Symmetry-breaking Bifurcations of Wreath Product Systems

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Abstract. Patterns formed through steady-state and Hopf bifurcations in wreath product systems depend on both the internal and global symmetries. In this paper we explore some features of this dependence related to general constraints on commuting matrices. We describe the stability of steady states and periodic solutions of wreath product systems obtained from the Equivariant Branching Lemma and the Equivariant Hopf Theorem.

1 Introduction

Symmetries of nonlinear dynamical systems constrain their behaviour in significant ways [12, 14]. In particular, the typical phenomena differ according to the group of symmetries that is present. Symmetries complicate the analysis in some respects, but simplify it in others. Simple representation-theoretic principles often help to organise and simplify calculations for symmetric systems - for example, the commuting matrices with the group action have block-diagonal form. In this paper we investigate the implications of such a strategy for steady-state and Hopf bifurcation in 'wreath product' systems [9, 13]. Such systems arise, in particular, when modelling symmetric networks of coupled identical cells where each individual cell has its own 'local' symmetry and the coupling is invariant under local symmetries. However, they also arise in many other contexts, for example lattice dynamics.

Suppose that we are interested in bifurcating branches of steady state solutions to a dynamical system of the form

$$\frac{dX}{dt} = F(X, \lambda)$$

where F commutes with a symmetry group Γ that acts on the state variables X. (The Hopf bifurcation case is analogous, but with more technical complications.) Equivariant bifurcation theory provides conditions for the existence of such branches. A key concept is the *isotropy subgroup* Σ of the solution - the subgroup of the symmetry group Γ of the system consisting of all symmetries that leave the solution invariant. When Σ is a proper subgroup of Γ we say that the bifurcating branch *breaks symmetry* to Σ. The stability of this branch is determined by the eigenvalues of the Jacobian dF, evaluated along the branch. The most obvious symmetry constraint on dF is that it should commute with the action of Σ. This constraint alone can greatly simplify stability calculations.

In this paper we study this constraint in the context of wreath product systems (see Section 2) where the rich algebraic context has surprisingly strong implications for the form of the matrices that commute with the action of an isotropy subgroup Σ. We assume throughout that Σ is 'axial' ('$\mathbf{C}$-axial' for the Hopf case), a group-theoretic condition that guarantees the existence of bifurcating branches provided that certain technical conditions on F are valid (Sections 3.1, 3.2). Such isotropy subgroups have been classified, for wreath product groups, by Dionne, Golubitsky and Stewart [9] and Dias [4]; we make essential use of this classification. The main results of Section 3 are Theorems 3.4 and 3.12, which describe symmetry constraints on dF that arise in wreath product systems.

In Section 4 we apply these ideas to systems that are written in 'coupled network' form $F_j(X, \lambda) = f(X_j, \lambda) + h_j(X, \lambda)$, where X_j determines the state of cell j, the vector field f is the 'internal' dynamic of the cell, and h_j describes the coupling between cell j and all other cells. It turns out that when the system is expressed in coupled cell form, many of the necessary bifurcation-theoretic calculations can be performed very easily - provided the symmetry constraints

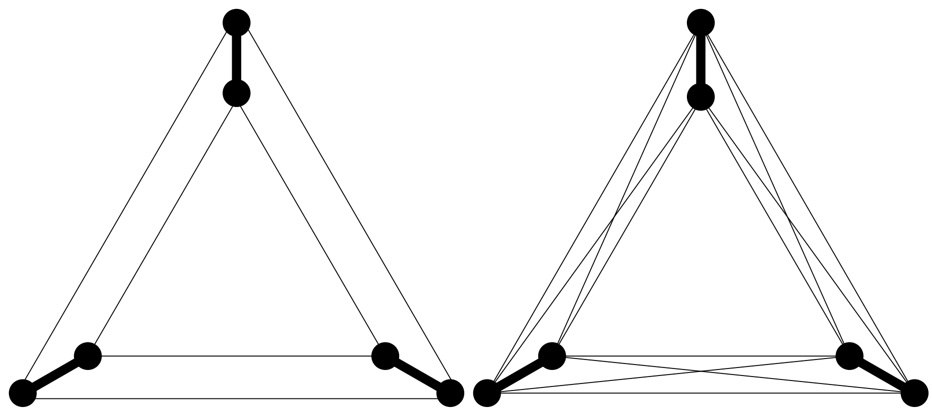

Figure 1: System of six identical oscillators indicated by dots, to be thought of as three 2-oscillators sets as indicated by heavy lines. The other lines show couplings and all couplings are bidirectional and identical. (a) Symmetry group $\mathbf{Z}_2 \times \mathbf{S}_3$: internal symmetries must be applied to all three oscillator pairs simultaneously. (b) Symmetry group $\mathbf{Z}_2 \wr \mathbf{S}_3$: internal symmetries can be applied independently to any oscillator pair.

on dF are fully exploited. For example the hypotheses of the standard existence theorems (Equivariant Branching Lemma and Equivariant Hopf Theorem) reduce to straightforward conditions on the coupling function h_1 and the vector field f, and the stabilities of bifurcating branches can be computed quite easily. Finally, in Section 5, we give some examples to illustrate how simple the calculations can become using our methods; in part, we re-derive results obtained by previous authors.

Before getting down to details, we make some remarks to motivate the use of wreath product systems, and to provide further orientation. The symmetries of coupled cell systems are determined by the symmetries of the individual cells and of the coupling between cells. Two natural types of coupling giving rise to *wreath product* and *direct product* groups have been studied [13, 9, 10]. In both cases, the total symmetry group Γ of the system depends on the group of *local* internal symmetries $\mathcal{L}$ of the dynamics in each individual cell, and on the group $\mathcal{G}$ of *global* permutations of the cells. Figure 1 shows schematically six identical oscillators coupled in three 2-oscillators sets. Thus three coupled identical cells where each cell is formed by two coupled oscillators. The two types of coupling are shown: in (a) the coupling leads to the direct product $\mathbf{Z}_2 \times \mathbf{S}_3$ and in (b) to the wreath product $\mathbf{Z}_2 \wr \mathbf{S}_3$.

Considering a network of N cells where each cell is a k-dimensional system of ODEs, the group $\mathcal{L}$ is a subgroup of $\mathbf{O}(k)$ and $\mathcal{G}$ is a transitive subgroup of the permutation group $\mathbf{S}_N$. We focus on the wreath product case when the coupling is invariant under any local symmetry of any cell: here the notation used for Γ is $\mathcal{L} \wr \mathcal{G}$.

We concentrate on steady-state and Hopf bifurcations. In steady-state bifurcations, the basic existence result is the Equivariant Branching Lemma [14]. It states that generically branches of steady states with an isotropy subgroup whose fixed-point subspace is one-dimensional bifurcate from a steady state with the full symmetry Γ of the system. These isotropy subgroups are called *axial.* Recall that the *isotropy subgroup* of an equilibrium consists of all symmetries of that equilibrium. In Hopf bifurcation, that is, bifurcation to periodic solutions, we have the Equivariant Hopf Theorem [14], which states that under certain non-degeneracy conditions, branches of periodic solutions with isotropy subgroups having two-dimensional fixed-point subspaces bifurcate from the trivial steady state. These groups are called $\mathbf{C}$-*axial* and are subgroups of Γ extended by an extra group of symmetries called phase-shift symmetries. This extra group can be identified with the circle group $\mathbf{S}^1$.

There are many naturally occurring nonlinear systems with wreath product structure. Golubitsky, Stewart and Dionne [13] list four examples: coupled arrays of Josephson junctions, discretizations of PDEs with gauge symmetry, molecular dynamics and heteroclinic cycles. Bifurcations for specific wreath product groups have also been studied. The Weyl group of type B_N denoted by $W(B_N)$ [1] can be viewed as the wreath product $\mathbf{Z}_2 \wr \mathbf{S}_N$ [11]. Steady-state bifurcation with this symmetry group can be found in [11] and Hopf bifurcation for $N = 3$ in [21]. Moreover, the group $\mathbf{Z}_2 \wr \mathbf{S}_N$ is the holohedry of a lattice in dimension N and when extended by the N-torus $\mathbf{T}^N$, we obtain the group $\mathbf{O}(2) \wr \mathbf{S}_N$. For $N = 3$, $\mathbf{Z}_2 \wr \mathbf{S}_N$ is a subgroup of the Euclidean group $E(3)$ that leaves invariant the space of spatially periodic functions with respect to a simple cubic lattice [7, 8]. Callahan and Knobloch [3] have studied steady-state bifurcation and Dias and Stewart [5] study Hopf bifurcation on this lattice. For $N = 2$, the lattice involved is a planar square lattice. Silber and Knobloch [19] have studied Hopf bifurcation on this lattice.

Our aim is to understand the effect of wreath product symmetries on steady-state and Hopf bifurcations that can occur for coupled cells with wreath product coupling. For wreath product groups $\mathcal{L} \wr \mathcal{G}$ the axial groups are described in [9] and the $\mathbf{C}$-axial groups in [9, 4]. Their structure depends on the axial ($\mathbf{C}$-axial) subgroups of $\mathcal{L}$ ($\mathcal{L} \times \mathbf{S}^1$) and on the possible *blocks* that can be obtained from the permutation group $\mathcal{G}$. Specifically, we are interested in the stability properties of the steady states and periodic solutions obtained by the Equivariant Branching Lemma and Equivariant Hopf Theorem respectively. We show that even though wreath product groups can have a complicated structure, the calculations for the stability of these solutions can be very straightforward and the decomposition of the state space into invariant spaces for the dynamics is clear. In many cases, block diagonalization of the Jacobian calculated at the steady state (or, in the Hopf case, at the zero of the reduced vector field obtained by Liapunov-Schmidt reduction that corresponds to the periodic solution) can be inferred. Moreover, from this decomposition we can conclude in some cases that no Hopf bifurcation can occur as a secondary bifurcation from these branches.

Keeping in mind the association between wreath product systems and coupled cell networks, we also describe in this paper symmetry-breaking for wreath product systems, where we can naturally recognize the equations governing the dynamics of one individual cell and the coupling that produces the full coupled system. Here, we suppose in general that the dynamics of one cell is known, and we wish to use this information to study the global behaviour of the entire system.

We begin in Section 2 by clarifying the natural association of coupling functions and individual cell dynamics with systems of ODEs commuting with wreath product groups. In Section 3 we use the structure of the axial groups obtained in [9] and the **C**-axial groups obtained in [9, 4] to describe the general form of the matrices commuting with these groups. In Section 4 we show that when we apply the Equivariant Branching Lemma and the Equivariant Hopf Theorem, then we can reduce the conditions on the entire system to a set of conditions on the one-cell system and on the coupling function. Moreover, we can apply the results of Section 3 for calculating the stability of the steady states and periodic solutions guaranteed by the Equivariant Branching Lemma and Equivariant Hopf Theorem respectively. Finally, in Section 5 we illustrate our results with some examples.

2 Coupled Cell Systems and Wreath Product Groups

Dynamics with wreath product symmetry can be studied abstractly, but it is often convenient to work with a more concrete interpretation in terms of coupled cells.

Following [9], we say that a system of ODEs $\dot{X} = F(X)$ is *written in coupled cell form* if $F_j(X) = f(X_j) + h_j(X)$, where h_j governs the coupling between cells and f governs the dynamics of one individual cell (we are assuming that the cells are identical and for the moment suppress any dependence on a bifurcation parameter λ). We show above that there is no loss of generality in the association of identical coupled cells systems with systems that are symmetric under wreath product groups.

The action of $\mathcal{L} \wr \mathcal{G}$

Let $\mathcal{L}$ be a subgroup of $\mathbf{O}(k)$ with an action defined on $V = \mathbf{R}^k$ and $\mathcal{G}$ a transitive subgroup of $\mathbf{S}_N$. We consider here the action of the group $\mathcal{L} \wr \mathcal{G}$ on V^N given by

$$(l, \sigma).(X_1, \ldots, X_N) = (l_1 \cdot X_{\sigma^{-1}(1)}, \ldots, l_N \cdot X_{\sigma^{-1}(N)})$$

for $l = (l_1, \ldots, l_N) \in \mathcal{L}^N$, $\sigma \in \mathcal{G}$ and $(X_1, \ldots, X_N) \in V^N$. The permutations act on $l \in \mathcal{L}^N$ by

$$\sigma(l) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(N)})$$

and it follows that the group multiplication in $\mathcal{L} \wr \mathcal{G}$ is given by

$$(h,\tau)(l,\sigma) = (h\tau(l), \tau\sigma).$$

For general information on wreath products see for example [18] page 215.

Let now $F = (F_1, \ldots, F_N)$ be a vector field defined on the vector space V^N with $V = \mathbf{R}^k$, and commuting with $\mathcal{L} \wr \mathcal{G}$. Since F commutes with $\mathcal{G}$ and we are assuming that it acts transitively on $\{1, \ldots, N\}$, there are permutations σ_j for $j = 2, \ldots, N$ such that $\sigma_j^{-1}(1) = j$ and so

$$F_j(X) = F_1(\sigma_j(X)) = F_1(X_j, X_{\sigma_j^{-1}(2)}, \ldots, X_{\sigma_j^{-1}(N)}).$$

There is no loss of generality in assuming that we can distinguish a part of F_1 that depends only on X_1 and that we can write F_1 as

$$F_1(X) = f(X_1) + h_1(X)$$

for some mappings $f : V \to V$ and $h_1 : V^N \to V$. We get

$$F_j(X) = f(X_j) + h_1(\sigma_j(X)),$$

for $j = 2, \ldots, N$. In general, the mapping f is not unique. Thus we have the following lemma:

Lemma 2.1 *Let $F = (F_1, \ldots, F_N)$ be a vector field defined on a vector space V^N with $V = \mathbf{R}^k$. Consider the action of $\Gamma = \mathcal{L} \wr \mathcal{G}$ on V^N (as above), for $\mathcal{L} \subseteq \mathbf{O}(k)$ (acting on V) and $\mathcal{G}$ a transitive subgroup of $\mathbf{S}_N$. Suppose that $F_1(X) = f(X_1) + h_1(X)$ and write $H(X) = (h_1(X), h_1(\sigma_2(X)), \ldots, h_1(\sigma_N(X)))$, where the permutations σ_i for $j = 2, \ldots, N$ are as above. Then F commutes with $\mathcal{L} \wr \mathcal{G}$ if and only if f commutes with $\mathcal{L}$, the vector field H commutes with $\mathcal{G}$ and h_1 is equivariant in X_1 and invariant in $X_2, \ldots, X_N$, under $\mathcal{L}$.*

It is often assumed that the effect of coupling on the jth cell is the sum of the effects of all the cells that are coupled to jth cell, that is, if $H(X) = (h_1(X), \ldots, h_N(X))$, then

$$h_j(X) = \sum_{i=1}^{N} C(i,j) k_{ij}(X_i, X_j),$$

where $C(i,j) = 1$ if cell i is coupled to cell j and zero otherwise. The coupling is *identical*, if $k_{ij} = k$ for all i and j. In this case, it follows that:

Lemma 2.2 *With the conditions of Lemma 2.1, if $h_j(X) = \sum_{i=1}^{N} C(i,j)k(X_i, X_j)$ and $C = (C(i,j))$, then H commutes with $\mathcal{L} \wr \mathcal{G}$ if and only if $\sigma C \sigma^{-1} = C$, for all $\sigma \in \mathcal{G}$, and $k(y_1, y_2)$ is invariant in y_1 and equivariant in y_2 under $\mathcal{L}$.*

More on invariant theory for wreath product groups can be found in [6].

3 Isotropy Restrictions on Commuting Matrices

We now come to the core of this paper: isotropy restrictions on dF imposed by wreath product symmetry. We develop the theory for the steady-state case in Section 3.1; the analogous Hopf case is more complicated and occupies Section 3.2.

First, some terminology. If Γ is a Lie group acting on V and g *commutes* with Γ, that is,

$$g(\gamma \cdot x) = \gamma \cdot g(x)$$

for all $\gamma \in \Gamma$ and $x \in V$, then $(dg)_x$ commutes with $\Sigma = \Sigma_x \subset \Gamma$, where Σ_x is the *isotropy subgroup* of $x \in V$:

$$\Sigma_x = \{\gamma \in \Gamma \ : \ \gamma \cdot x = x\}.$$

The *fixed-point subspace* of Σ_x in V is defined by

$$\mathrm{Fix}(\Sigma_x) = \{x \in V \ : \ \gamma \cdot x = x, \ \forall\, \gamma \in \Sigma_x\}.$$

If we decompose V into isotypic components for the action of Σ, say $V = W_1 \oplus \cdots \oplus W_k$, then $(dg)_x$ is invariant for each W_j. See [20] where the block diagonalization of Γ-symmetric matrices is explored for the theoretical and the numerical treatment of bifurcation problems with symmetry Γ.

We are interested now in the case where $\Gamma = \mathcal{L} \wr \mathcal{G}$ and Σ is a maximal isotropy subgroup of Γ (or $\Gamma \times \mathbf{S}^1$). Recall that steady-state bifurcation to equilibria with symmetry given by axial subgroups is guaranteed by the Equivariant Branching Lemma, and Hopf bifurcation to periodic solutions with symmetry given by $\mathbf{C}$-axial subgroups can be justified by the Equivariant Hopf Theorem. The axial and $\mathbf{C}$-axial groups, up to conjugacy, of groups $\mathcal{L} \wr \mathcal{G}$ are described in [9, 4] and it is clear how the structure of these groups depends both on $\mathcal{L}$ and $\mathcal{G}$.

3.1 Axial Groups

We begin by using the axial subgroups of a general wreath product group $\mathcal{L} \wr \mathcal{G}$ obtained in [9] to get the general form of the commuting matrices for these groups. We show that this form depends both on the groups $\mathcal{L}$ and $\mathcal{G}$, where each one imposes constraints in a systematic way.

As mentioned before, if Γ has an action defined in a (real) vector space V:

Definition 3.1

A subgroup $\Sigma \subseteq \Gamma$ is *axial* (on V) if it is an isotropy subgroup of Γ having a one-dimensional fixed-point subspace (over $\mathbf{R}$).

Recall that a space U is Γ-*absolutely irreducible* if it is irreducible for Γ and the only matrices that commute with Γ on U are the scalar multiples of the identity.

Details on wreath products and their representations can be found in [16]. For absolutely irreducible representations of wreath product groups we have:

Proposition 3.2 *Suppose that $\mathcal{L}\wr\mathcal{G}$ acts on W, such that $\mathcal{L}^N$ acts nontrivially and $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$. Then W is absolutely irreducible for $\mathcal{L}\wr\mathcal{G}$ if and only if $W\cong V^N$ where V is an absolutely irreducible representation of $\mathcal{L}$.*

Proof. See [9] Lemma 3.1 and Lemma 3.2. □

A subset $J \subseteq \{1,\dots,N\}$ is called a *block* if there is a subgroup $\mathcal{H}$ of $\mathcal{G}$ that leaves J invariant and acts transitively on it.

The group $\Sigma(A,J)$

As in [9], for a block $J \subseteq \{1,\dots,N\}$ and an axial subgroup $A \subset \mathcal{L}$ acting on V, let the subgroup $\Sigma(A,J) \subset \mathcal{L}\wr\mathcal{G}$ be defined by

$$\Sigma(A,J) = (B_1 \times \cdots \times B_N)\dot{+}Q_J,$$

where

$$B_j = \begin{cases} A & \text{if } j \in J, \\ \mathcal{L} & \text{if } j \notin J, \end{cases}$$

and

$$Q_J = \{\sigma \in \mathcal{G} : \sigma(J) = J\}.$$

Theorem 3.3 *An isotropy subgroup Σ of $\mathcal{L}\wr\mathcal{G}$ is axial (on V^N) if and only if it is conjugate to an (axial) group of the type $\Sigma(A,J)$, for some block $J \subseteq \{1,\dots,N\}$ and an axial group A of $\mathcal{L}$ (on V).*

Proof. See [9] Lemma 4.1, Lemma 4.2 and Proposition 4.3. □

The main result of this subsection is:

Theorem 3.4 *Assume that $\mathcal{L} \subseteq \mathbf{O}(k)$ is acting nontrivially and absolutely irreducibly on $V = \mathbf{R}^k$. Let $\mathcal{G}$ be a transitive subgroup of $\mathbf{S}_N$. Let $G \in \mathbf{M}_{Nk\times Nk}(\mathbf{R})$. Let $\Sigma(A,J)$ be an axial subgroup of $\mathcal{L}\wr\mathcal{G}$ as defined above for an axial subgroup A of $\mathcal{L}$ and for some block $J \subseteq \{1,\dots,N\}$. Suppose that $J = \{1,\dots,s\}$ and G commutes with $\Sigma(A,J)$. Then there exists a basis of V^N such that*

$$G = \mathrm{Diag}(G_1, G_2),$$

with

$$G_1 = \mathrm{Diag}(C, \underbrace{C_1,\dots,C_1}_{s \text{ times}})$$

and

$$G_2 = \mathrm{Diag}(\lambda_{s+1} Id_{k\times k},\dots,\lambda_N Id_{k\times k}),$$

where $c_{i,j},\dots,\lambda_i$ are real, $C_1 \in \mathbf{M}_{(k-1)\times(k-1)}(\mathbf{R})$, and the matrices

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,s} \\ c_{2,1} & c_{1,1} & \dots & c_{2,s} \\ \dots & \dots & \dots & \dots \\ c_{s,1} & c_{s,2} & \dots & c_{1,1} \end{bmatrix}$$

and G_2 commute with Q_J. More precisely, C commutes with $Q_J \mid_J$ and G_2 with $Q_J \mid_{\{s+1,\ldots,N\}}$.

Remark 3.5 *In Theorem 3.4, one more observation can be made. Since the representation involved is derived from an absolutely irreducible representation of the wreath product and an axial subgroup is defined as maximal subgroup having a one-dimensional fixed-point subspace, it is possible to transform the matrix C such that a 1×1 block is included. To be more precise, denote by $b_1, \ldots, b_k$ the basis of V, and by $b_1^1, \ldots, b_1^s,\ b_2^1, \ldots, b_k^s$ the corresponding basis for V^s considered in Theorem 3.4. If we define now a new basis $v_1 = b_1^1 + \cdots + b_1^s,\ v_2 = \frac{1}{s}v_1 - b_1^2, \ldots, v_s = \frac{1}{s}v_1 - b_1^s$, then $v_1, v_2, \ldots, v_s$ generate two invariant subspaces. This yields a 1×1-block and a $(s-1) \times (s-1)$-block for the matrix C in this new basis. Depending on the representations of the global permutation group, the second block may decompose even further.*

Before proving Theorem 3.4 we state a simple lemma. The result is an easy consequence of Schur's Lemma [17], but for completeness we spell out the proof.

Lemma 3.6 *Suppose that a group H acts on a space $U_1 \oplus U_2$ where each U_j is H-invariant. Suppose that U_1 has no H-irreducible component that is H-isomorphic to an H-irreducible component of U_2. Let P be a linear transformation on $U_1 \oplus U_2$ that commutes with H. Then $P = \mathrm{Diag}(\mathrm{P}_1, \mathrm{P}_2)$ where P_1 is a linear transformation of U_1 and P_2 is a linear transformation of U_2. Moreover, both P_j commute with H.*

Proof Commuting matrices correspond to H-module homomorphisms. Write

$$P = \begin{pmatrix} P_1 & Q_1 \\ Q_2 & P_2 \end{pmatrix}$$

with respect to $U_1 \oplus U_2$. Then Q_1 defines an H-module homomorphism $U_2 \to U_1$, and Q_2 defines one from $U_1 \to U_2$. Since $U_1,\ U_2$ have no common irreducible component, then $Q_1 = 0$ and $Q_2 = 0$ (see [14] Lemma XII 3.4). So

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

as claimed. □

Proof of Theorem 3.4 Since we are assuming $J = \{1, \ldots, s\}$, we have

$$\Sigma = \Sigma(A, J) = (A^s \times \mathcal{L}^{N-s}) \dot{+} Q_J.$$

Since V is an $\mathcal{L}$-irreducible space and $\mathcal{L}$ acts nontrivially on V, we have $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$. Thus if $M \in \mathbf{M}_{k\times k}(\mathbf{R})$ and $l \cdot M = M,\ \forall\, l \in \mathcal{L}$ or $M \cdot l = M,\ \forall\, l \in \mathcal{L}$, then $M = 0_{k\times k}$. To show this suppose $l \cdot M = M,\ \forall\, l \in \mathcal{L}$. Let $v \in V$. Then

$$l \cdot Mv = Mv,\ \forall\, l \in \mathcal{L}.$$

Thus $Mv \in \mathrm{Fix}_V(\mathcal{L})$ and so $Mv = 0$, $\forall\, v \in V$. Therefore $M = 0$. Now suppose $M \cdot l = M$, $\forall\, l \in \mathcal{L}$. Then

$$l^\top \cdot M^\top = M^\top, \ \forall\, l \in \mathcal{L}.$$

So by same argument, $M^\top = 0$. Note that the $\mathcal{L}$-action is orthogonal and so $l^\top = l^{-1}$, $\forall\, l \in \mathcal{L}$.

Since G commutes with $(\mathbf{1}^s, \mathcal{L}^{N-s})$, by Lemma 3.6,

$$G = \mathrm{Diag}(G_1, G_2),$$

where $G_1 \in \mathbf{M}_{sk\times sk}$ and $G_2 \in \mathbf{M}_{(N-s)k\times(N-s)k}$. Note that the group $(\mathbf{1}^s, \mathcal{L}^{N-s})$ is acting on $V^s \times V^{N-s}$, where the action on $V^s \times \{0\}^{N-s}$ is trivial and the action on $\{0\}^s \times V^{N-s}$ is by $\mathcal{L}^{N-s}$. The latter, being a diagonal action of $\mathcal{L}$, has no trivial component. Again, since $\mathcal{L}$ acts absolutely irreducibly on V and $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$, using Lemma 3.6, it follows that

$$G_2 = \mathrm{Diag}(\lambda_{s+1} Id_{k\times k}, \ldots, \lambda_N Id_{k\times k})$$

for some constants $\lambda_{s+1}, \ldots, \lambda_N \in \mathbf{R}$. Note that for each $j > s$, the spaces

$$\begin{array}{c}(0,\ldots,0;0, \quad \ldots,0,\underset{\substack{\uparrow \\ j^{th}\text{ place}}}{V},0,\ldots, \quad 0)\end{array}$$

are $(\mathbf{1}^s, \mathcal{L}^{N-s})$-irreducible and non $(\mathbf{1}^s, \mathcal{L}^{N-s})$-isomorphic.

Since G commutes with $(A^s, \mathbf{1}^{N-s})$, the matrix G_1 commutes with A^s. Let $G_1 = (g_{i,j})$, where $g_{i,j} \in \mathbf{M}_{k\times k}(\mathbf{R})$, for $i, j = 1, \ldots, s$. Then

$$g_{i,i} \cdot a = a \cdot g_{i,i}, \quad \forall\, a \in A, \quad (i = 1, \ldots, s),$$

and

$$g_{i,j} \cdot a = a \cdot g_{i,j} = g_{i,j}, \quad \forall\, a \in A, \quad (i \neq j,\ i, j = 1, \ldots, s).$$

Since G commutes with Q_J and Q_J is transitive on J, then

$$g_{1,1} = \cdots = g_{s,s}.$$

Since A is axial, it has a one-dimensional fixed-point subspace. We can choose a basis $\{b_1, \ldots, b_k\}$ for V such that $\mathrm{Fix}_V(\mathrm{A}) = < b_1 >$. Suppose that $V = W_1 \oplus \ldots \oplus W_p$, where W_i for $i = 1, \ldots, p$ are the isotypic components of V for the action of A, and let $W_1 = \mathrm{Fix}_V(\mathrm{A})$. Since $g_{1,1}$ commutes with A, there exists $c_{1,1} \in \mathbf{R}$ and a matrix $C_1 \in \mathbf{M}_{(k-1)\times(k-1)}(\mathbf{R})$ such that $g_{1,1}$ is written with respect to this basis $\{b_1, \ldots, b_k\}$ as

$$g_{1,1} = \mathrm{Diag}(c_{1,1}, C_1)$$

(this follows from [14] Theorem XII 3.5). Since $g_{i,j}$ for $i \neq j$ also commutes with A, we can use the same argument and write

$$g_{i,j} = \mathrm{Diag}(c_{i,j}, C_{i,j})$$

for constants $c_{i,j} \in \mathbf{R}$ and matrices $C_{i,j} \in \mathbf{M}_{(k-1)\times(k-1)}$. Since $a \cdot g_{i,j} = g_{i,j}$, $\forall\ a \in A$ and $i \neq j$, then $C_{i,j} = 0_{(k-1)\times(k-1)}$ because $\mathrm{Fix}_{W_2\oplus\cdots\oplus W_p}(\mathrm{A}) = \{0\}$ (recall the observation at the beginning of the proof). Thus

$$g_{i,j} = \mathrm{Diag}(c_{i,j}, 0)$$

and so

$$G_1 = \begin{bmatrix} c_{1,1} & & c_{1,2} & & \dots & c_{1,s} & \\ & C_1 & & 0 & \dots & & 0 \\ c_{2,1} & & c_{1,1} & & \dots & c_{2,s} & \\ & 0 & & C_1 & \dots & & 0 \\ \dots & & \dots & & \dots & \dots & \\ c_{s,1} & & c_{s,2} & & \dots & c_{1,1} & \\ & 0 & & 0 & \dots & & C_1 \end{bmatrix}$$

where the blank entries are zero matrices. Now if we change the order of the basis the result follows. $\square$

Remark 3.7

(a) Since C commutes with Q_J and Q_J acts transitively on the block $\{1, \dots, s\}$, this symmetry will strongly restrict the form of C. In particular, if C is symmetric, then it is diagonalizable and has only real eigenvalues. However, this is not always the case. For example, if we take $Q_J = \mathbf{Z}_4$, then the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

commutes with $\mathbf{Z}_4$ (since $\mathbf{Z}_4$ is an abelian group) and it has non real eigenvalues. Of course it is diagonalizable over $\mathbf{C}$.

(b) From Theorem 3.4, it follows that the problem of finding the eigenvalues of a matrix G that commutes with $\Sigma(A, J)$ is reduced to the problem of finding the eigenvalues of the two matrices $C \in \mathbf{M}_{s\times s}(\mathbf{R})$ and $C_1 \in \mathbf{M}_{(k-1)(k-1)}(\mathbf{R})$. Moreover, each eigenvalue of C_1 has multiplicity at least s.

Corollary 3.8 *With the conditions of Theorem 3.4, if Q_J acts transitively on $\{s+1, \dots, N\}$, then $\lambda_{s+1} = \cdots = \lambda_N$ and so $G_2 = \lambda_{s+1} Id_{(N-s)k\times(N-s)k}$.*

Proof. This follows since G_2 commutes with $Q_J\mid_{\{s+1,\dots,N\}}$. $\square$

Remark 3.9

Since V is $\mathcal{L}$-absolutely irreducible and if Q_J acts transitively on $\{s+1, \dots, N\}$, the space V^{N-s} is $\mathcal{L} \wr Q_J$-absolutely irreducible. Thus any linear mapping commuting with $\mathcal{L} \wr Q_J$ (considering the action of this group on V^{N-s})

must be a scalar multiple of $Id_{V^{N-s}}$. Applying this to the matrix G_2 in the previous corollary, we obtain the same result.

In general, if $\mathcal{G} \subseteq \mathbf{S}_N$ and $\{1, \ldots, N\} = J \cup J^{'}$, where J is a block, then Q_J acts transitively on $J^{'}$ if and only if $J^{'}$ is a block: since Q_J leaves J invariant, then it leaves $J^{'}$ invariant. So if Q_J acts transitively on $J^{'}$, then $J^{'}$ is a block. On the other hand, if J and $J^{'}$ are both blocks, as Q_J leaves $J^{'}$ invariant then $Q_J \subseteq Q_{J'}$, and as $Q_{J'}$ leaves J invariant, then $Q_{J'} \subseteq Q_J$. Thus $Q_J = Q_{J'}$ and Q_J acts transitively on $J^{'}$.

Examples (a) Let $\mathcal{G} = \mathbf{Z}_3 \subset \mathbf{S}_3$. Then $J = \{1\}$ is a block, where $Q_J = \{1\}$ and Q_J does not act transitively on $\{2, 3\}$.

(b) Let $\mathcal{G} = \mathbf{Z}_4 \subset \mathbf{S}_4$. Then $\mathcal{G}$ is generated by the permutation (1234) and $J = \{1, 3\}$ is a block. In this case the group $Q_J = \{1, \sigma^2\}$ acts transitively on $\{2, 4\}$ and $\{2, 4\}$ is also a block.

Corollary 3.10 *With the conditions of Theorem 3.4, if* $\mathcal{G} = \mathbf{S}_N$, *then* $G_2 = \lambda_{s+1} Id_{(N-s)k \times (N-s)k}$ *and* $c_{i,j} = c_{j,i}$ *for all* $i, j = 1, \ldots, s$, *i.e.,*

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,2} \\ c_{1,2} & c_{1,1} & \cdots & c_{1,2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1,2} & c_{1,2} & \cdots & c_{1,1} \end{bmatrix}.$$

Proof. For all $1 \leq s \leq N$, the set $J = \{1, \ldots, s\}$ is a block and $Q_J = \mathbf{S}_s \times \mathbf{S}_{N-s}$. For G_2, as Q_J is transitive on $\{s+1, \ldots, N\}$, Corollary 3.8 holds. Since C commutes with $Q_J \mid_J = \mathbf{S}_s$ and this group is generated by $\{(1k),\ k = 2, \ldots, s\}$, the result follows. □

3.2 C-axial Groups

The Hopf case is complicated by the richer structure available for the **C**-axial groups, but the basic approach is similar. Again we begin with some technical background.

We say that a space U is Γ-*simple* if either $U \cong W \oplus W$ where W is absolutely irreducicle for Γ, or U is non-absolutely irreducible for Γ. Let V be a $\mathcal{L}$-simple space where $\mathcal{L}$ acts nontrivially and $\mathcal{G}$ is a transitive group of $\mathbf{S}_N$. By [9] (recall Proposition 3.2), the space V^N is $\mathcal{L} \wr \mathcal{G}$-simple. Consider the natural action of $\mathbf{S}^1$ on V^N obtained by giving a complex structure to this space as in [14].

Definition 3.11

Let Γ be a Lie group with an action defined on a space V. A subgroup $\Sigma \subseteq \Gamma \times \mathbf{S}^1$ is **C**-*axial* (on V) if it is an isotropy subgroup having a two-dimensional fixed-point subspace (over **R**).

The group $\Sigma(B^{\psi}, J)$

Consider a block J and let B^{ψ} be a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$ (acting on V) where $\psi : B \to \mathbf{S}^1$ is an homomorphism and

$$B^{\psi} = \{(b, \psi(b)) : b \in B\}.$$

Following [14] we call the group B^{ψ} a *twisted* subgroup of $\mathcal{L} \times \mathbf{S}^1$. The image $\psi(B)$ is a closed subgroup of $\mathbf{S}^1$. The closed subgroups of $\mathbf{S}^1$ are $\mathbf{1}$, $\mathbf{Z}_n$ ($n = 2, 3, 4, \ldots$) and $\mathbf{S}^1$. We say that B^{ψ} is of *finite* twist type if the image $\psi(B)$ is not $\mathbf{S}^1$. Recall that proper isotropy subgroups of $\Gamma \times \mathbf{S}^1$ acting on a Γ-simple space are twisted subgroups [14].

Consider

$$\Sigma(B^{\psi}, J) = (\mathbf{1}^N, Q_J, 0) + ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) + ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi)$$

where $+$ indicates 'group generated by' as in [9]. Here it is assumed that $J = \{1, \ldots, s\}$ and the subgroup $\hat{B}$ is defined by

$$\hat{B} = \{(b_1, \ldots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Dionne, Golubitsky and Stewart [9] prove that $\Sigma(B^{\psi}, J)$ is a $\mathbf{C}$-axial subgroup of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$.

As in Theorem 3.4, we can use the structure of these groups to describe the general form of the commuting matrices for these groups.

Theorem 3.12 *Assume that $V = \mathbf{R}^k$ is a $\mathcal{L}$-simple space where $\mathcal{L} \subseteq \mathbf{O}(k)$ acts nontrivially on V. Let $\mathcal{G}$ be a transitive subgroup of $\mathbf{S}_N$. Let $G \in \mathbf{M}_{Nk \times Nk}(\mathbf{R})$ and $\Sigma(B^{\psi}, J)$ be a $\mathbf{C}$-axial subgroup of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ on V^N, where $B^{\psi} \subset \mathcal{L} \times \mathbf{S}^1$ is $\mathbf{C}$-axial (on V) and J is a block of $\{1, \ldots, N\}$. Suppose that $J = \{1, \ldots, s\}$ and G commutes with $\Sigma(B^{\psi}, J)$. Then there exists a basis of V^N for which*

$$G = \mathrm{Diag}(G_1, G_2),$$

with

$$G_1 = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,s} \\ g_{2,1} & g_{1,1} & \cdots & g_{2,s} \\ \cdots & \cdots & \cdots & \cdots \\ g_{s,1} & g_{s,2} & \cdots & g_{1,1} \end{bmatrix},$$

and where $g_{i,j} \in \mathbf{M}_{k \times k}(\mathbf{R})$ commutes with B^{ψ}. The matrix

$$G_2 = \mathrm{Diag}(g_{s+1}, \ldots, g_N),$$

is formed of matrices $g_{s+1}, \ldots, g_N \in \mathbf{M}_{k \times k}(\mathbf{R})$ commuting with $\mathcal{L} \times \psi(B)$. Moreover, as $\mathrm{Fix}_V(\mathrm{B}^{\psi})$ is two-dimensional, for some basis B of V^N the matrix

$$g_{1,1} = \mathrm{Diag}(c_{1,1}, C_1)$$

where $c_{1,1} \in \mathbf{M}_{2 \times 2}(\mathbf{R})$ and $C_1 \in \mathbf{M}_{(k-2) \times (k-2)}(\mathbf{R})$, and for $i \neq j$ we have

$$g_{i,j} = \mathrm{Diag}(c_{i,j}, C_{i,j})$$

for $c_{i,j} \in \mathbf{M}_{2 \times 2}(\mathbf{R})$ and $C_{i,j} \in \mathbf{M}_{(k-2) \times (k-2)}(\mathbf{R})$. Also the matrices G_1 and G_2 commute with Q_J.

Proof. The proof follows as in the proof of Theorem 3.4 where now $\mathrm{Fix}_V(\mathrm{B}^\psi)$ is two-dimensional and V is $\mathcal{L}$-simple. □

Remark 3.13

(a) We have stated Theorem 3.12 for a special but common class of **C**-axial subgroups, the groups $\Sigma(B^\psi, J)$. Similar but more complicated results can be derived for other **C**-axial subgroups, as classified by Dias [4]. These subgroups are denoted by $\Sigma(B^\psi, J, \sigma, J_1, p)$. Dias [4] proves that these subgroups are **C**-axial and that any **C**-axial subgroup is conjugate to one of these.

We describe these groups, and briefly indicate how Theorem 3.12 should be modified to apply to them.

The group $\Sigma(B^\psi, J, \sigma, J_1, p)$

Again consider a block $J \subseteq \{1, \ldots, N\}$ and let Q_J be the subgroup of $\mathcal{G}$ that leaves J invariant. Suppose

$$J = \{1, \ldots, s\}.$$

Let J_1 be a subset of J such that for some permutation $\sigma \in Q_J$

$$J = J_1 \;\dot\cup\; \sigma(J_1) \;\dot\cup\; \cdots \;\dot\cup\; \sigma^{s'-1}(J_1),$$

where $\dot\cup$ is disjoint union and

$$\sigma^{s'}(J_1) = J_1.$$

In particular it follows that $|J| = s'|J_1|$.

Choose indices such that

$$J_{i+1} = \sigma^i(J_1), \quad i = 1, \ldots, s' - 1$$

and let

$$Q_{J,J_k} = \{\tau \in Q_J : \tau(J_j) = \sigma^{k-1}(J_j),\ j = 1, \ldots, s'\},$$

for $k = 1, 2, \ldots, s'$. That is, each permutation in Q_{J,J_k} interchanges the subsets J_i of J in the same way as σ^{k-1} (where σ^0 is the identity in $\mathcal{G}$).

Suppose Q_{J,J_1} acts transitively on J_1. This implies that Q_{J,J_1} acts transitively on all J_i. Note that by definition of block the group Q_J acts transitively on J. Therefore $\sigma = 1$ and $J_1 = J$ are under those conditions.

$$Q_{J,J_k} = \{\tau \in Q_J : \tau(J_j) = \sigma^{k-1}(J_j),\ j = 1, \ldots, s'\},$$

for $k = 2, \ldots, s'$. That is, each permutation in Q_{J,J_k} interchanges the subsets J_i of J in the same way as σ^{k-1}.

Let B^ψ be a **C**-axial subgroup of $\mathcal{L}\times\mathbf{S}^1$ of finite twist type $\mathbf{Z}_r$, and let $\hat{B}$ be the subgroup of B^s defined by

$$\hat{B} = \{(b_1, \cdots, b_s) \in B^s : \psi(b_1) = \cdots = \psi(b_s)\}.$$

Let $\mathbf{Z}_p =< \xi_p >$ be a cyclic subgroup of $\mathbf{S}^1$ such that

$$s^{'} = \min_{i>0} \{\xi_p^i \in \mathbf{Z}_r\}.$$

Call $\xi_{r'} = \xi_p^{s^{'}}$. It follows that $\mathbf{Z}_p = \mathbf{Z}_{s' r'}$ where $\mathbf{Z}_{r'} \subseteq \mathbf{Z}_r$.

Define B_k the subgroup of B^s by

$$B_k = \left\{ \quad (b_1, \dots, b_s) \in B^s : \psi(b_j) = \quad \begin{cases} \xi_{r'}, & \text{if } j \in J_1 \cup \cdots \cup J_{k-1}, \\ 0, & \text{if } j \in J_k \cup \cdots \cup J_{s'} \end{cases} \quad \right\}$$

Finally denote by $\Sigma(B^\psi, J, \sigma, J_1, p)$ the subgroup of $\Gamma \times \mathbf{S}^1$ generated by the following groups:

$$\Sigma(B^\psi, J, \sigma, J_1, p) = ((\mathbf{1}^s, \mathcal{L}^{N-s}), \mathbf{1}, 0) \ + \ ((\hat{B}, \mathbf{1}^{N-s}), \mathbf{1}, \psi) \ +$$
$$+ \ (\mathbf{1}^N, Q_{J,J_1}, 0) \ + \bigcup_{k=2,\dots,s'} ((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1}).$$

Note that this group depends on the block J, the permutation σ (and so on J_1). Also the group Q_{J,J_1} has to act transitively on J_1. Finally, it depends on B^ψ (a $\mathbf{C}$-axial subgroup of $\mathcal{L} \times \mathbf{S}^1$) and on the cyclic subgroup $\mathbf{Z}_p$ of $\mathbf{S}^1$ (where some divisor $r^{'}$ of r divides p).

In [4] it is proved that an isotropy subgroup Σ of $\Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-axial if and only if it is conjugate to a ($\mathbf{C}$-axial) group of the type $\Sigma(B^\psi, J, \sigma, J_1, p)$, for some $\mathbf{C}$-axial group B^ψ of $\mathcal{L} \times \mathbf{S}^1$, a block J, a permutation σ of $\mathcal{G}$, a subset J_1 of J, and a nonnegative integer p. In the above notation, we have $\Sigma(B^\psi, J) = \Sigma(B^\psi, J, 1, J, r)$ if $\psi(B) = \mathbf{Z}_r$. Moreover, it is proved that a $\mathbf{C}$-axial group $\Sigma = \Sigma_w$ of $(\mathcal{L} \wr \mathcal{G}) \times \mathbf{S}^1$ is conjugate to a group of the type $\Sigma(B^\psi, J)$ (as defined before) for some block J (where w_1 is assumed nonzero and $B^\psi = \Sigma_{w_1}$) if and only if Σ and Σ_{w_1} are of the same twist type. This case happens for example when Σ_{w_1} is of twist type $\mathbf{S}^1$. Also, a $\mathbf{C}$-axial group Σ_w of $\Gamma \times \mathbf{S}^1$ is of type $\mathbf{S}^1$ if and only if Σ_{w_1} of $\mathcal{L} \times \mathbf{S}^1$ is of type $\mathbf{S}^1$.

In order to take into account all possible $\mathbf{C}$-axial subgroups, we must modify Theorem 3.12. Briefly, the resulting changes are as follows. With the conditions of Theorem 3.12, where now Σ is of type $\Sigma(B^\psi, J, \sigma, J_1, p)$, the matrix G is also $G = \mathrm{Diag}(G_1, G_2)$. However, the matrices $g_{i,i}$ are not necessarily equal. This is because of the more complicated structure of Σ: the matrices G_1 and G_2 commute only with Q_{J,J_1} (and not Q_J); in addition, G commutes with

$$\bigcup_{k=2,\dots,s'} ((B_k, \mathbf{1}^{N-s}), Q_{J,J_k}, \xi_p^{k-1})$$

The rest holds.

(b) If $\psi(B) = \mathbf{S}^1$, then with the conditions of the previous theorem, the matrices $g_{s+1}, \dots, g_N$ commute with $\mathcal{L} \times \mathbf{S}^1$ and so are of 'complex type' by [14] Lemma XVI 3.4.

4 Equivariant Branching Lemma and Equivariant Hopf Theorem in Wreath Product Systems

As motivation for the work done in this section, we start with a well known example in symmetric dynamical systems theory: the system abstracted by Guckenheimer and Holmes [15] from a model by Busse and Heikes [2] on rotating convection. Guckenheimer and Holmes observed that the model in [2] can be abstracted using a symmetry group which, in our terminology, is $\mathbf{Z}_2 \wr \mathbf{Z}_3$. Moreover, Golubitsky *et al.* [13] observed that, with an appropriate choice of coupling, this example can be viewed as a system of three identical cells with one internal state variable and one nontrivial symmetry $\mathbf{Z}_2$, coupled in a directed ring. Guckenheimer and Holmes [15] prove that for an open set of cubic order coefficients in a vector field commuting with $\mathbf{Z}_2 \wr \mathbf{Z}_3$ (acting on $\mathbf{R}^3$), there is an asymptotically stable (structurally stable) heteroclinic cycle connecting three equilibria representing states where one cell is active and the other two are quiescent. Here, we recall Section 3.1, where from the classification of the axial groups of wreath product groups, we expect generic steady-state bifurcation to equilibria where the state variables have one nonzero component and the others are zero. In fact, the phenomena of heteroclinic cycles is common in systems with wreath product symmetries.

We return to the example. Consider $\mathbf{Z}_2$ acting on $\mathbf{R}$ as multiplication by ∓ 1, and $\mathbf{Z}_3$ acting on $\mathbf{R}^3$, with the action generated by the permutation (123). To third order the differential equations with $\mathbf{Z}_2 \wr \mathbf{Z}_3$ symmetry are:

$$\begin{array}{l} \dot{x}_1 = (\lambda + \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2)x_1 \\ \dot{x}_2 = (\lambda + \alpha x_2^2 + \beta x_3^2 + \gamma x_1^2)x_2 \\ \dot{x}_3 = (\lambda + \alpha x_3^2 + \beta x_1^2 + \gamma x_2^2)x_3. \end{array}$$

As pointed by Dionne *et al.* [9], in order to write the system in a form that has identical coupling between identical cells, as defined in Section 2, we set $\gamma = 0$, and so we can write the system as $\dot{X} = F(X, \lambda)$, where $X = (x_1, x_2, x_3)$ and $F_j(X, \lambda) = f(x_j, \lambda) + h_j(X)$, for

$$f(x_j, \lambda) = (\lambda + \alpha x_j^2)x_j$$

and

$$h_1(x_1, x_2, x_3) = \beta x_2^2 x_1.$$

Using the results of [15], heteroclinic cycles exist when $\lambda > 0$ and $\beta < \alpha << 0$. Note that the internal cell dynamics, with α negative, are governed by a pitchfork bifurcation (related with the internal $\mathbf{Z}_2$ symmetry). Thus, as λ crosses zero (from negative to positive), a bifurcation from the trivial equilibrium to nontrivial equilibria, say x_0, occurs and these bifurcating equilibria are stable for the internal cell dynamics. With the results of Guckenheimer and Holmes [15], we conclude that when the strength of the coupling is large and negative, an asymptotically stable heteroclinic cycle connecting the equilibria $(x_0, 0, 0)$, $(0, x_0, 0)$, $(0, 0, x_0)$ exists. The internal symmetry forces, for example, the plane x_1x_2 to be invariant for the dynamics, and it is proved that there is a saddle-sink connection in this plane between the equilibria of type $(x_0, 0, 0)$ and $(0, x_0, 0)$. The global symmetry $\mathbf{Z}_3$ guarantees the required connections on the other planes.

We now set up a generalisation of the Guckenheimer and Holmes example and apply the methods of Section 3 to it. Let $\dot{X} = F(X, \lambda)$ be a system of ODEs, where $X \in V^N$, the bifurcation parameter is $\lambda \in \mathbf{R}$ and F commutes with $\Gamma = \mathcal{L} \wr \mathcal{G}$. Suppose that $F_j(X, \lambda) = f(X_j, \lambda) + h_j(X, \lambda)$, as in Section 2. In this section we explore only one aspect of the above example. We are interested in relating the conditions of the Equivariant Branching Lemma and the Equivariant Hopf Theorem between the coupled cell system $\dot{X} = F(X, \lambda)$ and the one-cell system $\dot{x} = f(x, \lambda)$. Proposition 4.1 of this section describes steady-state bifurcation and Proposition 4.3 describes Hopf bifurcation.

In both of these results we impose one condition on h_1. The condition is $\left(\frac{\partial h_1}{\partial X_1}\right)_{0,\lambda} \equiv 0$, which is natural for wreath product couplings. In particular, this condition is satisfied by the Guckenheimer and Holmes system. Note that since the coupling h_1 is invariant in x_2 by $\mathbf{Z}_2$, it is an even polynomial in x_2. As stated in [13, 9], a natural example of wreath product coupling is the case when h_1 can be written as the sum of terms like $|X_i|^2 X_1$, for $i \neq 1$ (and depending on the global symmetry $\mathcal{G}$). This absence of linear terms contrasts with the direct product case, where in the natural example h_1 is given by linear terms like $X_i - X_1$.

The absence of linear terms in wreath product systems is not only natural; with the mild technical hypothesis Fix $(\Gamma) = \{0\}$, it is automatic. To be specific, the actions we are considering are irreducible under Γ (or Γ-simple) and nontrivial. Thus for these cases we have Fix $(\Gamma) = \{0\}$. From [14] Proposition XIII 2.2, the only Γ-invariant linear function is the zero function. As explained in Section 2, the coupling function h_1 is Γ-invariant in the variables $X_2, \ldots, X_N$. Moreover, we can assume that h_1 has no linear terms in X_1 (we suppose that they are included in f). It follows that there is no loss of generality in assuming that h_1 has no linear terms.

We present our methods here in a special, simple case, but we can apply the results of Section 3 to any system that has wreath product symmetries. That is, the results of Section 3 can be used to simplify calculations of the stability of the steady states and periodic solutions obtained by the Equivariant Branching Lemma and Equivariant Hopf Theorem.

For the Equivariant Branching Lemma we have:

Proposition 4.1 *Suppose that $\mathcal{L} \subseteq \mathbf{O}(k)$ acts nontrivially and absolutely irreducibly on $V \equiv \mathbf{R}^k$, and that $\mathcal{G} \subseteq \mathbf{S}_N$ acts transitively on V^N. Consider*

$$\dot{X} = F(X, \lambda),$$

with $X = (X_1, \ldots, X_N) \in V^N$ and $\lambda \in \mathbf{R}$, where:
1) F commutes with the action of $\mathcal{L} \wr \mathcal{G}$ on V^N.

2) $F_j(X,\lambda) = f(X_j,\lambda) + h_j(X,\lambda)$ with $f : V \times \mathbf{R} \to V$ a $\mathcal{L}$-equivariant mapping on V and $h_j : V^N \times \mathbf{R} \to V$ satisfying $\left(\frac{\partial h_1}{\partial X_1}\right)_{0,\lambda} = 0$.
Moreover, suppose that $(x,\lambda) = (0,0)$ is a steady-state bifurcation point of $\dot{x} = f(x,\lambda)$ (i.e. $f(0,0) = 0$ and $(df)_{0,0} = 0$). If J is a block of $\{1,\ldots,N\}$, $A \subset \mathcal{L}$ is an axial group on V and, for $(df)_{0,\lambda} = C(\lambda)Id_V$, $C'(0) \neq 0$, then we have the conditions of the Equivariant Branching Lemma satisfied for F, i.e., there is branch of equilibria of $\dot{X} = F(X,\lambda)$ bifurcating from $(X,\lambda) = (0,0)$ with symmetry $\Sigma(A,J)$.

Proof. Since $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$, and as $f(\mathrm{Fix}_V(\mathcal{L})) \subseteq \mathrm{Fix}_V(\mathcal{L})$, then $f(0,\lambda) \equiv 0,\ \forall\ \lambda \in \mathbf{R}$. Similarly, since $\mathrm{Fix}_{V^N}(\mathcal{L} \wr \mathcal{G}) = \{0\}$, we have $F(0,\lambda) \equiv 0,\ \forall\ \lambda \in \mathbf{R}$. Because $(df)_{0,0} = 0$ and $(df)_{0,\lambda} = C(\lambda)Id_V$, then $C(0) = 0$. Thus $C(0) = 0$ and $C'(0) \neq 0$.

The space V^N is a $\mathcal{L} \wr \mathcal{G}$-absolutely irreducible (by Proposition 3.2) and $(dF)_{0,\lambda}$ commutes with this group. Thus $(dF)_{0,\lambda}$ is scalar multiple of Id_{V^N}. As

$$\left(\frac{\partial F_1}{\partial X_1}\right)_{0,\lambda} = (df)_{0,\lambda} + \left(\frac{\partial h_1}{\partial X_1}\right)_{0,\lambda} = C(\lambda)\ Id_V,$$

it follows that

$$(dF)_{0,\lambda} = C(\lambda)\ Id_{V^N}.$$

From [9], the group $\Sigma(A,J)$ is axial (recall Theorem 3.3). Thus, there is a unique branch of solutions $\dot{X} = F(X,\lambda)$ with symmetry $\Sigma(A,J)$ guaranteed by the Equivariant Branching Lemma. □

Remark 4.2

With the conditions of Proposition 4.1, where $V = \mathbf{R}^k$, if $\Sigma(A,J) = \Sigma_{X_0}$, then the derivative $(dF)_{X_0}$ viewed as a matrix in $(\mathbf{R}^k)^N$ is of the form presented in Theorem 3.4 for some basis of V^N.

For the Equivariant Hopf Theorem we have:

Proposition 4.3 *Suppose that $\mathcal{L} \subseteq \mathbf{O}(k)$ acts nontrivially and $\mathcal{L}$-simply on $V \equiv \mathbf{R}^k$, and that $\mathcal{G} \subseteq \mathbf{S}_N$ acts transitively on V^N. Consider*

$$\dot{X} = F(X,\lambda),$$

with $X = (X_1,\ldots,X_N) \in V^N$ and $\lambda \in \mathbf{R}$, where:
1) F commutes with the action of $\mathcal{L} \wr \mathcal{G}$ on V^N.
2) $F_j(X,\lambda) = f(X_j,\lambda) + h_j(X,\lambda)$ with $f : V \times \mathbf{R} \to V$ a $\mathcal{L}$-equivariant mapping on V and $h_j : V^N \times \mathbf{R} \to V$ satisfying $\left(\frac{\partial h_1}{\partial X_1}\right)_{0,\lambda} = 0$.
Moreover, suppose that $(df)_{0,0} = D$, where

$$D = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix},$$

$m = \dim V/2$ and I_m is the $m \times m$ identity matrix. Denote the eigenvalues of $(df)_{0,\lambda}$ by $\sigma(\lambda) \mp i\rho(\lambda)$ (and so $\sigma(0) = 0$ and $\rho(0) = 1$). If J is a block of $\{1, \ldots, N\}$, $B^{\psi} \subseteq \mathcal{L}$ is a $\mathbf{C}$-axial group of $\mathcal{L} \times \mathbf{S}^1$ on V and $\Sigma = \Sigma(B^{\psi}, J, \sigma, J_1, p)$ is a $\mathbf{C}$-axial group of $\Gamma \times \mathbf{S}^1$ on V^N, and $\sigma'(0) \neq 0$, then we have that the conditions of the Equivariant Hopf Theorem are satisfied for F and thus there exists a unique branch of small-amplitude periodic solutions of $\dot{X} = F(X, \lambda)$ with period near 2π, having Σ as their group of symmetries.

Proof. Note that since V is $\mathcal{L}$-simple, then V^N is $\mathcal{L} \wr \mathcal{G}$-simple (recall Proposition 3.2). Also $\mathrm{Fix}_V(\mathcal{L}) = \{0\}$ and $\mathrm{Fix}_{V^N}(\mathcal{L} \wr \mathcal{G}) = \{0\}$. As h_1 is invariant under $\mathcal{L}$ in $X_2, \ldots, X_N$, then $\left(\frac{\partial F_1}{\partial X_i}\right)_{0,\lambda} \equiv 0$ for $i \geq 2$: from $h_1(l \cdot X_1, X_2, \ldots, X_N, \lambda) = l \cdot h_1(X, \lambda)$, $\forall\, l \in \mathcal{L}$, it follows that

$$\left(\frac{\partial h_1}{\partial X_j}\right)_{0,\lambda} = l \cdot \left(\frac{\partial h_1}{\partial X_j}\right)_{0,\lambda}, \ \forall\, l \in \mathcal{L} \text{ and } j \geq 2,$$

and so $\left(\frac{\partial h_1}{\partial X_j}\right)_{0,\lambda} \equiv 0$. In general, $\left(\frac{\partial F_j}{\partial X_i}\right)_{0,\lambda} \equiv 0$ if $i \neq j$. Since $\left(\frac{\partial h_1}{\partial X_1}\right)_{0,\lambda} \equiv 0$ by hypothesis and so $\left(\frac{\partial h_j}{\partial X_j}\right)_{0,\lambda} \equiv 0$, then

$$\left(\frac{\partial F_j}{\partial X_j}\right)_{0,\lambda} = (df)_{0,\lambda}$$

for $j = 1, \ldots, N$. Thus

$$(dF)_{0,0} = \mathrm{Diag}(D, \ldots, D),$$

(and the eigenvalues of this matrix are $\mp i$) and by [14] Lemma XVI 1.5 the eigenvalues of $(\partial F)_{0,\lambda}$ are complex conjugate $\sigma_a st(\lambda) \mp i\rho_a st(\lambda)$, where $\sigma_a st(0) = 0$, $\rho_a st(0) = 1$ and $\sigma_a' st(0) = \sigma'(0) \neq 0$. $\square$

Remark 4.4

(a) Under the conditions of Proposition 4.3, Liapunov-Schmidt reduction gives a reduced map commuting with $\Gamma \times \mathbf{S}^1$ on V^N, whose zeros are in one-to-one correspondence with periodic solutions of $\dot{X} = F(X, \lambda)$ of period near 2π. Moreover, if we assume that F also commutes with $\mathbf{S}^1$, then by [14] Theorem XVI 10.1 the reduced map has an explicit form, and the asymptotic stability of a bifurcating solution is given by the linearized stability of the corresponding zero of that reduced map ([14] Corollary XVI 10.2). From Proposition 4.3, if $\Sigma = \Sigma(B^{\psi}, J)$ and F commutes also with $\mathbf{S}^1$, it follows that we can use Theorem 3.12 to deduce the form of the derivative of the reduced vector field at the zeros corresponding to the periodic solutions with symmetry Σ.

(b) With the conditions of Proposition 4.3, and supposing that the vector fields f and h_1 commute with $\mathcal{L} \times \mathbf{S}^1$ (to be more precise h_1 is $\mathcal{L} \times \mathbf{S}^1$ invariant in $X_2, \ldots, X_N$ and $\mathcal{L} \times \mathbf{S}^1$-equivariant in X_1), so that they are in Birkhoff normal form, then the vector field F is also in Birkhoff normal form.

5 Examples

Our aim now is to illustrate the ideas of Section 3 for four examples: $\mathbf{Z}_2 \wr \mathbf{S}_3$, $\mathbf{Z}_2 \wr \mathbf{Z}_3$, $\mathbf{D}_3 \wr \mathbf{S}_3$ and $\mathbf{O}(2) \wr \mathbf{S}_2$. We use Theorem 3.4 to find the most general matrices commuting with the corresponding axial groups of $\mathbf{Z}_2 \wr \mathbf{S}_3$, $\mathbf{Z}_2 \wr \mathbf{Z}_3$ and $\mathbf{D}_3 \wr \mathbf{S}_3$, and Theorem 3.12 and Remark 3.13 to find the most general matrices commuting with the $\mathbf{C}$-axial groups of $\mathbf{O}(2) \wr \mathbf{S}_2$.

Example 1: $\mathbf{Z}_2 \wr \mathbf{S}_3$.

Consider the group $\mathbf{Z}_2$ acting on $V = \mathbf{R}$ as multiplication by ∓ 1 and the group $\Gamma = \mathbf{Z}_2 \wr \mathbf{S}_3$ acting on V^3 as defined in Section 2. As noticed in [11], this group is called the *hyperoctahedral group* and it is the full symmetry group of the cube (it is the Weyl group of type B_3). Note that V^3 is absolutely irreducible for Γ (and V is $\mathbf{Z}_2$-absolutely irreducible). The axial subgroups of Γ are well known. Using the notation of Section 3, these are presented in table 1. Note that for $\mathcal{G} = \mathbf{S}_3$, the possible blocks we need to consider are $\{1\}$, $\{1,2\}$ and $\{1,2,3\}$, and for $\mathcal{L} = \mathbf{Z}_2$, the trivial group denoted by $\mathbf{1}$ is the only axial group of $\mathcal{L}$ (acting on V). The most general matrices commuting with the axial groups of Γ and the corresponding eigenvalues and eigenvectors are obtained in table 2. Here a, b, c denote real entries and the matrices are represented in the canonical basis of $\mathbf{R}^3$; namely, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Thus if $\dot{X} = F(X,\lambda)$ is a bifurcation problem with symmetry $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $X \in \mathbf{R}^3$ (conditions of Proposition 4.1), then the derivative $(dF)_{X_0}$ for X_0 a zero of $F(X,\lambda)$ with symmetry each one of the Σ_i, is given by the corresponding commuting matrix (in the canonical basis) listed in table 2. For Σ_1 we have $a = \left(\frac{\partial F_1}{\partial X_1}\right)_{X_0}$ and $b = \left(\frac{\partial F_2}{\partial X_2}\right)_{X_0}$. For Σ_2, $a = \left(\frac{\partial F_1}{\partial X_1}\right)_{X_0}$, $b = \left(\frac{\partial F_1}{\partial X_2}\right)_{X_0}$ and $c = \left(\frac{\partial F_3}{\partial X_3}\right)_{X_0}$. Finally for Σ_3, $a = \left(\frac{\partial F_1}{\partial X_1}\right)_{X_0}$ and $b = \left(\frac{\partial F_1}{\partial X_2}\right)_{X_0}$.

Orbit representative	Isotropy subgroup
$(x,0,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_1 = \Sigma(\mathbf{1},\{1\}) = \mathbf{1} \times (\mathbf{Z}_2)^2 \dot{+} \mathbf{S}_1 \times \mathbf{S}_2$
$(x,x,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_2 = \Sigma(\mathbf{1},\{1,2\}) = \mathbf{1}^2 \times \mathbf{Z}_2 \dot{+} \mathbf{S}_2 \times \mathbf{S}_1$
(x,x,x) $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_3 = \Sigma(\mathbf{1},\{1,2,3\}) = \mathbf{1}^3 \dot{+} \mathbf{S}_3$

Table 1: Orbit representatives and axial subgroups of $\mathbf{Z}_2 \wr \mathbf{S}_3$.

Example 2: $\mathbf{Z}_2 \wr \mathbf{Z}_3$.

Consider now the group $\Gamma = \mathbf{Z}_2 \wr \mathbf{Z}_3$, where again $\mathbf{Z}_2$ acts on $V = \mathbf{R}$ as multiplication by ∓ 1. This group Γ was studied by Guckenheimer and Holmes [15], where they show that it is possible for structurally stable, asymptotically

Axial group	Commuting matrix	Eigenvalues	Eigenvectors
Σ_1	$\begin{pmatrix} a & & \\ & b & \\ & & b \end{pmatrix}$	a b (two times)	e_1 e_2, e_3
Σ_2	$\begin{pmatrix} a & b & \\ b & a & \\ & & c \end{pmatrix}$	$a+b$ $a-b$ c	e_1+e_2 e_1-e_2 e_3
Σ_3	$\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$	$a+2b$ $a-b$ (two times)	$e_1+e_2+e_3$ $-2e_1+e_2+e_3$ $e_1-2e_2+e_3$

Table 2: General matrices commuting with the axial subgroups of $\mathbf{Z}_2 \wr \mathbf{S}_3$, together with the corresponding eigenvalues and eigenvectors.

Orbit representative	Isotropy subgroup
$(x,0,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_1 = \Sigma(\mathbf{1}, \{1\}) = \mathbf{1} \times (\mathbf{Z}_2)^2$
(x,x,x) $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_3 = \Sigma(\mathbf{1}, \{1,2,3\}) = \mathbf{1}^3 \dot{+} \mathbf{Z}_3$

Table 3: Orbit representatives and axial subgroups of $\mathbf{Z}_2 \wr \mathbf{Z}_3$.

Axial group	Commuting matrix	Eigenvalues	Eigenvectors
Σ_1	$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$	a b c	e_1 e_2 e_3
Σ_3	$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$	$a+b+c$ Two real or complex eigenvalues	$e_1+e_2+e_3$

Table 4: General matrices commuting with the axial subgroups of $\mathbf{Z}_2 \wr \mathbf{Z}_3$, corresponding eigenvalues and eigenvectors.

stable, cycles of saddle connections to be created by bifurcation when a group invariant equilibrium in a Γ-symmetric system loses stability. See also Section 4.

Orbit representative	Isotropy subgroup
$(x,0,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_1 = \Sigma(\mathbf{Z}_2(\kappa), \{1\}) = \mathbf{Z}_2(\kappa) \times (\mathbf{D}_3)^2 \dot{+} \mathbf{S}_1 \times \mathbf{S}_2$
$(x,x,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_2 = \Sigma(\mathbf{Z}_2(\kappa), \{1,2\}) = (\mathbf{Z}_2(\kappa))^2 \times \mathbf{D}_3 \dot{+} \mathbf{S}_2 \times \mathbf{S}_1$
(x,x,x) $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_3 = \Sigma(\mathbf{Z}_2(\kappa), \{1,2,3\}) = (\mathbf{Z}_2(\kappa))^3 \dot{+} \mathbf{S}_3$

Table 5: Orbit representatives and axial subgroups of $\mathbf{D}_3 \wr \mathbf{S}_3$.

Using the notation of Section 3, the axial groups are presented in table 3. Note that for $\mathcal{G} = \mathbf{Z}_3$ the only blocks we need to consider are $\{1\}$ and $\{1,2,3\}$. Also $\{1,2\}$ is not a block as it was for the previous group $\mathbf{Z}_2 \wr \mathbf{S}_3$. The most general matrices commuting with these axial groups, and the corresponding eigenvalues are obtained in table 4. The letters $a,\ b,\ c$ denote real entries and again $e_1,\ e_2, e_3$ denote the vectors of the canonical basis of $\mathbf{R}^3$. Again, under the conditions of Proposition 4.1, for $\mathcal{L} = \mathbf{Z}_2,\ \mathcal{G} = \mathbf{S}_3$ and $V = \mathbf{R}$, consider each matrix of table 4 representing $(dF)_{X_0}$ where X_0 represents a zero of $F(X,\lambda)$ with symmetry each of the groups Σ_i. For $\Sigma_1,\ a = \left(\frac{\partial F_1}{\partial X_1}\right)_{X_0},\ b = \left(\frac{\partial F_2}{\partial X_2}\right)_{X_0},\ c = \left(\frac{\partial F_3}{\partial X_3}\right)_{X_0}$, and for $\Sigma_3,\ a = \left(\frac{\partial F_1}{\partial X_1}\right)_{X_0},\ b = \left(\frac{\partial F_1}{\partial X_2}\right)_{X_0},\ c = \left(\frac{\partial F_1}{\partial X_3}\right)_{X_0}$.

Example 3: $\mathbf{D}_3 \wr \mathbf{S}_3$.

Consider the standard action of $\mathbf{D}_3$ acting on $V = \mathbf{C}$ generated by

$$\kappa \cdot z = \overline{z},$$
$$\xi \cdot z = e^{i\frac{2\pi}{3}} z.$$

The only axial group of $\mathbf{D}_3$ that we need to consider, up to conjugacy, is $\mathbf{Z}_2(\kappa) = \{1,\kappa\}$. Consider the corresponding action of $\Gamma = \mathbf{D}_3 \wr \mathbf{S}_3$ on V^3 (as defined in Section 2). Note that $\mathbf{D}_3$ and $\mathbf{S}_3$ are isomorphic as abstract groups. The space V^3 is absolutely irreducible for Γ (and V is $\mathbf{D}_3$-absolutely irreducible). Using the notation of Section 3, the axial groups of Γ are shown in table 5. We take coordinate functions on V^3:

$$z_1,\ \overline{z}_1,\ z_2,\ \overline{z}_2,\ z_3,\ \overline{z}_3.$$

These correspond to a basis B for V^3, the elements of which we denote by:

$$l_1,\ \bar{l}_1,\ l_2,\ \bar{l}_2,\ l_3,\ \bar{l}_3.$$

The commuting matrices for the axial groups, and the corresponding eigenvalues and eigenvectors, are listed in table 6. Here $a,\ a',\ b,\ c$ denote real numbers. Also, with the conditions of Proposition 4.1, if the matrices of table 6 represent

Axial group	Commuting matrix	Eigenvalues	Eigenvectors
Σ_1	$\begin{pmatrix} a & a' & & & & \\ a' & a & & & & \\ & & c & & & \\ & & & c & & \\ & & & & c & \\ & & & & & c \end{pmatrix}$	$a+a'$ $a-a'$ c (four times)	$l_1+\bar{l}_1$ $l_1-\bar{l}_1$ $l_2,\ \bar{l}_2,\ l_3,\ \bar{l}_3$
Σ_2	$\begin{pmatrix} a & a' & b & b & & \\ a' & a & b & b & & \\ b & b & a & a' & & \\ b & b & a' & a & & \\ & & & & c & \\ & & & & & c \end{pmatrix}$	$a+a'+2b$ $a-a'$ (two times) $a+a'-2b$ c (two times)	$l_1+l_2+\bar{l}_1+\bar{l}_2$ $l_1+l_2-\bar{l}_1-\bar{l}_2$ $l_1-l_2-\bar{l}_1+\bar{l}_2$ $l_1-l_2+\bar{l}_1-\bar{l}_2$ $l_3,\ \bar{l}_3$
Σ_3	$\begin{pmatrix} a & a' & b & b & b & b \\ a' & a & b & b & b & b \\ b & b & a & a' & b & b \\ b & b & a' & a & b & b \\ b & b & b & b & a & a' \\ b & b & b & b & a' & a \end{pmatrix}$	$a+a'+4b$ $a-a'$ (three times) $a+a'-2b$ (two times)	$l_1+l_2+l_3+\bar{l}_1+\bar{l}_2+\bar{l}_3$ $l_1+l_2+l_3-\bar{l}_1-\bar{l}_2-\bar{l}_3$ $l_1-l_2-\bar{l}_1+\bar{l}_2$ $l_1-l_3-\bar{l}_1+\bar{l}_3$ $l_1-l_2+\bar{l}_1-\bar{l}_2$ $l_1-l_3+\bar{l}_1-\bar{l}_3$

Table 6: General matrices commuting with the axial subgroups of $\mathbf{D}_3 \wr \mathbf{S}_3$, together with the corresponding eigenvalues and eigenvectors.

the derivative $(dF)_{X_0}$ (if X_0 is a zero of $F(X,\lambda)$ with symmetry Σ_i where F commutes with Γ), then for Σ_1, $a = \left(\frac{\partial F_1}{\partial z_1}\right)_{X_0}$, $a' = \left(\frac{\partial F_1}{\partial \bar{z}_1}\right)_{X_0}$ and $c = \left(\frac{\partial F_2}{\partial z_2}\right)_{X_0}$. For Σ_2, $a = \left(\frac{\partial F_1}{\partial z_1}\right)_{X_0}$, $a' = \left(\frac{\partial F_1}{\partial \bar{z}_1}\right)_{X_0}$, $b = \left(\frac{\partial F_1}{\partial z_2}\right)_{X_0}$, $c = \left(\frac{\partial F_3}{\partial z_3}\right)_{X_0}$. For Σ_3, $a = \left(\frac{\partial F_1}{\partial z_1}\right)_{X_0}$, $a' = \left(\frac{\partial F_1}{\partial \bar{z}_1}\right)_{X_0}$ and $b = \left(\frac{\partial F_1}{\partial z_2}\right)_{X_0}$.

Example 4: $\mathbf{O}(2) \wr \mathbf{S}_2$.
Consider the standard action of $\mathbf{O}(2) \times \mathbf{S}^1$ on $V = \mathbf{C} \oplus \mathbf{C}$:

$$\begin{array}{ll} \theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2) & (\theta \in \mathbf{S}^1), \\ \kappa \cdot (z_1, z_2) = (z_2, z_1) & (\kappa = \text{flip in } \mathbf{O}(2)), \\ \psi \cdot (z_1, z_2) = (e^{-i\psi} z_1, e^{i\psi} z_2) & (\psi \in \mathbf{SO}(2)). \end{array}$$

By [14] for example (Proposition XVII 1.1.) we have (up to conjugacy) two types of $\mathbf{C}$-axial subgroups of $\mathbf{O}(2) \times \mathbf{S}^1$ that are listed in table 7.

Orbit representative	Isotropy subgroup
$(x,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\widetilde{\mathbf{SO}}(2) = \{(\theta,\theta)\}$
(x,x) $(x \in \mathbf{R} \setminus \{0\})$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c = \{(1,0),\ (\kappa,0),\ (\pi,\pi),\ (\kappa\pi,\pi)\}$

Table 7: Orbit representatives and **C**-axial groups for the standard action of $\mathbf{O}(2) \times \mathbf{S}^1$.

Orbit representative	Isotropy subgroup	Generators
$(x,0,0,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_1 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1\})$	$\mathbf{1} \times \mathbf{O}(2),\ \{((\theta,1),1,\theta)\}$
$(x,x,0,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_2 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1\})$	$\mathbf{1} \times \mathbf{O}(2),\ (\kappa,1),\ ((\pi,1),1,\pi)$
$(x,0,x,0)$ $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_3 = \Sigma(\widetilde{\mathbf{SO}}(2), \{1,2\})$	$\{((\theta,\theta),1,\theta)\},\ \mathbf{S}_2$
(x,x,x,x) $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_4 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\})$	$(\kappa,1),\ ((\pi,\pi),1,\pi),\ \mathbf{S}_2$
(x,x,ix,ix) $(x \in \mathbf{R} \setminus \{0\})$	$\Sigma_5 = \Sigma(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c, \{1,2\}, (12), \{1\}, 4)$	$(\kappa,1),\ (1,\kappa),\ ((\pi,\pi),1,\pi)$ $((\pi,1),(12),\frac{\pi}{2})$

Table 8: Orbit representatives and **C**-axial groups of $(\mathbf{O}(2) \wr \mathbf{S}_2) \times \mathbf{S}^1$.

Consider the corresponding action of $\Gamma \times \mathbf{S}^1 = (\mathbf{O}(2) \wr \mathbf{S}_2) \times \mathbf{S}^1$ on V^2 (as defined in Section 2). This action of $\Gamma \times \mathbf{S}^1$ on V^2 is isomorphic to the action of $(\mathbf{D}_4 \dot{+} \mathbf{T}^2) \times \mathbf{S}^1$ presented in [19].

Note that as V is $\mathbf{O}(2)$-simple, also V^2 is Γ-simple.

Using the notation of Section 3, the **C**-axial groups of $\Gamma \times \mathbf{S}^1$ are shown in table 8. See [4] for details. The subgroups $\Sigma_1, \ldots, \Sigma_5$ correspond with the **C**-axial subgroups obtained in [19].

We take coordinate functions on V^2:

$$z_1,\ \overline{z}_1,\ z_2,\ \overline{z}_2,\ z_3,\ \overline{z}_3,\ z_4,\ \overline{z}_4.$$

These correspond to a basis B for V^2, the elements of which we denote by:

$$l_1,\ \bar{l}_1,\ l_2,\ \bar{l}_2,\ l_3,\ \bar{l}_3,\ l_4,\ \bar{l}_4.$$

Recall that an **R**-linear mapping on **C** has the form

$$w \longmapsto \alpha w + \beta \overline{w},$$

C-axial group	Commuting matrix		New basis	Blocks
Σ_1	$\begin{pmatrix} A & & & \\ & B & & \\ & & C & \\ & & & C \end{pmatrix}$	$B = \begin{pmatrix} b_1 & 0 \\ 0 & \overline{b}_1 \end{pmatrix}$ $C = \begin{pmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{pmatrix}$		A B C C
Σ_2	$\begin{pmatrix} A & B & & \\ B & A & & \\ & & C & D \\ & & D & C \end{pmatrix}$	$C = \begin{pmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{pmatrix}$ $D = \begin{pmatrix} 0 & d_1^{'} \\ \overline{d}_1^{'} & 0 \end{pmatrix}$	$l_1 + l_2,\ c.c.$ $l_1 - l_2,\ c.c.$ $l_3 + l_4,\ c.c.$ $l_3 - l_4,\ c.c.$	$A + B$ $A - B$ $C + D$ $C - D$
Σ_3	$\begin{pmatrix} A & & B & \\ & C & & D \\ B & & A & \\ & D & & C \end{pmatrix}$	$C = \begin{pmatrix} c_1 & 0 \\ 0 & \overline{c}_1 \end{pmatrix}$ $D = \begin{pmatrix} d_1 & 0 \\ 0 & \overline{d}_1 \end{pmatrix}$	$l_1 + l_3,\ c.c.$ $l_2 + l_4,\ c.c.$ $l_1 - l_3,\ c.c.$ $l_2 - l_4,\ c.c.$	$A + B$ $C + D$ $A - B$ $C - D$
Σ_4	$\begin{pmatrix} A & B & C & C \\ B & A & C & C \\ C & C & A & B \\ C & C & B & A \end{pmatrix}$		$l_1 - l_2,\ c.c.$ $l_3 - l_4,\ c.c.$ $l_1 + l_2 + l_3 + l_4,\ c.c.$ $l_1 + l_2 - l_3 - l_4,\ c.c.$	$A - B$ $A - B$ $A + B + 2C$ $A + B - 2C$
Σ_5	$\begin{pmatrix} A & B & C^{'} & C^{'} \\ B & A & C^{'} & C^{'} \\ C & C & A^{'} & B^{'} \\ C & C & B^{'} & A^{'} \end{pmatrix}$	$A^{'} = \begin{pmatrix} a_1 & -a_1^{'} \\ -\overline{a}_1^{'} & \overline{a}_1 \end{pmatrix}$ $B^{'} = \begin{pmatrix} b_1 & -b_1^{'} \\ -\overline{b}_1^{'} & \overline{b}_1 \end{pmatrix}$ $C^{'} = \begin{pmatrix} -c_1 & c_1^{'} \\ \overline{c}_1^{'} & -\overline{c}_1 \end{pmatrix}$ $C^{''} = \begin{pmatrix} -c_1 & -c_1^{'} \\ \overline{c}_1^{'} & \overline{c}_1 \end{pmatrix}$	$l_1 - l_2$ $c.c.$ $l_3 - l_4$ $c.c.$ $l_1 + l_2 + i(l_3 + l_4)$ $c.c.$ $l_1 + l_2 - i(l_3 + l_4)$ $c.c.$	$A - B$ $A^{'} - B^{'}$ $A + B + 2iC^{''}$ $A + B - 2iC^{''}$

Table 9: General matrices commuting with the **C**-axial subgroups of $(\mathbf{O}(2) \wr \mathbf{S}_2) \times \mathbf{S}^1$.

where α and β are complex and the matrix of this mapping in $z,\ \overline{z}$ coordinates is $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$. See for example [14] Lemma XIV 3.1.

The commuting matrices for the **C**-axial groups, a new basis with respect to which the new matrix is block diagonal where each block is a 2×2 matrix, and the corresponding blocks are presented in table 9. The matrices $A, B, \ldots,$ always denote 2×2 matrices of the form

$$\begin{pmatrix} a_1 & a_1^{'} \\ \overline{a}_1^{'} & \overline{a}_1 \end{pmatrix}, \ \ldots$$

with $a_1, \overline{a}_1, \ldots \in \mathbf{C}$, unless otherwise stated. Blank entries are equal to zero. Also "c.c." denotes the conjugate of the vector that precedes it. Recalling Remark 4.4, with the conditions of Proposition 4.3, we can consider each commuting matrix in table 9 as representating the derivative of the reduced map $G(X, \lambda, \tau)$ (after Liapunov-Schmidt reduction of $\dot{X} = F(X, \lambda)$ where F commutes with $\Gamma = \mathbf{O}(2) \wr \mathbf{S}_2$ and $\mathbf{S}^1$) at each zero (X_0, λ_0, τ_0) of $G(X, \lambda, \tau)$ corresponding to a periodic solution of $\dot{X} = F(X, \lambda)$ with symmetry Σ_i. For $\Sigma_1,\ a_1 = \left(\frac{\partial F_1}{\partial z_1}\right)_{X_0},\ a_1^{'} = \left(\frac{\partial F_1}{\partial \overline{z}_1}\right)_{X_0},\ b_1 = \left(\frac{\partial F_2}{\partial z_2}\right)_{X_0},\ c_1 = \left(\frac{\partial F_3}{\partial z_3}\right)_{X_0}$. For $\Sigma_2,\ a_1 = \left(\frac{\partial F_1}{\partial z_1}\right)_{X_0},\ a_1^{'} = \left(\frac{\partial F_1}{\partial \overline{z}_1}\right)_{X_0},\ b_1 = \left(\frac{\partial F_1}{\partial z_2}\right)_{X_0},\ b_1^{'} = \left(\frac{\partial F_1}{\partial \overline{z}_2}\right)_{X_0},\ c_1 = \left(\frac{\partial F_3}{\partial z_3}\right)_{X_0},\ d_1^{'} = \left(\frac{\partial F_3}{\partial \overline{z}_4}\right)_{X_0}$, etc.

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