SPECTRUM OF THE ELIMINATION OF LOOPS AND MULTIPLE ARROWS IN COUPLED CELL NETWORKS

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Abstract. A *uniform lift* of a given network is a network with no loops and no multiple arrows that admits the first network as quotient. Given a regular network (in which all cells have the same type and receive the same number of inputs and all arrows have the same type) with loops or multiple arrows, we prove that it is always possible to construct a uniform lift whose adjacency matrix has only two possible eigenvalues, namely, 0 and -1, in addition to all eigenvalues of the initial network adjacency matrix. Moreover, this uniform lift has the minimal number of cells over all uniform lifts. We also prove that if a non-vanishing eigenvalue of the initial adjacency matrix is fixed then it is always possible to construct a uniform lift that preserves the number of eigenvalues with the same real part of that eigenvalue. Finally, for the eigenvalue zero we show that such a construction is not always possible proving that there are networks with multiple arrows whose uniform lifts all have the eigenvalue 0, in addition to all eigenvalues of the initial network adjacency matrix.

Using the concept of ODE-equivalence, we prove then that it is always possible to study a degenerate bifurcation arising in a system whose regular network has multiple arrows as a bifurcation of a bigger system associated with a regular uniform network.

1. Introduction

We start by recalling briefly a few facts concerning the theory of (regular) coupled cell networks and quotients developed by Stewart, Golubitsky and coworkers [13, 8]. A *cell* is a system of ordinary differential equations and a *coupled cell system* is a finite collection of interacting cells. A coupled cell system can be associated to a *network*,

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a directed graph whose nodes represent cells and whose arrows represent couplings between cells. The general theory allows for *loops* and *multiple arrows*. All couplings of the same type between two cells are represented by a single arrow with the number of couplings attached to it, unless this number is equal to 1, in which case it is simply omitted. The general theory associates a class of admissible vector fields to each network.

In this paper we restrict attention to *regular networks*, that is, networks associated with coupled cell systems where all cells have the same differential equation (up to reordering of coordinates) and one kind of coupling. In this case, the state spaces of the cells are all identical, say a Euclidean space R^k, with $k \geq 1$, and so, if the network has n cells then the *total phase space* is $(R^k)^n$. The *valency* of a regular network is the number of arrows that input to each cell. The j-th coordinate of an admissible vector field of a regular network with valency m has the form

$$\dot{x}_j = f(x_j; \overline{x_{i_1}, \cdots, x_{i_m}}),$$

where the overbar indicates that the coupling coordinates are invariant under permutations of the coupling cells, which is due to the uniqueness of the kind of coupling.

A *polydiagonal* is a subspace of the network phase space that is defined by the equalities of certain cell coordinates. A *synchrony subspace* is a polydiagonal that is flow invariant for every admissible vector field. In Golubitsky *et al.* [8] it is proved that every coupled cell system associated with a network when restricted to a synchrony subspace corresponds to a coupled cell system associated to a smaller network, called the *quotient network*. If Q is a quotient network of a network G then we say that G is a *lift* of Q.

1.1. **Spectrum of regular networks.** The *adjacency matrix* of a n-cell regular network is a $n \times n$ matrix $A = [a_{ij}]$ where a_{ij} is the number of arrows from cell j to cell i. Note that as the network is regular, each row sum of the adjacency matrix equals the network valency.

The *spectrum of a matrix* is the multiset of its eigenvalues. The *spectrum of a regular network* G is the spectrum of its adjacency matrix and it is denoted by S_G. We also refer to the eigenvalues of the adjacency matrix as the *network eigenvalues*.

Observe that the network adjacency matrix can be seen as the matrix of a linear admissible vector field. Thus, from the definition of *quotient* it follows that all eigenvalues of a (quotient) regular network are eigenvalues of any lift (including multiplicities).

Definition 1.1. Consider a lift G of a regular network Q. The elements of $S_G - S_Q$ are the *extra eigenvalues* (of G with respect to Q). Given an eigenvalue λ of Q, the lift G is λ-*preserving* (for Q) if the real part of any extra eigenvalue is distinct from the real part of λ. The lift G is *spectrum-preserving* (for Q) if G is λ-preserving for every $\lambda \in S_Q$.

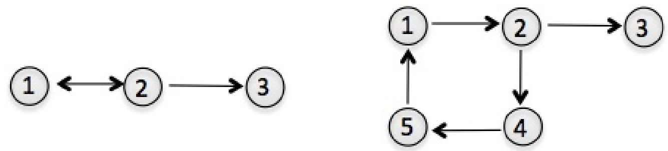


FIGURE 1. The 5-cell network on the right is a lift of the 3-cell network on the left that is not spectrum-preserving.

In Figure 1 we present a 3-cell network Q and one of its 5-cell lifts, G. For these networks, $S_G = [-1, -i, 0, i, 1]$ and $S_Q = [-1, 0, 1]$ and so, the extra eigenvalues are i and $-i$ and G is not 0-preserving (therefore, G is not spectrum-preserving).

In this paper we consider the issue of spectrum-preserving on the construction of uniform lifts (in which all arrows are simple and loops do not occur) of a given regular network.

We finish the introduction by presenting two problems that motivated our work on the issue of spectrum-preserving.

1.2. **Motivation 1.** Given a n-cell regular network G, consider a one-parameter system of ordinary differential equations

$$\dot{x} = F(x, \lambda), \tag{1}$$

where $F : (R^k)^n \times R \to (R^k)^n$ is a smooth G-admissible vector field and λ is the bifurcation parameter. Suppose that $F(0, 0) = 0$, that is, for $\lambda = 0$, the origin is a full synchronous equilibrium and let $J = (dF)_{0,0}$. Codimension 1 local bifurcations of (1) divide into *steady-state bifurcations* (if J has a zero eigenvalue) and *Hopf bifurcations* (if J has purely imaginary eigenvalues). These bifurcation types are *synchrony preserving* or *synchrony breaking*, depending whether the center subspace is contained or not, respectively, in the synchrony subspace. Bifurcation theory has been applied to the theory of coupled cell networks in various ways. See for example [9, 7, 5, 12].

Assume Q is a quotient of G determined by a synchrony subspace Δ and impose a degeneracy condition on F, implying that the center subspace of $J|_\Delta$ is nontrivial. We may now ask about the impact of that degeneracy condition at the bifurcation problem (1). This issue was raised by Aguiar *et al.* [5], and examples are given where, assuming a bifurcation occurs for a coupled cell system restricted to fixed (quotient) network, new branches of solutions occur in some of the quotient lifts. A necessary condition for new branches of bifurcating solutions to exist is the increasing of the dimension of the center subspace of J comparatively to $J|_\Delta$.

The results given by Leite and Golubitsky [9] and Golubitsky and Lauterbach [7] relate the eigenvalues of the Jacobian J of a system with the eigenvalues of the adjacency matrix of the system's network. More specifically, if $\mu_1, \ldots, \mu_n$ are the eigenvalues of the adjacency matrix of the network then the kn eigenvalues of the Jacobian J are the union of the eigenvalues of the $k \times k$ matrices $\alpha + \beta\mu_i$, for $1 \leq i \leq n$, where α is the $k \times k$ matrix of the linearized internal dynamics at the origin and β is the $k \times k$ matrix of the linearized coupling at the origin, both matrices found by differentiating F.

In [7] it is proved that when the dimension k of the internal dynamics of each cell is at least 2, then generically the center subspace at a synchrony breaking bifurcation is isomorphic to the real part of a generalized eigenspace of the adjacency matrix. However, for $k = 1$, every eigenvalue of the Jacobian J has the form $\alpha + \beta\mu_i$, for $1 \leq i \leq n$, where now $\alpha, \beta \in R$, and so, two eigenvalues $\alpha + \beta\mu_i$ and $\alpha + \beta\mu_i$ of the Jacobian J lie on the imaginary axis if and only if the eigenvalues μ_i and μ_j of the adjacency matrix have the same real part. In this sense, the issue of preserving the number of eigenvalues in the imaginary axis is translated, in terms of the adjacency matrix, into the preservation of the number of eigenvalues with a specific real part. Hence, if a lift of a regular network is spectrum-preserving, then no new branches of solutions can occur for the lift equations, assuming that a synchrony breaking bifurcation occurs for the quotient equations.

1.3. **Motivation 2.** This research was motivated by Stewart and Golubitsky [12]. In [12], examples are given where degenerate bifurcations occur in networks with few cells, but arrows of high multiplicity. They explain that multiple arrows can be removed by appealing to the Lifting Theorem, see Stewart [10, 11], which proves that any network with loops and multiple arrows is a quotient of a network with no loops and no multiple arrows. Therefore, the application of the Lifting Theorem to any of their examples leads to a conventional single-arrow network. They argue that the corresponding bifurcation for the lifted network remains degenerate saying that "even if that eigenvalue is not simple in the lifted network, the lift will have a degenerate branch of equilibria because the eigenvalue in the original network is simple". Here we prove, using ODE-equivalence, that the original bifurcation can in fact be studied as a bifurcation of a bigger system. Therefore, we prove that all examples of degenerate bifurcations constructed in [12] can indeed arise in conventional uniform networks. However, if the bifurcation is associated with the eigenvalue 0 of the quotient (adjacency matrix), instead of applying the Lifting Theorem directly to the multiple-arrow network that is being considered, we apply the Lifting Theorem to another network that is obtained from the previous by adding a suitable number of loops.

1.4. **Structure of the paper.** In Section 2 we introduce a crucial idea of this paper: all lifts of a given network can be interpreted as resulting from a cellular splitting of the original network. Our methods use Proposition 2.2 of [5] concerning the construction of the lifts of a quotient network. In Section 3 we prove the main theorem of this work, namely, Theorem 3.4, which says that, given a regular network with loops or multiple arrows, it is always possible to construct a uniform lift having only two possible extra eigenvalues, 0 and -1, and that these lifts can have the minimal number of cells over all uniform lifts. We also prove, in Theorem 3.7, that given an eigenvalue λ of a regular network with loops or multiple arrows lying outside the imaginary axis, it is possible to construct a λ-preserving uniform lift. Finally, for eigenvalues on the imaginary axis we prove that such a construction is not always possible, showing that there are networks with multiple arrows whose uniform lifts all have 0 as extra eigenvalue (Theorem 3.9). In Section 4 we apply our results to bifurcation analysis obtaining Theorem 4.1, which answers a question raised by Stewart and Golubitsky in [12]: we prove that any bifurcation arising in a system associated to a network with loops or multiple arrows can always be seen as a bifurcation arising in a bigger system associated to a uniform network.

2. Lift as splitting of cells

Recall that given networks G and Q, we say that G is a *lift* of Q when Q is a quotient of G, that is, when there is a synchrony subspace Δ such that the restrictions of the admissible vector fields for G to Δ are the admissible vector fields for Q.

In this work, a lift is interpreted as resulting from the cellular splitting of the initial network. For example, all lifts with exactly one more cell than the initial network result from the splitting of exactly one of its cells into two cells. A cell that splits is called a *splitting cell* and every cell resulting from this splitting is called a *splitted cell*. Sometimes we are referring to a splitting cell as the *old* cell and to the splitted cells as the *new* cells.

The term "splitting of cells" is new but the concept is not and, in fact, it was already used in some of the referenced literature. Indeed, in [5] it is developed an algorithm that enumerates all networks, defined by their adjacency matrices, that admit a given quotient regular network. Aguiar *et al.* [4] and Agarwal [1] present the concept of "p-fold inflation of a cell" that corresponds to the splitting of that cell into p cells. In [11], in order to eliminate loops and multiple arrows, Stewart "expands a cell to a set of cells" which also corresponds to the splitting of a cell.

2.1. **The Reduction Method.** We describe now a systematic method to construct lifts from regular networks. As mentioned above, a polydiagonal subspace of a network phase space is a synchrony subspace if it is flow-invariant for every admissible vector field for the network. We can associate with a polydiagonal subspace an equivalence relation on the set of cells of the network where two cells i and j are in the same class when $x_i = x_j$ is one of the conditions defining the polydiagonal. Theorem 4.3 of [8] states that a polydiagonal is a synchrony subspace if and only if the equivalence relation is *balanced*, that is, two cells in the same class receive the same number of inputs from cells that are in the same class. It follows then that the description of the synchrony spaces for a network can be given by enumerating these balanced equivalence relations. This enumeration depends only on the network and it is equivalent to the enumeration of the polydiagonal invariant spaces of the network adjacency matrix (interpreted as a linear operator on R^n, if n is the number of cells). Having this in mind, we recall Proposition 2.2 of [5] that we will use when enumerating adjacency matrices of networks that admit a fixed quotient network:

Proposition 2.1 ([5]). *Let G be a n-cell regular network with adjacency matrix A_G, whose i-th column we denote by A_{G_i}. Let Δ be a polydiagonal of the total phase space and $I_1, \ldots, I_p$ be the classes of the equivalence relation defined on the set of cells by setting two cells i and j in the same class when $x_i = x_j$ is one of the conditions defining Δ. Consider the $(n \times p)$-matrix $\overline{A_G}$ whose j-th column (for $j = 1, \ldots, p$) is defined by*

$$C_j = \sum_{i \in I_j} A_{G_i}.$$

The space Δ is a synchrony subspace if and only if for all j_1, j_2 such that $x_{j_1} = x_{j_2}$ is in Δ, the rows j_1 and j_2 of $\overline{A_G}$ are identical.

This proposition provides a very easy method to test if a given polydiagonal is a synchrony subspace. We call it the *Reduction method*. For example, using this method, to verify that $\Delta = \{x_1 = x_4, x_2 = x_5\}$ is a synchrony subspace for the 5-cell network G on the right of Figure 1, as the adjacency matrix is

$$A_G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

it amounts to verify that the lines of each pair, $1, 4$ and $2, 5$, of the matrix

$$\overline{A_G} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

are identical. Observe that the 3-cell network on the left of Figure 1 is the quotient network of G determined by the synchrony subspace Δ.

Consider a regular network Q and assume that some of its cells splits. In order to obtain a lift G after the splitting, the cellular splitting has to satisfy the following property:

> **Fundamental Property of the splittings**: Assume that i is a cell that receives k arrows from cell j and that at least one of these two cells splits. There are three cases to be considered:
> **(1)** i **splits but** j **does not:** after the splitting, each splitted cell associated to cell i receives k arrows from cell j.
> **(2)** j **splits but** i **does not:** after the splitting, cell i receives k cells from the set of splitted cells associated to cell j.
> **(3) both** i **and** j **split:** after the splitting, each splitted cell associated to cell i receives k cells from the set of splitted cells associated to cell j.

Remark 2.2. If $\tilde{A} = [\tilde{a}_{ij}]$ is the adjacency matrix of the given regular network Q and $A = [a_{ij}]$ is the adjacency matrix of the corresponding lifted network G, the Fundamental Property of the splittings described above relates $\tilde{A}$ and A. In fact, assume cell i receives k arrows from cell j in Q, that is, $\tilde{a}_{ij} = k$. Then:

(1) if i and j are not splitting cells then $k = \tilde{a}_{ij} = a_{ij}$, that is, cell i continues to receive k arrows from cell j.
(2) if i is not a splitting cell and j is a splitting cell then $k = \tilde{a}_{ij} = \sum_{k=1}^{m} a_{ij_k}$, where $j_1, \ldots, j_m$ are the splitted cells associated to

j. This means that i receives k arrows from the set of splitted cells associated to j

(3) if i is a splitting cell and j is not a splitting cell then $k = \tilde{a}_{ij} = a_{i_1 j} = \cdots = a_{i_n j}$, where $i_1, \ldots, i_n$ are the splitted cells associated to i. This means that each splitted cell associated to i receives k arrows from j.

(4) if i and j are both splitting cells then $k = \tilde{a}_{ij} = \sum_{k=1}^{m} a_{i_1 j_k} = \cdots = \sum_{k=1}^{m} a_{i_n j_k}$, where $i_1, \ldots, i_n$ and $j_1, \ldots, j_m$ are defined as previously. This means that each splitted cell associated to cell i receives k arrows from the set of splitted cells associated to cell j.

See also Theorem 2.5 of [5].

Example 2.3. In Figure 2 there is a 3-cell regular network Q and one of its 6-cell lifts, obtained by splitting both cells 1 and 2.

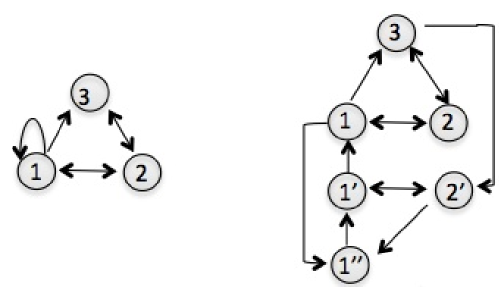


FIGURE 2. The 6-cell network is a lift of the 3-cell regular network on the left.

2.2. **Result.** We show now that the extra eigenvalues of a lift G with respect to a (quotient) regular network Q can be described using one subnetwork of G and one subnetwork of Q.

Definition 2.4. Consider a network and a subset C of cells. The *subnetwork* consisting of all cells in C is the digraph whose nodes are all cells in C and whose arrows are all existing arrows in the original network whose heads and tails are cells in C.

Example 2.5. In Figure 3 we present two subnetworks of the 6-cell network in Figure 2, one of them consisting of cells 1, $1'$, 2 and $2'$ and the other one consisting of cells 1, 2 and $2'$.

Note that it is possible for a subnetwork to be a disconnected digraph.

FIGURE 3. Two subnetworks of the 6-cell network in Figure 2.

Theorem 2.6. *Consider a lift G of a regular network Q. Let S be the subnetwork of Q of all splitting cells and let S' be the subnetwork of G of all splitted cells. The extra eigenvalues of G with respect to Q are precisely the extra eigenvalues of S' with respect to S.*

Proof. Let p and n be the number of cells in Q and G, respectively, and denote by A_G the adjacency matrix of G.

Let $\bowtie$ be the balanced equivalence relation on the set of cells of G with p classes $I_1, \ldots, I_p$. Choose $s_j \in I_j$, $j = 1, \ldots, p$, and let $S_j = I_j \backslash \{s_j\}$. Observe that cell s_j is a splitting cell if and only if $S_j \neq \emptyset$.

Following the lines of Theorem 2.9 of [5], we can interpret A_G as the matrix of a linear G-admissible vector field with respect to the canonical basis of R^n, say $b_c = \{e_1, \cdots, e_n\}$. In this case, $\left\{\sum_{i \in I_1} e_i, \ldots, \sum_{i \in I_p} e_i\right\}$ is a basis of the synchrony subspace Δ corresponding to the cellular splitting originating the lift G of Q. Observe that if cell j is a non-splitting cell of Q then $I_j = \{j\}$ and $\sum_{i \in I_j} e_i = e_j$. We have then that $A_G(\Delta) \subseteq \Delta$. Moreover,

$$B = \left\{ \sum_{i \in I_1} e_i, \ldots, \sum_{i \in I_p} e_i \right\} \bigcup \left\{ e_i : i \in \bigcup_{j=1}^{p} S_j \right\}$$

is a basis of R^n and A_G is similar to a matrix with the block form:

$$\begin{bmatrix} A_Q & B \\ 0_{(n-p)\times p} & M \end{bmatrix}$$

where A_Q is the adjacency matrix of Q. More precisely,

$$\begin{bmatrix} A_Q & B \\ 0_{(n-p)\times p} & M \end{bmatrix} = P^{-1} A_G P$$

where P is the change of basis (from B to b_c) matrix. Equivalently, the matrix A_G is transformed by doing inverse elementary operations over A_G:

First: For all j, $1 \leq j \leq p$, the column relative to s_j is replaced by the sum of all columns relative to the elements of I_j (multiplication of A_G by P on the right);

Second: For all j, $1 \leq j \leq p$, and for all $l \in S_j$, the l-th row is replaced by its difference with the row relative to s_j (multiplication of $A_G P$ by P^{-1} on the left).

We have then that the extra eigenvalues of A_G are the eigenvalues of M. Moreover, the above transformations show that the columns and lines in A_G relative to all non-splitting cells in Q play no role in the calculation of the matrix M and so, these lines and columns can be ignored in the calculation of all extra eigenvalues. That is, this calculation can be simplified by considering only the subnetwork of splitted cells of G, instead of all cells of this lift.

Formally, consider the subnetworks S and S' of splitting cells of Q and of splitted cells of G, respectively. Let k and m be the number of cells in S and S', respectively, and denote by A_S and $A_{S'}$ the adjacency matrices of S and S', respectively. We transform the matrix $A_{S'}$ into the matrix

$$A' = \begin{bmatrix} A_S & B \\ 0_{(m-k)\times k} & M \end{bmatrix}$$

by doing the above two inverse elementary transformations over $A_{S'}$.

Therefore, the extra eigenvalues of S' with respect to S are also the eigenvalues of M and the conclusion is straightforward. □

Example 2.7. Let Q and G be the networks in Figure 2 with 3 and 6 cells, respectively. The characteristic polynomial of the adjacency matrices for Q and G are $x^3 - x^2 - 2x$ and

$$x^6 - 3x^4 - 3x^3 + x^2 + 2x = (x^3 - x^2 - 2x)(x^3 + 2x - 1),$$

respectively. Hence, the extra eigenvalues of G with respect to Q are the three complex roots of the cubic $x^3 + 2x - 1$. In Figure 4 we present the subnetworks S and S' of splitting cells and of splitted cells, respectively. The characteristic polynomial of the corresponding adjacency matrices for S and S' are $x^2 - x - 1$ and

$$x^5 - 2x^3 - 2x^2 + x + 1 = (x^2 - x - 1)(x^3 + 2x - 1),$$

respectively. Hence, the extra eigenvalues of S' with respect to S are also the three roots of the previous cubic.

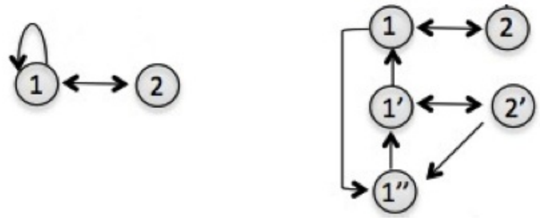


FIGURE 4. The subnetworks consisting of all splitting cells (on the left) and all splitted cells (on the right) for the networks in Figure 2.

3. ELIMINATION OF LOOPS AND MULTIPLE ARROWS

In this section we consider a regular non-uniform network, that is, a regular network with loops or multiple arrows. The Lifting Theorem proves that any non-uniform network is a quotient of a uniform network. In this paper, the process consisting in finding lifts with no loops of a given non-uniform network is called *elimination of loops* and, likewise, the process consisting in finding lifts with no multiple arrows is called *elimination of multiple arrows*.

We start explaining the reasoning behind the process of elimination of loops and multiple arrows. For more details see [11]. So, consider a regular network with loops and multiple arrows and let $A = [a_{ij}]$ be its adjacency matrix.

Consider a cell i with a loop and let $m = a_{ii} > 0$. The only possible way to eliminate this loop is splitting cell i and, by the Fundamental Property of the splittings, each splitted cell has to receive m arrows from the set of splitted cells. Therefore, if we require the minimality of cells, no loops and no multiple arrows, cell i has to be splitted into exactly $m + 1$ cells, say i (new), i', ..., $i^{(m)}$, in such a way that each splitted cell receives exactly one arrow from each of the other m distinct new cells (and, obviously, respecting all the other conditions coming from the Fundamental Property of the splittings).

Note that the minimal number of cells requires the splitting of cell i into $m+1$ cells, that is, any uniform lift has at least m cells more than the initial network. For instance, Figure 5 illustrates a network with a loop and an example (network on the center) of a uniform lift with the minimal number of cells resulting from the elimination of the loop.

Consider now a cell i sending multiple arrows and let $M = \max_l a_{li}$. The only possible way to eliminate all multiple arrows sent by this cell is splitting cell i and, by the Fundamental Property of the splittings, every cell that receives k arrows from cell i has to receive, after the splitting,

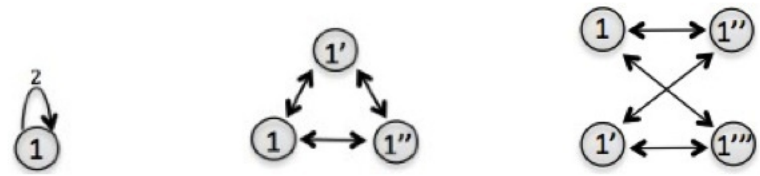


Figure 5. Two examples of lifts with no loops of the 1-cell network on the left.

k arrows from the set of splitted cells associated to i. Therefore, if we require the minimality of cells, no loops and no multiple arrows, cell i has to be splitted into exactly M cells, say i (new), i', ..., $i^{(M-1)}$, in such a way that all cells that used to receive a multiple arrow k from the old cell i, receive now exactly one arrow from k distinct new cells associated to the old cell i (and, obviously, respecting all the other conditions coming from the Fundamental Property of the splittings).

Note that the minimal number of cells requires the splitting of cell i into M cells, that is, any uniform lift has at least $M-1$ cells more than the initial network. For instance, Figure 6 illustrates a network with a multiple arrow and an example (network on the center) of a uniform lift with the minimal number of cells resulting from the elimination of the multiple arrow.

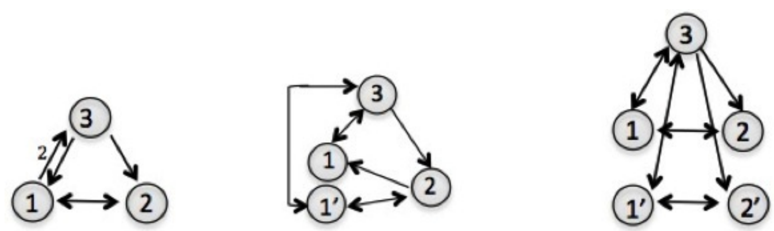


Figure 6. Two examples of lifts with no multiple arrows of the 3-cell network on the left.

Remark 3.1. Consider a regular non-uniform network with adjacency matrix $A = [a_{ij}]$. If there is a cell i that simultaneously has a loop and sends multiple arrows then the only possible way to eliminate both the loop and all multiple arrows associated to this cell is to split cell i into at least

$$\max\{M, a_{ii} + 1\}$$

cells, where $M = \max_l a_{li}$.

Proposition 3.2. *Given a regular network with loops, it is always possible to construct a lift with no loops and with the minimal number of cells over all lifts with no loops, where -1 is the unique extra eigenvalue.*

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of the given network. Consider a cell i with a loop and, to simplify the notation, let $m = a_{ii}$. As said before, in order to eliminate the loop we must split cell i into at least $m+1$ new cells. We split this cell into exactly $m+1$ cells, i (new), i', ..., $i^{(m)}$, and so, by the Fundamental Property of the splittings, up to renumeration, the adjacency matrix of the subnetwork of splitted cells is:

$$\begin{array}{c} \\ i \\ i' \\ i'' \\ \vdots \\ i^{(m)} \end{array} \begin{array}{c} \begin{array}{ccccc} i & i' & i'' & \cdots & i^{(m)} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ & & & \ddots & \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \end{array}.$$

This matrix has two distinct eigenvalues: m and -1, this last one with multiplicity m: this matrix is $J - I$, where J is the all-1 matrix an I is the identity matrix; as J has only $m + 1$ and 0 as eigenvalues, this last one with multiplicity m, and since λ is an eigenvalue of $J - I$ if and only if $\lambda + 1$ is an eigenvalue of J, we have that the eigenvalues of $J - I$ are as stated.

Besides, there is a unique splitting cell and so, the corresponding adjacency matrix is $[a_{ii}] = [m]$ which has a unique eigenvalue, m. Therefore, Theorem 2.6 guarantees that there is a unique extra eigenvalue: -1, with multiplicity m. The Optimal Lifting Theorem [11] guarantees that these lifts have the minimal number of cells over all uniform lifts and the first part is proved. □

This result guarantees that given a network with loops it is always possible to construct lifts with no loops that only have -1 as extra eigenvalue. However, the elimination of loops can be done in many different ways and so it is also possible to construct lifts with no loops that do not have -1 as extra eigenvalue, as we can see by the following example.

Example 3.3. In Column 2 from Table 1 we present networks with loops. In Column 3 we present examples of uniform lifts whose elimination of loops is done according to the proof of the previous proposition and so, this process produces the extra eigenvalue -1 in the lift. In Column 4 we present examples of uniform lifts that do not have -1 as extra eigenvalue.

Theorem 3.4. *Given a regular network with loops or multiple arrows, it is always possible to construct a uniform lift with the minimal number* *of cells over all uniform lifts where* -1 *and* 0 *are the unique possible extra eigenvalues.*

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of the given network. By Proposition 3.2 it is possible to eliminate all loops by introducing only -1 as extra eigenvalue. Therefore, we just have to prove that it is possible to eliminate all multiple arrows by introducing only -1 or

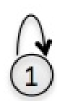
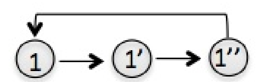
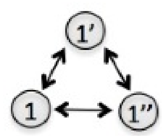
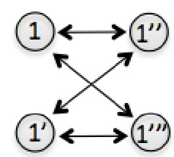
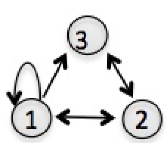
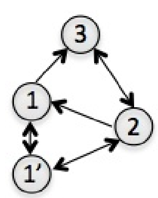
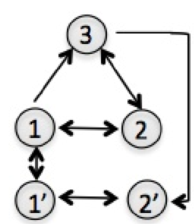
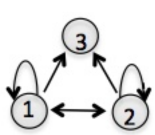
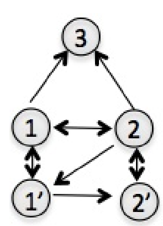
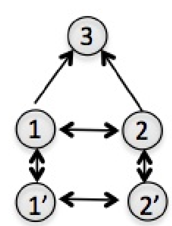
No.	Non-uniform network	Uniform lift: Example 1	Uniform lift: Example 2
1	 Eig.: 1	 Extra Eig.: -1	 Extra Eig.: $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
2	 Eig.: 2	 Extra Eig.: -1,-1	 Extra Eig.: -2, 0, 0
3	 Eig.: 2,0,-1	 Extra Eig.: -1	 Extra Eig.: $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$
4	 Eig.: 2,0,0	 Extra Eig.: -1, -1	 Extra Eig.: 0, -2

TABLE 1. Examples of uniform lifts where -1 is / is not an extra eigenvalue.

0 as extra eigenvalues. Consider a cell i sending multiple arrows. Let

$$M = \max_l a_{li} > 1.$$

As said before, in order to eliminate this multiple arrow we must split cell i into at least M new cells.

First case: $a_{ii} = 0$

In this case we can split i into exactly M cells, j (new), j', ..., $j^{(M-1)}$, in such a way that all cells that used to receive a multiple arrow $k \leq M$ from the old cell i depend now on k distinct new cells associated to the old cell i (and, obviously, respecting all the other conditions coming from the Fundamental Property of the splittings). Hence, because cell i does not have any loop, the adjacency matrix of the set of splitted cells is the zero matrix which has the unique zero eigenvalue with multiplicity M. Proceeding as before (Theorem 2.6) to determine the extra eigenvalues we obtain that 0 is the unique extra eigenvalue (with multiplicity $M-1$). Also in this case, the Optimal Lifting Theorem guarantees that these lifts have the minimal number of cells over all uniform lifts, that is, they have $(M-1)$ more cells.

Second case: $a_{ii} = m > 0$

In this case we use Proposition 3.2 to eliminate the loop by splitting cell i into exactly $m+1$ cells. In this process we can require the distribution of the tails of multiple arrows sent by i over different splitted cells because this request does not change the extra eigenvalue (Theorem 2.6) and permits to reduce the number associated to each multiple arrow. Doing this distribution we eliminate all multiple arrows if $m+1 \geq M$. However, if $m+1 < M$ then the splitted cells introduced to eliminate the loop are not sufficient to eliminate all multiple arrows and so we need more $M-(m+1)$ splitted cells. We introduce these new splitted cells according to the previous case and so, we introduce only 0 as extra eigenvalue. □

Remark 3.5. Theorem 3.4 implies that, for every eigenvalue λ of a regular network with loops or multiple arrows with real part distinct from 0 and -1, it is possible to construct a λ-preserving uniform lift. Hence, if the network has no eigenvalues with real part 0 or -1 then it is possible to construct a spectrum-preserving uniform lift.

The previous theorem guarantees that it is always possible to find uniform lifts that only have 0 or -1 as extra eigenvalue. However, the elimination of loops and multiple arrows can be done in many different ways and so, in many situations it is also possible to find uniform lifts that have different extra eigenvalues, as we can see by the following examples.

Example 3.6. In Column 2 from Table 2 we present networks with loops or multiple arrows. In Column 3 we present examples of uniform lifts that only have 0 or -1 as extra eigenvalues. In Column 4 we present examples of uniform lifts that have different extra eigenvalues.

Theorem 3.7. *Given an eigenvalue λ lying outside the imaginary axis of a regular network with loops or multiple arrows, it is possible to construct a λ-preserving uniform lift.*

Proof. Let $A = [a_{ij}]$ be the adjacency matrix of the given regular network with loops or multiple arrows. From Theorem 3.4 if λ has real part distinct from 0 and -1 then it is possible to construct a λ-preserving uniform lift. We prove now that if λ has real part -1 then it is possible to construct a -1-preserving uniform lift.

Consider a cell i with a loop and, to simplify the notation, let $m_i = a_{ii}$. As said before, in order to eliminate the loop we must split cell i into at least $m_i + 1$ new cells. This case is divided into two subcases: $m_i = 1$ and $m_i \neq 1$.

If $m_i = 1$ then we can split cell i into exactly three cells i, i' and i'' where the adjacency matrix of the subnetwork of splitted cells is:

$$\begin{array}{c} \\ i \\ i' \\ i'' \end{array} \begin{array}{c} \begin{array}{ccc} i & i' & i'' \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \end{array}.$$

By Theorem 2.6, the elimination of this loop introduces only $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ as extra eigenvalues (the non real roots of $z^3 - 1 = 0$).

If $m_i \neq 1$ then we split cell i into exactly $2m_i$ cells, i, i', ..., $i^{(2m_i-1)}$, where the adjacency matrix of the subnetwork of splitted cells is

$$\begin{array}{c} \\ i \\ \vdots \\ i^{(m_i-1)} \\ i^{(m_i)} \\ \vdots \\ i^{(2m_i-1)} \end{array} \begin{array}{c} \begin{array}{cccccc} i & \cdots & i^{(m_i-1)} & i^{(m_i)} & \cdots & i^{(2m_i-1)} \end{array} \\ \left(\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ & \vdots & & & \vdots & \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ & \vdots & & & \vdots & \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{array} \right) \end{array}.$$

By Theorem 2.6, the elimination of this loop introduces only $-m_i$ and 0 as extra eigenvalues, this last one with multiplicity $2m_i - 2$.

Consider a cell j sending multiple arrows and let

$$M = \max_{l} a_{lj} > 1. \qquad (2)$$

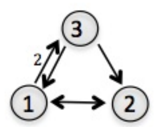
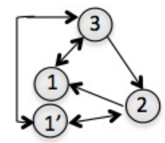
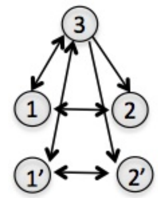
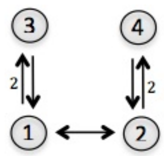
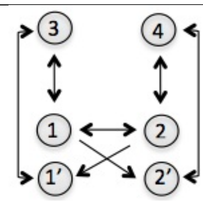
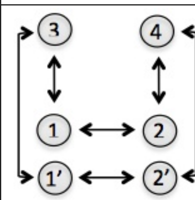
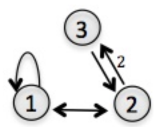
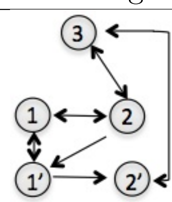
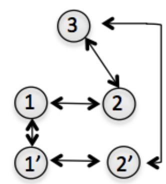
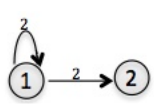
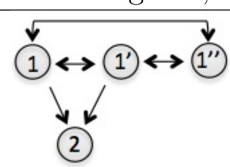
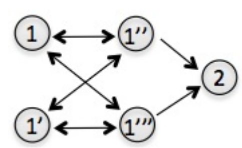
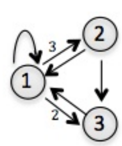
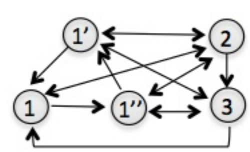
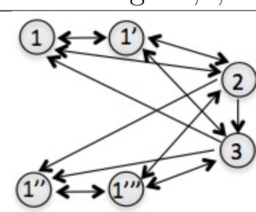
No.	Non-uniform network	Uniform lift: Example 1	Uniform lift: Example 2
1	 Eig.: 2,-1,-1	 Extra Eig.: 0	 Extra Eig.: ± 1
2	 Eig.: ± 2, ± 1	 Extra Eig.: 0,0	 Extra Eig.: ± 1
3	 Eig.: 2, $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$	 Extra Eig.: -1,0	 Extra Eig.: $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$
4	 Eig.: 2, 0	 Extra Eig.: -1,-1	 Extra Eig.: 0,0,-2
5	 Eig.: 3,-1,-1	 Extra Eig.: -1,0	 Extra Eig.: 1,-1,-1

Table 2. Examples of uniform lifts where -1 and 0 are and are not the unique possible extra eigenvalues.

If $a_{jj} = 0$ then, as shown in the proof of Theorem 3.4, it is possible to eliminate all multiple arrows sent by this cell by introducing only the eigenvalue 0. However, if $a_{jj} = m_j > 0$ then we eliminate the loop using the first part of this proof by splitting cell j into exactly s cells, where $s = 3$ if $m_j = 1$, and $s = 2m_j$ otherwise. In this process we can require the distribution of the tails of multiple arrows sent by j over different splitted cells. This request does not change the extra eigenvalue (Theorem 2.6) and permits to reduce the number associated to each multiple arrow. So, if $s \geq M$ then all multiple arrows are also eliminated. If $s > M$, the remaining multiple arrows are eliminated using the process described in the first case of the proof of Theorem 3.4, which only introduces the eigenvalue 0. $\square$

Remark 3.8. Given a regular network with loops or multiple arrows and a non-vanishing eigenvalue of the respective adjacency matrix, Theorem 3.7 implies that it is possible to construct a uniform lift where the multiplicity of the eigenvalue is preserved. In particular, if the eigenvalue is simple, it remains simple in the lift.

The following theorem identifies a class of networks whose uniform lifts all have 0 as extra eigenvalue. It allows, therefore, to conclude that in some cases it is impossible to construct λ-preserving uniform lifts if λ lies on the imaginary axis.

Theorem 3.9. *Given a regular network with loops or multiple arrows, if there is a cell that has no loops, that sends multiple arrows and that forms a trivial strongly connected component then all uniform lifts have 0 as extra eigenvalue.*

Proof. Consider a cell i in the conditions of this theorem, that is, it has no loops, it sends multiple arrows and it forms a trivial strongly connected component.

In order to eliminate the referred multiple arrow, cell i has to be splitted. Because i forms a strongly connected component, each splitted cell associated to this splitting cell forms a trivial strongly connected component which has no loops and, therefore, 0 is an extra eigenvalue of the corresponding lift. $\square$

Example 3.10. The network in Figure 7 has two distinct eigenvalues, namely, 2 and 0, the last one with multiplicity 3. Because cell 3 has no loops, forms a trivial strongly connected component and sends multiple arrows, all uniform lifts of this network have 0 as extra eigenvalue.

4. ODE-equivalence

Two networks are ODE-*equivalent* if they give rise to the same space of admissible vector fields (for a suitable choice of cell phase spaces)

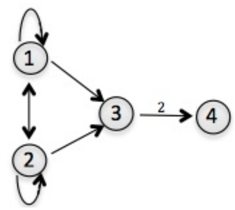


FIGURE 7. All uniform lifts of this network have 0 as extra eigenvalue.

([8, 6]). Therefore, given two ODE-equivalent networks, the set of bifurcations of the admissible vector fields for each of these two networks is the same. (See also the concept of *network equivalence* in [4] and Agarwal and Field [2, 3].)

We have seen that in the elimination of loops or multiple arrows it is not always possible to preserve the multiplicity of the eigenvalue 0, that is, given a non-uniform regular network having 0 as eigenvalue, there are situations where all uniform lifts have this eigenvalue as extra. In this section we prove that up to ODE-equivalence this impossibility can be overcome: we consider another non-uniform regular network which is ODE-equivalent to the previous one and which has eigenvalues with real part distinct from -1 and 0, for which it is possible to construct a uniform lift whose possible extra eigenvalues are only -1 and 0.

Theorem 4.1. *Given a regular network with loops or multiple arrows, there is always an ODE-equivalent network that admits a spectrum-preserving uniform lift.*

Proof. Let Q be the given n-cell regular network. Suppose $\mu_1, \ldots, \mu_n$ are the eigenvalues of its adjacency matrix A_Q. Take a nonnegative integer α such that

$$\alpha + \mathrm{Re}(\mu_i) \neq -1, 0, \qquad \text{for } i = 1, \ldots n.$$

Consider the network $\tilde{Q}$ obtained from Q adding α loops to each cell (if $\alpha = 0$ then $\tilde{Q} = Q$). Now $\tilde{Q}$ is a n-cell regular network with adjacency matrix

$$A_{\tilde{Q}} = A_Q + \alpha I_n,$$

and the eigenvalues of $A_{\tilde{Q}}$ are $\alpha + \mu_i$, $i = 1, \ldots, n$, which all have real part distinct from -1 and 0. By Dias and Stewart [6], the networks Q and $\tilde{Q}$ are ODE-equivalent since they are linearly independent. By Theorem 3.4, we conclude that $\tilde{Q}$ admits a spectrum-preserving uniform lift. $\square$

Example 4.2. In the previous example we presented a network (Figure 8 (a)) with eigenvalue 0 whose uniform lifts all have 0 as extra eigenvalue. Adding one loop to all cells in that network we obtain an ODE-equivalent network (Figure 8 (b)) whose eigenvalues are 3 and 1, the last one with multiplicity 3. Using Theorem 3.4 we can construct a spectrum preserving uniform lift and the network in Figure 8 (c) is an example of such a uniform lift.

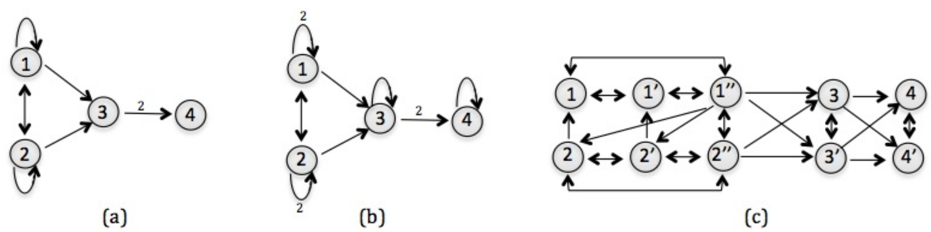


FIGURE 8. All bifurcations occurring in a coupled cell system associated to network (a) can be studied as bifurcations of coupled cell systems associated to networks (b) and (c).

One of the main applications of this result is in bifurcation analysis: assuming a bifurcation occurs for a coupled cell system defined by a regular network with loops or multiple arrows, the above result shows that the same bifurcation can be studied in a bigger coupled cell system defined by a regular network where arrows are single and loops do not occur. This matter appeared in [12] and was one of the main motivations of this work.

CONCLUSIONS

We show that it is always possible to construct uniform lifts of regular networks with loops or multiple arrows that only have 0 and -1 as extra eigenvalues. Besides, we can choose such lifts to have the minimal number of cells over all uniform lifts.

As a consequence, it is always possible to construct λ-preserving uniform lifts of regular networks, if λ has real part distinct from 0 and -1. We have also shown that the construction is possible when λ has real part -1. When λ lies on the imaginary axis we showed that it is not always possible to construct λ-preserving uniform lifts.

Nevertheless, for every regular network with loops or multiple arrows it is possible to consider an ODE-equivalent network that admits a spectrum-preserving uniform lift. This means that any steady-state or Hopf bifurcation problem associated to a network with loops or multiple arrows can be studied as a bifurcation problem associated to a uniform network. This result answers a question raised by Stewart and Golubitsky in [12].

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