

The Steady-state Lifting Bifurcation Problem Associated with the Valency on Networks

Manuela A D Aguiar^{a,c}, Ana P S Dias^{b,c}, Pedro Soares^{b,c}

^a*Faculdade de Economia, Universidade do Porto, Rua Dr Roberto Frias, 4200-464 Porto, Portugal*

^b*Dep. Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*

^c*Centro de Matemática da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*

Abstract

In this paper we consider homogeneous coupled cell networks with asymmetric inputs – networks where each cell receives exactly one input of each edge type. The coupled cell systems associated with a network are the dynamical systems that respect the network structure. There are subspaces, determined solely by the network structure, that are flow-invariant under any such coupled cell system – the synchrony subspaces. For a homogeneous network with asymmetric inputs, one of the eigenvalues of the Jacobian matrix of any coupled cell system at an equilibrium in the full-synchrony subspace corresponds to the valency of the network. In this work, we study the codimension-one steady-state bifurcations of coupled cell systems with a bifurcation condition associated with the valency. We start by giving an adaptation of the Perron–Frobenius Theorem for the eigenspace associated with the valency showing that the dimension of that eigenspace equals the number of the network source components. A network source component is a strongly connected component of the network whose cells receive inputs only from cells in the component. Each synchrony subspace determines a smaller network called quotient network. The lifting bifurcation problem addresses the issue of understanding when the bifurcation branches of a network can be lifted from one of its quotient networks. We consider the lifting bifurcation problem when the bifurcation condition is associated with the valency. We give sufficient conditions on the number of source components for the answer to the lifting bifurcation problem to be positive and prove that those conditions are necessary and sufficient for a class of networks.

Keywords: Coupled cell network, Steady-state bifurcation, Lifting bifurcation problem

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Email addresses: maguiar@fep.up.pt (Manuela A D Aguiar), apdias@fc.up.pt (Ana P S Dias), ptcsoares@fc.up.pt (Pedro Soares)

1. Introduction

A coupled cell network is a directed graph with labels on the cells and edges, which describe their types. A coupled cell system is a dynamical system given by an admissible vector field for the network, that is, it must respect the graph structure. More precisely, the admissible vector fields for a network determine that the dynamics of each cell is affected by its own state and the state of the cells with an input edge directed to that cell. Moreover, the admissible vector fields must respect the type of the cells and edges. In [1, 2], the authors formalize the concepts of coupled cell network and coupled cell system and enlighten about their intrinsic relation. They prove the existence of synchrony subspaces that are flow invariant for any coupled cell system. The synchrony subspaces are given by the state's equality of some cells of the network. The restriction of a network admissible vector field to a synchrony subspace is an admissible vector field for a smaller network, called a quotient network. A quotient network is obtained by merging cells that have the same state in the corresponding synchrony subspace. The original network is said to be a lift of the smaller network.

We focus on homogeneous networks with asymmetric inputs, where all cells have the same type and each cell has exactly one input of each type. These kind of networks have been studied in [3, 4, 5, 6]. For each type of input, there is an adjacency matrix that represents the inputs of that type. Moreover, the Jacobian matrix of an admissible vector field at a full-synchrony point can be expressed using the adjacency matrices of the network, and it has a constant row-sum called the valency of the network. Thus, the valency of the network is an eigenvalue of the Jacobian matrix. In this paper, we study codimension-one steady-state bifurcations for coupled cell systems of homogeneous networks where the bifurcation condition corresponds to the network valency and address the respective lifting bifurcation problem.

Bifurcation problems on coupled cell systems have been previously studied by different authors, see for example [7, 8, 9, 10, 11, 12]. These include specific network examples, classes of networks that have an additional structure such as (partial) symmetries or a feed-forward structure or even some conditions about the bifurcation condition which need to be verified in a case-by-case scenario. Here, we address a bifurcation problem that it is transverse to every homogeneous network with asymmetric inputs. We first describe the kernel of generic coupled cell systems with a bifurcation condition corresponding to the network valency. This is a technical step where we use a novel recursive argument on the number of cells of the network. After that, the standard methods of bifurcation theory are applied to describe the bifurcation branches. Next, we analyze the lifting bifurcation problem which has been studied in [8, 13, 14, 5, 15]. We observe that this problem is closely related with the study of synchrony-breaking bifurcations, since bifurcation branches which do not break the synchrony associated with a quotient network are lifted from that quotient network. Despite we give a complete description of the bifurcation branches for every generic coupled cell system with a bifurcation condition associated with the network valency, we present examples suggesting that it is not trivial to fully understand the

lifting bifurcation problem. Nevertheless, we are able to give a complete answer to the lifting bifurcation problem for a class of networks. Below, we make this discussion more precise.

In order to study the bifurcation problem where the bifurcation condition corresponds to the network valency, we first give an adaptation of the Perron-Frobenius Theorem to generic coupled cell systems. For real square matrices with non-negative entries and constant row-sum, it is well-known in stochastic theory that the row-sum is the greater eigenvalue in absolute value and how to describe the eigenspace associated with that eigenvalue. See for example [16, Chapter 4]. This applies to every adjacency matrix of an homogeneous network with asymmetric inputs since they have constant row-sum 1 and entries 0 or 1. The Jacobian matrix of a coupled cell system at a full synchrony subspace is not in general non-negative, but it has constant row-sum that we call the network valency. Then the valency of the network is an eigenvalue of the Jacobian matrix. In this paper, we describe the eigenspace corresponding to the valency of the network for generic coupled cell system in Propositions 5.1, 5.5 and 5.6. The dimension of this eigenspace is equal to the number of source components in the network. Every network can be partitioned into its strongly connected components. We say that a component is a source if every input targeting a cell in that component starts in a cell also inside that component. After this first step, we use well-known methods of bifurcation theory to describe the codimension-one steady-state bifurcations of generic coupled cell systems where the bifurcation condition corresponds to the network valency, see Proposition 5.7. In particular, we show that there exists a synchrony-breaking bifurcation branch if and only if the network has at least two source components.

Given a network and a lift network, the solutions (and the bifurcation branches) of a coupled cell system in the quotient network lift to solutions of the corresponding coupled cell system in the lift network. The lifting bifurcation problem addresses the issue of whether all bifurcation branches occurring in a coupled cell system of the lift network are lifted from the smaller network. In the last part of this paper, we study the lifting bifurcation problem of generic coupled cell systems where the bifurcation condition corresponds to the network valency. We obtain sufficient conditions, given by the number of source components in the lift and quotient network, to answer the lifting bifurcation problem, Proposition 6.1. More precisely, if the lift and quotient networks have the same number of source components then all the bifurcation branches on the lift network are lifted from the quotient network. On the other hand, if the quotient network has exactly one source component and the lift network has at least two source components, then there is a synchrony-breaking bifurcation branch on the lift network which is not lifted from the quotient network. This result is expected, since the number of source components equals the dimension of the kernel of the Jacobian matrix at a full synchronous point associated with a coupled cell system where the bifurcation condition corresponds to the network valency. Thereby, it would be expected that if the number of source components increases for the lift network, then some bifurcation branch on the lift network would not be lifted from the quotient network. We present, how-

ever, two examples where this does not hold (Examples 6.3 and 6.4). Despite the number of source components increases in those examples, every bifurcation branch in each lift network is lifted from the respective quotient network. In one of the examples, a condition on the partial symmetries of the quotient network, called transitive, is broken. In the other example, a condition on the connectivity of the lift network, called backward connected, is broken. Networks that are backward connected and transitive have received an extra attention in [17, 18]. Restricting the lifting bifurcation problem to transitive quotient networks and backward connected lift networks, we prove that every bifurcation branch on the lift network is lifted from the quotient network if and only if the quotient and lift network have the same number of source components, Theorem 6.5.

The structure of this paper is the following. In Section 2, we recall some notions about coupled cell networks such as quotient network, backward connected network and transitive network. In Section 3, we review the definition of coupled cell systems. In Section 4, we describe coupled cell systems having a bifurcation condition corresponding to the network valency and the respective lifting bifurcation problem. In Section 5, we study the codimension-one steady-state bifurcation problem for those coupled cell systems. In Section 6, we discuss the lifting bifurcation problem.

2. Coupled cell networks

In this section, we recall definitions and results concerning coupled cell networks, connectivity of networks, balanced colorings, quotient networks and network fibrations. We follow the presentation given in [1, 2, 3, 19].

Definition 2.1. A *network* N is defined by a directed graph with a finite set of cells C and finite sets of directed edges divided by types E_1, \dots, E_k such that each edge $e \in E_i$ starts in a cell s and targets a cell t , where $1 \leq i \leq k$ and $s, t \in C$. We denote by $|N|$ the number of cells in the network N . In this work we will assume that all networks are *homogeneous with asymmetric inputs* in the sense that each cell c is target by exactly one edge of each type. \diamond

Graphically, we use different symbols to distinguish the edge types. As an example, the network in Figure 1 has two types of edges.

Let N be a network and E_1, \dots, E_k the sets of edges in N . By relabeling the cells, we can assume that $C = \{1, \dots, n\}$, with $n = |N|$.

Given $1 \leq a_1, \dots, a_n \leq m$, we denote by $\sigma = [a_1 \dots a_n]$ the function $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ such that $\sigma(j) = a_j$, $1 \leq j \leq n$. The identity function on $\{1, \dots, n\}$ is denoted by σ_0 , i.e., $\sigma_0(j) = j$, for $1 \leq j \leq n$.

As pointed out by Rink and Sanders [3], a homogeneous network with asymmetric inputs can be represented using a collection of functions. For each $1 \leq i \leq k$, consider the function $\sigma_i = [s_i(1) \dots s_i(n)]$ such that there exists an edge $e \in E_i$ from $s_i(c)$ to c for $1 \leq c \leq n$. In fact a homogeneous network with asymmetric inputs is uniquely determined by the functions $(\sigma_i)_{i=1}^k$ and we say that N is *represented by* $(\sigma_i)_{i=1}^k$. See the network in Figure 1.

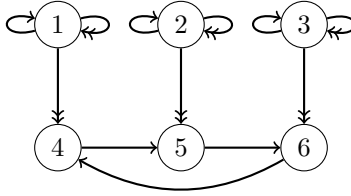


Figure 1: A homogeneous network with asymmetric inputs represented by $\sigma_1 = [1\ 2\ 3\ 6\ 4\ 5]$ and $\sigma_2 = [1\ 2\ 3\ 1\ 2\ 3]$. The strongly connected components of this network are $\{1\}$, $\{2\}$, $\{3\}$ and $\{4, 5, 6\}$.

A network can be also represented by its adjacency matrices. For each $1 \leq i \leq k$, the $n \times n$ -matrix A_i is the *adjacency matrix of type i* , if $(A_i)_{c\sigma_i(c)} = 1$ and $(A_i)_{c\sigma_i(c')} = 0$, when $\sigma_i(c) \neq c'$. A network N is uniquely represented by its adjacency matrices $(A_i)_{i=1}^k$. We denote the identity matrix by A_0 .

Given cells c and d of N , we say that c and d are *connected* if there exists a sequence of cells $c_0, c_1, \dots, c_{l-1}, c_l$ such that $c_0 = c$, $c_l = d$ and there is an edge from c_{j-1} to c_j or an edge from c_j to c_{j-1} , for every $1 \leq j \leq l$. In this work, we always consider *connected* networks, i.e., networks where every two distinct cells are connected.

Definition 2.2. Let N be a network. A *path* in N from the cell c to the cell d is a sequence of cells $c_0, c_1, \dots, c_{l-1}, c_l$ such that $c_0 = c$, $c_l = d$ and there is an edge from c_{j-1} to c_j , for every $1 \leq j \leq l$. We say that cells c and d are *strongly connected*, if there are paths from c to d and from d to c . A subset B of cells is a *strongly connected component* of N , if any two distinct cells $c, d \in B$ are strongly connected and B is a maximal subset of strongly connected cells, i.e., for every strongly connected cells $c \in B$ and d we have that $d \in B$. \diamond

The set of cells of a network can be partitioned into its strongly connected components, see e.g. [20, Theorem 2.4].

In the network example of Figure 1, the strongly connected components are $\{1\}$, $\{2\}$, $\{3\}$ and $\{4, 5, 6\}$.

Definition 2.3. Let N be a network. The network N is *strongly connected*, if N has exactly one strongly connected component given by the set of cells. A strongly connected component S is a *source component*, if every edge targeting a cell of S starts in a cell of S . We denote by $s(N)$ the *number of source components* of N . \diamond

The network in Figure 2 is an example of a strongly connected network and its unique source component is the set of cells. The networks in Figure 1 and Figure 3 have three source components: $\{1\}$, $\{2\}$ and $\{3\}$.

Definition 2.4. A network N is *backward connected* for a cell c , if for every other cell c' there exists a path from c' to c . A network N is *backward connected* if it is backward connected for some cell. \diamond

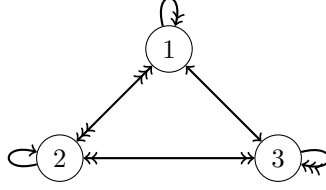


Figure 2: A strongly connected network represented by $\sigma_1 = [321]$, $\sigma_2 = [132]$ and $\sigma_3 = [213]$ which is backward connected for every cell.

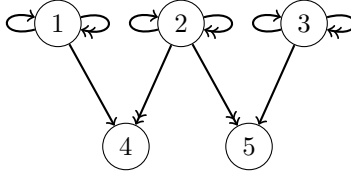


Figure 3: A network represented by $\sigma_1 = [1\ 2\ 3\ 1\ 3]$ and $\sigma_2 = [1\ 2\ 3\ 2\ 2]$ with three source components. This network is not backward connected, as there is no path from cell 4 to cell 5 and vice versa.

Every strongly connected network is backward connected for every cell. Figure 2 shows an example of a backward connected network. The network in Figure 1 is backward connected for the cells 4, 5 and 6. On the other hand, Figure 3 shows an example of a network which is not backward connected.

Let N be a network represented by the functions $(\sigma_i)_{i=1}^k$ such that $|N| = n$. A *coloring* of the set of cells of N is an equivalence relation on the set of cells. A coloring \bowtie is *balanced* if $\sigma_i(c) \bowtie \sigma_i(c')$, for every $1 \leq i \leq k$ and $1 \leq c, c' \leq n$ such that $c \bowtie c'$. It follows from [4, Proposition 7.2] that this definition coincides with the definition of balanced coloring given in [2, Definition 4.1]. Given a subset of cells $S \subseteq \{1, \dots, n\}$, we denote by $[S]_{\bowtie}$ the set of \bowtie -classes containing the cells in S , i.e., $[S]_{\bowtie} = \{[c]_{\bowtie} : c \in S\}$.

Definition 2.5 ([2, Section 5]). Let N be a network represented by the functions $(\sigma_i)_{i=1}^k$ such that $|N| = n$ and \bowtie a balanced coloring in N . The *quotient network* of N associated with \bowtie is the network where the set of cells are the \bowtie -classes and the edges are represented by the functions $(\sigma_i^{\bowtie})_{i=1}^k$ such that

$$\sigma_i^{\bowtie}([c]_{\bowtie}) = [\sigma_i(c)]_{\bowtie}, \quad 1 \leq i \leq k, 1 \leq c \leq n.$$

We denote the quotient network by N/\bowtie . We also say that a network L is a *lift* of N , if N is a quotient of L for some balanced coloring in L . \diamond

The networks in Figures 4 and 5 are examples of quotient networks of the networks described in Figures 1 and 3, respectively.

Remark 2.6. Let L be a lift of the network N . Any path in L between two cells in L induces a path in N between the corresponding cells in N . Hence, $s(N) \leq s(L)$. Furthermore, N is backward connected, whenever L is backward connected. \diamond

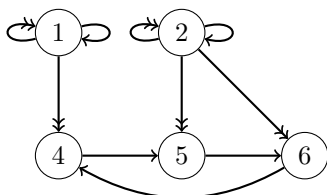


Figure 4: The quotient network of the network in Figure 1 associated with the balanced coloring \bowtie such that $2 \bowtie 3$. This quotient network is backward connected, since the network in Figure 1 is also backward connected.

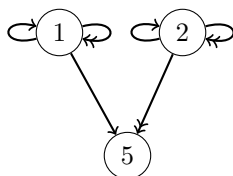


Figure 5: The quotient network of the network in Figure 3 associated with the balanced coloring \bowtie such that $1 \bowtie 3$ and $4 \bowtie 5$. This network is backward connected for the cell 5.

Next, we define network fibrations following [19, Definition 2.1.4] and [17, Proposition 5.3]. Network fibrations are functions between networks that respect the type of edges and the extreme cells of any edge. For homogeneous networks with asymmetric inputs, it is sufficient to describe the network fibration in their sets of cells as each cell receives exactly one edge of each type. In order for a function between sets of cells to be a network fibration we impose next that the extreme cells of any edge are preserved.

Definition 2.7. Let N and N' be the networks with the set of cells C , C' and represented by the functions $(\sigma_i)_{i=1}^k$ and $(\sigma'_i)_{i=1}^k$. A function $\varphi : C \rightarrow C'$ is a *network fibration from N to N'* , if

$$\varphi \circ \sigma_i = \sigma'_i \circ \varphi, \quad i = 1, \dots, k.$$

We denote a network fibration from N to N' by $\varphi : N \rightarrow N'$. We say that N and N' are equal and write that $N = N'$ if there exists a bijective network fibration $\varphi : N \rightarrow N'$. \diamond

If N is a network and \bowtie a balanced coloring of the set of cells of N , the *network fibration induced by \bowtie* is the function $\varphi_{\bowtie} : N \rightarrow N/\bowtie$ given for every cell c of N by

$$\varphi_{\bowtie}(c) = [c]_{\bowtie}.$$

Example 2.8. Let N be the network in Figure 5. There are three network fibrations from N to itself: $\varphi_1 = [1 \ 1 \ 1]$; $\varphi_2 = [2 \ 2 \ 2]$ and $\varphi_3 = [1 \ 2 \ 5]$.

Let L be the network in Figure 3 and \bowtie the balanced coloring in L such that $1 \bowtie 3$ and $4 \bowtie 5$. The network L is a lift of $N = L/\bowtie$ and $\varphi_{\bowtie} = [1 \ 2 \ 1 \ 5 \ 5]$

is a network fibration from L to N , where 1, 2 and 5 are representatives of the classes $[1]_{\boxtimes}$, $[2]_{\boxtimes}$ and $[5]_{\boxtimes}$, respectively. \diamond

Remark 2.9. Let $\varphi : N \rightarrow N'$ be a network fibration between two networks N and N' . If S is a source component of N , then $\varphi(S)$ is a source component of N' . This follows from the fact that each path in the network N is projected by the network fibration $\varphi : N \rightarrow N'$ into a path in the network N' . \diamond

The evaluation of a network fibration at each cell of a path can be determined by the evaluation of the network fibration at the end cell of that path. Since for a backward connected network there is a cell such that there is a path from every other cell to it, the network fibrations from such network are uniquely determined by their evaluation at that cell:

Remark 2.10. Let $\varphi : N \rightarrow N'$ be a network fibration between the networks N and N' . If N is backward connected for c , then the network fibration is uniquely determined by its evaluation at c , $\varphi(c)$. \diamond

The self-fibrations of a network are an indicator of the partial symmetries of that network. Next, we define a class of networks with sufficient partial symmetries. In particular, this means that there is a cell which can be permuted with any other cell using self-fibrations.

Definition 2.11. Let N be a network and c a cell of N . We say that N is *transitive for c* if for every cell d in N , there is a network fibration $\phi_d : N \rightarrow N$ such that $\phi_d(c) = d$. We say that N is *transitive*, if it is transitive for some cell. \diamond

The network in Figure 5 is an example of a transitive network, since it is transitive for the cell 5, see Example 2.8. The network in Figure 4 is not transitive, because there are only three self-fibrations. Those self-fibrations are the identity, a self-fibration that sends every cell to the cell 1 and a self-fibration that sends every cell to the cell 2. In [17], the authors have defined fundamental networks. A network is fundamental if and only if it is backward connected and transitive, see [18, Theorem 5.16]. Figure 5 is an example of a fundamental network.

3. Coupled cell systems

In this section, we recall concepts and results about coupled cell systems, synchrony subspaces and conjugacies induced by network fibrations, following [1, 2, 19]. We restrict the phase space of each cell to be the one-dimensional real space, however the definitions and results are valid for any differential manifold, see [1, 2, 19].

Let N be a network represented by the functions $(\sigma_i)_{i=1}^k$ and $|N| = n$. For each cell c of the network, we associate a coordinate $x_c \in \mathbb{R}$. We say that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an *admissible vector field* for N , if there is $f : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ such that

$$(F(x))_c = f(x_c, x_{\sigma_1(c)}, \dots, x_{\sigma_k(c)}),$$

for every cell c of N . The admissible vector fields for N are uniquely defined by such function $f : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$. We denote by f^N the admissible vector field for N defined by f .

Let $f : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a smooth function. A *coupled cell system* associated with a network N is a system of ordinary differential equations

$$\dot{x} = f^N(x), \quad x \in \mathbb{R}^n.$$

Let $(A_i)_{i=1}^k$ be the adjacency matrices of N . The *Jacobian matrix* of f^N at the origin is

$$J_f^N := (Df^N)_0 = \sum_{i=0}^k f_i A_i,$$

where

$$f_i := \frac{\partial f}{\partial x_i}(0, 0, \dots, 0),$$

for $0 \leq i \leq k$ and A_0 is the identity $n \times n$ matrix. Since $(1, \dots, 1)$ is an eigenvalue of A_0, \dots, A_k , then

$$J_f^N \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \left(\sum_{i=0}^k f_i \right) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

and $\sum_{i=0}^k f_i$ is always an eigenvalue of J_f^N that we call the *network valency*.

A *polydiagonal subspace* is a subspace of \mathbb{R}^n given by the equalities of some cell coordinates. Given a coloring \bowtie on the set of cells of N , the *polydiagonal subspace associated with \bowtie* is

$$\Delta_{\bowtie} := \{x : c \bowtie d \Rightarrow x_c = x_d\} \subseteq \mathbb{R}^n.$$

We say that a polydiagonal subspace $\Delta \subseteq \mathbb{R}^n$ is a *synchrony subspace* of a network N if the polydiagonal subspace is invariant by any admissible vector field of N , i.e., $f^N(\Delta) \subseteq \Delta$, for every $f : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$. There is a one-to-one correspondence between balanced colorings \bowtie and synchrony subspaces Δ_{\bowtie} , see [2, Theorem 4.3]. More specifically, the polydiagonal Δ_{\bowtie} is a synchrony subspace of N if and only if \bowtie is a balanced coloring. For homogeneous networks, the coloring with only one color is always balanced and the corresponding synchrony subspace is called the *full-synchrony subspace*:

$$\Delta_0 := \{(x, \dots, x) \in \mathbb{R}^n : x \in \mathbb{R}\}.$$

Since a synchrony subspace Δ_{\bowtie} is invariant by every admissible vector field f^N , the coupled cell systems of N can be restricted to Δ_{\bowtie} . The restricted systems are coupled cell systems of N/\bowtie given by admissibles $f^{N/\bowtie}$, see [2, Theorem 5.2].

Let N' be a network and $n' = |N'|$. Following [19], every network fibration $\varphi : N \rightarrow N'$ induces a map between the phase spaces of those networks, $P\varphi : \mathbb{R}^{n'} \rightarrow \mathbb{R}^n$ such that

$$(P\varphi(x))_c = x_{\varphi(c)}, \quad 1 \leq c \leq n.$$

Moreover, the coupled cell systems defined by f^N and $f^{N'}$ are conjugated

$$P\varphi \circ f^{N'} = f^N \circ P\varphi.$$

Any solution $y(t) \in \mathbb{R}^{n'}$ of $\dot{y} = f^{N'}(y)$, induces a solution $x(t) = P\varphi(y(t))$ of $\dot{x} = f^N(x)$. In particular, for any balanced coloring \bowtie in N , the solutions of every coupled cell system on N/\bowtie are lifted by $P\varphi_{\bowtie}$ to solutions of the corresponding coupled cell system on N and those solutions belong to the synchrony subspace Δ_{\bowtie} .

4. Steady-state bifurcations

In this section, we review some concepts related with steady-state bifurcations on coupled cell systems and the lifting bifurcation problem is formulated.

Let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}$ be a one-parameter family of smooth functions and consider the family of coupled cell systems, depending on the parameter λ ,

$$\dot{x} = f^N(x, \lambda) \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}. \quad (1)$$

Assume that the origin is an *equilibrium point* of (1) for every $\lambda \in \mathbb{R}$, i.e. $f^N(0, 0, \dots, 0, \lambda) = 0$ for every $\lambda \in \mathbb{R}$. If the Jacobian matrix of f^N at $(x, \lambda) = (0, 0)$, $J_f^N := (Df^N)_{(0,0)}$, is invertible, then the origin is the unique equilibrium point of (1) in a sufficient small neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}$. We say that a *steady-state bifurcation* occurs if there exists an equilibrium point of (1) different from the origin in any small neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}$. Hence a necessary condition for a steady-state bifurcation to occur is that J_f^N is non-invertible.

Recall that the network valency $\sum_{i=0}^k f_i$ is an eigenvalue of J_f^N . In this paper, we study steady-state bifurcation where the *bifurcation condition is given by the network valency*. Let $\mathcal{V}(N)$ be the set of smooth functions $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$\mathcal{V}(N) := \left\{ f : \sum_{i=0}^k f_i = 0, \quad f(0, 0, \dots, 0, \lambda) = 0, \quad \lambda \in \mathbb{R} \right\}.$$

Since our study is local, we recall the definition of germ. Let $U_1, U_2 \subset \mathbb{R}$ be open neighborhoods of 0. We say that two smooth functions $b_1 : U_1 \rightarrow \mathbb{R}^n$ and $b_2 : U_2 \rightarrow \mathbb{R}^n$ are *germ equivalents* if $b_1(\lambda) = b_2(\lambda)$, for every $\lambda \in U_1 \cap U_2$. Given a smooth function b , we use the term *germ b* to refer to a representative element of the equivalence class of b with respect to germ equivalence.

Let $f \in \mathcal{V}(N)$. We say that a germ $b : U \rightarrow \mathbb{R}^n$ is an *equilibrium branch of f on N* , if

$$f^N(b(\lambda), \lambda) = 0,$$

for every $\lambda \in U$. As $f(0, 0, \dots, 0, \lambda) = 0$, we have that $x(\lambda) = (0, \dots, 0)$ is an equilibrium branch of f on N , called *the trivial branch of f on N* . The

equilibrium branches of f on N different from the trivial branch are called the *bifurcation branches of f on N* .

As usual in bifurcation theory, the study of the steady-state bifurcation problem is posed for a large class of functions called generic functions. The generic functions are defined using non-degenerated conditions. A *non-degenerated condition* is given by a polynomial p on some partial derivatives of a function evaluated at the bifurcation point. Given a function f , we denote by $p(f)$ the evaluation of the polynomial p at that function and we say that a function f satisfies the non-degenerated condition given by p , if $p(f) \neq 0$.

Given $f \in \mathcal{V}(N)$, the value of its first partial derivative with respect to x_0 at the origin, f_0 , is given by its first partial derivatives with respect to x_i at the origin, f_i , for $i = 1, \dots, k$. Also, partial derivatives of any order $l > 0$ with respect to λ at the origin, $\partial^l f / \partial \lambda^l$, are zero. Hence, we do not use non-degenerated conditions which depend on f_0 and $\partial^l f / \partial \lambda^l$ for any $l > 0$. We say that an assertion holds for *generic functions* in $\mathcal{V}(N)$, if there exists a finite number of non-degenerated conditions such that this assertion holds for any function in $\mathcal{V}(N)$ satisfying those non-degenerated conditions.

Let N be a network and L a lift of N . If f^N is a coupled cell system on N with a bifurcation condition corresponding to the network valency, then f^L is a coupled cell system on L with a bifurcation condition corresponding to the network valency. Thus

$$\mathcal{V}(N) = \mathcal{V}(L).$$

In the end of the previous section, it was stated how to lift solutions of a coupled cell system associated with N to the corresponding coupled cell system associated with L using network fibrations. In the same way, we can lift bifurcation branches of a coupled cell system to another using network fibrations.

Definition 4.1. Let N be a network, L a lift of N and $f \in \mathcal{V}(N)$. We say that a *bifurcation branch b of f on L is lifted from N* , if there exists a network fibration $\varphi : L \rightarrow N$ and a bifurcation branch b' of f on N such that

$$b = P\varphi(b'). \quad \diamond$$

Given a network N and a lift network L of N , the *lifting bifurcation problem* asks when every bifurcation branch of L is lifted from N .

5. Steady-state bifurcations associated with the valency

In this section, we study the bifurcations branches of (1) where $f \in \mathcal{V}(N)$ and N is a homogeneous network with asymmetric inputs. We start by describing the kernel of the Jacobian matrix J_f^N , when the network N is strongly connected. By the Perron-Frobenius Theorem ([20, Theorem 0.3]), the kernel of J_f^N is equal to the full-synchrony subspace, when the network N is strongly connected, $f \in \mathcal{V}(N)$ and $f_i > 0$ for every $1 \leq i \leq k$. We show next that this holds for generic functions $f \in \mathcal{V}(N)$.

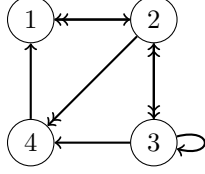


Figure 6: A strongly connected network.

Proposition 5.1. *Let N be a strongly connected network. For generic $f \in \mathcal{V}(N)$, the kernel of J_f^N is the full-synchrony subspace*

$$\ker(J_f^N) = \Delta_0.$$

We will prove this result by recursion on the number of cells. In order to do the recursive step, we create a new network with one cell less. Before the proof, we present a concrete example to illustrate how the recursive step is done.

Example 5.2. Let N be the network in Figure 6 and $f : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$. The network N is represented by (σ_1, σ_2) , where $\sigma_1 = [4 \ 1 \ 3 \ 3]$ and $\sigma_2 = [2 \ 3 \ 2 \ 2]$. Recall that $\sigma_0 = [1 \ 2 \ 3 \ 4]$ corresponds to the hidden self-dependence.

The Jacobian matrix of f^N at the origin is

$$J_f^N = \begin{bmatrix} f_0 & f_2 & 0 & f_1 \\ f_1 & f_0 & f_2 & 0 \\ 0 & f_2 & f_0 + f_1 & 0 \\ 0 & f_2 & f_1 & f_0 \end{bmatrix},$$

where each f_i is the partial derivative of f with respect to x_i at the origin. The eigenvalues of J_f^N are $f_0 + f_1 + f_2$, f_0 (twice) and $f_0 - f_2$. Assume f has a bifurcation condition associated with the valency, that is $f_0 + f_1 + f_2 = 0$ and f satisfies the non-degenerated conditions $f_1 + f_2 \neq 0$ and $f_1 + 2f_2 \neq 0$. Then the kernel of J_f^N is the eigenspace associated with $f_0 + f_1 + f_2$, that is,

$$\ker(J_f^N) = \Delta_0.$$

For concrete coupled cell systems, we can explicitly calculate the required non-degenerated conditions in Proposition 5.1 by computing the eigenvalues of J_f^N . Another approach that we present, goes through considering a new network with one less cell such that we can derive the kernel of the Jacobian matrix on the original network using the kernel of the Jacobian matrix on this new network.

Suppose that $f \in \mathcal{V}(N)$, i.e., $f_0 + f_1 + f_2 = 0$. Then the kernel of J_f^N is characterized by the following system:

$$\begin{bmatrix} f_0 & f_2 & 0 & f_1 \\ f_1 & f_0 & f_2 & 0 \\ 0 & f_2 & f_0 + f_1 & 0 \\ 0 & f_2 & f_1 & f_0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0 \Leftrightarrow \begin{cases} f_0 v_1 + f_2 v_2 + f_1 v_4 = 0 \\ f_1 v_1 + f_0 v_2 + f_2 v_3 = 0 \\ f_2 v_2 + (f_0 + f_1) v_3 = 0 \\ f_2 v_2 + f_1 v_3 + f_0 v_4 = 0 \end{cases}.$$

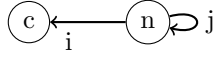

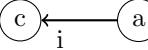

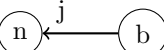
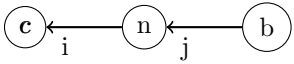
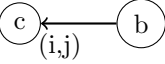
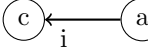
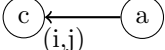

N	M
	
	
	
	
	
	

Table 1: Given an n -cell network N with set of cells $\{1, \dots, n\}$ and k types of edges $\{1, \dots, k\}$, we define the $(n-1)$ -cell network M with set of cells $\{1, \dots, n-1\}$ and $k^2 + 2k$ types of edges $\{(0, 1), \dots, (0, k), (1, 0), (1, 1), \dots, (1, k), \dots, (k, 0), (k, 1), \dots, (k, k)\}$ using the rules given by the table. Each edge in the table is annotated with its type and the cells $a, b, c \in \{1, \dots, n-1\}$. Given a cell $c \neq n$ in N , the left hand side of the table displays the edge of type i that targets c and the edge of type j that targets n where $0 \leq i, j \leq k$. Depending on the configuration of the left hand side of the table, we give the corresponding edge in M of type (i, j) that targets c .

Assume the non-degenerated condition $f_0 = -f_1 - f_2 \neq 0$. From the last equality in the previous system, we have that

$$v_4 = -\frac{f_2 v_2 + f_1 v_3}{f_0}.$$

Replacing v_4 in the other equalities and multiplying by f_0 , we obtain the system

$$\begin{cases} f_0^2 v_1 + (f_0 f_2 - f_1 f_2) v_2 - f_1^2 v_3 = 0 \\ f_0 f_1 v_1 + f_0^2 v_2 + f_0 f_2 v_3 = 0 \\ f_0 f_2 v_2 + (f_0^2 + f_0 f_1) v_3 = 0 \end{cases} . \quad (2)$$

As the variable v_4 does not appear in (2), we remove this cell from the network N . In order to apply a recursive argument on the number of cells, we find a network without cell 4 such that (2) defines the kernel of the Jacobian matrix for a coupled cell system of this new network with a bifurcation condition associated with the valency. We use the rules described on Table 1 to remove cell 4 and define the new network.

Fixing the cell $n = 4$ of N , we define the network M with 3 cells, $\{1, 2, 3\}$ and 8 edge's types, $(\gamma_{(0,1)}, \gamma_{(0,2)}, \gamma_{(1,0)}, \gamma_{(1,1)}, \gamma_{(2,1)}, \gamma_{(2,0)}, \gamma_{(2,1)}, \gamma_{(2,2)})$. The edges of M are given by the rules presented in Table 1. See Figure 7. Following the first

row of the table, the cell 1 receives in N an input of type 1 from the cell 4 and the cell 4 receives in N a self-input of type 0, then the cell 1 receives in M a self-input of type $(1, 0)$. Following the second row, the cells 1 and 4 receive in N an input of type 2 from the cell 2, then the cell 1 receives in M a self-input of type $(2, 2)$. Following the third row, the cell 1 receives in N an input of type 1 from the cell 4 and the cell 4 receives in N an input of type 1 from the cell 3, then the cell 1 receives in M an input of type $(1, 1)$ from the cell 3. Following the fourth row, the cell 1 receives in N an input of type 2 from the cell 2 and the cell 4 receives in N a self-input of type 0, then the cell 1 receives in M an input of type $(2, 0)$ from the cell 2. Doing the same for the other cells and inputs, we see that $\gamma_{(0,1)} = \gamma_{(0,2)} = \gamma_{(2,1)} = \gamma_{(2,2)} = [1 \ 2 \ 3]$, $\gamma_{(1,0)} = [1 \ 1 \ 3]$, $\gamma_{(1,1)} = [3 \ 2 \ 3]$, $\gamma_{(1,2)} = [2 \ 2 \ 3]$ and $\gamma_{(2,0)} = [2 \ 3 \ 2]$. Note that the network M is also strongly connected.

Looking to the system of equations (2), we see now that the type of inputs (i, j) adapt to that system. We find a coupled cell system g^M such that the system (2) corresponds to the kernel of J_g^M . We define the function $g : \mathbb{R} \times \mathbb{R}^8 \times \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$g(x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}, \lambda) =$$

$$f(f(x_{00}, -x_{01}, -x_{02}, \lambda), f(x_{10}, -x_{11}, -x_{12}, \lambda), f(x_{20}, -x_{21}, -x_{22}, \lambda), \lambda),$$

where $f \in \mathcal{V}(N)$. The partial derivative of g with respect to x_{ij} at the origin is

$$\begin{cases} g_{i0} = f_i f_0, & i = 0, 1, 2 \\ g_{ij} = -f_i f_j, & i = 0, 1, 2, j = 1, 2 \end{cases}.$$

Note that g^M has a bifurcation condition associated with the valency of M :

$$\sum_{i,j=0}^2 g_{ij} = f_0 \sum_{i=0}^2 f_i - \sum_{j=1}^2 f_j \sum_{i=0}^2 f_i = 0.$$

The Jacobian matrix of J_g^M at the origin is

$$J_g^M = \begin{bmatrix} a + g_{10} & g_{12} + g_{20} & g_{11} \\ g_{10} & a + g_{11} + g_{12} & g_{20} \\ 0 & g_{20} & a + g_{10} + g_{11} + g_{12} \end{bmatrix},$$

where $a = g_{00} + g_{01} + g_{02} + g_{21} + g_{22}$. Recalling that $f \in \mathcal{V}(N)$ and so $f_0 + f_1 + f_2 = 0$,

$$J_g^M = \begin{bmatrix} f_0^2 & f_2 f_0 - f_1 f_2 & -f_1 f_1 \\ f_1 f_0 & f_0^2 & f_2 f_0 \\ 0 & f_2 f_0 & f_0^2 + f_1 f_0 \end{bmatrix}.$$

Thus $(v_1, v_2, v_3) \in \ker(J_g^M)$ if and only if (v_1, v_2, v_3) satisfies the system (2). We could further reduce the network M . After two reductions, we would obtain a network with only one cell where the kernel of the Jacobian matrix of a coupled

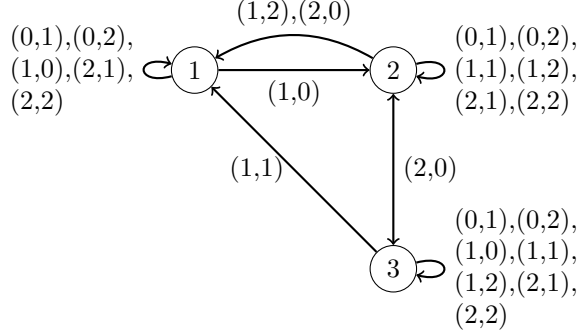


Figure 7: A three-cell homogeneous network with asymmetric inputs obtained from the network in Figure 6 by removing cell 4 and applying the rules in Table 1. The edges are annotated with their edge types and edges with more than one label represent multiple edges. This network is strongly connected.

cell system with a bifurcation condition associated with the valency is the full synchrony subspace. Instead, we assume that $\ker(J_g^M) = \{(x, x, x) : x \in \mathbb{R}\}$, for generic functions $g \in \mathcal{V}(M)$. Therefore,

$$(v_1, v_2, v_3, v_4) \in \ker(J_f^N) \Leftrightarrow \begin{cases} (v_1, v_2, v_3) \in \ker(J_g^M) \\ v_4 = -\frac{f_2 v_2 + f_1 v_3}{f_0} \end{cases} \Leftrightarrow v_1 = v_2 = v_3 = v_4.$$

Thus $\ker(J_f^N) = \Delta_0$, for generic functions $f \in \mathcal{V}(N)$. \diamond

Proof of Proposition 5.1. Let N be a strongly connected network with n cells and represented by $(\sigma_i)_{i=1}^k$ and $f \in \mathcal{V}(N)$ generic. Recall that σ_0 is the identity function on $\{1, \dots, n\}$ and it corresponds to the hidden self-dependence. We recursively prove that

$$\ker(J_f^N) = \{(x, \dots, x) \in \mathbb{R}^n : x \in \mathbb{R}\}.$$

Suppose that the network N has one cell, $n = 1$. Since $\sum_{i=0}^k f_i = 0$, we have that $J_f^N = [0]$ and $\ker(J_f^N) = \mathbb{R} = \Delta_0$.

Suppose that the network N has $m + 1$ cells $\{1, \dots, m, n\}$ with $n = m + 1$. Since $f \in \mathcal{V}(N)$, we have that $\ker(J_f^N)$ is nontrivial. Take $v = (v_1, \dots, v_m, v_n) \in \mathbb{R}^n$ such that $J_f^N v = 0$. Denote by J_{cd} the (c, d) entry of J_f^N , i.e.,

$$J_{cd} := \sum_{\sigma_i(c)=d} f_i.$$

Thus, using this notation, we have that

$$J_f^N v = 0 \Leftrightarrow \begin{cases} J_{11}v_1 + J_{12}v_2 + \dots + J_{1n}v_n = 0 \\ \vdots \\ J_{n1}v_1 + J_{n2}v_2 + \dots + J_{nn}v_n = 0 \end{cases}. \quad (3)$$

Since N is strongly connected, the cell n receives an edge from some other cell. Thus $J_{nn} \neq \sum_{i=0}^k f_i$ and so, we can generically assume on f that

$$J_{nn} = \sum_{\sigma_i(n)=n}^k f_i = - \sum_{\sigma_i(n) \neq n}^k f_i \neq 0.$$

Moreover,

$$v_n = - \frac{J_{n1}v_1 + J_{n2}v_2 + \cdots + J_{nm}v_m}{J_{nn}}.$$

Replacing v_n in the first m equations of the system (3), we obtain

$$\sum_{d=1}^m (J_{nn}J_{cd} - J_{cn}J_{nd})v_d = 0, \quad 1 \leq c \leq m.$$

Let J' be the $m \times m$ -matrix with entries

$$J'_{cd} = J_{nn}J_{cd} - J_{cn}J_{nd},$$

where $1 \leq c, d \leq m$. Next, we define a network M with m cells and a function $g \in \mathcal{V}(M)$ such that

$$J_g^M = J'.$$

In order to remove cell n from network N and define the network M , we use the rules presented in Table 1. The type of edges in M are (i, j) where $0 \leq i, j \leq k$ and $(i, j) \neq (0, 0)$, and the edges of type (i, j) are represented by the function γ_{ij} . Following Table 1, each function γ_{ij} is given by

$$\gamma_{ij}(c) = \begin{cases} c, & \sigma_i(c) = \sigma_j(n) = n \\ c, & \sigma_i(c) \neq n, \sigma_j(n) \neq n \\ \sigma_j(n), & \sigma_i(c) = n, \sigma_j(n) \neq n \\ \sigma_i(c), & \sigma_i(c) \neq n, \sigma_j(n) = n \end{cases},$$

where $1 \leq c \leq m$ and each case corresponds to the corresponding row in the table.

Each path in N induces a path in M by removing any transition by cell n . Therefore M is strongly connected, because N is strongly connected.

In order to use a recursive argument, we define now a function g such that $g \in \mathcal{V}(M)$ and $J' = J_g^M$. Let $g : \mathbb{R} \times \mathbb{R}^{(k+1)^2-1} \times \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$g(x_{00}, \dots, x_{0k}, x_{10}, \dots, x_{1k}, \dots, x_{k0}, \dots, x_{kk}, \lambda) = f(f(y_{00}, \dots, y_{0k}, \lambda), f(y_{10}, \dots, y_{1k}, \lambda), \dots, f(y_{k0}, \dots, y_{kk}, \lambda), \lambda),$$

where $f \in \mathcal{V}(N)$, $y_{ij} = \beta_j x_{ij}$, $\beta_j = 1$, if $\sigma_j(n) = n$, and $\beta_j = -1$, if $\sigma_j(n) \neq n$, for $0 \leq i, j \leq k$. For $1 \leq i, j \leq k$, we have that

$$g_{ij} = \frac{\partial g}{\partial x_{ij}}(0, 0, \dots, 0, 0) = \beta_j f_i f_j.$$

Now, we prove that g satisfies the required conditions: $J' = J_g^M$ and $g \in \mathcal{V}(M)$. Let $1 \leq c, d \leq m$. If $c \neq d$, then

$$(J_g^M)_{cd} = \sum_{\gamma_{ij}(c)=d} g_{ij} = \sum_{\sigma_i(c)=d} \sum_{\sigma_j(n)=n} f_i f_j - \sum_{\sigma_i(c)=n} \sum_{\sigma_j(n)=d} f_i f_j = J'_{cd}.$$

Recall that $\sum_{\sigma_i(c)<n} f_i + \sum_{\sigma_i(c)=n} f_i = 0$ as $f \in \mathcal{V}(N)$. Then

$$\begin{aligned} (J_g^M)_{cc} &= \sum_{\substack{\sigma_i(c)=c \\ \sigma_j(n)=n}} f_i f_j - \sum_{\substack{\sigma_i(c)=n \\ \sigma_j(n)=c}} f_i f_j + \sum_{\substack{\sigma_i(c)=n \\ \sigma_j(n)=n}} f_i f_j - \sum_{\substack{\sigma_i(c)<n \\ \sigma_j(n)<n}} f_i f_j = \\ &= J'_{cc} + \sum_{\sigma_i(c)=n} f_i \sum_{\sigma_j(n)=n} f_j - \sum_{\sigma_i(c)=n} f_i \sum_{\sigma_j(n)=n} f_j = J'_{cc}. \end{aligned}$$

Hence

$$J' = J_g^M.$$

Note that $g \in \mathcal{V}(M)$, since

$$\sum_{i=0}^k \sum_{j=0}^k g_{ij} = \sum_{j=0}^k \beta_j f_j \sum_{i=0}^k f_i = 0.$$

Before we apply the recursive argument, we emphasize that, when $m > 1$, the generic condition on g can be regarded as a generic condition on f :

$$\sum_{\gamma_{ij}(m)=m} g_{ij} = \sum_{\sigma_i(m)=m} f_i \sum_{\sigma_j(n)=n} f_j - \sum_{\sigma_i(m)=n} f_i \sum_{\sigma_j(n)=m} f_j \neq 0,$$

We can repeat the previous reduction to the network M and the function g . After a finite number of steps, we obtain a network with only one cell where the kernel of the Jacobian matrix is the full synchrony subspace. So we assume that $\ker(J_g^M)$ is the full-synchrony subspace and prove that $\ker(J_f^N) = \Delta_0$. We have that $v \in \ker(J_f^N)$ if and only if $(v_1, \dots, v_m) \in \ker(J_g^M)$ and

$$v_n = -\frac{J_{n1}v_1 + J_{n2}v_2 + \dots + J_{nm}v_m}{J_{nn}} = -\frac{\sum_{c=1}^m J_{nc}}{J_{nn}}v_1 = v_1,$$

because $v_1 = \dots = v_m$. Therefore $v \in \ker(J_f^N)$ if and only if $v \in \Delta_0$. \square

In the following example, we present a strongly connected network N and a degenerated function $f \in \mathcal{V}(N)$ for which the kernel of J_f^N is not the full-synchrony subspace.

Example 5.3. Let N be the strongly connected network represented in Figure 2 and $f \in \mathcal{V}(N)$ such that $f_1 = f_2 = 1$ and $f_3 = -1/2$. Then $f_0 = -3/2$,

$$J_f^N = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and

$$\Delta_0 \subsetneq \ker(J_f^N) = \{(2x, 2y, x + y) : x, y \in \mathbb{R}\}. \quad \diamond$$

Next, we describe the codimension-one steady-state bifurcation of coupled cell systems associated with strongly connected networks where the bifurcation condition corresponds to the network valency. As shown below, this bifurcation does not break the full-synchrony. This result follows from the previous result and well-know methods in bifurcation theory.

Proposition 5.4. *Let N be a strongly connected network and $f \in \mathcal{V}(N)$ generic. Then, there exist a neighborhood $U \subset \mathbb{R}$ of 0 and a germ $b_f : U \rightarrow \mathbb{R}$ such that if $b : U \rightarrow \mathbb{R}^{|N|}$ is a bifurcation branch of f on N then*

$$b(\lambda) = (b_f(\lambda), \dots, b_f(\lambda)) \in \Delta_0, \quad \lambda \in U.$$

Proof. Let N be a strongly connected network and $f \in \mathcal{V}(N)$ generic. By Proposition 5.1, we know that $\ker(J_f^N) = \Delta_0$. Applying the Lyapunov-Schmidt Reduction ([21, Chapter VII]), there exists $W : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{|N|}$ such that

$$f^N(b, \lambda) = 0 \Leftrightarrow b = (x, \dots, x) + W(x, \lambda) \wedge f(x, x, \dots, x, \lambda) = 0.$$

If $f(x, x, \dots, x, \lambda) = 0$, then $f^N((x, \dots, x), \lambda) = 0$. By uniqueness of W , $W \equiv 0$. So $b \in \Delta_0$.

Assuming non-degenerated conditions on the function f , the equation $f(x, x, \dots, x, \lambda) = 0$ has a transcritical bifurcation, see e.g., [22]. There exist a neighborhood $U \subseteq \mathbb{R}$ of 0 and a non-zero germ $b_f : U \rightarrow \mathbb{R}$ such that

$$f(x, x, \dots, x, \lambda) = 0 \Leftrightarrow x = 0 \vee x = b_f(\lambda), \quad \lambda \in U.$$

Moreover

$$f^N(b, \lambda) = 0 \Leftrightarrow b = (0, \dots, 0) \vee b = (b_f(\lambda), \dots, b_f(\lambda)), \quad \lambda \in U.$$

Therefore, if $b : U \rightarrow \mathbb{R}^{|N|}$ is a bifurcation branch of f on N then $b(\lambda) = (b_f(\lambda), \dots, b_f(\lambda))$, $\lambda \in U$. \square

Now, we address the same bifurcation problem assuming that the network is not necessarily strongly connected. Let N be a network and $f \in \mathcal{V}(N)$ generic. We start by describing the kernel of J_f^N . Reordering the cells in the network by its strongly connected components, we have that the eigenvalues of J_f^N are the union of the eigenvalues of J_f^B for each strongly connected component B of N . Here J_f^B is the submatrix of J_f^N with columns and rows corresponding to the cells in B . We prove now that the kernel of J_f^B is trivial, if B is not a source.

Proposition 5.5. *Let N be a network, $f \in \mathcal{V}(N)$ generic and B a strongly connected component of N which is not a source. Then $\ker(J_f^B) = \{0\}$.*

Proof. Let N be a network represented by $(\sigma_i)_{i=1}^k$, $f \in \mathcal{V}(N)$ generic and B a strongly connected component of N which is not a source. Like in the proof of Proposition 5.1, we use a recursive argument on the number of cells of B to prove the result. Denote by c_1, \dots, c_n the cells of B and by $J_{p,q}$ the (c_p, c_q) -entry of J_f^B . Define $\theta = (\theta_1, \dots, \theta_n)$, where

$$\theta_p := \sum_{\sigma_i(c_p) \notin B} f_i.$$

Generically on f , we can assume that $\theta \neq (0, \dots, 0)$, i.e., $\theta_p \neq 0$ for some $1 \leq p \leq n$, as B is not a source. Since $f \in \mathcal{V}(N)$,

$$\sum_{q=1}^n J_{p,q} = -\theta_p, \quad 1 \leq p \leq n.$$

Let $v = (v_1, \dots, v_n) \in \mathbb{R}^{|B|}$ be such that $J_f^B v = 0$.

Suppose that $n = 1$. Then $J_f^B = [-\theta_1] \neq 0$, generically, and so

$$\ker(J_f^B) = \{0\}.$$

Suppose now that $n = m + 1$. We can assume generically on f that $J_{n,n} \neq 0$, since B is a strongly connected component. Let J' be the $m \times m$ -matrix and let $\theta' = (\theta'_1, \dots, \theta'_m)$ which are, respectively, defined by

$$\begin{aligned} J'_{p,q} &= J_{n,n} J_{p,q} - J_{p,n} J_{n,q}, \\ \theta'_p &= J_{n,n} \theta_p - J_{p,n} \theta_n, \end{aligned}$$

for $1 \leq p, q \leq m$. Generically on f , we assume that $\theta' \neq (0, \dots, 0)$, and if $m > 1$ we also assume that $J'_{m,m} \neq 0$. Note that $(v_1, \dots, v_m) \in \ker(J')$ and

$$\begin{aligned} \sum_{q=1}^m J'_{p,q} v_q &= J_{n,n} \sum_{q=1}^m J_{p,q} v_q - J_{p,n} \sum_{q=1}^m J_{n,q} v_q \\ &= J_{p,n} (J_{n,n} + \theta_n) - J_{n,n} (J_{p,n} + \theta_p) = -\theta'_p. \end{aligned}$$

As in the proof of Proposition 5.1, we can remove the cell n belonging to B from the network N and define a network M with a strongly connected component B' and a function $g \in \mathcal{V}(M)$ such that $B' = B \setminus \{n\}$ has m cells and $J' = J_g^{B'}$. Hence we can apply the same recursive argument and conclude that

$$\ker(J_f^B) = \{0\}. \quad \square$$

In the next result, we describe the kernel of the Jacobian matrix of a coupled cell system with the bifurcation condition corresponding to the network valency.

Proposition 5.6. *Let N be a network and $f \in \mathcal{V}(N)$ generic. Then $\ker(J_f^N)$ has dimension equal to $s(N)$. Moreover, if S is a source component of N and $v \in \ker(J_f^N)$, then*

$$v_c = v_d, \quad c, d \in S.$$

Proof. Let N be a network and $f \in \mathcal{V}(N)$ generic such that $s(N) = s$. Denote the source components of N by S_1, \dots, S_s and order the other strongly connected components by B_1, \dots, B_r such that any edge targeting a cell in B_i starts in a cell of $S_1 \cup \dots \cup S_s \cup B_1 \cup \dots \cup B_i$. Reordering the cells of N by its strongly connected components, we see that the matrix J_f^N has the following block form

$$J_f^N = \left[\begin{array}{cccc|cccc} J_f^{S_1} & 0 & \dots & 0 & & & & \\ 0 & J_f^{S_2} & \dots & 0 & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \dots & J_f^{S_s} & & & & \\ \hline & & & & J_f^{B_1} & 0 & \dots & 0 \\ & & & & R_{21} & J_f^{B_2} & \dots & 0 \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & R_{r1} & R_{r2} & \dots & J_f^{B_r} \end{array} \right].$$

By Propositions 5.1 and 5.5, $\ker(J_f^{S_i})$ is one-dimensional for every $1 \leq i \leq s$ and $\ker(J_f^{B_j})$ is trivial for every $1 \leq j \leq r$. So $\ker(J_f^N)$ is $s = s(N)$ -th dimensional.

Let $v \in \ker(J_f^N)$ and $1 \leq i \leq s$. Then $v_{S_i} = (v_c)_{c \in S_i} \in \ker(J_f^{S_i})$ and, it follows from Proposition 5.1 that $v_c = v_d$, for $c, d \in S_i$. \square

Using the previous results, we describe next the codimension-one steady-state bifurcations of coupled cell systems associated with a network and imposing the bifurcation condition corresponding to the network valency.

Proposition 5.7. *Let N be a network and $f \in \mathcal{V}(N)$ generic. Then there are $2^{s(N)}$ equilibrium branches of f on N with the following properties:*

- (i) *For every equilibrium branch b , if $c, d \in S$, for some source component S , then $b_c = b_d = 0$ or $b_c = b_d = b_f$, where b_f is defined by Proposition 5.4.*
- (ii) *Given two equilibrium branches b and b' , if $b_S = b'_S$ for every source component S , then $b = b'$.*

Proof. Let N be a network, $s = s(N)$ and $f \in \mathcal{V}(N)$ generic. Denote by S_1, \dots, S_s the source components of N and by B the set of cells not belonging to $S_1 \cup \dots \cup S_s$.

It follows from the proof of Proposition 5.6 that J_f^B is invertible. By the Implicit Function Theorem, there exists $W : \mathbb{R}^{|S_1|} \times \dots \times \mathbb{R}^{|S_s|} \times \mathbb{R} \rightarrow \mathbb{R}^{|B|}$ such that

$$f^N(x, \lambda) = 0 \Leftrightarrow \begin{cases} f^{S_i}(x_{S_i}, \lambda) = 0, & i = 1, \dots, s \\ x_B = W(x_{S_1}, \dots, x_{S_s}, \lambda) \end{cases}.$$

Using Proposition 5.4, it follows that, for each source S_i , $1 \leq i \leq s$, we can solve as:

$$f^{S_i}(x_{S_i}, \lambda) = 0 \Leftrightarrow x_{S_i} = (0, \dots, 0) \vee x_{S_i} = (b_f(\lambda), \dots, b_f(\lambda)),$$

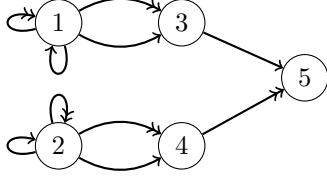


Figure 8: Homogeneous network with asymmetric inputs and two source components: $\{1\}$ and $\{2\}$. Given a generic coupled cell system with a bifurcation condition associated with the network valency, there are four bifurcation branches.

where $b_f : U \rightarrow \mathbb{R}$ is defined in Proposition 5.4 and it does not depend on the source component. Hence

$$f^N(x, \lambda) = 0 \Leftrightarrow \begin{cases} x_{S_i} = (0, \dots, 0) \vee x_{S_i} = (b_f(\lambda), \dots, b_f(\lambda)), & i = 1, \dots, s \\ x_B = W(x_{S_1}, \dots, x_{S_s}, \lambda) \end{cases}.$$

For each source component we have two choices in the previous equation, then there are $2^{s(N)}$ equilibrium branches of f on N .

If b is an equilibrium branch of f on N and c, d belong to the same source component, then $b_c = b_d = 0$ or $b_c = b_d = b_f$. This proves (i).

Let b and b' be two equilibrium branches of f on N . If $b_S = b'_S$ for every source component, then $b_B = W(b_{S_1}, \dots, b_{S_s}, \lambda) = W(b'_{S_1}, \dots, b'_{S_s}, \lambda) = b'_B$ and $b = b'$. Proving (ii). \square

Example 5.8. Let N be the network in Figure 8 and $f \in \mathcal{V}(N)$ generic. The network N has two source components. By Proposition 5.7, there are 4 bifurcation branches of f on N . The bifurcation branches are $(0, 0, 0, 0, 0)$, $(b_f, 0, b_f, 0, b_1)$, $(0, b_f, 0, b_f, b_2)$ and $(b_f, b_f, b_f, b_f, b_f)$, where b_f is defined by Proposition 5.4 applied to (any) of the source components of N , that is $f(b_f(\lambda), b_f(\lambda), b_f(\lambda), \lambda) = 0$, b_1 is the unique solution of $f(x, b_f(\lambda), 0, \lambda) = 0$ and b_2 is the unique solution of $f(x, 0, b_f(\lambda), \lambda) = 0$. \diamond

Remark 5.9. Let N be a network and $f \in \mathcal{V}(N)$ generic. Denote by S_1, \dots, S_s the source components of N . The cells inside a source component receive every input from a cell inside that source component. Then the coloring that assigns a different color for each source component and it assigns the same color only for cells inside the same source component is balanced. The corresponding synchrony subspace is

$$\Delta_{S_1} \times \dots \times \Delta_{S_s} \times \mathbb{R}^{|N| - (|S_1| + \dots + |S_s|)},$$

where $\Delta_{S_i} = \{x \in \mathbb{R}^{|S_i|} : x_c = x_d, c, d \in S_i\}$ is the full synchrony subspace in the source component S_j , for $j = 1, \dots, s$.

By Proposition 5.7, if b is a bifurcation branch of f on N , then

$$b \in \Delta_{S_1} \times \dots \times \Delta_{S_s} \times \mathbb{R}^{|N| - (|S_1| + \dots + |S_s|)}. \quad \diamond$$

6. The lifting bifurcation problem associated with the valency

In this section, we give conditions that characterize the lifting bifurcation problem for generic coupled cell systems with a bifurcation condition associated with the valency. Those conditions only depend on the number of source components. The results follow from the characterization of the bifurcation branches obtained in Section 5.

Proposition 6.1. *Let N be a network, L a lift of N and $f \in \mathcal{V}(N)$ generic. Then:*

- (i) *If $s(N) = s(L)$, then every bifurcation branch of f on L is lifted from N .*
- (ii) *If $1 = s(N) < s(L)$, then there exists at least one bifurcation branch of f on L not lifted from N .*

Proof. Let N be a network, L a lift of N , \boxtimes a balanced coloring in L such that $N = L/\boxtimes$ and $f \in \mathcal{V}(N)$ generic. Denote by $\varphi_{\boxtimes} : L \rightarrow N$ the network fibration induced by \boxtimes and by $S_1, \dots, S_{s(L)}$ the source components of L . Note that the source components of N are $\varphi_{\boxtimes}(S_1), \dots, \varphi_{\boxtimes}(S_{s(L)})$.

(i) Suppose that $s(N) = s(L)$. Let b be a bifurcation branch of f on L . Using Proposition 5.7, we define the bifurcation branch a of f on N such that

$$a_{\varphi_{\boxtimes}(c)} := b_c, \quad c \in S_i, 1 \leq i \leq s(L).$$

The bifurcation branch a is defined for each source component because the network fibration φ_{\boxtimes} sends each source component of L into a different source component of N . Therefore the bifurcation branch a is well defined. Note that

$$b_{S_i} = (P\varphi_{\boxtimes}(a))_{S_i}, \quad 1 \leq i \leq s(L),$$

where $P\varphi_{\boxtimes}$ is the map between the phase spaces of N and L induced by φ_{\boxtimes} . So the bifurcation branches b and $P\varphi_{\boxtimes}(a)$ coincide on the source components and $b = P\varphi_{\boxtimes}(a)$, by Proposition 5.7 (ii).

(ii) Suppose that $1 = s(N) < s(L)$. Denote by S the unique source component of N . Let b' be a bifurcation branch of f on N . By Proposition 5.7 (i), we know that $b'_S = (0, \dots, 0)$ or $b'_S = (b_f, \dots, b_f)$. Returning to the proof of Proposition 5.7, we have that $W(b'_S, \lambda) = b'_s(1, \dots, 1)$ for any $s \in S$, because $W : \mathbb{R}^{|S|} \times \mathbb{R} \rightarrow \mathbb{R}^{|N|-|S|}$ is the unique solution of the system $f^N(b'_S + W(b'_S, \lambda), \lambda) = 0$. So $b'_c = b'_d$, for any cells c and d of N and

$$b' \in \Delta_N \subseteq \mathbb{R}^{|N|},$$

where Δ_N is the full synchrony subspace associated with the network N . For any network fibration $\varphi : L \rightarrow N$, $(P\varphi b')_c = b'_{\varphi(c)} = b'_{\varphi(d)} = (P\varphi b')_d$ for any cells c and d of L and

$$P\varphi b' \in \Delta_L \subseteq \mathbb{R}^{|L|},$$

where Δ_L is the full synchrony subspace associated with the network L . By Proposition 5.7 there exists a bifurcation branch b of f on L such that $b \notin \Delta_L$, because $s(L) > 1$. Moreover b is not lifted from N , because any bifurcation branch lifted from N belongs to the full-synchrony subspace, Δ_L . \square

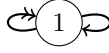


Figure 9: Network with a single cell and two edges types.

The previous result shows that the bifurcation branches of a generic coupled cell system, associated with a network with only one source component and the bifurcation condition correspondent to the network valency, are lifted from the trivial quotient network with a single cell and the same number of edge types. Thus those bifurcation branches do not break the full-synchrony.

Example 6.2. The network N in Figure 9 is the trivial quotient network of every network with two types of edges associated with the balanced coloring with exactly one color. Consider the lifts L_1 and L_2 of N given in Figures 6 and 8, respectively and $f \in \mathcal{V}(N)$ generic. Note that $1 = s(N) = s(L_1) < s(L_2) = 2$.

By Proposition 6.1 (i), the bifurcations branches of f on L_1 are lifted from N . On the other hand, by Proposition 6.1 (ii) there exists a bifurcation branch of f on L_2 which is not lifted from N . \diamond

In Proposition 6.1 (ii) we assume that $1 = s(N) < s(L)$. If we change that assumption to $1 < s(N) < s(L)$, then it may happen that all the bifurcation branches of f on L are lifted from N , as illustrated in the following two examples.

Example 6.3. Let N be the network described in Figure 5 which has 2 source components, L the lift network of N described in Figure 3 with 3 source components and $f \in \mathcal{V}(N)$ generic. Consider the network fibrations from L to N given by: $\varphi_{1,2} = [2\ 2\ 1\ 2\ 5]$; $\varphi_{1,3} = [1\ 2\ 1\ 5\ 5]$; and $\varphi_{2,3} = [1\ 2\ 2\ 5\ 2]$. Let b be a bifurcation branch of f on L . According to Proposition 5.7 (i), the bifurcation branch b can take one of two possible values on the coordinates of each of the cells 1, 2 and 3. Then at least one of the equalities $b_1 = b_2$, $b_1 = b_3$, $b_2 = b_3$ holds. Suppose that $b_i = b_j$, for some $1 \leq i < j \leq 3$. Let b' be the bifurcation branch of f on N such that $b'_{\varphi_{i,j}(c)} = b_c$, for $c \in \{1, 2, 3\}$. Then $b = P\varphi_{i,j}b'$ and it is lifted from N .

The networks L and N satisfy $1 < s(N) < s(L)$ and every bifurcation branch of f on L is lifted from N . \diamond

Example 6.4. Let N be the network in Figure 4, L the lift network in Figure 1 and $f \in \mathcal{V}(N)$ generic. Consider the network fibrations from L to N given by: $\varphi_{1,2} = [221564]$; $\varphi_{1,3} = [212645]$; and $\varphi_{2,3} = [122456]$. Let b be a bifurcation branch of f on L . By Proposition 5.7, we know that $b_1 = b_2$, $b_2 = b_3$ or $b_1 = b_3$. Suppose that $b_i = b_j$, for some $1 \leq i < j \leq 3$. Then b is lifted from N using $\varphi_{i,j}$.

Again, we have that $1 < s(N) < s(L)$ and every bifurcation branch of f on L is lifted from N . \diamond

In the previous examples, we saw that increasing the number of source components on the lift network is not sufficient to ensure that some bifurcation

branch on the lift network is not lifted from the quotient network. The lift network in Figure 3 and considered in Example 6.3 is not backward connected. In Example 6.4, we consider the quotient network given by Figure 4 which is not transitive. The next result shows that increasing the number of source components on the lift network with respect to the quotient network is a necessary and sufficient condition for the existence of some bifurcation branch on the lift network that is not lifted from the quotient network, provided that the lift network is backward connected and the quotient network is transitive.

Theorem 6.5. *Let N be a transitive homogeneous network with asymmetric inputs, L a backward connected lift of N and $f \in \mathcal{V}(N)$ a generic function with a bifurcation condition associated with the network valency. Then every bifurcation branch of f on L is lifted from N if and only if $s(N) = s(L)$.*

Proof. Let N be a transitive network for the cell t in N and represented by the functions $(\sigma_i)_{i=1}^k$ and $f \in \mathcal{V}(N)$ generic. Let L be a backward connected network for the cell l , \bowtie a balanced coloring in L such that $N = L/\bowtie$.

If $s(N) = s(L)$, then every bifurcation branch of f on L is lifted from N , by Proposition 6.1(i).

Next, we suppose that $s(N) < s(L)$ and prove that there is a bifurcation branch of f on L that is not lifted from N . Denote by $\varphi_{\bowtie} : L \rightarrow N$ the network fibration induced by \bowtie . Note that the network N is backward connected for the cell $l' = \varphi_{\bowtie}(l)$. Denote by $\phi_c : N \rightarrow N$ the network fibrations for each cell c in N such that $\phi_c(t) = c$.

Since N is backward connected for l' , for every cell c in N there exist $1 \leq i_1, \dots, i_m \leq k$ such that $\phi_{l'}(\sigma_{i_1} \circ \dots \circ \sigma_{i_m}(t)) = \sigma_{i_1} \circ \dots \circ \sigma_{i_m}(\phi_{l'}(t)) = \sigma_{i_1} \circ \dots \circ \sigma_{i_m}(l') = c$. Then $\phi_{l'}$ is surjective and it is also bijective, since N is finite. Applying the inverse of $\phi_{l'}$ to ϕ_c , we see that N is transitive for the cell l' . Assume that $l' = t$.

From $s(N) < s(L)$, it follows that there exist two source components S_1, S_2 of L such that $\varphi_{\bowtie}(S_1) = \varphi_{\bowtie}(S_2)$ is a source component of N . Let $\varphi : L \rightarrow N$ be any network fibration from L to N . By Remark 2.10 and $\varphi(l) = \phi_{\varphi(l)} \circ \varphi_{\bowtie}(l)$, we have that $\varphi = \phi_{\varphi(t)} \circ \varphi_{\bowtie}$. Hence $\varphi(S_1) = \varphi(S_2)$, for every network fibration $\varphi : L \rightarrow N$. If b' is a bifurcation branch of f on N , then

$$P\varphi(b')_{c_1} = P\varphi(b')_{c_2},$$

for $c_1 \in S_1$ and $c_2 \in S_2$. However, we know from Proposition 5.7 (i) that there exists a bifurcation branch b of f on L such that $b_{c_1} \neq b_{c_2}$, for $c_1 \in S_1$ and $c_2 \in S_2$. So b is not lifted from N . \square

Example 6.6. Let N be the transitive network described in Figure 5 and $f \in \mathcal{V}(N)$ generic. Consider the lift networks L_1 and L_2 of N described in the Figures 8 and 10, respectively. Note that $s(N) = s(L_1) = 2$, $s(L_2) = 3$ and the network L_1 and L_2 are backward connected for the cell 5. Using Theorem 6.5, we know that every bifurcation branch of f on L_1 is lifted from N but there exists a bifurcation branch of f on L_2 that is not lifted from N . \diamond

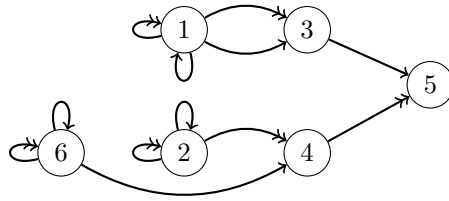


Figure 10: Homogeneous network with three source components: $\{1\}$, $\{2\}$ and $\{6\}$. This network is a lift of the network in Figure 5, taking the balanced coloring \bowtie such that $1 \bowtie 3$ and $2 \bowtie 4 \bowtie 6$.

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