\gamma} = 1\}. \tag{4.18}$$

Proof: Let $\mathcal{C}$ be the connected component of cell e, that is, the set of cells γ that are connected to cell e. We show that $\mathcal{C} =< S >$ and the lemma then follows.

It is obvious that $S \subset \mathcal{C}$.

Next we show that $\mathcal{C} \subseteq \Gamma$ is a subgroup and hence that $< S >\subseteq \mathcal{C}$, that is, we show that the subgroup generated by S is contained in $\mathcal{C}$. Note that if $\delta \in \mathcal{C}$ and $\delta^k = e$, then $\delta, \ldots, \delta^{k-1}$ are all connected to cell e. Thus $\delta^{-1} = \delta^{k-1}$ is connected to e and that e is connected to δ^{-1}; so $\delta^{-1} \in \mathcal{C}$. Suppose that $\gamma, \delta \in \mathcal{C}$. We must show that $\gamma\delta \in \mathcal{C}$. By assumption there is a path of coupled cells from e to δ. It follows that there is a path of coupled cells from γ to $\gamma\delta$. Since there is also a path of coupled cells from e to γ there is a path from e to $\gamma\delta$. A similar argument shows that there is a path of coupled cells from $\gamma\delta$ to e and $\gamma\delta \in \mathcal{C}$.

Finally, we show that $\mathcal{C} \subseteq< S >$. Let $\delta \in \mathcal{C}$ and let $e, \delta_1, \ldots, \delta_s = \delta$ be a directed path of coupled cells, which exists because Γ is transitive. It follows that $\delta_1 \in S$. In addition, $1 = C_{\delta_1,\delta_2} = C_{e,\delta_1^{-1}\delta_2}$. So $\delta_1^{-1}\delta_2 \in S$ and $\delta_2 = \delta_1(\delta_1^{-1}\delta_2)$ is in $< S >$. By induction, $\delta \in< S >$. □

By Corollary 3.8, the matrix $\mathbf{M}$ satisfies the following conditions (corresponding to the absence of the connections between cells $\gamma \in \Gamma \setminus S$ with e):

$$\sum_{j \in \{1,\ldots,n\}} \chi_j(\gamma)\mathbf{M}^j_{it} = 0, \; \forall i, t \in \{1, \ldots, n\}, \qquad \forall \gamma \in \Gamma \setminus S. \tag{4.19}$$

Here $\mathbf{M}^j_{it} = \big(c_i \otimes P^j(e_1), \mathbf{M}^j(c_t \otimes P^j(e_1))\big)$ where $\mathbf{M}^j$ is the restriction of $\mathbf{M}$ to the isotypic component $\mathbb{C}^l \otimes V_j$.

Lemma 4.2 *Let S denote the set of group elements corresponding to the present couplings, so that if $\gamma \in S$ there is a coupling from cell γ to cell e. Then for all $i, j \in \{1, \ldots, l\}$*

$$(\mathbf{M}^1_{it}, \ldots, \mathbf{M}^n_{it}) = \sum_{\gamma \in S} c_{it}(\gamma) \left(\overline{\chi_1(\gamma)}, \ldots, \overline{\chi_n(\gamma)}\right) \tag{4.20}$$

where $c_{it} : \; S \to \mathbb{C}$ are arbitrary.

Proof: By standard application of the character theory for compact Lie groups it is known that the character vectors $\chi(\gamma) = (\chi_1(\gamma), \ldots, \chi_n(\gamma))$, $\gamma \in \Gamma$ form an orthonormal basis of $\mathbb{C}^n$. Hence, the solution space to (4.19) is spanned by all character vectors corresponding to group elements whose couplings are present. □

We incorporate now the fact that in the context of coupled cell networks we work with real linear maps, rather than with their complexification. We thus need to decomplexify the answer obtained above.

Let V_j be an isotypic component for the action of Γ on V. Then:

- If V_j is of real type, then $\mathbf{M}^j$ should be interpreted as a matrix in $\mathrm{gl}(\mathbb{R}^l)$.
- If V_j is of complex type, then there is another isotypic component $V_i = \overline{V_j}$, so that $\mathbf{M}^j = \overline{\mathbf{M}^i}$. The decomplexification acts on $\mathbb{R}^l \otimes \widehat{V}_j$ where $\widehat{V}_j = V_j \oplus V_i \cong \mathbb{R}^2$, as

$$\widehat{\mathbf{M}}^j = \begin{pmatrix} \mathbf{M}^j_R & \mathbf{M}^j_I \\ -\mathbf{M}^j_I & \mathbf{M}^j_R \end{pmatrix}$$

where

$$\mathbf{M}^j = \mathbf{M}^j_R + i\mathbf{M}^j_I.$$

Accordingly, let us write $\chi_j \in \mathbb{C}$ as $\chi_j = (\chi_j)_R + i(\chi_j)_I$ with $(\chi_j)_R, (\chi_j)_I \in \mathbb{R}$. In (4.19) the sum $\chi_j \mathbf{M}^j + \overline{\chi_j}\overline{\mathbf{M}^j}$ yields $2(\chi_j)_R \mathbf{M}^j_R - 2(\chi_j)_I \mathbf{M}^j_I$, so that

$$\sum_j \chi_j(\gamma)\mathbf{M}^j = 0 \;\Leftrightarrow$$

$$\sum_{V_j \text{ real}} \chi_j(\gamma)\mathbf{M}^j + 2 \sum_{V_j \text{ complex}} \Big((\chi_j)_R(\gamma)\mathbf{M}^j_R - (\chi_j)_I(\gamma)\mathbf{M}^j_I \Big) = 0 \qquad (4.21)$$

where we note that in the latter sum, for each real invariant $\widehat{V}_j$, only one irreducible representation is taken.

In general equivariant systems with Abelian symmetry, we have the following result [13]: If in a one-parameter family of real linear equivariant vector fields, an eigenvalue crosses the imaginary axis then, typically, one of the following scenarios applies:

(a) The eigenvalues are restricted to the real axis, crossing the imaginary axis at zero, and the associated eigenvectors lie in one absolutely irreducible representation of Γ. The number of eigenvalues simultaneously crossing the imaginary axis is equal to the dimension of the irreducible representation, and they all have the same value.

(b) The eigenvalues are not restricted to lie on the real axis, crossing the imaginary axis at $\pm i\omega$ ($\omega \neq 0$). The associated eigenvectors lie in the direct sum of two isomorphic absolutely irreducible representations of Γ. The number of eigenvalues simultaneously crossing the imaginary axis is equal to twice the (complex) dimension of the irreducible representation, half taking one same value and the remaining half its complex conjugate.

(c) The eigenvalues are not restricted to lie on the real axis, crossing the imaginary axis at $\pm i\omega$ ($\omega \neq 0$). The associated eigenvectors lie in one irreducible representation of Γ of complex type. The number of eigenvalues simultaneously crossing the imaginary axis is equal to twice the (complex) dimension of the irreducible representation, half assuming one value and the remaining half its complex conjugate.

The eigenvalue movement in (a) is associated with *steady-state* bifurcation, and the remaining cases (b) and (c) with *Hopf* bifurcation.

We now make the following observation:

Lemma 4.3 *Codimension one movements of eigenvalues crossing the imaginary axis in an Abelian symmetric coupled cell network, are identical to the corresponding eigenvalue movements in general equivariant vector fields, if the conditions (4.21) for $\gamma \in \Gamma \setminus S$ on the $\mathbf{M}^j$ (if V_j is of real type), $\mathbf{M}^j_R$, $\mathbf{M}^j_I$ (if V_j is of complex type) imposed by the absence of connections, do not imply one of the following relations:*

(i) $\mathbf{M}^j = c\mathbf{M}^i$, *where V_j and V_i are distinct absolutely irreducible representations and $c \in \mathbb{R}$.*

(ii) $\mathbf{M}^j = c\mathbf{M}^i_R$ *where V_i is absolutely irreducible, and V_j is an irreducible representation of complex type and $c \in \mathbb{R}$.*

(iii) $\mathbf{M}^j_R = c\mathbf{M}^i_R$, *where V_j, V_i are distinct irreducible representations of complex type and $c \in \mathbb{R}$.*

Proof: If we have linear relations between more than two of the matrices $\mathbf{M}^j \in \mathrm{gl}(\mathbb{R}^l)$ (where V^j is absolutely irreducible) and $\mathbf{M}^i_R \in \mathrm{gl}(\mathbb{R}^l)$ and/or $\mathbf{M}^i_I \in \mathrm{gl}(\mathbb{R}^l)$ (where $\widehat{V}^i$ is irreducible of complex type), there are no forced degenerate codimension one eigenvalue movements through the imaginary axis.

Before we demonstrate this, we first remark that if we make perturbations of the form $\mathbf{M}^i_R + \varepsilon_R \mathrm{Id}$ and $\mathbf{M}^i_I + \varepsilon_I \mathrm{Id}$, then the eigenvalues λ of $\widehat{\mathbf{M}}^i$ change to $\lambda + \varepsilon_R \pm i\varepsilon_I$.

We assert that if we have a relation between four or more matrices, no forced degenerate codimension one eigenvalue movements across the imaginary axis will arise.

To illustrate this, suppose we have a relation of the type

$$\sum_{j=1}^{4} a_j \mathbf{A}_j = 0 \tag{4.22}$$

where $a_j \in \mathbb{R}$ for all j, and the $A_j \in \mathrm{gl}(\mathbb{R}^l)$ are of the above mentioned types.

First suppose that in this equation $\mathbf{M}^i_I$ features, but not $\mathbf{M}^i_R$. Then it is immediate that if $\widehat{\mathbf{M}}^i$ has eigenvalues on the imaginary axis, they can be moved off this axis by a small perturbation of the form $\mathbf{M}^i_R + \varepsilon_R \mathrm{Id}$ without affecting the relation.

Now suppose that $\mathbf{M}^i_I$ and $\mathbf{M}^i_R$ both appear. Then if $\widehat{\mathbf{M}}^i$ has eigenvalues on the imaginary axis, they can be moved off this axis by a small perturbation of the form $\mathbf{M}^i_R + \varepsilon_R \mathrm{Id}$ and $\mathbf{M}^i_I + \varepsilon_I \mathrm{Id}$, where ε_R and ε_I can now be chosen such that the relation holds without changing any of the other matrices involved.

Let us assume that the relation involves no matrix of the type $\mathbf{M}^i_I$. Suppose there is more than one isotypic component containing eigenvectors with corresponding eigenvalues on the imaginary axis that are involved in the relation. Then there exists a perturbation after which eigenvectors with corresponding eigenvalues on the imaginary axis occur in only one isotypic component. For instance, we can fix $\mathbf{A}_1$ and make the perturbation $\mathbf{A}_2 + \varepsilon \mathrm{Id}$ and $\mathbf{A}_3 - \varepsilon a_2/a_3 \mathrm{Id}$. The latter perturbations change the real parts of eigenvalues of the other isotypic components involved (and ε can always be chosen such that they come to lie off the imaginary axis) while leaving $\mathbf{A}_1$ invariant.

Now suppose that eigenvectors with corresponding eigenvalues on the imaginary axis occur in only one isotypic component, but that the centre subspace is not Γ-simple or absolutely irreducible. Then, by [13] there exists a perturbation of $\mathbf{M}$ restricted to this isotypic component such that the centre subspace is of the desired form. By adjusting the size of this perturbation, we can adjust $\mathbf{M}$ so as to satisfy the relation without enlarging the centre subspace.

Subsequently, the obtained linear system $\mathbf{M}$ can be unfolded by the perturbation $\mathbf{M} + \mu \mathrm{Id}$, yielding the desired codimension one eigenvalue crossing through the imaginary axis. $\square$

We now will state two lemmas that, in connection with Lemma 4.3 lead to a proof of Theorem 1.3 of the Introduction.

Lemma 4.4 *Suppose a coupled cell network is active, then in Lemma 4.3 we have* $c = 1$.

Proof: If the network is active ($e \in S$), then the identity vector field Id is admissible as a coupled cell network. For this vector field, $\mathbf{M}^j$ is the identity on $\mathbb{C}^l \otimes V_j$ for all j, from which the values of c directly follow. $\square$

Lemma 4.5 *Suppose a coupled cell network is connected and active. Then the conditions (4.21) for* $\gamma \in \Gamma \setminus S$ *on the* $\mathbf{M}^j$ *(if* V_j *is of real type),* $\mathbf{M}_R^j$, $\mathbf{M}_I^j$ *(if* V_j *is of complex type) imposed by the absence of connections, do not imply the conditions (i), (ii) or (iii) of Lemma 4.3.*

Proof: By Corollary 3.5, the character vectors $\chi(\gamma) = \left(\overline{\chi_1(\gamma)}, \ldots, \overline{\chi_n(\gamma)}\right)$ with $\gamma \in S$ must satisfy the above conditions. All the components of these vectors have modulus one. A condition of type $\mathbf{M}^j = \mathbf{M}^i$, where V^j, V_i are of type $\mathbb{R}$, implies that the corresponding components of the character vectors must satisfy

$$\chi_j(\gamma) = \chi_i(\gamma)$$

for all $\gamma \in S$. As S generates Γ since the network is connected it follows that $\chi_j = \chi_i$, a contradiction. In the condition of type $\mathbf{M}^j = \mathbf{M}_R^i$, where V^j is of type $\mathbb{R}$ and V_i is of complex type, this implies that the corresponding components of the character vectors must satisfy

$$\pm 1 = (\chi_i)_R(\gamma)$$

where $\chi_i(\gamma) = (\chi_j)_R(\gamma) + i(\chi_j)_I(\gamma) \in \mathbb{C}$ is the character, which has modulus one. In turn this implies that $(\chi_j)_I(\gamma) = 0$. Consequently, as this property holds for all $\gamma \in S$ and S generates Γ, it follows that χ_i is of real type, a contradiction.

In a similar way, the condition of type $\mathbf{M}_R^j = \mathbf{M}_R^i$, leads to the equation

$$(\chi_j)_R(\gamma) = (\chi_i)_R(\gamma)$$

for $\gamma \in S$, which implies that $\chi_j = \chi_i$ or $\chi_j = \overline{\chi_i}$, equivalently, $\widehat{V}_i = \widehat{V}_j$, a contradiction. $\square$

Proof of Theorem 1.3: From the above lemmas we see that with active cells, conditions (i), (ii), (iii) of Lemma 4.3 leading to degenerate codimension one eigenvalue behaviour, can not happen, unless the network is disconnected.

By Lemma 4.3, hence with active cells, in one-parameter families, we generically have eigenvalue movements through the imaginary axis, following the codimension one eigenvalue behaviour in generic (general) equivariant vector fields. $\square$

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